MATRIX TRANSFORMATIONS ASSOCIATED WITH NON-ARCHIMEDEAN SEQUENCE SPACES

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CHAPTER I

INTRODUCTION

A very interesting area in mathematics is the study of convergence of sequences and series. When a sequence converges it is possible to associate a number with this sequence by using the concept of the limit of a sequence. The theory of matrix transformations has been developed in an attempt to have some meaningful way of assigning a number to sequences which do not converge. The theory of matrix transformations in the case of real sequences has been studied extensively. The books by Hardy [7] and Cooke [5] are classics in this area.

Very little has been written for the student interested in the study of matrix transformations in a non-Archimedean setting. What work there is in this area has been published since 1950. Dorleijn [6], in 1955 published a paper based on his doctoral dissertation at the University of Amsterdam. In this article he introduces various sequence spaces and related matrix transformations. Andree and Petersen [2] in 1956, published a short article concerning regularity conditions for matrices in the p-adic field. Monna [9], who has written quite extensively in non-Archimedean analysis, published a paper in 1963 concerning matrix transformations on the space of convergent sequences. In addition to these, a few other articles written on matrix transformations in a non-Archimedean setting are listed in the bibliography.

Most of the articles mentioned above are collections of results with at best quite sketchy proofs and would be difficult reading for the beginning graduate or advanced undergraduate student. It is the purpose of the author in writing this paper to make matrix transformations in a non-Archimedean field accessible to these individuals. To read this paper the student should have a good knowledge of sequences, such as would be obtained in an advanced calculus course where a strong treatment of sequences and series was given. Furthermore, he should have at least a basic understanding of linear algebra.

A discussion concerning matrix transformations should begin with a discussion of the sequence spaces to which the transformations are to be applied. In Chapter II, the definition of a sequence space is given. The usual norm convergence of a sequence of points in a sequence space is introduced. Associated with each sequence space is another sequence space known as the dual space. Another type of convergence in a sequence space is defined in terms of the dual space. This form of convergence is useful in the study of matrix transformations. Relationships between the two types of convergence are investigated.

Chapter III is a study of matrix transformations which have the property that convergent sequences are transformed into convergent sequences. Necessary and sufficient conditions for a matrix to have this property are determined. Later in the chapter these conditions are used in investigating matrix transformations on a power series.

The set of matrices, $\Sigma(\alpha)$, that transform a particular sequence space α into itself is introduced in Chapter IV. Necessary and sufficient conditions, when they exist, for a matrix A to belong to $\Sigma(\alpha)$

are determined. These conditions are used to determine for what sequence spaces α, the set $\Sigma(\alpha)$ will be a ring.

The concluding chapter is used to investigate two types of convergence of a sequence of matrices belonging to $\Sigma(\alpha)$. The two convergence criteria introduced are the usual norm convergence, and the convergence defined in terms of the sequence space α and its dual space. Relationships between these two types of convergences are obtained.

Preliminary Definitions and Results

It is important that the reader have a clear idea of a non-Archimedean valuation and a non-Archimedean field. For this reason the definitions are repeated here.

Definition 1.1: A non-Archimedean valuation on a field K is a real-valued function, $| \; |$, on K such that

(1) $|a| \geq 0$ and $|a| = 0$ if and only if $a = 0$,

(2) $|ab| = |a| \, |b|$,

(3) $|a+b| \leq \text{Max}(|a|, |b|)$,

A field K with a non-Archimedean valuation, $| \; |$, is called a non-Archimedean field.

Theorem 1.2: If $| \; |$ is a non-Archimedean valuation on K and if $|a| > |b|$ then $|a+b| = |a|$.

The definitions of a sequence and the limit of a sequence are very important in the work that will follow.

Definition 1.3: A sequence is a function whose domain is the set P of positive integers.

The sequence x will be denoted by $\{x_n\}$, $(x_1, x_2, \ldots, x_n, \ldots)$, or $\{x_n : n = 1, 2, 3, \ldots\}$ where $x_n = x(n)$. The sequence $\{x_n\}$ is said to be a sequence in K if x_n is an element in K for all n.

Definition 1.4: A sequence $\{x_n\}$ is said to converge to a if for every $\varepsilon > 0$ there corresponds an integer N such that $|x_n - a| < \varepsilon$ for all $n \geq N$.

A familiar definition to the student is that of a Cauchy sequence.

Definition 1.5: A sequence $\{x_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists an integer N such that $|x_n - x_m| < \varepsilon$ for all n and $m \geq N$.

The following theorem gives a useful characterization of a Cauchy sequence in a non-Archimedean field.

Theorem 1.6: A sequence $\{x_n\}$ in a non-Archimedean field K is a Cauchy sequence if and only if for each $\varepsilon > 0$ there exists an integer $N > 0$ such that $|x_n - x_{n+1}| < \varepsilon$ when $n \geq N$.

The non-Archimedean field to be used in this work is a complete non-Archimedean field.

Definition 1.7: A field K is said to be complete with respect to the valuation $| \ |$ if every Cauchy sequence in K has a limit in K.

Definition 1.8: Let K be a non-Archimedean field which is complete with respect to the valuation $| \ |$. A vector space V over

K is called a <u>non-Archimedean normed vector space</u> if there exists a real valued function $\| \;\|$ on V, such that

(1) $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$,

(2) $\|v+w\| \leq \text{Max}(\|v\|, \|w\|)$, for all v and w in V,

(3) $\|av\| = |a| \, \|v\|$ where $|a|$ denotes the valuation in K.

All of the examples in this thesis are taken from the field of p-adic numbers, Q_p. Some of the properties of Q_p needed in this paper are listed below.

(1) Each a in Q_p can be expressed uniquely in the form

$$a = p^k \sum_{n=0}^{\infty} a_n p^n$$

where $0 \leq a_n \leq p-1$ for each n, $a_0 \neq 0$, and k is an integer. This form is referred to as the <u>canonical representation</u> of a.

(2) If a is given in canonical form, the valuation $| \;|$ on Q_p is defined by

$$|a| = \left(\frac{1}{p}\right)^k .$$

(3) The function $| \;|$ is a non-Archimedean valuation on Q_p.

(4) Q_p is complete with respect to the valuation $| \;|$.

Any additional understanding of p-adic numbers needed can be obtained by reading Chapter 8 of Agnew's book, <u>Explorations in Number Theory</u>, [1] or the first three chapters of Snook's thesis [14]. Some of the basic ideas needed concerning the nature of the non-Archimedean field can be obtained by reading Chapter II of Palmer's dissertation, [10].

Series and the convergence of series play an important role in this paper. The usual definition of a series is given here.

Definition 1.9: Let $\{a_n\}$ be a sequence in the non-Archimedean field K. Form a new sequence $\{s_n\}$ as follows:

$$s_n = a_1 + a_2 + \ldots + a_n = \sum_{i=1}^{n} a_i \quad (n = 1, 2, 3, \ldots) .$$

A sequence $\{s_n\}$ formed in this way is called a series or an infinite series. The number s_n is called the nth partial sum of the series and a_n is called the nth term of the series. The series is said to converge or to diverge according as $\{s_n\}$ is convergent or divergent. The series is usually denoted by $\sum_{n=1}^{\infty} a_n$.

The next theorem gives a characterization of a convergent series in a non-Archimedean field that is not valid in an Archimedean field.

Theorem 1.10: The series $\sum_{n=1}^{\infty} a_n$ in a non-Archimedean field K converges if and only if $\lim_{n \to \infty} a_n = 0$.

For other properties of convergent series the reader is referred to Chapter III in Palmer's thesis [10].

Using the non-Archimedean property it can be shown that for any integer n, $|\sum_{i=1}^{n} a_i| \leq \text{Max}(|a_1|, |a_2|, \ldots, |a_n|)$. The non-Archimedean property can now be extended to infinite series by using the concept of the supremum of a set of real numbers.

Theorem 1.11: If $\sum_{n=1}^{\infty} a_n$ is a convergent series in K then $|\sum_{n=1}^{\infty} a_n| \leq \text{Sup}_n |a_n|$.

Proof: Let $a = \text{Sup}_n |a_n|$. By the non-Archimedean property it follows that

$$|\sum_{i=1}^{n} a_i| \leq \text{Max}(|a_1|, |a_2|, \ldots, |a_n|) \leq a$$

for all n. Since

$$|\sum_{i=1}^{\infty} a_i| = \lim_{n\to\infty} |\sum_{i=1}^{n} a_i| ,$$

it follows that $|\sum_{n=1}^{\infty} a_n| \leq \text{Sup}_n |a_n|$. △

Double Sequences

Since matrix transformations are naturally associated with double sequences, in this section some elementary concepts of double sequences and series are developed.

Definition 1.12: A double sequence is a function whose domain is the set $P \times P$, where P is the set of positive integers. A double sequence a is denoted by $\{a_{mn}\}$ or $\{a_{mn} : m, n = 1, 2, 3, \ldots\}$.

Another way of writing the double sequence a is as an array. That is, a is the array

$$\begin{array}{cccccc}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & \cdots \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & \cdots \\
\vdots & \vdots & \vdots & & \vdots & \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & \cdots \\
\vdots & \vdots & \vdots & & \vdots &
\end{array}$$

Note: For a fixed m the sequence $\{a_{mn} : n = 1, 2, 3, \ldots\}$ is called the mth row of the sequence, while for a fixed n the sequence $\{a_{mn} : m = 1, 2, 3, \ldots\}$ is called the nth column.

The double sequences to be considered in the remainder of this thesis are those whose range is a subset of a non-Archimedean field K. With the valuation in K, the limit of a double sequence can be defined in the usual way.

Definition 1.13: A double sequence $\{a_{mn}\}$ is said to converge to the element a in K, if for every $\varepsilon > 0$ there corresponds an integer N such that $|a_{mn} - a| < \varepsilon$ for $m \geq N$ and $n \geq N$. If $\{a_{mn}\}$ converges to a then this is indicated by $\lim_{m,n\to\infty} a_{mn} = a$.

A natural question is the following. "If for each m, $\lim_{n\to\infty} a_{mn} = a_m$, and if $\lim_{m\to\infty} a_m$ exists, does $\lim_{m,n\to\infty} a_{mn}$ exist?" The next example indicates that the answer to this question may be no.

Example 1.14: Let $\{a_{mn}\}$ be the following double sequence in Q_p.

$$\begin{matrix} 1 & p & p^2 & p^3 & \cdots \\ 0 & 1 & p & p^2 & \cdots \\ 0 & 0 & 1 & p & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix} .$$

That is,

$$a_{mn} = \begin{cases} 0 & \text{if } n < m \\ p^{n-m} & \text{if } n \geq m . \end{cases}$$

Now, $\lim_{n\to\infty} a_{mn} = \lim p^{n-m} = 0$ for each fixed m. Thus $\lim_{m\to\infty} (\lim_{n\to\infty} a_{mn}) = 0$. However, $\lim_{m,n\to\infty} a_{mn}$ does not exist since $a_{mm} = 1$ while $a_{m,m-1} = 0$.

If $\lim_{n\to\infty} a_{mn}$ exists for each m then the limit, $\lim_{m\to\infty} (\lim_{n\to\infty} a_{mn})$ is called an <u>iterated limit</u>. By interchanging the order of taking limits another iterated limit is obtained; that is, $\lim_{n\to\infty} (\lim_{m\to\infty} a_{mn})$. The following theorem relating the limit of a double sequence and an iterated limit can be proved in the non-Archimedean case. The proof is completely analogous to the case for real sequences and the reader is referred to Apostol for its proof, [4, p. 371].

<u>Theorem 1.15</u>: If $\lim_{m,n\to\infty} a_{mn} = a$ and if for each fixed m, $\lim_{n\to\infty} a_{mn}$ exists then $\lim_{m\to\infty} (\lim_{n\to\infty} a_{mn})$ exists and is equal to a.

The definition of a double series in a non-Archimedean field is the same as in the case of a double series of real numbers.

Definition 1.16: Let $\{a_{mn}\}$ be a double sequence. The double sequence S defined by the equation

$$S_{mn} = \sum_{p=1}^{m} \sum_{q=1}^{n} a_{pq}$$

is called a double series and is denoted by $\sum_{m,n} a_{mn}$. The double series is said to converge to a if $\lim_{m,n\to\infty} S_{mn} = a$. The sequence $\{S_{mn}\}$ is called the sequence of partial sums.

Let $\sum_{m,n} a_{mn}$ be a double series. For each fixed m, consider the series $\sum_{n=1}^{\infty} a_{mn}$. If this series converges for each m, form the series $\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{mn}\right)$. This series is called a repeated series or an iterated series. In a similar way the iterated series $\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{mn}\right)$ can be formed.

Before the main theorem in this section is stated and proved it is necessary to define what is meant by the rearrangement of a double sequence $\{a_{mn}\}$.

Definition 1.17: Let $\{a_{mn}\}$ be a double sequence and let g be a one-to-one function defined on P with a range of $P \times P$. Let b be the function defined by $b_i = a_{g(i)}$ for all i in P. Then g is said to be a rearrangement of the double sequence $\{a_{mn}\}$ into the sequence $\{b_i\}$.

This next theorem is quite important in some of the work to be done in this paper. It gives a sufficient conditions for the convergence of the double series $\sum_{m,n} a_{mn}$, the iterated series, and any simple series formed by a rearrangement of the sequence $\{a_{mn}\}$. This theorem is stated in a slightly different form in Borevich and Shafarevich, [4].

<u>Theorem 1.18</u>: If for any $\varepsilon > 0$, there exist only a finite number of elements a_{mn} with the property that $|a_{mn}| \geq \varepsilon$, then the double series $\sum_{m,n} a_{mn}$ converges and its sum equals the sum of each of the iterated series which also converge. Furthermore, if g is any rearrangement of the sequence $\{a_{mn}\}$, then the simple series determined by this rearrangement converges to the same sum as $\sum_{m,n} a_{mn}$.

<u>Proof</u>: Let $\varepsilon > 0$. Let $N_1 = \text{Max}\{n : |a_{mn}| \geq \varepsilon\}$ and $N_2 = \text{Max}\{m : |a_{mn}| \geq \varepsilon\}$. N_1 and N_2 exist since only a finite number of elements a_{mn} are such that $|a_{mn}| \geq \varepsilon$. Let $N_3 = \text{Max}(N_1 + 1, N_2 + 1)$. Then,

(1.18.1) $$|a_{mn}| < \varepsilon \quad \text{for} \quad m \geq N_3 \quad \text{and all} \quad n,$$

and

(1.18.2) $$|a_{mn}| < \varepsilon \quad \text{for} \quad n \geq N_3 \quad \text{and all} \quad m.$$

By (1.18.2), $\sum_{n=1}^{\infty} a_{mn}$ exists for each m. Let $A_m = \sum_{n=1}^{\infty} a_{mn}$. Now

$$|A_m| = |\sum_{n=1}^{\infty} a_{mn}| \leq \sup_n |a_{mn}| .$$

Thus by (1.18.1), $|A_m| < \varepsilon$ if $m \geq N_3$. Therefore, $\sum_{m=1}^{\infty} A_m$

exists. Hence, $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}$ exists. Let

$$S = \sum_{m=1}^{\infty} A_m = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{mn}\right)$$

and let $S_m = \sum_{i=1}^{m} A_i$. Form the partial sum $S_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$. Now, $|S_m - S| < \varepsilon$ for $m \geq N_3$. Also,

$$\begin{aligned} S_m - S_{mn} &= \sum_{i=1}^{m}\left(\sum_{j=1}^{\infty} a_{ij}\right) - \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}\right) \\ &= \sum_{i=1}^{m}\left(\sum_{j=1}^{\infty} a_{ij} - \sum_{j=1}^{n} a_{ij}\right) \\ &= \sum_{i=1}^{m}\left(\sum_{j=n+1}^{\infty} a_{ij}\right) . \end{aligned}$$

Thus, $|S_m - S_{mn}| \leq \operatorname{Sup}\{|a_{ij}| : 1 \leq i \leq m,\ j \geq n+1\}$. Hence, by (1.18.2), $|S_m - S_{mn}| < \varepsilon$, whenever $n \geq N_3$. Therefore, by the non-Archimedean property

$$(1.18.3) \qquad |S_{mn} - S| \leq \operatorname{Max}(|S_{mn} - S_m|, |S_m - S|) < \varepsilon$$

whenever $m, n \geq N_3$. Hence, $\lim_{m,n\to\infty} S_{mn} = S$, and $\sum_{m,n} a_{mn}$ converges to S. In a similar way it can be shown that $\sum_{m,n} a_{mn} = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{mn}\right)$.

To prove the last statement of the theorem let $b_i = a_{g(i)}$, and let $\varepsilon > 0$ be given. By the proof of the first half of the theorem there is an integer N_3 such that

(1) $|S_{mn} - S| < \varepsilon$ for $m, n \geq N_3$,

(2) $|a_{mn}| < \varepsilon$ for $m \geq N_3$ and all n,

(3) $|a_{mn}| < \varepsilon$ for $n \geq N_3$ and all m.

Let $N = N_3$. Since g is a one-to-one function of P onto $P \times P$ there is an integer M such that

$$\{a_{mn} : 1 \leq m \leq N,\ 1 \leq n \leq N\} \subseteq \{g(1), g(2), \ldots, g(M)\} .$$

Note by (2) and (3), if $a_{m'n'}$ is not in $\{a_{mn} : 1 \leq m \leq N,\ 1 \leq n \leq N\}$, then $|a_{m'n'}| < \varepsilon$. Now for all $i > M$, $|a_{g(i)}| = |b_i| < \varepsilon$ since $a_{g(i)}$ does not belong to $\{g(1), g(2), \ldots, g(M)\}$. By the choice of N it follows from (1) that $|S_{NN} - S| < \varepsilon$. Thus, if $T_i = \sum_{j=1}^{i} b_j$, it follows from the non-Archimedean property that

$$|T_i - S| \leq \text{Max}\,(|T_i - S_{NN}|, |S_{NN} - S|) < \varepsilon$$

if $i > M$. Thus, $\lim_{i\to\infty} T_i = S$ and $\sum_{i=1}^{\infty} b_i = S$. △

The last example of this chapter presents an interesting use of the preceding theorem to find the value of the series $\sum_{n=1}^{\infty} n p^{n-1}$ in Q_p.

Example 1.19: Let $K = Q_p$ and let $\{a_{mn}\}$ be given by the following array.

$$\begin{matrix} 1 & p & p^2 & p^3 & p^4 & \cdots \\ 0 & p & p^2 & p^3 & p^4 & \cdots \\ 0 & 0 & p^2 & p^3 & p^4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

It is clear that the double series $\sum_{m,n} a_{mn}$ converges. Thus

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{mn}\right) \quad \text{and} \quad \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{mn}\right)$$

have the same sum. Now, $\sum_{m=1}^{\infty} a_{mn} = n p^{n-1}$ for each n, and

$$\sum_{n=1}^{\infty} a_{mn} = p^{m-1} + p^{m} + p^{m+1} + \ldots = p^{m-1}\left(\frac{1}{1-p}\right),$$

for each m. Thus, $\sum_{n=1}^{\infty} n p^{n-1} = \sum_{m=1}^{\infty} p^{m-1}\left(\frac{1}{1-p}\right) = \frac{1}{(1-p)^2}$.

CHAPTER II

SEQUENCE SPACES IN A NON-ARCHIMEDEAN FIELD

A sequence is a function whose domain is the set of positive integers. A sequence x will be denoted by $x = \{x_n\}$ or by $x = \{x_n : n = 1, 2, 3, \ldots\}$ or $x = (x_1, x_2, x_3, \ldots\}$ where $x_n = x(n)$. The sequences considered in this paper will be sequences, the range of which is a subset of a complete non-Archimedean valued field K; that is, x is a function from the set P of positive integers into K, a complete non-Archimedean field.

Example 2.1: Let $K = Q_p$, where Q_p is the field of p-adic numbers. Let x be the function defined by $x(n) = p^{n-1}$. Then the sequence x is

$$x = (1, p, p^2, p^3, \ldots, p^{n-1}, \ldots) .$$

Sequence Spaces

There are times when it is necessary to consider a set of sequences that possess a particular property. In this case the property may be that all the sequences are convergent sequences or that all the sequences are sequences which converge to zero. Sets of sequences are called sequence spaces if they are closed under addition and scalar multiplication.

Definition 2.2: A sequence space α is a set of sequences $x = \{x_n\}$, x_n in K, that satisfy the following two properties:

(1) If $x = \{x_n\}$ and $y = \{y_n\}$ belong to α then the sequence $x+y = \{x_n + y_n\}$ belongs to α.

(2) If $x = \{x_n\}$ belongs to α and a is any element in K then the sequence $ax = \{ax_n\}$ belongs to α.

Note that condition (2) implies that if α is a sequence space then the sequence consisting of zeros, $(0, 0, 0, \ldots)$, belongs to α and that if $x = (x_1, x_2, x_3, \ldots)$ belongs to α then the sequence $-x$, $-x = (-x_1, -x_2, -x_3, \ldots)$, belongs to α.

The following definitions and theorems identify particular types of sequences and related sequence spaces. The first type of sequence to be considered is a finite sequence. A sequence $x = \{x_n\}$ is a finite sequence if there exists an integer N such that $x_n = 0$ for all $n \geq N$. For example, the sequence $x = (1, 2, 3, 0, 0, 0, \ldots)$ is a finite sequence with $N = 4$. The set of finite sequences will be denoted by φ.

Theorem 2.3: φ is a sequence space.

Proof: Let $x = \{x_n\}$ and $y = \{y_n\}$ be finite sequences, Then there exists an integer N such that for all $n \geq N$, both $x_n = 0$ and $y_n = 0$. Hence, $x_n + y_n = 0$ for all $n \geq N$. Thus, the sequence $x+y = \{x_n + y_n\}$ belongs to φ. Furthermore, if a is any element of K then $ax_n = 0$ for all $n \geq N$. Therefore, ax belongs to φ. Consequently, φ is a sequence space. $\triangle$

Using the concept of the limit of a sequence one can obtain another sequence space. That is, consider the set of all sequences $x = \{x_n\}$ such that $\lim_{n\to\infty} x_n$ exists. Let this set be denoted by (c).

<u>Theorem 2.4</u>: The set (c) is a sequence space.

<u>Proof</u>: This is immediate from the properties of limits; that is,

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$$

and

$$\lim_{n\to\infty} (ax_n) = a \lim_{n\to\infty} x_n .$$

Thus, $x + y$ and ax belong to (c). △

Other sets of sequences of interest in this study are the following: The set (c_0) of null sequences, that is convergent sequences whose limit is 0; the set (m) of bounded sequences; the set θ of all sequences $\{x_n\}$ such that $x_{n+1} = x_n$ for all $n \geq N(x)$; the set ω of all sequences. The following theorems can be proved by simple arguments.

<u>Theorem 2.5</u>: The set (c_0) is a sequence space.

<u>Theorem 2.6</u>: The set (m) is a sequence space.

<u>Theorem 2.7</u>: The set θ is a sequence space.

Norm Convergence

In any space it is natural to question whether some concept of convergence can be defined for a sequence of points. If the space has a topology associated with it, then the topology can be used to decide whether a sequence of points in α will be convergent. It should be noted at this time that a point in a sequence space is actually a sequence from a non-Archimedean field K. Thus, a sequence in α will be denoted by $\{x^{(n)}\}$ where for each n, $x^{(n)}$ is in α; that is, $x^{(n)}$ is the sequence $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots, x_m^{(n)}, \ldots)$ where $x_m^{(n)}$ belongs to K for each m.

Just as in the case of a sequence of real numbers a norm can be defined on a sequence in a non-Archimedean field. With this norm a convergence criterion can be defined for a sequence in a sequence space α. This convergence will be referred to as norm convergence.

In Chapter I a non-Archimedean norm is defined as a real valued function on a vector space V over a non-Archimedean field K. By the definition of a sequence space α, it is apparent that α is a vector space over the non-Archimedean field K. If for any x in α the value $\|x\|$ is defined by $\|x\| = \operatorname{Sup}_n |x_n|$, then the function $\| \;\|$ is a norm on α, and α is a non-Archimedean vector space.

Theorem 2.8: $\| \;\|$ is a non-Archimedean norm on the sequence space α.

This proof is straightforward and is left to the reader. The norm of this theorem is called the supremum norm or the Sup norm. This supremum norm will be referred to as the "norm" since it is the only norm of a sequence to be considered in this study. With this norm it is possible to define a convergence criterion for a sequence space α.

Definition 2.9: Let $\{x^{(n)}\}$ be a sequence in α. The sequence, $\{x^{(n)}\}$ is said to be norm convergent to x if for every $\varepsilon > 0$ there is an integer N such that

$$\|x^{(n)} - x\| = \operatorname*{Sup}_m |x_m^{(n)} - x_m| < \varepsilon \quad \text{if} \quad n \geq N .$$

The sequence $\{x^{(n)}\}$ is said to be <u>norm convergent</u> if for each $\varepsilon > 0$ there is an integer N such that

$$\|x^{(n)} - x^{(n+1)}\| = \operatorname*{Sup}_m |x_m^{(n)} - x_m^{(n+1)}| < \varepsilon \quad \text{if} \quad n \geq N .$$

The next example is a sequence in (c_0) that is norm convergent.

<u>Example 2.10</u>: Let $K = Q_p$ and let $\{x^{(n)}\}$ be given as follows.

$$x^{(1)} = (1, p, p^2, p^3, \ldots, p^{m-1}, \ldots)$$

$$x^{(2)} = (p, p^2, p^3, p^4, \ldots, p^m, \ldots)$$

$$x^{(3)} = (p^2, p^3, p^4, p^5, \ldots, p^{m+1}, \ldots)$$

$$\ldots$$

$$x^{(n)} = (p^{n-1}, p^n, p^{n+1}, p^{n+2}, \ldots, p^{(k=n)+(m-1)}, \ldots)$$

$$\ldots \qquad .$$

It is clear that $x^{(n)}$ is in (c_0) for each n. Furthermore,

$$x^{(n)} - x^{(n+1)} = (p^{n-1} - p^n, p^n - p^{n+1}, \ldots, p^{n+m-2} - p^{n+m-1}, \ldots) .$$

Thus, $\|x^{(n)} - x^{(n+1)}\| = \frac{1}{p^{n-1}}$. Therefore, if $\varepsilon > 0$ choose N large enough so that $\frac{1}{p^{N-1}} < \varepsilon$. Then if $n \geq N$, $\|x^{(n)} - x^{(n+1)}\| < \varepsilon$. Hence, $\{x^{(n)}\}$ is norm convergent. In fact, it can be shown that $\{x^{(n)}\}$ is norm convergent to the zero sequence.

The next example gives a sequence of null sequences that is not norm convergent.

Example 2.11: Let $K = Q_p$ and let $\{x^{(n)}\}$ be given as follows.

$$x^{(1)} = (1, p, p^2, \ldots, p^{m-1}, \ldots)$$

$$x^{(2)} = (2, 2p, 2p^2, \ldots, 2p^{m-1}, \ldots)$$

$$\ldots$$

$$x^{(n)} = (n, np, np^2, \ldots, np^{m-1}, \ldots)$$

$$\ldots$$

For each n, it is clear that $x^{(n)}$ is in (c_0). However, $\{x^{(n)}\}$ is not norm convergent. This follows from the fact that

$$x^{(n)} - x^{(n+1)} = (1, p, p^2, \ldots, p^{m-1}, \ldots) .$$

Thus, $\|x^{(n)} - x^{(n+1)}\| = 1$, for all n. Hence, $\{x^{(n)}\}$ is not norm convergent.

Since K is a complete space it can be shown that if $\{x^{(n)}\}$ is a norm convergent sequence in α then $\{x^{(n)}\}$ is coordinate convergent; that is, $\lim_{n\to\infty} x_m^{(n)}$ exists for each m.

Theorem 2.12: If $\{x^{(n)}\}$ is a norm convergent sequence in the sequence space α then $\{x^{(n)}\}$ is coordinate convergent.

Proof: Let $\varepsilon > 0$ be given. Since $\{x^{(n)}\}$ is norm convergent there is an integer N such that

$$\|x^{(n)} - x^{(n+1)}\| = \sup_m |x_m^{(n)} - x_m^{(n+1)}| < \varepsilon \quad \text{if} \quad n \geq N .$$

Thus for all m it is true that

$$|x_m^{(n)} - x_m^{(n+1)}| < \varepsilon \quad \text{for} \quad n \geq N .$$

Since K is complete, $\lim_{n\to\infty} x_m^{(n)} = x_m$ in K, for each m. △

With the concept of norm convergence it is now possible to prove that the sequence spaces (c), (c_0), and (m) are complete spaces with respect to norm convergence. That is, if $\{x^{(n)}\}$ is a Cauchy sequence, with respect to the norm, in (c), (c_0), or (m) then there is a sequence x in (c), (c_0), or (m), respectively, such that $\{x^{(n)}\}$ is norm convergent to x. These proofs are the same as in the real case and will not be given here.

Theorem 2.13: The sequence spaces (c), (c_0) and (m) are complete spaces with respect to norm convergence.

Dual Spaces

Associated with each sequence space α is another sequence space known as the dual of α. This space is denoted by α^*, and is defined in terms of convergence of a series. Recall, in a non-Archimedean space, a series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim_{n\to\infty} a_n = 0$. The dual space will be used to define another type of convergence that will be more useful than norm convergence in working with matrix transformations.

Definition 2.14: Let α be a sequence space. The space α^*, called the dual of α or the dual space of α, is the set of all sequences $u = \{u_n\}$ such that for every $x = \{x_n\}$ in α the series $ux = \sum_{n=1}^{\infty} u_n x_n$ converges.

The following theorem demonstrates that α^* is a sequence space. It should be noted that the proof does not depend on the fact that the sequences are contained in a non-Archimedean field.

Theorem 2.15: If α is a sequence space then α^* is a sequence space.

Proof: Let $u = \{u_n\}$ and $v = \{v_n\}$ belong to α^*. Then by definition the series $ux = \sum_{n=1}^{\infty} u_n x_n$ and $vx = \sum_{n=1}^{\infty} v_n x_n$, converge for every $x = \{x_n\}$ in α. Hence,

$$(u+v)x = \sum_{n=1}^{\infty} (u_n + v_n) x_n$$

converges and equals $\sum_{n=1}^{\infty} u_n x_n + \sum_{n=1}^{\infty} v_n x_n$ for every x in α. Therefore, the sequence $u+v$ belongs to α^*. Similarly, for any a in K, the series,

$$(au)x = \sum_{n=1}^{\infty} (au_n) x_n = a \sum_{n=1}^{\infty} u_n x_n$$

converges. So au belongs to α^*. Hence, α^* is a sequence space. △

Since sequence spaces are sets of sequences it is possible to order sequence spaces by set inclusion.

Definition 2.16: Let α and β be sequence spaces. If for every $x = \{x_n\}$ in α, x is also in β then α is contained in β and this is denoted by $\alpha \subseteq \beta$.

An immediate result of this definition is the fact that "$\alpha = \beta$ if and only if $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$." Other facts that are easy to check are

(1) $\varphi \subseteq (c_0)$

(2) $(c_0) \subseteq (c)$

(3) $(c) \subseteq (m)$

(4) $(m) \subseteq \omega$

(5) $\varphi \subseteq \theta \subseteq (c)$.

It is also possible to prove the fact, "if $\alpha \subseteq \beta$ then the dual space of β will be contained in the dual space of α."

Theorem 2.17: If $\alpha \subseteq \beta$ then $\beta^* \subseteq \alpha^*$.

Proof: Suppose u is in β^*. Then for every x belonging to β and hence, for every x belonging to α the series $ux = \sum_{n=1}^{\infty} u_n x_n$ converges. Thus, u is in α^*. Hence, $\beta^* \subseteq \alpha^*$. Δ

Since α^* is a sequence space it is possible to consider the space dual to α^*; that is, the space α^{**}. The following relationship exists between the spaces α and α^{**}.

Theorem 2.18: For any sequence space α, $\alpha \subseteq \alpha^{**}$.

Proof: Let x belong to α. Then for every x in α and every u in α^* the series $ux = \sum_{n=1}^{\infty} u_n x_n$ converges. Therefore, x is in α^{**}.

Hence, $\alpha \subseteq \alpha^{**}$. Δ

It is possible that $\alpha^{**} = \alpha$. The following definition refers to this case.

Definition 2.19: A sequence space α is said to be perfect if $\alpha^{**} = \alpha$.

The dual spaces associated with the spaces φ, (c_0), (c), and (m) will now be derived. Although the definitions and existence of the dual space is the same for any field K, the identification of the dual space for the spaces discussed above depends on the fact that they are defined over a non-Archimedean field K.

Theorem 2.20: $\omega^* = \varphi$ and $\varphi^* = \omega$.

Proof: If $x = \{x_n\}$ is in ω and $u = \{u_n\}$ is in φ then

$$ux = \sum_{n=1}^{\infty} x_n u_n = \sum_{n=1}^{m} x_n u_n$$

for some $m = m(u)$. Thus, ux converges and $\varphi \subseteq \omega^*$. To prove that $\omega^* \subseteq \varphi$, assume by way of contradiction that $u = \{u_n\}$ is not in φ. Then there exists an increasing sequence, $\{M_i\}$ of positive integers such that $u_{M_i} \neq 0$. Define the sequence $x = \{x_n\}$ by

$$x_n = \begin{cases} 0 & \text{if } n \neq M_i \\ \dfrac{1}{u_{M_i}} & \text{if } n = M_i \end{cases} .$$

Since ω is the space of all sequences, $x = \{x_n\}$ is in ω.

Furthermore,

$$ux = \sum_{n=1}^{\infty} u_n x_n = \sum_{i=1}^{\infty} u_{M_i} x_{M_i} = \sum_{i=1}^{\infty} 1$$

which diverges. Thus, u does not belong to ω^*, and it follows that $\omega^* = \varphi$.

To prove the second part of the theorem, note that it has been shown for any u in φ and any x in ω that $ux = \sum_{i=1}^{\infty} u_i x_i$ converges. Thus $\omega \subseteq \varphi^*$. The reverse inclusion follows from the fact that ω is the space of all sequences. Hence, $\omega = \varphi^*$. △

Corollary 2.21: ω and φ are perfect.

The following corollary gives a relationship between the space of finite sequences and any perfect sequence space.

Corollary 2.22: If α is perfect then $\varphi \subseteq \alpha$.

Proof: Since ω is the space of all sequences $\alpha^* \subseteq \omega = \varphi^*$. Thus, Theorem 2.17 implies that $\varphi^{**} \subseteq \alpha^{**}$; that is, $\varphi \subseteq \alpha$. △

Theorem 2.23: $(m)^* = (c_0)$ and $(c_0)^* = (m)$. Thus, (c_0) and (m) are perfect.

Proof: If must first be shown that if x is in (m) and u is in (c_0) then $ux = \sum_{n=1}^{\infty} u_n x_n$ converges. This fact is immediate if one shows that $\lim_{n\to\infty} x_n u_n = 0$. Now, since $x = \{x_n\}$ is in (m) there exists a number $M > 0$ such that $|x_n| < M$ for all n. Therefore,

$$0 \leq |x_n u_n| = |u_n|\,|x_n| \leq M|u_n| .$$

Now, since $u = \{u_n\}$ belongs to (c_0)

$$0 \leq \lim_{n\to\infty} |x_n u_n| \leq M \lim_{n\to\infty} |u_n| = 0 .$$

Thus, $\lim_{n\to\infty} x_n u_n = 0$ and the series $ux = \sum_{n=1}^{\infty} u_n x_n$ converges. Therefore, $(c_0) \subseteq (m)^*$, Next suppose that u is not in (c_0). Now either $\lim_{n\to\infty} u_n$ exists and is non-zero or $\lim_{n\to\infty} u_n$ does not exist. In either case the series $ux = \sum_{n=1}^{\infty} u_n x_n$ does not converge when $x_n = 1$ for all n. Hence, u is not in $(m)^*$ and it follows that $(m)^* = (c_0)$.

To prove the second half of the theorem first note that it has been shown that for any u in (m) and any x in (c_0) that $ux = \sum_{n=1}^{\infty} u_n x_n$ converges. Thus, $(m) \subseteq (c_0)^*$. To show the reverse includion, suppose u is not in (m). For the sequence $\{2^i\}$ there exists an increasing sequence of integers $\{M_i\}$ such that,

$$|u_{M_i}| \geq 2^i$$

where $\{u_{M_i}\}$ is a subsequence of u. Using this subsequence define a sequence $y = \{y_n\}$ by

$$y_n = \begin{cases} 0 & \text{if } n \neq M_i \\ \dfrac{1}{u_{M_i}} & \text{if } n = M_i \end{cases} .$$

Now $y = \{y_n\}$ is in (c_0) since

$$0 \leq \lim_{n\to\infty} |y_n| = \lim_{i\to\infty} \frac{1}{|u_{M_i}|} \leq \lim_{i\to\infty} \frac{1}{2^i} = 0 .$$

Furthermore, $u_n y_n$ does not approach zero and so the series $uy = \sum_{n=1}^{\infty} u_n y_n$ does not converge. Therefore, $u = \{u_n\}$ is not in $(c_0)^*$. Consequently, $(c_0)^* \subseteq (m)$. So $(c_0)^* = (m)$. △

The spaces considered in the previous theorems have been perfect. The next theorem gives a space that is not perfect.

Theorem 2.24: $(c)^* = (c_0)$.

Proof: Let u be in (c) such that u is not in (c_0). Then, $\lim_{n\to\infty} u_n x_n = 0$ if and only if $\lim_{n\to\infty} x_n = 0$. Therefore, if x is in $(c)^*$, x is also in (c_0). Thus, $(c)^* \subseteq (c_0)$. The reverse inclusion follows from the fact that if $u = \{u_n\}$ is in (c_0) then the series $ux = \sum_{n=1}^{\infty} u_n x_n$ converges where $x = \{x_n\}$ is in $(c) \subseteq (m)$. Hence, $(c)^* = (c_0)$. △

Corollary 2.25: (c) is not perfect.

Proof: By the preceding theorem $(c)^* = (c_0)$. But by Theorem 2.23 $(c)^{**} = (c_0)^* = (m)$. Thus $(c) \neq (c)^{**}$ and (c) is not perfect. △

Another example of a sequence space that is not perfect is the space θ. The following theorem shows that $\theta^* = (c_0)$, so $\theta^{**} = (m)$.

Theorem 2.26: $\theta^* = (c_0)$.

Proof: Since $\theta \subseteq (c)$ then $\theta^* \supseteq (c_0)$ by Theorem 2.24 and Theorem 2.17. Now let $x = (1, 1, 1, 1, \ldots)$. Then x belongs to θ. For $u = \{u_n\}$ in θ^*, $\sum_{n=1}^{\infty} u_n$ converges. Therefore, $\lim_{n\to\infty} u_n = 0$. Thus u is in (c_0). So $\theta^* \subseteq (c_0)$. Hence, $\theta^* = (c_0)$. △

There is one other concept to be discussed in this section. This concept is that of a normal sequence space. The definition of a normal sequence space follows.

Definition 2.27: A sequence space α is normal if whenever x is in α and $y = \{y_n\}$ is a sequence such that $|y_n| \leq |x_n|$ for every n, then the sequence y is in α.

It is clear that φ, ω, (c_0), (c), and (m) are all normal sequence spaces. An example of a sequence space that is not normal is the space θ.

Theorem 2.28: The sequence space θ is not a normal space.

Proof: Let $x = \{x_n\}$ be the sequence with $x_n = 1$ for all n. Let $y = \{y_n\}$ be the sequence such that $y_n = 1$ if n is even and $y_n = 0$ if n is odd. Now x belongs to θ, but y does not belong to θ. However, $|y_n| \leq |x_n|$ for all n. Hence, θ is not normal. Δ

(α,β)-Convergence

If α is a sequence space and β is a sequence space so that $\beta \subseteq \alpha^*$, then another type of convergence can be defined for a sequence of points in α. This convergence, in some instances, is weaker than norm convergence. However, as will become evident in Chapters III and IV this convergence will be useful in discussing matrix transformations on sequence spaces.

Definition 2.29: The sequence of points $\{x^{(n)}\}$ of α is said to be (α,β)-convergent, $\varphi \subseteq \beta \subseteq \alpha^*$, if to every $u = \{u_m\}$ of β and to every $\varepsilon > 0$, there corresponds a number $N = N(\varepsilon, u) > 0$, such that

$$|u(x^{(n)} - x^{(n+1)})| = \left| \sum_{m=1}^{\infty} u_m(x_m^{(n)} - x_m^{(n+1)}) \right| < \varepsilon$$

for $n \geq N$. If $\beta = \alpha^*$ then $\{x^{(n)}\}$ is said to be <u>α-convergent</u>.

The following is an example of a sequence $\{x^{(n)}\}$ that is $((c_0), (m))$-convergent.

<u>Example 2.30</u>: Let $K = Q_p$ and let $\{x^{(n)}\}$ be given as follows.

$$x^{(1)} = (1, p, p^2, p^3, \ldots)$$

$$x^{(2)} = (p, p^2, p^3, p^4, \ldots)$$

$$x^{(3)} = (p^2, p^3, p^4, p^5, \ldots)$$

$$\ldots$$

$$x^{(n)} = (p^{n-1}, p^n, p^{n+1}, p^{n+2}, \ldots)$$

$$\ldots \qquad .$$

It is clear that $\{x^{(n)}\}$ is a sequence from (c_0). Now let $u = \{u_m\}$ be a bounded sequence. Then there exists a positive number $M = M(u)$ such that $|u_m| < M$ for all m. Thus,

$$|u_m(p^{n+m-2} - p^{n+m-1})| \leq \frac{M}{p^{n+m-2}} \quad .$$

Hence, $\lim_{m \to \infty} u_m(p^{n+m-2} - p^{n+m-1}) = 0$ and

$$\sum_{m=1}^{\infty} u_m(p^{n+m-2} - p^{n+m-1})$$

converges. It can now be shown that $\{x^{(n)}\}$ is $((c_0),(m))$-convergent. To do this it must be shown that

$$\left| \sum_{m=1}^{\infty} u_m (p^{n+m-2} - p^{n+m-1}) \right| < \varepsilon \quad \text{for} \quad n \geq N .$$

But

$$\left| \sum_{m=1}^{\infty} u_m (p^{n+m-2} - p^{n+m-1}) \right| = \left| p^{n-1} \sum_{m=1}^{\infty} u_m (p^{m-1} - p^m) \right| .$$

Now $\sum_{m=1}^{\infty} u_m (p^{m-1} - p^m)$ converges, since u belongs to (m) and $\lim_{m\to\infty} p^{m-1} - p^m = 0$. Thus $\left| \sum_{m=1}^{\infty} u_m (p^{m-1} - p^m) \right| = M'$ for some number M'. Hence,

$$\left| \sum_{m=1}^{\infty} u_m (p^{n+m-2} - p^{n+m-1}) \right| = \frac{M'}{p^{n-1}} .$$

Thus, since $\lim_{n\to\infty} \frac{M'}{p^{n-1}} = 0$, $\{x^{(n)}\}$ is $((c_0),(m))$-convergent. Since $(m) = (c_0)^*$, the sequence $\{x^{(n)}\}$ is (c_0)-convergent.

The following theorem gives an interesting result about an (α,β)-convergent sequence.

<u>Theorem 2.31</u>: Every (α,β)-convergent sequence $\{x^{(n)}\}$ of α, $(\varphi \subseteq \beta \subseteq \alpha^*)$, is coordinate convergent; that is, $\lim_{n\to\infty} x_m^{(n)}$ exists for every m.

<u>Proof</u>: Let $\{x^{(n)}\}$ be an (α,β)-convergent sequence and let $\varepsilon > 0$ be given. Then for a fixed m, since $\varphi \subseteq \beta$, there is a sequence $u = \{u_m\}$ such that $u_m = 1$ while $u_i = 0$ for all $i \neq m$. Since

$\{x^{(n)}\}$ is (α,β)-convergent there exists a positive number $N = N(\varepsilon, u) > 0$, such that

$$|u(x^{(n)} - x^{(n+1)})| = \left| \sum_{i=1}^{\infty} u_i(x_i^{(n)} - x_i^{(n+1)}) \right| < \varepsilon$$

for $n \geq N$. Therefore, for $n \geq N$

$$|x_m^{(n)} - x_m^{(n+1)}| < \varepsilon$$

Hence, for each fixed m, $\{x_m^{(n)}\}$ is a Cauchy sequence in K and since K is complete it follows that $\lim_{n\to\infty} x_m^{(n)}$ exists for each m. Δ

Example 2.32: Let $\{x^{(n)}\}$ be the sequence defined in Example 2.30. Then for a fixed m,

$$x_m^{(n)} = (1^{m-1}, p^m, p^{m+1}, p^{m+2}, \ldots, p^{m+(n-2)}, \ldots) .$$

Thus, $\lim_{n\to\infty} x_m^{(n)} = \lim_{n\to\infty} p^{m+(n-2)} = 0$.

The following lemma will be useful in determining necessary and sufficient conditions for a sequence $\{x^{(n)}\}$ in α to be (α,β)-convergent. This lemma will also be used again in Chapter IV in connection with determining necessary and sufficient conditions for a set of matrices to be a ring.

Lemma 2.33: Let $\{a_{mn}\}$ be a double sequence with the properties:

(1) For each fixed n, $\lim_{m\to\infty} a_{mn} = 0$.

(2) For each fixed m, $\lim_{n\to\infty} a_{mn} = 0$.

(3) There exists an $\varepsilon > 0$ such that for every N there is an $n_0 \geq N$ such that $|a_{m_0 n_0}| \geq \varepsilon$ for some m_0.

Then there exists strictly increasing sequences $(n_0, n_1, n_2, \ldots)$, $(N_0', N_1', N_2', \ldots)$, $(m_0, m_1, m_2, \ldots)$ and $(m_0', m_1', m_2', \ldots)$ such that $m_i < m_i' < m_{i+1}$ and $n_i \geq N_i'$ for every i, and $|a_{m_k n_k}| \geq \varepsilon$ for $k = 0, 1, 2, \ldots$. Furthermore, the sets

$$R_k = \{a_{mn} : 1 \leq m \leq m_{k-1}', \ n \geq N_k'\}$$

have the property that for every a_{mn} in R_k, $|a_{mn}| < \varepsilon$.

Proof: The sequences will be constructed by induction. By condition (2), for each m there exists an integer N_m such that

$$|a_{mn}| < \varepsilon \quad \text{for } n \geq N_m . \tag{2.33.1}$$

To begin the construction let $N_0' = 1$. Then by (3), there exists an integer $n_0 \geq N_0'$ such that $|a_{m_0 n_0}| \geq \varepsilon$ for some m_0. Using condition (1), there exists an integer m_0' such that $|a_{mn_0}| < \varepsilon$ if $m \geq m_0'$. It is clear that $m_0 < m_0'$. [See Figure 1. For purposes of illustration suppose $m_0 = 2$, $n_0 = 4$, and $m_0' = 3$]. Now determine the integers $N_1, N_2, \ldots, N_{m_0'}$ of (2.33.1) so that

$$\begin{cases} |a_{1n}| < \varepsilon & \text{for } n \geq N_1 \\ |a_{2n}| < \varepsilon & \text{for } n \geq N_2 \\ \quad \cdots & \\ |a_{m_0' n}| < \varepsilon & \text{for } n \geq N_{m_0'} . \end{cases} \tag{2.33.2}$$

Let $N_1' = \mathrm{Max}(N_0, N_1, \ldots, N_{m_0'})$. Then define R_1 to be $R_1 = \{a_{mn} : 1 \leq m \leq m_0',\ n \geq N_1'\}$. Now by (2.33.2), $|a_{mn}| < \varepsilon$ for all a_{mn} in R_1. Apply condition (3) again to obtain an integer $n_1 \geq N_1'$ so that $|a_{m_1 n_1}| \geq \varepsilon$ for some m_1. Now $m_1 > m_0'$ since $n_1 \geq N_1'$ and because $a_{m_0 n_1}$ belongs to R_1. Using condition (1) there exists an integer m_1' such that $|a_{mn_1}| < \varepsilon$ for $m \geq m_1'$. Clearly $m_1' > m_1$. Now determine the integers $N_{m_0'+1}, N_{m_0'+2}, \ldots, N_{m_1'}$ of (2.33.1) so that

$$\text{(2.33.3)} \quad \begin{cases} |a_{m_0'+1,n}| < \varepsilon & \text{for } n \geq N_{m_0'+1} \\ |a_{m_0'+2,n}| < \varepsilon & \text{for } n \geq N_{m_0'+2} \\ \quad \cdots & \\ |a_{m_1'n}| < \varepsilon & \text{for } n \geq N_{m_1'} . \end{cases}$$

Let $N_2' = \mathrm{Max}(N_{m_0'+1}, N_{m_0'+2}, \ldots, N_{m_1'})$. Define R_2 by $R_2 = \{a_{mn} : 1 \leq m \leq m_1',\ n \geq N_2'\}$. [See Figure 1]. Then by (2.33.3) $|a_{mn}| < \varepsilon$ for all a_{mn} in R_2. Suppose now that the sets of integers $\{N_0', N_1', \ldots, N_{k-1}'\}$, $\{n_0, n_1, \ldots, n_{k-1}\}$, $\{m_0, m_1, \ldots, m_{k-1}\}$ and $\{m_0', m_1', \ldots, m_{k-1}'\}$ have been determined and that R_{k-1} has been defined so that $R_{k-1} = \{a_{mn} : 1 \leq m \leq m_{k-1}',\ n \geq N_{k-1}'\}$. Suppose further that $n_j \geq N_j'$ and $m_{j-1} < m_{j-1}' < m_j$ for $1 \leq j \leq k-1$. Now determine the integers N_j, $m_{k-2}' \leq j \leq m_{k-1}'$, of (2.33.1). Let $N_k' = \mathrm{Max}\{N_j : m_{k-2}' \leq j \leq m_{k-1}'\}$. Define $R_k = \{a_{mn} : 1 \leq m \leq m_{k-1}',\ n \geq N_k'\}$. Apply condition (3) to obtain an integer $n_k \geq N_k'$ such that

$$|a_{m_k n_k}| < \varepsilon \quad \text{for some} \quad m_k .$$

Clearly $m_k > m'_{k-1}$. Now $a_{m_k n_k}$ does not belong to R_k since all terms in R_k are such that $|a_{mn}| < \varepsilon$. By condition (1) there exists an integer m'_k such that $|a_{mn_k}| < \varepsilon$ for all $m \geq m'_k$. By the choice of m_k it follows that $m_k < m'_k$. Hence, the kth terms in the sequence have been determined given that the $k-1$ st term were given. Thus, by induction the sequences have been constructed so that $n_k \geq N'_k$ and $m_k < m'_k < m_{k+1}$. Furthermore for all terms a_{mn} in R_k, $|a_{mn}| < \varepsilon$. △

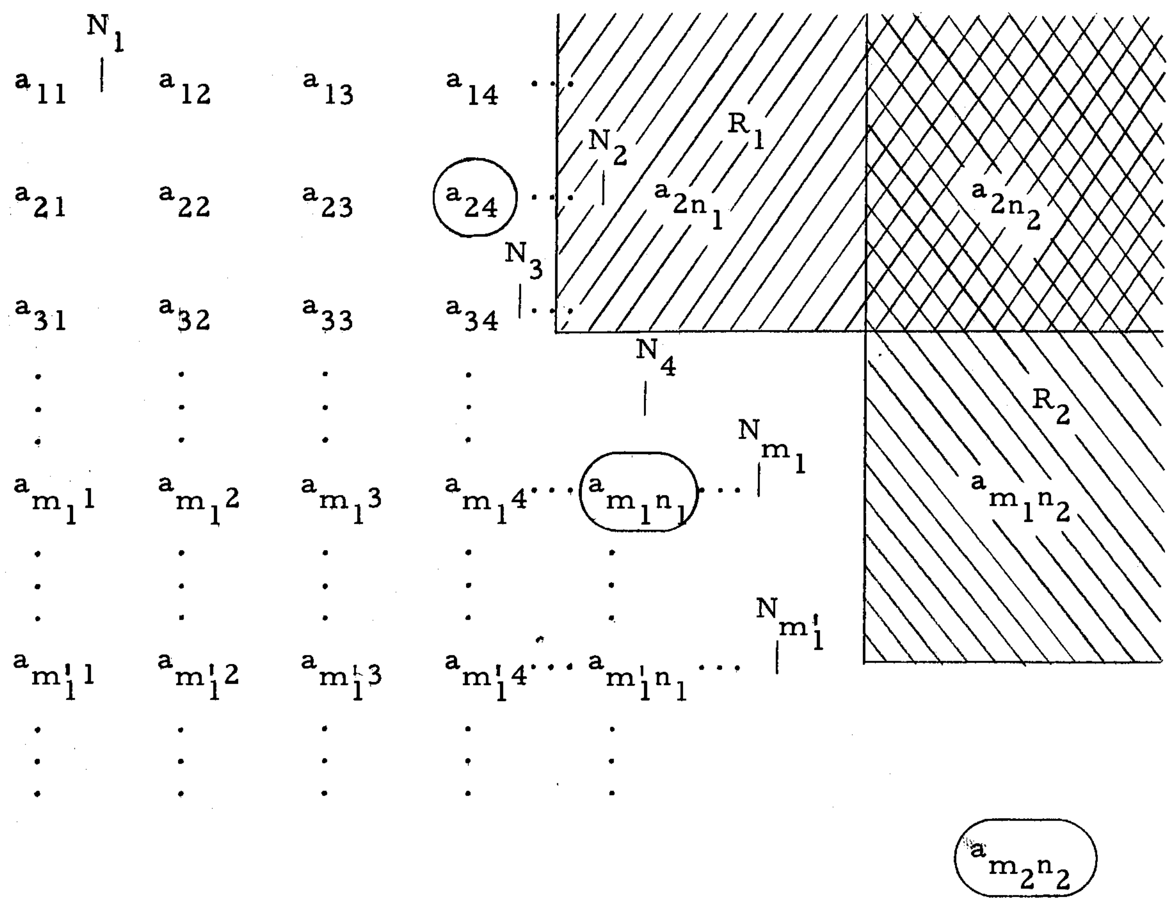

Figure 1. Subsequence ; Lemma 2.33

The following theorem gives necessary and sufficient conditions for a sequence $\{x^{(n)}\}$ in the sequence space α to be (α,β)-convergent where $\varphi \subseteq \beta \subseteq \alpha^*$, and β is a normal space.

Theorem 2.34: A necessary and sufficient condition for the (α,β)-convergence, $(\varphi \subseteq \beta \subseteq \alpha^*, \beta$ normal$)$, of a sequence of points $\{x^{(n)}\}$ of α is that to every $u = \{u_m\}$ of β and to every $\varepsilon > 0$, there corresponds a number $N = N(\varepsilon, u) > 0$, such that for every m

$$|u_m(x_m^{(n)} - x_m^{(n+1)})| < \varepsilon \quad \text{for } n \geq N.$$

Proof: Suppose first that the condition is satisfied. Since K is non-Archimedean it follows that

$$\left| \sum_{m=1}^{\infty} u_m(x_m^{(n)} - x_m^{(n+1)}) \right| \leq \operatorname{Sup}_m \left| u_m(x_m^{(n)} - x_m^{(n+1)}) \right| .$$

Thus, for $\varepsilon > 0$ and for every u of β there exists a positive number $N = N(\frac{\varepsilon}{2}, u)$ such that for $n \geq N$

$$|u_m(x_m^{(n)} - x_m^{(n+1)})| < \frac{\varepsilon}{2} \quad \text{for every } m .$$

Therefore, for $n \geq N$

$$\begin{aligned} \left| u(x^{(n)} - x^{(n+1)}) \right| &= \left| \sum_{m=1}^{\infty} u_m(x_m^{(n)} - x_m^{(n+1)}) \right| \\ &\leq \operatorname{Sup}_m \left| u_m(x_m^{(n)} - x_m^{(n+1)}) \right| . \end{aligned}$$

Thus,

$$|u(x^{(n)} - x^{(n+1)})| \leq \frac{\varepsilon}{2} < \varepsilon \quad \text{for} \quad n \geq N .$$

Therefore, $\{x^{(n)}\}$ is (α,β)-convergent.

To prove the converse an indirect proof will be used. That is, suppose that there exists an $\varepsilon > 0$, an $x = \{x_n\}$ in α, and $u = \{u_m\}$ in β such for all N there is an integer $n_0 \geq N$ such that

$$|u_m(x_m^{(n_0)} - x_m^{(n_0+1)})| \geq \varepsilon .$$

[For notational purposes let $y_{mn} = x_m^{(n)} - x_m^{(n+1)}$]. Since $\{x^{(n)}\}$ is an (α,β)-convergent sequence, $\{x^{(n)}\}$ is coordinate convergent. That is, for each fixed m $\lim_{n\to\infty} y_{mn} = 0$. Hence, $\lim_{n\to\infty} u_m y_{mn} = 0$ for each fixed m. Furthermore, since $u = \{u_m\}$ belongs to $\beta \subseteq \alpha^*$, the series $\sum_{m=1}^{\infty} u_m x_m^{(n)}$ and $\sum_{m=1}^{\infty} u_m x_m^{(n+1)}$ converge for each n. Thus $\lim_{m\to\infty} u_m y_{mn} = 0$ for each fixed n. Therefore $\{u_m y_{mn}\}$ is a double sequence that satisfies the three properties of Lemma 2.33. [The contradiction will be obtained by defining a sequence $v = \{v_j\}$ in terms of the sequence $u = \{u_m\}$ so that v is in β and $|v(x^{(n)} - x^{(n+1)})| \geq \varepsilon$ for $n = n_k \geq N'_k$. This will then contradict the fact that $\{x^{(n)}\}$ is (α,β)-convergent.]

From Lemma 2.33 there exist sequences $(n_0, n_1, \ldots, n_k, \ldots)$, $(N'_0, N'_1, \ldots, N'_k, \ldots)$, $(m_0, m_1, \ldots, m_k, \ldots)$, and $(m'_0, m'_1, \ldots, m'_k, \ldots)$ such that $m_k < m'_k < m_{k+1}$, $n_k \geq N'_k$ and $|u_{m_k} y_{m_k n_k}| \geq \varepsilon$ for all k. The sequence $v = \{v_j\}$ id defined as follows:

$$v_j = \begin{cases} 0 & \text{if } j \neq m_k \quad (k = 0, 1, 2, \dots) \\ u_{m_k} & \text{if } j = m_k \quad (k = 0, 1, 2, \dots) . \end{cases}$$

Now $v = \{v_j\}$ has the properties

(2.34.1) $$|v_{m_k} y_{m_k n_k}| \geq \varepsilon \quad \text{for all } k$$

and

(2.34.2) $$|v_{m_k} y_{m_k n_j}| < \varepsilon \quad \text{if } j \neq k .$$

The property in (2.34.1) follows since $v_{m_k} = u_{m_k}$ and $|u_{m_k} y_{m_k n_k}| \geq \varepsilon$ by construction. The property in (2.34.2) is true since for any $j > k$ $v_{m_k} y_{m_k n_j}$ belongs to R_k, and for any $j < k$, $m_k > m_j^!$ and $|v_m y_{m n_j}| < \varepsilon$ for $m \geq m_j^!$. Now by definition of $v = \{v_j\}$, $|v_j| \leq |u_j|$ for all j. Thus, since β is a normal space v belongs to β. Furthermore,

$$|v(x^{(n)} - x^{(n+1)})| \geq \varepsilon \quad \text{for } n = n_k, \ k = 0, 1, 2, \dots .$$

For consider $n = n_k$. Then,

$$v(x^{(n_k)} - x^{(n_k+1)}) = v_{m_k} y_{m_k n_0} + v_{m_k} y_{m_k n_1} + \dots + v_{m_k} y_{m_k n_k}$$
$$+ y_{m_k} y_{m_k n_{k+1}} + \dots .$$

Now by (2.34.1), $|v_{m_k} y_{m_k n_k}| \geq \varepsilon$. But by (2.34.2), $|v_{m_k} y_{m_k n_j}| < \varepsilon$ if $j \neq k$. Thus by the non-Archimedean property

$$|v(x^{(n_k)} - x^{(n_k+1)})| = |v_{m_k} y_{m_k n_k}| \geq \varepsilon .$$

Hence for any integer N, there exists an integer $n_k \geq N$ such that

$$|v(x^{(n_k)} - x^{(n_k+1)})| \geq \varepsilon ,$$

Thus, $\{x^{(n)}\}$ is not (α,β)-convergent. This contradiction proves the condition is necessary. △

This section concludes with a theorem which is an easy consequence of the preceding theorem and which is quite useful in the work to be done in finding necessary and sufficient conditions for a matrix to belong to a particular set of matrices. It states that an (m)-convergent sequence is uniformly bounded.

Theorem 2.35: If $\{x^{(n)}\}$ is an (m)-convergent sequence then there exists a positive number M such that $|x_m^{(n)}| < M$ for all m and n; that is, $\|x^{(n)}\| < M$ for all n.

Proof: Let $u = \{u_m\}$ belong to (c_0) so that $\|u\| \leq 1$. Pick $\varepsilon = 1$. Since $\{x^{(n)}\}$ is (m)-convergent there exists an integer n' such that for $n \geq n'$

$$|u_m(x_m^{(n)} - x_m^{(n+1)})| < 1 \quad \text{for all } m .$$

Now by the non-Archimedean property it follows that for all $k \geq 0$

$$|u_m(x_m^{(n')} - x_m^{(n'+k)})| < 1 \quad \text{for all } m .$$

Therefore for all $k \geq 0$ and for all m

$$|u_m x_m^{(n'+k)}| < 1 + |u_m x_m^{(n')}| .$$

Since $x^{(n')}$ is a bounded sequence $\|x_m^{(n')}\| = M_{n'}$. Thus, since $|u_m| \leq 1$ for all m

$$|u_m x_m^{(n'+k)}| < 1 + M_{n'}$$

for all $k \geq 0$ and all m. Therefore,

$$\operatorname*{Sup}_m |u_m x_m^{(n'+k)}| \leq 1 + M_{n'} \quad \text{for all } k \geq 0 .$$

Thus, $\|u\| M_{n'+k} \leq 1 + M_{n'}$ for all $k \geq 0$, where $M_{n'+k}$ is $\|x^{(n'+k)}\|$. Therefore,

$$\|x^{(n'+k)}\| \leq M_{n'} + 1 \quad \text{for } k \geq 0 ,$$

Now, let $M = \text{Max}(M_1, M_2, \ldots, M_{n'}+1)$. Then for all n, $\|x^{(n)}\| < M$. Thus, the theorem is proved. △

Comparisons of Norm Convergence

and α-Convergence

The remaining part of this chapter is devoted to the investigation of the relationship between norm convergence and α-convergence. Two types of convergence are equivalent if the same sequences converge for each method of convergence.

The next theorem demonstrates that (c_0)-convergence and norm convergence are equivalent.

<u>Theorem 2.36</u>: Let $x^{(n)}$ belong to (c_0), for every n; the sequence $\{x^{(n)}\}$ is (c_0)-convergent if and only if $\{x^{(n)}\}$ is norm convergent.

<u>Proof</u>: Let $\{x^{(n)}\}$ be a (c_0)-convergent sequence in (c_0). By Theorem 2.34 for every u in (m) and every $\varepsilon > 0$ there is an integer N such that for every m

$$(2.36.1) \qquad |u_m(x_m^{(n)} - x_m^{(n+1)})| < \varepsilon \quad \text{for} \quad n \geq N .$$

Let $u = (1, 1, 1, \dots)$. Then u is in (m) and it follows from (2.36.1) that for every m

$$(2.36.2) \qquad |x_m^{(n)} - x_m^{(n+1)} < \varepsilon \quad \text{for} \quad n \geq N .$$

Thus

$$(2.36.3) \qquad \|x^{(n)} - x^{(n+1)}\| = \operatorname{Sup}_m |x_m^{(n)} - x_m^{(n+1)}| < \varepsilon$$

for $n \geq N$. Hence, $\{x^{(n)}\}$ is norm convergent. Conversely, suppose $\{x^{(n)}\}$ is a norm convergent sequence in (c_0). To show that $\{x^{(n)}\}$ is (c_0)-convergent let $\varepsilon > 0$ and let $u = \{u_m\}$ belong to (m). Since u is in (m) there is anumber $M > 0$ such that $|u_m| < M$ for all m. Furthermore, since $\{x^{(n)}\}$ is norm convergent there exists an integer N such that

$$\|x^{(n)} - x^{(n+1)}\| < \frac{\varepsilon}{M} \quad \text{for} \quad n \geq N .$$

Thus,

$$\sup_m |x_m^{(n)} - x_m^{(n+1)}| < \frac{\varepsilon}{M} \quad \text{for} \quad n \geq N .$$

Therefore for every m it follows that

$$|u_m(x_m^{(n)} - x_m^{(n+1)}| < \varepsilon \quad \text{for} \quad n \geq N .$$

Hence, $\{x^{(n)}\}$ is (c_0)-convergent. △

In the next theorem it is shown that a norm convergent sequence from (c) is also (c)-convergent.

Theorem 2.37: Let $\{x^{(n)}\}$ be a sequence of convergent sequences. If $\{x^{(n)}\}$ is norm convergent then $\{x^{(n)}\}$ is (c)-convergent.

Proof: Let $u = \{u_m\}$ belong to $(c_0) = (c)^*$. If u is identically zero then clearly $|u(x^{(n)} - x^{(n+1)}| < \varepsilon$ for all n. Thus, suppose there is an entry u_m of u that is non-zero. Then $\|u\| \geq |u_m| > 0$. Now, since $\{x^{(n)}\}$ is a norm convergent sequence, for each $\varepsilon > 0$, there is an integer N such that

$$\|x^{(n)} - x^{(n+1)}\| = \sup_m |x_m^{(n)} - x_m^{(n+1)}| < \frac{\varepsilon}{\|u\|}$$

for $n \geq N$. Thus, for every m

$$|u_m(x_m^{(n)} - x_m^{(n+1)}| \leq \|u\| \sup_m |x_m^{(n)} - x_m^{(n+1)}| < \varepsilon ,$$

if $n \geq N$. Therefore, by Theorem 2.34, $\{x^{(n)}\}$ is (c)-convergent.△

The converse of the preceding theorem is not true. That is, there exists a sequence $\{x^{(n)}\}$ of convergent sequences that is (c)-convergent but is not norm convergent.

<u>Example 2.38</u>: Let $K = Q_p$ and let $\{x^{(n)}\}$ be given as follows.

$$x^{(1)} = (1, 0, p, p^2, p^3, \dots)$$

$$x^{(2)} = (0, 1, 0, p, p^2, \dots)$$

$$x^{(3)} = (0, 0, 1, 0, p, \dots)$$

$$\dots \qquad .$$

In general $x^{(n)}$ is given by $x^{(n)} = \{x_m^{(n)} : m = 1, 2, 3, \dots\}$ where

$$(2.38.1) \qquad x_m^{(n)} = \begin{cases} 0 & \text{if } m < n \text{ or } m = n+1 \\ 1 & \text{if } m = n \\ p^{m-n-1} & \text{if } m \geq n+2 \, . \end{cases}$$

To show that $\{x^{(n)}\}$ is (c)-convergent let $\varepsilon > 0$ be given and $u = \{u_m\}$ belong to (c_0). Then there exists an m_0 such that $|u_m| < \varepsilon$ if $m \geq m_0$. Now by (2.38.1) the sequence $\{x^{(n)} - x^{(n+1)}\}$ is given by

$$(2.38.2) \qquad x_m^{(n)} - x_m^{(n+1)} = \begin{cases} 0 & \text{if } m < n \\ 1 & \text{if } m = n \\ -1 & \text{if } m = n+1 \\ p & \text{if } m = n+2 \\ p^{m-n-1} - p^{m-n-2} & \text{if } m > n+2 \, . \end{cases}$$

Thus, $|x_m^{(n)} - x_m^{(n+1)}| \leq 1$ for all n and m. Therefore, if $n \geq m_0$

$$|u_m(x_m^{(n)} - x_m^{(n+1)})| < \varepsilon \quad \text{for all } m \, .$$

Hence, $\{x^{(n)}\}$ is (c)-convergent. However, by (2.38.2)

$$\|x^{(n)} - x^{(n+1)}\| = \operatorname*{Sup}_m |x_m^{(n)} - x_m^{(n+1)}| = 1$$

for all n. Therefore, $\{x^{(n)}\}$ is not norm convergent.

Notice in the proof of Theorem 2.37 that no use is made of the fact that the sequences $x^{(n)}$ are convergent sequences. The fact crucial to the proof is that $u = \{u_m\}$ belong to (c_0). Therefore, since $(m)^* = (c_0)$, it follows that a norm convergent sequence of bounded sequences is also (m)-convergent. Furthermore, the sequence in Example 2.38 furnishes an instance where (m)-convergence does not imply norm convergence. This observation concludes Chapter II.

CHAPTER III

MATRIX TRANSFORMATIONS IN A NON-ARCHIMEDEAN FIELD

Basic Definitions and Examples

Let K be a non-Archimedean field and let $\{x_n\}$ be a sequence of elements from K. If $x = \{x_n\}$ is a convergent sequence then a number can be associated with the sequence x by using the concept of the limit of a sequence. However, if the sequence is not convergent there is no immediate method of assigning a number to the sequence. The methods that have been developed to assign numbers to divergent sequences are known as summability methods or matrix transformations.

The student of linear algebra encounters matrix transformations when he considers the product of an $m \times n$ matrix A and a column vector x with n-components. In this instance, the product is the m-tuple Ax, and the $m \times n$ matrix A is a transformation from the vector space $K^{(n)}$ into the vector space $K^{(m)}$ where $A = (a_{mn})$, a_{mn} belongs to K, and $x = \{x_j : 1 \leq j \leq n\}$ is in $K^{(n)}$. Summability methods can be viewed as involving a generalization of this idea.

In summability methods the matrix A is an infinite matrix and the vector x is a sequence of elements from K. The product Ax is a sequence of terms of the form $\sum_{n=1}^{\infty} a_{mn}x_n$. Thus to consider the product Ax convergence of the series $\sum_{n=1}^{\infty} a_{mn}x_n$ must be assured. Recall that in the non-Archimedean field the series converges if and only if $\lim_{n\to\infty} a_{mn}x_n = 0$.

Some of the basic definitions in summability methods do not depend on whether K is an Archimedean field or a non-Archimedean field. Two of these definitions with illustrative examples will now be given.

Definition 3.1: The matrix $A = (a_{mn})$, $m, n = 1, 2, 3, \ldots$, transforms any given sequence $x = \{x_n\}$ into a sequence $y = \{y_m\}$ defined by the equations

$$y_m = A_m(x) = \sum_{n=1}^{\infty} a_{mn}x_n \quad (m = 1, 2, 3, \ldots) .$$

The sequence $y = \{y_m\}$ is called the A-transform of the sequence x. The sequence $y = \{y_m\}$ is sometimes written as $y = Ax$.

Notice that in this definition it is assumed that $y_m = \sum_{n=1}^{\infty} a_{mn}x_n$ exists for each m; i.e., the series $\sum_{n=1}^{\infty} a_{mn}x_n$ converges for each m.

Since $y = \{y_m\}$ is a sequence in K it is possible to decide whether $\lim_{m\to\infty} y_m$ exists. If $\lim_{m\to\infty} y_m = t$ then the number t can be associated with the sequence y and therefore with the sequence x. The following definition is made in the light of these ideas.

Definition 3.2: If A is an infinite matrix and $y = \{y_m\}$ is the A-transform of $x = \{x_n\}$ and if $\lim_{m\to\infty} y_m = t$ exists, then x is said to be A-summable to t or equivalently, A is said to sum x to t.

In the first example a matrix A is given and its transform of a sequence is determined. The second example illustrates a sequence that is not A-summable even though its A-transform exists. Also in the second example a sequence is given that is not convergent but it does have a convergent A-transform.

Example 3.3: Let $K = Q_p$ and let A be the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & p & 0 & 0 & \cdots \\ 1 & p & p^2 & 0 & \cdots \\ 1 & p & p^2 & p^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} .$$

Consider the sequence $x = (p-1, p-1, p-1, p-1, \ldots)$. The sequence x is A-summable to -1 since $y_m = \sum_{n=1}^{m} (p-1)p^{n-1}$, and $\lim_{m\to\infty} y_m = \sum_{n=1}^{\infty} (p-1)p^{n-1}$, which is the canonical representation of -1.

Example 3.4: Let A be the matrix of Example 3.3. Let x' be the sequence $(1, \frac{1}{p}, \frac{1}{p^2}, \frac{1}{p^3}, \ldots)$ and x'' be the sequence $(1, 0, 1, 0, 1, 0, \ldots)$. The A-transform of x' is the sequence $y' = (1, 2, 3, \ldots, m, \ldots)$. Now $\lim_{m\to\infty} y'_m$ does not exist. Thus x' is not A-summable. The sequence x'' is not a convergent sequence but the A-transform of x'' is the sequence

$$y'' = (1, 1, 1+p^2, 1+p^2, 1+p^2+p^4, 1+p^2+p^4, \ldots) ,$$

or in general, $y_{2m-1} = y_{2m} = (1 - p^{2m})/(1 - p^2)$. Therefore, $\lim_{m\to\infty} y_m = \frac{1}{1-p^2}$. So A sums x'' to the number $\frac{1}{1-p^2}$.

In the preceding examples since A is a lower triangular matrix (all entries were zero above the diagonal), there is no concern over the existence of y_m. In each instance the series $\sum_{n=1}^{\infty} a_{mn}x_n$ reduces to the finite sum $\sum_{n=1}^{m} a_{mn}x_n$ since $a_{mn} = 0$ for $n > m$. In the next example a matrix B and a sequence x are given for which the B-transform is not defined.

Example 3.5: Let $K = Q_p$ and let B be given by

$$B = \begin{bmatrix} 1 & p & p^2 & p^3 & p^4 & \cdots \\ 0 & 1 & p & p^2 & p^3 & \cdots \\ 0 & 0 & 1 & p & p^2 & \cdots \\ 0 & 0 & 0 & 1 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} .$$

Let $x = (1, 1/p, 1/p^2, 1/p^3, \ldots)$. The B-transform of x does not exist since $y_m = \sum_{n=1}^{\infty} \frac{1}{p^{m-1}}$, which does not converge for any m.

Conservative Matrix Transformations

In the preceding section the matrix transformations were applied to convergent sequences as well as divergent sequences. In some instances the sequence was transformed to a convergent sequence and in others it was transformed to a divergent sequence. The work to be done in this section will be concerned with finding necessary and sufficient conditions on a matrix A so that the A-transform of any convergent sequence will be a convergent sequence. The following example illustrates that such matrices exist.

Example 3.6: Let $K = Q_p$ and let A be the matrix of Example 3.3. Let $x = \{x_n\}$ be any convergent sequence. Then there is a positive number M such that $|x_n| < M$ for all n. Now the A-transform of x is the sequence $y = \{y_m\}$ where $y_m = \sum_{n=1}^{m} x_n p^{n-1}$. To see that $y = \{y_m\}$ is a convergent sequence, notice that y_m is the mth partial sum of the series $\sum_{n=1}^{\infty} x_n p^{n-1}$, which converges. Hence, $\{y_m\}$ is a convergent sequence.

The next example illustrates that there exist matrices that will transform a convergent sequence to a sequence that is divergent.

Example 3.7: Let $K = Q_p$ and let C be the matrix

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & \frac{1}{p} & 0 & 0 & 0 & \cdots \\ 1 & \frac{1}{p} & \frac{1}{p^2} & 0 & 0 & \cdots \\ 1 & \frac{1}{p} & \frac{1}{p^2} & \frac{1}{p^3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Let $x = \{x_n\}$ be a convergent sequence such that $\lim_{n\to\infty} x_n \neq 0$. Then there exists an integer N such that for $n \geq N$ $|x_n| = b$ for some constant b. Now the C-transform of x is the sequence $y = \{y_m\}$ where $y_m = \sum_{n=1}^{m} x_n p^{1-n}$, which does not converge.

The next definition is the same as the one given for summability methods over the real numbers.

Definition 3.8: A matrix A that transforms every convergent sequence into a convergent sequence is called a conservative matrix or convergence preserving matrix.

The proof of Example 3.6 indicates that the matrix of Example 3.3 is a conservative matrix. The following theorem characterizes a convergence preserving matrix. This theorem is very important and will be used quite extensively in the work that follows. Monna [9] proved this theorem in a more general setting in 1963.

Theorem 3.9: In order that $A = (a_{mn})$ be a convergence preserving matrix it is necessary and sufficient that the following conditions are satisfied:

(1) There exists a number $M > 0$ such that

$\sup_{m,n} |a_{mn}| < M$,

(2) $\lim_{m\to\infty} a_{mn} = \delta_n \quad (n = 1, 2, 3, \ldots)$, and

(3) $\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn} = \delta$.

In this case if $\lim_{n\to\infty} x_n = t$ then $\lim_{m\to\infty} y_m = t + \sum_{n=1}^{\infty} \delta_n (x_n - t)$.

Proof: To prove the conditions are sufficient let $z_n = x_n - t$. Then $\lim_{n\to\infty} z_n = \lim_{n\to\infty} (x_n - t) = 0$. So for $\varepsilon > 0$ there is a positive integer n_0 such that

(3.9.1) $$|z_n| < \frac{\varepsilon}{M} \quad \text{for} \quad n \geq n_0 .$$

Furthermore, there is a positive number Q such that

(3.9.2) $$|a_n| < Q \quad \text{for all} \quad n .$$

Now by conditions (1) and (2) it follows that

(3.9.3) $$\lim_{m\to\infty} a_{mn} = \delta_n \leq M \quad \text{for each} \quad n .$$

Also since $\lim_{m\to\infty} a_{mn} = \delta_n$, there exists a positive integer N_n such that

$$|a_{mn} - \delta_n| < \frac{\varepsilon}{Q} \quad \text{for} \quad m \geq N_n .$$

Let $N = \text{Max}(N_1, N_2, \ldots, N_{n_0})$. Then for $m \geq N$ and $1 \leq n \leq n_0$

(3.9.4) $$|a_{mn} - \delta_n| < \frac{\varepsilon}{Q} .$$

Let $S_1 = \sum_{n=1}^{n_0} (a_{mn} - \delta_n)z_n$ and $S_2 = \sum_{n=n_0+1}^{\infty} (a_{mn} - \delta_n)z_n$. Then

$$|S_1| = \left|\sum_{n=1}^{n_0} (a_{mn} - \delta_n)z_n\right| \leq \max_{1 \leq n \leq n_0} (|a_{mn} - \delta_n| \, |z_n|).$$

Thus, by (3.9.2) and (3.9.4)

(3.9.5) $$|S_1| < Q \frac{\varepsilon}{Q} = \varepsilon \quad \text{if} \quad m \geq N .$$

Now considering $|S_2|$ one obtains

$$|S_2| = \left| \sum_{n=n_0+1}^{\infty} (a_{mn} - \delta_n) z_n \right| \leq \sup_{n_0+1 \leq n} (|a_{mn} - \delta_n| \, |z_n|)$$

$$\leq \sup_{n_0+1 \leq n} (|a_{mn}| \, |z_n|, |\delta_n| \, |z_n|).$$

Thus by condition (1), (3.9.1), and (3.9.3) it follows that

$$(3.9.6) \qquad |S_2| < M \frac{\varepsilon}{M} = \varepsilon .$$

Because K is a non-Archimedean field,

$$\left| \sum_{n=1}^{\infty} (a_{mn} - \delta_n) z_n \right| = \left| \sum_{n=1}^{n_0} (a_{mn} - \delta_n) z_n + \sum_{n=n_0+1}^{\infty} (a_{mn} - \delta_n) z_n \right|$$

$$\leq \operatorname{Max} \left(\left| \sum_{n=1}^{n_0} (a_{mn} - \delta_n) z_n \right|, \left| \sum_{n=n_0+1}^{\infty} (a_{mn} - \delta_n) z_n \right| \right)$$

$$= \operatorname{Max} (|S_1|, |S_2|).$$

Thus using (3.9.5) and (3.9.6) it follows that

$$(3.9.7) \qquad \left| \sum_{n=1}^{\infty} (a_{mn} - \delta_n) z_n \right| < \varepsilon \quad \text{if} \quad m \geq N .$$

The required conclusion, that is A is a conservative matrix, would follow from the statement

$$\sum_{n=1}^{\infty} (a_{mn} - \delta_n) z_n = \sum_{n=1}^{\infty} a_{mn} z_n - \sum_{n=1}^{\infty} \delta_n z_n .$$

This statement is true if the two series on the right converge. But because $z = \{z_n\}$ belongs to (c_0) and for a fixed m,

$(a_{m1}, a_{m2}, a_{m3}, \ldots)$ belongs to $(m) = (c_0)^*$, the definition of $(c_0)^*$ implies that $\sum_{n=1}^{\infty} a_{mn} z_n$ converges. Similarly, the sequence $\{\delta_n\}$ belongs to (m) and so $\sum_{n=1}^{\infty} \delta_n z_n$ converges. Hence,

$$\sum_{n=1}^{\infty} (a_{mn} - \delta_n) z_n = \sum_{n=1}^{\infty} a_{mn} z_n - \sum_{n=1}^{\infty} \delta_n z_n \quad \text{for each } m .$$

Thus, (3.9.7) implies that

$$\left| \sum_{n=1}^{\infty} a_{mn} z_n - \sum_{n=1}^{\infty} \delta_n z_n \right| < \varepsilon \quad \text{for} \quad m \geq N .$$

That is,

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn} z_n = \sum_{n=1}^{\infty} \delta_n z_n . \tag{3.9.8}$$

But,

$$y_m = \sum_{n=1}^{\infty} a_{mn} x_n = \sum_{n=1}^{\infty} a_{mn}(z_n + t) = \sum_{n=1}^{\infty} (a_{mn} z_n + a_{mn} t) .$$

Thus, $y_m = \sum_{n=1}^{\infty} a_{mn} t + \sum_{n=1}^{\infty} a_{mn} z_n$, since these series converge. So $\lim_{m\to\infty} y_m$ exists and

$$\lim_{m\to\infty} y_m = t \lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn} + \lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn} z_n .$$

Therefore, by (3.9.8) $\lim_{m\to\infty} y_m = \delta t + \sum_{n=1}^{\infty} \delta_n z_n$. Substituting $x_n - t$ for z_n one obtains

$$\lim_{m\to\infty} y_m = \delta t + \sum_{n=1}^{\infty} \delta_n (x_n - t) .$$

Thus, $y = \{y_m\}$ belongs to (c) and A is a convergence preserving matrix if (1), (2), and (3) are true.

To prove the necessity of the conditions, it is convenient to prove (2) and (3) first, and then prove (1) using (2) and (3) along with the fact that A is a convergence preserving matrix.

To prove (2), for a fixed n consider the sequence $x = \{x_k\}$ where $x_k = 0$ if $k \neq n$ and $x_n = 1$. Then the A-transform of x is $y = \{y_m\}$ where $y_m = a_{mn}$. Therefore, since A is conservative, $\lim_{m\to\infty} y_m = \lim_{m\to\infty} a_{mn}$ exists for each n.

To prove (3) consider the sequence $x = \{x_n\}$ where $x_n = 1$ for all n. Then the A-transform of x is $y = \{y_m\}$, where $y_m = \sum_{n=1}^{\infty} a_{mn}$. Hence, $\lim_{m\to\infty} y_m = \lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn}$ exists.

In the proof of the necessity of (3) since $y_m = \sum_{n=1}^{\infty} a_{mn}$ exists for each m, $\lim_{m\to\infty} a_{mn} = 0$ for each m. Therefore for each m, there is a number M_m such that $|a_{mn}| < M_m$ for all n. Thus, the sequence $a_m = \{a_{mn} : n = 1, 2, 3, \ldots\}$ is a bounded sequence. [Note: The sequence a_m is the mth row of A]. This means that the sequence $\{a_m : m = 1, 2, 3, \ldots\}$ is a sequence in the sequence space (m). It will be shown that the sequence $\{a_m\}$ is an (m)-convergent sequence.

Let $x = \{x_n\}$ be a null sequence and hence a convergent sequence. Let $y = \{y_m\}$ be the A-transform of x. Then $y = \{y_m\}$ is a convergent sequence since A is conservative. For each $\varepsilon > 0$ and the sequence x there corresponds a number $N > 0$ such that $|y_m - y_{m+1}| < \varepsilon$ for $m \geq N$. Thus,

$$\left| \sum_{n=1}^{\infty} a_{mn}x_n - \sum_{n=1}^{\infty} a_{m+1,n}x_n \right| < \varepsilon \quad \text{for } m \geq N .$$

Since each of the series in the last inequality converges, it follows that for $m \geq N$

$$\left| \sum_{n=1}^{\infty} (a_{mn} - a_{m+1,n}) x_n \right| < \varepsilon .$$

Therefore, by the definition of (α)-convergence in Chapter II, the sequence $\{a_m\}$ is (m)-convergent. By Theorem 2.35, there is a number $M > 0$ such that $\|a_m\| < M$ for all m and n; that is, $\sup_{m,n} |a_{mn}| < M$. Hence, condition (1) is necessary. $\triangle$

The final statement in Theorem 3.9 is "If $\lim_{n\to\infty} x_n = t$ then $\lim_{m\to\infty} y_m = \lim_{m\to\infty} \left(\sum_{n=1}^{\infty} a_{mn} x_n \right) = \delta t + \sum_{n=1}^{\infty} \delta_n (x_n - t)$." From this statement it is apparent that a conservative matrix A might sum a sequence $x = \{x_n\}$ to a convergent sequence $y = \{y_m\}$ and these two sequences might have limits that are not equal. However, it is also possible for the sequence $x = \{x_n\}$ and the transformed sequence $y = \{y_m\}$ to have the same limit. With these ideas in mind the following definition is given.

Definition 3.10: A matrix A is said to be regular if it sums any convergent sequence $x = \{x_n\}$ to a sequence $y = \{y_m\}$ such that $\lim_{n\to\infty} x_n = \lim_{m\to\infty} y_m$.

The following theorem characterizes a regular matrix. Andree and Petersen [2, p. 250] proved this theorem when K is the field of p-adic numbers in 1956.

Theorem 3.11: In order that $A = (a_{mn})$ be a regular matrix it is necessary and sufficient that the following conditions are satisfied:

(1) There exists a number $M > 0$ such that $\operatorname{Sup} |a_{mn}| < M$,

(2) $\lim_{m\to\infty} a_{mn} = 0 \quad (n = 1, 2, 3, \ldots)$, and

(3) $\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn} = 1$.

Proof: In Theorem 3.9, if $\delta_n = 0$ for all n and $\delta = 1$ then

$$\lim_{m\to\infty} y_m = t + \sum_{n=1}^{\infty} 0(x_n - t) = t = \lim_{n\to\infty} x_n .$$

Thus, A is regular. Conversely, if A is regular then for any fixed n let $x = \{x_k\}$ where $x_n = 1$ and $x_k = 0$ if $k \neq n$. Now the A-transform of this sequence is the nth column of A and so $\lim_{m\to\infty} a_{mn} = \lim_{n\to\infty} x_n = 0$. Likewise, the sequence $x = \{x_n\}$, where $x_n = 1$ for all n, has the A-transform $y = \{y_m\}$ where $y_m = \sum_{n=1}^{\infty} a_{mn}$. Since A is regular and $\lim_{n\to\infty} x_n = 1$ it follows that $\lim_{m\to\infty} y_m = 1$; that is, $\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn} = 1$. Δ

Matrix Transformations in Q_p

Using Theorem 3.9 it is now possible to determine whether or not a matrix is a convergence preserving matrix. Furthermore, using Theorem 3.11 one can decide if a conservative matrix is a regular matrix. In each of these instances the sequences involved are convergent sequences and the idea is to determine conditions on a matrix so that the transform of a convergent sequence exists and is a convergent sequence. A somewhat different problem is the following: "For a fixed matrix, can one find conditions on a sequence $x = \{x_n\}$ so that the transform of that sequence is a convergent sequence?" Two specific matrices will be introduced in this section and the letter used to designate each matrix will refer only to that matrix throughout the section. The first matrix to be considered is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & p & 0 & 0 & 0 & \cdots \\ 1 & p & p^2 & 0 & 0 & \cdots \\ 1 & p & p^2 & p^3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} .$$

That is, $A = (a_{mn})$ where

$$a_{mn} = \begin{cases} 0 & \text{if } n > m \\ p^{n-1} & \text{if } n \leq m . \end{cases}$$

This matrix transforms all bounded sequences into convergent sequences.

<u>Theorem 3.12</u>: The sequence $x = \{x_n\}$ is A-summable if and only if $\lim\limits_{n\to\infty} \frac{|x_n|}{p^{n-1}} = 0$.

<u>Proof</u>: Since A is a lower triangular matrix, the A-transform $y = \{y_m\}$ of x exists and is the sequence

$$(x_1, x_1 + x_2 p, x_1 + x_2 p + x_3 p^2, \ldots, \sum_{n=1}^{m} x_n p^{n-1}, \ldots) .$$

Note that $y_m = \sum_{n=1}^{m} x_n p^{n-1}$ is the mth partial sum of the p-series $\sum_{n=1}^{\infty} x_n p^{n-1}$. Thus, $y = \{y_m\}$ is a convergent sequence if and only if the series $\sum_{n=1}^{\infty} x_n p^{n-1}$ converges. This series converges if and only if $\lim_{n\to\infty} |x_n p^{n-1}| = 0$. That is, if and only if $\lim_{n\to\infty} \frac{|x_n|}{p^{n-1}} = 0$. Thus, the theorem is proved. $\triangle$

<u>Corollary 3.13</u>: If $x = \{x_n\}$ is a bounded sequence then the A-transform of x exists.

<u>Proof</u>: Since $x = \{x_n\}$ is bounded there is a number $M > 0$ such that $|x_n| < M$ for all n. Thus, $\lim_{n\to\infty} \frac{|x_n|}{p^{n-1}} = 0$. Hence, the A-transform of x exists. $\triangle$

The next example indicates that one can have an unbounded sequence with an A-transform that is a convergent sequence.

<u>Example 3.14</u>: Let

$$x = \left(\frac{1}{p}, \frac{1}{p}, \frac{1}{p}, \frac{1}{p^2}, \frac{1}{p^2}, \frac{1}{p^2}, \frac{1}{p^2}, \frac{1}{p^2}, \ldots, \frac{1}{p^{[\sqrt{n}]}}, \ldots\right)$$

where $[\sqrt{n}]$ denotes the greatest integer less than or equal to $\sqrt{n}$. The A-transform of $x = \{x_n\}$ is the sequence $y = \{y_m\}$ where $y_m = \sum_{n=1}^{m} \frac{p^{n-1}}{p^{[\sqrt{n}]}}$. To show that $\{y_m\}$ is a convergent sequence observe that y_m is the mth partial sum of the series $\sum_{n=1}^{\infty} \frac{p^{n-1}}{p^{[\sqrt{n}]}}$, and that

$$0 \leq \lim_{n\to\infty} \left| \frac{p^{n-1}}{p^{[\sqrt{n}]}} \right| \leq \lim_{n\to\infty} \frac{p^{[\sqrt{n}]}}{p^{n-1}} \leq \lim_{n\to\infty} \frac{1}{p^{n-1-\sqrt{n}}} = 0 .$$

Thus, $\lim_{m\to\infty} y_m = \sum_{n=1}^{\infty} \frac{p^{n-1}}{p^{[\sqrt{n}]}}$. However, $|x_n| = p^{[\sqrt{n}]}$. So x is unbounded.

To complete the discussion of the matrix A it is necessary to discuss the A-transform of a series $\sum_{n=0}^{\infty} a_n$. Since the series $\sum_{n=0}^{\infty} a_n$ is the sequence $\{s_n\}$ of partial sums, where $s_n = \sum_{i=0}^{n} a_i$, the series $\sum_{n=0}^{\infty} a_n$ is R-summable if the sequence of partial sums $\{s_n\}$ is R-summable, where R is any infinite matrix. The following theorem gives a condition on the terms of the series so that the sequence of partial sums is A-summable.

<u>Theorem 3.15</u>: $\sum_{n=0}^{\infty} a_n$ is A-summable if there is a number $M > 0$ such that $|a_n| < M$ for all n.

<u>Proof</u>: Suppose there is a number M such that $|a_n| < M$ for all n. Then $|s_n| \leq \text{Max}(|a_0|, |a_1|, \ldots, |a_n|) < M$. Thus, $\{s_n\}$ is a bounded sequence. Therefore, by Corollary 3.13, $\sum_{n=0}^{\infty} a_n$ is A-summable. $\triangle$

The next theorem offers a slightly weaker converse result in that it shows that if $\sum_{n=0}^{\infty} a_n$ is A-summable, then $|a_n| < Mp^n$ for some number $M > 0$.

<u>Theorem 3.16</u>: If $\sum_{n=0}^{\infty} a_n$ is A-summable then there is a number $M > 0$ such that $|a_n| < Mp^n$.

<u>Proof</u>: Let $\{s_n\}$ be the sequence of partial sums of the series $\sum_{n=0}^{\infty} a_n$. Since the series is A-summable the A-transform of $\{s_n\}$ is a convergent sequence. That is, the sequence $\{y_m\}$ is convergent where

$$y_m = s_0 + ps_1 + p^2 s_2 + \dots + p^m s_m .$$

Therefore, there exists a number $M > 0$ such that $|y_m| < M$ for all m. Now by the non-Archimedean property, $|y_m - y_{m-1}| \leq \text{Max}(|y_m|, |y_{m-1}|) < M$. Thus, $|p^m s_m| < M$ for all m. Hence, $|s_m| < Mp^m$. Therefore, $|a_m| \leq \text{Max}(|s_m|, |s_{m-1}|) \leq Mp^m$. △

The next matrix to be considered is one that has the effect of averaging consecutive terms of a sequence. This matrix is

$$B = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} .$$

It is easy to show that B is a regular matrix. The following theorem gives necessary and sufficient conditions on a sequence $\{x_n\}$ so that the B-transform will be a convergent sequence.

Theorem 3.17: The B-transform of $x = \{x_n\}$ is convergent if and only if $\lim_{m\to\infty} x_m - x_{m-2} = 0$.

Proof: Suppose the B-transform of x is a convergent sequence and let $y = \{y_m\}$ be the B-transform of x. Then $\{y_m\}$ is a Cauchy sequence. That is, for $\varepsilon > 0$ there is an integer N such that

$$|y_m - y_{m-1}| < \varepsilon \quad \text{for} \quad m \geq N .$$

Therefore,

$$\left| \frac{x_m + x_{m-1}}{2} - \frac{x_{m-1} + x_{m-2}}{2} \right| < \varepsilon \quad \text{for} \quad m \geq N .$$

Thus,

$$\left| \frac{x_m - x_{m-2}}{2} \right| < \varepsilon \quad \text{for} \quad m \geq N .$$

Therefore, $\lim_{m\to\infty} x_m - x_{m-2} = 0$. Conversely, suppose $\lim_{m\to\infty} x_m - x_{m-2} = 0$. Then for $\varepsilon > 0$ there is an integer N such that $|x_m - x_{m-2}| < \varepsilon$ for $m \geq N$. Hence,

$$\left| \frac{x_m}{2} - \frac{x_{m-2}}{2} \right| < \varepsilon \quad \text{for} \quad m \geq N .$$

Adding and subtracting $\frac{x_{m-1}}{2}$ one obtains,

$$\left| \frac{x_m + x_{m-1}}{2} - \frac{x_{m-1} + x_{m-2}}{2} \right| < \varepsilon \quad \text{for} \quad m \geq N .$$

Thus, $|y_m - y_{m-1}| < \varepsilon$ for $m \geq N$. Therefore, $\{y_m\}$ is a Cauchy sequence and since Q_p is a complete space the B-transform of x is a convergent sequence. $\triangle$

Using the result of the preceding theorem it is possible to obtain a divergent sequence that B will transform to a convergent sequence. To accomplish this all one needs to do is to make alternate terms of the sequence equal. That is, let $x = (1, 0, 1, 0, \ldots)$. This example is one that is rather trivial. To obtain an example that is not so trivial consider two sequences $x = \{x_n\}$ and $y = \{y_n\}$ that converge to different limits. Then define another sequence $z = \{z_m\}$ by letting $z_{2m} = x_m$ and $z_{2m+1} = y_m$. With this definition for z, it follows that $\lim_{m\to\infty} z_m - z_{m-2} = 0$, since $x = \{x_n\}$ and $y = \{y_n\}$ are both Cauchy sequences and $z_m - z_{m-2}$ is either the difference of two consecutive terms in the sequence x or in the sequence y.

Using the result of Theorem 3.17 it is possible to prove a necessary and sufficient condition on the terms of the series $\sum_{n=0}^{\infty} a_n$ so that the series will be B-summable. This condition is that $\lim_{n\to\infty} a_n + a_{n-1} = 0$.

<u>Theorem 3.18</u>: The series $\sum_{n=0}^{\infty} a_n$ is B-summable if and only if $\lim_{n\to\infty} a_n + a_{n-1} = 0$.

<u>Proof</u>: The series $\sum_{n=0}^{\infty} a_n$ is B-summable if and only if the B-transform of the sequence of partial sums $\{s_n\}$ is a convergent sequence. But by Theorem 3.17 the B-transform of $\{s_n\}$ is a convergent sequence if and only if $\lim_{n\to\infty} s_n - s_{n-2} = 0$. Thus the B-transform of $\{s_n\}$ is a convergent sequence if and only if $\lim_{n\to\infty} a_n + a_{n-1} = 0$. △

In contrast to the result of Theorem 3.16 (If $\sum_{n=0}^{\infty} a_n$ is A-summable then there is a number $M > 0$ such that $|a_n| < Mp^n$) it is possible to show that if $\sum_{n=1}^{\infty} a_n$ is B-summable then the terms of $\sum_{n=0}^{\infty} a_n$ are bounded.

<u>Theorem 3.19</u>: If $\sum_{n=0}^{\infty} a_n$ is B-summable then there is a number $M > 0$ such that $|a_n| < M$ for all n.

Proof: Suppose $\sum_{n=0}^{\infty} a_n$ is B-summable. Then by Theorem 3.18, $\lim_{n\to\infty} a_n + a_{n-1} = 0$. That is, the sequence $\{b_n\}$, where $b_0 = a_0$ and $b_n = a_n + a_{n-1}$ if $n \geq 1$, is a null sequence. Thus $\{b_n\}$ is a bounded sequence. So there is a number $M > 0$ such that $|b_n| < M$ for all n. But,

$$(-1)^n a_n = a_0 - (a_1 + a_0) + (a_2 + a_1) - \ldots + (-1)^n (a_n + a_{n-1}) .$$

That is,

$$(-1)^n a_n = b_0 - b_1 + b_2 - \ldots + (-1)^n b_n .$$

Thus,

$$|a_n| \leq \mathrm{Max}(|b_0|, |b_1|, |b_2|, \ldots, |b_n|) < M , \qquad \triangle$$

The following example illustrates that the converse of Theorem 3.19 is not true. That is, if $\sum_{n=0}^{\infty} a_n$ is a series with the sequence $\{a_n\}$ bounded then it does not follow that the series is B-summable.

Example 3.20: The series $\sum_{n=0}^{\infty} a_n$, where $a_n = 1$ for all n is not B-summable. To show this consider the sequence $(1, 2, 3, 4, \ldots, n, \ldots)$ of partial sums. The B-transform of this sequence is $(\frac{1}{2}, \frac{3}{2}, \ldots, \frac{2n-1}{2}, \ldots)$. Now from this sequence one can choose a subsequence with terms of p-value one $(p \neq 2)$, and another subsequence that is a null sequence. If $p \neq 2$ choose the subsequence $(\frac{p}{2}, \frac{p^2}{2}, \ldots, \frac{p^n}{2}, \ldots)$. This subsequence is a null sequence. Also choose the subsequence $(\frac{q}{2}, \frac{q^2}{2}, \ldots, \frac{q^n}{2}, \ldots)$ where q is a prime and $q \neq p$. Then $|\frac{q^n}{2}| = 1$ for all n. Therefore, the B-transform of $\sum_{n=0}^{\infty} a_n$ does not exist when $p \neq 2$. If $p = 2$ then choose the sequences $(\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \ldots, \frac{2^n - 1}{2}, \ldots)$ and $(\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots, \frac{2^n + 1}{2}, \ldots)$. The first of these sequences converges to $-\frac{1}{2}$ while the second converges to $\frac{1}{2}$. Thus, the sequence $(\frac{1}{2}, \frac{3}{2}, \ldots, \frac{2n-1}{2}, \ldots)$ does not converge. In either case, $p = 2$ or $p \neq 2$, the series $\sum_{n=0}^{\infty} a_n$ is not B-summable.

This section will be concluded by considering matrix transformations on a power series. Although from calculus the student is familiar with a power series, the definition is repeated here.

<u>Definition 3.21</u>: If $\{a_n\}$ is a sequence the series $\sum_{n=0}^{\infty} a_n x^n$ is called a power series in x.

For particular values of x the series $\sum_{n=1}^{\infty} a_n x^n$ may converge or may diverge. However, those values of x for which the series converge can be identified. Palmer [10, p. 62] proves the following theorem which gives a means of determining the values of x for which the series $\sum_{n=0}^{\infty} a_n x^n$ converges.

<u>Theorem 3.22</u>: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series in a complete field K and let $a = \limsup \sqrt[n]{|a_n|}$. If $a \neq 0$ let $\rho = \frac{1}{a}$; if $a = +\infty$ let $\rho = 0$; and if $a = 0$ let $\rho = +\infty$. Then the series $\sum_{n=0}^{\infty} a_n x^n$

(a) converges for $|x| < \rho$,

(b) diverges for $|x| > \rho$,

(c) may converge or diverge for $|x| = \rho$.

The value ρ is called the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ and the set of all x such that $|x| < \rho$ is called the domain of convergence of $\sum_{n=0}^{\infty} a_n x^n$. The set of all x such that $|x| = \rho$ is called the circle of convergence of $\sum_{n=0}^{\infty} a_n x^n$.

The circle of convergence of a power series in a non-Archimedean field presents an interesting contrast with the circle of convergence in an Archimedean field. In the Archimedean case a power series may converge for all values of x on the circle of convergence, may converge for no values of x on the circle of convergence, or may converge for some but not all values of x on the circle. In the non-Archimedean case only two possibilities exist; either a power series converges for no values of x on the circle of convergence or it converges for all x on the circle of convergence. That is, if $\sum_{n=0}^{\infty} a_n x^n$ converges for one value of x such that $|x| = \rho$ then the series converges for all x such that $|x| = \rho$. This interesting behavior is due to the fact that a series $\sum_{n=0}^{\infty} b_n$ converges in a non-Archimedean field if and only if $\lim_{n\to\infty} b_n = 0$.

Theorem 3.23: Let $\sum_{n=0}^{\infty} a_n x^n$ have a radius of convergence of ρ. Then either $\sum_{n=0}^{\infty} a_n x^n$ converges for all x such that $|x| = \rho$ or diverges for all x such that $|x| = \rho$.

Proof: Let x_0 be such that $|x_0| = \rho$ and $\sum_{n=0}^{\infty} a_n x_0^n$ converges. Then $\lim_{n\to\infty} |a_n x_0^n| = \lim_{n\to\infty} |a_n| \rho^n = 0$. Then for any x such that $|x| = \rho$ it is true that $\lim_{n\to\infty} |a_n x^n| = 0$. Thus, $\sum_{n=0}^{\infty} a_n x^n$ converges for all x such that $|x| = \rho$. $\triangle$

The familiar geometric series $\sum_{n=0}^{\infty} x^n$ offers an example of a series that diverges at every point on its circle of convergence. The following example is of a series that converges at every point on the circle of convergence.

Example 3.24: Consider the power series $\sum_{n=0}^{\infty} p^{[\sqrt{n}]} x^n$. Since

$$\frac{1}{p^{\sqrt{n}}} \leq \frac{1}{p^{[\sqrt{n}]}} \leq \frac{1}{p^{\sqrt{n}-1}}$$

it follows that

$$\frac{1}{p^{\sqrt{n}/n}} \leq \frac{1}{p^{[\sqrt{n}]/n}} \leq \frac{1}{p^{(\sqrt{n}-1)/n}} .$$

Hence,

$$\limsup \sqrt[n]{|p^{[\sqrt{n}]}|} = 1 .$$

Thus $\sum_{n=0}^{\infty} p^{[\sqrt{n}]} x^n$ has radius of convergence equal to 1. Since

$$\lim_{n\to\infty} |p^{[\sqrt{n}]}| = \lim_{n\to\infty} \frac{1}{p^{[\sqrt{n}]}} = 0 .$$

$\sum_{n=0}^{\infty} p^{[\sqrt{n}]} x^n$ converges for all x on its circle of convergence.

Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be power series with domains of convergence D_a and D_b, respectively. Then in a non-Archimedean field the intersection of D_a and D_b is either empty or one of the domains is contained in the other. [See Snook, p. 80]. This property makes the continuation of a power series beyond the circle of convergence by usual methods of calculus impossible. Thus it becomes necessary to resort to other techniques. Continuation of a power series by matrix methods is one of these other techniques. This topic will not be pursued in detail in this study although it is related to one under consideration in this paper in that it relies on having a matrix that will sum a power series outside its circle of convergence.

The remainder of this section will be devoted to investigating the matrices A and B to determine if either of these matrices will sum a power series outside the domain of convergence. The next theorem shows that A may sum a power series outside its domain of convergence.

Theorem 3.25: A may sum a power series outside its domain of convergence.

Proof: Suppose $\sum_{n=0}^{\infty} a_n x_0^n$ is A-summable. Then by Theorem 3.16 there is a number $M > 0$ such that $|a_n x_0^n| < Mp^n$. Thus, by taking nth roots of both sides of this inequality it follows that

$$|a_n|^{1/n} < \frac{M^{1/n} p}{|x_0|} .$$

Thus,

$$\frac{1}{\rho} = \lim\sup |a_n|^{1/n} \leq \lim_{n\to\infty} \frac{M^{1/n} p}{|x_0|} = \frac{p}{|x_0|} .$$

Therefore, $|x_0| \leq p\rho$. Now since $p\rho > \rho$, it follows that A may sum $\sum_{n=0}^{\infty} a_n x^n$ outside its domain of convergence. Δ

It can be shown that the matrix B will not sum a power series outside its domain of convergence. Comparing this to the matrix A it is noted that A will sum outside the domain of convergence of a power series while B will not.

<u>Theorem 3.26</u>: A power series $\sum_{n=0}^{\infty} a_n x^n$ is not B-summable outside its domain of convergence.

<u>Proof</u>: Let $\sum_{n=0}^{\infty} a_n x^n$ be B-summable at $x = x_0$. Then by Theorem 3.19, there is a number $M > 0$ such that $|a_n x_0^n| < M$. Hence,

$$|a_n|^{1/n} < \frac{M^{1/n}}{|x_0|} .$$

Thus,

$$\frac{1}{\rho} = \lim\sup |a_n|^{1/n} \leq \lim \frac{M^{1/n}}{|x_0|} = \frac{1}{|x_0|} .$$

Hence $|x_0| \leq \rho$. △

The next example indicates that there is a power series that B will sum at one point on the circle of convergence but not at another point.

<u>Example 3.27:</u> Consider the geometric series $\sum_{n=0}^{\infty} x^n$, $|x| < 1$. For $x = -1$ the series becomes $\sum_{n=0}^{\infty} (-1)^n$. The sequence of partial sums of this series is $(1, 0, 1, 0, \ldots)$. The B-transform of this sequence is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots)$. Hence $\sum_{n=0}^{\infty} x^n$ is B-summable to $\frac{1}{2}$ when $x = -1$. However, for $x = 1$ the series is not B-summable. [See Example 3.20].

CHAPTER IV

ALGEBRAIC STRUCTURE OF SETS OF MATRIX TRANSFORMATIONS

In Chapter III matrix transformations on the space of convergent sequences are considered. Necessary and sufficient conditions on a matrix A are obtained so that A transforms a convergent sequence into a convergent sequence. In Chapter II the concept of (α,β)-convergence of a sequence $\{x^{(n)}\}$ is introduced. Necessary and sufficient conditions are obtained so that the sequence $\{x^{(n)}\}$ is $(\alpha$-$\beta)$-convergent. The purpose of the first section of this chapter is to show a relationship between matrix transformations on a sequence space and the (α,β)-convergence of a sequence of points. The remainder of the chapter is devoted to the algebraic structure of certain sets of infinite matrices.

Matrix Transformations and (α,β)-Convergence

Let α and β be sequence spaces. Further suppose that $A = (a_{mn})$ is an infinite matrix such that for any sequence x in α the sequence Ax belongs to the sequence space β. Then the matrix A is said to transform sequences of α into sequences of β. That is, A maps the sequence space α into the sequence space β. Theorem 3.12 of Chapter III gives an instance where α is the space of bounded sequences and β is the space of convergent sequences in Q_p.

A connection between (α,β)-convergence and matrix transformations can be demonstrated by using the definitions of (α,β)-convergence and the dual space of α. To keep the matrix notation $A = (a_{mn})$, where m represents the row and n represents the column to which a_{mn} belongs, it will be necessary to switch the roles of m and n in the definition of (α,β)-convergence. This definition is restated using this change.

A sequence of points $\{x^{(m)}\}$ of α is said to be (α,β)-convergent, $(\varphi \subseteq \beta \subseteq \alpha^*)$, if for every $\varepsilon > 0$ and every u of β there is a number $N = N(\varepsilon, u) > 0$ such that

$$(4.0.1) \qquad |u(x^{(m)} - x^{(m+1)})| = \left| \sum_{n=1}^{\infty} u_n(x_n^{(m)} - x_n^{(m+1)}) \right| < \varepsilon$$

for $m \geq N$. Further, remember that a sequence $u = \{u_m\}$ belongs to α^* if and only if the series $\sum_{n=1}^{\infty} u_n x_n$ converges for all $x = \{x_n\}$ in α. Thus, if $\{x^{(m)}\}$ is a sequence in α and $\varphi \subseteq \beta \subseteq \alpha^*$ then for every u in β and for each m the series $\sum_{n=1}^{\infty} u_n x_n^{(m)}$ converges, since u is also in α^*. Therefore if $y_m = \sum_{n=1}^{\infty} u_n x_n^{(m)}$ then the sequence $y = \{y_m\}$ is defined for each m.

Let X be the matrix whose rows are the sequences $x^{(n)}$. That is,

$$(4.0.2)\qquad X = \begin{bmatrix} x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & \cdots & x_n^{(1)} & \cdots \\ x_1^{(2)} & x_2^{(2)} & x_3^{(2)} & \cdots & x_n^{(2)} & \cdots \\ x_1^{(3)} & x_2^{(3)} & x_3^{(3)} & \cdots & x_n^{(3)} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ x_1^{(m)} & x_2^{(m)} & x_3^{(m)} & \cdots & x_n^{(m)} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \end{bmatrix}$$

The following theorem relates the matrix X and the sequence spaces β and (c).

Theorem 4.1: Let X be a matrix whose rows belong to α. Then X maps β into (c) if and only if the rows of X form an (α,β)-convergent sequence.

Proof: Let $x^{(m)}$ be an (α,β)-convergent sequence and let X be the matrix of (4.0.2). Also, let $u = \{u_n\}$ belong to β and let $y = \{y_m\}$ be the X-transform of u; i.e., $y_m = \sum_{n=1}^{\infty} u_n x_n^{(m)}$. Now let $\varepsilon > 0$ be given. Since $x^{(m)}$ is (α,β)-convergent, by (4.0.1) there is a number N such that

$$(4.1.1)\qquad |u(x^{(m)} - x^{(m+1)})| = \left| \sum_{n=1}^{\infty} u_n(x_n^{(m)} - x_n^{(m+1)}) \right| < \varepsilon$$

for $m \geq N$. Thus by definition of $\{y_m\}$ it follows that

$$(4.1.2) \qquad |y_m - y_{m+1}| = \left| \sum_{n=1}^{\infty} u_n(x_n^{(m)} - x_n^{(m+1)}) \right| < \varepsilon$$

if $m \geq N$. Hence, $\{y_m\}$ is a Cauchy sequence and since K is complete, $\{y_m\}$ is a convergent sequence.

Conversely, let u belong to β and let $z = \{z_m\}$ be the X-transform of u. Then $Xu = z$ is a convergent sequence since X maps β into (c). Thus, for $\varepsilon > 0$, there exists an integer $N = N(\varepsilon, u)$ such that

$$|z_m - z_{m+1}| < \varepsilon \quad \text{for} \quad m \geq N.$$

Since $z_m = \sum_{n=1}^{\infty} x_n^{(m)} u_n$, it follows that

$$|u(x^{(m)} - x^{(m+1)})| = \left| \sum_{n=1}^{\infty} u_n(x_n^{(m)} - x_n^{(m+1)}) \right| < \varepsilon$$

for $m \geq N$. Therefore, $\{x^{(m)}\}$ is an (α, β)-convergent sequence. Δ

Sets of Matrix Transformations

Definition 4.2: The matrix A is said to belong to the sequence space α if for every x of α the sequence Ax is also in α. The set of all matrices that belong to a sequence space α is denoted by $\Sigma(\alpha)$.

Theorem 3.9 gives necessary and sufficient conditions for a matrix A to belong to $\Sigma(c)$. Similar results can be proved for the sequence spaces φ, (c_0), (m), and ω. A theorem that will be useful in deriving these results will now be proved. This theorem gives necessary conditions for a matrix A to belong to $\Sigma(\alpha)$. These conditions are that the columns of A belong to α and the rows of A belong to α^*. Notice the relationship between the sequence space α, its dual space α^*, and the matrix A belonging to $\Sigma(\alpha)$.

Theorem 4.3: If α is a sequence space $(\varphi \subseteq \alpha)$ and the matrix A belongs to $\Sigma(\alpha)$ then

(1) for each n, the sequence $(a_{1n}, a_{2n}, \ldots, a_{mn}, \ldots)$ belongs to α, that is, the columns of A belong to α;

(2) for each m, the sequence $(a_{m1}, a_{m2}, \ldots, a_{mn}, \ldots)$ belongs to α^*, that is, the rows of A belong to α^*.

Proof: Let n be a positive integer and let $x = \{x_j\}$ be the sequence where $x_n = 1$ and $x_j = 0$ if $j \neq n$. Then x belongs to α since $\varphi \subseteq \alpha$ and x is in φ. The A-transform of x must be in α since A is in $\Sigma(\alpha)$. But the A-transform of $x = \{x_j\}$ is the sequence $(a_{1n}, a_{2n}, a_{3n}, \ldots, a_{mn}, \ldots)$. Thus (1) has been established. To prove (2) note that the series $\sum_{n=1}^{\infty} a_{mn} x_n$ must converge for each $x = \{x_n\}$ in α. By definition of α^* the sequence $(a_{m1}, a_{m2}, \ldots, a_{mn}, \ldots)$ must belong to α^*. △

Corollary 4.4: If the infinite matrix A belongs to $\Sigma(m)$ then

(1) $\lim_{n\to\infty} a_{mn} = 0$ for $m = 1, 2, 3, \ldots$, and

(2) for each n there exists a positive number M_n such that $\operatorname{Sup}_m |a_{mn}| < M_n$.

Corollary 4.5: If the infinite matrix A belongs to $\Sigma(c_0)$ then

(1) $\lim_{m\to\infty} a_{mn} = 0$ for $n = 1, 2, 3, \ldots$, and

(2) for each m there exists a positive number M_m such that $\sup_n |a_{mn}| < M_m$.

The two previous corollaries indicate necessary conditions on a matrix for the matrix to belong to $\Sigma(m)$ or $\Sigma(c_0)$. The conditions are not sufficient conditions however, as the following two examples indicate.

Example 4.6: Let the matrix B be

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & \frac{1}{p} & 0 & 0 & \cdots \\ 1 & p & \frac{1}{p^2} & 0 & \cdots \\ 1 & p & p^2 & \frac{1}{p^3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Now, $\lim_{n\to\infty} b_{mn} = 0$ for each m and $\sup_m |b_{mn}| = p^{n-1}$ for $n = 1, 2, \ldots$. Thus, B satisfies both conditions of Corollary 4.4. To show that B is not an element of $\Sigma(m)$, consider the sequence $x = (1, 1, 1, \ldots)$. The B-transform of x is the sequence

$$y = \left(1,\ 1 + \frac{1}{p},\ 1 + p + \frac{1}{p^2}, \ldots, \sum_{n=1}^{m-1} p^{n-1} + \frac{1}{p^{m-1}}, \ldots\right).$$

Now y does not belong to (m) since for any positive number M there is an integer m_0 such that $|y_{m_0}| = p^{m_0 - 1} > M$.

<u>Example 4.7</u>: Let the matrix C be

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & \frac{1}{p} & p & p & p & \cdots \\ 0 & 0 & \frac{1}{p^2} & p^2 & p^2 & \cdots \\ 0 & 0 & 0 & \frac{1}{p^3} & p^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} .$$

Now, $\lim_{m\to\infty} c_{mn} = 0$ for each n and $\sup_n |c_{mn}| = p^{m-1}$ for $m = 1, 2, 3, \ldots$. Thus, C satisfies the conditions of Corollary 4.5. However, C does not belong to $\Sigma(c_0)$. To show this, let x be the null sequence $(1, p, p^2, p^3, \ldots)$. The C-transform of x is the sequence

$$y = \left(1 + \sum_{n=1}^{\infty} p^n, 1 + \sum_{n=3}^{\infty} p^n, 1 + \sum_{n=5}^{\infty} p^n, \ldots, 1 + \sum_{n=2m-1}^{\infty} p^n, \ldots\right) .$$

Thus, $\lim_{m\to\infty} y_m = 1$ and y is not in (c_0).

The question that now arises is, "Do conditions exist on a matrix similar to the conditions of Corollaries 4.4 and 4.5 that will guarantee that the matrix belongs to either $\Sigma(c_0)$ or $\Sigma(m)$?" An answer to this question can be found by requiring that the set of numbers $\{M_1, M_2, \ldots\}$ is bounded above by some real number. This requirement is equivalent to requiring that the entries of the matrix be uniformly bounded. In Examples 4.6 and 4.7, notice that the numbers M_n or M_m were not themselves bounded sequences but increased with n or m.

Theorem 4.8: The infinite matrix $A = (a_{mn})$ belongs to $\Sigma(c_0)$ if and only if

(1) $\lim_{m\to\infty} a_{mn} = 0$ for $n = 1, 2, 3, \ldots$; that is, the columns of A belong to (c_0), and

(2) there is a real number $M > 0$ such that $\sup_{m,n} |a_{mn}| < M$.

Proof: Suppose (1) and (2) are satisfied and that $x = \{x_n\}$ belongs to (c_0). Let $y = \{y_m\}$ be the A-transform of x. To prove that y belongs to (c_0) it must be shown that $\lim_{m\to\infty} y_m = 0$. Let $\varepsilon > 0$ be given. Then since x is in (c_0) there is an integer n_0 such that

(4.8.1) $$|x_n| < \frac{\varepsilon}{M} \quad \text{if } n \geq n_0 .$$

Furthermore there is a number $Q > 0$ such that

(4.8.2) $$|x_n| < Q \quad \text{for all } n .$$

By condition (1), for each n there is an integer N_n such that $|a_{mn}| < \frac{\varepsilon}{Q}$ for $m \geq N_n$. Thus there are integers $N_1, N_2, \ldots, N_{n_0}$ such that

(4.8.3) $$\begin{cases} |a_{m1}| < \frac{\varepsilon}{Q} & \text{if } m \geq N_1 \\ |a_{m2}| < \frac{\varepsilon}{Q} & \text{if } m \geq N_2 \\ \quad \cdots & \\ |a_{mn_0}| < \frac{\varepsilon}{Q} & \text{if } m \geq N_{n_0} . \end{cases}$$

Let $N = \text{Max}(N_1, N_2, \ldots, N_{n_0})$. Then

$$(4.8.4) \qquad |a_{mn}| < \frac{\varepsilon}{Q} \quad \text{if} \quad m \geq N \quad \text{and} \quad 1 \leq n \leq n_0 .$$

Now let $S_1 = \sum_{n=1}^{n_0} a_{mn}x_n$ and $S_2 = \sum_{n=n_0+1}^{\infty} a_{mn}x_n$. Then by (4.8.2) and (4.8.4), if $m \geq N$,

$$(4.8.5) \qquad |S_1| \leq \text{Max}(|a_{m1}x_1|, |a_{m2}x_2|, \ldots, |a_{mn_0}x_{n_0}|)$$

$$< \frac{\varepsilon}{Q} Q = \varepsilon .$$

Also by (4.8.1) and condition (2)

$$(4.8.6) \qquad |S_2| \leq \sup_{n_0+1 \leq n} (|a_{mn}|\, |x_n|) < M \frac{\varepsilon}{M} = \varepsilon .$$

Thus, by (4.8.5), (4.8.6), and the non-Archimedean property

$$|y_m| \leq \left| \sum_{n=1}^{n_0} a_{mn}x_n + \sum_{n=n_0+1}^{\infty} a_{mn}x_n \right|$$

$$\leq \text{Max}(|S_1|, |S_2|) < \varepsilon \quad \text{if} \quad m \geq N .$$

Therefore, $\lim_{m\to\infty} y_m = 0$ and y is in (c_0). Hence, A is in $\Sigma(c_0)$.

To prove the converse, suppose that $A = (a_{mn})$ belongs to $\Sigma(c_0)$. By Corollary 4.5

(a) $\lim_{m\to\infty} a_{mn} = 0$ for $n = 1, 2, 3, \ldots$, and

(b) for each m there exists a positive number M_m such that $\sup_n |a_{mn}| < M_m$.

Statement (a) is the same as statement (1) in the theorem so (1) is a necessary condition.. Statement (b) means that the rows of A considered as sequences are bounded. That is, if $a_m = \{a_{mn} : n = 1, 2, 3, \ldots\}$ then $\{a_m\}$ is a sequence in (m). It will be shown that the sequence $\{a_m\}$ is an (m)-convergent sequence.

Let $x = \{x_n\}$ be in (c_0), let $\varepsilon > 0$ be given, and let $y = \{y_m\}$ be the A-transform of x. Then since A belongs to $\Sigma(c_0)$, y is in (c_0). Thus, there exists an integer N such that

$$|y_m - y_{m+1}| < \varepsilon \quad \text{for} \quad m \geq N .$$

Hence,

$$\left| \sum_{n=1}^{\infty} a_{mn} x_n - \sum_{n=1}^{\infty} a_{m+1,n} x_n \right| < \varepsilon \quad \text{for} \quad m \geq N .$$

Since each of the series converge it follows for $m \geq N$ that

$$\left| \sum_{n=1}^{\infty} (a_{mn} - a_{m+1,n}) x_n \right| < \varepsilon .$$

Therefore by definition of (m)-convergence it follows that $\{a_m\}$ is an (m)-convergent sequence. Thus, by Theorem 2.35, there is a positive number M such that $|a_{mn}| < M$ for all m and n. Hence, condition (2) is necessary. $\triangle$

Necessary and sufficient conditions for a matrix A to belong to $\Sigma(m)$ exist. These conditions are given in the next theorem. The proof of this theorem is omitted because is quite involved and tedious.

Theorem 4.9: The infinite matrix $A = (a_{mn})$ belongs to $\Sigma(m)$ if and only if

(1) $\lim_{n\to\infty} a_{mn} = 0$ for $m = 1, 2, 3, \ldots$; that is, the rows of A belong to (c_0), and

(2) there is a number $M > 0$ such that $\sup_{m,n} |a_{mn}| < M$.

Turning attention now to the sequence space θ, Theorem 4.3 leads easily to necessary conditions for a matrix to belong to $\Sigma(\theta)$.

Theorem 4.10: If the infinite matrix $A = (a_{mn})$ belongs to $\Sigma(\theta)$ then

(1) $\lim_{n\to\infty} a_{mn} = 0 \quad m = 1, 2, 3, \ldots$, and

(2) for each n there exists an integer N_n such that $a_{m+1,n} = a_{mn}$ for $m \geq N_n$.

Proof: Since $\theta^* = (c_0)$ by Theorem 2.26 statement (1) follows from Theorem 4.3. Statement (2) also follows from Theorem 4.3 since the columns of any matrix in $\Sigma(\theta)$ must belong to θ. Δ

The following example will illustrate that conditions (1) and (2) of Theorem 4.10 are not sufficient conditions for a matrix to belong to $\Sigma(\theta)$.

Example 4.11: Let $K = Q_p$ and let the matrix A be

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & p & 0 & 0 & \cdots \\ 1 & p & p^2 & 0 & \cdots \\ 1 & p & p^2 & p^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} .$$

It is clear that A satisfies the two conditions in Theorem 4.10. However, the sequence $x = (1, 1, 1, 1, \ldots)$ belongs to θ but the A-transform of x is the sequence

$$y = \left(1, 1+p, 1+p+p^2, \ldots, \sum_{n=1}^{m} p^{n-1}, \ldots\right) .$$

Thus for any integer m,

$$y_m = 1 + p + p^2 + \ldots + p^{m-1} \neq 1 + p + p^2 + \ldots + p^m = y_{m+1} .$$

So y is not in θ.

Note in the preceding example that the integers of condition (2) are $N_1 = 1, N_2 = 2, \ldots, N_n = n, \ldots$. Therefore there is no integer N such that for all n, $a_{m+1,n} = a_{m,n}$ when $m \geq N$. One might suspect that replacing condition (2) with the condition, "there is an integer N such that for all n, $a_{mn} = a_{m+1,n}$ for $m \geq N$" would make A belong to $\Sigma(\theta)$. This is the case as is indicated in the next theorem.

<u>Theorem 4.12</u>: If $A = (a_{mn})$ is an infinite matrix such that

(1) $\lim_{n\to\infty} a_{mn} = 0 \quad m = 1, 2, 3, \ldots$, and

(2) there is an integer N such that for all n

$$a_{mn} = a_{m+1,n} \quad \text{for} \quad m \geq N$$

then A belongs to $\Sigma'(\theta)$.

Proof: Let $x = \{x_n\}$ belong to θ and let $y = \{y_m\}$ be the A-transform of x. Then

$$y_m = \sum_{n=1}^{\infty} a_{mn} x_n = \sum_{n=1}^{\infty} a_{m+1,n} x_n = y_{m+1}$$

if $m \geq N$. Thus A belongs to $\Sigma(\theta)$. Δ

The converse of Theorem 4.12 is not true. The next example gives a matrix A that belongs to $\Sigma(\theta)$ but there is no N such that for all n, $a_{mn} = a_{m+1,n}$ for $m \geq N$.

Example 4.13: Let A be the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 1 & -1 & 0 & 0 & \cdots \\ 1 & -1 & 1 & -1 & 0 & 0 & \cdots \\ 1 & -1 & 1 & -1 & 1 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} .$$

Now for each m, $\lim_{n\to\infty} a_{mn} = 0$. Also the integers N_n of condition (2) in Theorem 4.10 are

$N_1 = N_2 = 1$, $N_3 = N_4 = 3, \ldots, N_{2n-1} = N_{2n} = 2n-1, \ldots$. Thus there does not exist an integer N that satisfies condition (2) of Theorem 4.12. To show that A belongs to $\Sigma(\theta)$ let $x = \{x_n\}$ belong to θ. Then there is an integer k such that $x_k = x_{k+i}$ for $i \geq 1$. Let $y = \{y_m\}$ be the A-transform of x. Then

$$y_1 = y_2 = x_1 - x_2$$

$$y_3 = y_4 = x_1 - x_2 + x_3 - x_4$$

$$\cdots$$

$$y_{2j-1} = y_{2j} = x_1 - x_2 + x_3 - x_4 + \ldots + x_{2j-1} - x_{2j}$$

$$\cdots \qquad .$$

Now if $k = 2j$ for some j then $y_m = x_1 - x_2 + x_2 + \ldots + x_{k-1} - x_k$ for $m \geq \frac{k}{2}$. If $k = 2j - 1$ for some j then $y_m = x_1 - x_2 + \ldots + x_{k-1} - x_k$ for $m \geq \frac{(k-1)}{2}$. Thus in either case y belongs to θ. So A belongs to $\Sigma(\theta)$.

If necessary and sufficient conditions exist for a matrix A to belong to $\Sigma(\theta)$ they are not known to the author.

The situation is much simpler in the case of the sequence space φ. Here a necessary and sufficient condition that a matrix A belongs to $\Sigma(\varphi)$ is simply that the columns of A belong to φ.

Theorem 4.14: $A = (a_{mn})$ belongs to $\Sigma(\varphi)$ if and only if for each n there is an integer N_n such that $a_{mn} = 0$ for $m \geq N_n$.

Proof: Suppose $A = (a_{mn})$. Then the condition is a result of statement (1) of Theorem 4.3. Conversely, suppose that A is a matrix with the given condition. Let $x = \{x_n\}$ belong to φ and let $y = \{y_m\}$ be the A-transform of x. Then there is an integer n_0 such that $x_n = 0$ for all $n \geq n_0$. For $n = 1, 2, 3, \ldots, n_0$ determine the integers $N_1, N_2, \ldots, N_{n_0}$ so that $a_{mn} = 0$ for $m \geq N_n$, $1 \leq n \leq n_0$. Let $N = \text{Max}\{N_n : 1 \leq n \leq n_0\}$. Then $a_{mn} = 0$ for $m \geq N$ and $1 \leq n \leq n_0$. Since $y_m = \sum_{n=1}^{\infty} a_{mn} x_n$, $y_m = 0$ for $m \geq N$ since either $a_{mn} = 0$ or $x_n = 0$. So A belongs to $\Sigma(\varphi)$. Δ

This section is concluded by determining a characterization of a matrix A that belongs to $\Sigma(\omega)$. As in the case for $\Sigma(\varphi)$, this is a rather simple case. The necessary and sufficient condition is that the rows of A belong to φ.

Theorem 4.15: $A = (a_{mn})$ belongs to $\Sigma(\omega)$ if and only if the rows of A belong to φ.

Proof: Let A belong to $\Sigma(\omega)$. Since $\omega^* = \varphi$, the rows of A belong to φ by statement (2) of Theorem 4.3. Now suppose the rows of A belong to φ and let $x = \{x_n\}$ be any sequence. To prove that A belongs to $\Sigma(\omega)$ one need only show that the A-transform of x exists for each m. But since $\varphi = \omega^*$, $\sum_{n=1}^{\infty} a_{mn} x_n$ exists for each m. Hence, A belongs to $\Sigma(\omega)$. Δ

Rings of Matrix Transformations

It is natural to ask whether the sets of matrices $\Sigma(\alpha)$ possess an interesting algebraic structure. For this investigation the binary operations of addition and multiplication of infinite matrices are defined. The definitions are completely analogous to these for finite matrices.

<u>Definition 4.16</u>: Let $A = (a_{mn})$ and $B = (b_{mn})$ be two infinite matrices. <u>$A = B$</u> if and only if $a_{mn} = b_{mn}$ for all m and n. The <u>sum</u> of A and B is the matrix, $A + B = (a_{mn} + b_{mn})$ and the <u>product</u> of A and B (if it is defined) is the matrix $AB = (c_{mn})$ where $c_{mn} = \sum_{k=1}^{\infty} a_{mk} b_{kn}$.

The main theme of this section is the identification of the sequence spaces α for which the set of matrices $\Sigma(\alpha)$ will be a ring with the operations of addition and multiplication. The next theorem discusses the operation of addition in the set of matrices $\Sigma(\alpha)$, for any sequence space α. The proof of this theorem is straightforward and is left to the reader.

<u>Theorem 4.17</u>: (a) If A and B belong to $\Sigma(\alpha)$ then $A + B$ belongs to $\Sigma(\alpha)$ and $A + B = B + A$.

(b) The matrix, Θ, all of whose entries are 0, belongs to α and $A + \Theta = A$.

(c) If A belongs to $\Sigma(\alpha)$ then $-A = (-a_{mn})$ belongs to $\Sigma(\alpha)$ and $A + (-A) = (-A) + A = \Theta$.

(d) If A, B, and C belong to $\Sigma(\alpha)$ then $A + (B + C) = (A + B) + C$.

Theorem 4.17 indicates that $\Sigma(\alpha)$ is an Abelian group under the operation of addition.

The situation with regard to multiplication of matrices is complicated by the fact that in the definition each entry of the matrix AB is a series. Thus it is necessary to insure the existence of the product. The following example indicates two matrices for which the product does not exist.

Example 4.18: Let the matrix A be

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

and let B be the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} .$$

Note now that the term c_{11} of the product AB is $\sum_{k=1}^{\infty} 1$ and therefore does not exist. Thus AB is not defined.

Using the result of Theorem 4.3 and the definition of a dual space it can be shown that when A and B belong to $\Sigma(\alpha)$ then AB and BA are defined.

Theorem 4.19: If A and B belong to $\Sigma(\alpha)$ then AB and BA exist.

Proof: Since the rows of A considered as sequences belong to α^* and the columns of B considered as sequences belong to α then

$$c_{mn} = \sum_{k=1}^{\infty} a_{mk} b_{kn}$$

exists for each m and n. Thus the product AB exists. Similarly, BA exists. △

Matrix multiplication distributes over addition when the products are defined. This is an easy exercise and is left to the reader.

<u>Theorem 4.20</u>: If A, B, and C belong to $\Sigma(\alpha)$ then $A(B+C)$ and $(B+C)A$ are defined. Furthermore, $A(B+C) = AB+AC$ and $(B+C)A = BA + CA$.

Theorems 4.17, 4.19, and 4.20 give properties needed when the attempt is made to determine if $\Sigma(\alpha)$ is a ring under the operations of addition or multiplication. The only two properties that remain to be shown are:

(1) If A and B are in $\Sigma(\alpha)$ then AB is in $\Sigma(\alpha)$.

(2) If A, B, and C are in $\Sigma(\alpha)$ then $A(BC) = (AB)C$.

Note that Theorem 4.21 states only that AB exists when A and B belong to $\Sigma(\alpha)$ and not that AB belongs to $\Sigma(\alpha)$ when A and B belong to $\Sigma(\alpha)$. Thus, to decide if $\Sigma(\alpha)$ is a ring it is still necessary to show that $\Sigma(\alpha)$ is closed with respect to multiplication. The following theorem gives a sufficient condition for $\Sigma(\alpha)$ to be a ring. That is, a sufficient condition for AB to belong to $\Sigma(\alpha)$ when A and B belong to $\Sigma(\alpha)$.

<u>Theorem 4.21</u>: Let α be a sequence space such that $\varphi \subseteq \alpha$. If for every x of α, every A of $\Sigma(\alpha)$, every u of α^*, and every

$\varepsilon > 0$, there corresponds a positive integer N such that for $n \geq N$

$$|u_m a_{mn} x_n| < \varepsilon \quad (m = 1, 2, 3, \ldots)$$

then, AB belongs to $\Sigma(\alpha)$ for any A and B in $\Sigma(\alpha)$ and (AB)C = A(BC) for any A, B, and C in $\Sigma(\alpha)$.

Proof: Let A and B belong to $\Sigma(\alpha)$ and let x belong to α. Since Bx is a sequence in α then A(Bx) is a sequence in α. Thus, if it can be shown that the AB-transform of x is the same as the A-transform of Bx then it can be concluded that AB belongs to $\Sigma(\alpha)$. Now let $y = \{y_m\}$ be the B-transform of x, let $z = \{z_k\}$ be the A-transform of y, and let $w = \{w_k\}$ be the AB-transform of x. Then $y_m = \sum_{n=1}^{\infty} b_{mn} x_n$, $z_k = \sum_{m=1}^{\infty} a_{km} y_m$, and $w_k = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_{km} b_{mn} \right) x_n$.

Let $\varepsilon > 0$ be given. Since the rows of A belong to α and B belongs to $\Sigma(\alpha)$ the condition of the theorem shows that for each row of A, that $|a_{km} b_{mn} x_n| < \varepsilon$ for $n \geq N$ and $m = 1, 2, 3, \ldots$. Thus from Theorem 1.18, the double series $\sum_{m,n} a_{km} b_{mn} x_n$ converges and has the same sum as the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{km} b_{mn} x_n \quad \text{and} \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{km} b_{mn} x_n .$$

Furthermore, any rearrangement of the double series will have the same sum. Thus,

$$\sum_{m=1}^{\infty} a_{km} \left(\sum_{n=1}^{\infty} b_{mn} x_n \right) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_{km} b_{mn} \right) x_n .$$

Now, $\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{km}b_{mn}\right)x_n = w_k$ is the kth term of the AB-transform of x while

$$\sum_{m=1}^{\infty} a_{km}\left(\sum_{n=1}^{\infty} b_{mn}x_n\right) = \sum_{m=1}^{\infty} a_{km}y_m = z_k .$$

Thus, $w_k = z_k$ and $(AB)x = A(Bx)$. Hence, AB belongs to $\Sigma(\alpha)$.

Now let A, B, C belong to $\Sigma(\alpha)$ and let $x = \{x_n\}$ belong to α. Since BC belongs to $\Sigma(\alpha)$ it follows from the previous part of the theorem that

$$[A(BC)]x = A[(BC)x] .$$

Thus,

$$\begin{aligned}[A(BC)]x &\doteq A[(BC)x] \\ &= A[B(Cx)] \\ &= (AB)(Cx) \\ &= [(AB)C]x .\end{aligned}$$

Therefore, $(AB)C = A(BC)$. △

The following theorem is a summary of the results of Theorems 4.17, 4.19, 4.20, and 4.21.

Theorem 4.22: Let α be a sequence space with $\varphi \subseteq \alpha$. If for every $\varepsilon > 0$, every u of α^*, every A of $\Sigma(\alpha)$, and every x in α there corresponds an integer N such that for $n \geq N$

$$|u_m a_{mn} x_n| < \varepsilon \quad \text{for } m = 1, 2, 3, \ldots$$

then $\Sigma(\alpha)$ is a ring.

Using Lemma 2.33, the converse of the preceding theorem can be proved.

Theorem 4.23: Let α be a sequence space with $\varphi \subseteq \alpha$. If $\Sigma(\alpha)$ is a ring then for every $\varepsilon > 0$, every u of α^*, every A of $\Sigma(\alpha)$, and every x in α there corresponds an integer N such that for $n \geq N$

$$|u_m a_{mn} x_n| < \varepsilon \quad \text{for} \quad m = 1, 2, 3, \ldots .$$

Before proving Theorem 4.23 the following lemma will be proved. This lemma will create the double sequence needed in Lemma 2.33.

Lemma 4.24: Let $u = \{u_m\}$ be in α^*, $x = \{x_n\}$ be in α, and $A = (a_{mn})$ belong to $\Sigma(\alpha)$. Form the double sequence $\{u_m a_{mn} x_n\}$. Then for each fixed n, $\lim_{m\to\infty} u_m a_{mn} x_n = 0$; and for each fixed m, $\lim_{n\to\infty} u_m a_{mn} x_n = 0$.

Proof: By Theorem 4.3, for each fixed n, the sequence $(a_{1n}, a_{2n}, \ldots, a_{mn}, \ldots)$ belongs to α. Thus the series $\sum_{m=1}^{\infty} u_m a_{mn}$ exists for each fixed n. Therefore, $\lim_{m\to\infty} u_m a_{mn} = 0$ and $\lim_{m\to\infty} u_m a_{mn} x_n = 0$. To prove the other limit is zero, note that since A belongs to $\Sigma(\alpha)$ the series $\sum_{n=1}^{\infty} a_{mn} x_n$ exists for each m. Therefore $\lim_{n\to\infty} a_{mn} x_n = 0$ and $\lim_{n\to\infty} u_m a_{mn} x_n = 0$. $\triangle$

Proof of Theorem 4.23: Suppose that the condition is not satisfied. That is, suppose there exists an $\varepsilon > 0$, an $x = \{x_n\}$ in α, a $u = \{u_m\}$ in α^*, and an $A = (a_{mn})$ in $\Sigma(\alpha)$ such that for every N there exists an $n_0 \geq N$ such that

$$|u_{m_0} a_{m_0 n_0} x_{n_0}| \geq \varepsilon \quad \text{for some} \quad m_0 .$$

Thus, by this supposition and Lemma 4.24, $\{u_m a_{mn} x_n\}$ is a double sequence that satisfies the three conditions of Lemma 2.33. Thus there exist strictly increasing sequences $(n_0, n_1, n_2, \ldots)$, $(N_0', N_1', N_2', \ldots)$, $(m_0, m_1, m_2, \ldots)$, and $(m_0', m_1', m_2', \ldots)$ such that $m_i < m_i' < m_{i+1}$ and $n_i \geq N_i'$ for every i. Furthermore,

$$(4.23.1) \qquad |u_{m_k} a_{m_k n_k} x_{n_k}| \geq \varepsilon \quad \text{for} \quad k = 0, 1, 2, \ldots$$

and

$$(4.23.2) \qquad |u_{m_j} a_{m_j n_k} x_{n_k}| < \varepsilon \quad \text{for} \quad j \neq k .$$

Now define the sequence $v = \{v_j\}$ by

$$v_j = \begin{cases} 0 & \text{if} \quad j \neq m_k \quad (k = 0, 1, 2, \ldots) \\ u_{m_k} & \text{if} \quad j = m_k . \end{cases}$$

Then $v = \{v_j\}$ satisfies the properties

$$(4.23.3) \qquad \begin{cases} |v_{m_j} a_{m_j n_k} x_{n_k}| < \varepsilon & \text{if} \quad j \neq k \quad (k = 0, 1, 2, \ldots) \\ |v_{m_k} a_{m_k n_k} x_{n_k}| \geq \varepsilon & \end{cases}$$

Now v belongs to α^* since u belongs to α^*. The remainder of the proof is to define a matrix V that belongs to $\Sigma(\alpha)$ but such that the product of V and A does not belong to $\Sigma(\alpha)$.

Define the matrix V by letting the entries in the first row of V be the entries in v and by letting the remaining entries of V be zero. Since $\varphi \subseteq \alpha$ and v is in α^*, V belongs to $\Sigma(\alpha)$. The product of V and A is the matrix

$$VA = \begin{bmatrix} \sum_{k=0}^{\infty} v_{m_k} a_{m_k,1} & \sum_{k=0}^{\infty} v_{m_k} a_{m_k,2} & \cdots & \sum_{k=0}^{\infty} v_{m_k} a_{m_k,n} & \cdots \\ 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & & \vdots & \end{bmatrix}.$$

Now VA does not belong to $\Sigma(\alpha)$ since the VA-transform of x is undefined. To see this consider the VA-transform of x. It should be the sequence $\left(\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty} v_{m_k} a_{m_k n}\right)x_n, 0, 0, 0, \ldots\right)$, if the series in the first position converges. However, by (4.23.3),

$$\left|\sum_{k=0}^{\infty} (v_{m_k} a_{m_k n}) x_n\right| = |v_{m_k} a_{m_k n_k} x_{n_k}|$$

when $n = n_k$. Thus $\lim_{n\to\infty} \sum_{k=0}^{\infty} (v_{m_k} a_{m_k n}) x_n \neq 0$. Therefore the series $\sum_{n=1}^{\infty}\left(\sum_{k=0}^{\infty} v_{m_k} a_{m_k n}\right)x_n$ does not converge. Hence, the VA-transform of x does not exist and consequently, VA does not belong to $\Sigma(\alpha)$. Thus $\Sigma(\alpha)$ is not closed with respect to multiplication and is therefore not a ring. This contradiction proves the theorem. $\triangle$

Theorem 4.22 and its converse, Theorem 4.23, are useful in deciding for what sequence spaces α, $\Sigma(\alpha)$ will be a ring. Because of the importance of the necessary and sufficient condition for $\Sigma(\alpha)$

to be a ring, this condition is given a name. That is, the matrix A belonging to $\Sigma(\alpha)$, $(\varphi \subseteq \alpha)$ is said to satisfy the "ring condition" if to every $\varepsilon > 0$ and to every x of α and u of α^*, there corresponds a number N, such that for $n \geq N$, $|u_m a_{mn} x_n| < \varepsilon$ for $m = 1, 2, 3, \ldots$.

The remainder of this section will be devoted to determining for what sequence spaces α, $\Sigma(\alpha)$ will be a ring.

Since $(c)^* = (c_0) = (m)^*$ and $(c) \subseteq (m)$ it can be shown simultaneously that $\Sigma(c)$ and $\Sigma(m)$ are rings.

<u>Theorem 4.25</u>: The sets of matrices $\Sigma(m)$ and $\Sigma(c)$ are rings.

<u>Proof</u>: Let $\varepsilon > 0$, $x = \{x_n\}$ belong to (m) or (c), $u = \{u_m\}$ be in (c_0), and let A belong to $\Sigma(m)$ or $\Sigma(c)$. There is a number M_1 such that $|u_m| < M_1$ for all m and a number M_2 such that $|x_n| < M_2$ for all n. From Theorem 3.9 if A is in $\Sigma(c)$, or Theorem 4.9 if A is in $\Sigma(m)$, there is a number M such that $|a_{mn}| < M$ for all m and n. Now because $\{u_m\}$ belongs to (c_0) there is a number m_0 such that

(4.25.1) $$|u_m| < \frac{\varepsilon}{M_2 M} \quad \text{for } m > m_0 .$$

But by (4.25.1) it follows that for all n

(4.25.2) $$|u_m a_{mn} x_n| < \varepsilon \quad \text{for } m > m_0 .$$

Since $\lim_{n\to\infty} a_{mn} = 0$, for each m there is a number N_m such that

$$|a_{mn}| < \frac{\varepsilon}{M_1 M_2} \quad \text{for} \quad n \geq N_m .$$

Determine the integers $N_1, N_2, \ldots, N_{m_0}$ and let $N = \text{Max}(N_1, N_2, \ldots, N_{m_0})$. Then for $n \geq N$ and $1 \leq m \leq m_0$

$$|a_{mn}| < \frac{\varepsilon}{M_1 M_2} .$$

Therefore for $n \geq N$,

(4.25.3) $$|u_m a_{mn} x_n| < \varepsilon \quad \text{for} \quad 1 \leq m \leq m_0 .$$

The combination of (4.25.2) and (4.25.3) yields for $n \geq N$

$$|u_m a_{mn} x_n| < \varepsilon \quad \text{for} \quad m = 1, 2, 3, \ldots .$$

Therefore, $\Sigma(m)$ and $\Sigma(c)$ are rings. △

<u>Theorem 4.26</u>: The set of matrices $\Sigma(c_0)$ is a ring.

<u>Proof</u>: Let $\varepsilon > 0$, $x = \{x_n\}$ be in (c_0), $u = \{u_m\}$ be in (m) and, A belong to $\Sigma(c_0)$. Since u is in (m) there is a number M_1 such that $|u_m| < M_1$ for all m. From Theorem 4.8, since A is in $\Sigma(c_0)$, there is a number $M > 0$ such that $|a_{mn}| < M$ for all m and n. Now since x is in (c_0) there is an integer N such that for $n \geq N$, $|x_n| < \frac{\varepsilon}{M_1 M}$. Thus, for $n \geq N$

$$|u_m a_{mn} x_n| < M M_1 |x_n| < \varepsilon \quad \text{for all} \quad m .$$

Hence, $\Sigma(c_0)$ is a ring. △

It is a very easy task to show that $\Sigma(\varphi)$ and $\Sigma(\omega)$ are rings and these cases are left to the reader.

Theorem 4.27: The sets of matrices $\Sigma(\varphi)$ and $\Sigma(\omega)$ are rings.

The remaining part of this section is devoted to showing that $\Sigma(\theta)$ is not a ring. This is accomplished in two examples. The first of these examples is a matrix A in $\Sigma(\theta)$ that does not satisfy the ring condition. Since the ring condition is a necessary condition, this example shows that $\Sigma(\theta)$ is not a ring. The second example is a matrix B in $\Sigma(\theta)$ such that BA is not in $\Sigma(\theta)$. This shows that $\Sigma(\theta)$ is not even closed with respect to multiplication.

Example 4.28: Let $K = Q_p$ and let A be the matrix

$$A = \begin{bmatrix} 1 & p & p^2 & p^3 & p^4 & \cdots \\ -\frac{1}{p} & 1 & p & p^2 & p^3 & \cdots \\ \frac{1}{p^2} & -\frac{1}{p^2} & 1 & p & p^2 & \cdots \\ 0 & \frac{1}{p^3} & -\frac{1}{p^3} & 1 & p & \cdots \\ 0 & 0 & \frac{1}{p^4} & -\frac{1}{p^4} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

In general $A = (a_{mn})$ where

$$a_{mn} = \begin{cases} p^{n-1} & \text{if } n \geq m \\ \dfrac{-1}{p^{m-1}} & \text{if } n = m-1 \\ \dfrac{+1}{p^{m-1}} & \text{if } n = m-2 \\ 0 & \text{if } n < m-2 \end{cases} .$$

To show that A belongs to $\Sigma(\theta)$ consider $x = \{x_n\}$ in θ. Then there is an integer N such that for $n \geq N$, $x_n = x_{n+k}$, $k = 0, 1, 2, \ldots$. Now let $y = \{y_m\}$ be the A-transform of x. Then

$$y_m = \frac{x_{n-2}}{p^{n-1}} - \frac{x_{n-1}}{p^{n-1}} + \sum_{j=0}^{\infty} x_{n+j} p^j, \quad \text{for } m \geq 2 .$$

So $y_m = x_N \sum_{j=0}^{\infty} p^{j-1}$ when $m \geq N+2$. That is, $y_m = \frac{x_N}{1-p}$ for $m \geq N+2$. Thus y belongs to θ, and so A belongs to $\Sigma(\theta)$.

To show that A does not satisfy the ring condition let $u = (1, p, p^2, p^3, \ldots)$ and let $x = (1, 1, 1, 1, \ldots)$. Now u belongs to (c_0) and x belongs to θ. However, for $m = n+2$,

$$|u_m a_{mn} x_n| = |p^{m-1} \frac{1}{p^{m-1}}| = 1 .$$

Thus, $|u_m a_{mn} x_n|$ cannot be made arbitrarily small for all m when n is sufficiently large. Therefore A does not satisfy the ring condition.

Example 4.29: Define the matrix B as follows: the first row is the sequence $(1, p, p^2, p^3, \ldots)$ and the remaining entries are zero. Then B is clearly in $\Sigma(\theta)$, since the B-transform of any x in θ

is $\left(\sum_{n=0}^{\infty} x_n p^{n-1}, 0, 0, 0, \ldots \right)$.

To show that BA is not in $\Sigma(\theta)$ first find the product BA.

$$BA = \begin{bmatrix} 3 & 2p-2 & 3p^2+2 & 4p^3-2 & \cdots & (n+1)p^n+(-1)^n 2 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \end{bmatrix}.$$

Now let $x = (1, 1, 1, \ldots)$. Then the BA-transform of x is

$$\left(\sum_{n=0}^{\infty} [(n+1)p^n + (-1)^n 2], 0, 0, 0, \ldots \right).$$

But $\sum_{n=0}^{\infty} [(n+1)p^n + (-1)^n 2]$ does not converge since

$$\lim_{n \to \infty} |(n+1)p^n + (-1)^n 2| = 1 \neq 0.$$

Therefore, the BA-transform of x does not exist and consequently BA does not belong to $\Sigma(\theta)$. Therefore, $\Sigma(\theta)$ is not a ring.

Transpose of a Matrix

The last idea to be discussed in this chapter is the transpose of a matrix. From linear algebra, if $A = (a_{mn})$ then the transpose of A is the matrix $A' = (a_{nm})$; that is, A' is obtained from A by interchanging the rows and columns of A. It can be observed by Theorem 4.3 that there is a possibility that if A belongs to $\Sigma(\alpha)$, for some sequence space $\alpha \subseteq \varphi$, then A' belongs to $\Sigma(\alpha^*)$. The reason for this possibility is that the rows of A' considered as sequences are elements of α and the columns of A' belong to α^*. Recall, however, that these conditions were necessary conditions and not sufficient conditions. The next two examples will indicate that A' may or may not belong to $\Sigma(\alpha^*)$.

Example 4.30: Let $\alpha = (c_0)$ and let A be the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ p & p & 0 & 0 & \cdots \\ p^2 & p^2 & p^2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Theorem 4.8 implies that A belongs to $\Sigma(c_0)$. It is clear that the matrix A' belongs to $\Sigma(m)$. Recall that $(c_0)^* = (m)$.

Example 4.31: Consider the sequence space θ and let A be the matrix of Example 4.28. It was shown in this example that A belongs to $\Sigma(\theta)$. However, it is clear that A' does not belong to $\Sigma(c_0)$ since the entries of A are unbounded. Thus A' does not belong to $\Sigma(\theta^*) = \Sigma(c_0)$.

Definition 4.32: The set of matrices $\Sigma'(\alpha)$ is the set of all transposes of matrices belonging to $\Sigma(\alpha)$. That is
$\Sigma'(\alpha) = \{A' : A \text{ belongs to } \Sigma(\alpha)\}$.

Recall that in the proof of Example 4.28 the matrix A did not satisfy the "ring condition". In view of this fact it might cause one to wonder if the "ring condition" is the characteristic of A needed to insure that A' belong to $\Sigma(\alpha^*)$ when A belongs to $\Sigma(\alpha)$. The next theorem indicates that this is precisely the case.

Theorem 4.33: A necessary and sufficient condition that the transpose A' of a matrix A of $\Sigma(\alpha)$, $\alpha \supseteq \varphi$, belongs to $\Sigma(\alpha^*)$ is that A satisfy the ring condition.

Proof: Suppose that A belongs to $\Sigma(\alpha)$ and that A satisfies the ring condition; i.e., for every $\varepsilon > 0$, for every u of α^*, and for every x in α there corresponds an N such that for $n \geq N$

(4.33.1) $$|u_m a_{mn} x_n| < \varepsilon \quad (m = 1, 2, 3, \ldots)$$

Thus,

(4.33.2) $$|u_m a_{nm} x_n| < \varepsilon \quad \text{for} \quad n \geq N .$$

Now consider the sequence $\{ \sum_{m=1}^{\infty} u_m a_{nm} : n = 1, 2, 3, \ldots \}$. This sequence is defined since for each fixed n, $\{a_{mn} : m = 1, 2, 3, \ldots\}$ belongs to α. Therefore by (4.33.2) the series

(4.33.3) $$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} u_m a_{nm} \right) x_n$$

converges. Therefore, since $x = \{x_n\}$ is an arbitrary element of α, the sequence $\{ \sum_{m=1}^{\infty} a_{nm} u_m : n = 1, 2, 3, \ldots\}$ belongs to α^*, by definition of α^*. Thus A' belongs to $\Sigma(\alpha^*)$.

The proof of the converse uses the method of contradiction. That is, suppose that A does not satisfy the ring condition. Hence there exists an $\varepsilon > 0$, an x in α, and a u of α^* such that for every N there is an integer $n_0 \geq N$ such that

$$|u_{m_0} a_{m_0 n_0} x_{n_0}| \geq \varepsilon \quad \text{for some } m_0 .$$

Now by interchanging the roles of m and n, u and x in the proof of Theorem 4.23 it is possible to construct a sequence x' (corresponds to v) in α so that x'(A'u) does not converge. Hence, A'u does not belong to α^* and A' does not belong to $\Sigma(\alpha^*)$. This contradiction proves that the ring condition is necessary for A' to belong to $\Sigma(\alpha^*)$ when A belongs to $\Sigma(\alpha)$. △

<u>Corollary 4.34</u>: If for every A in $\Sigma(\alpha)$, A satisfies the ring condition then $\Sigma'(\alpha)$ is contained in $\Sigma(\alpha^*)$ where "contained" is the usual subset relation between sets.

Suppose now that $\Sigma(\alpha)$ is a ring. Then by Theorem 4.23 for every A of $\Sigma(\alpha)$ A satisfies the ring condition. Hence, by Corollary 4.34 the following theorem is proved.

<u>Theorem 4.35</u>: If $\Sigma(\alpha)$, $\alpha \supseteq \varphi$, is a ring then $\Sigma'(\alpha) \subseteq \Sigma(\alpha^*)$.

The final theorem of this chapter indicates a particular instance where $\Sigma'(\alpha) = \Sigma(\alpha^*)$.

<u>Theorem 4.36</u>: $\Sigma'(c_0) = \Sigma(m)$ and $\Sigma'(m) = \Sigma(c_0)$.

<u>Proof</u>: By Theorem 4.37, $\Sigma'(c_0) \subseteq \Sigma(m)$. The reverse inclusion is obtained by using Theorem 4.8. Let A belong to $\Sigma(m)$. Then by

Theorem 4.8, A' belongs to $\Sigma(c_0)$. Hence, $(A')' = A$ belongs to $\Sigma'(c_0)$. The other equality is proved in a similar fashion using Theorem 4.9. Δ

An example will now be given to show that $\Sigma'(\alpha)$ might be properly contained in $\Sigma(\alpha^*)$. Because of the additional requirement for the matrix A to belong to $\Sigma(c)$, the obvious place to look for such an example is in the set $\Sigma(c_0)$.

<u>Example 4.37</u>: Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} .$$

Now A belongs to $\Sigma(c_0)$ by Theorem 4.8. However, A' does not belong to $\Sigma(c)$ since $\lim_{n\to\infty} \sum_{m=1}^{\infty} a_{nm} = \lim_{n\to\infty} n$, does not exist. Therefore, A' does not satisfy condition (3) of Theorem 3.9.

CHAPTER V

SEQUENCES OF MATRICES

In Chapter II norm convergence and (α,β)-convergence are defined for a sequence $\{x^{(n)}\}$ in a sequence space α. In Chapter IV a set of matrices, $\Sigma(\alpha)$, is introduced. This chapter is devoted to defining and investigating convergence criteria for a sequence of matrices $\{A^{(k)}\}$ belonging to $\Sigma(\alpha)$. Two types of convergence are introduced, one defined in terms of the norm, and the other defined in terms of the sequence space and its dual.

Norm Convergence

Since the concept of norm convergence is based on the definition of a norm it is first necessary to define the norm of a matrix. For the infinite matrix $A = (a_{mn})$ the function $\| \ \|$ is given by

$$\|A\| = \operatorname*{Sup}_{m,n} |a_{mn}| .$$

The necessary and sufficient conditions for A to belong to either $\Sigma(c)$, $\Sigma(c_0)$, or $\Sigma(m)$ determined in Chapters III and IV indicate that $\|A\|$ is finite for each A in $\Sigma(c)$, $\Sigma(c_0)$, or $\Sigma(m)$. The next theorem states the function $\| \ \|$ is a non-Archimedean norm for the set of infinite matrices. Its proof is straightforward and is left for the reader.

Theorem 5.1: The function $\| \ \|$ is a non-Archimedean norm for the set of infinite matrices.

Norm convergence of a sequence of matrices is defined in the usual way.

Definition 5.2: The sequence of matrices $A^{(k)} = (a_{mn}^{(k)})$ belonging to $\Sigma(\alpha)$ is said to be norm convergent to $A = (a_{mn})$ if for each $\varepsilon > 0$ there is an integer N such that for $k \geq N$

$$\|A^{(k)} - A\| = \sup_{m,n} |a_{mn}^{(k)} - a_{mn}| < \varepsilon .$$

In the following series of lemmas and theorems [5.3 - 5.6] it is shown that $\Sigma(c)$, $\Sigma(c_0)$ and $\Sigma(m)$ are complete with respect to norm convergence.

Lemma 5.3: If $\{A^{(k)}\}$ is a Cauchy sequence in $\Sigma(c)$, $\Sigma(c_0)$, or $\Sigma(m)$ then there is a matrix $A = (a_{mn})$ such that $\{A^{(k)}\}$ is norm convergent to A and $\sup_{m,n} |a_{mn}| = M$ for some number $M > 0$.

Proof: Since $\{A^{(k)}\} = \{(a_{mn}^{(k)})\}$ is a Cauchy sequence it follows that for $\varepsilon > 0$, there is an integer N' such that

$$\|A^{(k)} - A^{(j)}\| < \varepsilon$$

whenever $j, k \geq N'$. Thus,

$$\sup_{m,n} |a_{mn}^{(k)} - a_{mn}^{(j)}| < \varepsilon \tag{5.3.1}$$

if $j, k \geq N'$. Therefore, for each fixed m and n the sequence

$\{a_{mn}^{(k)} : k = 1,2,3,\ldots\}$ is a Cauchy sequence in K. Hence, there exists a_{mn} in K such that $\lim_{k\to\infty} a_{mn}^{(k)} = a_{mn}$. Let $A = (a_{mn})$. Now, by (5.3.1), for all m and n it is true that

$$|a_{mn}^{(k)} - a_{mn}^{(j)}| < \varepsilon \tag{5.3.2}$$

if j, $k \geq N'$. Thus, letting k approach ∞, it follows that for all m and n

$$|a_{mn} - a_{mn}^{(j)}| \leq \varepsilon \tag{5.3.3}$$

if $j \geq N'$. Therefore, $\{A^{(k)}\}$ is norm convergent to A. Furthermore, by (5.3.3),

$$|a_{mn}| \leq \varepsilon + |a_{mn}^{(N')}| \leq \varepsilon + M_{N'}$$

where $M_{N'} = \operatorname{Sup}_{m,n} |a_{mn}^{(N')}|$. Thus, $\operatorname{Sup}_{m,n} |a_{mn}| = M$ for some number $M > 0$. △

Lemma 5.4: (a) If $\{A^{(k)}\}$ is a Cauchy sequence belonging to $\Sigma(c)$ or $\Sigma(m)$ then the rows of the matrix A of Lemma 5.3 are null sequences.

(b) If $\{A^{(k)}\}$ is a Cauchy sequence belonging to $\Sigma(c_0)$ then the columns of the matrix A of Lemma 5.3 are null sequences.

Proof: By Theorem 3.9 or 4.9, for each fixed m and k, the sequence $(a_{m1}^{(k)}, a_{m2}^{(k)}, \ldots, a_{mn}^{(k)}, \ldots)$ is a null sequence. [Note, this sequence is the mth row of the kth matrix]. Therefore,

$$\{(a_{m1}^{(k)}, a_{m2}^{(k)}, \ldots, a_{mn}^{(k)}, \ldots) : k = 1,2,3,\ldots\} \tag{5.4.1}$$

is a sequence of null sequences. For each fixed m, by (5.3.2) it follows that

$$\sup_n |a_{mn}^{(k)} - a_{mn}^{(j)}| \leq \varepsilon$$

for $j, k \geq N_0$. Thus the sequence (5.4.1), is a Cauchy sequence in (c_0). Since (c_0) is complete with respect to norm convergence it follows that the sequence $(a_{m1}, a_{m2}, \ldots, a_{mn}, \ldots)$ is a null sequence. This concludes the proof of statement (a). Statement (b) is proved in a similar manner. △

Using Lemmas 5.3 and 5.4, it is now possible to show that $\Sigma(c_0)$ and $\Sigma(m)$ are complete with respect to norm convergence.

<u>Theorem 5.5</u>: $\Sigma(c_0)$ and $\Sigma(m)$ are complete with respect to norm convergence.

<u>Proof</u>: This theorem follows firectly from the two lemmas and Theorems 4.8 and 4.9. △

Because of the added conditions for a matrix A to belong to $\Sigma(c)$, the proof that $\Sigma(c)$ is complete is a little more difficult.

<u>Theorem 5.6</u>: $\Sigma(c)$ is complete with respect to norm convergence.

<u>Proof</u>: Let $\{A^{(k)}\}$ be a Cauchy sequence belonging to $\Sigma(c)$. Then by Lemmas 5.3 and 5.4 there is a matrix $A = \{a_{mn}\}$ with the properties

(1) there is a number $M > 0$ such that $\sup_{m,n} |a_{mn}| = M$, and

(2) $\lim_{n\to\infty} a_{mn} = 0, \quad m = 1, 2, 3, \ldots$.

Therefore, to show that A belongs to $\Sigma(c)$ it remains to show

(3) $\lim_{m\to\infty} a_{mn}$ exists for each n, and

(4) $\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn}$ exists.

To prove (3) note that for each fixed k and n the sequence $(a_{1n}^{(k)}, 2_{an}^{(k)}, \ldots, a_{mn}^{(k)}, \ldots)$ belongs to (c), by Theorem 3.9. Thus, the sequence

$$(5.6.1) \qquad \{(a_{1n}^{(k)}, a_{2n}^{(k)}, \ldots, a_{mn}^{(k)}, \ldots) : k = 1, 2, 3, \ldots\}$$

is a sequence in (c). Now for each fixed n, by (5.3.2)

$$\operatorname*{Sup}_{m} |a_{mn}^{(k)} - a_{mn}^{(j)}| \leq \varepsilon$$

if $j, k \geq N'$. Thus the sequence (5.6.1), is a Cauchy sequence in (c). Since (c) is complete with respect to norm convergence it follows that the sequence $(a_{1n}, a_{2n}, \ldots, a_{mn}, \ldots)$ belongs to (c). Thus for each n, $\lim_{m\to\infty} a_{mn}$ exists.

To prove (4), note first that (2) implies $\sum_{n=1}^{\infty} a_{mn}$ exists for each m. Let

$$\sigma_m = \sum_{n=1}^{\infty} a_{mn} \quad \text{and} \quad \sigma_m^{(k)} = \sum_{n=1}^{\infty} a_{mn}^{(k)} .$$

By the non-Archimedean property

$$(5.6.2) \qquad |\sigma_r - \sigma_s| \leq \operatorname{Max}(|\sigma_r - \sigma_r^{(k)}|, |\sigma_r^{(k)} - \sigma_s^{(k)}|, |\sigma_s^{(k)} - \sigma_s|).$$

But by (5.3.3) and the definition of σ_r,

$$(5.6.3) \qquad |\sigma_r - \sigma_r^{(k)}| = \left| \sum_{n=1}^{\infty} (a_{rn} - a_{rn}^{(k)}) \right| \leq \operatorname{Sup}_n |a_{rn} - a_{rn}^{(k)}|$$

$$< \varepsilon \quad \text{for all } r, \text{ if } k \geq N' .$$

For each fixed n, since $\lim_{m\to\infty} a_{mn}^{(N')}$ exists, there is an integer N such that for $r, s \geq N$

$$(5.6.4) \qquad |a_{rn}^{(N')} - a_{sn}^{(N')}| < \varepsilon .$$

Thus, by (5.6.4) it follows that

$$(5.6.5) \qquad |\sigma_r^{(N')} - \sigma_s^{(N')}| = \left| \sum_{n=1}^{\infty} (a_{rn}^{(N')} - a_{sn}^{(N')}) \right|$$

$$\leq \operatorname{Sup}_n |a_{rn}^{(N')} - a_{sn}^{(N')}|)$$

$$\leq \varepsilon \quad \text{if } r, \; s \geq N .$$

Therefore by (5.6.2), (5.6.3), and (5.6.5), and for $k = N'$,

$$|\sigma_r - \sigma_s| \leq \varepsilon \quad \text{if } r, \; s \geq N .$$

Hence, $\{\sigma_r\}$ is a Cauchy sequence in K and $\lim_{r\to\infty} \sigma_r$ exists. Therefore, $\lim_{r\to\infty} \sum_{n=1}^{\infty} a_{rn}$ exists and (4) is proved. Hence, $\Sigma(c)$ is complete with respect to norm convergence. △

(α, β)-Convergence

Let $\{A^{(k)}\}$ be a sequence of matrices belonging to $\Sigma(\alpha)$. Another type of convergence of this sequence can be defined in terms of the sequence space α and its dual space. To aid in defining this convergence, for each x of α and each u of $\beta \subseteq \alpha^*$ consider the sequence $y = \{y_k\}$ defined as follows:

$$y_1 = \sum_{m=1}^{\infty} u_m \left(\sum_{n=1}^{\infty} a_{mn}^{(1)} x_n \right)$$

$$y_2 = \sum_{m=1}^{\infty} u_m \left(\sum_{n=1}^{\infty} a_{mn}^{(2)} x_n \right)$$

$$\cdots$$

$$y_k = \sum_{m=1}^{\infty} u_m \left(\sum_{n=1}^{\infty} a_{mn}^{(k)} x_n \right)$$

$$\cdots \quad .$$

For this definition to be meaningful y_k must exist for each k; that is, the series $\sum_{m=1}^{\infty} u_m \left(\sum_{n=1}^{\infty} a_{mn}^{(k)} x_n \right)$ must converge for each k. Since $A^{(k)}$ belongs to $\Sigma(\alpha)$ for each k the sequence $\{ \sum_{n=1}^{\infty} a_{mn}^{(k)} : m = 1, 2, 3, \ldots \}$ belongs to α. Thus, since u belongs to β, and hence to α^*, the series $\sum_{m=1}^{\infty} u_m \left(\sum_{n=1}^{\infty} a_{mn}^{(k)} x_n \right)$ converges for each k.

The sequence $y = \{y_k\}$ is generally written as $\{uA^{(k)}x\}$. It should be noted that for each k, $y_k = uA^{(k)}x$ is an element in K. That is, the sequence $\{y_k\}$ is a sequence of elements in K. Since this is the case, the question of whether $\{y_k\}$ converges or diverges can be answered in terms of the usual metric in K.

To each sequence of matrices $\{A^{(k)}\}$ of $\Sigma(\alpha)$, each u of $\beta \subseteq \alpha^*$, and each x of α, there corresponds a sequence $\{uA^{(k)}x\}$ in K. The definition of (α, β)-convergence of the sequence $\{A^{(k)}\}$ is given in terms of this sequence.

Definition 5.7: A sequence of matrices $\{A^{(k)}\}$, belonging to $\Sigma(\alpha)$ is said to be (α,β)-convergent, $(\varphi \subseteq \beta \subseteq \alpha^*)$, if for every x of α and u of β the sequence $\{uA^{(k)}x\}$ is convergent. If $\beta = \alpha^*$, then $A^{(k)}$ is said to be α-convergent. The sequence $\{A^{(k)}\}$ is said to be (α,β)-convergent to A, where A belongs to $\Sigma(\alpha)$, if the sequence $\{uA^{(k)}x\}$ converges to uAx for every u of β and x of α.

The following example gives a sequence of matrices that is (c_0)-convergent to the matrix $A = (a_{mn})$ where

$$a_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$

Example 5.8: Let $K = Q_p$. Define $A^{(k)} = (a_{mn}^{(k)})$ by

$$a_{mn}^{(k)} = \begin{cases} p^{k(n-m)} & \text{if } n \geq m \\ 0 & \text{if } n < m. \end{cases}$$

That is, the matrix $A^{(k)}$ is given by

$$A^{(k)} = \begin{bmatrix} 1 & p^k & p^{2k} & p^{3k} & \cdots \\ 0 & 1 & p^k & p^{2k} & \cdots \\ 0 & 0 & 1 & p^k & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

By Theorem 4.9, the matrix $A^{(k)}$ belongs to $\Sigma(c_0)$ for every k.

To show $\{A^{(k)}\}$ is (c_0)-convergent, let u belong to (m) and $x = \{x_n\}$ belong to (c_0). The sequence $\{uA^{(k)}x\}$ is

$$\left\{ \sum_{m=1}^{\infty} u_m \left(\sum_{n=m}^{\infty} p^{k(n-m)} x_n \right) : k = 1, 2, 3, \ldots \right\} .$$

Also, $uAx = \sum_{m=1}^{\infty} u_m x_m$ since $a_{mm} = 1$ and $a_{mn} = 0$ if $n \neq m$. Thus,

$$(5.8.1) \qquad |uA^{(k)}x - uAx| = \left| \sum_{m=1}^{\infty} u_m \left(\sum_{n=m}^{\infty} p^{k(n-m)} x_n \right) - \sum_{m=1}^{\infty} u_m x_m \right| .$$

Since each of the series in (5.8.1) converges,

$$(5.8.2) \qquad |uA^{(k)}x - uAx| = \left| \sum_{m=1}^{\infty} u_m \left(\sum_{n=m+1}^{\infty} p^{k(n-m)} x_n \right) \right| .$$

Thus, from (5.8.2) it follows that

$$|uA^{(k)}x - uAx| \leq \sup_m |u_m| \left| \sum_{n=m+1}^{\infty} p^{k(n-m)} x_n \right|$$

$$= \sup_m |u_m| \left| p^k \sum_{n=m+1}^{\infty} p^{k(n-m-1)} x_n \right|$$

$$\leq \frac{\|u\|}{p^k} \sup_m \left(\sup_{n \geq m+1} \left(|p^{k(n-m-1)} x_n| \right) \right)$$

$$\leq \frac{\|u\| \; \|x\|}{p^k} \sup_m \left(\sup_{n \geq m+1} \frac{1}{p^{k(n-m-1)}} \right)$$

$$\leq \frac{\|u\| \; \|x\|}{p^k} .$$

Therefore, $\lim_{k\to\infty} uA^{(k)}x = uAx$, for every x of α and u of β.

After the definition of (α,β)-convergence of a sequence of matrices it is natural to ask whether a necessary and sufficient condition exists for a sequence $\{A^{(k)}\}$ to be (α,β)-convergent. The next theorem gives such a characterization. The proof uses the fact that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge then $\sum_{n=1}^{\infty} (a_n + b_n)$ converges and is equal to

$$\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n .$$

Theorem 5.9: A necessary and sufficient condition for the (α,β)-convergence of a sequence of matrices $\{A^{(k)}\}$ belonging to $\Sigma(\alpha)$ is that the sequence $\{A^{(k)} - A^{(k+1)}\}$ is (α,β)-convergent to the zero matrix.

Proof: Suppose $\{A^{(k)}\}$ is (α,β)-convergent. Let $\{y_k\}$ be the sequence $\{uA^{(k)}x\}$ and $\{z_k\}$ be the sequence $\{u(A^{(k)} - A^{(k+1)})x\}$. By hypothesis $\{y_k\}$ is a convergent sequence in K. That is, $\{y_k - y_{k+1}\}$ is a null sequence. Thus, for every $\varepsilon > 0$ there is an integer N such that

$$|y_k - y_{k+1}| < \varepsilon \quad \text{if} \quad k \geq N .$$

In terms of the original notation,

$$(5.9.1) \quad \left| \sum_{m=1}^{\infty} u_m \left(\sum_{n=1}^{\infty} a_{mn}^{(k)} x_n \right) - \sum_{m=1}^{\infty} u_m \left(\sum_{n=1}^{\infty} a_{mn}^{(k+1)} x_n \right) \right| < \varepsilon \text{ if } k \geq N .$$

Hence, since each of the outer series in (5.9.1) converge

$$(5.9.2) \quad \left| \sum_{m=1}^{\infty} u_m \left(\sum_{n=1}^{\infty} a_{mn}^{(k)} x_n - \sum_{n=1}^{\infty} a_{mn}^{(k+1)} x_n \right) \right| < \varepsilon \quad \text{if} \quad k \geq N .$$

Now since the series involved converge,

$$\sum_{n=1}^{\infty} a_{mn}^{(k)} x_n - \sum_{n=1}^{\infty} a_{mn}^{(k+1)} x_n = \sum_{n=1}^{\infty} (a_{mn}^{(k)} - a_{mn}^{(k+1)}) x_n ,$$

and

$$(5.9.3) \qquad \left| \sum_{m=1}^{\infty} u_m \left[\sum_{n=1}^{\infty} a_{mn}^{(k)} x_n - a_{mn}^{(k+1)} x_n \right] \right| < \varepsilon \quad \text{if } k \geq N .$$

Thus, $|z_k| < \varepsilon$ if $k \geq N$. That is, $\{u(A^{(k)} - A^{(k+1)})x\}$ converges to zero for every u of β and x of α. Thus $\{A^{(k)} - A^{(k+1)}\}$ converges to the zero matrix.

To prove the converse, reverse the above argument to obtain

$$|y_k - y_{k+1}| < \varepsilon \quad \text{if } k \geq N .$$

Therefore, $\{y_k\}$ is a Cauchy sequence in K. Since K is complete, $\{y_k\}$ converges so $\{A^{(k)}\}$ is (α,β)-convergent. Δ

Comparison of Norm Convergence and (α,β)-Convergence

The purpose of this section is to investigate the relationship between norm convergence and α-convergence where α is either (c_0), (c), or (m).

Theorem 5.10: (a) If $\{A^{(k)}\}$ is a norm convergent sequence in $\Sigma(c)$ or $\Sigma(m)$ then $\{A^{(k)}\}$ is (c)-convergent or (m)-convergent, respectively.

(b) If $\{A^{(k)}\}$ is a norm convergent sequence in $\Sigma(c_0)$ then $\{A^{(k)}\}$ is (c_0)-convergent.

Proof of (a): Let $u = \{u_m\}$ belong to (c_0) and $x = \{x_n\}$ belong to (c) or to (m). Then,

$$|u(A^{(k)} - A^{(k+1)})x| = \left| \sum_{m=1}^{\infty} u_m \left[\sum_{n=1}^{\infty} (a_{mn}^{(k)} - a_{mn}^{(k+1)}) x_n \right] \right|$$

$$\leq \operatorname{Sup}_m \left| u_m \sum_{n=1}^{\infty} (a_{mn}^{(k)} - a_{mn}^{(k+1)}) x_n \right|$$

$$\leq \|u\| \operatorname{Sup}_m \left| \sum_{n=1}^{\infty} (a_{mn}^{(k)} - a_{mn}^{(k+1)}) x_n \right|$$

$$\leq \|u\| \operatorname{Sup}_m (\operatorname{Sup}_n |a_{mn}^{(k)} - a_{mn}^{(k+1)}| \, |x_n|)$$

$$\leq \|u\| \, \|x\| \operatorname{Sup}_m (\operatorname{Sup}_n |a_{mn}^{(k)} - a_{mn}^{(k+1)}|$$

$$= \|u\| \, \|x\| \operatorname{Sup}_{m,n} |a_{mn}^{(k)} - a_{mn}^{(k+1)}| .$$

Now, since $\{A^{(k)}\}$ is norm convergent, for $\varepsilon > 0$, there is an integer N such that for $k \geq N$

$$\operatorname{Sup}_{m,n} |a_{mn}^{(k)} - a_{mn}^{(k+1)}| < \frac{\varepsilon}{(\|u\| + 1)(\|x\| + 1)} .$$

Thus, for $k \geq N$

$$|u(A^{(k)} - A^{(k+1)})x| \leq \|u\| \, \|x\| \operatorname{Sup}_{m,n} |a_{mn}^{(k)} - a_{mn}^{(k+1)}| < \varepsilon .$$

Hence, $\{A^{(k)}\}$ is (c)-convergent or (m)-convergent, depending on whether $\{A^{(k)}\}$ belongs to $\Sigma(c)$ or $\Sigma(m)$.

The only change needed to prove part (b) is to let $\{u_m\}$ belong to (m), $\{x_n\}$ belong to (c_0), and $\{A^{(k)}\}$ be a sequence in $\Sigma(c_0)$. The rest of the argument follows as in the proof of (a). △

Using Example 2.35 it is possible to construct a sequence $\{A^{(k)}\}$ of matrices belonging to $\Sigma(c_0)$ that are (c_0)-convergent but are not norm convergent. Recall, Example 2.35 illustrated a sequence of convergent sequences that was (c)-convergent but not norm convergent. The general term, $x^{(n)}$, of this sequence is $x^{(n)} = \{x_m^{(n)} : m = 1, 2, 3, \ldots\}$ where

$$x_m^{(n)} = \begin{cases} 0 & \text{if } n < m \text{ or } n = m+1 \\ 1 & \text{if } n = m \\ p^{m-n-1} & \text{if } m \geq n+2 . \end{cases}$$

Let the matrix $A^{(k)}$ be the matrix whose first row is $x^{(k)}$ and has the remaining entries 0. It is clear that $A^{(k)}$ belongs to $\Sigma(c_0)$ for every k.

<u>Example 5.11</u>: $\{A^{(k)}\}$ is (c_0)-convergent but not norm convergent. In fact $\{A^{(k)}\}$ is (c_0)-convergent to the zero matrix. That $\{A^{(k)}\}$ is not norm convergent follows from Example 2.35. Since the first row of $A^{(k)} - A^{(k+1)}$ is the sequence $x^{(k)} - x^{(k+1)}$ and $\|x^{(k)} - x^{(k+1)}\| = 1$, it follows that $\|A^{(k)} - A^{(k+1)}\| = 1$ for all k.

To show that $\{A^{(k)}\}$ is (c_0)-convergent let $u = \{u_m\}$ belong to (m) and $x = \{x_n\}$ belong to (c_0). Now the sequence $\{uA^{(k)}x\}$ is $\{y_k\}$ where

$$y_k = u_1(x_k + p x_{k+2} + p^2 x_{k+3} + \ldots + p^j x_{k-j+1} + \ldots) .$$

If $u_1 = 0$ then $y_k = 0$ for all k. Suppose then that $u_1 \neq 0$. Since $x = \{x_n\}$ is in (c_0) there is an integer N such that for $n \geq N$,

$$|x_n| < \frac{\varepsilon}{|u_1|} .$$

Hence $|y_k| < \varepsilon$ if $k \geq N$. Thus y_k is a null sequence for every $x = \{x_n\}$ in (c_0). Therefore, $\{A^{(k)}\}$ is (c_0)-convergent to the zero matrix.

Considering the sequence of matrices $\{A^{(k)}\}$ of the preceding example it is possible to obtain a sequence $\{B^{(k)}\}$ of matrices, that belongs to $\Sigma(m)$, which is (m)-convergent but not norm convergent. Likewise it is possible to obtain a sequence $\{C^{(k)}\}$ of matrices, that belongs to $\Sigma(c)$, which is (c)-convergent but not norm convergent. To accomplish this let $B^{(k)} = C^{(k)} = A^{(k)'}$ where $A^{(k)'}$ is the transpose of $A^{(k)}$.

An interesting application of the work on the norm convergence of a sequence of matrices belonging to $\Sigma(c)$ is the proof of a p-adic analogue of Mercer's theorem [11]. For the student interested in further work in non-Archimedean sequence spaces, Dorleijn [6] has some additional work on choosing an (α, β)-convergent subsequence from an (α, β)-bounded sequence. This is analogous to the Bolzano-Weierstrass theorem for a bounded sequence of real numbers.

Since the Hahn-Banach theorem is available in a non-Archimedean field with a discrete valuation (See Snook [14, p. 106]), an individual interested in functional analysis can use this powerful theorem in further study of matrix transformations. The necessary and sufficient conditions obtained for a matrix to belong to $\Sigma(c)$, $\Sigma(c_0)$, or $\Sigma(m)$

can be used to show that if a matrix A belongs to one of these sets, it is a continuous linear functional. Rangachari and Srinivasan [11], state some results that can be obtained from the Hahn-Banach theorem, considering the matrix as a linear functional.

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