INTERSECTIONAL PROPERTIES OF FAMILIES OF COMPACT CONVEX SETS

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PREFACE

This thesis is the study of certain intersectional properties of families of compact convex subsets of finite dimensional normed real linear spaces. Some of the results, especially those in Chapter I, are stated in a more general setting since it is just as easy to do so. The problems which are considered are centered around a theorem setting forth conditions under which the intersection of a family of convex sets cannot be empty. This famous theorem of Eduard Helly was discovered by him in 1913 and is referred to as Helly's theorem. A form of Helly's theorem is stated in Chapter II using the notation developed in this study.

The symbol R^n will denote the n-dimensional real linear space which consists of all n-tuples of real numbers. The symbol E^n denotes n-dimensional Euclidean space. The terminology, $\mathfrak{F}$ is a family of subsets of X, is used to mean that $\mathfrak{F} = \{A_\alpha : \alpha \in \Lambda\}$ for some index set Λ, and $A_\alpha \subset X$ for each $\alpha \in \Lambda$. Moreover, it is possible to have $A_\alpha = A_\beta$ for $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$. The rest of the symbolism and terminology used is either defined or is the same as that in Valentine [10]. The end of a proof is marked by the symbol ■ .

The first chapter concerns itself with a generalization of a well-known theorem about families of closed and compact sets with the finite intersectional property. In Chapter II certain intersectional properties of families of mutually parallel parallelotopes in E^n are studied. It should be pointed out that in some mathematical writings the term "parallelotope" always implies the existence of an interior point; however, this is not the case in this thesis. Chapter III is a study similar to that of Chapter II, except more general families of compact convex sets are considered.

I wish to express my appreciation to all those who assisted me in the preparation of this thesis. In particular, I would like to thank Professor E. K. McLachlan for his inspiration, advice and assistance. For their encouragement and cooperation while serving as members of my committee, my thanks goes to Professors Forrest Whitfield, Joe Howard and Earl Ferguson. Also, I want to thank the typist, Cynthia Wise, for the typing of this thesis, fellow student Gloria Gautier for proof reading a rough draft, and my wife, Judy, without whose help I could never have finished my work.

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CHAPTER I

FAMILIES OF COMPACT SETS

Let X be a topological space and r a positive integer. A nonempty family $\mathfrak{F}$ of subsets of X is said to have the r-intersectional property if there exist nonempty subfamilies $\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_r$ of $\mathfrak{F}$ such that

$$\text{(a)} \quad \mathfrak{F} = \bigcup_{i=1}^{r} \mathfrak{F}_i \quad \text{and}$$

$$\text{(b)} \quad \bigcap\{A: A \in \mathfrak{F}_i\} \neq \emptyset \quad \text{for} \quad 1 \leq i \leq r.$$

The families $\mathfrak{F}_1, \mathfrak{F}_2, \ldots, \mathfrak{F}_r$ give a finite partition of $\mathfrak{F}$ into r subfamilies. For example, if $\mathfrak{F}$ contains n sets, r could be n, and each $\mathfrak{F}_i$ could contain a single set of $\mathfrak{F}$. Thus, in general a finite family $\mathfrak{F}$ will always have the r-intersectional property with r equal to the cardinality of the family $\mathfrak{F}$. Notice that nothing requires that the subfamilies $\mathfrak{F}_i$ be disjoint. Consequently, $\mathfrak{F}_i$ might indeed be identical to $\mathfrak{F}_j$, $1 \leq i < j \leq r$. Thus, even if $\mathfrak{F}$ contains a finite number of sets, r can be arbitrarily large. Moreover, if n is a positive integer with $n \geq r$, then $\mathfrak{F}$ also has the n-intersectional property.

However, the minimum value for r is well-defined since the infimum of the possible numbers r is a positive integer. Let $|\mathfrak{F}|$ be the minimum value of r such that $\mathfrak{F}$ has the r-intersectional property.

If $|\mathfrak{F}| = 1$, the sets of $\mathfrak{F}$ have a nonempty intersection. It is also clear that if $\mathcal{G}$ is a nonempty subfamily of $\mathfrak{F}$, then $|\mathcal{G}| \leq |\mathfrak{F}|$. Thus, if $\mathfrak{F}$ has the r-intersectional property, then every nonempty subfamily of $\mathfrak{F}$ also has the r-intersectional property. If a family $\mathfrak{F}$ fails to have the r-intersectional property for any r, then $|\mathfrak{F}|$ is defined to be ∞. This is the case for example when $\mathfrak{F}$ is the family of all subsets of an infinite set.

Choose a point x_i from each of the nonempty sets $\cap\{A: A \in \mathfrak{F}_i\}$ in (b) and let $D = \{x_1, x_2, \ldots, x_r\}$. Then the set D may contain less than r distinct points. Moreover, given any $A \in \mathfrak{F}$ we have that A contains at least one point of the set D. Conversely, if $\mathfrak{F}$ is a nonempty family of subsets of X such that there exists a nonempty subset D of X containing no more than r points with the property that $D \cap A \neq \emptyset$ for all $A \in \mathfrak{F}$, then $\mathfrak{F}$ has the r-intersectional property. To see this, let $D \cap (\cup\{A: A \in \mathfrak{F}\}) = \{x_1, x_2, \ldots, x_k\}$. Then $k \leq r$ and each of the subfamilies $\mathfrak{F}_i = \{A \in \mathfrak{F}: x_i \in A\}$ are nonempty for $1 \leq i \leq k$. Moreover, $x_i \in \cap\{A: A \in \mathfrak{F}_i\}$. Thus, $\mathfrak{F}$ has the k-intersectional property, and since $k \leq r$, $\mathfrak{F}$ also has the r-intersectional property. Moreover, $|\mathfrak{F}| = r$ if and only if there exists a subset D of X containing r points such that $A \cap D \neq \emptyset$ for all $A \in \mathfrak{F}$, and no set with fewer than r points has this property. Grünbaum and others have defined a family $\mathfrak{F}$ to be r-pierceable if there exists a set D containing r points such that $A \cap D \neq \emptyset$ for all $A \in \mathfrak{F}$. Thus, a family $\mathfrak{F}$ of sets is r-pierceable if and only if $\mathfrak{F}$ has the r-intersectional property.

A well-known theorem about families of closed and compact subsets of X with the finite intersection property is as follows: Let $\mathfrak{F}$ be a family of closed and compact subsets of X such that each finite subfamily of $\mathfrak{F}$ has a nonempty intersection. Then the family $\mathfrak{F}$ has a nonempty intersection. In order to motivate the following two theorems, we state this theorem as follows: If $\mathfrak{F}$, a family of closed compact subsets of X, is such that $|\mathfrak{G}| \leq 1$ for each nonempty finite subfamily $\mathfrak{G}$ of $\mathfrak{F}$, then $|\mathfrak{F}| \leq 1$.

Really $|\mathfrak{G}|$ cannot be less than one, but the less or equal symbol is used to show a pattern that will appear in the next theorem.

<u>Theorem</u> 1-1. Let $\mathfrak{F}$ be a nonempty countable family of closed compact subsets of X and m_0 a positive integer. If $|\mathfrak{G}| \leq m_0$ for each nonempty finite subfamily $\mathfrak{G}$ of $\mathfrak{F}$, then $|\mathfrak{F}| \leq m_0$.

Proof: The set $\{|\mathfrak{G}| : \mathfrak{G}$ is a nonempty finite subfamily of $\mathfrak{F}\}$ of positive integers is bounded above by m_0; thus it has a maximum, and the maximum is attained on some nonempty finite subfamily $\mathfrak{F}'$ of $\mathfrak{F}$. Since $|\mathfrak{F}'| \leq m_0$, it suffices to show that $|\mathfrak{F}| = |\mathfrak{F}'|$. Let $\mathfrak{F} = \{C_i : i = 1,2, \ldots\}$ and $\mathfrak{F}_k = \{C_i : 1 \leq i \leq k\}$ for $k = 1,2, \ldots$. For each integer k we have that $\mathfrak{F}' \subset \mathfrak{F}' \cup \mathfrak{F}_k$; thus, $|\mathfrak{F}'| \leq |\mathfrak{F}' \cup \mathfrak{F}_k|$. However, $\mathfrak{F}' \cup \mathfrak{F}_k$ is a finite subfamily of $\mathfrak{F}$, so the definition of $\mathfrak{F}'$ implies that $|\mathfrak{F}' \cup \mathfrak{F}_k| \leq |\mathfrak{F}'|$. Thus, $|\mathfrak{F}'| = |\mathfrak{F}' \cup \mathfrak{F}_k|$ for each k. Let $m = |\mathfrak{F}'|$. Then the definition of $|\mathfrak{F}' \cup \mathfrak{F}_k|$ implies that for each k there exists a set D_k of m distinct points of X such that each set of $\mathfrak{F}_k \cup \mathfrak{F}'$ contains a point of D_k. Let K denote the compact set $\bigcup\{A : A \in \mathfrak{F}'\}$. If there exists a point $x \in D_k \smallsetminus K$, then there would exist a subset of D_k consisting of fewer than m points such that each set of $\mathfrak{F}'$ contains one of these points. This would imply that $\mathfrak{F}'$ has the r-intersectional property for some $r < m$. This would then contradict the definition of m. Therefore, $D_k \subset K$ for each k.

Let $x_i^1 \in D_i$, for $i = 1,2, \ldots$. Then the sequence $\{x_i^1\}$ contains a convergent subsequence

$$\left\{ x_{i_j}^1 \right\} \text{ in } K \text{ such that } x_{i_j}^1 = x_1^1.$$

Let $D_1^1 = D_1$, and for each k and $j \geq 2$ with $i_{j-1} < k \leq i_j$ let

$$D_k^1 = D_{i_j};$$

also, for $i_{j-1} < k \leq i_j$ let

$$y_k^1 = x_{i_j}^1 \text{ and } y_1^1 = x_{i_j}^1 .$$

Then for $i_{j-1} < k \leq i_j$ we have

$$\mathfrak{F}_k \subset \mathfrak{F}_{i_j},$$

which implies each set in $\mathfrak{F}_k$ contains a point of D_k^1 for $k = 1,2, \ldots$. Also, the sequence $\{y_k^1\}$ is a convergent sequence.

Suppose now for $1 \leq r < m$ it has been shown that for each integer k there exists a set D_k^r containing m points with the following properties:

(a) Each set of $\mathfrak{F}_k$ contains a point of D_k^r.

(b) There exist r convergent sequences $\{x_i^j\}$, $1 \leq j \leq r$, with $x_i^j \in D_i^r$ such that for $j \neq t$, $x_i^j \neq x_i^t$.

(c) $D_k^r = D_i$ for some i, and $D_1^r = D_1$.

Now for each positive integer i choose

$$x_i^{r+1} \in D_i^r \sim \{x_i^1, x_i^2, \ldots, x_i^r\}.$$

Then the sequence $\{x_i^{r+1}\}$ contains a convergent subsequence

$$\left\{x_{i_j}^{r+1}\right\} \quad \text{such that} \quad x_{i_1}^{r+1} = x_1^{r+1}.$$

Let $D_1^{r+1} = D_1^r$, and for each k and $j \geq 2$ with $i_{j-1} < k \leq i_j$ let

$$D_k^{r+1} = D_{i_j};$$

also, for $i_{j-1} < k \leq i_j$ let

$$y_k^t = x_{i_j}^t \quad \text{and} \quad y_1^t = x_{i_1}^t$$

for $t = 1, 2, \ldots, r+1$. Then for $i_{j-1} < k \leq i_j$ we have

$$\mathfrak{I}_k \subset \mathfrak{I}_{i_j},$$

which implies each set of $\mathfrak{I}_k$ contains a point of D_k^{r+1} for $k = 1, 2, \ldots$.

The $r+1$ sequences $\{y_k^j\}$, $1 \le j \le r+1$, and the sets D_k^{r+1} satisfy properties (a), (b) and (c), with r replaced by $r+1$.

Thus, by induction there exist sets D_k^m, $k = 1,2, \ldots,$ containing m distinct points, and sequences $\{x_i^j\}$, $1 \le j \le m$, which satisfy properties (a), (b) and (c) with $r = m$.

Let x^j denote a point to which the sequence $\{x_i^j\}$, $1 \le j \le m$, converges. It will be shown that each element of $\mathfrak{F}$ contains one of the points of the set $D = \{x^1, x^2, \ldots, x^m\}$. Suppose that this is not the case. Then there exists a set $A \varepsilon \mathfrak{F}$ such that A contains no point of D. Since A is a closed set, for each j with $1 \le j \le m$, there exists an integer n_j such that $i \ge n_j$ implies that $x_i^j \notin A$. Let $n_0 = \max\{n_1, \ldots, n_m\}$. Then for $i \ge n_0$, $x_i^j \notin A$ for all j with $1 \le j \le m$. Now $A \varepsilon \mathfrak{F}_i$ for some $i \ge n_0$; thus, A contains no point of $D_i^m = \{x_i^1, x_i^2, \ldots, x_i^m\}$. This is a contradiction of property (a). Hence, each set of $\mathfrak{F}$ contains one of the points of the set D. This implies that $\mathfrak{F}$ has the m-intersectional property. Hence, $|\mathfrak{F}| \le m$; however, $\mathfrak{F}' \subset \mathfrak{F}$ implies that $|\mathfrak{F}| = |\mathfrak{F}'| = m$. ■

<u>Theorem</u> 1-2. Let $\mathfrak{F} = \{A_\beta : \beta \varepsilon \Lambda\}$ be a nonempty family of closed compact subsets of a space X. If $|\mathfrak{G}| \le m_0$ for each nonempty finite subfamily $\mathfrak{G}$ of $\mathfrak{F}$, then $|\mathfrak{F}| \le m_0$.

Proof: The proof proceeds by transfinite induction on the cardinality of Λ. If the cardinality of Λ is the same as that of the positive integers, then the desired result follows from Theorem 1-1. Let Σ be a cardinal number and suppose that if the cardinality of Λ is less than Σ, then the desired conclusion holds. To prove the theorem when the cardinality of Λ is equal to Σ, well-order Λ by a relation $\le'$

such that for each $\lambda \in \Lambda$ the set $\{\beta : \beta \leq' \lambda\}$ has cardinality less than Σ. For each $\lambda \in \Lambda$ let $\mathfrak{F}_\lambda = \{A_\beta : \beta \leq' \lambda\}$. The induction hypothesis implies that $|\mathfrak{F}_\lambda| \leq m_0$ for each $\lambda \in \Lambda$. The proof now follows in a similar manner as that of Theorem 1-1 using nets with the directed set $(\Sigma, \leq')$ instead of sequences. The theorem follows by transfinite induction. ■

We now state and prove a corollary to Theorem 1-2 which will be used in the sequel.

<u>Corollary</u> 1-2.1. Let $\mathfrak{F}$ be a nonempty family of compact subsets of E^n. If $|\mathfrak{G}| \leq m_0$ for each nonempty finite subfamily $\mathfrak{G}$ of $\mathfrak{F}$, then $|\mathfrak{F}| \leq m_0$.

Proof: The corollary follows from Theorem 1-2 after noticing that every compact subset of E^n is closed. ■

CHAPTER II

FAMILIES OF MUTUALLY PARALLEL PARALLELOTOPES

In this chapter, families of mutually parallel parallelotopes in E^n will be defined and certain intersectional properties of these families considered.

Definition 2-1. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for E^n. A subset P of E^n is called a parallelotope with respect to the basis $\{x_1, x_2, \ldots, x_n\}$ if there exist scalars λ_i, $i = 1, \ldots, n$, and β_i, $i = 1, \ldots, n$, with $\lambda_i \leq \beta_i$ such that

$$P = \left\{ \sum_{i=1}^{n} \alpha_i x_i : \lambda_i \leq \alpha_i \leq \beta_i, \quad i = 1, \ldots, n \right\}.$$

A nonempty family $\mathfrak{F}$ of subsets of E^n is called a family of mutually parallel parallelotopes in E^n if there exists a basis $\{x_1, x_2, \ldots, x_n\}$ for E^n such that each set in $\mathfrak{F}$ is a parallelotope with respect to the basis $\{x_1, x_2, \ldots, x_n\}$.

Note that if P is a parallelotope in E^n and A is a set of points in E^n, then the family $\mathfrak{F} = \{x + P: x \in A\}$ is a family of mutually parallel parallelotopes in E^n. Moreover, if to each $x \in A$ there is associated a scalar α_x, then the family $\{x + \alpha_x P: x \in A\}$ is also a family of mutually parallel parallelotopes in E^n.

Bounds on $|\mathfrak{F}|$

The following theorem is due to Edward Helly. It is stated here without proof and shall be referred to in the sequel as Helly's theorem. A proof can be found in Valentine [p. 70, 10].

Helly's Theorem. Let $\mathfrak{F}$ be a family of compact convex sets in E^n containing at least $n+1$ sets. A necessary and sufficient condition that $|\mathfrak{F}| = 1$ is that $|\mathfrak{G}| = 1$ for every subfamily $\mathfrak{G}$ of $\mathfrak{F}$ which contains $n+1$ sets.

The following theorem is due to B. Sz.-Nagy [9]; however, to be complete we shall give a proof.

Theorem 2-1. Let $\mathfrak{F} = \{P_\alpha : \alpha \in \Lambda\}$ be a family of mutually parallel parallelotopes in E^n. If each two sets of $\mathfrak{F}$ have a common point, then $|\mathfrak{F}| = 1$.

Proof: We proceed by induction on n, the dimension of E^n. If $n = 1$, then the result follows from Helly's theorem. So assume the theorem holds for all k with $1 \le k < n$. Due to the compactness of the sets of $\mathfrak{F}$ it suffices to assume that $\mathfrak{F}$ is finite. Thus, assume $\mathfrak{F} = \{P_1, \ldots, P_m\}$ for some integer $m \ge 1$. Let $\{x_1, \ldots, x_n\}$ be a basis for E^n such that each set in $\mathfrak{F}$ is a parallelotope with respect to the basis $\{x_1, \ldots, x_n\}$. Each $P_i \in \mathfrak{F}$ has the form

$$P_i = \left\{ \sum_{j=1}^{n} \alpha_j x_j : \lambda_j^i \le \alpha_j \le \beta_j^i, \quad j = 1, \ldots, n \right\}$$

for some set

$$\{\lambda_1^i, \lambda_2^i, \ldots, \lambda_n^i, \beta_1^i, \beta_2^i, \ldots, \beta_n^i\}$$

of scalars. Let $\beta_1^i = \min\{\beta_1^j : j = 1, \ldots, n\}$. Then let H be the hyperplane

$$\left\{ \sum_{j=1}^{n} \epsilon_j x_j : \epsilon_1 = \beta_1^i \right\}.$$

For P_k, P_j in $\mathfrak{F}$ there exists a point $\alpha_1 x_1 + \cdots + \alpha_n x_n$ in $P_k \cap P_j$. Also, $P_k \cap P_i \neq \emptyset$ and $P_j \cap P_i \neq \emptyset$ implies that

$$\lambda_1^k \leq \beta_1^i \leq \beta_1^k \quad \text{and} \quad \lambda_1^j \leq \beta_1^i \leq \beta_1^j.$$

Let $y = \beta_1^i x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$. Then y belongs to $P_k \cap P_j \cap H$. Thus, the family $\mathfrak{F}' = \{P_r \cap H : r = 1, \ldots, m\}$ pairwise intersect in the (n-1)-dimensional hyperplane H. The theorem follows by induction. ∎

<u>Definition</u> 2-2. A family $\mathfrak{F}$ of nonempty convex sets in E^n has the (p,q)-property, where p and q are integers with $p \geq q \geq 2$, if $\mathfrak{F}$ contains at least p sets and from each p sets of $\mathfrak{F}$ some q have a common point.

Relating this definition to Theorem 2-1 we see that if the family $\mathfrak{F}$ given in the hypothesis has at least two sets, then $\mathfrak{F}$ has the (2,2)-property. Thus, Theorem 2-1 implies that a sufficient condition that a family $\mathfrak{F}$ of mutually parallel parallelotopes in E^n, with at least two sets, have a common point so that $\mathfrak{F}$ have the (2,2)-property.

<u>Example</u> 2-1. The four sets illustrated in Figure 1 is a family of sets in E^n with the (2,2)-property; however, there is no point which is

common to all four sets. Thus, we see that a theorem such as Theorem 2-1 cannot hold for arbitrary families of convex sets with the (2,2)-property. Figure 2 illustrates a family of five translates of a given parallelotope in E^2 with the (3,2)-property. Figure 3 illustrates a family of seven translates of a given parallelotope in E^2 with the (4,2)-property.

Definition 2-3. Let k be an odd positive integer such that $k \geq 5$. Then a family $\mathfrak{F}$ of mutually parallel parallelotopes in E^n is said to be a k-cycle if $\mathfrak{F}$ consists of k distinct translates of a parallelotope no three of which have a common point and $\mathfrak{F}$ has the $(1/2(k+1),2)$-property.

Figure 2 then represents a 5-cycle in E^2, and Figure 3 represents a 7-cycle in E^2. It is a simple exercise to construct figures such as Figures 2 and 3 to show that a k-cycle exists in E^n for all odd integers k with $k \geq 5$ and all n with $n \geq 2$.

The following theorem is original; however, the proof is similar to the proof of Theorem 85 of [5].

Theorem 2-2. Let $\mathfrak{F}$ be a family of mutually parallel parallelotopes in E^n, and let k and q be integers such that $k \geq 1 - q$ where $q \geq 3$. If $\mathfrak{F}$ has the $(2q+k,q)$-property and does not have the $(2(q-1)+k,q-1)$-property, then $|\mathfrak{F}| \leq q + k + 1$.

Notice that the restrictions on q and k make the two mentioned properties meaningful with regard to their definition.

Proof: Since $\mathfrak{F}$ does not have the $(2(q-1)+k,q-1)$-property, there is a subcollection $\mathfrak{G}$ of $\mathfrak{F}$ containing $2(q-1)+k$ sets such that no $q-1$

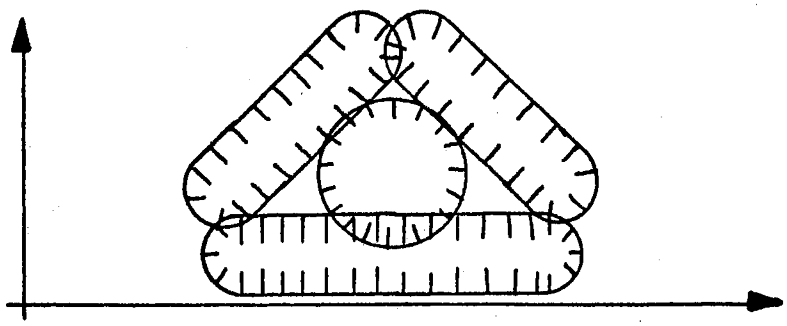

Figure 1. Sets with the (2,2)-property

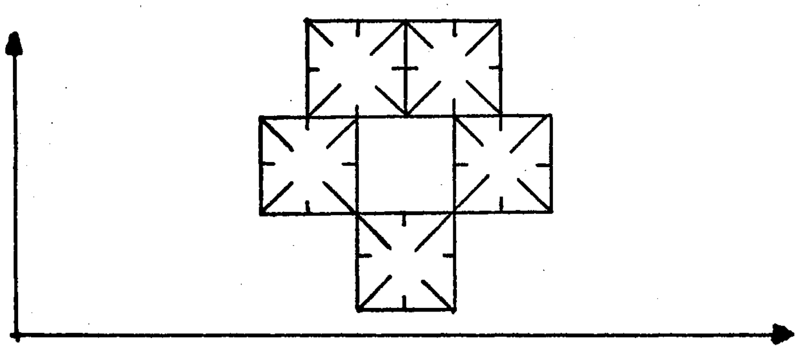

Figure 2. Translates with the (3,2)-property.

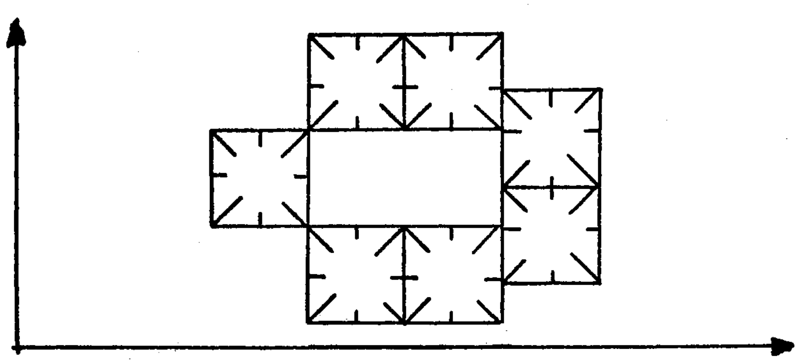

Figure 3. Translates with the (4,2)-property.

sets of $\mathfrak{G}$ have a common point. Let D and E be distinct sets of $\mathfrak{F} \setminus \mathfrak{G}$. Then $\mathfrak{G} \cup \{D,E\}$ is a collection of $2q+k$ sets of $\mathfrak{F}$, and hence, some q of the sets in $\mathfrak{G} \cup \{D,E\}$ must have a common point. Recall that no $q-1$ sets of $\mathfrak{G}$ have a common point. Hence, any q of the sets of $\mathfrak{G} \cup \{D,E\}$ which have a nonempty intersection cannot contain more than $q-2$ sets from $\mathfrak{G}$. Thus, any such q sets with a nonempty intersection must contain both D and E. Thus, $\mathfrak{F} \setminus \mathfrak{G}$ has the $(2,2)$-property. Moreover, every set in $\mathfrak{F} \setminus \mathfrak{G}$ must have a nonempty intersection with some $q-2$ sets of $\mathfrak{G}$.

Let $\mathfrak{H}$ be any collection of $q-2$ sets of $\mathfrak{G}$ which have a nonempty intersection. Let $A_1, A_2, \ldots, A_{q+k}$ denote the sets in $\mathfrak{G} \setminus \mathfrak{H}$. Define subcollections $\mathfrak{M}_i$, $1 \leq i \leq q+k+1$, of $\mathfrak{F}$ as follows:

$$\mathfrak{M}_i = (\{A \in \mathfrak{F}: A \cap A_i \neq \emptyset\} \setminus \mathfrak{G}) \cup \{A_i\}, \quad 1 \leq i \leq q+k$$

$$\mathfrak{M}_{q+k+1} = \{A \in \mathfrak{F}: A \cap (\cap\{B: B \in \mathfrak{H}) \neq \emptyset\}.$$

We shall show that

(a) $\mathfrak{F} = \bigcup_{i=1}^{q+k+1} \mathfrak{M}_i$ and that

(b) $\cap\{A: A \in \mathfrak{M}_i\} \neq \emptyset$ for $1 \leq i \leq q+k+1$.

Proof of (a): Suppose

$$\mathfrak{F} \neq \bigcup_{i=1}^{q+k+1} \mathfrak{M}_i,$$

that is, there exists

$$A' \in \mathfrak{F} \setminus \bigcup_{i=1}^{q+k+1} \mathfrak{M}_i.$$

Then A' is not in $\mathfrak{G}$; hence, A' belongs to $\mathfrak{F} \setminus \mathfrak{G}$. The definition of $\mathfrak{M}_i$ for $1 \leq i \leq q+k+1$ implies $A' \cap A_i = \emptyset$ for $1 \leq i \leq q+k$ and $A' \cap (\cap\{B: B \in \mathfrak{H}\}) = \emptyset$. We know that A' belongs to $\mathfrak{F} \setminus \mathfrak{G}$; hence, it must intersect the intersection of some $q-2$ sets of $\mathfrak{G}$. Since $A' \cap A_i = \emptyset$, $1 \leq i \leq q+k$, any such $q-2$ sets of $\mathfrak{G}$ which intersect A' must come from the collection $\mathfrak{G} \setminus \{A_1, \ldots, A_{q+k}\} = \mathfrak{H}$. However, this implies A' must intersect the intersection of the $q-2$ sets of $\mathfrak{H}$. This is a contradiction since A' does not belong to $\mathfrak{M}_{q+k+1}$. Therefore,

$$\mathfrak{F} = \bigcup_{i=1}^{q+k+1} \mathfrak{M}_i.$$

Proof of (b): Let $\mathfrak{M}_i$ be such that $1 \leq i \leq q+k$. If $\mathfrak{M}_i$ consists of only the single set A_i, then clearly $\cap\{A: A \in \mathfrak{M}_i\} \neq \emptyset$. So assume $\mathfrak{M}_i$ contains at least two sets. Let C and D be two arbitrary sets of $\mathfrak{M}_i$. If $C \in \mathfrak{G}$, the definition of $\mathfrak{M}_i$ implies $C = A_i$. Then $D \in \mathfrak{M}_i$ implies that $D \cap C = D \cap A_i \neq \emptyset$. Similarly, if $D \in \mathfrak{G}$, $D \cap C \neq \emptyset$. So assume D and C belong to $\mathfrak{F} \setminus \mathfrak{G}$. Since $\mathfrak{F} \setminus \mathfrak{G}$ has the (2,2)-property, $D \cap C \neq \emptyset$. Hence, $\mathfrak{M}_i$, $1 \leq i \leq q+k$, is a family of mutually parallel parallelotopes in E^n such that each two sets of $\mathfrak{M}_i$ have a common point. Theorem 2-1 implies that $\cap\{A: A \in \mathfrak{M}_i\} \neq \emptyset$, $1 \leq i \leq q+k$.

Let C and D be two arbitrary sets of $\mathfrak{M}_{q+k+1}$. Then neither C nor D can be in $\mathfrak{G} \setminus \mathfrak{H}$ since no $q-1$ sets of $\mathfrak{G}$ have a common point. Thus, C and D belong to $(\mathfrak{F} \setminus \mathfrak{G}) \cup \mathfrak{H}$. If one of these sets, say

C, belongs to $\mathfrak{H}$, then $D \cap (\bigcap\{B: B \in \mathfrak{H}\}) \neq \emptyset$ implies $C \cap D \neq \emptyset$. So assume that C and D both belongs to $\mathfrak{F} \setminus \mathfrak{G}$. Since $\mathfrak{F} \setminus \mathfrak{G}$ has the (2,2)-property, $C \cap D \neq \emptyset$. Therefore, $\mathfrak{M}_{q+k+1}$ has the (2,2)-property. Theorem 2-1 implies that $\bigcap\{A: A \in \mathfrak{M}_{q+k+1}\} \neq \emptyset$. Thus, $\mathfrak{F}$ has the q+k+1-intersectional property which implies that $|\mathfrak{F}| \leq q + k + 1$. ■

<u>Theorem</u> 2-3. If $\mathfrak{F}$ is a family of sets in E^n with the (p+k,q+k)-property for $p \geq q \geq 2$ and $k \geq 1$, then $\mathfrak{F}$ has the (p,q)-property.

Proof: Suppose $\mathfrak{F}$ fails to have the (p,q)-property. Then there exists a subcollection $\mathfrak{G}$ of $\mathfrak{F}$ with p sets such that no q of them have a common point. Let $\mathfrak{M}$ be an arbitrary subcollection of $\mathfrak{F} \setminus \mathfrak{G}$ containing k sets. Then $\mathfrak{G} \cup \mathfrak{M}$ is a family of $p+k$ sets such that no $q+k$ of them have a common point. This is a contradiction. ■

The following theorem is equivalent to Theorem 78 of [5]. However, we will give a proof to be complete.

<u>Theorem</u> 2-4. Let $\mathfrak{F}$ be a family of mutually parallel parallelotopes in E^1 with the (p,q)-property for some $p \geq q \geq 2$. Then $|\mathfrak{F}| \leq p - q + 1$.

Proof: Now $\mathfrak{F}$ has the (p-q+2+(q-2),2+(q-2))-property. Theorem 2-3 implies that $\mathfrak{F}$ has the (p-q+2,2)-property. Hence, it suffices to prove the theorem for $q = 2$. We now proceed by induction on p to show that if $\mathfrak{F}$ is any family of mutually parallel parallelotopes in E^1 with the (p,2)-property, then $|\mathfrak{F}| \leq p - 2 + 1$. For $p = 2$ the result follows from Theorem 2-1. Suppose now that the result holds for $p \geq 2$. Let $\mathfrak{F}$ be a family of mutually parallel parallelotopes in E^1 with the (p+1,2)-property. Corollary 1-2.1 implies that it suffices to assume that $\mathfrak{F}$ is finite. Then $\mathfrak{F} = \{P_1, \ldots, P_m\}$ where each P_i is of the form $P_i = \{\epsilon : \alpha_i \leq \epsilon \leq \beta_i\}$. Without loss of generality, assume $\beta_1 = \min\{\beta_i : 1 \leq i \leq m\}$. If $P_j \in \mathfrak{F}$, then either $\beta_1 \in P_j$ or P_j fails to intersect P_1. Let $\mathfrak{F}_1 = \{P \in \mathfrak{F} : \beta_1 \in P\}$ and $\mathfrak{F}' = \{P \in \mathfrak{F} : P \cap P_1 = \emptyset\}$. Then $\mathfrak{F} = \mathfrak{F}_1 \cup \mathfrak{F}'$. If $\mathfrak{F}' = \emptyset$, then $\mathfrak{F} = \mathfrak{F}_1$ implies $|\mathfrak{F}| = 1$. If $\mathfrak{F}' \neq \emptyset$, then either $\mathfrak{F}'$ has the (p,2)-property, or $\mathfrak{F}'$ fails to contain p sets. In either case, we have $|\mathfrak{F}'| \leq p - 2 + 1$. It is clear that

$$|\mathfrak{F}| \leq |\mathfrak{F}'| + |\mathfrak{F}_1| \leq (p + 1) - 2 + 1.$$

Hence, by induction the theorem follows. ∎

<u>Theorem</u> 2-5. Let k and q be integers such that $k \geq 1 - q$, where $q \geq 3$, and $\mathfrak{F}$ is a family of mutually parallel parallelotopes in E^n. Suppose that $\mathfrak{F}$ has the (2q+k,q)-property and that there exists an integer m with $0 < m \leq \min\{q-2, q+k\}$ such that $\mathfrak{F}$ does not have the (2(q-m)+k,q-m)-property. Then $|\mathfrak{F}| \leq q + k + 1$.

Notice here again that the restrictions on q, k and m make the two mentioned properties meaningful with regard to their definition.

Proof: Let m_0 be the smallest positive integer such that $\mathfrak{F}$ does not have the $(2(q-m_0)+k, q-m_0)$-property. Such an integer exists by hypothesis; moreover, $0 < m_0 \leq \min\{q-2, q+k\}$. Let $r = m_0 - 1$. Then the definition of m_0 implies that $\mathfrak{F}$ has the (2(q-r)+k,q-r)-property but not the $(2[(q-r)-1]+k,(q-r)-1)$-property. Theorem 2-2 implies that $|\mathfrak{F}| \leq q - r + k + 1 = q - m_0 + k + 2$. However, $q - m_0 + k + 2 \leq q + k + 1$, since $m_0 \geq 1$. ■

Lemma 2-6.1. Let $\mathfrak{F}$ be a family of mutually parallel parallelotopes in E^n with the $(p,2)$-property for some $p \geq 2$. Then

$$|\mathfrak{F}| \leq \binom{p-2+n}{n}.$$

Proof: We proceed by induction. For $n = 1$ the result follows from Theorem 2-4. So assume the lemma holds in E^n for $n \geq 1$. To prove that the lemma holds in E^{n+1} we proceed by induction on p. For $p = 2$ the result follows from Theorem 2-1. Thus, assume for $p \geq 2$ the result holds in E^{n+1}. To prove the result holds for $p + 1$, let $\mathfrak{F} = \{P_1, P_2, \ldots, P_m\}$ be a finite family of mutually parallel parallelotopes in E^{n+1} with the $(p+1,2)$-property. Let $\{x_1, x_2, \ldots, x_{n+1}\}$ be a basis for E^{n+1} such that each set in $\mathfrak{F}$ is a parallelotope with respect to the basis $\{x_1, x_2, \ldots, x_{n+1}\}$. Each $P_i \in \mathfrak{F}$ has the form

$$P_i = \left\{ \sum_{j=1}^{n+1} \alpha_j x_j : \lambda_j^i \leq \alpha_j \leq \beta_j^i, \quad j = 1, \ldots, n+1 \right\}$$

for some set

$$\{\lambda_1^i, \lambda_2^i, \ldots, \lambda_{n+1}^i, \beta_1^i, \beta_2^i, \ldots, \beta_{n+1}^i\}$$

of scalars. Let $\beta_1^i = \min\{\beta_1^j : j = 1, \ldots, m\}$. Then let H be the hyperplane

$$\left\{ \sum_{j=1}^{n+1} \epsilon_j x_j : \epsilon_1 = \beta_1^i \right\}.$$

Let $\mathfrak{I}_1 = \{P \in \mathfrak{I} : P \cap H \neq \emptyset\}$ and $\mathfrak{I}_2 = \mathfrak{I} \setminus \mathfrak{I}_1$. Now the family $\mathfrak{I}_1' = \{P \cap H : P \in \mathfrak{I}_1\}$ either fails to contain $p+1$ sets or has the $(p+1,2)$-property on the n-dimensional hyperplane H. If $\mathfrak{I}_1'$ fails to contain $p+1$ sets, then clearly $|\mathfrak{I}_1'| \leq \binom{(p+1)-2+n}{n}$. If $\mathfrak{I}_1'$ contains $p+1$ sets, then our induction hypothesis on n implies that $|\mathfrak{I}_1'| \leq \binom{(p+1)-2+n}{n}$. This then implies that $|\mathfrak{I}_1| \leq \binom{(p+1)-2+n}{n}$. Now the definition of H implies each set of $\mathfrak{I}_2$ fails to intersect the set P_i. Hence, either $\mathfrak{I}_2$ fails to contain p sets, or $\mathfrak{I}_2$ has the $(p,2)$-property. If $\mathfrak{I}_2$ has the $(p,2)$-property, then our induction hypothesis on p implies that $|\mathfrak{I}_2| \leq \binom{p-2+(n+2)}{n+1}$. If $\mathfrak{I}_2$ fails to contain p sets, then again we clearly have that $|\mathfrak{I}_2| \leq \binom{p-2+(n+1)}{n+1}$ or $\mathfrak{I}_2 = \emptyset$. In either case, we have

$$|\mathfrak{I}| = |\mathfrak{I}_1 \cup \mathfrak{I}_2| \leq \binom{(p+1)-2+n}{n} + \binom{p-2+(n+1)}{n+1}$$

$$= \frac{(p-1+n)!}{n!(p-1)!} + \frac{(p-1+n)!}{(n+1)!(p-2)!} = (p-1+n)! \left[\frac{(n+1) + (p-1)}{(n+1)!(p-1)!} \right]$$

$$= \frac{(p+n)!}{(n+1)!(p-1)!} = \binom{(p+1)-2+(n+1)}{n+1}.$$

Thus, for each finite family $\mathfrak{I}$ of mutually parallel parallelotopes in E^{n+1} with the $(p+1,2)$-property we have $|\mathfrak{I}| \leq \binom{(p+1)-2+(n+1)}{n+1}$. Corollary 1-2.1 implies the same is true for infinite families. By induction on p we have for all $p \geq 2$ the desired result in E^{n+1}.

By induction on n we conclude that the theorem is true in E^n for all $n \geq 1$. ■

Theorem 2-6. Let k and q be integers such that $2 \leq q \leq 2q + k$. Then for any family $\mathfrak{F}$ of mutually parallel parallelotopes in E^n with the $(2q+k,q)$-property we have

$$|\mathfrak{F}| \leq \max\left\{q+k+1, \binom{2+k+n}{n}\right\},$$

where we take the standard convention by defining $\binom{m}{n} = 0$ if $m < n$.

Proof: The proof consists of two cases: (a) $k \leq -2$ and (b) $k > -2$.

Proof of (a): If $k = -q$, then $\mathfrak{F}$ has the (q,q)-property. Theorem 2-3 implies that $\mathfrak{F}$ has the $(2,2)$-property. Thus, for $k = -q$, Theorem 2-1 implies the desired result. If $k > -q$, let $m = q + k$. Since $-q < k \leq -2$, we have that $0 < m \leq \min\{q-2,q+k\}$. If $\mathfrak{F}$ has the $(2(q-m)+k,q-m)$-property, that is, the $(-k,-k)$-property, then as before Theorems 2-3 and 2-1 imply that $|\mathfrak{F}| = 1$. If $\mathfrak{F}$ fails to have the $(2(q-m)+k,q-m)$-property, Theorem 2-5 implies that $|\mathfrak{F}| \leq q + k + 1$. Hence, in either case we have $|\mathfrak{F}| \leq q + k + 1$. Thus, for $k \leq -2$,

$$|\mathfrak{F}| \leq \max\left\{q+k+1, \binom{2+k+n}{n}\right\} = q + k + 1.$$

Proof of (b): If $q = 2$, then the result follows from Lemma 2-6.1. If $q \geq 3$, let $m = q - 2$. Then $0 < m \leq \min\{q-2,q+k\}$. If $\mathfrak{F}$ has the $(2(q-m)+k,q-m)$-property, that is, the $(4+k,2)$-property, then as above Lemma 2-6.1 implies the desired result. If $\mathfrak{F}$ fails to have the

(2(q-m)+k,q-m)-property, Theorem 2-5 implies that $|\mathfrak{F}| \leq q + k + 1$. Thus, in either case we have

$$|\mathfrak{F}| \leq \max\left\{q+k+1,\ \binom{2+k+n}{n}\right\}. \blacksquare$$

The Functions $N_n(p,q)$ and $T_n(p,q)$

Two functions of three variables $N_n(p,q)$ and $T_n(p,q)$ will now be defined. Properties of these functions will then be studied in detail. The function $N_n(p,q)$ has been defined by Hadwiger and Debrunner [p. 32, 5]. The definition of $T_n(p,q)$ is similar to that of $N_n(p,q)$ and is due to the author.

Definition 2-4. Let $p \geq q \geq 2$, and $n \geq 1$. Then $N_n(p,q)$ is defined to be the maximum value of $|\mathfrak{F}|$ where $\mathfrak{F}$ ranges over all families of mutually parallel parallelotopes in E^n with the (p,q)-property. Also, define $T_n(p,q)$ to be the maximum value of $|\mathfrak{F}|$ where $\mathfrak{F}$ ranges over all families of mutually parallel parallelotopes in E^n with the (p,q)-property, and each set in $\mathfrak{F}$ is a translate of every other set in $\mathfrak{F}$.

Theorem 2-6 implies that the number $N_n(p,q)$ is well defined and is a positive integer. Since we clearly have $T_n(p,q) \leq N_n(p,q)$, the same is true of $T_n(p,q)$.

Theorem 2-6 implies that $N_2(3,2) \leq 3$; however, by considering the family $\mathfrak{F}$ in E^2 illustrated in Figure 2, we see that $3 \leq T_2(3,2)$. Thus, $T_2(3,2) = N_2(3,2) = 3$. Also, by considering Figure 3 we see that $4 \leq T_n(4,2) \leq N_n(4,2)$. Theorem 2-1 implies that

$N_n(2,2) = 1$ for all $n \geq 1$, from which it follows that $N_n(p,p) = T_n(p,p) = 1$ for all $p \geq 2$ and $n \geq 1$.

To determine exactly what one of the numbers $N_n(p,q)$ is, it appears that in most cases two things are required. One must prove that for some known integer r, $N_n(p,q) \leq r$, and construct a family $\mathfrak{F}$ of mutually parallel parallelotopes in E^n with the (p,q)-property such that $|\mathfrak{F}| \geq r$. A similar statement holds for the numbers $T_n(p,q)$.

Some of the results in the following theorem are equivalent to results of Theorems 79, 80 and 81 of [5].

<u>Theorem</u> 2-7. Let $p \geq q \geq 2$, and $n \geq 1$. Then

$$p - q + 1 \leq T_n(p,q) \leq N_n(p,q) \leq \max\left\{p-q+1, \binom{p-2q+2+n}{n}\right\}$$

and

$$T_1(p,q) = N_1(p,q) = p - q + 1.$$

If the supplementary condition $2 \leq q \leq p \leq 2q - 2$ is satisfied, then

$$T_n(p,q) = N_n(p,q) = p - q + 1.$$

Proof: Let $x_1, x_2, \ldots, x_{p-q+1}$ be $p-q+1$ distinct points of E^n. Let $P_i = \{x_i\}$ for $1 \leq i \leq p - q$ and $P_j = \{x_{p-q+1}\}$ for $p - q + 1 \leq j \leq p$. Then $\mathfrak{F} = \{P_1, \ldots, P_p\}$ is a family of mutually parallel parallelotopes in E^n with the (p,q)-property. Moreover, each set of $\mathfrak{F}$ is a translate of any other set in $\mathfrak{F}$. It is also clear that $|\mathfrak{F}| \geq p - q + 1$. Hence,

$$p - q + 1 \leq T_n(p,q) \leq N_n(p,q).$$

Let $k = p - 2q$. Then $2 \leq q \leq 2q + k$. Theorem 2-6 implies that

$$N_n(2q+k,q) \leq \max\left\{q+k+1,\ \binom{2+k+n}{n}\right\},$$

that is,

$$N_n(p,q) \leq \max\left\{p-q+1,\ \binom{p-2q+2+n}{n}\right\}.$$

The inequality of the theorem now follows.

Theorem 2-4 implies that $N_1(p,q) \leq p - q + 1$. This fact and the first part of the theorem implies that

$$T_1(p,q) = N_1(p,q) = p - q + 1.$$

If $2 \leq q \leq p \leq 2q - 2$, then $p - 2q \leq -2$. This implies that $\binom{p-2q+2+n}{n} \leq 1$. Thus, $T_n(p,q) = N_n(p,q) = p - q + 1$ for $2 \leq q \leq p \leq 2q - 2$. ∎

From the discussion following Definition 2-4 we have that $T_2(3,2) = N_2(3,2) = 3$. From this it follows that the equation $N_n(p,q) = p - q + 1$ is not always true. In fact, it will be shown later that $T_n(p,2) - (p - 1)$ becomes infinite as p becomes large and $n \geq 2$.

<u>Theorem</u> 2-8. Let k be a fixed integer such that $k \geq -2$, and $t_m = \min\{N_n(2m+k,m)\colon m = 2, \ldots, q\}$. Then for $q \geq 2$, we have $N_n(2q+k,q) \leq \max\{t_q, q+k+1\}$.

Proof: To prove this we must show that if $\mathfrak{F}$ is any family of mutually parallel parallelotopes in E^n with the $(2q+k,q)$-property, then $|\mathfrak{F}| \leq \max\{t_q, q+k+1\}$.

We proceed by induction on q. If $q = 2$, then $t_q = N_n(4+k,2)$. Thus, $N_n(4+k,2) \leq \max\{t_2, k+3\}$.

Suppose now the result holds for $q - 1 \geq 2$. Then if $\mathfrak{F}$ has the $(2(q-1)+k, q-1)$-property, our induction hypothesis implies that $|\mathfrak{F}| \leq \max\{t_{q-1}, q+k\}$. If $t_{q-1} > t_q$, then we must have $t_q = N_n(2q+k,q)$, and the required inequality is satisfied. If $t_{q-1} = t_q$, then we have

$$|\mathfrak{F}| \leq \max\{t_{q-1}, q+k\} \leq \max\{t_q, q+k+1\},$$

and again the desired result follows. If $\mathfrak{F}$ does not have the $(2(q-1)+k, q-1)$-property, then Theorem 2-5 implies that $|\mathfrak{F}| \leq q + k + 1$. Therefore, in each case we have

$$|\mathfrak{F}| \leq \max\{t_q, q+k+1\}. \blacksquare$$

<u>Lemma</u> 2-9.1. Let $p \geq q \geq 2$. Then $N_n(p,q) + 1 \leq N_n(p+1,q)$ and $T_n(p,q) + 1 \leq T_n(p+1,q)$.

Proof: We prove only the first inequality since the second follows by a similar argument. Definition 2-4 and Corollary 1-2.1 imply that there exists a finite family $\mathfrak{F}$ of mutually parallel parallelotopes in E^n with the (p,q)-property such that $|\mathfrak{F}| = N_n(p,q)$. Let $x \in E^n$ such that $x \notin P$ for all $P \in \mathfrak{F}$. Then the family $\mathfrak{F}' = \mathfrak{F} \cup \{\{x\}\}$ of mutually parallel parallelotopes in E^n has the $(p+1,q)$-property. Since $|\mathfrak{F}| = N_n(p,q)$ and $x \notin P$ for all $P \in \mathfrak{F}$, it is clear that

$$|\mathfrak{I}'| \geq |\mathfrak{I}| + 1 = N_n(p,q) + 1.$$

Thus, $N_n(p,q) + 1 \leq N_n(p+1,q)$. ■

In the above theorem, we note that equality need not hold since $N_2(2,2) = T_n(2,2) = 1$ and $N_2(3,2) = T_2(3,2) = 3$. However, at this time it is unknown whether equality holds or not for $q = 3$ and $n \geq 2$. If it was known that equality was true for $q \geq 3$ and $n \geq 2$, then one would be able to obtain all the numbers $N_n(p,q)$ and $T_n(p,q)$ for $p \geq q \geq 3$ and $n \geq 2$. The values of $N_n(p,q)$ and $T_n(p,q)$ obtained in Theorem 2-7 do satisfy the equations $N_n(p,q) + 1 = N_n(p+1,q)$ and $T_n(p,q) + 1 = T_n(p+1,q)$.

<u>Lemma</u> 2-9.2. Let $p \geq q \geq 2$. Then $N_n(p,q) \geq N_n(p,q+1) + 1$ and $T_n(p,q) \geq T_n(p,q+1) + 1$.

Proof: Here again, we prove only the first inequality since the second follows by a similar argument. Theorem 2-3 implies that $N_n(p+1,q+1) \leq N_n(p,q)$. Lemma 2-9.1 implies that $N_n(p,q+1) + 1 \leq N_n(p+1,q+1)$. Thus, $N_n(p,q) \geq N_n(p,q+1) + 1$. ■

Since $T_n(3,3) = N_n(3,3) = 1$ and for $n \geq 2$

$$3 \leq T_2(3,2) \leq T_n(3,2) \leq N_n(3,2),$$

we see that equality in the above theorem need not hold. However, here again the values of $T_n(p,q)$ and $N_n(p,q)$ obtained in Theorem 2-7 do satisfy the equations $N_n(p,q) = N_n(p,q+1) + 1$ and $T_n(p,q) = T_n(p,q+1) + 1$.

<u>Theorem</u> 2-9. Let t be a fixed integer with $t \geq 1$. Then for each integer $m \geq 1$ we have the following: Any family $\mathfrak{F}$ of mutually parallel parallelotopes in E^n (with the property that each set of $\mathfrak{F}$ is a translate of any other set in $\mathfrak{F}$) with the $(p+t+m,q+t)$-property which fails to have the (p,q)-property satisfies the inequality $|\mathfrak{F}| \leq p - q + 1 + N_n(t+m,t+1)$ $(|\mathfrak{F}| \leq p - q + 1 + T_n(t+m,t+1))$.

Proof: The proof of the statement in parentheses is almost identical to the following proof, so we omit it.

We proceed by induction on m. Let $p \geq q \geq 2$ and $\mathfrak{F}$ be any family of mutually parallel parallelotopes in E^n with the $(p+t+1,q+t)$-property but not the (p,q)-property. Let $k = p + t + 1 - 2(q + t)$. Then $p \geq q$ implies that $k = p + 1 - 2q - t \geq 1 - (q + t)$. Thus, $\mathfrak{F}$ has the $(2(q+t)+k,q+t)$-property with $k \geq 1 - (q + t)$. If $\mathfrak{F}$ has the $(2(q+t-2)+k,q+t-1)$-property, that is, the $(p+t-1,q+t-1)$-property, then Theorem 2-3 would imply that $\mathfrak{F}$ has the (p,q)-property, a contradiction. Therefore, $\mathfrak{F}$ has the $(2(q+t)+k,q+t)$-property and does not have the $(2(q+t-1)+k,q+t-1)$-property. Theorem 2-2 implies that $|\mathfrak{F}| \leq (q + t) + k + 1$. However, $N_n(t+1,t+1) \geq 1$ implies that

$$p - q + 2 \leq p - q + 1 + N_n(t+1,t+1).$$

Thus, the theorem holds for $m = 1$.

Suppose now that for all $p \geq q \geq 2$ the theorem holds for m_1 with $1 \leq m_1 < m$. Let $\mathfrak{F}$ be any family of mutually parallel parallelotopes in E^n with the $(p+t+m,q+t)$-property but not the (p,q)-property. If $\mathfrak{F}$ also fails to have the $(p+1,q)$-property, then we have that $\mathfrak{F}$ has the $((p+1)+t+(m-1),q+t)$-property and not the $(p+1,q)$-property. The induction hypothesis implies that $|\mathfrak{F}| \leq (p+1) - q + 1 + N_n(t+(m-1),t+1)$. Lemma 2-9.1 implies that $N_n(t+(m-1),t+1) \leq N_n(t+m,t+1) - 1$. Thus,

$$(p+1) - q + 1 + N_n(t+(m-1),t+1) \leq p - q + 1 + N_n(t+m,t+1).$$

Hence, if $\mathfrak{F}$ fails to have the $(p+1,q)$-property, the desired conclusion follows. So assume that $\mathfrak{F}$ has the $(p+1,q)$-property. To complete the proof we shall use an argument similar to the proof of Theorem 2-2.

Since $\mathfrak{F}$ does not have the (p,q)-property, there is a subfamily $\mathfrak{G}$ of $\mathfrak{F}$ containing p sets such that no q sets of $\mathfrak{G}$ have a common point. Let $\mathcal{J}$ be a subcollection of $\mathfrak{F} \setminus \mathfrak{G}$ containing $t+m$ sets. Then $\mathcal{J} \cup \mathfrak{G}$ consists of $p+t+m$ sets of $\mathfrak{F}$. Thus, the hypothesis of the theorem implies that some $q+t$ sets of $\mathcal{J} \cup \mathfrak{G}$ have a common point. Since no q sets of $\mathfrak{G}$ have a common point, some $t+1$ sets of $\mathcal{J}$ must have a common point. Hence, $\mathfrak{F} \setminus \mathfrak{G}$ has the $(t+m,t+1)$-property. So there exist nonempty subfamilies $\mathfrak{C}_i$, $1 \leq i \leq N_n(t+m,t+1)$ of $\mathfrak{F} \setminus \mathfrak{G}$, such that

(a) $\mathfrak{F} \setminus \mathfrak{G} = \cup\{\mathfrak{C}_i\colon 1 \leq i \leq N_n(t+m,t+1)\}$

(b) $\cap\{P\colon P \in \mathfrak{C}_i\} \neq \emptyset$ for $1 \leq i \leq N_n(t+m,t+1)$.

Let $A \in \mathfrak{C}_1$. Then $\{A\} \cup \mathfrak{G}$ is a subfamily of $\mathfrak{F}$ consisting of $p+1$ sets. Since $\mathfrak{F}$ has the $(p+1,q)$-property, $\{A\} \cup \mathfrak{G}$ contains q sets with a nonempty intersection. Now no q sets of $\mathfrak{G}$ have a common point, so A must have a point in common with $q-1$ sets of $\mathfrak{G}$. Let

$\mathfrak{H}$ be any subfamily of $\mathfrak{G}$ which consists of $q-1$ sets and $|\mathfrak{H}| = 1$. Then for each $A \in \mathfrak{C}_1$ we have either the sets of $\{A\} \cup \mathfrak{H}$ have a common point, or A intersects some set in $\mathfrak{G} \setminus \mathfrak{H}$.

Let $A_1, A_2, \ldots, A_{p-q+1}$ denote the sets in $\mathfrak{G} \setminus \mathfrak{H}$. Define subfamilies $\mathfrak{M}_i$, $1 \leq i \leq p - q + 2$, of $\mathfrak{C}_1 \cup \mathfrak{G}$ as follows:

$$\mathfrak{M}_i = (\{A \in \mathfrak{C}_1 : A \cap A_i \neq \emptyset\} \setminus \mathfrak{G}) \cup \{A_i\}, \quad 1 \leq i \leq p - q + 1,$$

$$\mathfrak{M}_{p-q+2} = \{A \in \mathfrak{C}_1 : A \cap (\cap\{B : B \in \mathfrak{H}\}) \neq \emptyset\} \cup \mathfrak{H}.$$

Then each set of $\mathfrak{C}_1 \cup \mathfrak{G}$ belongs to some $\mathfrak{M}_i$. Since each two sets in $\mathfrak{C}_1$ have a common point, it can be shown as in the proof of Theorem 2-2 that $|\mathfrak{M}_i| = 1$ for $1 \leq i \leq p - q + 2$. Now

$$\mathfrak{C}_1 \cup \mathfrak{G} = \cup\{\mathfrak{M}_i : 1 \leq i \leq p - q + 2\}$$

and

$$\mathfrak{F} \setminus (\mathfrak{G} \cup \mathfrak{C}_1) \subset \cup\{\mathfrak{C}_i : 2 \leq i \leq N_n(t+m, t+1)\}.$$

Hence,

$$\mathfrak{F} = \cup(\{\mathfrak{C}_i : 2 \leq i \leq N_n(t+m,t+1)\}) \cup (\cup\{\mathfrak{M}_i : 1 \leq i \leq p - q + 2\})$$

and $\mathfrak{M}_j$, $1 \leq j \leq p - q + 2$, is such that $|\mathfrak{M}_j| = 1$. Thus,

$$\begin{aligned} |\mathfrak{F}| &= |(\mathfrak{F} \setminus (\mathfrak{G} \cup \mathfrak{C}_1)) \cup (\mathfrak{G} \cup \mathfrak{C}_1)| \\ &\leq p - q + 2 + N_n(t+m,t+1) - 1 \\ &= p - q + 1 + N_n(t+m,t+1). \blacksquare \end{aligned}$$

We shall now illustrate how Theorem 2-9 may be used to determine certain values of $N_n(p,q)$ when others are known. It will be shown later that $N_2(17,5) = 13$. Assuming that this is true, let $\mathfrak{F}$ be a family of mutually parallel parallelotopes in E^2 with the $(34,9)$-property. Also let $t = 4$, $m = 13$, $p = 17$ and $q = 5$. Then $\mathfrak{F}$ has the $(p+t+m,q+t)$-property. If $\mathfrak{F}$ also has the $(17,5)$-property, then $|\mathfrak{F}| \leq N_2(17,5) = 13$. If $\mathfrak{F}$ fails to have the $(17,5)$-property, then Theorem 2-9 implies that $|\mathfrak{F}| \leq 17 - 5 + 1 + N_n(17,5) = 13 + 13 = 26$. Hence, in either case $|\mathfrak{F}| \leq 26$. Thus, $N_2(34,9) \leq 26$. Theorem 2-7 implies that $N_2(34,9) \geq 26$. Therefore, $N_2(34,9) = 26$.

<u>Theorem</u> 2-10. Let $n \geq 2$ and $k \geq 1$. Then $T_n(2k+1,2) \geq 3k$ and $T_n(2k,2) \geq 3k - 2$.

Proof: Let $L_1, L_2, \ldots, L_k$ be k 5-cycles in E^n such that $\mathfrak{F} = \{P : P \in L_i,\ 1 \leq i \leq k\}$ is a family of translates of a given parallelotope, and $P \in L_i$, $P' \in L_j$, with $i \neq j$, implies that $P \cap P' = \emptyset$. Let $\mathcal{G}$ be a subfamily of $\mathfrak{F}$ consisting of $2k+1$ sets. Then $\mathcal{G}$ contains at least three sets from some L_i. However, each L_i has the $(3,2)$-property. Thus, two elements of $\mathcal{G}$ have a common point. This implies that $\mathfrak{F}$ has the $(2k+1,2)$-property. Moreover, the definition of $\mathfrak{F}$ and the fact that $|L_i| \geq 3$ for each i, $1 \leq i \leq k$, implies that

$$|\mathfrak{F}| = \sum_{i=1}^{k} |L_i| \geq 3k.$$

Thus, $T_n(2k+1,2) \geq 3k$.

Now let $L_1, L_2, \ldots, L_{k-2}$ be k-2 5-cycles in E^n and L_{k-1} a 7-cycle in E^n such that $\mathfrak{F} = \{P: P \in L_i,\ 1 \leq i \leq k-1\}$ is a family of translates of a given parallelotope in E^n, and $P \in L_i$, $P' \in L_j$, with $i \neq j$, implies that $P \cap P' = \emptyset$. Let $\mathfrak{G}$ be a subfamily of $\mathfrak{F}$ containing $2k$ sets. Then either $\mathfrak{G}$ contains three sets from some L_i, with $1 \leq i \leq k-2$, or $\mathfrak{G}$ contains four sets from L_{k-1}. However, each L_i, with $1 \leq i \leq k-2$, has the (3,2)-property, and L_{k-1} has the (4,2)-property. Thus, in either case some two elements of $\mathfrak{G}$ have a common point. Hence, $\mathfrak{F}$ has the (2k,2)-property. Now for each i, with $1 \leq i \leq k-2$, we have $|L_i| \geq 3$ and $|L_{k-1}| \geq 4$. The definition of $\mathfrak{F}$ implies that

$$|\mathfrak{F}| = \sum_{i=1}^{k-1} |L_i| = (k-2)3 + 4 = 3k - 2.$$

Thus, $T_n(2k,2) \geq 3k - 2$. ■

<u>Corollary</u> 2-10.1. Let $n \geq 2$ and $k \geq 1$. Then there exists a family $\mathfrak{F}$ of translates of a given parallelotope in E^n with the following property: $\mathfrak{F}$ contains $5k$ sets ($5k-3$ sets) with the $(2k+1,2)$-property ($(2k,2)$-property) no three of which have a common point.

Proof: The corollary follows by taking $\mathfrak{F}$ to be the families of sets used in the proof of Theorem 2-10. ■

The lower bounds given in Theorem 2-10 for $T_n(p,2)$ are clearly lower bounds for the number $N_n(p,2)$. From this theorem it follows that $T_n(p,2) \neq p - 2 + 1$ for $n \geq 2$ and $p \geq 3$.

Theorem 2-11. Let $n \geq 2$ and $k \geq 3$. Then $N_n(k,2) \geq 2k - 4$.

Proof: Since any family of mutually parallel parallelotopes in E^2 may also be considered as a family of mutually parallel parallelotopes in E^n for $n \geq 2$, it suffices to prove the theorem for $n = 2$. For each positive integer i define four sets as follows:

$$G_i = \{(x,y): 3i \leq x \leq 3i + 1, \ 0 \leq y \leq 3i\},$$

$$S_i = \{(x,y): 0 \leq x \leq 3i, \ 3i \leq y \leq 3i + 1\},$$

$$R_i = \{(x,y): 3i - 2 \leq x \leq 3i - 1, \ 3i - 3 \leq y < \infty\},$$

$$B_i = \{(x,y): 3i - 3 \leq x < \infty, \ 3i - 2 \leq y \leq 3i - 1\}.$$

For each positive integer k with $k \geq 3$, let

$$\mathfrak{G}_k = \{G_i: 1 \leq i \leq k - 2\},$$

$$\mathfrak{S}_k = \{S_i: 1 \leq i \leq k - 2\},$$

$$\mathfrak{R}_k = \{R_i: 1 \leq i \leq k - 2\},$$

$$\mathfrak{B}_k = \{B_i: 1 \leq i \leq k - 2\},$$

$$\mathfrak{F}_k = \mathfrak{G}_k \cup \mathfrak{S}_k \cup \mathfrak{R}_k \cup \mathfrak{B}_k.$$

Note that each family $\mathfrak{F}_k$ contains $4k-8$ sets. Figure 4 represents the family $\mathfrak{F}_7$. Note also that each of the following hold:

(1) For $i \leq j$, $(3j,3i-1) \in B_i \cap G_j$, and for $i < j$, $B_j \cap G_i = \emptyset$.

(2) For $i \leq j$, $(3i-1,3j) \in R_i \cap S_j$, and for $i < j$, $R_j \cap S_i = \emptyset$.

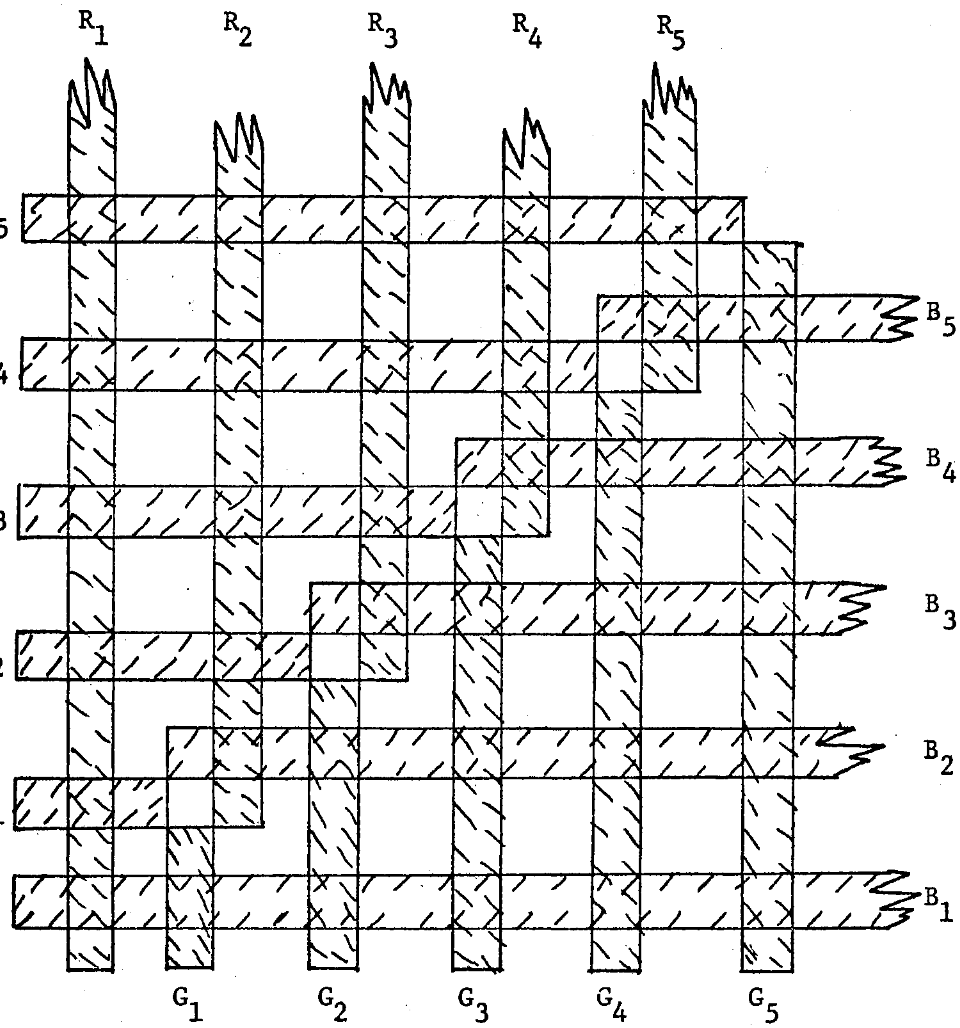

Figure 4. The Family $\mathfrak{F}_7$.

(3) For all i, $(3i,3i) \in G_i \cap S_i$, and for $i \neq j$, $G_i \cap S_j = \emptyset$.

(4) For all i, $(3i,3i+1) \in S_i \cap B_{i+1}$, and for $j \neq i+1$, $S_i \cap B_j = \emptyset$.

(5) For all i, $(3i+1,3i) \in G_i \cap R_{i+1}$, and for $j \neq i+1$, $G_i \cap R_j = \emptyset$.

(6) For all i, $(3i-1,3i-1) \in B_i \cap R_i$, and for $i \neq j$, $B_i \cap R_j = \emptyset$.

Moreover, each of the families $\mathfrak{G}_k$, $\mathfrak{S}_k$, $\mathfrak{R}_k$ and $\mathfrak{B}_k$ consists of pairwise disjoint sets.

Suppose that for some $k \geq 3$, $\mathfrak{F}_k$ contains three sets with a common point. Then either two of these sets are of the form S_i and B_j or of the form G_i and R_j for some i and j. Without loss of generality, assume two of the sets are of the form G_i and R_j. Then (5) implies that $j = i + 1$. The third set must either be a B_m or a S_m for some m. If the third set is a B_m for some m, then (1) implies that $m \leq i$ and (6) implies that $m = j$. This contradicts the fact that $j = i + 1$. A similar argument yields a contradiction when the third set is a S_m for some m. Hence, for each $k \geq 3$, $\mathfrak{F}_k$ fails to contain three sets with a common point. This implies that if $\mathfrak{H}$ is any subfamily of $\mathfrak{F}$ such that $|\mathfrak{H}| = 1$, then $\mathfrak{H}$ contains at most two sets.

We now proceed by induction on k to show that each of the families $\mathfrak{F}_k$ have the $(k,2)$-property. For $k = 3$, we have that $\mathfrak{F}_3 = \{G_1,S_1,R_1,B_1\}$. Moreover, $G_1 \cap B_1 \neq \emptyset$, $G_1 \cap S_1 \neq \emptyset$, $B_1 \cap R_1 \neq \emptyset$, and $R_1 \cap S_1 \neq \emptyset$. Thus, $\mathfrak{F}_3$ fails to contain three pairwise disjoint sets. Hence, $\mathfrak{F}_3$ has the (3,2)-property.

Suppose now that $\mathfrak{F}_k$ has the $(k,2)$-property for $k \geq 3$. To prove that $\mathfrak{F}_{k+1}$ has the $(k+1,2)$-property, suppose this were false. Then $\mathfrak{F}_{k+1}$ contains a subfamily $\mathfrak{H}$ consisting of $k+1$ sets, no two of which have a common point. Now $\mathfrak{F}_{k+1} = \mathfrak{F}_k \cup \{G_{k-1}, S_{k-1}, R_{k-1}, B_{k-1}\}$. Since $\mathfrak{F}_k$ has the $(k,2)$-property, $\mathfrak{H}$ must contain at least two sets from the family $\{G_{k-1}, S_{k-1}, R_{k-1}, B_{k-1}\}$. Moreover, since $G_{k-1} \cap B_{k-1} \neq \emptyset$, $S_{k-1} \cap R_{k-1} \neq \emptyset$, $G_{k-1} \cap S_{k-1} \neq \emptyset$, and $B_{k-1} \cap R_{k-1} \neq \emptyset$, $\mathfrak{H}$ contains no more than two sets of $\{G_{k-1}, S_{k-1}, R_{k-1}, B_{k-1}\}$. We have that either

$$\mathfrak{H} \cap \{G_{k-1}, S_{k-1}, R_{k-1}, B_{k-1}\} = \{G_{k-1}, R_{k-1}\}$$

or

$$\mathfrak{H} \cap \{G_{k-1}, S_{k-1}, R_{k-1}, B_{k-1}\} = \{S_{k-1}, B_{k-1}\}.$$

Without loss of generality, assume that

$$\mathfrak{H} \cap \{G_{k-1}, S_{k-1}, R_{k-1}, B_{k-1}\} = \{S_{k-1}, B_{k-1}\}.$$

From (2), we have that $R_j \cap S_{k-1} \neq \emptyset$ for all $j \leq k-1$; thus, $\mathfrak{H} \cap \mathfrak{R}_k = \emptyset$. From (4), we have that $S_{k-2} \cap B_{k-1} \neq \emptyset$. Hence,

$$\mathfrak{H} \subset (\mathfrak{F}_k \setminus \mathfrak{R}_k \cup \{S_{k-2}\}) \cup \{S_{k-1}, B_{k-1}\}.$$

Thus, $\mathfrak{F}_k \setminus \mathfrak{R}_k \cup \{S_{k-2}\}$ must contain $k-1$ sets from $\mathfrak{H}$. Figure 5 represents the family $\mathfrak{F}_k \setminus \mathfrak{R}_k \cup \{S_{k-2}\}$ for $k + 1 = 7$.

We shall now prove by induction on r that $\mathfrak{F}_r \setminus \mathfrak{R}_r \cup \{S_{r-2}\}$ has the $(r-1,2)$-property for $r \geq 3$. This will then contradict the fact

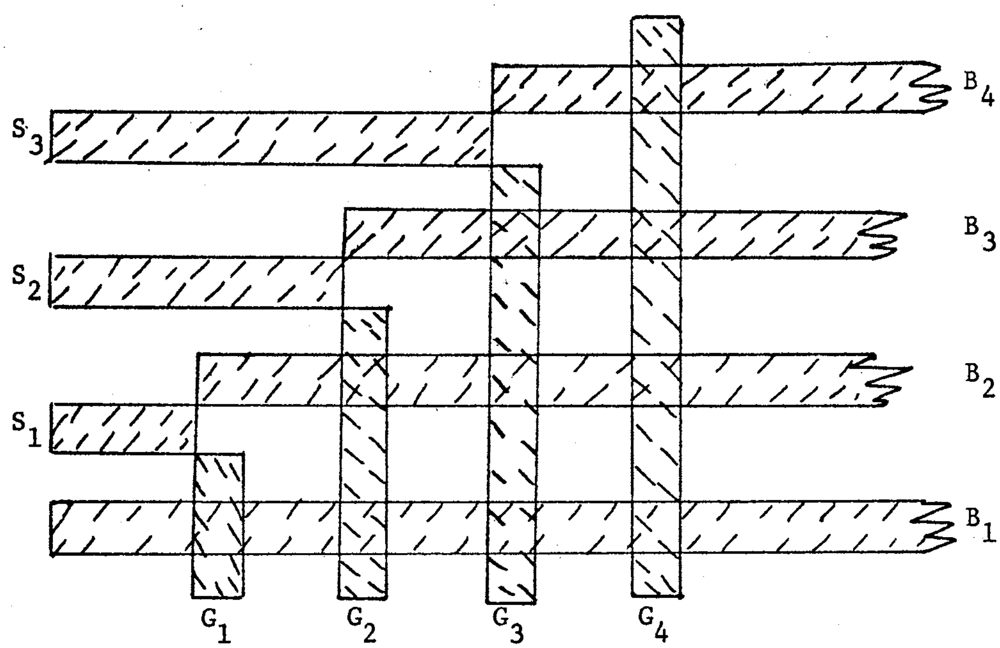

Figure 5. The Family $\mathfrak{J}_k \setminus \mathfrak{R}_k \cup \{S_{k-2}\}$.

that $\mathfrak{F}_k \setminus \mathfrak{R}_k \cup \{S_{k-2}\}$ contains k-1 pairwise disjoint sets of $\mathfrak{H}$. This will then contradict the original assumption that $\mathfrak{F}_{k+1}$ fails to have the (k+1,2)-property.

For $r = 3$, we have $\mathfrak{F}_3 \setminus \mathfrak{R}_3 \cup \{S_1\} = \{B_1, G_1\}$, which has the (2,2)-property by (1). Now suppose for $r \geq 3$, it has been shown that $\mathfrak{F}_r \setminus \mathfrak{R}_r \cup \{S_{r-2}\}$ has the (r-1,2)-property. To prove that $\mathfrak{F}_{r+1} \setminus \mathfrak{R}_{r+1} \cup \{S_{r-1}\}$ has the (r,2)-property, suppose this were false. Then $\mathfrak{F}_{r+1} \setminus \mathfrak{R}_{r+1} \cup \{S_{r-1}\}$ contains a subfamily $\mathfrak{M}$ consisting of r pairwise disjoint sets. Now

$$\mathfrak{F}_{r+1} \setminus \mathfrak{R}_{r+1} \cup \{S_{r-1}\} = (\mathfrak{F}_r \setminus \mathfrak{R}_r \cup \{S_{r-2}\}) \cup \{B_{r-1}, G_{r-1}, S_{r-2}\}.$$

Our induction hypothesis on r implies that $\mathfrak{M}$ contains two sets from the family $\{B_{r-1}, G_{r-1}, S_{r-2}\}$. However, (1) and (4) implies that $\{G_{r-1}, S_{r-2}\} \subset \mathfrak{M}$. Since $G_{r-1} \in \mathfrak{M}$, we have by (1) that $B_i \notin \mathfrak{M}$ for all $i < r - 1$. Moreover, since $S_{r-2} \in \mathfrak{M}$, we have by (3) that $G_{r-2} \notin \mathfrak{M}$. Hence, $\mathfrak{M}$ is contained in

$$(\mathfrak{F}_r \setminus \mathfrak{R}_r \cup \mathfrak{B}_r \cup \{G_{r-2}\}) \cup \{G_{r-1}, S_{r-2}\}.$$

Since $\mathfrak{M}$ contains r sets, $\mathfrak{M}$ contains r-2 sets from

$$\mathfrak{F}_r \setminus (\mathfrak{R}_r \cup \mathfrak{B}_r \cup \{G_{r-2}, S_{r-2}\}) = \{G_1, \ldots, G_{r-3}, S_1, \ldots, S_{r-3}\}.$$

Thus, for some i, $1 \leq i \leq r - 3$, $\mathfrak{M}$ contains G_i and S_i. However, (3) implies that $G_i \cap S_i \neq \emptyset$. This is a contradiction to the definition of $\mathfrak{M}$. Therefore, $\mathfrak{F}_{r+1} \setminus \mathfrak{R}_{r+1} \cup \{S_{r-1}\}$ has the (r,2)-property. By induction, we now have that $\mathfrak{F}_r \setminus \mathfrak{R}_r \cup \{S_{r-2}\}$ has the (r-1,2)-property for all $r \geq 3$. This implies, by induction on k, that $\mathfrak{F}_k$ has the $(k,2)$-property for all $k \geq 3$.

Note that the sets in $\mathfrak{F}_k$ are not parallelotopes; thus, $\mathfrak{F}_k$ needs to be adjusted so that the sets considered are parallelotopes. Let $H_k = \{(x,y): 0 \leq x \leq 3k - 4, \ 0 \leq y \leq 3k - 4\}$, $k \geq 3$, a square set. We note that for $i \leq k - 2$ and $j \leq k - 2$ the points in (1) through (6) all belong to H_k. Thus, if $A,C \in \mathfrak{F}_k$ with $A \cap C \neq \emptyset$, then $(A \cap H_k) \cap (C \cap H_k) \neq \emptyset$. Let $\mathfrak{F}_k' = \{A \cap H_k: A \in \mathfrak{F}_k\}$. Then $\mathfrak{F}_k'$ is a family of mutually parallel parallelotopes in E^2 with the $(k,2)$-property. Moreover, $\mathfrak{F}_k'$ contains $4k-8$ sets, no three of which have a common point. Thus, if $\mathfrak{N}$ is any nonempty subfamily of $\mathfrak{F}_k'$ such that $|\mathfrak{N}| = 1$, then $\mathfrak{N}$ contains at most two sets. Hence,

$$|\mathfrak{F}_k'| \geq 1/2(4k - 8) = 2k - 4.$$

Therefore, $N_2(k,2) \geq 2k - 4$. Figure 6 illustrates the family $\mathfrak{F}_7'$. ■

<u>Corollary</u> 2-11.1. Let $n \geq 2$ and $k \geq 3$. Then there exists a family $\mathfrak{F}$ of mutually parallel parallelotopes in E^n with the following property: $\mathfrak{F}$ contains $4k-8$ sets with the $(k,2)$-property, no three of which have a common point.

Proof: The proof is contained in the proof of Theorem 2-11. ■

We note that the lower bound of 2 given in Theorem 2-11 for $N_n(3,2)$, $n \geq 2$, is not the best result we have, since from Theorem 2-10 it follows that $3 \leq T_n(3,2) \leq N_n(3,2)$ for $n \geq 2$.

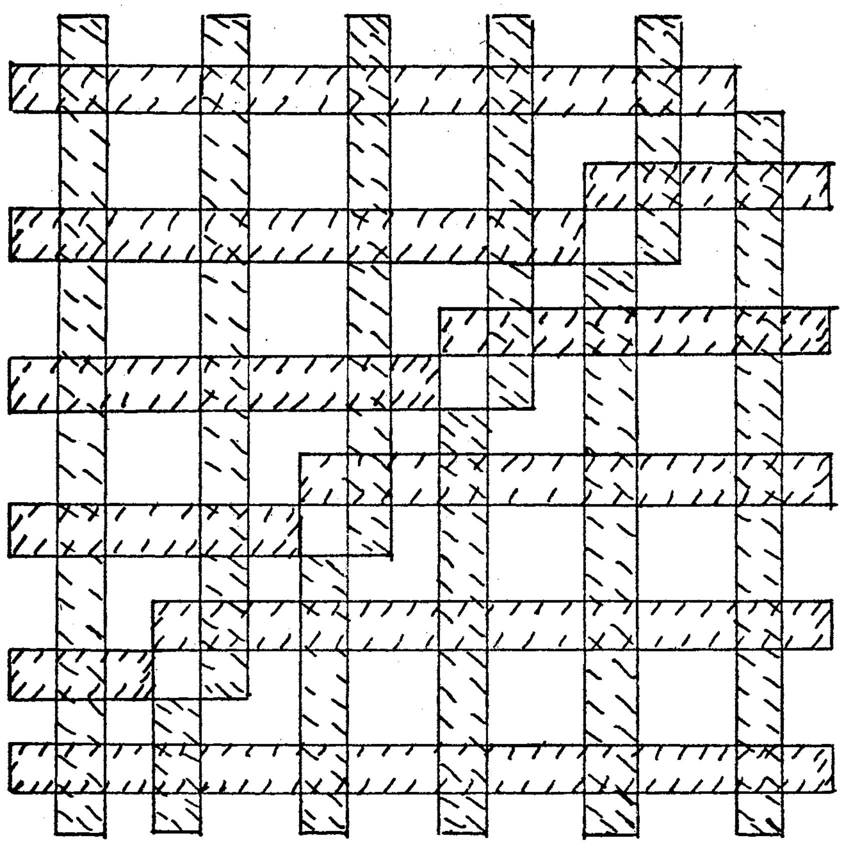

Figure 6. The Family $\mathfrak{F}_7'$.

<u>Theorem</u> 2-12. For $k \geq 2$ and $n \geq 2$,

$$N_n(k+2,2) \leq N_n(k,2) + N_{n-1}(k+1,2) + N_{n-1}(k+2,2) - k + 2.$$

Proof: Let $\mathfrak{F} = \{P_i\colon i = 1,2, \ldots, m\}$ be a finite family of mutually parallel parallelotopes in E^n with the $(k+2,2)$-property. Then there exists a basis $\{x_1, \ldots, x_n\}$ for E^n such that each P_i has the form

$$P_i = \left\{ \sum_{j=1}^{n} \alpha_j x_j \colon \lambda_j^i \leq \alpha_j \leq \delta_j^i \right\}.$$

Without loss of generality, assume that $\delta_1^1 = \min\{\delta_1^i\colon i = 1, \ldots, m\}$. Let H_1 be the hyperplane

$$\left\{ \sum_{j=1}^{n} \alpha_j x_j \colon \alpha_j = \delta_1^1 \right\}.$$

If each set of $\mathfrak{F}$ intersects H_1, then $|\mathfrak{F}| \leq N_{n-1}(k+2,2)$. If there exists a set in $\mathfrak{F}$ which fails to intersect H_1, then the set $\{\delta_1^i\colon \lambda_1^i > \delta_1^1\}$ is nonempty. Without loss of generality, assume that $\delta_1^2 = \min\{\delta_1^i\colon \lambda_1^i > \delta_1^1\}$. Let H_2 be the hyperplane

$$\left\{ \sum_{j=1}^{n} \alpha_j x_j \colon \alpha_1 = \delta_1^2 \right\}.$$

Define subcollections $\mathfrak{F}_i$, $i = 1, \ldots, 5$, of $\mathfrak{F}$ as follows:

$$\mathfrak{F}_1 = \{P_i \in \mathfrak{F}: \delta_1^i < \delta_1^2\},$$

$$\mathfrak{F}_2 = \{P_i \in \mathfrak{F}: \lambda_1^i \leq \delta_1^2 \leq \delta_1^i\},$$

$$\mathfrak{F}_3 = \{P_i \in \mathfrak{F}: \lambda_1^i > \delta_1^2\},$$

$$\mathfrak{F}_4 = \{P_i \in \mathfrak{F}: \delta_1^1 < \lambda_1^i \leq \delta_1^2 \leq \delta_1^i\},$$

$$\mathfrak{F}_5 = \{P_i \in \mathfrak{F}: \lambda_1^i \leq \delta_1^1\}.$$

We note that $P_1 \in \mathfrak{F}_1$, $P_1 \in \mathfrak{F}_5$, $P_2 \in \mathfrak{F}_2$, $P_2 \in \mathfrak{F}_4$, and $\mathfrak{F}_3$ may be empty. If $\mathfrak{F}_3 = \emptyset$, then the definitions of H_1 and H_2 imply that $\mathfrak{F} = \mathfrak{F}_4 \cup \mathfrak{F}_5$. Moreover, each set of $\mathfrak{F}_4$ intersects H_2 and fails to intersect P_1. Thus, either $\mathfrak{F}_4$ has the (k+1,2)-property or fails to contain k+1 sets. In either case, $|\mathfrak{F}_4| \leq N_{n-1}(k+1,2)$. Now $\mathfrak{F}_5$ either fails to contain k+2 sets, or $\mathfrak{F}_5$ has the (k+2,2)-property. Also, since each set in $\mathfrak{F}_5$ intersects H_1, we conclude that $|\mathfrak{F}_5| \leq N_{n-1}(k+2,2)$. Thus, if $\mathfrak{F}_3 = \emptyset$, it follows that

$$|\mathfrak{F}| = |\mathfrak{F}_4 \cup \mathfrak{F}_5| \leq |\mathfrak{F}_4| + |\mathfrak{F}_5| \leq N_{n-1}(k+1,2) + N_{n-1}(k+2,2).$$

Theorem 2-7 implies that $N_n(k,2) - k + 1 \geq 0$. Hence,

$$|\mathfrak{F}| \leq N_{n-1}(k+1,2) + N_{n-1}(k+2,2) + N_n(k,2) - k + 1.$$

If $\mathfrak{F}_3 \neq \emptyset$, let $\mathfrak{F}' = \{P_i \in \mathfrak{F}: \delta_1^i \leq \delta_1^2\}$, and let h be the maximum number of pairwise disjoint sets in $\mathfrak{F}'$. Since $\mathfrak{F}_3 \neq \emptyset$, we have that $h \leq k$. Moreover, $P_1, P_2 \in \mathfrak{F}'$ implies that $h \geq 2$. The definition of h implies that either $\mathfrak{F}'$ has the (h+1,2)-property, or $\mathfrak{F}'$ fails to contain h+1 sets. Since each set of $\mathfrak{F}_2$ intersects

H_2, it follows that $|\mathfrak{F}_2| \leq N_{n-1}(k+2,2)$. The fact that $\mathfrak{F}_1 \subset \mathfrak{F}'$ implies that either $\mathfrak{F}_1$ has the $(h+1,2)$-property, or $\mathfrak{F}_1$ fails to contain $h+1$ sets. Since each set of $\mathfrak{F}_1$ intersects H_1, we have in either case $|\mathfrak{F}_1| \leq N_{n-1}(h+1,2)$. Since each set of $\mathfrak{F}_3$ fails to intersect all sets of $\mathfrak{F}'$, we have either $\mathfrak{F}_3$ has the $(k+2-h,2)$-property, or $\mathfrak{F}_3$ fails to contain $(k+2-h)$ sets. In either case, $|\mathfrak{F}_3| \leq N_n(k+2-h,2)$. The definitions of H_1 and H_2 imply that $\mathfrak{F} = \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$. Hence,

$$|\mathfrak{F}| \leq |\mathfrak{F}_1| + |\mathfrak{F}_2| + |\mathfrak{F}_3| \leq N_{n-1}(h+1,2) + N_{n-1}(k+2,2) + N_n(k+2-h,2).$$

Lemma 2-9.1 with $h-2$ iterations implies that

$$N_n(k+2-h,2) + h - 2 \leq N_n(k,2).$$

Also, Lemma 2-9.1 with $k-h$ iterations implies that

$$N_{n-1}(h+1,2) + k - h \leq N_{n-1}(k+1,2).$$

Thus,

$$|\mathfrak{F}| \leq N_{n-1}(k+1,2) + N_{n-1}(k+2,2) + N_n(k,2) - k + 2.$$

Corollary 1-2.1 implies that

$$N_n(k+2,2) \leq N_{n-1}(k+1,2) + N_{n-1}(k+2,2) + N_n(k,2) - k + 2. \blacksquare$$

<u>Corollary</u> 2-12.1. If $k \geq 1$, then $N_2(2k,2) \leq k^2 + 2k - 2$ and $N_2(2k+1,2) \leq k^2 + 3k - 1$.

Proof: For $p \geq q \geq 2$, Theorem 2-7 implies that $N_1(p,q) = p - q + 1$. For $r \geq 2$, Theorem 2-12 implies that

$$N_2(r+2,2) \leq N_1(r+1,2) + N_1(r+2,2) + N_2(r,2) + 2 - r.$$

Thus, for $r \geq 2$, $N_2(r+2,2) \leq N_2(r,2) + r + 3$. The proof will now be completed by induction. For $k = 1$, Theorem 2-7 implies that the equation $N_2(2k,2) \leq k^2 + 2k - 2$ holds. So suppose the result holds for $k \geq 1$. Now $N_2(2(k+1),2) \leq N_2(2k,2) + 2k + 3$. Since the result holds for k, we have $N_2(2k,2) \leq k^2 + 2k - 2$. Hence,

$$\begin{aligned} N_2(2(k+1),2) &\leq N_2(2k,2) + 2k + 3 \\ &\leq k^2 + 2k - 2 + 2k + 3 \\ &= (k+1)^2 + 2(k+1) - 2. \end{aligned}$$

Thus, $N_2(2k,2) \leq k^2 + 2k - 2$ for all $k \geq 1$.

The discussion following Theorem 2-7 implies that the inequality $N_2(2k+1,2) \leq k^2 + 3k - 1$ holds for $k = 1$. So suppose the result holds for $k \geq 1$. We have $N_2(2(k+1)+1,2) \leq N_2(2k+1,2) + 2k + 1 + 3$. Since the result holds for k, we have $N_2(2k+1,2) \leq k^2 + 3k - 1$. Hence,

$$\begin{aligned} N_2(2(k+1)+1,2) &\leq N_2(2k+1,2) + 2k + 4 \\ &\leq k^2 + 3k - 1 + 2k + 4 \\ &= (k+1)^2 + 3(k+1) - 1. \end{aligned}$$

Thus,

$$N_2(2k+1,2) \leq k^2 + 3k - 1 \quad \text{for all } k \geq 1. \blacksquare$$

Results in E^2 and E^3

The following sequence of theorems gives deeper results for the lower dimensional spaces.

Theorem 2-13. If $\mathfrak{F}$ is a family of mutually parallel parallelotopes in E^2 with both the (3,2)-property and the (5,3)-property, then $|\mathfrak{F}| \le 2$.

Proof: Corollary 1-2.1 implies that it suffices to prove the theorem when $\mathfrak{F}$ is finite. So let $\mathfrak{F} = \{P_1, \ldots, P_m\}$ be a finite family of mutually parallel parallelotopes in E^2 with both the (3,2)-property and the (5,3)-property. Then there exists a basis $\{x,y\}$ for E^2 such that each P_i has the form

$$P_i = \{\alpha x + \beta y: \lambda_i \le \alpha \le \delta_i, \quad \epsilon_i \le \beta \le \eta_i\}.$$

Without loss of generality, assume that $\delta_1 = \min\{\delta_i: i = 1, \ldots, m\}$. Let H_1 denote the line $\{\delta_1 x + \beta y: \beta$ is real$\}$. If every element of $\mathfrak{F}$ intersects the line H_1, then it follows from Theorem 2-7 that $|\mathfrak{F}| \le 2$. Thus, assume that some set in $\mathfrak{F}$ fails to intersect H_1. Then without loss of generality, assume that

$$\lambda_2 = \max\{\lambda_i: i = 1,2, \ldots, m\} \text{ and } \lambda_2 > \delta_1.$$

Let H_2 be the line $\{\lambda_2 x + \beta y: \beta$ is real$\}$. By a similar argument it may be assumed that there exists sets P_k and P_h in $\mathfrak{F}$ such that $\epsilon_k = \max\{\epsilon_i: i = 1, \ldots, m\}$ and $\eta_h = \min\{\eta_i: i = 1, \ldots, m\}$, with $\epsilon_k > \eta_h$. Let H'_k and H'_h denote the lines $\{\alpha x + \epsilon_k y: \alpha$ is real$\}$ and $\{\alpha x + \eta_h y: \alpha$ is real$\}$, respectively. Let $a = \delta_1 x + \eta_h y$,

$b = \delta_1 x + \epsilon_k y$, $c = \lambda_2 x + \epsilon_k y$ and $d = \lambda_2 x + \eta_h y$. It will now be shown in detail how to deal with one of the four cases $h = 1$, $h = 2$, $k = 1$ or $k = 2$. The other three cases follow similarly. If $h = 1$, let $\mathfrak{F}_1 = \{P_i \in \mathfrak{F}: a \in P_i\}$ and $\mathfrak{F}_2 = \mathfrak{F} \setminus \mathfrak{F}_1$ (c.f. Figure 7). If $P_j \in \mathfrak{F}_2$, then either $\lambda_j > \delta_1$ or $\epsilon_j > \eta_1$. Consequently, P_j fails to intersect the set P_1. Thus, either $\mathfrak{F}_2$ has the (2,2)-property, or $\mathfrak{F}_2$ contains fewer than two sets. In either case, $|\mathfrak{F}_2| = 1$. Thus, $|\mathfrak{F}| = |\mathfrak{F}_1 \cup \mathfrak{F}_2| \leq |\mathfrak{F}_1| + |\mathfrak{F}_2| = 2$.

Now assume that $h \notin \{1,2\}$ and $k \notin \{1,2\}$ (c.f. Figure 8). Since $\mathfrak{F}$ has the (3,2)-property, every set in $\mathfrak{F}$ intersects P_1 or P_2 and also P_h or P_k. Thus, every set in $\mathfrak{F}$ contains one of the four points from the set $\{a,b,c,d\}$. Suppose that the set $\{a,b,c,d\}$ fails to contain two points such that each set in $\mathfrak{F}$ contains one of these two points. It will now be shown that one of the four cases $P_k \cap P_1 = \emptyset$, $P_k \cap P_2 = \emptyset$, $P_h \cap P_1 = \emptyset$, or $P_h \cap P_2 = \emptyset$ leads to a contradiction. The other three also lead to a contradiction by a similar argument. If $P_k \cap P_1 = \emptyset$, then from the (3,2)-property it follows that $P_k \cap P_2 \neq \emptyset$. By the assumptions there exists a set $P_i \in \mathfrak{F}$ such that $P_i \cap \{a,c\} = \emptyset$. Since each set in $\mathfrak{F}$ contains at least one point from the set $\{a,b,c,d\}$, it follows that $P_i \cap \{b,d\} \neq \emptyset$. If $b \in P_i$, then $a \notin P_i$ implies that $P_i \cap P_h = \emptyset$. Moreover, $c \notin P_i$ implies that $P_2 \cap P_i = \emptyset$. Consequently, no three of the five sets $\{P_1,P_2,P_k,P_h,P_i\}$ have a common point. If $d \in P_i$, then $a \notin P_i$ implies that $P_i \cap P_1 = \emptyset$. Moreover, $c \notin P_i$ implies that $P_i \cap P_k = \emptyset$. Consequently, no two of the three sets $\{P_1,P_k,P_i\}$ have a common point. Hence, $P_k \cap P_1 = \emptyset$ leads to a contradiction.

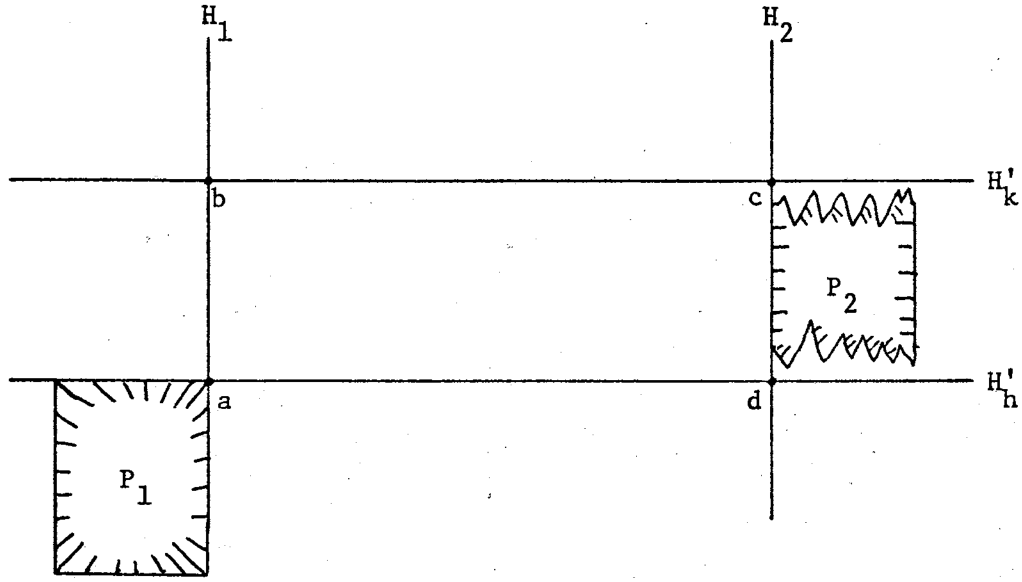

Figure 7. The Case of h = 1.

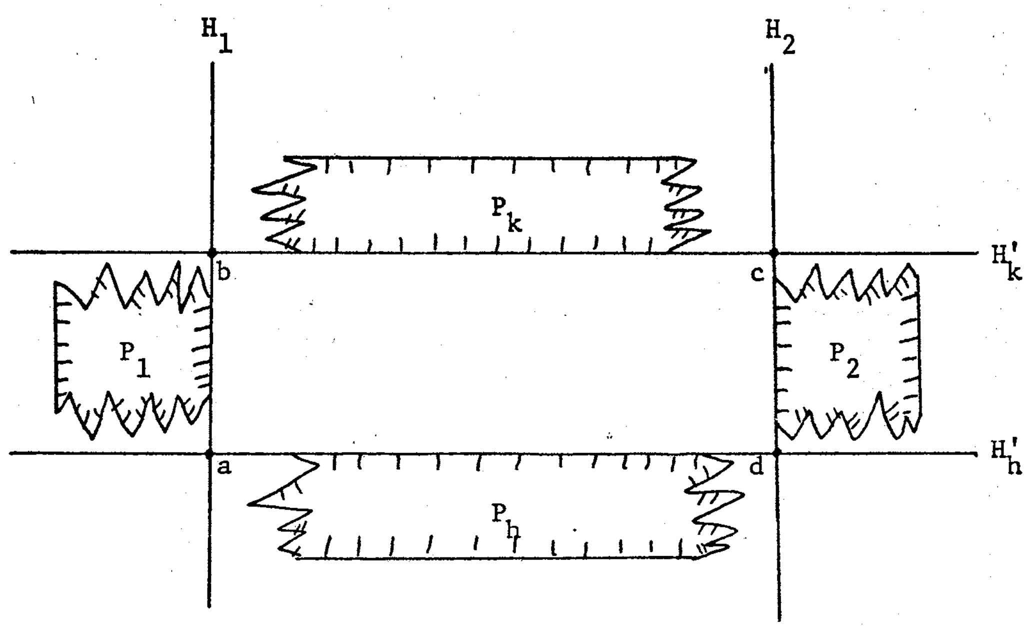

Figure 8. The Case of $h \notin \{1,2\}$ and $k \notin \{1,2\}$.

Now assume that $P_k \cap P_1 \neq \emptyset$, $P_k \cap P_2 \neq \emptyset$, $P_h \cap P_1 \neq \emptyset$ and $P_h \cap P_2 \neq \emptyset$. By the assumptions, there exist sets $P_i, P_j \in \mathfrak{F}$ such that $P_i \cap \{a,c\} = \emptyset$ and $P_j \cap \{b,d\} = \emptyset$. Without loss of generality, assume that $a \in P_j$. Then $P_j \cap \{b,d\} = \emptyset$ implies that $P_j \cap P_2 = \emptyset$ and $P_j \cap P_k = \emptyset$. Now either $b \in P_i$ or $d \in P_i$. If $d \in P_i$, then it follows that $P_i \cap P_1 = \emptyset$ and $P_i \cap P_k = \emptyset$. Consequently, no three of the five sets $\{P_1,P_2,P_k,P_i,P_j\}$ have a common point. If $b \in P_i$, then one argues similarly that no three of the five sets $\{P_2,P_h,P_k,P_i,P_j\}$ have a common point. Thus, the assumption that the set $\{a,b,c,d\}$ fails to contain two points such that each set in $\mathfrak{F}$ contains one of these two points leads to a contradiction. Thus, $|\mathfrak{F}| \leq 2$. ∎

It follows from Theorems 2-13, 2-2 and 2-7 that $N_2(5,3) = 3$. Moreover, Theorem 2-13 implies that the maximum value of $|\mathfrak{F}|$, where $\mathfrak{F}$ ranges over all families of mutually parallel parallelotopes in E^2, is not taken on when $\mathfrak{F}$ also has the (3,2)-property. However, the following example shows that this is not the case in E^n for $n \geq 3$.

<u>Example</u> 2-2. Define sets P_i, $i = 1,2, \ldots, 7$, in E^3 as follows:

$$P_1 = \{(x,y,z): -1 \leq x \leq 1, \quad -1 \leq y \leq 1, \quad -1 \leq z \leq 1\},$$

$$P_2 = P_1 + (2,2,0), \qquad P_3 = P_1 + (2,1,-1),$$

$$P_4 = P_1 + (1,-1,1), \qquad P_5 = P_1 + (2,-2,0),$$

$$P_6 = P_1 + (4,0,0), \qquad P_7 = P_1 + (3,0,2).$$

Figure 9 represents the projection of P_i, $1 \leq i \leq 6$, into the xy-plane, and the sets marked with an I are the sets which P_7 intersects.

Let $\mathfrak{F} = \{P_1, P_2, \ldots, P_7\}$. Then $\mathfrak{F}$ is a family of mutually parallel parallelotopes in E^3 with both the (5,3)-property and the (3,2)-property. Moreover, $\mathfrak{F}$ is a family of translates of P_1 such that no four sets in $\mathfrak{F}$ have a common point. Consequently, $|\mathfrak{F}| \geq 3$.

<u>Corollary</u> 2-13.1. $N_3(5,3) = 3$.

Proof: Let $\mathfrak{F} = \{P_1, P_2, \ldots, P_m\}$ be a finite family of mutually parallel parallelotopes in E^3 with the (5,3)-property. If $\mathfrak{F}$ fails to have the (3,2)-property, then Theorem 2-2 implies that $|\mathfrak{F}| \leq 3$. So assume that $\mathfrak{F}$ also has the (3,2)-property.

There exists a basis $\{x,y,z\}$ for E^3 such that each $P_i \in \mathfrak{F}$ has the form

$$P_i = \{\alpha_1^i x + \alpha_2^i y + \alpha_3^i z \colon \lambda_j^i \leq \alpha_j^i \leq \beta_j^i,\quad j = 1,2,3\}.$$

Without loss of generality, assume that $\beta_1^1 = \min\{\beta_1^i \colon i = 1,2, \ldots, m\}$. Let H denote the 2-dimensional hyperplane

$$\{\beta_1^1 x + \beta y + \epsilon z \colon \beta, \epsilon \text{ are real}\}.$$

If every set in $\mathfrak{F}$ intersects H, then Theorem 2-13 implies that $|\mathfrak{F}| \leq 2$. If some set in $\mathfrak{F}$ fails to intersect H, let $\mathfrak{F}_2 = \{P \in \mathfrak{F} \colon P \cap H = \emptyset\}$ and $\mathfrak{F}_1 = \{P \in \mathfrak{F} \colon P \cap H \neq \emptyset\}$. Now either $\mathfrak{F}_1$ has the (5,3)-property and the (3,2)-property, or $\mathfrak{F}_1$ fails to contain five sets. Suppose $\mathfrak{F}_1$ contains fewer than five sets. If $\mathfrak{F}_1$

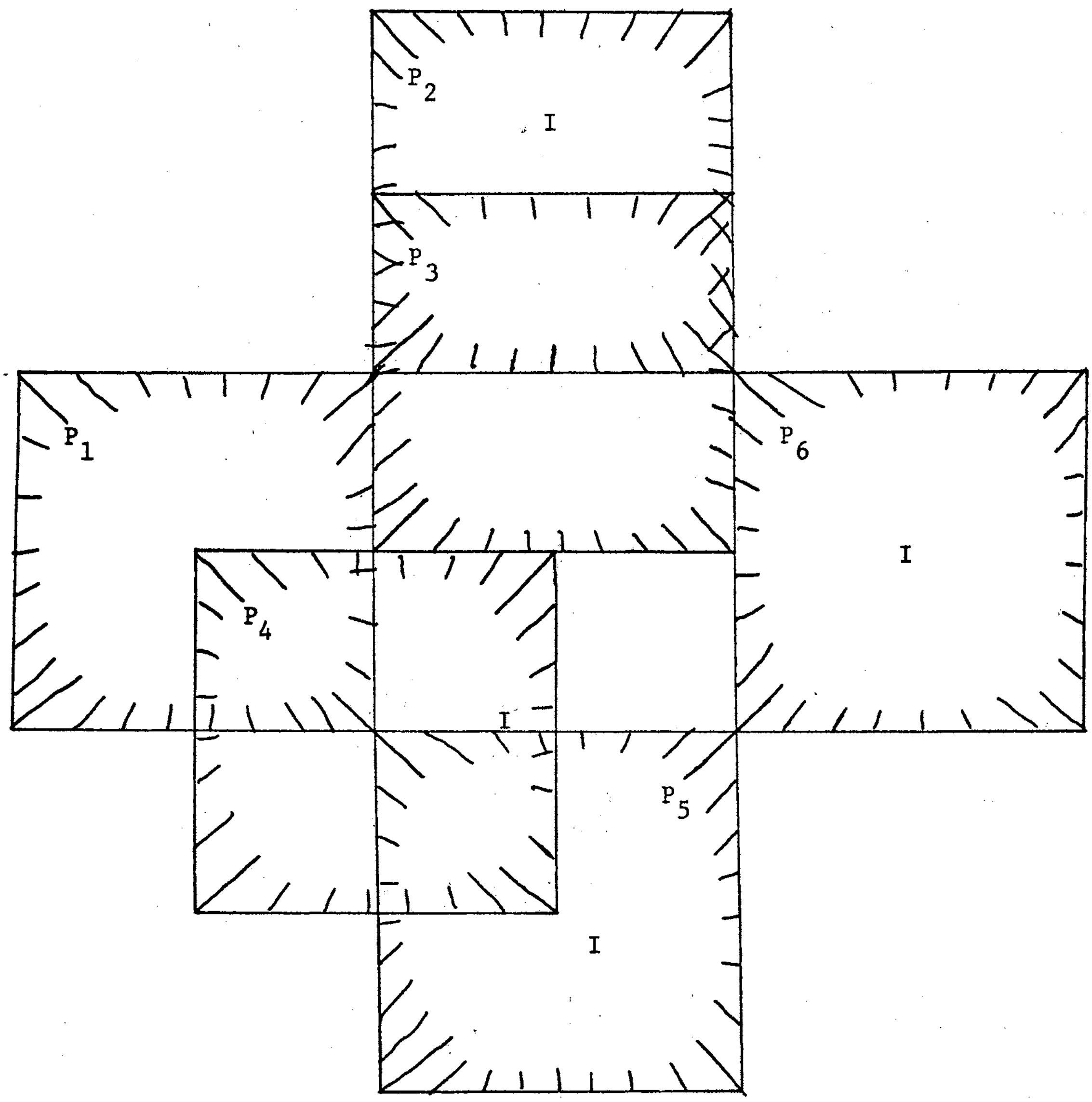

Figure 9. The Projections of the Sets P_i.

contains fewer than four sets, it follows from the (3,2)-property of $\mathfrak{F}$ that $|\mathfrak{F}_1| \leq 2$. If $\mathfrak{F}_1 = \{P_1, P_i, P_j, P_k\}$, that is $\mathfrak{F}_1$ consists of four sets, then we may assume $P_1 \cap P_i \neq \emptyset$. If $P_j \cap P_k \neq \emptyset$, then clearly $|\mathfrak{F}_1| \leq 2$. If $P_j \cap P_k = \emptyset$, then because of the (3,2)-property it may be assumed that $P_1 \cap P_j \neq \emptyset$. Now also because of the (3,2)-property, we have either $P_i \cap P_j \neq \emptyset$ or $P_i \cap P_k \neq \emptyset$. If $P_i \cap P_k \neq \emptyset$, then clearly $|\mathfrak{F}_1| \leq 2$. If $P_i \cap P_j \neq \emptyset$, then $|\{P_1, P_i, P_j\}| = 1$. Consequently, $|\mathfrak{F}_1| \leq 2$. Thus, if $\mathfrak{F}_1$ fails to contain five sets, $|\mathfrak{F}_1| \leq 2$. If $\mathfrak{F}_1$ contains at least five sets, then Theorem 2-13 together with the fact that each set in $\mathfrak{F}_1$ intersects H implies that $|\mathfrak{F}_1| \leq 2$.

Since each set in $\mathfrak{F}_2$ fails to intersect P_1, we have either $\mathfrak{F}_2$ has the (2,2)-property, or $\mathfrak{F}_2$ fails to contain two sets. Theorem 2-1 implies $|\mathfrak{F}_2| \leq 1$. Hence, $|\mathfrak{F}| = |\mathfrak{F}_1 \cup \mathfrak{F}_2| \leq 3$. Therefore, $|\mathfrak{F}| \leq 3$ for any finite family $\mathfrak{F}$ of mutually parallel parallelotopes in E^3 with the (5,3)-property. Corollary 1-2.1 implies the same result for infinite families. Thus, $N_3(5,3) \leq 3$. Theorem 2-7 implies that $N_3(5,3) \geq 3$; hence, $N_3(5,3) = 3$. ∎

Example 2-2 implies that there exists a family $\mathfrak{F}$ of mutually parallel parallelotopes in E^3 with both the (5,3)-property and the (3,2)-property such that $|\mathfrak{F}| \geq 3$. On the other hand, Theorem 2-13 implies that this is not the case in E^2. Thus, in some sense the situation in E^3 is quite different from that of E^2. However, this difference is not expressed in the equation $N_2(5,3) = N_3(5,3) = 3$.

<u>Corollary</u> 2-13.2. $N_2(6,3) = 4$.

Proof: Let $\mathfrak{J} = \{P_1, P_2, \ldots, P_m\}$ be a finite family of mutually parallel parallelotopes in E^2 with the (6,3)-property. If $\mathfrak{J}$ fails to have the (4,2)-property, then Theorem 2-2 implies $|\mathfrak{J}| \leq 4$. So assume that $\mathfrak{J}$ also has the (4,2)-property.

There exists a basis $\{x,y\}$ for E^2 such that each $P_i \in \mathfrak{J}$ has the form

$$P_i = \{\alpha_1^i x + \alpha_2^i y\colon \lambda_j^i \leq \alpha_j^i \leq \beta_j^i, \quad j = 1,2\}.$$

Without loss of generality, assume that $\beta_1^1 = \min\{\beta_1^i\colon i = 1, \ldots, m\}$. Let H denote the line $\{\beta_1^1 x + \alpha y\colon \alpha$ is real$\}$. If every set in $\mathfrak{J}$ intersects H, then it follows from Theorem 2-7 that $|\mathfrak{J}| \leq 3$. So assume that some set in $\mathfrak{J}$ fails to intersect H. Let $\mathfrak{J}_1 = \{P \in \mathfrak{J}\colon P \cap P_1 \neq \emptyset\}$ and $\mathfrak{J}_2 = \{P \in \mathfrak{J}\colon P \cap P_1 = \emptyset\}$. The definition of H implies that each set in $\mathfrak{J}_1$ intersects H. Also, $\mathfrak{J}_2 \neq \emptyset$, since there exists a set in $\mathfrak{J}$ which fails to intersect H. Since each set in $\mathfrak{J}_2$ fails to intersect P_1, we have either $\mathfrak{J}_2$ has the (5,3)-property and the (3,2)-property, or $\mathfrak{J}_2$ fails to contain five sets. In either case, it follows from the proof of Corollary 2-13.1 and the conclusion of Theorem 2-13 that $|\mathfrak{J}_2| \leq 2$.

There are two cases $|\mathfrak{J}_2| = 1$ or $|\mathfrak{J}_2| = 2$. If $|\mathfrak{J}_2| = 2$, then Theorem 2-7 implies that $\mathfrak{J}_2$ contains sets P_i and P_j with $P_i \cap P_j = \emptyset$. Suppose now that $\mathfrak{J}_1$ contains a subfamily $\mathcal{G}$ consisting of three pairwise disjoint sets. Then $\mathcal{G} \cup \{P_1\} \cup \{P_i, P_j\}$ is a family of six sets from $\mathfrak{J}$ no three of which have a common point. This contradicts the fact that $\mathfrak{J}$ has the (6,3)-property. Hence, either $\mathfrak{J}_1$ has the (3,2)-property, or $\mathfrak{J}_1$ fails to contain three sets. If $\mathfrak{J}_1$ has the (3,2)-property, it follows from Theorem 2-7 and the fact that each set of $\mathfrak{F}_1$ intersects H, that $|\mathfrak{F}_1| \leq 2$. If $\mathfrak{F}_1$ fails to contain three sets, then clearly $|\mathfrak{F}_1| \leq 2$. Thus,

$$|\mathfrak{F}| = |\mathfrak{F}_1 \cup \mathfrak{F}_2| \leq 2 + 2 = 4.$$

Suppose now $|\mathfrak{F}_2| = 1$. Recall $\mathfrak{F}_1$ has the (4,2)-property, or $\mathfrak{F}_1$ fails to contain four sets. In either case, it follows that $|\mathfrak{F}_1| \leq 3$. Thus, $|\mathfrak{F}| = |\mathfrak{F}_1 \cup \mathfrak{F}_2| \leq 3 + 1 = 4$. Therefore, $|\mathfrak{F}| \leq 4$ for any finite family $\mathfrak{F}$ of mutually parallel parallelotopes in E^2 with the (6,3)-property. Corollary 1-2.1 implies the same result for infinite families. Thus, $N_2(6,3) \leq 4$. Theorem 2-7 implies that $N_2(6,3) \geq 4$; hence, $N_2(6,3) = 4$.■

A technique will not be illustrated which allows one to determine some of the numbers $N_2(p,q)$ and upper bounds on others which are smaller than those given previously in this chapter.

First, it is shown that $N_2(10,4) = 7$. Theorem 2-7 implies that $7 \leq N_2(10,4)$. Thus, it suffices to show that if $\mathfrak{F}$ is a family of mutually parallel parallelotopes in E^2 with the (10,4)-property, then $|\mathfrak{F}| \leq 7$. If $\mathfrak{F}$ fails to have the (4,2)-property, then Theorem 2-9 with $p = 4$, $q = 2$, $t = 2$ and $m = 4$ implies that $|\mathfrak{F}| \leq 4 - 2 + 1 + N_2(6,3)$. However, Corollary 2-13.2 implies that $N_2(6,3) = 4$. Thus, $|\mathfrak{F}| \leq 7$. If $\mathfrak{F}$ has the (4,2)-property, then $|\mathfrak{F}| \leq N_2(4,2)$. Corollary 2-12.1 implies that $N_2(4,2) \leq 6$. Consequently, $|\mathfrak{F}| \leq 7$ which implies that $N_2(10,4) = 7$.

Now it will be shown that $N_2(17,5) = 13$. The technique used to do this may also be used together with the theorems of this chapter to determine the values of $N_2(p,q)$ listed in Table I. Again, by

TABLE I

$N_2(p,q)$

q \ p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
2	1	3																																															
3		1	2	3	4																																												
4			1	2	3	4	5	6	7																																								
5				1	2	3	4	5	6	7	8	9	10	11	12	13	14	15																															
6					1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25																				
7						1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39					
8							1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43
9								1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
10									1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
11										1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
12											1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
13												1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38
14													1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
15														1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
16															1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
17																1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
18																	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
19																		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
20																			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
21																				1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
22																					1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
23																						1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
24																							1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
25																								1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
26																									1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
27																										1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
28																											1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
29																												1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
30																													1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
31																														1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
32																															1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

Theorem 2-7, $13 \leq N_2(17,5)$. Thus, it suffices to show that if $\mathfrak{F}$ is a family of mutually parallel parallelotopes in E^2 with the (17,5)-property, then $|\mathfrak{F}| \leq 13$. If $\mathfrak{F}$ fails to have the (7,2)-property, then Theorem 2-9 with $p = 7$, $q = 2$, $m = 7$ and $t = 3$ implies that $|\mathfrak{F}| \leq 6 + N_2(10,4) = 13$. If $\mathfrak{F}$ fails to have the (11,3)-property, then Theorem 2-9 implies that $|\mathfrak{F}| \leq 9 + N_2(6,3)$. However, Corollary 2-13.2 implies that $N_2(6,3) = 4$; thus, $|\mathfrak{F}| \leq 13$. If $\mathfrak{F}$ fails to have the (15,4)-property, then Theorem 2-9 implies that $|\mathfrak{F}| \leq 12 + N_2(2,2) = 13$. Thus, assume that $\mathfrak{F}$ has the (7,2), (11,3), (15,4) and (17,5)-property.

Suppose that $\mathfrak{F}$ fails to have the (14,4)-property. Then there exists a subfamily $\mathfrak{G}$ of $\mathfrak{F}$ containing 14 sets, no four of which have a common point. Consequently, $\mathfrak{F} \sim \mathfrak{G}$ has the (3,2)-property. The discussion following Theorem 2-7 implies that $N_2(3,2) = 3$; thus, $|\mathfrak{F} \sim \mathfrak{G}| \leq 3$. Since $\mathfrak{F}$ has the (11,3)-property, some three sets from $\mathfrak{G}$ have a common point. Hence, $\mathfrak{G}$ contains a subfamily $\mathfrak{H}$ which contains three sets with a common point. The family $\mathfrak{G} \sim \mathfrak{H}$ is a family of 11 sets and thus must contain a subfamily $\mathcal{J}$ which contains three sets with a common point. Consequently,

$$\begin{aligned} |\mathfrak{F}| = |(\mathfrak{F} \sim \mathfrak{G}) \cup \mathfrak{G}| &\leq |\mathfrak{F} \sim \mathfrak{G}| + |\mathfrak{G}| \\ &\leq 3 + |\mathfrak{H}| + |\mathcal{J}| + |\mathfrak{G} \sim (\mathfrak{H} \cup \mathcal{J})| \\ &\leq 3 + 1 + 1 + (14 - 6) = 13. \end{aligned}$$

Thus, assume that $\mathfrak{F}$ also has the (14,4)-property.

If $\mathfrak{F}$ fails to have the (10,3)-property, then there exists a subset $\mathfrak{G}$ of $\mathfrak{F}$ containing 10 sets, no three of which have a common point. Since $\mathfrak{F}$ has the (14,4)-property, the family $\mathfrak{F} \sim \mathfrak{G}$ has the

(4,2)-property. Corollary 2-12.1 implies that $|\mathfrak{F} \setminus \mathcal{G}| \leq 6$. Since $\mathfrak{F}$ has the (7,2)-property, $\mathcal{G}$ contains two distinct subfamilies $\mathcal{H}$ and $\mathcal{J}$ each of which contains two sets with a common point. The family $\mathcal{G} \setminus (\mathcal{H} \cup \mathcal{J})$ then consists of 6 sets. If some two sets of the family $\mathcal{G} \setminus (\mathcal{H} \cup \mathcal{J})$ have a common point, then $|\mathcal{G} \setminus \mathcal{H} \cup \mathcal{J}| \leq 5$. Consequently,

$$\begin{aligned} |\mathfrak{F}| &\leq |\mathfrak{F} \setminus \mathcal{G}| + |\mathcal{H}| + |\mathcal{J}| + |\mathcal{G} \setminus (\mathcal{H} \cup \mathcal{J})| \\ &\leq 6 + 1 + 1 + 5 = 13. \end{aligned}$$

Thus, assume $\mathcal{G} \setminus (\mathcal{H} \cup \mathcal{J})$ consists of 6 pairwise disjoint sets. Let $\mathcal{G} \setminus \mathcal{H} \cup \mathcal{J} = \{A_1, \ldots, A_6\}$ and $\mathcal{J} = \{B_1, B_2\}$. Since $\mathfrak{F}$ has the (7,2)-property, some two of the sets from the family $\{A_1, \ldots, A_6, B_1\}$ have a common point. Without loss of generality, $B_1 \cap A_1 \neq \emptyset$. Similarly, $B_2 \cap A_i \neq \emptyset$ for some i, $1 \leq i \leq 6$. If $i = 1$, then $B_1 \cap B_2 \cap A_1 \neq \emptyset$. Consequently,

$$|\mathcal{G} \setminus \mathcal{H}| \leq |\{A_2, \ldots, A_6\}| + |\{A_1, B_1, B_2\}| = 6.$$

If $i \neq 1$, then

$$\begin{aligned} |\mathcal{G} \setminus \mathcal{H}| \leq |\{A_2, \ldots, A_{i-1}, A_{i+1}, \ldots, A_6\}| &+ |\{A_1, B_1\}| \\ &+ |\{A_i, B_2\}| = 6. \end{aligned}$$

Thus, in either case $|\mathcal{G} \setminus \mathcal{H}| \leq 6$. Consequently,

$$|\mathfrak{F}| \leq |\mathfrak{F} \setminus \mathcal{G}| + |\mathcal{G} \setminus \mathcal{H}| + |\mathcal{H}| \leq 6 + 6 + 1 = 13.$$

Thus, assume that $\mathfrak{F}$ also has the (10,3)-property.

If $\mathfrak{F}$ fails to have the (6,2)-property, then since $\mathfrak{F}$ has the (10,3)-property, Theorem 2-9 implies that $|\mathfrak{F}| \leq 5 + N_2(4,2)$. However, Corollary 2-12.1 implies that $N_2(4,2) \leq 6$. Consequently, $|\mathfrak{F}| \leq 11$. If $\mathfrak{F}$ has the (6,2)-property, then Corollary 2-12.1 implies that $|\mathfrak{F}| \leq 13$. Thus, this reduction process yields $|\mathfrak{F}| \leq 13$ in each case. Therefore, $N_2(17,5) = 13$.

The number $N_2(19,5)$ may also be evaluated by this technique. However, due to the length of the reduction process, $N_2(17,5)$ was evaluated in detail instead of $N_2(19,5)$. The evaluation of $N_2(19,5)$ proceeds as follows. First, show that if $\mathfrak{F}$ is a family of mutually parallel parallelotopes in E^2 with the (19,5)-property which fails to have the (9,2), (13,3) or (17,4)-property, then $|\mathfrak{F}| \leq 15$. Then show that if $\mathfrak{F}$ fails to have the (16,4), (12,3) or (8,2)-property, then $|\mathfrak{F}| \leq 15$. The proof must show that $\mathfrak{F}$ fails to have these properties in the order in which they are listed. Next show that if $\mathfrak{F}$ fails to have the (7,2), (15,4) or (11,3)-property, then $|\mathfrak{F}| \leq 15$. Here again the proof must follow the order in which the properties are listed. Now if $\mathfrak{F}$ fails to have the (6,2)-property, Theorem 2-9 with $p = 6$, $q = 2$, $t = 1$ and $m = 4$ implies that $|\mathfrak{F}| \leq 5 + N_2(5,2)$. However, Corollary 2-12.1 implies that $N_2(5,2) \leq 9$. Thus, $|\mathfrak{F}| \leq 14$. If $\mathfrak{F}$ has the (6,2)-property, then Corollary 2-12.1 implies that $|\mathfrak{F}| \leq 13$. Thus, in each case $|\mathfrak{F}| \leq 15$. Theorem 2-7 implies that $N_2(19,5) \geq 15$; thus, $N_2(19,5) = 15$.

If one tries to evaluate the number $N_2(20,5)$ by this technique, he obtains the following: If $\mathfrak{F}$ is a family of mutually parallel parallelotopes in E^2 with the (20,5)-property which fails to have the (7,2), (12,3) or (16,4)-property, then $|\mathfrak{F}| \leq 16$. Moreover, no further reduction of these properties is possible with the previous technique. Thus, one is only able to conclude that $N_2(20,5) \leq \max\{16, N_2(7,2)\}$.

Since Corollary 2-12.1 only implies that $N_2(7,2) \leq 17$, this technique fails to determine $N_2(20,5)$ since Theorem 2-7 says that $N_2(20,5) \geq 16$.

The values of $N_2(p,q)$ which appear to the left of the dark stair step curve in Table I are those given in Theorem 2-7 and were determined prior to this study by Hadwiger and Debrunnor [5]. The values which appear to the right of the stair step curve are those which the author has determined in this study.

CHAPTER III

FAMILIES OF COMPACT CONVEX SETS

In this chapter, the problem of determining whether or not $|\mathfrak{F}|$ is finite for a family $\mathfrak{F}$ of compact convex subsets of E^n with the (p,q)-property will be considered. This problem is motivated by a theorem of Hadwiger and Debrunner [4], which implies that for certain pairs of natural numbers p and q there exists a smallest natural number $M_n(p,q)$ such that $|\mathfrak{F}| \leq M_n(p,q)$, for any finite family $\mathfrak{F}$ of compact convex subsets of E^n with the (p,q)-property.

The (p,q,k)-property

The space L^n is defined to be R^n with the norm whose unit ball is the set

$$B^n = \{(\alpha_1, \ldots, \alpha_n): -1 \leq \alpha_i \leq 1, \quad i = 1, \ldots, n\}.$$

The norm for $(\alpha_1, \ldots, \alpha_n) \in L^n$ is then given by

$$||(\alpha_1, \ldots, \alpha_n)|| = \max\{|\alpha_i|: i = 1, \ldots, n\}.$$

Since all norms on R^n give an equivalent topology, the results of the previous chapter remain true when E^n is replaced by L^n. Henceforth,

L^n will denote the above described normed linear space and B^n the unit ball of this space.

The following theorem is stated in L^n; however, the theorem also holds in any complete normed linear space, a Banach space. Note also that the sets in the family $\mathfrak{F}$ need not be required to be convex.

<u>Theorem</u> 3-1. Let $\mathfrak{F} = \{A_\alpha : \alpha \in \Lambda\}$ be a family of compact sets in L^n with the (2,2)-property. If $\inf\{\mathrm{diam}(A_\alpha) : \alpha \in \Lambda\} = 0$, then $|\mathfrak{F}| = 1$.

Proof: For $A_\alpha \in \mathfrak{F}$, let $\beta_\alpha = \mathrm{diam}(A_\alpha)$. There exists a sequence $\{\beta_{\alpha_i}\}$ such that $\beta_{\alpha_i} \to 0$ and $\beta_{\alpha_{i+1}} \leq \beta_{\alpha_i}$. Choose $x_i \in A_{\alpha_i}$ and $w_{ij} \in A_{\alpha_i} \cap A_{\alpha_j}$. Then

$$\|x_i - x_j\| \leq \|x_i - w_{ij}\| + \|w_{ij} - x_j\| \leq \beta_{\alpha_i} + \beta_{\alpha_j}.$$

Since $\beta_{\alpha_i} + \beta_{\alpha_j} \to 0$ as i and j become large, it follows that $\{x_i\}$ is a Cauchy sequence. Let x denote the point in L^n to which the sequence $\{x_i\}$ converges.

Suppose that there exists $\lambda \in \Lambda$ such that $x \notin A_\lambda$. Since A_λ is a closed set, there exists an $\epsilon > 0$ such that $\|x - a\| > \epsilon$ for all $a \in A_\lambda$. There exists an integer N_1 such that if $k \geq N_1$, then $\beta_{\alpha_k} < \epsilon/2$. Also, there exists an integer N_2 such that $\|x - x_k\| < \epsilon/2$ for all $k \geq N_2$. Let $k \geq \max\{N_1, N_2\}$. Then for $y \in A_k$, $\|x_k - y\| \leq \beta_{\alpha_k} < \epsilon/2$. Thus,

$$\|y - x\| \leq \|y - x_k\| + \|x_k - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, $a \in A_\lambda$ implies that $\|x - a\| > \epsilon$, and $y \in A_{\alpha_k}$ implies that

$\|y - x\| < \epsilon$. Consequently, $A_{\alpha_k} \cap A_\lambda = \emptyset$, a contradiction. Therefore, $x \in A_\lambda$ for all λ, and hence it follows that $|\mathfrak{F}| = 1$. ∎

The remainder of this study concerns certain real numbers which can be associated with a compact convex set A. These numbers are defined below.

Definition 3-1. Let A be a nonempty compact convex set in L^n. For $x \in A$, let $I(x,A) = \sup\{\lambda: x + \lambda B^n \subset A\}$ and $E(x,A) = \inf\{\lambda \geq 0: A \subset x + \lambda B^n\}$.

The sets $\{\lambda: x + \lambda B^n \subset A\}$ and $\{\lambda \geq 0: A \subset x + \lambda B^n\}$ are both nonempty since $0 \in \{\lambda: x + \lambda B^n \subset A\}$, and $\operatorname{diam} A \in \{\lambda \geq 0: A \subset x + \lambda B^n\}$. Thus, $I(x,A)$ and $E(x,A)$ are both nonnegative real numbers.

Since the set A is compact, a simple sequence argument can be constructed to prove that $x + I(x,A)B^n \subset A$ and $A \subset x + E(x,A)B^n$, for each $x \in A$. These facts will be used extensively in the sequel.

A set A which has an empty interior will have the property $I(x,A) = 0$ for all $x \in A$, while on the other hand if A has a nonempty interior, A will contain at least one point x such that $I(x,A) > 0$. If A is a nondegenerate set, that is, A contains at least two points, then $E(x,A) > 0$ for all $x \in A$. Thus, if A is a nondegenerate set, the ratio

$$\frac{I(x,A)}{E(x,A)}$$

is defined for each $x \in A$. This ratio will be of interest in the sequel. Note that if A is a nondegenerate set, then the inequality

$$0 \leq \frac{I(x,A)}{E(x,A)} \leq 1$$

is valid for all $x \varepsilon A$.

If $A = x + \alpha B^n$ for some $\alpha > 0$, then it follows that $I(x,A) = \alpha$, $E(x,A) = \alpha$ and $(I(x,A)/E(x,A)) = 1$ (c.f. Figure 10 for the case $n = 2$). If A denotes a circular disk in L^2 of radius $\alpha > 0$ with center x, then $I(x,A) = (\sqrt{2}/2)\alpha$, $E(x,A) = \alpha$ and $(I(x,A)/E(x,A)) = \sqrt{2}/2$ (c.f. Figure 11). However, if $y \neq x$, then it is clear that $I(y,A) < (\sqrt{2}/2)\alpha$, $E(y,A) > \alpha$ and $(I(y,A)/E(x,A)) < \sqrt{2}/2$. Thus, the ratio assumes its maximum at x. The following theorem implies the existence of such a point for each set A. However, Figure 12 gives an example of a compact convex body A where the point x_0 given in the following theorem is not unique.

<u>Theorem</u> 3-2. Let A be a nondegenerate compact convex set in L^n. There exists a point $x_0 \varepsilon A$ such that

$$\frac{I(x_0,A)}{E(x_0,A)} = \sup\left\{\frac{I(x,A)}{E(x,A)} : x \varepsilon A\right\}.$$

Proof: Let $b = \sup\{(I(x,A)/E(x,A)) : x \varepsilon A\}$. Then $0 \leq b \leq 1$. Since $\mathrm{diam}(A) \neq 0$, there exists a number $r > 0$ such that $r < E(x,A)$ for all $x \varepsilon A$. Without loss of generality, assume that there exists a sequence $\{x_i\}$ of points of A such that $(I(x_i,A)/E(x_i,A)) \to b$, $x_i \to x_0 \varepsilon A$, $I(x_i,A) \to I$ and $E(x_i,A) \to E$, for some real numbers I and E where $E > 0$. It will be shown that $I = I(x_0,A)$ and $E(x_0,A) = E$, which will complete the proof.

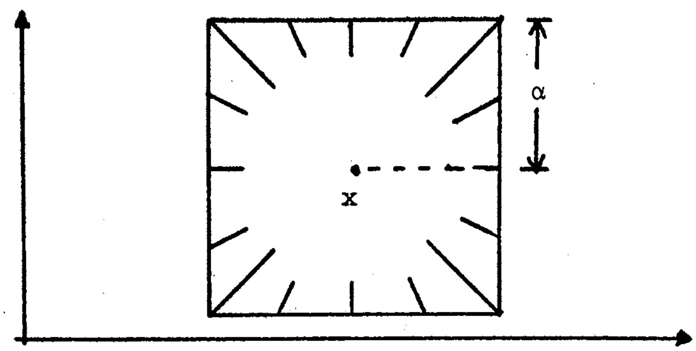

Figure 10. $I(x,A) = E(x,A) = \alpha$.

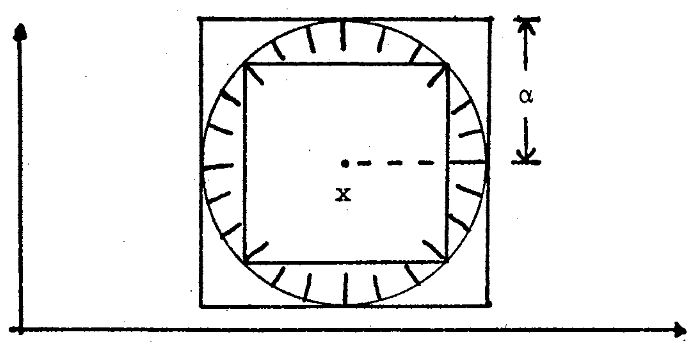

Figure 11. $I(x,A) = (\sqrt{2}/2)\alpha$, $E(x,A) = \alpha$.

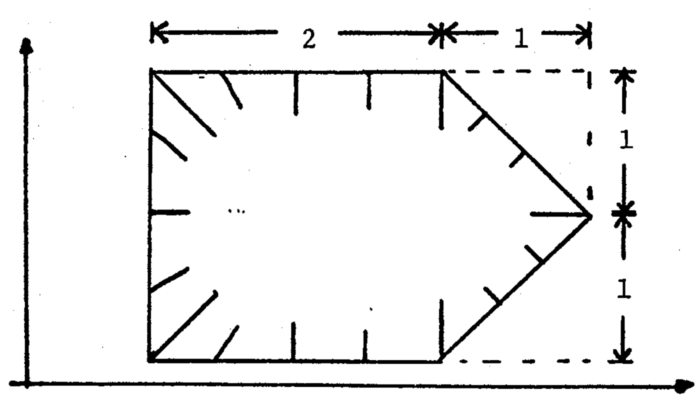

Figure 12. Nonuniqueness of x_0.

Suppose that $I < I(x_0,A)$. Let $\epsilon = I(x_0,A) - I$. Then there exists an integer N such that $i \geq N$ implies that $\|x_0 - x_i\| < \epsilon/2$. Let $k \geq N$ and $x \in x_k + (I + \epsilon 2^{-1})B^n$. Then

$$\|x - x_0\| \leq \|x - x_k\| + \|x_k - x_0\| \leq I + \frac{\epsilon}{2} + \frac{\epsilon}{2} = I + \epsilon = I(x_0,A).$$

Thus, $x \in x_0 + I(x_0,A)B^n \subset A$, which implies that $x \in A$. That is, $x_k + (I + \epsilon 2^{-1})B^n \subset A$. The definition of $I(x_k,A)$ implies that $I(x_k,A) \geq I + \epsilon 2^{-1}$ for $k \geq N$. This contradicts the convergence of the sequence $\{I(x_i,A)\}$ to I; therefore, $I(x_0,A) \leq I$.

Suppose now that $I(x_0,A) < I$. Let $\epsilon = I - I(x_0,A) > 0$. There exists an integer N_1 such that $k \geq N_1$ implies that $\|x_0 - x_k\| < \epsilon 4^{-1}$. Moreover, since $I(x_0,A) + \epsilon 2^{-1} < I$, there exists an integer N_2 such that $k \geq N_2$ implies that $I(x_k,A) > I(x_0,A) + \epsilon 2^{-1}$. Now for $k \geq \max\{N_1,N_2\}$ and $x \in x_0 + (I(x_0,A) + \epsilon 4^{-1})B^n$, it follows that

$$\begin{aligned}\|x - x_k\| &\leq \|x - x_0\| + \|x_0 - x_k\| \leq I(x_0,A) + \frac{\epsilon}{4} + \frac{\epsilon}{4} = I(x_0,A) + \frac{\epsilon}{2} \\ &\leq (I(x_k,A) - \frac{\epsilon}{2}) + \frac{\epsilon}{2} = I(x_k,A).\end{aligned}$$

Hence, $\|x - x_k\| \leq I(x_k,A)$. This implies that $x \in x_k + I(x_k,A)B^n \subset A$, and therefore, $x_0 + (I(x_0,A) + \epsilon 4^{-1})B^n \subset A$, a contradiction to the definition of $I(x_0,A)$. Thus, we obtain $I(x_0,A) = I$.

Since $(I(x_0,A)/E(x_0,A)) \leq b$ and $b = I/E$, it follows that $(I(x_0,A)/E(x_0,A)) \leq I/E$, that is, inverting gives $E \leq E(x_0,A)$, since $I = I(x_0,A)$. Suppose $E < E(x_0,A)$. Let $\epsilon = E(x_0,A) - E > 0$. Then there exists an integer N_1 such that $k \geq N_1$ implies that $\|x_0 - x_k\| < \epsilon/4$ and an integer N_2 such that $k \geq N_2$ implies that

$E(x_k,A) < E + \epsilon 4^{-1}$. Then for $k \geq \max\{N_1,N_2\}$ and $x \in A$, it follows that

$$\|x - x_0\| \leq \|x - x_k\| + \|x_k - x_0\| \leq E(x_k,A) + \frac{\epsilon}{4}$$

$$< E + \frac{\epsilon}{4} + \frac{\epsilon}{4} = E + \frac{\epsilon}{2} = E + \frac{1}{2}(E(x_0,A) - E)$$

$$= E(x_0,A) + \frac{1}{2}(E - E(x_0,A)) = E(x_0,A) - \frac{\epsilon}{2}.$$

This implies that $A \subset x_0 + (E(x_0,A) - \epsilon 2^{-1})B^n$, which is a contradiction to the definition of $E(x_0,A)$. Therefore, $E = E(x_0,A)$. ∎

Definition 3-2. Let p and q be integers with $p \geq q \geq 2$ and k a real number with $0 \leq k \leq 1$. A family $\mathfrak{J} = \{A_\alpha : \alpha \in \Lambda\}$ is said to have the (p,q,k)-property in L^n if and only if the following two conditions are satisfied:

(1) $\mathfrak{J}$ is a family of nondegenerate compact convex sets in L^n with the (p,q)-property; and

(2) $$\inf_{\alpha \in \Lambda} \left\{ \sup_{x \in A_\alpha} \left\{ \frac{I(x,A_\alpha)}{E(x,A_\alpha)} \right\} \right\} \geq k.$$

Theorem 3-3. A family $\mathfrak{J} = \{A_\alpha : \alpha \in \mathfrak{J}\}$ of nondegenerate compact convex subsets of L^n has the (p,q,k)-property in L^n if and only if $\mathfrak{J}$ has the (p,q)-property, and for each $A_\alpha \in \mathfrak{J}$ there exists a point $x_\alpha \in A_\alpha$ such that $(I(x_\alpha,A_\alpha)/E(x_\alpha,A_\alpha)) \geq k$.

Proof: The theorem follows directly from Definition 3-2 and Theorem 3-2. ∎

A family of circular disks in L^2 with the (p,q)-property will also have the $(p,q,\sqrt{2}/2)$-property in L^2 as seen from Figure 10. Thus,

Figure 13 represents a family of sets with the $(2,2,\sqrt{2}/2)$-property in L^2. If a family $\mathfrak{F}$ of subsets of L^n has the $(p,q,1)$-property, then it follows from Theorem 3-3 that for each $A \in \mathfrak{F}$ there exists a point $x \in A$ such that $(I(x,A)/E(x,A)) = 1$. Thus, $I(x,A) = E(x,A)$, from which it follows that $A = x + I(x,A)B^n$ (c.f. Figure 10 for the case $n = 2$). Hence, $\mathfrak{F}$ is a family of mutually parallel parallelotopes in L^n with the (p,q)-property. Figure 2 then represents a family with the $(3,2,1)$-property in L^2, and Figure 3 represents a family with the $(4,2,1)$-property in L^2. On the other hand, however, every family of mutually parallel parallelotopes in L^n with the (p,q)-property does not necessarily have the $(p,q,1)$-property.

Figure 14 represents the sets $\alpha\lambda B^2$ and $\epsilon\lambda B^2$, $0 < \alpha < \epsilon$. If A is any compact convex set which contains 0 in L^2 and whose boundary lies in the shaded portion of Figure 14, then it is true that $I(0,A) \geq \alpha\lambda$, $E(0,A) \leq \epsilon\lambda$ and $(I(0,A)/E(0,A)) \geq \alpha/\epsilon$. Thus, if $\mathfrak{F}$ is a family of compact convex subsets of L^2 with the (p,q)-property such that for each $A \in \mathfrak{F}$ there exists an $x \in A$ such that the boundary of $A - x$ lies in this shaded region, then $\mathfrak{F}$ has the $(p,q,\alpha/\epsilon)$-property in L^2.

If $0 \leq r \leq k \leq 1$, and $\mathfrak{F}$ is a family of subsets of L^n with the (p,q,k)-property, then it is clear that $\mathfrak{F}$ also has the (p,q,r)-property. Moreover, if $r > 0$, then $\mathfrak{F}$ is a family of compact convex bodies in L^n, that is, each set in $\mathfrak{F}$ has a nonempty interior.

The Function $P_n(p,q,k)$

A function $P_n(p,q,k)$ of four variables will now be defined which is similar to the aforementioned functions $N_n(p,q)$ and $T_n(p,q)$.

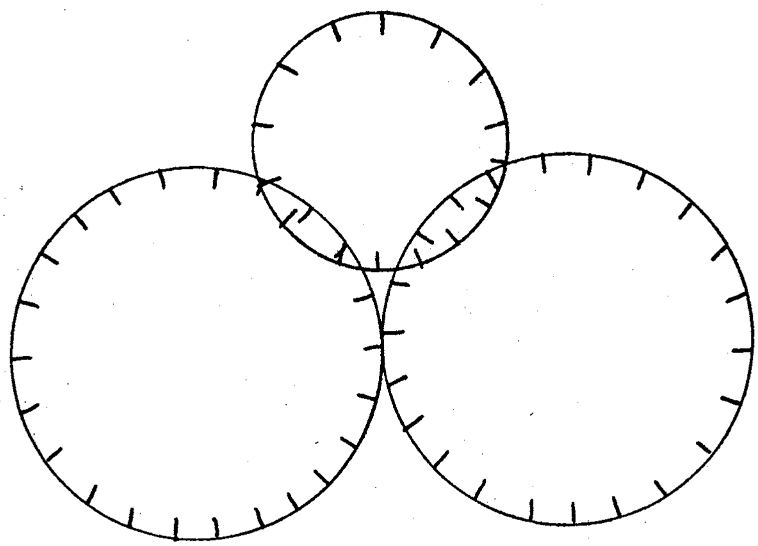

Figure 13. Sets with the $(2,2,\sqrt{2}/2)$-property.

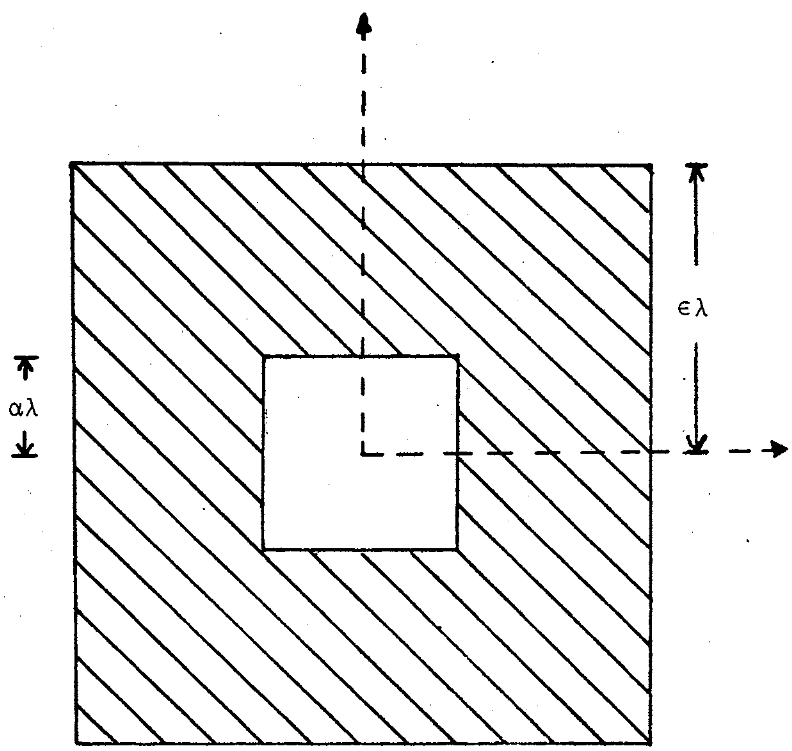

Figure 14. The Sets $\alpha\lambda B^2$ and $\epsilon\lambda B^2$.

Definition 3-2. Let p, q and n be positive integers with $p \geq q \geq 2$, and let k be a real number with $0 \leq k \leq 1$. Then

$$P_n(p,q,k) = \sup\{|\mathfrak{F}| : \mathfrak{F} \text{ has the } (p,q,k)\text{-property in } L^n\}.$$

Note that $P_n(p,q,k)$ need not be finite; in fact, it will be proven later that $P_n(n,n,0) = \infty$ for all $n \geq 2$. However, if $P_n(p,q,k)$ is finite, then $P_n(p,q,k)$ must be a positive integer.

If $n = 1$, it follows that any family $\mathfrak{F}$ in L^1 with the (p,q,k)-property is a family of mutually parallel parallelotopes in L^1. Hence, from Theorem 2-7, it follows that $P_1(p,q,k) = N_1(p,q) = p - q + 1$ for all k. By a similar argument as that used in the proof of Theorem 2-7, it can be shown that $P_n(p,q,k) \geq p - q + 1$ for all $k \in [0,1]$ and all $n \geq 1$.

It follows from Helly's theorem that $P_n(n+1,n+1,k) = 1$ for all $k \in [0,1]$. In the case when $k = 1$ it is not difficult to conclude that

$$T_n(p,q) \leq P_n(p,q,1) \leq N_n(p,q).$$

Thus, Theorem 2-7 implies that if $2 \leq q \leq p \leq 2q - 2$, then $P_n(p,q,1) = p - q + 1$.

The following two theorems determine whether or not $P_n(p,q,k)$ is finite for certain combinations of p, q, n and k.

Lemma 3-4.1. Let $\mathfrak{F} = \{A_\alpha : \alpha \in \Lambda\}$ be a family of sets with the (p,q,k)-property in L^n and x_0 be a fixed point in L^n. Then for any fixed real number $\beta > 0$, the family $\mathfrak{F}' = \{x_0 + \beta A_\alpha : \alpha \in \Lambda\}$ has the (p,q,k)-property in L^n.

Proof: Let $\alpha \varepsilon \Lambda$. Then Theorem 3-2 implies that there exists a point $x_\alpha \varepsilon A_\alpha$ such that $(I(x_\alpha,A_\alpha)/E(x_\alpha,A_\alpha)) \geq k$. Now $x_\alpha + I(x_\alpha,A_\alpha)B^n \subset A_\alpha$ implies that $x_0 + \beta(x_\alpha + I(x_\alpha,A_\alpha)B^n) \subset x_0 + \beta A_\alpha$; similarly, $x_0 + \beta A_\alpha \subset x_0 + \beta(x_\alpha + E(x_\alpha,A_\alpha)B^n)$. That is,

$$(x_0 + \beta x_\alpha) + \beta(I(x_\alpha,A_\alpha)B^n) \subset x_0 + \beta A_\alpha$$

and

$$x_0 + \beta A_\alpha \subset (x_0 + \beta x_\alpha) + \beta(E(x_\alpha,A_\alpha)B^n).$$

Hence, $I(x_0+\beta x_\alpha,x_0+\beta A_\alpha) \geq \beta I(x_\alpha,A_\alpha)$ and $E(x_0+\beta x_\alpha,x_0+\beta A_\alpha) \leq \beta E(x_\alpha,A_\alpha)$, from which it follows that

$$\frac{I(x_0+\beta x_\alpha,x_0+\beta A_\alpha)}{E(x_0+\beta x_\alpha,x_0+\beta A_\alpha)} \geq \frac{\beta I(x_\alpha,A_\alpha)}{\beta E(x_\alpha,A_\alpha)} \geq k.$$

It is also clear that $\mathfrak{J}'$ has the (p,q)-property. Thus, Theorem 3-2 implies that $\mathfrak{J}' = \{x_0 + \beta A_\alpha: \alpha \varepsilon \Lambda\}$ has the (p,q,k)-property in L^n.■

<u>Theorem</u> 3-4. Let p, q and n be positive integers with $p \geq q \geq 2$ and k a real number with $0 < k \leq 1$. Then $P_n(p,q,k)$ is finite. Moreover, the following inequality is satisfied:

$$\begin{aligned} p - q + 1 &\leq P_n(p,q,k) \\ &\leq (p - q + 1)\left[\left(\frac{1 + 2k}{k}\right)^n + 2n\left(\frac{1 + k + k^2}{k^2}\right)^{n-1}\right]. \end{aligned}$$

Proof: The fact that $p - q + 1 \leq P_n(p,q,k)$ was discussed after Definition 3-2. Let N_1 be the minimum number of translates of kB^n required to cover $(1 + k)B^n$ and N_2 the minimum number of translates of k^2B^n with centers on $bd[(1 + k)B^n]$ required to cover $bd[(1 + k)B^n]$. Then it is not difficult to show that

$$N_1 \leq \left(\frac{1 + 2k}{k} \right)^n \quad \text{and} \quad N_2 \leq 2n\left(\frac{1 + k + k^2}{k^2} \right)^{n-1} .$$

The theorem will now follow by showing that

$$P_n(p,q,k) \leq (p - q + 1)(N_1 + N_2).$$

To prove this, Corollary 1-2.1 implies that it suffices to show that if $\mathfrak{F}$ is any finite family of sets in L^n with the (p,q,k)-property, then $|\mathfrak{F}| \leq (p - q + 1)(N_1 + N_2)$.

Let $\mathfrak{F} = \{A_1, \ldots, A_m\}$ be a finite family of subsets of L^n with the (p,q,k)-property. Theorem 3-3 implies that for each $A_i \in \mathfrak{F}$ there exists a point $x_i \in A_i$ such that $(I(x_i,A_i)/E(x_i,A_i)) \geq k$. Without loss of generality, assume that $E(x_1,A_1) = \min\{E(x_i,A_i): i = 1, \ldots, m\}$. Lemma 3-4.1 implies that $\mathfrak{F}' = \{(E(x_1,A_1))^{-1}A_i: i = 1, \ldots, m\}$ has the (p,q,k)-property. It is also clear that $|\mathfrak{F}| = |\mathfrak{F}'|$. Hence, we may assume that $\mathfrak{F}$ has the property that

$$E(x_1,A_1) = \min\{E(x_i,A_i): i = 1, \ldots, m\} = 1.$$

By Lemma 3-4.1 we may also assume that $x_1 = 0$.

Let $\mathfrak{F}_1 = \{A_i \in \mathfrak{F}: A_i \cap A_1 \neq \emptyset\}$ and $\mathfrak{F}_2 = \{A_i \in \mathfrak{F}: A_i \cap A_1 = \emptyset\}$. We note that $\mathfrak{F}_2$ may be empty; however, $\mathfrak{F}_1 \neq \emptyset$ since $A_1 \in \mathfrak{F}_1$. It will not be shown that $|\mathfrak{F}_1| \leq N_1 + N_2$. Let

$$\mathcal{C}_1 = \{A_i \in \mathfrak{F}_1: \|x_i\| \leq 1 + k\} \quad \text{and} \quad \mathcal{C}_2 = \{A_i \in \mathfrak{F}_1: \|x_i\| > 1 + k\}.$$

There exist N_1 points $z_1, \ldots, z_{N_1}$ such that

$$(1 + k)B^n \subset \bigcup_{i=1}^{N_1} (z_i + kB^n).$$

For $A_i \in \mathcal{C}_1$, it follows that $x_i \in z_j + kB^n$ for some j, and $I(x_i,A_i) \geq k(E(x_i,A_i)) \geq k$. This implies that $x_i + kB^n \subset x_i + I(x_i,A_i)B^n \subset A_i$. Thus, $x_i \in z_j + kB^n$ implies that $z_j \in x_i + kB^n \subset A_i$. Hence, each set in $\mathcal{C}_1$ contains one of the N_1 points $z_1, z_2, \ldots, z_{N_1}$; consequently, $|\mathcal{C}_1| \leq N_1$.

If $\mathcal{C}_2 = \emptyset$, then clearly $|\mathfrak{F}_1| \leq N_1 + N_2$. If $\mathcal{C}_2 \neq \emptyset$, let $A_i \in \mathcal{C}_2$. Then $\|x_i\| > 1 + k$. Since $A_i \cap A_1 \neq \emptyset$ and $A_1 \subset B^n$, it follows that $A_i \cap B^n \neq \emptyset$. Let $y_i \in (\mathrm{bd}(B^n)) \cap A_i$. Then $k = 1 + k - 1 < \|x_i\| - \|y_i\| \leq \|x_i - y_i\| \leq E(x_i,A_i)$. Since $y_i \in B^n$ and $\|x_i\| > 1 + k$, the set $\{\lambda x_i + (1 - \lambda)y_i : 0 < \lambda < 1\}$ intersects the boundary of $(1 + k)B^n$ at some point $w_i = \lambda_0 x_i + (1 - \lambda_0)y_i$, for some $0 < \lambda_0 < 1$. The set

$$\lambda_0(x_i + I(x_i,A_i)B^n) + (1 - \lambda_0)y_i = w_i + \lambda_0 I(x_i,A_i)B^n$$

is contained in the convex hull of the set $\{x_i + I(x_i,A_i)B^n\} \cup \{y_i\}$, which is contained in A_i. Now

$$k \leq \|y_i - w_i\| = \|y_i - \lambda_0 x_i - (1 - \lambda_0)y_i\| = \lambda_0\|y_i - x_i\| \leq \lambda_0 E(x_i,A_i),$$

which implies that $(k/E(x_i,A_i)) \leq \lambda_0$. Hence,

$$\lambda_0 I(x_i,A_i) \geq \left(\frac{k}{E(x_i,A_i)}\right) I(x_i,A_i) \geq k^2.$$

Therefore, $w_i + k^2B^n \subset w_i + \lambda_0 I(x_i,A_i)B^n \subset A_i$. Let $u_1, \ldots, u_{N_2}$ be N_2 points such that $u_i \in \mathrm{bd}((1 + k)B^n)$ and

$$\mathrm{bd}((1 + k)B^n) \subset \bigcup_{i=1}^{N_2} (u_i + k^2B^n).$$

Then for $A_i \in \mathcal{C}_2$, it follows that $w_i \in u_j + k^2B^n$ for some j, which implies that $u_j \in w_i + k^2B^n \subset A_i$. Therefore, each set in $\mathcal{C}_2$ contains one of the N_2 points $u_1, \ldots, u_{N_2}$; consequently, $|\mathcal{C}_2| \leq N_2$. Since $\mathfrak{F}_1 = \mathcal{C}_1 \cup \mathcal{C}_2$, it follows that $|\mathfrak{F}_1| \leq N_1 + N_2$.

To complete the proof we proceed by induction on $s = p - q$. If $s = 0$, then $\mathfrak{F}$ has the (p,p,k)-property. In this case it clearly follows that $\mathfrak{F}_2 = \emptyset$. Thus, $|\mathfrak{F}| \leq N_1 + N_2$. Suppose now that it has been shown that $|\mathfrak{F}| \leq (p_0 - q_0 + 1)(N_1 + N_2)$ for all $0 \leq p_0 - q_0 < s$, and let p and q be such that $p - q = s$. Since each set in $\mathfrak{F}_2$ fails to intersect A_1, it follows that either $\mathfrak{F}_2$ has the $(p-1,q,k)$-property, or $\mathfrak{F}_2$ fails to contain $p-1$ sets. If $\mathfrak{F}_2$ has the $(p-1,q,k)$-property, then our induction hypothesis implies that $|\mathfrak{F}_2| \leq (p - q)(N_1 + N_2)$. In this case

$$\begin{aligned} |\mathfrak{F}| &\leq |\mathfrak{F}_1| + |\mathfrak{F}_2| \leq (N_1 + N_2) + (p - q)(N_1 + N_2) \\ &= (p - q + 1)(N_1 + N_2). \end{aligned}$$

If $\mathfrak{F}_2$ fails to contain $p-1$ sets, let j denote the cardinality of $\mathfrak{F}_2$. Then $0 \leq j < p - 1$. If $j = 0$, then $\mathfrak{F}_2 = \emptyset$, and it follows that $|\mathfrak{F}| = |\mathfrak{F}_1| \leq (p - q + 1)(N_1 + N_2)$. If $0 < j \leq p - q$, then $|\mathfrak{F}_2| \leq j \leq p - q \leq (p - q)(N_1 + N_2)$ and

$$|\mathfrak{F}| \leq |\mathfrak{F}_1| + |\mathfrak{F}_2| \leq N_1 + N_2 + (p - q)(N_1 + N_2)$$
$$= (p - q + 1)(N_1 + N_2).$$

Thus, assume that $p - q < j < p - 1$. Theorem 2-3 implies that $\mathfrak{F}$ has the $(j+1,q-p+j+1)$-property. Since each set of $\mathfrak{F}_2$ fails to intersect A_1, $\mathfrak{F}_2$ has the $(j,q-p+j+1)$-property. Hence, some subfamily $\mathfrak{G}$ of $\mathfrak{F}_2$ containing $q-p+j+1$ sets has a nonempty intersection. Hence,

$$|\mathfrak{F}_2| \leq |\mathfrak{G}| + |\mathfrak{F}_2 \setminus \mathfrak{G}| \leq 1 + j - (q - p + j + 1)$$
$$= p - q \leq (p - q)(N_1 + N_2).$$

Thus, $|\mathfrak{F}| \leq |\mathfrak{F}_1| + |\mathfrak{F}_2| \leq (p - q + 1)(N_1 + N_2)$. By induction, it now follows if $\mathfrak{F}$ is any finite family of sets in L^n with the (p,q,k)-property, then $|\mathfrak{F}| \leq (p - q + 1)(N_1 + N_2)$. The theorem now follows.■

Note that it is shown in the proof of Theorem 3-4 that $P_n(p,q,k) \leq (p - q + 1)(N_1 + N_2)$. In some cases, that is, when a particular k is given, the number $(p - q + 1)(N_1 + N_2)$, when calculated, may be smaller than the upper bound for $P_n(p,q,k)$ which is given in the theorem. It can also be concluded from the proof of Theorem 3-4 that if $p > q$, then $P_n(p,q,k) \leq P_n(p-1,q,k) + (N_1 + N_2)$, where N_1 and N_2 depend on k. These upper bounds are probably rather large as compared to the value of $P_n(p,q,k)$. The important thing, however, is the fact that $P_n(p,q,k)$ is <u>finite</u> for $0 < k \leq 1$.

It should also be noted that Theorem 3-4 fails to give any information about $P_n(p,q,0)$. The next theorem partially answers the question as to whether or not $P_n(p,q,0)$ is finite.

Theorem 3-5. Let $p \geq q \geq 2$ and $n \geq 2$ be integers. If $q \leq n$, then $P_n(p,q,0) = \infty$.

Proof: Suppose first that $p = q = n$. Let g_j denote the linear equation

$$\sum_{i=1}^{n} \alpha_i j^{i-1} = j^n,$$

where j is a positive integer. Let H_j denote the hyperplane

$$\left\{ (\alpha_1, \ldots, \alpha_n) \in L^n : \sum_{i=1}^{n} \alpha_i j^{i-1} = j^n \right\}$$

and $\mathfrak{F}' = \{H_j : j = 1,2, \ldots\}$. Given n distinct equations g_j, $i = i_1, \ldots, i_n$, the determinant formed by the coefficients of these equations is known as the Vandermonde determinant, which is never zero [p. 70, 7]. Hence, any n distinct equations g_j, $j = i_1, i_2, \ldots, i_n$, have a common solution. Hence, the family $\mathfrak{F}'$ has the (n,n)-property. Let N be any positive integer and $\mathfrak{F}'_N = \{H_j : j = 1,2, \ldots, nN\}$. Since $\mathfrak{F}'_N$ is finite, it is possible to find an integer m such that each n sets of $\mathfrak{F}'_N$ have a common point in the set mB^n. Let $\mathfrak{F} = \{H_j \cap mB^n : j = 1, \ldots, nN\}$. Then $\mathfrak{F}$ has the (n,n)-property. Moreover, $\mathfrak{F}$ consists of nondegenerate compact convex sets. Consequently, $\mathfrak{F}$ has the (n,n,0)-property in L^n. Suppose some $n+1$ sets in $\mathfrak{F}$ have a nonempty intersection. Then there exists a point

$(\alpha_1^0, \ldots, \alpha_n^0)$ such that the polynomial

$$\sum_{i=1}^{n} \alpha_i^0 y^{i-1} - y^n = 0$$

is satisfied for $n+1$ distinct values of y. However, this is impossible since an nth degree polynomial has at most n zeros. Thus, no $n+1$ sets of $\mathfrak{F}$ have a common point. Thus, if a subfamily $\mathfrak{G}$ of $\mathfrak{F}$ has a nonempty intersection, then $\mathfrak{G}$ contains at most n sets. This implies that $|\mathfrak{F}| \geq N$. Hence, $P_n(n,n,0) \geq N$ for every integer N. Thus, $P_n(n,n,0) = \infty$.

Theorem 2-3 implies that every family $\mathfrak{F}$ with the $(n,n,0)$-property in L^n also has the $(q,q,0)$-property in L^n where $q \leq n$. Thus, Definition 3-2 implies that $P_n(q,q,0) = \infty$ for all q with $2 \leq q \leq n$. Also, any family with the $(q,q,0)$-property in L^n has the $(p,q,0)$-property in L^n; thus, Definition 3-2 implies that $P_n(p,q,0) = \infty$ for $2 \leq q \leq p$ and $q \leq n$. ∎

Corollary 3-8.2 will further answer the question of the finiteness of $P_n(p,q,0)$ for some $q > n$. The following theorem describes how the function $P_n(p,q,k)$ behaves relative to increasing one of the variables while holding the others constant.

<u>Theorem</u> 3-6. Let $p \geq q \geq 2$ and $n \geq 1$ be integers and k and k_1 real numbers with $0 < k \leq 1$ and $0 \leq k_1 \leq 1$. Then the following results hold;

(a) If $k_1 \leq k_2 \leq 1$, then $P_n(p,q,k_1) \geq P_n(p,q,k_2)$.

(b) If p_1 is an integer with $p < p_1$, then $P_n(p,q,k) < P_n(p_1,q,k)$.

(c) If q_1 is an integer with $2 \leq q_1 < q$, then $P_n(p,q_1,k) > P_n(p,q,k)$.

(d) If d is an integer with $n \leq d$, then $P_n(p,q,k_1) \leq P_d(p,d,k_1)$.

(e) If $P_n(p,q,k_1)$ is finite, then (b) and (c) remain true when k is replaced by k_1.

Proof: The parts (a)-(d) will be proven in order, and on (b) and (c) part (e) will be checked.

Let $\mathfrak{F}$ be a family of sets in L^n. If $\mathfrak{F}$ has the (p,q,k_2)-property, then $\mathfrak{F}$ also has the (p,q,k_1)-property. Hence, Definition 3-2 implies that $P_n(p,q,k_1) \geq P_n(p,q,k_2)$, which proves (a). A proof similar to the proof of Lemma 2-9.1 can be used to show that $P_n(p,q,k_1) + (p_1 - p) \leq P_n(p_1,q,k_1)$. Thus, if $P_n(p,q,k_1)$ is finite, we have $P_n(p,q,k_1) < P_n(p_1,q,k_1)$. Since Theorem 3-4 implies that $P_n(p,q,k)$ is finite, (b) follows and also the part of (e) which concerns (b). Also a proof similar to the proof of Lemma 2-9.2 can be used to show that $P_n(p,q_1,k_1) \geq P_n(p,q,k_1) + (q - q_1)$. Thus, if $P_n(p,q,k_1)$ is finite, then $P_n(p,q_1,k_1) > P_n(p,q,k_1)$. Since Theorem 3-4 implies that $P_n(p,q,k)$ is finite, (c) follows and also the part of (e) which concerns (c).

It now remains to prove (d), and to do this it suffices to prove that $P_n(p,q,k_1) \leq P_{n+1}(p,q,k_1)$. From Definition 3-2, we see that this will follow if we can show that for each family $\mathfrak{F}$ of sets with the (p,q,k_1)-property in L^n, there exists a family $\mathfrak{F}'$ of sets with the (p,q,k_1)-property in L^{n+1} such that $|\mathfrak{F}| = |\mathfrak{F}'|$. So let $\mathfrak{F} = \{A_\alpha : \alpha \in \Lambda\}$ be a family of sets with the (p,q,k_1)-property in L^n.

Theorem 3-3 implies that for each $\alpha \in \Lambda$ there exists a point $x_\alpha \in A_\alpha$ such that

$$\frac{I(x_\alpha, A_\alpha)}{E(x_\alpha, A_\alpha)} \geq k_1.$$

For each $\alpha \in \Lambda$ define a set A'_α in L^{n+1} by

$$A'_\alpha = \{(\alpha_1, \ldots, \alpha_{n+1}) : (\alpha_1, \ldots, \alpha_n) \in A_\alpha \text{ and } |\alpha_{n+1}| \leq I(x_\alpha, A_\alpha)\}.$$

Then the family $\mathfrak{F}' = \{A'_\alpha : \alpha \in \Lambda\}$ is a family of nondegenerate compact convex sets in L^{n+1}. Since A_α can be considered as a subset of A'_α, it follows that $\mathfrak{F}'$ has the (p,q)-property. Let $x'_\alpha = (\beta_1, \ldots, \beta_n, 0)$. Then if $y = (\eta_1, \ldots, \eta_{n+1}) \in A'_\alpha$, we have

$$\|y - x'_\alpha\| = \max\left[\{\|\eta_i - \beta_i\| : i = 1, \ldots, n\} \cup \{|\eta_{n+1}|\}\right]$$

$$\leq E(x_\alpha, A_\alpha).$$

Hence, it follows that $E(x_\alpha, A_\alpha) = E(x'_\alpha, A'_\alpha)$. Also, it follows from the definition of A'_α that $I(x_\alpha, A_\alpha) = I(x'_\alpha, A'_\alpha)$. Consequently, Theorem 3-3 implies that $\mathfrak{F}'$ has the (p,q,k_1)-property. Since $(\eta_1, \ldots, \eta_{n+1}) \in A'_\alpha$ implies that $(\eta_1, \ldots, \eta_n) \in A_\alpha$, it follows that $|\mathfrak{F}| = |\mathfrak{F}'|$. The theorem now follows. ■

Theorem 3-7. Let $p \geq q \geq 2$ and $n \geq 1$ be integers, and $f(k) = P_n(p,q,k)$ for $0 \leq k \leq 1$. If $0 \leq k_0 < 1$, then

$$\lim_{k \to k_0^+} f(k) = f(k_0).$$

Proof: If $k \leq k'$, then any family $\mathfrak{F}$ with the (p,q,k')-property also has the (p,q,k)-property. Hence, $f(k') \leq f(k)$, that is f is a decreasing function. If $k > k_0$, then $k > 0$, and Theorem 3-4 implies that $f(k)$ is finite. The theorem will clearly follow if one can show that $f(k_0) = \sup\{f(k): k > k_0\}$. Suppose that this is not the case. Then $\sup\{f(k): k > k_0\} < f(k_0)$ (note, $f(k_0)$ may be infinity). Thus, $\{f(k): k > k_0\}$ is a set of positive integers which is bounded above. Let $m = \sup\{f(k): k > k_0\} = \max\{f(k): k > k_0\}$. Then m is a positive integer. Now $P_n(p,q,k_0) > m$ implies that there exists a family $\mathfrak{F}'$ of sets in L^n with the (p,q,k_0)-property such that $|\mathfrak{F}'| > m$. Corollary 1-2.1 implies that there exists a finite subfamily $\mathfrak{F} = \{A_i: i = 1, \ldots, h\}$ of $\mathfrak{F}'$ such that $|\mathfrak{F}| > m$. We may also assume that $h \geq p$, so that $\mathfrak{F}$ also has the (p,q,k_0)-property.

Theorem 3-3 implies that for each $i = 1,2, \ldots, h$ there exists a point $x_i \in A_i$ such that

$$\frac{I(x_i,A_i)}{E(x_i,A_i)} \geq k_0.$$

Let

$$\beta = \min\left[\{E(x_i,A_i) - I(x_i,A_i): E(x_i,A_i) - I(x_i,A_i) > 0 \text{ and } 1 \leq i \leq h\} \cup \{1\}\right].$$

Then $0 < \beta \leq 1$. Let $E = \max\{E(x_i,A_i): i = 1, \ldots, h\}$. Then $E > 0$. Choose α such that $0 < \alpha < \min\{\beta/E, 1-k_0\}$. For each positive integer j and $A_i \in \mathfrak{F}$ let

$$A_i^j = \operatorname{conv}(A_i \cup [x_i + (I(x_i,A_i) + \beta j^{-1})B^n]),$$

and

$$\mathfrak{F}_j = \{A_i^j : i = 1, \ldots, h\}.$$

Then $\mathfrak{F}_j$ is a family of compact convex sets in L^n. Moreover, since $A_i \subset A_i^j$, it follows that $\mathfrak{F}_j$ has the (p,q)-property. Let $A_i^j \in \mathfrak{F}_j$. Then $x_i + (I(x_i,A_i) + \beta j^{-1})B^n \subset A_i^j$; consequently, $I(x_i,A_i^j) \geq I(x_i,A_i) + \beta j^{-1}$. Also, if $E(x_i,A_i) - I(x_i,A_i) > 0$, then it follows that $E(x_i,A_i) - I(x_i,A_i) \geq \beta/j$, which implies that

$$x_i + (I(x_i,A_i) + \beta j^{-1})B^n \subset x_i + E(x_i,A_i)B^n.$$

Since we also have $A_i \subset x_i + E(x_i,A_i)B^n$, it follows that $A_i^j \subset x_i + E(x_i,A_i)B^n$; consequently, $E(x_i,A_i^j) \leq E(x_i,A_i)$. Thus,

$$\frac{I(x_i,A_i^j)}{E(x_i,A_i^j)} \geq \frac{I(x_i,A_i) + \beta j^{-1}}{E(x_i,A_i)} \geq k_0 + \frac{\beta}{jE} > k_0 + \frac{\alpha}{j}\ .$$

If $E(x_i,A_i) = I(x_i,A_i)$, then $A_i = x_i + I(x_i,A_i)$, which implies that $A_i^j = x_i + (I(x_i,A_i) + \beta j^{-1})B^n$. Hence,

$$\frac{I(x_i,A_i^j)}{E(x_i,A_i^j)} = 1 \geq k_0 + \alpha \geq k_0 + \frac{\alpha}{j}\ .$$

Let $k_j' = k_0 + \alpha/j$. Then Theorem 3-3 implies that $\mathfrak{F}_j$ has the (p,q,k_j')-property. Since $k_j' > k_0$, it follows from the definition of m that $f(k_j') \leq m$. Hence, $|\mathfrak{F}_j| \leq f(k_j') \leq m$ which implies that each family $\mathfrak{F}_j$ has the m-intersectional property. For each integer j there exists a set D_j containing m points such that if $A_i^j \in \mathfrak{F}_j$, then $A_i^j \cap D_j \neq \emptyset$. The points of D_j may also be chosen to lie in the compact set $\bigcup\{A_i^1: i = 1,2, \ldots, h\}$ since $A_i^1 \supset A_i^j$, for $1 \leq j \leq h$.

In a manner similar to the proof of Theorem 1-1, one can construct sets D_j^m, $j = 1,2, \ldots,$ containing m points such that:

(a) each set of $\mathfrak{F}_j$ contains a point of D_j^m;

(b) there exist m convergent sequences $\{x_j^i\}$, $1 \leq i \leq m$, with $x_j^i \in D_j^m$ such that for $i \neq t$, $x_j^i \neq x_j^t$;

(c) $D_j^m = D_i$ for some i.

Let x^i denote the point to which the sequence $\{x_j^i\}$ converges. It will be shown that each set in $\mathfrak{F}$ contains one of the points of the set $D = \{x^1, \ldots, x^m\}$. This will then imply that $|\mathfrak{F}| \leq m$, which contradicts the fact that $|\mathfrak{F}| > m$.

Suppose that for some set $A_i \in \mathfrak{F}$, $A_i \cap D = \emptyset$. Let $\delta = \inf\{\|x - y\|: x \in A_i$ and $y \in D\}$. Then $\delta > 0$, since A_i and D are disjoint compact sets. Let

$$C = \{y: \|y - x\| \leq \delta/2 \text{ for some } x \in A_i\}.$$

Then C is a compact convex set; moreover, the definition of δ implies that $C \cap D = \emptyset$. Let $y \in x_i + (I(x_i,A_i) + \beta j^{-1})B^n$, with $y \notin x_i + I(x_i,A_i)B^n$, and

$$\epsilon = 1 - \frac{I(x_i,A_i)}{\|x_i - y\|} .$$

Then $0 < \epsilon < 1$ and $\|\epsilon x_i + (1 - \epsilon)y - x_i\| = (1 - \epsilon)\|y - x_i\| = I(x_i,A_i)$. This implies that $\epsilon x_i + (1 - \epsilon)y \in x_i + I(x_i,A_i)B^n \subset A_i$. Also,

$\|\epsilon x_i + (1-\epsilon)y - y\| = \epsilon\|x_i - y\| = \|x_i - y\| - I(x_i,A_i) \leq \beta/j$. Thus, if $\beta/j \leq \delta/2$, then $x_i + (I(x_i,A_i) + \beta j^{-1})B^n \subset C$. Let N_1 be an integer such that $j > N_1$ implies that $\beta/j \leq \delta/2$. Since C is a compact set and $C \cap D = \emptyset$, there exists an integer N_2 such that if $j > N_2$, then $x_j^t \notin C$ for all $t = 1, \ldots, m$. If $j > \max\{N_1,N_2\}$, then it follows that $x_i + (I(x_i,A_i) + \beta j^{-1})B^n \subset C$, and clearly, $A_i \subset C$; hence, $A_i^j \subset C$. Also, $j > N_2$ implies that $x_j^t \notin C$ for all $t = 1, \ldots, m$; consequently, $x_j^t \notin A_i^j$ for $t = 1, \ldots, m$. From (b) it follows that $D_j^m = \{x_j^1, \ldots, x_j^m\}$; thus, $A_i^j \cap D_j^m = \emptyset$. However, since $A_i^j \in \mathfrak{F}_j$, it follows from (a) that $A_i^j \cap D_j^m \neq \emptyset$. Thus, the assumption that $A_i \cap D = \emptyset$ must be false. Therefore, each set in $\mathfrak{F}$ contains a point of the set D. The theorem now follows.■

For fixed $p \geq q \geq 2$ and $n \geq 1$, let $f(k) = P_n(p,q,k)$. Then f is a function defined for each $k \in [0,1]$. Theorem 3-4 implies that $f(k)$ is finite for all k with $0 < k \leq 1$; consequently, $f(k)$ is a positive integer for each $k \in (0,1]$. Theorem 3-6(a) and Theorem 3-7 imply that f is a decreasing function on $[0,1]$ and is continuous from the right at each $k \in [0,1)$. In particular, this is understood to mean that if $f(0) = \infty$, then $f(k) \to \infty$ as $k \to 0^+$. Thus, f can have at most a countable number of discontinuities in $[0,1]$, and each discontinuity of f is a jump discontinuity. (Note f is being considered as an extended real valued function on $[0,1]$ with its relative topology.) Let $D(p,q;n)$ denote the set of discontinuities of f. Note that since $P_1(p,q,k) = p - q + 1$ for all $k \in [0,1]$, $D(p,q;1) = \emptyset$. Also, Helly's theorem implies that $D(n+1,n+1;n) = \emptyset$ for $n \geq 1$. Theorem 3-5 implies that if $q \leq n$, then $D(p,q;n)$ is an infinite set. If $D(p,q;n) \neq \emptyset$, then the fact that f is continuous from the right at each point of $[0,1)$ implies that $D(p,q;n)$ can be written in the form $\{\xi_i(p,q;n): i \in J\}$, where J is either the positive integers or an initial segment of the positive integers, and $f(\xi_{i+1}(p,q;n)) > f(\xi_i(p,q;n))$. The set $D(p,q;n)$ is an infinite set if and only if $f(0) = \infty$ and is a finite set if and only if $f(0) < \infty$. In the case when $D(p,q;n) \neq \emptyset$, the exact location of the points $\xi_i(p,q;n)$ are unknown.

Let $f'(k) = P_2(2,2,k)$ for $k \in [0,1]$ and $\xi_i = \xi_i(2,2;2)$. Since $f'(0) = \infty$, the set $D(2,2;2) = \{\xi_i: i \in J\}$ is an infinite set. It will be shown later that if $k < 1$, then $f'(k) \geq 3$. Theorem 2-7 implies that $f'(1) = N_2(2,2) = 1$; consequently, $\xi_1 = 1$. Thus, f' is not an onto map. The graph of f' looks something like Figure 15.

The proof of the following theorem is similar to the proof of a theorem of Hadwiger and Debrunner [4].

<u>Theorem</u> 3-8. Let $p \geq q \geq 2$ and $n \geq 1$ be integers. If $\mathfrak{F}$ is a finite family of compact subsets of L^n with the $(p+n,q+n-1)$-property, there exists a subfamily $\mathfrak{H}$ of $\mathfrak{F}$ such that $|\mathfrak{F}| \leq |\mathfrak{H}| + 1$, and either $\mathfrak{H}$ has the (p,q)-property or $|\mathfrak{H}| \leq p - q + 1$.

Proof: Let $\mathfrak{F} = \{A_1, \ldots, A_m\}$ be a finite family of compact subsets of L^n with the $(p+n,q+n-1)$-property. Define a family $\mathcal{C}$ of subsets of L^n by $C \in \mathcal{C}$ if and only if C is a compact convex set and $\{A_i \cap C: i = 1, \ldots, m\}$ has the $(p+n,q+n-1)$-property. The set $\mathcal{C}$ is nonempty since $\mathfrak{F}$ is finite, and there exists a compact convex set C which contains each set in $\mathfrak{F}$. Let $\mathbf{S}$ be a nest in $\mathbf{C}$, that is, if $C,D \in \mathbf{S}$, then either $C \subset D$ or $D \subset C$. Let $C_0 = \bigcap\{C: C \in \mathbf{S}\}$. Then C_0 being the intersection of a monotonic family of compact sets is

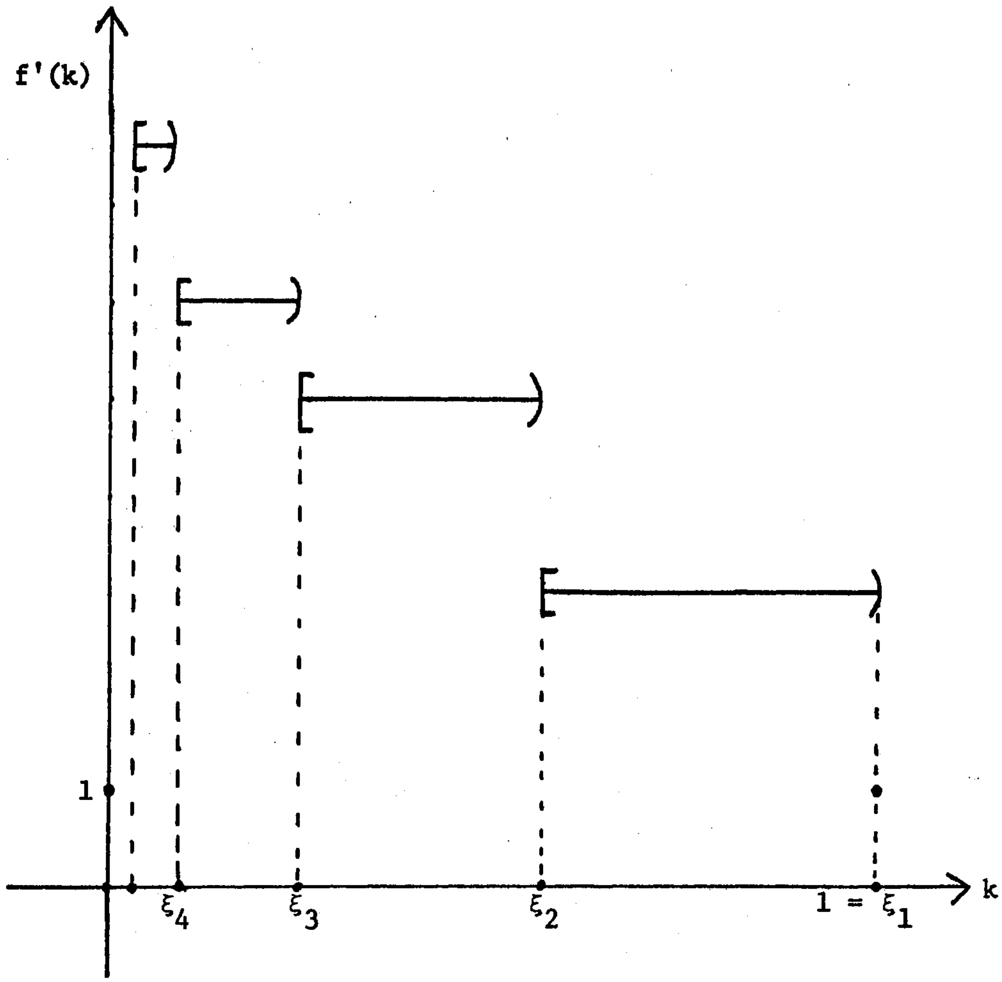

Figure 15. The Function $P_2(2,2,k)$.

nonempty. Let $\mathfrak{F}' = \{A_i \cap C_0: i = 1, \ldots, m\}$. Suppose that $\mathfrak{F}'$ fails to have the $(p+n,q+n-1)$-property. Then there exists a subfamily $\mathcal{G}'$ of $\mathfrak{F}'$ containing $p+n$ sets no $q+n-1$ of which have a common point. Let $\mathcal{G} = \{A_i: A_i \cap C_0 \in \mathcal{G}'\}$. Let $\mathcal{G}_1, \ldots, \mathcal{G}_r$ denote all possible subfamilies of $\mathcal{G}$ containing $q+n-1$ sets such that $H_i = \cap\{A: A \in \mathcal{G}_i\} \neq \emptyset$ for $i = 1, \ldots, r$. By our supposition, $H_i \cap C_0 = \emptyset$ for $i = 1, \ldots, r$. Thus, the family $\{H_i \cap C: C \in \mathcal{S}\}$ is a monotonic family of compact sets with an empty intersection; consequently, there exists a set $C_i \in \mathcal{S}$ such that $H_i \cap C_i = \emptyset$. Since $\mathcal{S}$ is a chain, there exists a set $C' \in \mathcal{S}$ such that $C' \subset C_i$ for $i = 1, \ldots, r$. The family $\{A_i \cap C': i = 1, \ldots, m\}$ has the $(p+n,q+n-1)$-property; thus, some $q+n-1$ sets from $\{A \cap C': A \in \mathcal{G}\}$ has a nonempty intersection. Any such $q+n-1$ sets must be of the form $\{A \cap C': A \in \mathcal{G}_i\}$ for some i. Consequently,

$$\emptyset \neq \cap\{A \cap C': A \in \mathcal{G}_i\} = C' \cap [\cap\{A: A \in \mathcal{G}_i\}] = C' \cap H_i,$$

for some i. However, $C' \cap H_i \subset C_i \cap H_i = \emptyset$, a contradiction. Thus, $\mathfrak{F}'$ has the $(p+n,q+n-1)$-property. Hence, $C_0 \in \mathcal{S}$. The minimal principle [p. 33, 8] implies that there exists a set $M \in \mathcal{C}$ such that if $C \in \mathcal{C}$ and $C \subset M$, then $C = M$. It will be shown next that M is a polytope.

Let $\mathfrak{F}_1 = \{A_i \cap M: i = 1, \ldots, m\}$. Then $\mathfrak{F}_1$ has the $(p+n,q+n-1)$-property. Choose a point $x_i \in A_i \cap M$ for each $i = 1, \ldots, m$. Let $\mathfrak{B}_1, \ldots, \mathfrak{B}_t$ denote the collection of all subfamilies of $\mathfrak{F}_1$ containing $q+n-1$ sets such that $\cap\{A: A \in \mathfrak{B}_i\} \neq \emptyset$ for $i = 1, \ldots, t$. Choose a point $y_i \in \cap\{A: A \in \mathfrak{B}_i\}$ for $i = 1, \ldots, t$. Let U denote the convex hull of the set

$\{x_1, \ldots, x_m, y_1, \ldots, y_t\}$. Then it is easy to see that $\{A_i \cap U: i = 1, \ldots, m\}$ has the $(p+n, q+n-1)$-property. Since $U \subset M$, it follows that $U = M$. Thus, M is a polytope.

Let x be a vertex of M. Then there exists a linear functional f on L^n such that $\{w: f(w) = 1\} \cap M = \{x\}$ and $f(y) \geq 1$, for all $y \in M$. Let J denote the support hyperplane $\{w: f(w) = 1\}$. It will now be shown that $\{x\} = \cap\{A: A \in \mathfrak{J}_1$ and $x \in A\}$. Let $J_r = \{w: f(w) \geq 1 + r^{-1}\}$. Then for each integer $r > 0$, $J_r \cap M$ is a proper subset of M; thus, $J_r \cap M \notin C$. Hence, for each r there exists a subfamily $\mathcal{G}_r$ of $\mathfrak{J}$ containing $p+n$ sets no $q+n-1$ of which have a common point in $J_r \cap M$. Since there are only a finite number of ways of choosing $p+n$ sets of $\mathfrak{J}$, there exists a subfamily $\mathcal{G}$ of containing $p+n$ sets such that $\mathcal{G}_r = \mathcal{G}$ for infinitely many r. Since $\{J_r \cap M: r = 1, \ldots\}$ is a monotonic family of sets, it follows that no $q+n-1$ sets of $\mathcal{G}$ have a common point in $J_r \cap M$ for all r. Hence, no $q+n-1$ set of $\mathcal{G}$ have a common point in $\cup\{J_i \cap M: i = 1, \ldots\} = \{w: f(w) > 1\} \cap M = M \smallsetminus \{x\}$. However, some $q+n-1$ sets of $\mathcal{G}$ must have a common point in M; consequently, the intersection of some $q+n-1$ sets of $\mathcal{G}$ is $\{x\}$. This clearly implies that $\{x\} = \cap\{A: A \in \mathfrak{J}_1$ and $x \in A\}$.

Let $\mathfrak{H}_1' = \{A_i \cap M: x \in A_i\}$ and $\mathfrak{H}_1 = \{A_i \cap M: x \notin A_i\}$. Then $\mathfrak{J}_1 = \mathfrak{H}_1' \cup \mathfrak{H}_1$ and $\cap\{A: A \in \mathfrak{H}_1'\} = \{x\}$. Suppose that each n sets of $\mathfrak{H}_1'$ have a common point in M which is different from x. Let $\mathfrak{R}_1, \ldots, \mathfrak{R}_k$ denote all subfamilies of $\mathfrak{H}_1'$ containing n sets. Choose a point $z_i \in \cap\{A \cap (M \smallsetminus \{x\}): A \in \mathfrak{R}_i\}$. Then there exists an integer $r > 0$ such that $f(z_i) > 1 + r^{-1}$ for $i = 1, \ldots, k$. Since $f(z_i) > 1 + r^{-1} > f(x)$ and x and z_i both belong to each set in $\mathfrak{R}_i$, it follows that the sets in $\mathfrak{R}_1$ have a common point in $\{w: f(w) = 1 + r^{-1}\} \cap M$. Hence, every n sets of the family $\mathcal{H}_1^{'}$ have a common point in the $n-1$ dimensional hyperplane $\{w: f(w) = 1 + r^{-1}\}$. Helly's theorem implies that there exists a point $y \in (\cap\{A: A \in \mathcal{H}_1^{'}\}) \cap \{f(w): f(w) = 1 + r^{-1}\}$. This is a contradiction since $\cap\{A: A \in \mathcal{H}_1^{'}\} = \{x\}$ and $x \notin \{f(w): f(w) = 1 + r^{-1}\}$. Thus, either $\mathcal{H}_1^{'}$ fails to contain n sets, or the intersection of some n sets of $\mathcal{H}_1^{'}$ is $\{x\}$. In either case, there exists a subfamily $\mathcal{G}_0$ of $\mathcal{H}_1^{'}$ containing j sets with $j \leq n$ such that $\cap\{A: A \in \mathcal{G}_0\} = \{x\}$. Theorem 2-3 implies that $\mathfrak{F}_1$ has the $(p+j,q+j-1)$-property. If one chooses p sets from $\{A: A \in \mathcal{H}_1\}$ and j sets from $\mathcal{G}_0$, some $q+j-1$ must have a common point. However, this point cannot be x; thus, the p sets from $\{A: A \in \mathcal{H}_1\}$ must contain some q sets with a common point. Thus, either $\mathcal{H}_1$ has the (p,q)-property, or $\mathcal{H}_1$ fails to contain p sets. If $\mathcal{H}_1 = \emptyset$, let $\mathcal{H}$ consist of a single set of $\mathfrak{F}$. Then $|\mathfrak{F}| = 1$; consequently, $|\mathfrak{F}| \leq |\mathcal{H}| + 1$ where $|\mathcal{H}| \leq p - q + 1$. If $\mathcal{H}_1 \neq \emptyset$, let $\mathcal{H}^{'} = \{A_i \in \mathfrak{F}: x \in A_i\}$ and $\mathcal{H} = \{A_i \in \mathfrak{F}: x \notin A_i\}$. Then each set of $\mathcal{H}$ contains a set in $\mathcal{H}_1$; consequently, $|\mathcal{H}| \leq |\mathcal{H}_1|$. If $\mathcal{H}_1$ fails to contain p sets, then it can be shown as in the proof of Theorem 3-4 that $|\mathcal{H}_1| \leq p - q + 1$. If $\mathcal{H}_1$ has the (p,q)-property, then the same is true of $\mathcal{H}$. Thus, $|\mathfrak{F}| \leq |\mathcal{H}^{'}| + |\mathcal{H}| = 1 + |\mathcal{H}|$, where $|\mathcal{H}| \leq p - q + 1$ or $\mathcal{H}$ has the (p,q)-property and $\mathcal{H} \subset \mathfrak{F}$. ■

<u>Corollary</u> 3-8.1. Let $p \geq q \geq 2$ and $n \geq 1$ be integers. Then $P_n(p+n,q+n-1,k) \leq P_n(p,q,k) + 1$ for all k with $0 \leq k \leq 1$.

Proof: If $P_n(p,q,k)$ is infinite, then the result clearly follows. So assume that $P_n(p,q,k)$ is finite. Let $\mathfrak{F}$ be any finite family of subsets of L^n with the $(p+n,q+n-1,k)$-property. Then Theorem 3-8 implies that there exists a subfamily $\mathfrak{H}$ of $\mathfrak{F}$ such that $|\mathfrak{F}| \leq |\mathfrak{H}| + 1$, and either $\mathfrak{H}$ has the (p,q,k)-property or $|\mathfrak{H}| \leq p - q + 1$. Since by Theorem 3-4 $P_n(p,q,k) \geq p - q + 1$, it follows that in either case $|\mathfrak{H}| \leq P_n(p,q,k)$. Thus, $|\mathfrak{F}| \leq P_n(p,q,k) + 1$. Corollary 1-2.1 implies that $P_n(p,q,k) \leq P_n(p,q,k) + 1$. ■

The following corollary is similar to a result of Hadwiger and Debrunner [4].

<u>Corollary</u> 3-8.2. If $p \geq q \geq n + 1 \geq 2$ and $nq \geq (n - 1)p + n + 1$, then $P_n(p,q,k) = p - q + 1$ for all $k \in [0,1]$.

Proof: Theorem 3-4 implies that $P_n(p,q,k) \geq p - q + 1$ for all $k \in [0,1]$; thus, it suffices to show that $P_n(p,q,k) \leq p - q + 1$.

We proceed by induction on $m = p - q$. If $m = 0$, then $p = q \geq n + 1$, and Helly's theorem implies that $P_n(p,p,k) = 1$ for all $k \in [0,1]$. Suppose now that if $p \geq q \geq n + 1 \geq 2$, $nq \geq (n - 1)p + n + 1$ and $0 \leq p - q < m$, then $P_n(p,q,k) = p - q + 1$ for all $k \in [0,1]$. Let p and q be such that $p \geq q \geq n + 1 \geq 2$, $nq \geq (n - 1)p + n + 1$ and $p - q = m$. Corollary 3-8.1 implies that $P_n(p,q,k) \leq P_n(p-n,q-(n-1),k) + 1$. Moreover,

$$n(q - n + 1) \geq (n - 1)(p - n) + n + 1.$$

The inequality $n(q - n + 1) \geq (n - 1)(p - n) + n + 1$ and the fact $m > 0$ can also be used to show that $p - n \geq q - (n - 1) \geq n + 1$. Since $0 \leq (p - n) - (q - n + 1) < m$, our induction hypothesis implies that $P_n(p-n,q-(n-1),k) = p - q$. Hence, $P_n(p,q,k) \leq p - q + 1$. Therefore, $P_n(p,q,k) = p - q + 1$.∎

From Corollary 2-8.2 and the fact that $P_1(p,q,0) = p - q + 1$, it follows that the simplest case, namely, the case with the smallest values for p, q and n, in which the question is unanswered as to whether or not $P_n(p,q,0)$ is finite or not is that of $p = 4$, $q = 3$, and $n = 2$. From Theorem 2-7 it follows that $N_2(4,3) = 2$. Moreover, since $2 \leq P_2(4,3,1) \leq N_2(4,3)$, it follows that $P_2(4,3,1) = 2$. An example will now be given which shows that $P_2(4,3,k) \geq 3$ and $P_2(2,2,k) \geq 3$ for all k with $0 \leq k < 1$. Consequently, this implies that the equation $P_n(p,q,k) = p - q + 1$ is not always valid for $q > n$. If $D(4,3;2)$ is the set defined in the discussion following Theorem 3-7, then this example will also imply that $D(4,3;2) \neq \emptyset$ and $\xi_1(4,3;2) = 1$.

Example 3-1. Define points in L^2 as follows:

$$A_n = \left(-\frac{1}{2n}, 0 \right), \qquad B_n = \left(\frac{1}{2n}, 0 \right),$$

$$C_n = \left(\frac{-1}{2n}, \frac{-1-2n}{2n^2} \right), \qquad D_n = \left(0, -\frac{1}{n} \right),$$

$$E_n = \left(\frac{1}{2n}, \frac{1-2n}{2n^2} \right), \qquad F = (1,0),$$

$$G = (-1,0), \qquad H_n = \left(1, \frac{-1}{n} \right),$$

$$I = (1,-1), \qquad J = (-1,-1),$$

$K_n = \left(\frac{1}{4n^2 - 4n + 2}, \frac{1 - 2n}{2n} \right)$, $\quad$ $L = (0,-1)$,

$M_n = \left(\frac{1 - n}{2n^2 - 2n - 2}, \frac{1 + 3n - 2n^2}{2n^3 - 2n^2 - 2n} \right)$, $\quad$ $N_n = \left(-1, \frac{1 - 2n}{2n} \right)$,

$R = (-1,2)$, $\quad$ $P = (1,2)$.

Using the above points, define sets in L^2 as follows:

$$S_n^1 = \operatorname{conv}\{R,P,F,D_n,G\},$$

$$S_n^2 = \operatorname{conv}\{E_n,F,I,L\},$$

$$S_n^3 = \operatorname{conv}\{D_n,H_n,I,L\},$$

$$S_n^4 = \operatorname{conv}\{G,M_n,L,J\},$$

$$S_n^5 = \operatorname{conv}\{G,C_n,K_n,N_n\},$$

$$S_n^6 = \operatorname{conv}\{A_n,B_n,E_n,C_n\}.$$

Moreover, let $\mathfrak{F}_n = \{S_n^i\colon i = 1, \ldots, 6\}$ for $n = 3,4, \ldots$. Figure 16 illustrates the family $\mathfrak{F}_n$ for $n = 4$. For each n with $n \geq 3$, the family $\mathfrak{F}_n$ has both the (2,2)-property and the (4,3)-property. Moreover, $|\mathfrak{F}_n| = 3$ for all $n \geq 3$. Note also that as n becomes large, S_n^i, $1 \leq i \leq 5$, approaches a square. Let $x_n = (0,-1/2n)$. Then it is not difficult to show that

$$\frac{I(x_n,S_n^6)}{E(x_n,S_n^6)} \to 1 \quad \text{as} \quad n \to \infty.$$

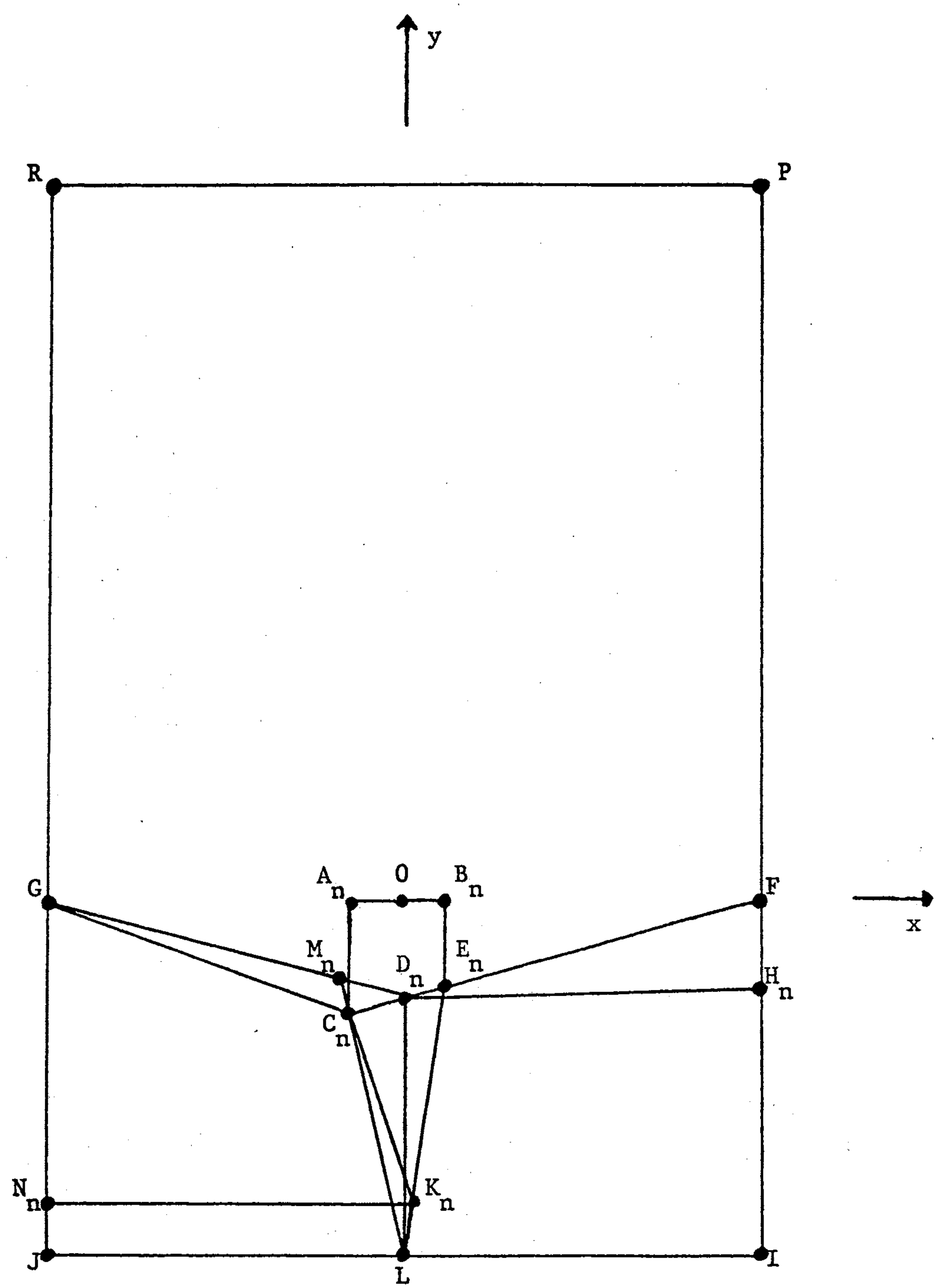

Figure 16. The Family $\mathfrak{F}_4$.

Thus, given any k with $0 \leq k < 1$, there exists an integer N such that $n \geq N$ implies that $\mathfrak{F}_n$ has both the $(2,2,k)$-property and $(4,3,k)$-property. Consequently, $P_2(2,2,k) \geq 3$ and $P_2(4,3,k) \geq 3$ for all k with $0 \leq k < 1$.

Families of Homotheties and Applications

The following section is concerned with families of homotheties in L^n with the (p,q)-property. At the end of the section are applications of the theorems to transversal type problems.

Definition 3-3. If B is a subset of a linear space, the set $x + \lambda B$, for $\lambda \neq 0$, is said to be homothetic to B and positively homothetic if $\lambda > 0$.

If $\mathfrak{F}$ is a family of sets with the $(p,q,1)$-property in L^n, then it has already been noted that each set in $\mathfrak{F}$ has the form $x + I(x,A)B^n$ for some x. Consequently, each set in $\mathfrak{F}$ is positively homothetic to B^n.

Let K be a compact convex subset of L^n. Grünbaum [3] defined $H(K) = \sup\{|\mathfrak{F}|\}$ where $\mathfrak{F}$ is a family of homotheties of K with the $(2,2)$-property and proved that $H_n = \max\{H(K): K \subset L^n\}$ is finite. The following theorem implies that the number H_n defined by Grünbaum satisfies the inequality $H_n \leq P_n(2,2,1/2n)$.

<u>Lemma</u> 3-9.1. If K is a compact convex body in L^n and $\mathfrak{F}$ a family of sets with the (p,q)-property such that each set in $\mathfrak{F}$ is homothetic to K, then $|\mathfrak{F}| \leq P_n(p,q,1/2n)$.

Proof: Corollary 1-2.1 implies that it suffices to prove the theorem for $\mathfrak{F}$ finite. Let $\mathfrak{F} = \{x_i + \alpha_i K: i = 1, \ldots, m\}$ be a finite family of homotheties of K with the (p,q)-property. It has been shown by Chakerian and Stein [2] that there exists a n-dimensional parallelotope P_1 such that $P_1 \subset K$ and $K \subset y + nP_1$ for some y. The parallelotope P_1 has the form $P_1 = x + P$ where P is the parallelotope

$$\left\{ \sum_{i=1}^{n} \alpha_i a_i : -1 \leq \alpha_i \leq 1 \right\}$$

with respect to some basis $\{a_1, \ldots, a_n\}$ of L^n. Let f be the linear isomorphism of L^n into L^n such that $f(a_i) = (0, \ldots, 1, 0, \ldots, 0)$ where the one is in the ith position. Then $f(P) = B^n$. Let $\mathfrak{F}' = \{f(x_i + \alpha_i K): i = 1, \ldots, m\}$. Since f is a linear isomorphism, it follows that $\mathfrak{F}'$ has the (p,q)-property and $|\mathfrak{F}'| = |\mathfrak{F}|$. Moreover, $\mathfrak{F}'$ is a family of homotheties of $f(K)$. Without loss of generality, assume that $x = 0$. Then $P = P_1$. Since $P \subset K$ and $K \subset y + nP$, it follows that $B^n \subset f(K)$ and $f(K) \subset f(y) + nB^n$. Let $z \in f(K)$. Then

$$\|z\| \leq \|z - f(y)\| + \|f(y)\| \leq n + \|f(y)\|.$$

Since $0 \in K$ and $f(K) \subset f(y) + nB^n$, it follows that $\|f(y)\| \leq n$. Thus, $\|z\| \leq 2n$ for all $z \in f(K)$. Consequently, $f(K) \subset 2nB^n$. Let $f(x_i) + \alpha_i f(K) \in \mathfrak{F}'$. Then $B^n \subset f(K)$ implies that

$$f(x_i) + |\alpha_i| B^n = f(x_i) + \alpha_i B^n \subset f(x_i) + \alpha_i f(K).$$

Moreover, $f(K) \subset 2nB^n$ implies that

$$f(x_i) + \alpha_i f(K) \subset f(x_i) + \alpha_i (2n)B^n = f(x_i) + (2n)|\alpha_i|B^n.$$

Thus,

$$\frac{I(f(x_i),f(x_i)+\alpha_i f(K))}{E(f(x_i),f(x_i)+\alpha_i f(K))} \geq \frac{|\alpha_i|}{2n|\alpha_i|} = \frac{1}{2n} .$$

Theorem 3-3 implies that $\mathfrak{F}'$ has the $(p,q,1/2n)$-property. Thus, $|\mathfrak{F}| = |\mathfrak{F}'| \leq P_n(p,q,1/2n)$. ∎

The proof of the above lemma leads to the following discussion: Let $\mathfrak{F}$ be a family of sets with the (p,q,k)-property in L^n and f be a linear isomorphism from L^n to L^n. Let $\mathfrak{F}' = \{f(A): A \in \mathfrak{F}\}$. Then $\mathfrak{F}'$ has the (p,q)-property and $|\mathfrak{F}'| = |\mathfrak{F}|$. It is possible that $\mathfrak{F}'$ has the (p,q,r)-property with $r > k$. If the set $D(p,q;n) \neq \emptyset$, let $\xi_i = \xi_i(p,q;n)$ for $i \in J$. If $\{\xi_i: P_n(p,q,\xi_i) < |\mathfrak{F}|\} \neq \emptyset$, let $\xi_j = \min\{\xi_i: P_n(p,q,\xi_i) < |\mathfrak{F}|\}$. Then $r < \xi_j$, for if this is not the case, it would follow from Theorem 2-7(a) that $P_n(p,q,\xi_j) \geq P_n(p,q,r) \geq |\mathfrak{F}|$, contradicting the fact that $|\mathfrak{F}| > P_n(p,q,\xi_j)$. Thus, by means of a linear isomorphism it is sometimes possible to transform a family $\mathfrak{F}$ with the (p,q,k)-property in L^n into a family $\mathfrak{F}'$ with the (p,q,r)-property in L^n with $r > k$. However, r must satisfy the inequality $r < \min\{\xi_i: P_n(p,q,\xi_i) < |\mathfrak{F}|\}$ whenever the set $\{\xi_i: P_n(p,q,\xi_i) < |\mathfrak{F}|\} \neq \emptyset$.

The following theorem removes the restriction that K be a body in the above lemma.

Theorem 3-9. If K is a compact convex set in L^n and $\mathfrak{J}$ a family of sets with the (p,q)-property such that each set in $\mathfrak{J}$ is homothetic to K, then $|\mathfrak{J}| \leq P_n(p,q,1/2n)$.

Proof: Corollary 1-2.1 implies that it suffices to prove the theorem for $\mathfrak{J}$ finite. Let $\mathfrak{J} = \{x_i + \alpha_i K: i = 1, \ldots, m\}$ be a finite family of homotheties of K with the (p,q)-property. Without loss of generality, assume that $0 \varepsilon K$. For each positive integer s and $i = 1, \ldots, m$, let $A_j^s = x_i + \alpha_i(K + s^{-1}B^n)$. Let $\mathfrak{J}_s = \{A_i^s: i = 1, \ldots, m\}$. Then $\mathfrak{J}_s$ is a family of homotheties of the compact convex body $K + s^{-1}B^n$. Moreover, since $x_i + \alpha_i K \subset A_i^s$, $\mathfrak{J}_s$ has the (p,q)-property. Lemma 3-9.1 implies that $|\mathfrak{J}_s| \leq P_n(p,q,1/2n)$ for all $s = 1, \ldots$. Theorem 3-4 implies that $P_n(p,q,1/2n)$ is finite. Thus, for each positive integer s there exists a set D_s containing $P_n(p,q,1/2n)$ points such that if $A_i^s \varepsilon \mathfrak{J}_s$, then $A_i^s \cap D_s \neq \emptyset$. The points of D_s may also be chosen to lie in the compact set $\cup\{A_i^1: i = 1, \ldots, m\}$.

Let $r = P_n(p,q,1/2n)$. In a manner similar to the proof of Theorem 1-1 one can construct sets D_s^r, $s = 1, \ldots,$ containing r points such that:

(a) each set of $\mathfrak{J}_s$ contains a point of D_s^r;

(b) there exists r convergent sequences $\{y_j^i\}$, $1 \leq i \leq r$, with $y_j^i \varepsilon D_j^r$ such that for $i \neq t$, $y_j^i \neq y_j^t$;

(c) $D_j^r = D_s$ for some s.

Let y^i denote the point to which the sequence $\{y_j^i\}$ converges. It will be shown that each set in $\mathfrak{F}$ contains one of the points of the set $D = \{y^1, \ldots, y^r\}$. This will then imply that $|\mathfrak{F}| \leq P_n(p,q,1/2n)$.

Suppose that for some set $x_i + \alpha_i K \in \mathfrak{F}$, $(x_i + \alpha_i K) \cap D = \emptyset$. Let $\delta = \inf\{\|x - y\|: x \in x_i + \alpha_i K$ and $y \in D\}$. Then $\delta > 0$ since $x_i + \alpha_i K$ and D are disjoint compact sets. There exists a positive integer s_0 such that $|\alpha_i|/s_0 < \delta$. If $x \in x_i + \alpha_i(K + s_0^{-1}B^n)$, then there exists a point $z \in x_i + \alpha_i K$ such that $\|x - z\| \leq |\alpha_i|/s_0 < \delta$. Consequently, $[x_i + \alpha_i(K + s_0^{-1}B^n)] \cap D = \emptyset$. Since $x_i + \alpha_i(K + s_0^{-1}B^n)$ is compact and contains no point of D, there exists an integer N such that if $j > N$, then $y_j^t \notin x_i + \alpha_i(K + s_0^{-1}B^n)$ for all $t = 1, \ldots, r$. Thus, if $j > N$, $x_i + \alpha_i(K + s_0^{-1}B^n)$ contains no points of the set D_j^r. Let $j \geq \max\{N,s_0\}$. Then

$$x_i + \alpha_i(K + j^{-1}B^n) \subset x_i + \alpha_i(K + s_0^{-1}B^n).$$

The definition of D_j^r implies that $x_i + \alpha_i(K + j^{-1}B^n) \cap D_j^r \neq \emptyset$. This contradicts the fact that $y_j^t \notin x_i + \alpha_i(K + s_0^{-1}B^n)$ for all $t = 1, \ldots, r$. Thus, if $x_i + \alpha_i K \in \mathfrak{F}$, then $(x_i + \alpha_i K) \cap D \neq \emptyset$. ■

Additional information on the r-intersectional properties of families of homotheties and translates of compact convex sets in E^n with the (2,2)-property can be found in Hanner [6] and Chakerian and Sallee [1].

The following theorem is an application of Theorem 3-9 to a transversal type problem.

<u>Theorem</u> 3-10. Let $p \geq q \geq 2$ and $1 \leq m \leq n$ be integers. Also, let L be an m-dimensional subspace of L^n and $\mathfrak{F} = \{A_\alpha: \alpha \in \Lambda\}$ a family of

homotheties of a compact convex set K in L^n with the property that for each subfamily $\mathcal{G}$ of $\mathfrak{F}$ which contains p sets, there exists a point $y \in L^n$ such that $y + L$ intersects some q sets in $\mathcal{G}$. Then there exist r points $x_1, \ldots, x_r$ in L^n with $0 < r \leq P_{n-m}(p,q,(2n-2m)^{-1})$ such that if $A_\alpha \in \mathfrak{F}$, then $(x_i + L) \cap A_\alpha \neq \emptyset$ for some i with $1 \leq i \leq r$.

Proof: There exists a linear isomorphism f from L^n into L^n such that $f(L)$ is the span of the set $\{\delta_{n-m+1}, \ldots, \delta_n\}$, where $\delta_i = (0, \ldots, 1, 0, \ldots)$ with the one in the ith position. Let $L' = f(L)$, $\mathfrak{F}' = \{f(A_\alpha): \alpha \in \Lambda\}$ and $K' = f(K)$. Then it follows that $\mathfrak{F}'$ is a family of homotheties of K', and from each p sets of $\mathfrak{F}'$ some q sets are intersected by a single translate of L'. Let S denote the span of the set $\{\delta_1, \ldots, \delta_{n-m}\}$, that is, $L' \oplus S = L^n$, the direct sum of L' and S. For each $\alpha \in \Lambda$ let $B_\alpha = \{x \in S: (x + L') \cap f(A\ ) \neq \emptyset\}$ and let $B = \{x \in S: (x + L') \cap K' \neq \emptyset\}$. If $\alpha \in \Lambda$, then $A_\alpha = y + \lambda K$ for some point y and some scalar $\lambda \neq 0$. Thus, $f(A_\alpha) = f(y) + \lambda K'$. Now $f(y) = u + v$ for some $u \in S$ and $v \in L'$. Thus, $f(A_\alpha) = u + v + \lambda K'$. Recall $x \in B$ if and only if $(x + L') \cap K \neq \emptyset$ and $x \in S$. However, $(x + L') \cap K' \neq \emptyset$ if and only if

$$(u + \lambda x + v + \lambda L') \cap (u + v + \lambda K') \neq \emptyset.$$

Since $v \in L'$, it follows that $u + \lambda x + v + \lambda L' = u + \lambda x + L'$. Thus, $x \in B$ if and only if $(u + \lambda x + L') \cap f(A_\alpha) \neq \emptyset$ and $x \in s$, that is, $x \in B$ if and only if $u + \lambda x \in B_\alpha$ and $x \in s$. Consequently, $B_\alpha = x + \lambda B$. Let $\mathcal{G} = \{B_\alpha: \alpha \in \Lambda\}$. Then $\mathcal{G}$ is a family of homotheties of the set B. Since K' is compact, a simple sequence argument can be constructed to show that B is compact. Moreover, since from each p sets of $\mathfrak{F}'$ some q sets are intersected by a single translate of L', it follows that $\mathcal{G}$ has the (p,q)-property. Since S is linearly isomorphic to L^{n-m}, it follows from Theorem 3-9 that there exist r points $x_1, \ldots, x_r$ with $r \leq P_{n-m}(p,q,(2n-2m)^{-1})$ such that if $\alpha \in \Lambda$, then $x_i \in B_\alpha$ for some i. Hence, if $\alpha \in \Lambda$, then $(x_i + L') \cap f(A_\alpha) \neq \emptyset$ for some i with $1 \leq i \leq r$. Since f is a linear isomorphism, it follows that if $\alpha \in \Lambda$, then $(f^{-1}(x_i) + L) \cap A_\alpha \neq \emptyset$ for some i with $1 \leq i \leq r$. ■

Let $\mathfrak{F}$ denote the family of circular disks in L^2 which are illustrated in Figure 17. Then it is clear from the figure that if $\mathcal{G}$ is a subfamily of $\mathfrak{F}$ containing three sets, then some two sets of $\mathcal{G}$ are intersected by a line with slope zero. Consequently, Theorem 3-10 implies that there exist $P_1(3,2,1/2) = 2$ lines with slope zero such that each set in $\mathfrak{F}$ is intersected by at least one of these lines. Two such lines are shown in the figure.

The following theorem yields results similar to those of Theorem 3-10 when $\mathfrak{F}$ is a family of parallel line segments in L^n.

<u>Theorem</u> 3-11. Let K be a compact line segment in L^n and $\mathfrak{F}$ a finite family of homotheties of K. If from each p sets of $\mathfrak{F}$ some q sets are intersected by a hyperplane, then there exist r hyperplanes with $r \leq P_n(p,q,0)$ such that each set in $\mathfrak{F}$ is intersected by one of these hyperplanes.

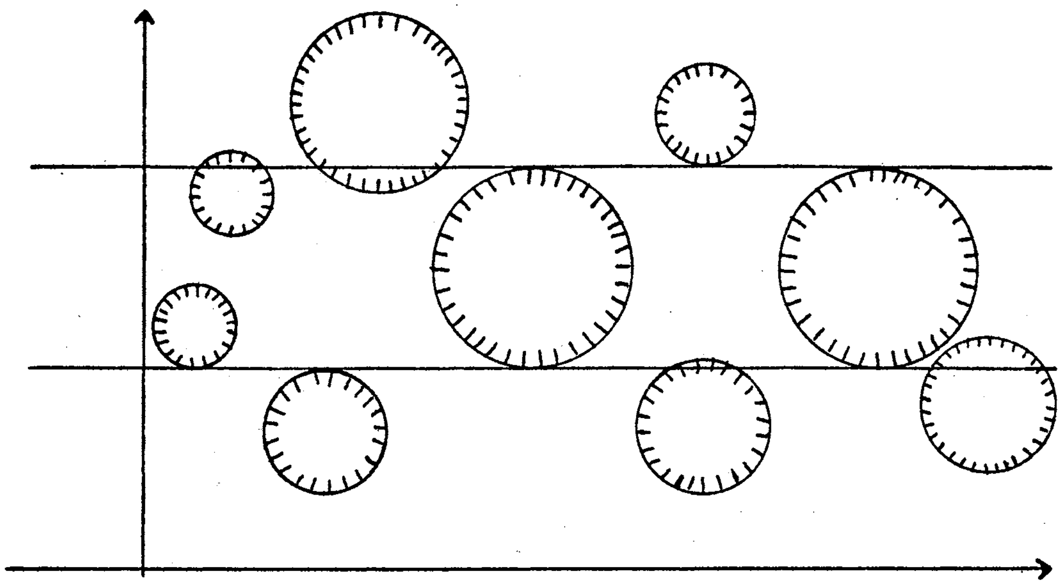

Figure 17. Homotheties of a Circular Disk in L^2.

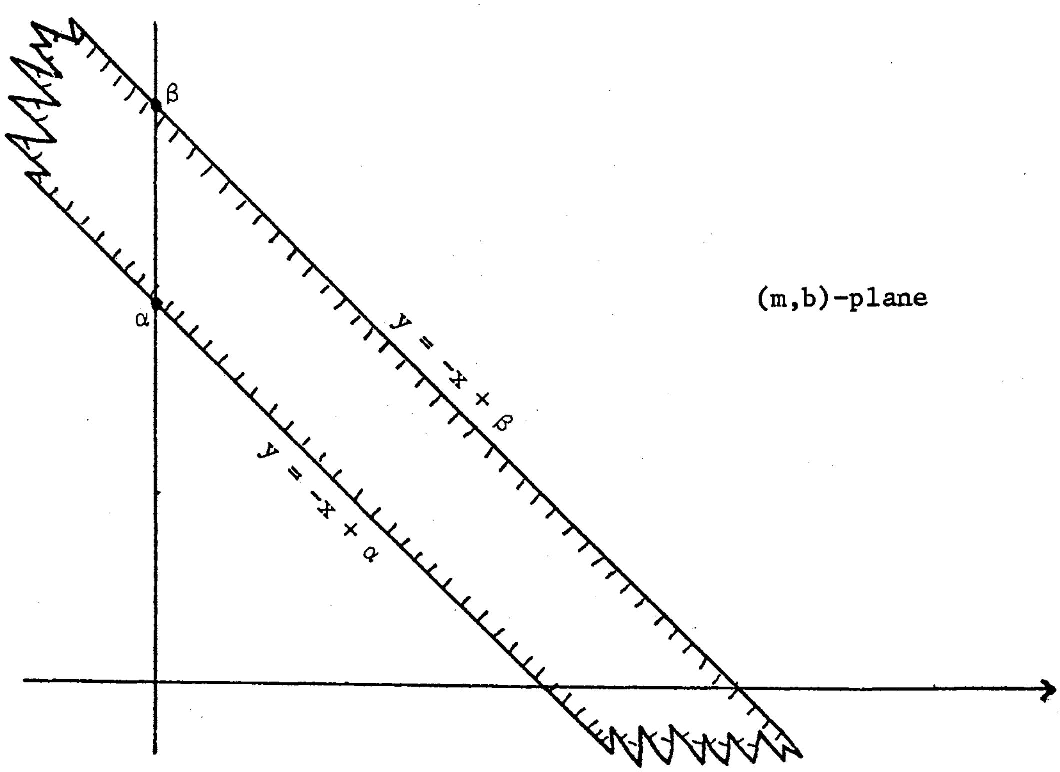

Figure 18. The Set C_i.

Proof: By means of an affine transformation it suffices to prove the theorem when $K = \{(0, \ldots, 0,y): -1 \leq y \leq 1\}$. Let $\mathfrak{F} = \{A_i: i = 1, \ldots, m\}$. Since each A_i is homothetic to K, each A_i has the form $A_i = \{(x_1^i, \ldots, x_{n-1}^i,y): \alpha_i \leq y \leq \beta_i\}$ for some point $(x_1^i, \ldots, x_{n-1}^i,0)$ in L^n and some real scalars α_i and β_i. For each i with $1 \leq i \leq m$ let

$$C_i = \left\{ (a_1, \ldots, a_n): \alpha_i \leq a_n + \sum_{j=1}^{n-1} a_j x_j^i \leq \beta_i \right\}.$$

The set C_i is clearly a closed convex set, a closed "strip" between two parallel hyperplanes. A point $(a_1, \ldots, a_n) \in L^n$ belongs to C_i if and only if the hyperplane

$$\{(x_1, \ldots, x_n): (x_1, \ldots, x_n)\cdot(a_1, \ldots, a_{n-1},-1) = a_n\}$$

$$= \left\{ (x_1, \ldots, x_n): x_n = a_n + \sum_{j=1}^{n-1} a_j x_j \right\}$$

intersects the set A_i. Consequently, the family $\mathcal{C}' = \{C_i: 1 \leq i \leq m\}$ has the (p,q)-property. Figure 18 represents the set C_i when $A_i = \{(1,y): \alpha \leq y \leq \beta\} \subset L^2$. Since the family $\mathcal{C}'$ is finite, there exists an integer k such that the family $\mathcal{C} = \{C_i \cap kB^n: 1 \leq i \leq m\}$ is a family of nondegenerate compact convex sets with the (p,q)-property in L^n. If $P_n(p,q,0) = \infty$, then the fact that $\mathfrak{F}$ is finite clearly implies the existence of the r in the theorem. So assume that $P_n(p,q,0) < \infty$. Then there exist r points $(a_1^j, \ldots, a_n^j)$, $1 \leq j \leq r$, with $r \leq P_n(p,q,0)$ such that if $C_i \cap kB^n \in \mathcal{C}$, then $(a_1^j, \ldots, a_n^j) \in C_i \cap kB^n$ for some j. For $1 \leq j \leq r$, let

$$H_j = \left\{ (x_1, \ldots, x_n): x_n = a_n^j + \sum_{j=1}^{n-1} a_i^j x_i \right\}.$$

Then each set in $\mathfrak{F}$ is intersected by one of the hyperplanes H_j. ■

It will now be illustrated how Theorem 3-11 can be applied to an approximation type problem. Let $S = \{(x_i,y_i): i = 1, \ldots, 8\}$ denote the set of points illustrated in Figure 19 where the points are indexed in order from left to right. Let $\epsilon > 0$ and $A_i = \{(x_i,y): |y - y_i| \leq \epsilon\}$, $1 \leq i \leq 8$. Figure 20 represents the sets A_i, $1 \leq i \leq 8$. One can now ask the following question: Does there exist a line $\{(x,y): y = ax + b\}$ such that $|ax_i + b - y_i| \leq \epsilon$ for all i with $1 \leq i \leq 8$? That is, is there a line L which intersects each of the sets A_i? Since no one line meets A_5, A_6 and A_7 the above question has a negative answer. However, one may extend the above question as follows: Is it possible to find <u>two</u> lines L_1 and L_2 such that each A_i is intersected by at least one of these two lines? Corollary 3-8.2 implies that $P_2(5,4,0) = 2$. Consequently, if from each five of the A_i some four are intersected by a single line, then Theorem 3-11 would imply the existence of two such lines L_1 and L_2. The sets A_i illustrated in Figure 20 do indeed have the property that from each five of the A_i some four are intersected by a line. Figure 21 illustrates two lines L_1 and L_2 such that each A_i is intersected by at least one of these lines.

A situation in which the above approximation type problem may arise would be as follows: Suppose that we are given the set of data points $S = \{(x_i,y_i): 1 \leq i \leq r\}$ in the plane, where $x_i = x_j$ if and

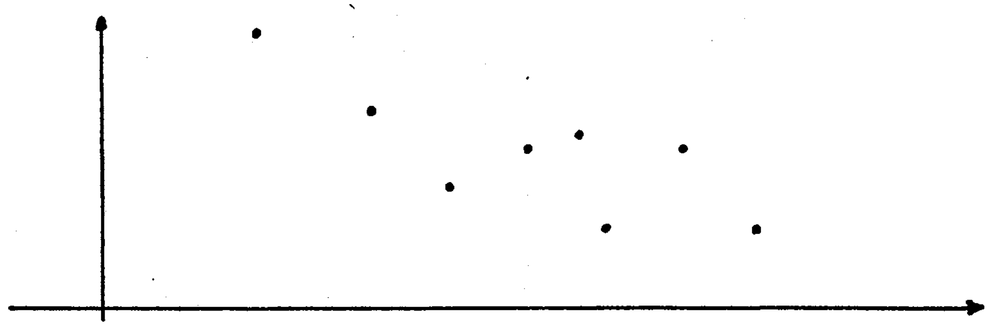

Figure 19. The Set S.

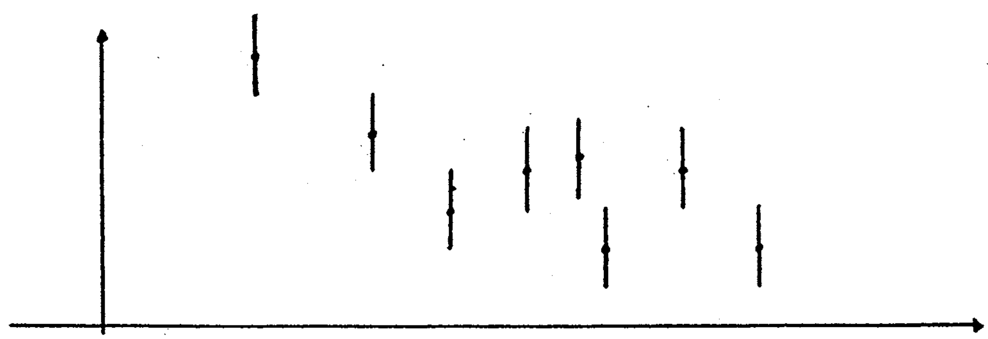

Figure 20. The Sets A_i.

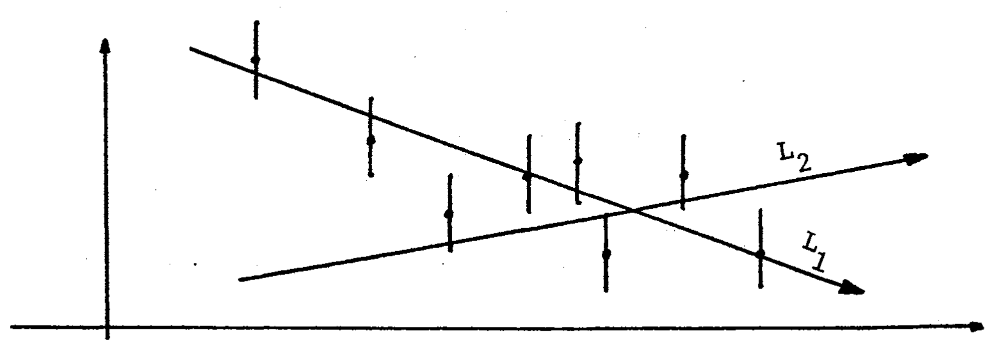

Figure 21. Lines L_1 and L_2.

only if $i = j$. The real numbers x_i are known with certainty but the numbers y_i are subject to an error of ϵ. The data may have been gathered by several persons or by a single person at different times; we do not know, but we would like to make a good guess about this. Somehow we feel the phenomenon is linear in nature and thus our problem is one of finding "linear patterns" in the data. Each linear pattern would represent a data gathering episode and, thus, give us clues about the number of sources from which the data came. Clearly, this is not the usual problem of linear regression whereby the method of least squares a line of best fit is found. Furthermore, whereas, any two points determine a line it would be unreasonable to think of just two points determining a "linear pattern". Intuitively we would feel like we had discovered a "linear pattern" if there was a line L given by $y = ax + b$ such that $|ax_i + b - y_i| \leq \epsilon$ for each (x_i, y_i) belonging to some suitably large subset S' of S. To get a well-defined problem we would need to specify the number m of elements required in S' and ask what is the minimum number of "linear patterns" determined by the data S. The theorem gives a sufficient criteria--one easily amenable to computers--for knowing that there exist less than $P_2(p,q,0)$ lines such that one of the lines passes sufficiently close to each data point. The problem of finding a set of such lines such that each line would pass close to a sufficiently large number of points of S would remain to be solved.

CHAPTER IV

SUMMARY AND CONCLUSIONS

The basic purpose of this study has been to examine certain intersectional properties which a family of subsets of a space X may possess. The problem which was of primary interest was formulated first by Hadwiger and Debrunner [4], and is as follows: Given a family $\mathfrak{F}$ of compact convex subsets of E^n with the (p,q)-property, what can be concluded about $|\mathfrak{F}|$, the minimum piercing number.

Corollary 1-2.1 implies that if $\mathfrak{F}$ is a family of compact subsets of E^n such that $|\mathfrak{G}| \leq m$ for each finite subfamily $\mathfrak{G}$ of $\mathfrak{F}$, then $|\mathfrak{F}| \leq m$. Consequently, many of the problems of the above type were reduced to considering finite families of sets.

In Chapter II families of mutually parallel parallelotopes are considered. The definition of the function $N_n(p,q)$ is due to Hadwiger and Debrunner [p. 32, 5], and the definition of $T_n(p,q)$ which is similar to that of $N_n(p,q)$ and is due to the author. Theorem 2-7 contains practically all the results which were known about the number $N_n(p,q)$ prior to this study. However, the upper bound given in Theorem 2-7 for $N_n(p,q)$ is considerably smaller than those previously known [p. 32, 5]. Corollary 2-12.1 gives even smaller upper bounds for the numbers $N_2(p,2)$. Theorems 2-10 and 2-11 give lower bounds for the numbers $N_n(p,2)$ and $T_n(p,2)$. These lower bounds are considerably larger than those obtained by Hadwiger and Debrunner.

Corollaries 2-10.1 and 2-11.1 were stated in order to give rise to problems for further study. That is, if $\mathfrak{F}$ is a family of mutually parallel parallelotopes in E^n with the (p,2)-property and no three sets in $\mathfrak{F}$ have a common point, then what is the maximal number $h_n(p)$ of elements $\mathfrak{F}$ can contain. The number $h_n(p)$, clearly, satisfies the inequality $h_n(p) \leq 2(N_n(p,2))$. Also, from Corollary 2-11.1 it follows that $h_n(p) \geq 4p - 8$. The author has shown that $h_n(3) = 5$ and $h_n(4) = 8$ for $n \geq 2$; however, the proof of these results does not appear in this study. It is conjectured that $h_2(p) = 4p - 8$ for all $p \geq 4$; that is, Corollary 2-11.1 is the best possible result in E^2 for $p \geq 4$. To go even farther the author conjectures that $N_2(p,2) = 1/2(h_2(p))$ for all $p \geq 4$.

The values of $N_n(p,q)$ for $q \geq 3$, which have been determined in this study, all satisfy the equation $N_n(p,q) = p - q + 1$. Theorems 2-5 and 2-9, which considered families of mutually parallel parallelotopes with the (p,q)-property which fail to have the (p_1,q_1)-property for some $p_1 \leq p - 1$ and $q_1 \leq q - 2$, strongly indicate that the equation $N_n(p,q) = p - q + 1$ may be valid for all $q \geq 3$. The difficulty in showing that $N_n(p,q) = p - q + 1$ for $q \geq 3$ seems to be in showing that $|\mathfrak{F}| \leq p - q + 1$ whenever $\mathfrak{F}$ is a family of mutually parallel parallelotopes with the (p,q)-property which also has the (p_1,q_1)-property for some $p_1 \leq p - 1$ and $q \leq q - 2$. One would think that the more properties $\mathfrak{F}$ had, the smaller $|\mathfrak{F}|$ would be; however, the author has not been able to conclude this in general. However, Theorem 2-13 gives a result of this type.

The problem of determining upper bounds for $|\mathfrak{F}|$ when $\mathfrak{F}$ is an arbitrary family of compact convex subsets of E^n with the

(p,q)-property is much more complicated than the one for mutually parallel parallelotopes. It was known prior to this study that if $q \leq n$ there exists families $\mathfrak{F}$ of compact convex sets in E^n with the (q,q)-property such that $|\mathfrak{F}|$ was arbitrarily large. Grünbaum [3] discovered that if in addition to the (q,q)-property $\mathfrak{F}$ was required to be a family of homotheties of a compact convex set K then $|\mathfrak{F}|$ was always bounded above by a fixed finite number. However, requiring $\mathfrak{F}$ to be a family of homotheties was a rather strong restriction. The search for a more general restriction leads the author to define the (p,q,k)-property. In some sense k is a measure of the "squareness" of the sets of a family $\mathfrak{F}$. The fact that $P_n(p,q,k)$ was finite for $0 < k \leq 1$ and $P_n(p,q,0) = \infty$ for $q \leq n$ seemed to imply that the definition of the (p,q,k)-property had some merit.

It was established that as a function of k, $P_n(p,q,k)$ is a decreasing function which is continuous from the right in k for $0 \leq k < 1$. Also, several problems have been raised by considering $P_n(p,q,k)$ as a function of k. For example, for what values of p, q and n is the set $D(p,q;n) = \emptyset$? Also, if $D(p,q;n) \neq \emptyset$, what are the values $\xi_i(p,q;n)$ and how do these values relate to the geometry of the sets in the families?

In the case $k = 0$, Corollary 3-8.2 can be shown to be equivalent to a theorem of Hadwiger and Debrunner [4], which contains practically all the earlier known results in E^n, $n \geq 2$, about arbitrary families of compact convex sets with the (p,q)-property.

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