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JAMES A. BELL

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TRENDS IN SECONDARY MATHEMATICS IN RELATION TO

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APPROVED BY

Thomas J. Hill

Donald Reynolds

Chipman G. Stuart

Gene Levy

DISSERTATION COMMITTEE

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CHAPTER I

INTRODUCTION

The purpose of this study is to determine the influence of psychology on the secondary mathematics curriculum from 1893 to 1970. The various aims and recommendations for secondary mathematics have been examined, and the psychological background, if any, of these recommendations has been considered. Along with the direct influence on the curriculum, this study will relate broad trends in psychology and the resulting changes to the mathematics curriculum. To this end, the various national committee reports of the period have been examined for evidence of psychological implications. Professional journal articles, yearbooks of the National Council of Teachers of Mathematics, textbooks for prospective teachers, and materials of important writing groups will also be considered.

Review of Related Research

While no studies have been found which have the same emphasis as the one described above, several related works are in existence. These works detail the development of the mathematics curriculum, but emphasize the effect on this development of areas other than psychology. Those which have mentioned psychological implications have not covered the period in question. The study described above would broaden the history of curriculum development in mathematics by describing influences which have not been formally developed before.

Betz pointed out several views in relation to psychology which he felt have been detrimental to the mathematics curriculum. Dewey's pragmatic theory of knowledge and his insistence on both immediate and personal experience and social utility have caused the "cultural" aims of mathematics to be ignored. This has caused academic mathematics to become an elective in many of our schools. Kilpatrick's emphasis on "individual needs and interests" served in a similar capacity. E. L. Thorndike's mechanistic psychology and his faith in measurement and fixed IQ's have had considerable influence on the mathematics curriculum. Thorndike's efforts in undermining the basic ideas of "transfer of training" greatly affected the position of geometry in the high school.[1]

A further hindrance occurred when the Committee of Seven on Grade Placement in Mathematics reported their findings. This committee demanded a drastic postponement of

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[1]William Betz, "Five Decades of Mathematical Reform--Evaluation and Challenge," The Mathematics Teacher, XLIII (December, 1950), 377-87.

arithmetic instruction based on their research in "number readiness." The results of this committee's work were evident during the service examinations during World War II. Although this committee dealt with elementary mathematics, the result in secondary mathematics was a lowering of the level of instruction in the junior high school.[2]

Birdwell examined the reports of the Committee of Ten and the Committee of Fifteen. He noted several recommendations which have a psychological background. Both committees showed a general rejection of mental discipline in favor of an inductive approach for elementary arithmetic.

Other recommendations also centered about the use of the number line and arrays in multiplication. Further discussion occurred in the "decimal versus common fraction" controversy. Stressed too was the avoidance of unnecessary mechanical training and excessive thoroughness. Educators felt that thoroughness hampered acceptance of higher mathematics and wider generalizations.[3]

Fishman attempted to identify trends in secondary mathematics within the context of social change and to assess the present reforms in mathematics with respect to their effectiveness in correcting the inadequacies of current practice. His study revealed that the curriculum did respond

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[2] Ibid.

[3] James K. Birdwell, "A New Look at Old Committee Reports," The Mathematics Teacher, IXI (April, 1968), 383-87.

to social pressures. Those who favored social adjustment and vocational training prevailed over those who favored disciplined academic training. This is seen in a decrease in mathematics enrollments and a dilution of the mathematics content in schools.[4]

The reform movement of secondary mathematics was sparked by the concern for the inadequate preparation of the high school graduate, the demands of the new technology for superior training in mathematics, and the demand for programs which would assure the scientific supremacy of the United States. The results of this reform movement were a narrowing of the gap between university mathematicians and secondary mathematics teachers, a reorganization of the mathematics curriculum, and an emphasis on the teaching of the fundamental structure of mathematics. It also produced an intellectual ferment among mathematics teachers and encouraged them to improve their background.[5]

Hancock (1961) sought to ascertain the various aims and recommendations for secondary mathematics as specified by various national committees, to trace the history of major individual aims throughout the period, and to find those aims which were active at that time. Hancock discovered that in periods when society does not see a pressing need

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[4]Joseph Fishman, "Trends in Secondary School Mathematics in Relation to Educational Theories and Social Change" (unpublished Ph.D. dissertation, New York University, 1965).

[5]*Ibid*.

for mathematics, emphasis is placed on utilitarian aims of instruction, provided the economy is active. Cultural aims were emphasized during periods of economic depression. When the need for mathematics was evident, there had been a tendency to assume that whatever mathematics could be taught could be justified.[6]

Sigurdson gives a chronological development of unified courses, beginning with Truman Harry Safford's description in 1887 of a course in which algebra and geometry were taught in a parallel manner. The English influence is shown along with the influence of the University of Chicago. Mention is made of the growing distrust of "mental discipline," which aided the development of "social" mathematics.[7]

Yasin presents the history of the reform movement in four periods. The period from 1900 to 1914 saw the birth of a reform movement with an eventual aim of correlating mathematics with science and other fields of knowledge. 1914 to 1928 saw the decline of mathematics in the secondary curriculum. A general relaxation occurred during this period along with a lessening of the mathematics requirements for graduation.

1928 to 1940 saw educators claiming that mathematics

[6] John Hancock, "The Evolution of the Secondary Mathematics Curriculum: A Critique" (unpublished Ed.D. dissertation, Stanford University, 1961).

[7] Solherg Einar Sigurdson, "The Development of the Idea of Unified Mathematics in the Secondary School" (unpublished Ph.D. dissertation, University of Wisconsin, 1962).

beyond the ninth grade was of no need to the majority of students. This period also saw courses being delayed in a desire to obtain a better understanding due to maturity. This period also saw the appearance of secondary mathematics in the colleges. Remedial courses were also being offered. Research in mathematics, recognition of the gifted student, and competition with Russia characterized the post-war period.[8]

Problems to be Investigated

The basic problems of this study may be outlined as follows:

1. In what ways have the recommendations and aims of national committees been influenced by psychology?
2. How has the mathematics curriculum been directly influenced by psychology?

Design of the Study

The study will consider the influences of faculty psychology, connectionism, Gestalt psychology, and developmental psychology on the various committee recommendations. As a service to the reader, a brief description of the committee recommendations will be included in the second chapter. A final chapter will consider the influence of modern psychologists, including Piaget, Bruner, Gagné, and Skinner.

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[8] Said Taha Yasin, "The Reform Movement in Secondary Mathematics--Its History and Present State" (unpublished Ph.D. dissertation, Indiana University, 1961).

The study will begin with the report of the Committee of Ten in 1893, as this date marks the beginning of large-scale investigation of the secondary curriculum. The period under investigation will end in 1970 with the publication of two important yearbooks, the Thirty-second Yearbook of the National Council of Teachers of Mathematics and the Sixty-ninth Yearbook of National Society for the Study of Education. The committee reports during the period from 1893 to 1970 will be examined along with reports of various writing groups such as the School Mathematics Study Group and the Secondary School Mathematics Curriculum Improvement Study. Other sources will be yearbooks of the National Council of Teachers of Mathematics and the National Society for the Study of Education, textbooks in mathematics education, and articles in periodicals.

Significance of this Study

The study will provide an examination of the evolution of the mathematics curriculum that differs significantly from previous efforts. While covering the same time period as other studies, the emphasis on psychological influences will provide today's educator with a more pedagogically oriented explanation of the trends in our mathematics offerings. This study, along with related studies, will provide the prospective teacher and modern educator with an understanding of why the curriculum has evolved in the manner that it has. Of significance too will be the explanation and examination of trends that have flowed throughout both fields. This will provide the educator with a historical source when problems similar to those examined in this paper arise in the future.

CHAPTER II

COMMITTEE RECOMMENDATIONS

The appointment of the Committee of Ten on Secondary School Studies marked the beginning of large scale curriculum investigation in the United States. The twentieth century was to see several committees formed with such a purpose in mind. The various committees and their sponsoring organizations played an important role in the development of the curriculum of the twentieth century. Widespread distribution and top calibre personnel helped bring about this influence. Their recommendations, whether accepted blindly or subjected to lengthy debate, were formidable stimuli for the evolving secondary curriculum.

The Committee of Ten was appointed by the National Education Association in 1892 and made its final report in 1893. The committee recommended a drastic revision of elementary arithmetic. They felt that the teaching should be more concrete, that more attention should be paid to correctness in work, and that portions of arithmetic should be eliminated. Some of the topics that they felt could be omitted included cube root, duodecimals, and compound proportion.[1]

To fill the gaps in the elementary grades, topics in concrete geometry were to be introduced. Algebraic expressions and algebraic symbols, later referred to as "literal arithmetic," were also to be introduced in the seventh and eighth grades. Formal algebra was to begin at age fourteen with five class periods per week of instruction. Beginning in the tenth grade, algebra and geometry were each to be studied two and one-half hours per week. This simultaneous teaching was to last for two years. The senior year consisted of trigonometry and advanced algebra.[2] The course of study for the non-college bound departed from the one described above after the ninth grade. Bookkeeping or commercial arithmetic provided training for this category of student.[3]

Correlation of studies was considered by the Committee of Ten. Arithmetic and geometry were the subjects where possibilities for correlation were mentioned.

> . . . The whole work in concrete geometry will connect itself on the one side with the work in arithmetic, and on the other with elementary instruction in physics. With the study of arithmetic is therefore to be intimately associated the study of algebraic signs and forms, of concrete geometry, and of elementary physics. Here is a striking instance of the interlacing of subjects

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[1]National Education Association, Report of the Committee of Ten on Secondary School Studies (New York: American Book Company, 1894), p. 105.

[2]Ibid., pp. 23-24.

[3]Ibid., p. 107.

which seems so desirable to every one of the nine conferences.[4]

It might be mentioned that the following sentence has been of considerable importance. "It believes that if the introductory course in geometry has been well taught, both plane and solid geometry can be mastered at this time."[5] Many have felt that this proposed fusion of solid and plane geometry.

The committee insisted on quickness and accuracy in written and oral work. Stress also was placed on "accuracy of statement and elegance of form" in the presentation of geometry theorems.[6]

The most important aspect of the Committee of Ten was that it marked the beginning of investigations of the mathematics curriculum. Along with being the "first," the committee made several recommendations which did influence mathematics education in the United States.

The Report of the Committee of Fifteen on Elementary Education of the National Education Association (1895) is included here because of the discussions concerning mathematics in grades seven and eight. With the feeling that the higher arithmetic of the day could be handled more easily by elementary algebra, the committee recommended that

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[4]*Ibid.*, p. 24.

[5]*Ibid.*, p. 106.

[6]*Ibid.*, pp. 106-112.

elementary algebra be introduced in the seventh grade. This presentation was to include the solution of first degree equations and solution of arithmetical problems in proportion. Quadratic equations were to be introduced in the eighth grade as were solutions of any arithmetic problems that could be solved in a more satisfactory manner by algebraic methods. This was also to provide an introduction to algebra, thus making the transition somewhat smoother.[7]

The National Education Association appointed the Committee on College Entrance Requirements at their Denver meeting in 1895. Their final report was published in 1899. The Committee on College Entrance Requirements cooperated with a committee of the Chicago Section of the American Mathematical Society. The point of interest in the American Mathematical Society committee was that non-Society members were utilized to cover all phases of mathematics teaching.[8]

The Committee on College Entrance Requirements admitted general agreement with the Committee of Ten. However, specific recommendations showed a slightly different approach.

Seventh grade - Concrete geometry and introductory algebra
Eighth grade - Introductory demonstrative geometry

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[7]National Education Association, Report of the Committee of Fifteen on Elementary Education (New York: American Book Company, 1895), pp. 55-58.

[8]National Education Association, Report of Committee on College Entrance Requirements (Chicago: The University of Chicago Press, 1899), p. 135.

and algebra
Ninth and tenth grades - Algebra and plane geometry
Eleventh grade - Solid geometry and plane trigonometry
Twelfth grade - Advanced algebra and mathematical reviews
(each four periods of at least forty-five minutes)[9]

The introductory algebra of the seventh and eighth grades could be generally called literal arithmetic. The eighth grade demonstrative geometry was basically restricted to the more easily proved theorems of plane geometry. The ninth and tenth grades saw algebra and plane geometry drawing equal time both years.[10]

Two comments from the report of the Committee of the Chicago Section of the American Mathematical Society were important. First, they felt that a student should study mathematics during all four years of high school. Second, the committee made this statement about solid geometry.

> The attempt has been successfully made to teach geometry by interweaving solid and plane geometry from the outset. While the committee is not prepared to commend this, there are advantages to be gained by beginning solid geometry before plane geometry is completed."[11]

The International Commission on the Teaching of Mathematics was created by the Fourth International Congress of Mathematicians in 1908. The American Commissioners, under the leadership of J. W. A. Young, Eugene Smith, and William F. Osgood, held their first meeting in 1909. The work of the American Commissioners and their appointed committees

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[9] Ibid., p. 21.

[10] Ibid., pp. 21-22.

[11] Ibid., p. 123.

was published by the United States Bureau of Education from 1911 to 1918.[12] The general purpose was not one of making recommendations, but one of producing a general survey of mathematics and mathematics teaching in the United States. Although differing from other curriculum study groups, their investigations were indicative of the curriculum of the day.

The commission mentioned placing elements of algebra and "intuitive" geometry in the seventh and eighth grades. With many topics in arithmetic being omitted, there was "room" for such action. Earlier committees had recommended that certain topics be dropped. The algebra mentioned for grades seven and eight could have been called literal arithmetic. The purpose of introducing literal arithmetic was to make many of the problems of arithmetic easier. A second reason for such an introduction was that a knowledge of simple formulas, such as could be obtained by the study of literal arithmetic, was becoming essential. Numerous handbooks and journals of the day had formulas in them.[13]

Introduction of "observational," "intuitive," "constructive," or "concrete" geometry in the eighth grade had

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[12] International Commission on the Teaching of Mathematics, The American Report, Report of the American Commissioners of the International Commission on the Teaching of Mathematics, United States Bureau of Education Bulletin 1912, no. 14 (Washington, D.C.: Government Printing Office, 1912), p. 5.

[13] Ibid., pp. 18-19.

not met success by 1910. Reasons given were that many teachers were poorly trained in geometry <u>or</u> that they had been informed by an "authority" that geometry had no place in the curriculum. However, the committee predicted that geometry would find a place in the seventh and eighth grades, and even in the elementary school.[14]

Caution was the watchword when the committee considered practical applications. While agreeing that the work should appeal to the student's interests and activities, the commission felt that this type of presentation did not contribute to broadening the mind or add to the general culture. They did say that this type of presentation was suitable in some cases, such as industrial schools.[15]

Simultaneous teaching of algebra and geometry received considerable mention. Criticism grew out of the inability of ninth graders to handle demonstrative geometry. Proponents felt that interaction between the subjects and the postponement of difficult portions of algebra outweighed such disadvantages.

Some presentations included three hours of algebra and two hours of geometry weekly over a period of two years, instead of a year of algebra and a year of geometry. This method was criticized by the commission because they felt it was too mechanized. Their feelings are shown below:

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[14] *Ibid.*, pp. 19-20.

[15] *Ibid.*, p. 22.

> It must not be understood that the simultaneous teaching of algebra and geometry means an equal distribution of the weekly time between the two subjects. Such a mechanical attempt at simultaneous teaching would offer the minimum promise of success. Simultaneous teaching as used above means rather a "block system," a distribution of the subject matter so that in each year blocks of geometry and algebra alternate, each block having a measure of unity and completeness in itself. The present curriculum for the first two years may be described as a system of two blocks. Those who may be disposed to try simultaneous teaching would find a system of four blocks for these years a conservative and advisable beginning. If this should prove successful, the blocks may gradually be made smaller as experience indicates.[16]

The American Commissioners were not yet ready for "fused" courses.

> Nor should simultaneous teaching be understood to imply a "fusion" of the two subjects into a homogeneous whole which is neither one subject nor the other. Such a fusion is impossible, owing to the inherently diverse characters of the two subjects. It would in any event be undesirable and attempts at approximation to it are not to be commended.[17]

A change was to be seen before the report of the National Committee on Mathematical Requirements in 1923.

The Commission on the Reorganization of Secondary Education was appointed by the National Education Association. Their findings were reported in the first Bulletin of the Bureau of Education in 1920. While the commission was unable to make final recommendations concerning the course of study, their report was of significance as they were the first curriculum study group to recommend teaching mathematics as a "laboratory" course. "Introductory mathematics--

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[16] Ibid., pp. 29-30.

[17] Ibid.

ordinarily conceived as separate courses in algebra, geometry, and trigonometry--should be given in connection with the solving of problems and the executing of projects in fields where the pupils already have knowledge and interest."[18] Here, the commission felt that utilization of the interests and needs of the child made for better learning.

The commission made a distinction between the groups of people who would use mathematics.

> (a) The "general readers," who will find their use of mathematics beyond arithmetic confined largely to the interpretative function described above.
> (b) Those whose work in certain trades will make limited, but still specific, demand for the "practical" use of mathematics.
> (c) Those whose practical work as engineers or as students of certain sciences requires considerable knowledge of mathematics.
> (d) Those who specialize in the study of mathematics with a view either to research or to teaching or to the mere satisfaction of extended study in the subject.[19]

Along with previous committees, the commission was in favor of removing topics from the curriculum. They felt that many topics were too difficult and were not necessary. "No item shall be retained for any specific group of pupils unless, in relation to other items and to time involved, its (probable) value can be shown."[20] They were specific as to

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[18] National Education Association, Commission on the Reorganization of Secondary Education, The Problem of Mathematics in Secondary Education, United States Bureau of Education Bulletin 1920, no. 1 (Washington, D.C.: Government Printing Office, 1920), p. 12.

[19] Ibid., p. 16.

[20] Ibid., p. 15.

items to be omitted. Among these topics were radical equations, least common multiple, and greatest common factor.[21]

Four recommendations were made concerning the work of grades seven, eight, and nine. They felt that these years should include:

> A. A body of processes and conceptions commonly called arithmetic, where the study, however is of social activities--trade or otherwise--which need mathematics, rather than of mathematical topics artificially motivated by social relationships.
> B. A body of mathematical symbols, concepts, information, and processes--commonly thought of as belonging to algebra and geometry or beyond--which the intelligent general reader of high school or college standing will need in order to meet the demands of his social and intellectual life.
> C. The opportunity for at least a preliminary testing of mathematical taste and aptitude.
> D. Such additional content--relatively small in amount--as may be needed to make effective the teaching of the foregoing.[22]

The National Committee on Mathematical Requirements, sponsored by the Mathematical Association of America, was organized during the summer of 1916. Although sponsored by the Mathematical Association of America, there was cooperation with such mathematics teacher organizations as The Association of Teachers of Mathematics in New England, The Association of Teachers of Mathematics in the Middle States and Maryland, and The Central Association of Science and Mathematics Teachers. The final report of this committee, published in 1923, had a tremendous influence on mathematics

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[21]*Ibid.*, p. 19.

[22]*Ibid.*, p. 22.

education in the United States.[23]

Significant recommendations were made by this committee in the areas of curriculum development and instructional aims. The aims of mathematics instruction were placed in three classes: (1) practical or utilitarian, (2) disciplinary, and (3) cultural. Practical aims were to provide the fundamental processes necessary in the life of the individual. Disciplinary aims were related to mental training and general character training. These were to also include formation of mental habits which are applicable not only to the setting in which they are developed, but to situations which are similar. This was the idea of "transfer of training." Cultural aims were to develop appreciation of beauty, ideals of perfection, and appreciation of the power of mathematics.[24]

At the time of the committee report, there was a movement toward removal of barriers between subject matter areas, thus seeking a more effective approach to the teaching of mathematics. Although the recommendation was not all encompassing, the committee recognized that such an arrangement could be beneficial in some situations. Their recommendation was centered on the junior high school, grades seven, eight, and nine. Here they proposed a course of study which

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[23] National Committee on Mathematical Requirements, The Reorganization of Mathematics in Secondary Education (Washington, D.C.: Mathematical Association of America, Inc., 1923), vii.

[24] Ibid., pp. 6-9.

was to be taken by all students, regardless of ability, social class, or vocational plans. This was a course in "general" mathematics. Topics included arithmetic, algebra, intuitive geometry, numerical trigonometry, demonstrative geometry, and history of mathematics.[25]

The program for grades ten through twelve was arranged in four plans, with no single plan being "better" than another. These plans provided for the student who was destined to take the complete course in high school mathematics. The four plans were:

PLAN A.

Tenth year: Plane demonstrative geometry, algebra.
Eleventh year: Statistics, trigonometry, solid geometry.
Twelfth year: The calculus, other elective.

PLAN B.

Tenth year: Plane demonstrative geometry, solid geometry.
Eleventh year: Algebra, trigonometry, statistics.
Twelfth year: The calculus, other elective.

PLAN C

Tenth year: Plane demonstrative geometry, trigonometry.
Eleventh year: Solid geometry, algebra, statistics.
Twelfth year: The calculus, other elective.

PLAN D

Tenth year: Algebra, statistics, trigonometry.
Eleventh year: Plane and solid geometry.
Twelfth year: The calculus, other elective.[26]

One point to consider is that these plans provided for teaching plane and solid geometry together or separately. The four plans also made provision for either a change from or an agreement with the "simultaneous" teaching of algebra

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[25] Ibid., pp. 12-13.

[26] Ibid., p. 40.

and geometry mentioned in the report of the Committee of Ten. Calculus as a high school subject was to include derivatives and their applications, summation methods, integrals, and applications of integrals to simple problems in motion, area, and volume.[27] Another important point to consider is the role given to functions. The function concept was to aid in unifying secondary mathematics. To this end, an entire chapter was devoted to functional relations in algebra and geometry.[28]

There was also provision for elective courses in the mathematics program. These were to serve the needs of students with special interests, along with additional "pure" mathematics electives. These included mathematics of investment, shop mathematics, surveying and navigation, descriptive geometry, and projective geometry.[29]

The decade of the thirties was later to be called "The Progressive Era." Two major committee reports done during the thirties showed the general influence of the theories of progressive education. The Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics published their report in the Fifteenth Yearbook of the National Council of Teachers of Mathematics. The Committee on the Function of Mathematics in

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[27] Ibid., p. 38.

[28] Ibid., p. 72-73.

[29] Ibid., p. 39.

General Education, a committee of the Progressive Education Association, presented their ideas in Mathematics in General Education in 1940. The degree to which the two committees stressed progressive ideas was different. Another fundamental difference in the two was the general nature of their recommendations. The Joint Commission set up a program similar to those in the better schools in the country. Many of its suggestions had already been tested. On the other hand, the Commission on the Function of Mathematics in General Education was more idealistic. They proposed a course that they felt mathematics education would follow in the years to come. Their recommendations were to be tested and verified by experimentation.[30]

Important considerations of the Committee on the Function of Mathematics in General Education were centered in two areas, the place of mathematics in the secondary curriculum and broad concepts in mathematics. The committee felt that the curriculum should be designed to contribute to the aims of general education. With the needs and interests of the child in mind, the committee placed needs of the student in four categories. These were:

1. Personal Living
2. Immediate Personal-Social Relationships
3. Social-Civic Relationships
4. Economic Relationships[31]

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[30] Committee on the Function of Mathematics in General Education, *Mathematics in General Education* (New York: D. Appleton-Century Company, Inc., 1940), p. 14.

[31] *Ibid.*, p. 20.

"Personal living" referred to permitting the individual to find his place in this world. By "immediate personal-social relationships," the committee meant forming meaningful relationships with peers and adults. Self-reliance was also to be established. "Social-civic" relationships were to include extending the student's concern for society, making him aware of the social implications of his own activities, and increasing his participation in social movements. "Economic relationships" were to encompass more than obtaining a livelihood. Here, the student was to see his place as a future consumer, and to see the important role of mathematics in our society. With these as needs, it remained to see the role mathematics was to play.[32]

Mathematics was to be a tool in solving basic problems in living, a bridge connecting various disciplines, a source of appreciation of the values of tolerance and cooperation, and a factor attributing to general mental discipline. To reach this goal, the committee sought to discuss broad areas of mathematics which were directly applicable to problem solving. These included the following major concepts: formulation and solution, data, approximation, function, operations, proof, and symbolism. No strict course outlines were presented to implement these ideas.[33]

The work of the Joint Commission had a progressive

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[32] Ibid., pp. 20-23.

[33] Ibid., pp. 59-63.

flavor. The Joint Commission's aims were to provide the child with the training to meet the demands of modern life and the appreciation to recognize the cultural contributions of mathematics. They recommended the utilization of "broad areas" or "centers of interest." By utilizing broad categories, it was felt that substantially more of the needs of the students could be anticipated.[34]

Two different curriculum plans were submitted by the commission. The first was designed for "pupils of normal ability who have had good training." The seventh and eighth grades were to be treated as a unit involving work in arithmetic, elementary algebra, and informal geometry. Grade nine was a course in general mathematics with most of the concentration being in algebra. Geometry was central in the tenth grade, along with some more work in algebra. Using the function as a unifying concept, algebra and trigonometry were correlated in the eleventh grade. The senior year was a "unified" course covering material in advanced algebra, solid geometry, analytic geometry, trigonometry, and differential calculus.

The second plan provided for more arithmetic in the ninth grade. The general course in the ninth grade

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[34]Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics, "The Mathematics Curriculum," The Place of Mathematics in Secondary Education, Fifteenth Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1940), pp. 58-59.

corrected weakness in arithmetic, provided training for other subjects, and offered broad training which would benefit all. Larger schools were to offer an algebra course in the ninth grade. The tenth grade was still essentially geometry, although this course could be taken by those who did not expect to go to college. Algebra was the central theme in the junior year, with electives available for the twelfth grade. These included semester courses in trigonometry, solid geometry, social-economic arithmetic, and college algebra.[35]

World War II saw several committees which concerned themselves with the emergency of wartime. The War Preparedness Committee (1941) represented a joint effort of the MAA and the AMS. Their recommendations were centered principally on military needs. Among further recommendations were:

> 1. We recommend that, in connection with emphasis on so-called socialized aspects of secondary curricula, a liberalized definition of socialized mathematics should be adopted for students at all ability levels, in contrast to more narrow definitions which give unique prominence to business applications and consumer interests.
> 2. We strongly recommend that a single set of courses be used in any high school for students of appropriate ability in attaining desired ends relating to industry, military service, or future college education.
> 3. At all stages of secondary mathematics we recommend emphasis on applications.
> 4. We advise the evening schools in cities to give new emphasis to courses in advanced high school mathematics through the stage of trigonometry.
> 5. The military, industrial, and scientific utility of a considerable quantity of space intuitions and at least a little spherical trigonometry, causes us to recommend that the high school work in solid geometry, both

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[35] *Ibid.*, pp. 72-119.

intuitional and demonstrative, be given more prominence than in recent years.[36]

The Committee on Essential Mathematics for Minimum Army Needs published a lengthy report in 1943. Again, as the title suggested, the aim had a military flavor. Two other important recommendations were centered on the mathematically talented student. Provisions were made in advanced courses for removal of deficiencies in prior training. These courses were also to be examined for irrelevant material and for wartime applications. Grouping was also recommended according to age, as age was the determinant in army induction.[37]

The Commission on Post-War Plans was organized by the NCTM. Their first report, published in 1944, proposed three series of mathematics courses, sequential mathematics, related mathematics, and social mathematics. Sequential mathematics was to be an improved version of the traditional four year high school sequence. Efforts were to be made to increase the mathematical rigor and to provide for more continuity. Related mathematics was to be the course designed to meet the needs of those people who were to enter industry

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[36]William L. Hart, "Progress Report of the Subcommittee on Education for Service of the War Preparedness Committee of the American Mathematical Society and the Mathematical Association of America," The Mathematics Teacher, XXXIV (November, 1941), 301-2.

[37]Committee on Essential Mathematics for Minimum Army Needs, "Essential Mathematics for Minimum Army Needs," The Mathematics Teacher, XXXVI (October, 1943), 281.

at an early age. Their courses were to be molded from the existing courses in general mathematics. Social mathematics was to ensure mathematical competency in everyday human affairs. These courses included problems in taxation, insurance and social security.[38]

The second report appeared in 1945. With two broad aims in mind, providing mathematics for scientific training and providing functional competence for all, the committee proposed thirty-four theses concerning the mathematics program.

There was a general opposition against using mathematics singly as a tool subject. Furthermore, they felt that teaching arithmetic through the pupils' "interests and needs" had several limitations. The junior high program in grades seven and eight was to be organized around broad categories, an idea proposed by the Joint Commission. In schools with enrollments over 200, it was recommended that both general mathematics and algebra be offered in the ninth grade.[39]

Concerning the high school curriculum, the committee stressed that sequential courses were only for those of high ability. With this in mind, these courses stressed

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[38] Commission on Post-War Plans, "The First Report of the Commission on Post-War Plans," The Mathematics Teacher, XXXVII (May, 1944), 228.

[39] Commission on Post-War Plans, "The Second Report of the Commission on Post-War Plans," The Mathematics Teacher, XXXVIII (May, 1945), 204-8.

understanding of concepts and principles, with the goal being application to social situations.[40]

Practical applications were to be used more frequently than in the past. This was to provide motivation, illustration, and transfer. Practical applications were present in the teacher-training program recommended by the committee.

> He should have opportunity for experience in such fields as the general shop, machine shop, the making and reading of simple blueprints, and surveying.[41]

All the committees during the forties had one general characteristic--they all tried to meet the needs of a nation at war. To accomplish this end meant that the "social utility" of the progressives had to be altered.

The Commission on Mathematics of the College Entrance Examination Board was appointed in 1955 as a result of concern about the mathematics curriculum. Increases in technology, research in mathematics, and the need for mathematics in other disciplines were stimuli for a critical survey of the secondary mathematics program. Four years of study culminated in their report in 1959 when they published their proposal for a new program for high school mathematics.[42]

While concerned for the college capable, the

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[40] Ibid., pp. 208-9.

[41] Ibid., p. 219.

[42] College Entrance Examination Board, Commission on Mathematics, Program for College Preparatory Mathematics (New York: College Entrance Examination Board, 1959), pp. 3-9.

commission did set forth a list of objectives of mathematics for general education.

> 1. An understanding of, and competence in, the processes of arithmetic and the use of formulas in elementary algebra. A basic knowledge of graphical methods and simple statistics is also important.
> 2. An understanding of the general properties of geometrical figures and the relationships among them.
> 3. An understanding of the deductive method as a method of thought. This includes the ideas of axioms, rules of inference, and methods of proof.
> 4. An understanding of mathematics as a continuing creative endeavor with aesthetic values similar to those found in art and music. In particular, it should be made clear that mathematics is a living subject, not one that has long since been embalmed in textbooks.[43]

The program proposed for the college-bound was not a radical change from the traditional content. The commission recommended modification and addition of topics instead of dropping the traditional curriculum. The discovery approach was stressed, time permitting, as was the importance of interrelationships between courses. The general characteristics of the recommendations were outlined by the commission as follows.

> 1. Strong preparation, both in concepts and in skills, for college mathematics at the level of calculus and analytic geometry.
> 2. Understanding of the nature and role of deductive reasoning--in algebra, as well as in geometry.
> 3. Appreciation of mathematical structure ("patterns")--for example, properties of natural, rational, real, and complex numbers.
> 4. Judicious use of unifying ideas--sets, variables, functions, and relations.
> 5. Treatment of inequalities along with equations.
> 6. Incorporation with plane geometry of some coordinate geometry, and essentials of solid geometry and space perception.

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[43] Ibid., p. 11.

7. Introduction in grade 11 of fundamental trigonometry--centered on coordinates, vectors, and complex numbers.
8. Emphasis in grade 12 on elementary functions (polynomial, exponential, circular.)
9. Recommendation of additional alternative units for grade 12: <u>either</u> introductory probability with statistical applications <u>or</u> an introduction to modern algebra.[44]

Probably because of the general nature of the College Entrance Examination Board, courses such as "consumer mathematics" were not mentioned.

For grades nine to twelve, the Commission proposed courses labeled Elementary Mathematics I, Elementary Mathematics II, Intermediate Mathematics, and Advanced Mathematics. The work of grades nine and ten was designated Elementary Mathematics I and II, while Intermediate Mathematics covered grade eleven. These courses were to serve as the entrance requirements for all college-bound students. Topics included were algebra, geometry, trigonometry, and introductory statistics.[45]

Specific recommendations were made covering grades nine to eleven. Stress in algebra was to be on structure rather than on manipulation. Geometry received considerable mention. Along with the introduction of coordinate geometry and the fusion of plane and solid, the commission recommended the reduction of the number of theorems and the

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[44] <u>Ibid.</u>, pp. 33-34.

[45] <u>Ibid.</u>, pp. 20-30.

introduction of "other geometries."[46]

Mathematics in grade twelve, Advanced Mathematics, was to be centered around the study of polynomial, exponential, logarithmic, and circular function. This course was to be one or two semesters in length, depending upon the school system. If a one semester presentation was used, the second semester was devoted to the study of modern algebra or probability with statistical applications.[47]

The commission also took a firm stand on calculus in the high school. They felt that calculus was a college-level subject. "The Commission cannot recommend the practice of exposing college preparatory students to formal calculus for a short time at the end of the twelfth grade. Such anticipation tends to breed overconfidence and blunt the exciting impact of a thorough presentation."[48]

The influence of the Commission on Mathematics was very great. Although the final report did not appear until 1959, preliminary reports were well circulated throughout the United States. E. G. Begle felt that the forming of the Commission on Mathematics was "probably the most important step in the improvement of the mathematics curriculum in the United States."[49] Of importance too was the fact that the

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[46] Ibid.

[47] Ibid., pp. 30-33.

[48] Ibid., pp. 14-15.

[49] Alan R. Osborne and F. Joe Crosswhite, "Reform, Revolution, and Reaction," A History of Mathematics Education

commission actually wrote curriculum materials, Introductory Probability and Statistical Inference, something that earlier national committees had not done.

The fifties and early sixties saw the birth of curriculum study groups which shared one characteristic--they produced their own materials. Some of these groups were the School Mathematics Study Group (SMSG), the University of Illinois Committee on School Mathematics (UICSM), the Boston College Mathematics Institute (BCMI), the Greater Cleveland Mathematics Program (GCMP), the Syracuse University--Webster College Madison Project, the University of Maryland Mathematics Project (UMMaP), the Ontario Mathematics Commission (OMC), and the Developmental Project in Secondary Mathematics at Southern Illinois University.[50] Those with considerable influence were SMSG, because of the vastness of their program; UICSM, being the "first" to attempt the task; and UMMaP, whose seventh and eighth grade materials and coordinator influenced the SMSG writing teams. By far the most important program was SMSG.

The School Mathematics Study Group came into being as a result of two conferences. These were the Chicago

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in the United States and Canada, Thirty-second Yearbook of the National Council of Teachers of Mathematics (Washington, D.C.: National Council of Teachers of Mathematics, 1970), p. 266.

[50] National Council of Teachers of Mathematics, An Analysis of New Mathematics Programs (Washington, D.C.: National Council of Teachers of Mathematics, 1963), pp. 1-2.

Conference on Research Potential and Training and the Mathematics Meeting of the National Science Foundation. The results of these meetings were: (1) a four or five week writing session was scheduled the next summer to prepare a detailed syllabus for a model secondary mathematics program; and (2) arrangement for publication of monographs on mathematical topics for the secondary school student.[51]

SMSG felt it was "important that mathematics be so taught that students will be able in later life to learn the new mathematical skills which the future will surely demand of them."[52] To meet this objective, SMSG listed three requirements.

> . . . First we need an improved curriculum which will offer students not only the basic mathematical skills but also a deeper understanding of the basic concepts and structure of mathematics. Second, mathematics programs must attract and train more of the students who are capable of studying mathematics with profit. Finally, all help possible must be provided for teachers who are preparing themselves to teach these challenging and interesting courses.[53]

Houston T. Karnes, a member of the seventh and eighth grade writing teams, mentioned these aims and characteristics of the SMSG materials in a seminar at Louisiana State University during the summer of 1967.

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[51] William Wooton, SMSG The Making of a Curriculum (New Haven, Connecticut: Yale University Press, 1965), pp. 9-11.

[52] Osborne and Crosswhite, "Reform, Revolution and Reaction," p. 275.

[53] *Ibid.*

1. Uses intuitive approach as opposed to rigid formal presentation.
2. Uses a discovery method.
3. Is written in a readable manner.
4. Is teachable.
5. Constantly reinforces concepts by use of the "spiral approach"--gradual introduction to major ideas.
6. Uses careful precise language in defining mathematical terms and phrases.
7. Presents an axiomatic approach to algebra.
8. Written for the "college capable" (Early editions).

After writing, testing, reviewing, and rewriting, texts were published for school use. These texts were to serve as guidelines for secondary mathematics until the commercial publishers began producing similar texts. After the original seven through twelve program, several other ventures were completed. An elementary series was published in 1964. Senior level courses covering analytic geometry and calculus were published in 1964 and 1965, respectively. Texts for the below-average student in junior high school were published in 1962.[54] A junior high series was published which was aimed at the student with a traditional background. Texts for slower students and culturally disadvantaged children, units on probability, films for elementary school teachers, enrichment materials, and other supplementary materials were also produced. Part of a new junior high school series is to be made available in September, 1970.[55]

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[54] Ibid., p. 274.

[55] School Mathematics Study Group, SMSG Publications, Newsletter no. 32 (Pasadena, California: A. C. Vroman, Inc., March, 1970), pp. 6-8.

The School Mathematics Study Group was influenced by the Commission on Mathematics. Their courses generally followed the recommendations of the commission. Major breaks from tradition came in the geometry course. Their first attempt utilized the real numbers and the ruler and protractor postulates. A later effort produced Geometry With Coordinates.[56]

Other departures from tradition came in the materials in the seventh and eighth grades. The writing teams felt that there was too much emphasis on social mathematics in these grades. They stressed structure and formed their course around topics from algebra, arithmetic, geometry, and trigonometry. Applications were to be secondary to structure, and drill on previous material was accomplished by introduction of new topics which utilized previous ideas. Although led by the UMMaP program, there was a significant difference. SMSG's course did not constitute, upon completion, a year of algebra.[57]

The writing teams of SMSG consisted of people in pure mathematics, mathematics education, and industry. Although not the first curriculum group to do so, SMSG sought diversification in the making of their writing teams. SMSG also evaluated their programs intensely and published extensive

[56]Osborne and Crosswhite, "Reform, Revolution, and Reaction," p. 278.

[57]Ibid., p. 277.

teacher's commentaries.[58]

The influence of the School Mathematics Study Group can be seen in the following statement. "The SMSG publications, without a doubt, constitute the most extensively used experimental materials in school mathematics at the present time."[59]

In regard to the future plans, SMSG will concentrate in three areas, research in learning theory in mathematics, production of more materials for students whose achievement in mathematics is below average, and increased emphasis on applications of mathematics in other disciplines.[60]

The Secondary School Curriculum Committee of the National Council of Teachers of Mathematics published their report in the May, 1959 issue of The Mathematics Teacher. While being sponsored by an important group, this report had relatively minor significance as its curricular recommendations were quite similar to the Commission on Mathematics of the College Entrance Examination Board.[61]

However, there were several recommendations that deserve mention here. The case for solid geometry was explained fully.

The conclusion that a full semester of deductive

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[58] Ibid., p. 281.

[59] Ibid.

[60] Ibid., p. 280.

[61] Ibid., p. 266.

> solid geometry is not desirable rests on two opinions, which represent the consensus of this committee, First, solid geometry is not a good place to study deductive proof. Many of the proofs tend to be long and cumbersome and their presentation does little to enhance the understanding of deductive methods that the pupil gained in plane geometry. Second, on the basis of relative values, deductive solid geometry is not worth a full semester of the pupil's time.[62]

With the "extra room" provided by the exclusion of solid geometry and the moving of trigonometry to the junior year, the committee provided the following program for the college-preparatory student.

> Ninth grade - algebra
> Tenth grade - geometry (plane with some solid)
> Eleventh grade - algebra, trigonometry
> Twelfth grade - any two of the following semester courses: Probability and statistics; analytic geometry; mathematical analysis based on a study of functions (algebraic, trigonometric, exponential, and logarithmic). Some schools might find desirable a strong course in analytic geometry and calculus as preparation for the Advanced Placement Examinations.[63]

An alternative to general mathematics in the ninth grade was mentioned. The committee cited cases in which first year algebra was taught more slowly for the slower group. The gifted child was allowed to do individual work or explore deeper into subject matter. Coordinate geometry was included in the geometry course and throughout the curriculum.[64]

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[62] National Council of Teachers of Mathematics, Secondary School Curriculum Committee, "The Secondary Mathematics curriculum," The Mathematics Teacher, LII (May, 1959), 405.

[63] Ibid., pp. 405-6.

[64] Ibid.

The common practice of omitting topics had a role in the committee's decision-making. They stressed the elimination of tricky factoring, simultaneous quadratics, business mathematics, and extensive computational problems in both algebra and geometry.[65]

The committee investigated areas which previous committees had not mentioned or had covered superficially. This was seen in the consideration of the entire program in mathematics, from the gifted to the below-average. The reasons for the lack of influence could be explained by the fact that it reported its findings amidst several other curriculum study groups. The committee cited the works of other study groups throughout their report, thus making the report seem like a review of other studies.

The Cambridge Conference on School Mathematics was held during the summer of 1963. This group consisted of twenty-five people. There were both pure and applied mathematicians, statisticians, physicists, economists, and a chemist at the conference. The conference differed from those committees which made direct recommendations and from those which produced curriculum materials. This conference put forth their views on the mathematics curriculum of the future. Their intention was to provoke experimentation and discussion of their ideas for the pre-college curriculum.[66]

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[65] Ibid., p. 402.

[66] Cambridge Conference on School Mathematics, Goals for School Mathematics (Boston: Houghton Mifflin Company,

The conference listed several broad goals of the mathematics curriculum. Among these were; (1) the acquisition of skills, (2) a familiarity with mathematics, (3) an understanding of the "power" of mathematics, (4) an appreciation of the humanistic value of mathematics, (5) the building of self-confidence, and (6) communication with precision.[67]

Concerning the presentation, the conference stressed the spiral curriculum, a method "abandoned" before the twentieth century. In this type of presentation, as the levels of abstraction rose, there were to be different approaches used. The feeling was that this would broaden those students who understood the material, while others were permitted to see a different approach.[68]

The elementary grades were to use a discovery approach, with teacher guidance. This was to evolve until the seventh grade, where the presentation would be more direct. Discovery was limited to exercises, thus still developing the student's independent thinking and creativity.[69]

There were two basic secondary curriculums presented. The total picture was one of moving the first three years of college mathematics into the secondary school. The two proposals are listed below.

1963), p. 3.

[67] Ibid., pp. 7-12.

[68] Ibid., p. 15.

[69] Ibid., p. 17.

PROPOSAL I

Grades 7 and 8	Algebra and probability
9	Geometry
10	Geometry, topology, and linear algebra
11 and 12	Analysis

PROPOSAL II

Grades 7 and 8	Algebra, geometry, and probability
9	Algebra, geometry, and calculus
10	Analysis, probability, and linear algebra
11 and 12	Analysis[70]

The two proposals represented somewhat different approaches. This difference ranged from the presentation of polynomials in algebra, as polynomial forms over a field or as polynomial functions, to calculus where one proposal stressed a straight-through two year course and the other offered a discovery-oriented introductory course in the ninth grade. Probability was presented in both programs in various stages of difficulty and abstraction. Geometry, in the first proposal was taught up to the Pythagorean theorem in grade nine. Then, with the needed background, analytic geometry was covered. Geometry was then returned to in grade ten. Linear algebra was also spiraled. The first contact was in Euclidean space and finite dimensional vector spaces. Later contact considered general linear spaces.[71]

This report was not without criticism. Some writers felt that the report "fell short of the recommendations

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[70] Ibid., pp. 43-46.

[71] Ibid., pp. 47-50.

already formulated in Europe." Others felt that not enough attention was paid to the work of the developmental psychologists. Still others felt that the recommendations were made without considering the problem of qualified teachers.[72]

Regardless of the criticisms, the Cambridge Conference stimulated experimentation and discussion in the area of secondary mathematics. Two separate groups are currently working in areas related to the Cambridge Report. The curriculum of the Cambridge Conference was introduced at Nova High School in Fort Lauderdale, Florida, under the direction of Burt Kaufman. The project, Comprehensive School Mathematics Project, has since been moved to Southern Illinois University.[73] The Secondary School Mathematics Curriculum Improvement Study, under the direction of Howard Fehr, is developing a program in which much of our current college mathematics is presented in the secondary school. The SSMCIS is attempting to see if recommendations such as the Cambridge Report are feasible and if teachers can be trained to present the program.[74]

[72]Osborne and Crosswhite, "Reform, Revolution, and Reaction," p. 295.

[73]Ibid., p. 296.

[74]Secondary School Mathematics Curriculum Improvement Study, "Objectives of SSMCIS as Stated July 1, 1966." New York, 1966, p. 1. (Mimeographed.)

CHAPTER III

FACULTY PSYCHOLOGY

Faculty psychology was one of two opposing theories in the eighteenth century. The second theory, associationism, will be discussed in a later chapter. The basic idea in faculty psychology was that the mind was composed of different faculties, and these faculties were strenghtened through exercise in much the same way as muscles are strengthened.[1]

The origins of faculty psychology can be found in Scotland and Germany. Thomas Reid (1710-1796) and his chief disciple, Dugald Stewart (1753-1828), listed twenty-four powers of the mind. These included active powers such as self-preservation, gratitude, pity, desire for power, and imagination. The intellectual powers included perception, judgement, memory, and conception. However, their efforts were re-directed under Thomas Brown (1778-1820) when Stewart's health failed, eventually approaching the views of the associationists.[2] Because of this later compromise, the

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[1]Ernest R. Hilgard, Introduction to Psychology (New York: Harcourt, Brace and World, Inc., 1962), p. 316.

[2]E. G. Boring, A History of Experimental Psychology (New York: Appleton-Century-Crofts, Inc., 1950), pp. 203-05.

development of faculty psychology was credited to Christian Wolff, a German mathematician and philosopher. Wolff also described the mind as being composed of distinct faculties such as imagination, memory, and reason.[3] The "will" faculty, for instance, was strengthened when the student completed difficult problems in mathematics. The study of Latin strengthened the "reason" faculty. Much of the education in the latter half of the nineteenth century showed the influence of faculty psychology. Some subjects, such as Latin and geometry, held their place in the secondary curriculum as a result of their purported disciplinary powers.

> . . . In fact, they said the old preparatory subjects were much better suited for such purposes. By their very nature they were best suited for developing the memory, reason, stick-to-it-iveness, patience, and neatness. Those were the powers most important to develop in young people. Later on they could develop initiative, judgement, and creativeness, it was said, if anything was said at all of those qualities. The fact that traditional subjects were hard and uninteresting made them peculiarly suitable for developing the desired traits and powers.[4]

> A public school is designed to train the mind, to develop powers of reasoning, to exercise the memory, to encourage the initiative, to cultivate the powers of expression, and in order to do all that, it uses as its chief instruments those subjects which long years of experience have shown to be the best for the purpose.[5]

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[3] M. Vere DeVault and J. Fred Weaver, "From Settlement to the End of the Nineteenth Century: 1607-1894," A History of Mathematics Education in the United States and Canada, Thirty-second Yearbook of the National Council of Teachers of Mathematics (Washington, D.C.: National Council of Teachers of Mathematics, 1970), p. 99.

[4] Harl R. Douglass and Calvin Grieder, American Public Education (New York: The Ronald Press Company, 1948), p. 289.

[5] Ibid., p. 290.

Those who advocated faculty psychology also felt that the general powers of the mind could be trained in certain specialized situations so that they could apply to all situations. This second belief soon became the important aspect of faculty psychology as the late 1880's saw several attacks made on the "faculties of the mind."[6] The work of Woodworth, Thorndike, and James weakened faculty psychology, and brought about the movement toward "transfer of training."

William James conducted experiments to see if the memorizing of certain kinds of poetry would improve the memory for poetry in general. The belief of the nineteenth century was that practice in one memory function aided other memory functions, and that the classics and ancient history strengthened the memory. James concluded that "general retentiveness could not be improved by training." Other studies also showed that there was no such thing as "memory training."[7]

Woodworth and E. L. Thorndike were also instrumental in discrediting faculty psychology. Their work, published in 1901, revealed that one skill influences another only to the extent that the two are similar.[8] Later work by Thorndike

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[6] Ibid.

[7] Gardner Murphy, Historical Introduction to Modern Psychology (New York: Harcourt, Brace and World, 1949), p. 200.

[8] Richard J. Herrnstein and Edwin G. Boring, eds., A Source Book in the History of Psychology (Cambridge, Massachusetts: Harvard University Press, 1965), p. 516.

investigated the effects of specific school subjects. The early committees held strongly to the doctrine of formal discipline, while later committees were influenced by experimentation on transfer of training. The first part of this chapter will deal with faculty psychology and its influence on the recommendations of the various committees. The latter part will be concerned with transfer of training.

The Committee of Ten reflected the influence of faculty psychology. The committee placed a great deal of importance on faculty psychology as their report preceded much of the experimentation concerning faculty psychology. Geometry was to serve an important role in the debate over mental discipline.

> . . . It insists on the importance of elegance and finish in geometrical demonstration, for the reason that the discipline for which geometrical demonstration is to be chiefly prized is a discipline in complete, exact, and logical statement. If slovenliness of expression, or awkwardness of form, is tolerated, this admirable discipline is lost.[9]

Further evidence of the committee's emphasis can be seen below.

> The opinion is widely prevalent that even if the subjects are totally forgotten, a valuable mental discipline is acquired by the efforts made to master them. While the Conference admits that, considered in itself, this discipline has a certain value, it feels that such a discipline is greatly inferior to that which may be gained by a different class of exercises, and bears the same relation to a really improving discipline that lifting exercises in an ill-ventilated room bear to

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[9]National Education Association, Committee of Ten, Report of the Committee of Ten on Secondary School Studies (New York: American Book Company, 1894), p. 25.

> games in the open air. . . . This end can be best gained by comparatively easy problems, involving interesting combinations of ideas.[10]

The passage above showed that the committee, having already recommended the omission of the more difficult areas of arithmetic, felt that difficult problems were not necessary for mental discipline.

The period preceding the organization of the International Commission was one which proved to be a transitional era in the field of psychology. The years from 1885 to 1904 saw the emergence of experimentation in learning. This sudden emergence was difficult to explain as the experimental techniques necessary were simple enough to have been invented one hundred years sooner. E. G. Boring felt the reason was a growing faith in scientific research and in scientific psychology.[11]

The International Commission was organized after the Thorndike study on how training in one skill influences the performance in another. With such a setting, the commission took an important stand in the area of faculty psychology. They were aware of the work of the psychologists and urged that attention be paid to their efforts.

In a general report, the American Commissioners felt that teaching should be modified to conform to the psychological research on the value of formal discipline. However,

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[10] *Ibid.*, p. 108.

[11] Herrnstein and Boring, *A Source Book*, p. 516.

they did throw out a word of caution.

> With respect to the so-called "doctrine of formal discipline" in particular there is danger that results of psychologic research may be misunderstood and misapplied by one who accepts it as a settled fact that these researches have "exploded the doctrine of formal discipline as generally stated," without any statement of what the doctrine is or how it is "exploded."[12]

Committee I, General Elementary Schools, made the following remarks.

> The doctrine of formal discipline is accepted now in only a modified form so that it is no longer deemed sufficient to claim for any subject that it has great disciplinary value. The educational value of all subjects, including mathematics, has been and is being subjected to close scrutiny with the result that subjects and subject matter long retained in the curriculum, through regard for tradition, are now being displaced by new material, equally valuable as means of training, but more representative of current life. When this is not the result, old material is frequently dropped without any such substitution, on the ground that it neither has valuable content nor is necessary as a means of training. As in all reforms, there is a tendency to go to extremes in this, especially with regard to mathematics. The willingness of the mathematicians to subject their science to the test of the new doctrine has invited the less sympathetic scrutiny of others who are not interested in the subject. The result is a tendency to demand more of mathematics in this respect than of other subjects.[13]

The various committees, twelve in all, expressed their

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[12]International Commission on the Teaching of Mathematics, The American Report, <u>Report of the American Commissioners of the International Commission on the Teaching of Mathematics</u>, United States Bureau of Education Bulletin 1912, no. 14 (Washington, D.C.: Government Printing Office, 1912), p. 31.

[13]International Commission on the Teaching of Mathematics, The American Report, <u>Mathematics in the Elementary Schools of the United States</u>, United States Bureau of Education Bulletin 1911, no. 13 (Washington, D.C.: Government Printing Office, 1911), p. 128.

own views, some of which were more radical than those mentioned above. Committee III made a bold statement about mental discipline. "The one-sided doctrine of mental discipline must go."[14] This committee felt strongly enough to discuss a new doctrine.

> . . . the new doctrine of mental discipline. Although not yet clearly formulated, its main contentions are: (1) That mental discipline as ordinarily conceived is a myth, in the sense that no "general training" is to be derived from the intensive study of one or more subjects, such as Latin, algebra, etc.; (2) that the disciplinary value of a subject is a function of the interest which it inspires and of the motive guiding the student; (3) that the cultural value of a subject depends on the extent to which that subject can be, and actually is, linked with the activities and the thought content of real life. In this way the old static, historic, idealistic conception of mental discipline is being replaced by a dynamic, realistic, practical view.[15]

Committee III also warned that hasty inferences made concerning the "new" theory could possibly be dangerous. Also mentioned was the fact that the "new" theory proved nothing concerning the disciplinary value of the curriculum of the time. Along this same line of thinking, the "new" theory destroyed the idea of cultural equivalence of all high school subjects. The committee felt that if "it can be shown that mathematics can be linked with a larger range of actual thought processes and activities than typewriting or

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[14] International Commission on the Teaching of Mathematics, The American Report, Mathematics in the Public and Private Secondary Schools of the United States, United States Bureau of Education Bulletin 1911, no. 16 (Washington, D.C.: Government Printing Office, 1912), p. 104.

[15] Ibid., p. 99.

bookkeeping, for example, then the greater disciplinary value of mathematics will have been established."[16]

The feeling among high school teachers at that time can be seen in the investigation of Committee IV. Of 136 replies to a questionnaire sent to high school teachers in private schools, all listed mental discipline as one of the aims of mathematics teaching. When queried as to their meaning of mental discipline, most held to the older doctrine. Their responses indicated they felt mental discipline to be

> . . . that which produces an improvement in intuition, judgment, memory, imagination, intelligence, reason, mental powers, reasoning powers; or an improvement in ability or power of mental concentration, initiative, sustained effort, analysis, generalization; or an improvement in ability to think rapidly, clearly, independently, logically; to recognize the essential elements in a problem, to note resemblances and relationships, to grasp and apply principles, to understand cause and effect.[17]

They also felt that the principal outcome of mental discipline was the ability to express thought clearly, concisely, and accurately. Some responses showed an awareness of the "new" doctrine.

> Mental discipline through mathematics for mathematical work consists in the acquirement of mathematical facts and ideas, processes, methods of solution, and standards of accomplishment. Mental discipline through mathematics for other kinds of work consists, I think, in development or strengthening of certain ideals of standards, such as logical perfection, mastery of difficulties, etc. General power is not necessarily gained by

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[16] Ibid., p. 99.

[17] Ibid., p. 100.

> the study of mathematics, but the student may be so impressed by the logical perfection of mathematical reasoning as to consciously test his thinking on other subjects by similar standards, and the sense of mastery experienced in mathematical victories may give confidence in attacking other difficulties.[18]

In the reports of both the American Commissioners of the International Commission and the Committee of Ten, faculty psychology was important. While the Committee of Ten placed a great deal of importance on faculty psychology, the American Commissioners took a cautious stand. They were not swept away by the enthusiasm of the "scientific experimenters." The report of the American Commissioners showed that the teachers of the time held quite firmly to faculty psychology. Because mathematics, especially algebra and geometry, was considered a good disciplinary subject, its place in the high school curriculum was justified. When experimental evidence showed that the mind was not composed of faculties, mathematics had to justify its place in the curriculum. The early part of the twentieth century saw record enrollments in the high schools. To add to the problems of the mathematics educator, many of these students were seeking practical training in their mathematics classes. Transfer of training, a modified view of formal discipline, was one area which mathematics educators felt would justify keeping algebra and geometry in the secondary curriculum.

One of the last mentions of the older doctrine of

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[18] _Ibid._, pp. 145-46.

formal discipline can be seen in The Problem of Mathematics in Secondary Education (1920). Following the cautious handling by the International Commission, the Commission on Reorganization of Secondary Education took a firmer stand. With the benefit of more research than the International Commission, the Commission on Reorganization did not use formal discipline as a factor in determining the content of the mathematics program.

> . . . All agree, none the less, in greatly reducing the old claim both as to the amount and as to the generality of conditions under which transfer may be expected. In accordance with these considerations the committee has not used the factor of "formal discipline" in determining the content of the mathematical courses to be recommended in this report.[19]

Although not using formal discipline in their recommendations, the commission's views about transfer of training deserve mention. The commission denied the existence of separate faculties of the mind, but held open the idea of transfer of training. "The psychologists, however, have so far found it difficult to agree upon any final situation as to the amount of transfer which in any particular situation may be a priori expected."[20]

Transfer of training, defined in the Fifth Yearbook of the NCTM as "The influence which an improvement or change

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[19]National Education Association, The Commission on the Reorganization of Secondary Education, The Problem of Mathematics in Secondary Education, United States Bureau of Education Bulletin 1920, no. 1 (Washington, D.C.: Government Printing Office, 1920), pp. 16-17.

[20]Ibid., p. 1.

in one mental function has upon other mental function," became an important issue in the first quarter of the twentieth century. Transfer of training still receives mention by such modern psychologists as Gagné and Bruner, and will be covered in Chapter V. The early psychologists, generally admitting that transfer occurred, were divided into two groups. One school, led by Thorndike, believed in transfer through identical elements. Thorndike suggested that in a new situation the learner draws upon previous experiences and uses the common elements. The second school, led by Judd, believed in transfer through principles. Judd proposed that transfer occurred when application was made in a new situation of principles learned in another situation.[21]

The transfer of training issue was clouded by two principal problems. First, there was the interpretation of the results of experimental studies. Some educators felt that the results were misinterpreted. The second problem was one of general distrust of the testing movement. The following passages indicate the feelings of educators of the time. Ernest C. Moore of Harvard University stated:

> . . . Some of those who have investigated the question whether training is transferred declare that it is not. Some affirm that under certain conditions it is sometimes and in some degree; but even when they declare that it is transferred the evidence of transfer is so slight and the expectation of it so uncertain, that it is the part of wisdom no longer to build houses of

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[21]Hilgard, Introduction to Psychology, p. 317.

learning upon the shifting sands of this doctrine.[22] Moore was also one who felt that the results of experiments were misinterpreted.

In Principles of Secondary Education, Inglis felt that educators should wait for more conclusive evidence for transfer.

> We may conclude that experimental evidence, while suggestive and indicative that older notions of general discipline are untenable, has as yet done relatively little to determine either the mode or extent of the transfer or spread of improved efficiency.[23]

J. W. A. Young injected still further doubt about transfer.

> . . . The results reached by psychologists working in the scientific spirit, when received and interpreted in the same spirit, can do only good. But danger lies in the accretions and distortions that the descriptions of these results undergo when handed down the line from science to rhetoric, from first to tenth hand.[24]

The last passages showed the reluctance of the educators to take a firm stand on the issue of transfer of training.

Throughout the literature can be found two central themes in articles on transfer of training. Transfer was not automatic, and teaching methods needed to be altered to

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[22] Ernest C. Moore, "Does the Study of Mathematics Train the Mind Specifically or Universally?," The Mathematics Teacher, X (September, 1917), 18.

[23] A. Inglis, Principles of Secondary Education (New York: Houghton-Mifflin Co., 1918), p. 409.

[24] J. W. A. Young, The Teaching of Mathematics (New York: Longmans, Greenard Co., 1924), p. 375.

provide for transfer. The belief in "teaching for transfer" began in the 1920's and was to become a central feature in the 1930's.

The question of transfer of training was discussed by the National Committee on Mathematical Requirements, the Joint Commission of the NCTM and MAA, and the Commission on Post-War Plans. An extensive chapter on transfer was included in the Fifth Yearbook of the NCTM. The National Committee on Mathematical Requirements was the first national committee to mention transfer to any great degree. One entire chapter of their report was devoted to transfer of training. The basic conclusions of the committee were:

> 1. The two extreme views for and against disciplinary values practically no longer exist. As the question now stands, as transfer of training, the psychologists quoted here almost unanimously agree that transfer does exist.
> 2. A large majority agree that there is a possibility of negative transfer, and of zero transfer, caused by interference effect.
> 3. Very few if any experiments have shown the full amount of transfer between the fields chosen for investigation.
> 4. The amount of transfer in any case where transfer is admitted at all, is very largely dependent upon methods of teaching.
> 5. A majority of the psychologists seem to believe that with certain restriction, transfer of training is a valid aim in teaching.
> 6. Transfer is most evident with respect to general elements--ideas, attitudes, and ideals.[25]

The above conclusions were drawn from twenty-four psychologists. Several prominent psychologists among those who

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[25]National Committee on Mathematical Requirements, _The Reorganization of Mathematics in Secondary Education_ (n.p.: Mathematical Association of America, 1923), pp. 95-96.

responded were: J. R. Angell, Edwin G. Boring, Stephen S. Colvin, Charles Judd, Lewis Terman, John B. Watson, R. S. Woodworth, and E. L. Thorndike.[26]

Perhaps the most significant result of their psychological investigations was that transfer was not automatic. Charles Judd did not think that "any subject transfers automatically and in every case. The real problem is a problem of so organizing training that it will carry over in the minds of students into other fields."[27] Woodworth felt that "to guarantee any transfer, the element to be transferred must be brought specifically to the pupil's attention, generalized into a principle, and the application of the principle to other fields made clear."[28] H. V. Bingham's view can be seen in the following.

> In view of the probability that the amount of transfer of specific training from one subject to another is relatively small, a mathematics curriculum should be almost exclusively constructed with a view to securing a wide range and variety of specific habits and items of knowledge whose value quite apart from any question of the development of general abilities is in each instance readily demonstrable.[29]

The Fifth Yearbook of the NCTM listed the following conclusions after investigations concerning transfer of training.

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[26] Ibid., pp. 96-104.

[27] Ibid., p. 99.

[28] Ibid., p. 103.

[29] Ibid., p. 97.

1. Training for transfer is a worth-while aim on instruction; from the standpoint of life it is the most important aim.
2. Transfer is not automatic.
3. Every type of specific training, if it is to rise above a purely mechanical level, should be used as a vehicle for generalized experience.
4. The cultivation of thinking is the central concern of education.[30]

Thorndike's 1923 study on transfer in high school studies dealt a blow to mathematics. "The expectation of any large differences in general improvement of the mind from one study rather than another seems doomed to disappointment."[31] However, this statement was not accepted by all who read it. Bagley and Colvin held firmly to the superior disciplinary powers of pure mathematics. Inglis felt that "mathematics possesses advantages over many subjects of study."[32]

The 1930's saw increased efforts toward reorganization of secondary mathematics to facilitate transfer. J. O. Hassler's text, The Teaching of Secondary Mathematics, mentioned transfer through identical elements.

> . . . As the theory stands today, we can believe that transfer exists to the extent that identical elements are found in different situations, and that transfer is

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[30]William Betz, "The Transfer of Training, With Particular Reference to Geometry," The Teaching of Geometry, Fifth Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1930), p. 197.

[31]E. L. Thorndike, "Mental Discipline in High School Studies," Journal of Educational Psychology, XV (February, 1924), 98.

[32]Betz, The Teaching of Geometry, pp. 188-89.

> more readily possible in the case of attitudes and ideals than in specific skills. For this reason, the modern movement toward reorganization of mathematics takes account of the disciplinary values of mathematics as well as the utilitarian values. If we wish transfer, however, we must teach with this idea definitely in mind.[33]

Hassler also offered these suggestions on how to teach geometry in order for maximum transfer to take place. First, the subject must be thoroughly understood. The second step was to discuss procedures in geometry with the aim in mind of lifting the method from the subject matter. This meant that if skill in reasoning was to transfer, then it should be introduced in the field of geometry. Finally, the teacher had to use illustrations from the student's other courses or his home life. The teacher had to make a conscious effort to show how a particular line of reasoning had applications outside the geometry classroom.[34]

Pedro Orata published several works concerning transfer. He defined transfer of training as "that process of using or applying previously acquired information, habit, skill, attitude, or ideal in dealing with a relatively new or novel situation."[35] His first work in 1927, The Theory of Identical Elements, covered transfer experiments from 1890

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[33]Jasper O. Hassler and Roland R. Smith, The Teaching of Secondary Mathematics (New York: The MacMillan Co., 1930), p. 123.

[34]Ibid., p. 371.

[35]Pedro Orata, "Transfer of Training and Reconstruction of Experience," The Mathematics Teacher, XXX (March, 1937), 99.

to 1927. His article in the May, 1935, issue of The Mathematics Teacher covered experiments from 1927 to 1935. His investigations found the following: nearly 30 per cent showed considerable transfer, nearly 50 per cent showed appreciable transfer, less than 10 per cent showed little transfer, and less than 4 per cent showed no transfer. The remainder, less than 10 per cent showed both transference and interference.

More important than the actual figures was the progress made in transfer investigation from 1927 to 1935. Four developments were mentioned for the period. First, the generalization theory of Judd had displaced the identical element theory of Thorndike. Second, Thorndike had changed his thinking on transfer, approaching the position of Judd. The third development derived from transfer experiments was a substantial weakening of Thorndike's mechanistic view of learning. The results leaned more toward the Gestaltist's viewpoints. Finally, there were efforts made by various subject matter specialists to determine the teaching methods which would bring about transfer.[36]

A later article by Orata showed the influence of the "progressives." He listed eight areas in which mathematics instruction could be reorganized to achieve the objectives of progressive education, while providing for transfer.

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[36]Pedro Orata, "Transfer of Training and Educational Psuedo-Science," The Mathematics Teacher, XXVIII (May, 1935), 267-74.

1. He [the teacher] should conceive of mathematics both as a mode of thinking and as a tool for thinking.
2. The subject of mathematics should be brought into relation with the other subjects of the curriculum in such a way as to permit the organization of the school program in terms of the child's daily life problems and activities.
3. While this will mean the abolition of mathematics as a special subject except for those who are preparing for specific professions that require mathematics, it does not necessarily preclude the specific learning of number concepts, combinations, relations, and other processes that are ordinarily regarded as belonging to the field of mathematics.
4. While the mathematical processes are to be taught as they are needed in the solution of a specific problem, provision should be made in the organization of what is learned so that it will transfer in a way that will deal with other problems as well as to learn related processes.
5. The problem of simplifying problems so as to present only the mathematical side of it and neglecting the other aspects which are likely as important if not more so, is, at best, a falsification of facts and, at worst, ineffective in providing for transfer to life situations.
6. Emphasis should be placed as much if not more in the process of thinking through the problem than in the correct answer.
7. Proper and valid instruments or measures of evaluation of results that are anticipated should be invented.
8. There should be proper and constant diagnosis of pupil's difficulties not only in computation but also, and perhaps more so, in problem solving.[37]

The Committee on Geometry of the National Council of Teachers of Mathematics (1935) placed a great deal of importance on transfer of training.

> There is almost unanimous agreement that demonstrative geometry can be so taught that it will develop the power to reason logically more readily than other school subjects, and that the degree of transfer of this logical training to situations outside geometry is a fair measure of the efficacy of the instruction.[38]

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[37] Orata, "Transfer of Training and Reconstruction of Experience," p. 105-08.

[38] Ralph Beatley, "Third Report of the Committee on Geometry," The Mathematics Teacher, XXVIII (October, 1935), 336.

The 1930's saw specific attempts to teach mathematics with transfer of training as a basic aim. Harold Fawcett's article, "Teaching for Transfer," in The Mathematics Teacher, was one which showed how instruction in geometry could be restructured to obtain transfer. In considering definitions, for instance, the game of football provided an outside area of discussion. Proof was discussed in terms of advertising and its claims.[39]

A similar article by Hedrick was published in the February, 1937, issue of The Mathematics Teacher. Hedrick suggested that transfer in the field of mathematics could be improved by the axiomatization of algebra. Hedrick placed the burden on the mathematics teacher to find those areas in mathematics which offered the best possibilities for transfer. Hedrick felt the student should be shown how each topic in mathematics could be transferred to other fields.

The Committee on the Function of Mathematics in General Education used social situations to emphasize its important mathematical concepts; formulation, data, approximation, function, operations, proof, and symbolism. "Opportunities for clarifying concepts of the fundamental operations occur in innumerable situations. For example, the problem of finding the number of city blocks in a district which is 17 blocks long and 14 blocks wide may be solved by counting

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[39] Harold Fawcett, "Teaching for Transfer," The Mathematics Teacher, XXVIII (December, 1935), 467-69.

the 'blocks' on a map."[40] The development of a clear understanding of the role of data in problem solving was to be expedited by permitting the students to collect data themselves. The committee mentioned that deductive reasoning should be introduced in all areas, not just geometry. "When the student has broad first-hand experience in deductive reasoning the likelihood of the transfer of this ability to other fields is increased."[41]

The Joint Commission had an Appendix concerning transfer. However, it only echoed the works already mentioned here. They also felt that the teacher was obligated to assist the pupil in forming general ideas and understanding concepts.

Faculty psychology, and its later reformulation, transfer of training, had a distinct influence on the secondary curriculum. The latter part of the nineteenth century saw faculty psychology as the chief justification for the inclusion of mathematics in the curriculum. When the doctrine of formal discipline was "exploded," tradition and distrust in "scientific" experimentation helped hold the place of mathematics in the secondary school. Some mathematics educators, needing justification for the inclusion of mathematics beyond simple arithmetic, moved toward the theory

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[40] Committee on the Function of Mathematics in General Education, Mathematics in General Education (New York: D. Appleton-Century Company, Inc., 1940), p. 172.

[41] *Ibid.*, p. 190.

of transfer of training. The disciplinary powers that mathematics was known for were now to be useful in transfer to other areas. Constant attacks on transfer experiments and further distrust in educational measurement did not help the mathematician's cause. Many prominent educators of the time, though admitting transfer existed, did not fully support transfer of training as a valid aim of teaching. Even the National Committee on Mathematical Requirements, while furnishing considerable supporting evidence, did not take a definite stand on the issue. After tracing the issue of transfer of training from its questionable status near the turn of the century to its acceptance as an aim of teaching just prior to World War II, we leave the issue, only to return to it in a different setting after 1950.

CHAPTER IV

THORNDIKE'S CONNECTIONISM

> At an average of 2 1/2 seconds per bond and 10 failures of the bond to act per thousand, we have 150 seconds and 0.6 errors.
>
> --Thorndike

Thorndike's connectionism had its roots in the associationism of the nineteenth century. The associationists "limited the mind's content to ideas coming by way of the senses, which then become associated through principles such as similarity, contrast, and contiguity."[1] Thorndike based his theory of learning on the association between "sense impressions and impulses to action."[2] He referred to this association as a "connection" or "bond." Thorndike's theory, the original stimulus-response psychology of learning, has been called "bond" psychology or "connectionism." His theory of learning, first published in the book, Animal Intelligence (1898), dominated thinking in learning theory in America for almost fifty years.[3] A more recent but related theory,

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[1]Ernest R. Hilgard, Introduction to Psychology (New York: Harcourt, Brace and World, Inc., 1962), p. 14.

[2]Ernest R. Hilgard and Gordon H. Bower, Theories of Learning (New York: Appleton-Century-Crofts, 1966), p. 15.

[3]Ibid.

Skinner's operant conditioning, will be discussed in Chapter V.

Three major laws and five subsidiary laws characterized Thorndike's theory of learning. The law of readiness stated the circumstances under which a learner tends to be satisfied or annoyed, to welcome or to reject.

> 1. When a conduction unit is ready to conduct, conduction by it is satisfying, nothing being done to alter its action.
> 2. For a conduction unit ready to conduct not to conduct is annoying, and provokes whatever response nature provides in connection with that particular annoying lack.
> 3. When a conduction unit unready for conduction is forced to conduct, conduction by it is annoying.[4]

The law of exercise referred to the strengthening of connections through practice and to the weakening of connections (forgetting) when practice is ceased. Thorndike defined strengthening as "the increase in probability that the response will be made when the situation recurs."[5]

The law of effect referred to the strengthening or weakening of a connection as a result of the consequences. "When a modifiable connection is made and is accompanied by a satisfying state of affairs, the strength of the connection is increased; if the connection is made and followed by an annoying state of affairs, its strength is decreased."[6]

The five subsidiary laws, occasionally omitted in

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[4] Ibid., p. 18.

[5] Ibid., p. 19.

[6] Ibid., p. 20.

later writings, will be discussed when references are made to them. For the sake of completeness, these laws are named below.

1. Multiple response
2. Set or attitude
3. Prepotency of elements
4. Response by analogy
5. Associative shifting[7]

Thorndike's theories were questioned by the Gestaltists in the 1920's. These attacks led Thorndike to revise his laws of exercise and effect in 1929.[8]

The Thirty-Second Yearbook of NCTM said that Thorndike's influence was most noticeable during the first half of the period from 1920 to 1945.[9] Thorndike's rise and fall in prominence in mathematics education can be seen by considering the first (1926), tenth (1935), and sixteenth (1941) yearbooks of the National Council of Teachers of Mathematics.

The First Yearbook devoted fifteen pages to the psychology of drill, something that became quite characteristic of the twenties. In its list of twenty principles concerning drill, several instances can be found which showed Thorndike's influence.

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[7] Ibid., pp. 21-22.

[8] Ibid., pp. 24-25.

[9] Phillip S. Jones and Arthur Coxford, Jr., "Abortive Reform--Depression and War: 1920-45," A History of Mathematics Education in the United States and Canada, Thirty-Second Yearbook of the National Council of Teachers of Mathematics (Washington, D.C.: National Council of Teachers of Mathematics, 1970), p. 49.

> A drill exercise must be specific. . . . If we desire to give our pupils greater skill in learning to add halves, quarters, and eights, then it is possible for us to set up a training series in which every pupil and the teacher will know that this is the specific thing to be practiced and similarly for every other bond that needs to be formed in mathematics.[10]

This proved to be typical of the twenties. The idea was to break subjects down to simple tasks. Then, each simple task was drilled on separately. At higher levels, a complex situation was broken down into a series of simple bonds. This was referred to as seriation by H. W. Fehr.[11]

"In the early stages in the fixing of a bond, progress should be relatively deliberate."[12] The teacher was cautioned to be sure that students thoroughly understood the material before "fixing the bond." Other principles also mentioned bonds. "Not all bonds should be given practice until high skill is obtained."[13] Principle Eleven stated "RIGHT practice makes perfect."[14]

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[10]Raleigh Schorling, "Suggestions for the Solution of an Important Problem That Has Arisen in the Last Quarter of a Century," A General Survey of Progress in the Last Twenty-five Years, First Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1926), p. 87.

[11]Howard Fehr, "Theories of Learning Related to the Field of Mathematics," The Learning of Mathematics: Its Theory and Practice, Twenty-first Yearbook of the National Council of Teachers of Mathematics (Washington, D.C.: National Council of Teachers of Mathematics, 1953), p. 17.

[12]Schorling, "Suggestions for Solution," p. 89.

[13]Ibid., p. 91.

[14]Ibid., p. 93.

The law of effect was mentioned. "School life should be staged so that all desirable activities will have pleasurable outcomes and all undesirable activities will eventuate in unpleasant results."[15] Here we see the use of the satisfier or annoyer in fixing a particular bond.

"Everything else being equal, a skill that is fixed in its natural setting will need less repetition."[16] This represented an area which many people felt Thorndike had neglected, the motivational side of learning.

However, in The Psychology of Learning, Thorndike presented the following principles concerning motivation: interest in the work, interest in improvement, significance, problem-attitude, attentiveness, absence of irrelevant emotion, and absence of worry.[17]

While the paragraphs above show the influences of E. L. Thorndike, the tenth and sixteenth yearbooks took a different position. The Tenth Yearbook (1935) devoted two chapters to learning theory in mathematics. Chapter I discussed three psychological theories which were important at the time. Drill theory, incidental learning theory, and meaning theory were covered. The basic nature of the chapter was not one of endorsement of any particular theory, but

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[15] *Ibid.*, p. 95.

[16] *Ibid.*, p. 93.

[17] E. L. Thorndike, Educational Psychology, Vol. II: The Psychology of Learning (New York: Teachers College, Columbia University, 1913), pp. 217-226.

one of exposition. Criticisms of all three theories were offered. In this chapter, several criticisms of drill theory were offered. First, it was felt that the isolation of each skill, especially in arithmetic, was impractical. Second, drill did not approach the desired accuracy in computation. Studies showed that students counted, guessed, or incidentally solved test problems more than 60 per cent of the time. The third criticism was that drill did not improve the student's ability to think quantitatively.[18]

While the relative merits and disadvantages of each theory were given in the first chapter, the yearbook contained an additional chapter on Gestalt psychology. The tone of that chapter and the treatment of meaning theory seemed to favor these related theories more than either drill theory or incidental learning theory. Hence, we see that Thorndike's hold on the mathematics curriculum, especially at the elementary level, had weakened considerably between 1926 and 1935.

The Sixteenth Yearbook took a firmer stand against Thorndike. A chapter devoted to psychology of learning in arithmetic favored Gestalt psychology. H. G. Wheat's quote showed the general tone of the chapter.

Unfortunately, arithmetic has been analyzed, as Wheat

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[18]William A. Brownell, "Psychological Considerations in the Learning and the Teaching of Arithmetic," The Teaching of Arithmetic, Tenth Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1953), pp. 6-12.

> explains, "into a multitude of combinations, processes, formulas, rules, types of problems, etc., and the pupil is taught each in turn as a separate item of experience. Often, when he has completed the course, he knows only those parts that he can still remember, and they all seem to him as separate and unrelated combinations, processes, formulas, rules, and types of problems to be solved. Finally, when his memory for these separate items fails him, he has nothing left to carry into his adult world but the meaningless, uninteresting, and unpleasant experiences that his classes in arithmetic have provided for him.[19]

More doubt was injected by the following remarks.

> . . . By decomposing it into a multitude of relatively unrelated connections or facts, psychologists have mutilated it mathematically, and, at the same time, by emphasizing or encouraging discreteness and specificity rather than relatedness and generalization, they have distorted it psychologically. They have obscured the systematic character of the subject, and have created a doubtful conception of how children learn it. Furthermore, the practice of connectionism in arithmetic leads almost inevitably to immediate emphasis on rapid and accurate computation rather than on the development of the ability to think quantitatively.[20]

Hence, we see from the NCTM yearbooks that Thorndike's influence was quite strong in the 1920's, questioned in the early 1930's, and weakened in the late 1930's.

Thorndike's influence in the 1920's can also be seen by the nature of the psychologically oriented articles in The Mathematics Teacher during the period. While several of these were written by Thorndike, the number of articles should reflect the psychological interests of the period.

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[19] T. R. McConnell, "Recent Trends in Learning Theory: Their Application to the Psychology of Arithmetic," Arithmetic in General Education, Sixteenth Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1941), p. 170.

[20] Ibid., p. 175.

Practically all the psychologically oriented articles were of a connectionist nature. A partial list of these articles is included in the appendix.

The 1920's also saw publication of two books by Thorndike, The Psychology of Algebra (1924) and The Psychology of Arithmetic (1922), which applied his psychology of learning to these subjects. In the preface of his algebra text, Thorndike states that

> . . . our work has been very different, inasmuch as we have resolutely applied to the pedagogy of algebra the facts and principles which recent work in the psychology of learning has established. The nature of this work will appear more suitably in the actual uses of it in this volume than in any brief account. Suffice to say here that it emphasizes the dynamic aspect of the mind as a system of connections between situations and responses; treats learning as the formation of such connections or bonds or elementary habits; and finds that thought and reasoning--the so-called higher powers--are not forces opposing these habits but are these habits organized to work together and selectively.[21]

Both texts thoroughly emphasized bonds. Thorndike felt learning algebraic computation to be

> . . . in large measure the formation and organization of a hierarchy of mental connections or bonds. The science of teaching algebra must consider what these bonds are, the amount of practice each should have, how this practice should be distributed, how the order and methods of formation of these bonds may provide the maximum of facilitation and the minimum of interference among them.[22]

Several recommendations by Thorndike deserve mention. He felt that in order to achieve 100 per cent accuracy in

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[21] E. L. Thorndike, The Psychology of Algebra (New York: The Macmillan Company, 1924), p. v.

[22] Ibid., p. 250.

some bonds, one must sacrifice others. Thorndike felt that the student should be permitted to write down various formulas instead of committing them to memory. He suggested that a "long list could be made of bonds between various disguises of $a^2 - b^2$, $a^2 + 2ab + b^2$, etc., and their factors where time might be saved."[23]

Thorndike did not recommend teaching several methods of solving a particular kind of problem. "Do not form two or more habits where one will do as well."[24] In considering the solving of quadratics, he felt that "for the purposes of real life the use of the formula serves not only as well as all three together but better."[25] This influence is seen in the comparison of the New York State Syllabus of 1910 with its 1930 counterpart. The 1910 syllabus called for the solution of quadratics by factoring, by completing the square, and by the quadratic formula. However, in 1930, the syllabus suggested that quadratics be optional, and when considered, that the quadratic formula be used.[26]

Concerning verbal problems, Thorndike made three

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[23] Ibid., p. 327.

[24] Ibid., p. 273.

[25] Ibid.

[26] William D. Reeve, "United States," Significant Changes and Trends in the Teaching of Mathematics Throughout the World Since 1910, Fourth Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1929), p. 161.

recommendations, all of which characterize his view. First, he felt that in verbal problems, the value was in setting up the equation, not in the actual solving.

> We do claim that the peculiar virtue of the verbal problem is in the framing, not the solving, and that problems should be selected and arranged from this point of view rather than as exercises to show that certain algebraic computational tasks can be used in problems and to give practice in their use.[27]

Second, Thorndike felt that verbal problems should be genuine. Finally, he felt that grouping problems according to some aspect of science, home life, or industry was advantageous.[28] A carryover of this may be seen in the popular texts of the 1960's which present word problems categorically.[29] Verbal problems are grouped into investment problems, work problems, mixture problems, age problems, and motion problems.

Two chapters of The Psychology of Algebra were devoted to drill. Distribution of practice was the central theme of one of the chapters as well as being a popular research topic of the period. In Chapter XIV several illustrations were given concerning the distribution of practice. The most effective distribution proved to be one which had decreasing amounts of drill placed at increasing time intervals. Prominent texts of the time were examined and criticized as to

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[27] Thorndike, Psychology of Algebra, p. 150.

[28] Ibid., p. 158.

[29] See, for example, Dolciani's Modern Algebra: Structure and Method, 1962.

their distributions of practice.[30]

Drill books and workbooks were used to a large extent in the 1920's. Thorndike offered these suggestions concerning drill work in algebra. First, he felt that the fact that the letters used really represented numbers had to be stressed in order to give meaning to the exercise. Second, connections were made with distinctness, emphasis, and satisfaction if the student was provided with answers to all or some of the problems. A third suggestion was to make the work interesting by charting the student's progress and permitting competition between the students.[31]

Thorndike's influence can also be seen in the various committee reports, especially those prior to 1930. Although not subscribing totally to his theory of learning, the Commission on the Reorganization of Secondary Education (1920) utilized two of Thorndike's "laws" in their description of how learning occurred.

> . . . How does learning in fact take place? (1) Repetition is a factor in learning known to all. (2) An inclusive "set" which shall predispose the attention, focus available inner resources, and secure repetition is a necessary condition less commonly considered. (3) The effect of accompanying satisfaction to foster habit formation is a third factor to be noted.[32]

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[30]Thorndike, Psychology of Algebra, pp. 369-85.

[31]Ibid., pp. 329-33.

[32]National Education Association, Commission on the Reorganization of Secondary Education, The Problem of Mathematics in Secondary Education, United States Bureau of Education Bulletin 1920, no. 1 (Washington, D.C.: Government Printing Office, 1920), p. 12.

The National Committee on Mathematical Requirements (1923) made a famous statement concerning drill which was quoted in committee reports for more than two decades.

> Drill in algebraic manipultaion should be limited to those processes and to the degree of complexity required for a thorough understanding of principles and for probable applications either in common life or in subsequent courses which a substantial proportion of the pupils will take.[33]

Other mentions of drill were made concerning arithmetic. In the junior high program, opportunities were provided for drill of material presented in the early grades. The committee mentioned that the amount of time devoted to arithmetic as a distinct subject was to be greatly reduced. Concerning this reduction in time, the committee stated that "this does not mean a lessening of emphasis on drill in arithmetic processes for the purpose of securing accuracy and speed."[34]

Thorndike's influence can also be seen in the Twenty-ninth Yearbook (1930) of the National Society for the Study of Education (NSSE), which was concerned with arithmetic. F. B. Knight, editor of the yearbook, felt that "teaching based upon felt needs and interest only is inadequate."[35]

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[33]National Committee on Mathematical Requirements, The Reorganization of Mathematics on Secondary Education (n.p.: Mathematical Association of America, 1923) p. 11.

[34]Ibid., pp. 21-22.

[35]F. B. Knight, "Introduction," Report of the Society's Committee on Arithmetic, Twenty-ninth Yearbook of the National Society for the Study of Education (Bloomington, Illinois: Public School Publishing Company, 1930), p. 5.

Knight also asserted that the psychological point of view of the yearbook was in contrast with the newer Gestalt psychology, which had not yet made significant contributions to mathematics education at the elementary level.[36]

The yearbook contained several articles that were concerned with connectionism. One chapter dealt with number combinations while a second considered mixed versus isolated organization of drill. A third chapter was concerned with errors in percentage. Chapter IV (122 pages), written by Knight, thoroughly analyzed several topics in arithmetic. Knight called for consideration of "higher decade addition combinations" (25 + 7, 56 + 5, etc.). He felt higher decade combinations up to 39 + 9 were useful in column additions and that there were eighty higher decade combinations which were useful in carrying in multiplication.[37]

Influences of connectionism were also seen in algebra. A quadratic equation was solved by considering the problem as being composed of several separate skills. Figure 1 shows the method of solution.[38] (It should also be noted that in each case the coefficient of the second degree term was one!)

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[36] Ibid.

[37] F. B. Knight, "Some Considerations of Method," Report of the Society's Committee on Arithmetic, Twenty-ninth Yearbook of the National Society for the Study of Education (Bloomington, Illinois: Public School Publishing Company, 1930), pp. 192-93.

[38] Reeve, "United States," p. 178.

The Quadratic Formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Taking $ax^2 + bx + c = 0$ as the type form of the general quadratic equation which has the two roots x_1 and x_2, insert the proper values in the following table, using the numbered columns which correspond with the eight given equations, as shown for the first five in Ex. 1.

1. $x^2+3x+2=0$. 3. $x^2+9x-36=0$. 5. $x^2-8x=-24$. 7. $x^2-10x=11$.
2. $x^2-3x+2=0$. 4. $x^2-13x-40=0$. 6. $x^2+10x=-24$. 8. $x^2+14x=32$.

	1.	2.	3.	4.	5.	6.	7.	8.
$a =$	1							
$b =$	3							
$c =$	2							
$-b =$	-3							
$b^2 =$	9							
$4ac =$								
$-4ac =$								
$2a =$								
$b^2 - 4ac =$								
$+\sqrt{b^2 - 4ac} =$								
$-\sqrt{b^2 - 4ac} =$								
$-b + \sqrt{b^2 - 4ac} =$								
$-b - \sqrt{b^2 - 4ac} =$								
$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} =$								
$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} =$								

Figure 1

Factoring problems were considered "from the ground up." The student learned factoring in the following order: the difference of squares, the square of the sum, the square of the difference, next the type $x^2 + ax + b$ and finally the type $ax^2 + bx + c$. Students were drilled on each type before moving on to the next type.[39]

One cannot appreciate Thorndike's influence without the following example. A study on division was published in the NSSE Yearbook in 1930. This study examined the difficulty of <u>all</u> long-division examples with double-digit divisors and non-zero single digit quotients. Omitted were those problems in which the unit's digit of the divisor was zero. These were considered on the basis of their carrying, borrowing, remainder, and quotient skills.

$24\overline{)48}$	Practices:	No carrying No borrowing No remainder No quotient difficulty
$76\overline{)161}$	Practices:	No carrying [<u>sic</u>] Borrowing Remainder No quotient difficulty
$27\overline{)172}$	Practices:	Carrying Borrowing Remainder A quotient difficulty

This study considered <u>40,095</u> such examples![40]

A popular text in 1847 considered 100 fundamental addition facts. The number of fundamental combinations was

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[39] Schorling, "Suggestions for Solution," pp. 96-97.

[40] Knight, "Consideration of Method," pp. 162-65.

reduced to eighty-one in a text published in 1895. This was accomplished by eliminating the combinations involving zero. Two texts, one published in 1916 and one published in 1918 utilized forty-five combinations in their addition tables, all of which had the larger number first. However, under the influence of the connectionists, the later 1920's saw the return of 100 addition combinations. Tables for the fundamental operations were not a popular item as tables emphasized relatedness and systematization.[41]

Thorndike's influence served to "mechanize" mathematics, both at the elementary and secondary levels. His followers, more than himself, brought about an emphasis on learning mathematics as a group of isolated, unrelated facts. The subject was analyzed into a number of units of elements of knowledge, each of which was to be mastered. As each stimulus-response connection was considered to exist independently of others, drill on these connections was the result. This belief led to a firm reliance on drill. Numerous investigations of drill methods were conducted to find the optimum approach. This caused a natural trend away from teaching for understanding. A young girl's response to her father's question about algebra was printed in the October,

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[41]E. A. Berto and Leo J. Brueckner, "A Measurement of Transfer in the Learning of Number Combinations," Report of the Society's Committee on Arithmetic, Twenty-ninth Yearbook of the National Society for the Study of Education (Bloomington, Ill.: Public School Publishing Company, 1930), pp. 569-70.

1925, issue of The Mathematics Teacher. "'You understand algebra now, don't you?' 'Oh, no, father, but you don't have to understand it. You only do it the way they show you.'"[42]

The relatedness of number facts and the structure of mathematics were often ignored by the connectionists. It would be unfair to blame Thorndike for this mechanization of a subject with such a consistent structure. His followers sought to reorganize mathematics to conform to the laws of exercise and to his belief in bonds. However, they did not read his works carefully enough. Thorndike mentioned structure and the systematic nature of mathematics in 1921, but these comments were overlooked much the same as his remarks about his classic transfer of training experiment in 1923.[43] "A repetition of this experiment with 16,000 or 18,000 more cases is needed before final conclusions should be stated."[44]

What caused mathematics to be influenced so greatly by the connectionist theory of learning? First, the very nature of mathematics was the cause. The basic nature of mathematics permitted the necessary classification and subdivision of habits and skills. Second, many educators were in favor of mathematics simply as a tool subject, and

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[42] Harry C. Barber, "Real Improvement in Algebra Teaching," The Mathematics Teacher, XVIII (October, 1925), 366.

[43] E. L. Thorndike, New Methods in Arithmetic (Chicago: Rand McNally and Co., 1921), pp. 58-59.

[44] E. L. Thorndike, "Mental Discipline in High School Studies," Journal of Educational Psychology, XV (February, 1924), 97.

computation, the end result, was their goal. Finally, connectionism was popularized just after the decline of faculty psychology. Faculty psychology and its frequent reliance on drill affected mathematics. Hence, the drill aspect of connectionism was not a revolutionary idea for educators to accept.

CHAPTER V

GESTALT PSYCHOLOGY: 1930-1954

Gestalt psychologists felt that "our experiences depend on the patterns that stimuli form, on the organization of experience."[1] The Gestaltists were the first to make the concept of organization the cornerstone of a psychological theory of learning. Several principles concerning organization were the following:

> nearness (grouping of things near together), quality (grouping of things qualitatively similar), closure (grouping of things enclosing a space), common destiny (grouping of things moving together), good continuation (grouping of things to make a symmetrical or simple figure), and good whole Gestalt or pregnance (grouping as good as conditions permit.)[2]

The German word _gestalt_ means _form or configuration_. The last principle mentioned above is often referred to as the _law_ of _Pragnaz_. Psychological organization, according to the Gestalt psychologist, tends to move in a general direction, that being toward a "good" gestalt. A "good" gestalt would then have the properties such as stability,

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[1]Ernest R. Hilgard, _Introduction to Psychology_ (New York: Harcourt, Brace and World, Inc., 1962). p. 19.

[2]R. S. Peters, ed., _Brett's History of Psychology_ (New York: The Macmillan Company, 1953), p. 712.

simplicity, and regularity.[3] With the concern for stability, symmetry, and equilibrium, the Gestaltist found an analogy in the "fields" mentioned in physics. The patterned character of physical fields was analogous to the patterns of psychological organization. Because of this relationship, Gestalt psychology and its derivatives are often referred to as field theories.[4]

Two principal ideas of Gestalt psychology are insight and the importance of the whole. "Parts are not isolated constituents but constituents which form a whole or have certain reciprocal relationships to each other."[5] Gestaltists felt that the individual perceives wholes and does not sense isolated elements.

The second important item to mention concerning Gestalt psychology is insight. Insight occurs when the learner grasps a relationship in the problem which ultimately leads to the solution. A mathematical example was given by Wertheimer in Productive Thinking. A child, knowing how to find the area of a rectangle, was given a parallelogram and was asked for the area. When considering the problem, the child referred to "troublesome" areas near the ends of the parallelogram. Then, asking for scissors, she cut the

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[3] Ernest R. Hilgard and Gordon H. Bower, Theories of Learning (New York: Appleton-Century-Crofts, 1966), p. 223.

[4] Hilgard, Psychology, p. 20.

[5] Peters, History of Psychology, p. 678.

parallelogram as shown in Figure 2, and moved the pieces,

Figure 2

arriving at an equivalent rectangle with which she was familiar. Insight had occurred! The field was essentially restructured from one with "troublesome" areas to one with a "good" gestalt.[6]

Gestalt psychology was originated in Germany in 1912 by Max Wertheimer (1880-1943). Two of his colleagues, Wolfgang Kohler (1887-1967) and Kurt Koffka (1886-1941), were also instrumental in the growth of Gestalt psychology. American learning theories were influenced by Kohler's Mentality of Apes (1925) and Koffka's Growth of the Mind (1914). Koffka had an important effect on American learning theories as he criticized Thorndike's trial and error learning. These attacks came during the 1920's when Thorndike held a prominent position in mathematics education.[7]

The influences of Gestalt psychology spread to America, with the theory being firmly established by 1933. Prior to

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[6]Max Wertheimer, Productive Thinking (New York: Harper and Row, 1945), p. 48.

[7]Hilgard and Bower, Theories, p. 229.

1933, Koffka had come to America in 1924, acting as a visiting professor at Cornell and Wisconsin. He later moved to Smith College (1927) and remained there until his death in 1941. Wertheimer came to the New School for Social Research in 1933 and was followed by Kohler who came to Harvard in 1934. Kohler later moved to Swarthmore. Kohler's Gestalt Psychology (1929) and Koffka's Principles of Gestalt Psychology (1935) were important works which expounded the Gestaltist's views. Published posthumously in 1945, Wertheimer's Productive Thinking served to point the role of Gestalt psychology in teaching.[8]

Principal proponents of Gestalt theory in America were R. M. Ogden, F. T. Perkins, and R. H. Wheeler. Wheeler and Perkins co-authored Principles of Mental Development in 1932. However, this work deviated somewhat from Gestalt principles and was referred to as "organismic" psychology. Kurt Lewin, a German who also moved to America, has been placed in the core group of Gestalt psychologists. Lewin's version of Gestalt psychology, usually referred to as field theory, placed more emphasis on the motivational and sociological factors than Wertheimer, Koffka, and Kohler.[9]

Gestalt psychology's entrance into mathematics education can be seen by examining yearbooks of NCTM and NSSE.

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[8] Edwin G. Boring, A History of Experimental Psychology (New York: Appleton-Century-Crofts, Inc., 1950), pp. 598-99.

[9] Hilgard and Bower, Theories, p. 232.

The Twenty-ninth Yearbook of NSSE (1930) offered their psychological viewpoint as being in contrast to the more recent Gestalt psychology.[10]

A chapter in the Tenth Yearbook of NCTM (1935) by R. H. Wheeler had a strong Gestalt influence when compared to the Twenty-ninth Yearbook of NSSE. While incorporating several ideas from his work Principles of Mental Development, Wheeler offers several ideas with a Gestalt flavor. ..

> 1. Learning is a function of maturation and insight. It is a growth process that follows laws of dynamics, that is, laws of structured, unitary, energy systems, or fields.
> 2. First impressions are of total situations, but are indifferentiated.
> 3. Learning is not exclusively an inductive process. . . . on the contrary, learning is a logical process and from the beginning characterized by a grasp of relations, no matter how vague. Progress is systematic; it is a logical expansion and differentiation of unitary grasps of total situations--of wholes. It is organized and insightful, creative response to stimulus-patterns.
> 4. Learning does not proceed by trial and error.
> 10. Learning depends on transposition, that is, discovery of form, system, order, pattern, logical relations, analogies, the repeated use of a hidden logical principle, and the making of relational judgements. It is not a matter of combining skills.[11]

Thorndike's mechanistic psychology was severly criticized by Wheeler.

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[10] F. B. Knight, "Introduction," Report of the Society's Committee on Arithmetic, Twenty-ninth Yearbook of the National Society for the Study of Education (Bloomington, Illinois: Public School Publishing Company, 1930), p. 5.

[11] Raymond Holder Wheeler, "The New Psychology of Learning," The Teaching of Arithmetic, Tenth Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1935), pp. 237-39.

> Learning, then is not a matter of forming bonds, a process of putting pieces of experience together. It is not based on drill and on repetition of response. Bond psychology is irrational and has never been required by the facts of observation. It is a mechanistic philosophy imposed on the facts.[12]

Wheeler also offered suggestions to teachers at various levels which he felt would be consistent with the "new" psychology.[13]

A chapter in the Sixteenth Yearbook of NCTM, written by T. R. McConnell, also favored Gestalt psychology. McConnell stressed the importance of organization and inherent relationships in arithmetic.

> In arithmetic, the number system provides the intrinsic relations which constitute the basis for understanding and organizing the multitude of specific skills and abilities which are included in it and controlled by it. This closely knit system of ideas, principles, and processes has a meaning which will not be revealed by dealing with the elements alone. By failing to teach the basic principles of the decimal system, and by requiring the pupil merely to memorize a host of discrete number facts, we deprive him of the only effective means of generalizing his number experiences and of applying his learning intelligently to new situations.[14]
> . . . The course of development in behavior is often from the whole to the part, from the general to the specific. This process of growth is called differentiation, which has been defined as the "emergence of a feature of detail of the original pattern out of its setting to become a new and particularized whole."[15]

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[12] Ibid., p. 238.

[13] Ibid., p. 243-45.

[14] T. R. McConnell, "Recent Trends in Learning Theory: Their Application to the Psychology of Arithmetic," Arithmetic in General Education, Sixteenth Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1941), p. 270.

[15] Ibid., p. 271.

In summarizing the chapter, McConnell took a stand against Thorndike's connectionism and the investigations of the Committee of Seven. Concerning the latter, he felt their findings were related to the mechanistic method by which the topics were presented. (A similar reaction can be seen in Bruner's response to the developmental psychologists of the 1960's.) Concerning the issue of which psychological theory best served mathematics, McConnell felt that

> The newer point of view emphasizes relatedness rather than itemization. It stresses generalization instead of extreme specificity. It conceives of learning as a meaningful, not a mechanical process. It considers understanding more important than mere repetition or drill. It looks upon learning as a developmental process, not one of fixation of stereotyped reactions. It encourages discovery and problem solving rather than rote learning and parrot-like repetition.[16]

To conclude that connectionism was no longer of importance to the mathematics educator of the middle and late thirties and early forties would be a mistake. While NCTM (1941) spoke of only Gestalt psychology, the NSSE yearbook (1942) included connectionism as one of three active theories of learning at that time.[17]

Butler and Wren's The Teaching of Secondary Mathematics (1941), a standard for many years, espoused principles of drill similar to those of Thorndike.

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[16] *Ibid.*, pp. 288-89.

[17] T. R. McConnell, "The Purpose and Scope of the Yearbook," *The Psychology of Learning*, Forty-first Yearbook of the National Society for the Study of Education, Part II (Chicago: University of Chicago Press, 1942), p. 3.

> In order to be most effective, drill must be specific. . . . The insistence upon right practice from the start cannot be too greatly stressed. . . . Whenever possible, drill materials should also be provided with some means whereby the student can score his own work and can compare his performance not only with that of the other members of the class but also with established standards and with his own performance on previous occasions. . . . But for the purposes of fixation, which is the object of all drill, the particular detail or skill should be for the moment disassociated from its setting and context and should be drilled upon per se. . . Drill, to be most effective, must be well motivated.[18]

Butler and Wren offered other comments which were in opposition to Gestalt theory.

> On the other hand, the "new pedagogy", in its extreme form, takes the position that meanings alone have value, and that whatever fails to contribute directly to the development of concepts and understandings has no legitimate place in the educative process. Obviously there would be little place for drill in a program of instruction based upon such a philosophy as this. This point of view overlooks the important element of fixation, without which it would be manifestly impossible to organize and relate concepts or to carry on any process at a reasonable level of efficiency.[19]

An article by B. P. Reinsch in The Texas Mathematics Teachers' Bulletin (1934) offered a connectionist view in discussing the psychological principles of learning. "To develop a habit, one act is not sufficient--constant repetitions are needed." "To develop certain specific habits, we must constantly practice those very abilities."[20]

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[18]Charles Butler and F. Lynwood Wren, The Teaching of Secondary Mathematics (New York: McGraw-Hill Book Company, Inc., 1941), pp. 151-153.

[19]Ibid., p. 150.

[20]B. P. Reinsch, "Psychological Principles Applied to the Learning Process and to Testing in Mathematics," The Texas Mathematics Teachers' Bulletin, XVIII (February 8, 1934), 14-15.

Concerning the latter quote, he felt that "the student who knows his laws of exponents for positive integers needs much additional practice in working with fractional exponents."[21] While characteristic of the fragmentization of the twenties, this was somewhat alien to the Gestalt theorists.

Journal articles in The Mathematics Teacher related Gestalt psychology to secondary mathematics. An article by B. R. Buckingham (1938) indicates the trend away from connectionism. "The atomistic psychology is giving way to one which emphasizes not the parts but the whole--in short, to an organismic psychology."[22]

Buckingham mentions significance, meaning, and insight in his article. Significance referred to the value of a concept to the social order. Meaning was concerned with mathematical meaning. Here, he stressed the systematic nature of mathematics. Insight, he felt, would best result when meaning and significance are given to the concepts.[23]

Hartmann offered a more specific view in 1937. He offered three propositions which he felt were basic to Gestalt psychology.

> 1. All experience or mental life implies a differentiation of the sensory or perceptual field to which the

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[21] Ibid.

[22] B. R. Buckingham, "Significance, Meaning, Insight--These Three," The Mathematics Teacher, XXXI (January, 1938), 26.

[23] Ibid., pp. 26-27.

organism can respond into some kind of figure-ground pattern.
2. The course of mental development is from a broad, vague and indefinite total to the particular and precise detail.
3. The properties of parts are functions of the whole or total system in which they are imbedded.[24]

Hartmann referred to the Gestalt principle of closure in explaining the difference in the number of errors in adding 9 + 7 and 7 + 9. (Figure 3 explains closure.)[25]

Figure 3

With the latter problem being missed more frequently than the former, Hartmann explained the difference to be in the greater "psychic distance" from 7 to the goal 16 than from 9 to 16.[26]

Euler circles were offered to aid the learner in discovering relationships and basic structures. Hartmann also proposed the usefulness of the principle of transformation. The example offered was the following. Here, the properties of the figure are restructured or transformed.[27]

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[24] George W. Hartmann, "Gestalt Psychology and Mathematical Insight," The Mathematics Teacher, XXX (October, 1937), 266-67.

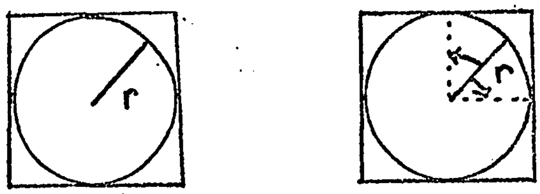
[25] Ibid., p. 268.

[26] Ibid.

[27] Ibid., p. 269.

Figure 4

What is the area of the square?



The following passage from Hartmann's article will throw some light on discovery teaching in secondary mathematics. "Perhaps if mathematics teachers act upon the recognition that the content of their discipline has to be rediscovered and created <u>de novo</u> by every learner, they will have provided themselves with one tool needed to make the Grand Tradition of rigorous thinking influential in the lives of our people."[28]

H. W. Fehr (1954) offers some Gestalt views on problem-solving with insight playing a large role in his article. Problem solving by trial-and-error was compared to problem-solving using insight. Insightful solutions were characterized by the following:

> 1. Surveying the problem situation as a whole.
> 2. Perceiving relationships between, and properties of, the elements to the whole situation and the whole situation to the elements.
> 3. Organizing and reorganizing the elements into patterns or gestalts.
> 4. Emergence of a complete solution from the structure of the situation.
> 5. Sudden discovery and high retention of the learning.[29]

Another key feature to learning was "generalization by

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[28] <u>Ibid</u>., p. 270.

[29] H. F. Fehr, "The Role of Insight in the Learning of Mathematics," <u>The Mathematics Teacher</u>, XLVII (October, 1954), 388.

extension." More meaning is given by considering generalizations of a problem, and more insight into the relationships of the various parts is obtained.[30]

Concerning problem-solving, Fehr felt that

> finding a solution to a problem is done through a type of thinking that consists in combining parts of previous experience with elements of a given situation into new patterns until insight or a total configurational relationship occurs.[31]

Training students for this type of solution involves the ability to "use (1) recall of past solutions, (2) key elements, (3) abstractions, (4) generalizations, (5) analyses, and (6) reconstructions or new patterns of all the elements of the situation."[32]

In this article is found a firmer stand on teaching by discovery.

> Insightful learning does not come about by telling students how to solve problems, and then having them reproduce the procedure on a given set of problems. Rather, it comes about by giving only hints and cues, guiding the selection of elements, and having students examine their own solutions. This calls for a heuristic approach to teaching.[33]

The teacher's responsibility in developing this type of learning was

> . . . to build many concepts through the use of concrete experience and encourage pupils to avoid wild guesses and to seek relationships, to seek to complete configurations,

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[30] Ibid., p. 387.

[31] Ibid., p. 389.

[32] Ibid.

[33] Ibid., p. 391.

> to select functional and pertinent elements related to the entire situation, to frame hypotheses, to ask pertinent questions, ever to generalize their partial solutions, and to check every answer.[34]

Several Gestalt principles are also clear in the concluding paragraph of the article.

> The solution is from "above," seeing the situation as a whole, recalling past similar situations, restructuring or reorganizing all the elements of the situation into new patterns, making analyses of the situation relating the elements to the whole situation, reformulating the problem, making generalizations, and abstracting common properties.[35]

Two yearbooks in the early fifties, NSSE (1950) and NCTM (1953), mentioned Gestalt psychology, or its principal derivative, field theory. However, both were of a rather expository nature, with no real endorsement of any approach. The NSSE yearbook mentioned the further trend toward the Gestalt viewpoint. Gestalt theorists capitalized on the systematic character of the number system and on the mental operations involved in problem-solving in much the same way as the connectionists found arithmetic quite susceptible for analysis into elements, bonds, and connections. The views of the Gestalt theorist on drill were mentioned. Drill was to follow understanding and emphasize the systematic character of the number system and its relations.[36]

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[34] Ibid., p. 392.

[35] Ibid.

[36] G. T. Buswell, "The Psychology of Learning in Relation to the Teaching of Arithmetic," The Teaching of Arithmetic, Fiftieth Yearbook of the National Society for the Study of Education, Part II (Chicago: The University of Chicago Press, 1951), pp. 146-147.

The NCTM yearbook covered the learning theories related to mathematics. Field theory, the derivative of Gestalt psychology due to Lewin, was discussed. The principal applications of this theory in mathematics were in plane geometry and problem-solving. Several problems were considered in which insight, analysis, and generalization were of prime importance.[37]

Learning, as explained by the field theorist, was characterized by the following:

> 1. Initial learnings come from experience (physical and mental experiments), constructive methods, not from definitions.
> 2. All parts related to the learning situation must be brought into focus to see the problem as a whole.
> 3. The analysis and obtaining of relations of parts to whole and whole to parts, the recalling of past patterns of learning, and blending of the given elements permit the restructuring of the elements into a new pattern.
> 4. After insight, the student practices the solution to smooth and clarify the new learning (structure).
> 5. The whole (configuration) is always a part of a greater whole.[38]

Major committee reports during the period from 1930 to 1954, the Commission on the Function of Mathematics in Secondary Education, the Joint Commission of the National Council of Teachers of Mathematics and the Mathematical Association of America, and the Commission on Post-War Plans, were somewhat influenced by Gestalt theory. Direct mention of

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[37] Howard F. Fehr, "Theories of Learning Related to the Field of Mathematics," The Learning of Mathematics: Its Theory and Practice, Twenty-first Yearbook of the National Council of Teachers of Mathematics (Washington, D.C.: National Council of Teachers of Mathematics, 1953), p. 28.

[38] Ibid., pp. 23-24.

Gestalt psychology is not found in these committee recommendations. The Committee on the Function of Mathematics in General Education offered the following:

> . . . But just as the specific-habit theory of learning superseded faculty psychology, more recent theories of learning are now superseding habit formation as the key to learning. . . . Evidence is accumulating to the effect that the individual responds as a whole to whole situations which confront him, and is to some degree remade as a person in the course of his experience.[39]

This wholeness was expanded to include the personality of the learner, his teacher, his classmates, and the school atmosphere. The recommendations of the committee did favor broad areas in mathematics and the needs of the student. Broad areas favored transfer more than the specificity of the connectionists.[40]

Problem-solving showed the influence of Gestalt psychology. The commission recommended a practice of ancient geometers when a theorem was proved. They presented only the figure and the word "Behold!" to their colleagues. The commission suggested that the student "attempt to discover all the facts and relationships which appear to hold true about it."[41]

The Joint Commission of NCTM and MAA mentioned the "old education" as that in which "facts and skills were

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[39] Committee on the Function of Mathematics in General Education, Mathematics in General Education (New York: D. Appleton-Century Company, Inc., 1940), pp. 8-9.

[40] Ibid.

[41] Ibid., pp. 52-53.

isolated, logically organized, and systematically taught." These same facts and skills, under the "new education," were to be "acquired informally in the course of experiences selected."[42] While this was oriented toward the incidental learning mentioned in the Tenth Yearbook of NCTM (1935), the commission was aware of the Gestalt theory.

> In the past, much dependence was placed on mere drill. Recent psychological investigations suggest that all techniques should be based on insight. This implies that adequate practice is to be provided, not mere drill, to lead the pupil to proper assimilation and master.[43]

The Commission on Post-War Plans felt that meaning and understanding were important.

> . . . Not possessing the basic understandings, they can do little more than to acquire skills in a mechanical fashion.
>
> Meanings do not just happen. Nor can they be imparted directly from teacher to pupils, as by having them memorize the language patterns in which meanings are couched. Instead, meanings grow out of experience, as that experience is analyzed and progressively re-organized in the thinking of the learner.[44]

Concerning drill, the commission followed the Gestalt line of thinking.

> From all this it follows that drill is to be prescribed, not in this or that grade, but at the critical

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[42] Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics, "The Mathematics Curriculum," The Place of Mathematics in Secondary Education, Fifteenth Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1940), p. 59.

[43] Ibid., p. 57.

[44] Commission on Post-War Plans, "Second Report of the Commission on Post-War Plans," The Mathematics Teacher, XXXVIII (May, 1945), 201.

> time with respect to each separate idea and skill. This critical time comes in the last stage of learning, when earlier steps have been completed and when mastery is the goal.[45]

The other wartime committees were oriented toward subject matter and practical training, and therefore did not mention Gestalt psychology.

In general, teacher training texts did not fully espouse Gestalt doctrines. Butler and Wren's editions in 1941, 1951, 1960, and 1965 held to the same connectionist viewpoint on drill mentioned earlier. Kinney and Purdy (1952) and Brown (1953) did not mention Gestalt psychology. One author, H. G. Wheat, previously mentioned in Chapter III, was very much against the connectionist's views. While not mentioning Gestalt psychology directly, he followed several Gestalt doctrines.

His solution to the problem of teaching arithmetic is presented in The Psychology and Teaching of Arithmetic (1937), a teacher training text with a great deal of concern for methods and structure. He stresses the importance of the "whole" besides attacking the analysis of arithmetic into its parts and the doctrine of social utility.

A concern for problem-solving and generality runs throughout his text. According to Wheat, the purpose of instruction in arithmetic

> is to provide them with methods of thinking, with ideas of procedure, with meanings inherent in number relations,

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[45] Ibid., p. 203.

> with general principles of combination and arrangement, in order that the quantitative situations of life may be handled intelligently and without doubt and uncertainty.[46]

Wheat uses "the idea of ten" and "the idea of position" to stress generality as these two ideas run throughout the course of arithmetic.

The importance of interrelationships in arithmetic combinations, both addition and subtraction, can be seen in the following illustration. Wheat refers to the approach as studying by "groups."[47]

Figure 5

4			5				6					7					
4	4	4	5	5	5	5	6	6	6	6	6	7	7	7	7	7	7
-1	-2	-3	-1	-2	-3	-4	-1	-2	-3	-4	-5	-1	-2	-3	-4	-5	-6
3	2	1	4	3	2	1	5	4	3	2	1	6	5	4	3	2	1
1	2	3	1	2	3	4	1	2	3	4	5	1	2	3	4	5	6
3	2	1	4	3	2	1	5	4	3	2	1	6	5	4	3	2	1
4	4	4	5	5	5	5	6	6	6	6	6	7	7	7	7	7	7

David Davis, in The Teaching of Mathematics (1951), offers a short section on Gestalt psychology. He cites the Gestalt influence in teaching as the reason for the shift in teaching technique toward that of the "discoverer and organizer of mathematics." "In practice it means that generalizations, forms and structural characteristics are the items to emphasize."[48]

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[46] Harry G. Wheat, The Psychology and Teaching of Arithmetic (New York: D. C. Heath and Company, 1937), p. 140.

[47] Ibid., p. 197.

[48] David Davis, The Teaching of Mathematics (Cambridge,

He then points out a difficulty in a discovery-oriented approach.

> The principal difficulty from the student's standpoint is that during the initial process of assimilation he is not often in the position of being able to fully appreciate the generalized concepts and their properties. He must acquire skills, processes, and a working knowledge of special phases of the subject matter before he can grasp the meaning of the more general principles.[49]

More pro-Gestalt thinking is exhibited in considering four common approaches to teaching the fundamental operations with signed numbers. The four procedures were the following:

> 1. The rules are merely stated, illustrated by examples, committed to memory by the students, and repeatedly applied to problems. Ordinary drill fixes the rules in mind.
> 2. The rules are stated and explained by means of problems based on profit and loss, credit and debit, temperature scales, etc. Repeated use of the rules established the necessary concepts.
> 3. The number scale is employed to explain the interpretation and meaning of these processes with respect to the relative positions of the real numbers on the straight line. Examples based on these interpretations are worked in order to rationalize the process. Drill is then used to reduce the responses to habit.
> 4. The number scale is employed, but additional interpretation is given each process as indicated by the signs used and the interrelation of the fundamental processes. Drill is then employed.[50]

Davis felt that the last method was preferred from the standpoint of the fundamental concepts.

Texts of the 1930's moved toward Gestalt viewpoints in

Massachusetts: Addison-Wesley Press, Inc., 1951), p. 91.

[49] *Ibid.*, p. 92.

[50] *Ibid.*, p. 93.

organization. Blackhurst's text, Humanized Geometry, went so far as to mention the monkey experiments of Kohler. However this text was an exception to the rule and was not a popular text.[51]

Hence, Gestalt psychology influenced a move away from the atomistic approach of the connectionists to an approach which stressed the structure of mathematics. The stress on meaning, understanding, and interrelationships inherent in Gestalt psychology went hand-in-hand with contemporary thinking. First, the later 1930's saw a concern for teaching for transfer. Gestalt psychology, with its stress on structure and generality, was a positive factor for those who favored "transfer through principles." With the emphasis on broad areas during the thirties, there came an interest in problem-solving. This too provided an area where Gestalt psychology, principally its idea of insight, offered a better explanation of learning.

Hilgard offered the following explanation of the receptiveness of educators concerning Gestalt psychology.

> . . . There had already been a rift growing between Thorndike and the more progressive group within education, who under Dewey's leadership, had made much more than he of the capacity of the individual for setting and solving his own problems. The new insight doctrine fitted nicely their slogan of freeing intelligence for

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[51]Alan R. Osborne and F. Joe Crosswhite, "Mathematics Education on the Defensive," A History of Mathematics Education in the United States and Canada, Thirty-second Yearbook of the National Council of Teachers of Mathematics (Washington D.C.: National Council of Teachers of Mathematics, 1970), p. 217.

creative activity.[52]

Further impact of Gestalt psychology on mathematics can be seen in the discovery method of teaching. Although interrupted by World War II, the idea survived, becoming an important factor in modern programs such as UICSM.

Although Gestalt psychology as a system has been absorbed into field theories, its impact must be recognized whenever references are made to wholes as different from parts, to structures as evolved from figure-ground relationships, and to cognitive processes (insight and understanding).[53] This impact has been shown in this chapter, and will be discussed further in Chapter VI.

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[52]Hilgard and Bower, Theories, p. 231.

[53]Ernest R. Hilgard, "The Place of Gestalt Psychology and Field Theories in Contemporary Learning Theory," Theories of Learning and Instruction, Sixty-third Yearbook of the National Society for the Study of Education, Part I (Chicago: The University of Chicago Press, 1964), p. 77.

CHAPTER VI

FACULTY PSYCHOLOGY, CONNECTIONISM, GESTALT PSYCHOLOGY, AND DEVELOPMENTAL PSYCHOLOGY--1970

In this chapter, the status of the theories covered in previous chapters--faculty psychology, connectionism, and Gestalt psychology--will be discussed as to their role in the mathematics curriculum of the decade of the 1960's. The approach will be similar to that of previous chapters, the examination of prominent committee reports of the times--Commission on Mathematics of the College Entrance Examination Board and the Cambridge Conference. Further investigation will be made of the texts of two important curriculum writing groups, School Mathematics Study Group (SMSG) and Secondary School Mathematics Curriculum Improvement Study (SSMCIS), which wrote materials which approximated the two committee reports.

Due to the influence of Jean Piaget in American education, the importance of developmental psychology will be considered. The same format will be used as in discussing the previous theories.

Transfer of Training

In Chapter II, the discussion of transfer of training was carried from the turn of the century to the early 1940's. In this section, the discussion of transfer of training will be carried through to the publication of the Sixty-ninth Yearbook of the National Society for the Study of Education.

By 1940, most educators had come to believe that in order for transfer of training to occur, one had to "teach for transfer." Butler and Wren (1941) felt that "in order to achieve the disciplinary values of any subject, it is necessary to teach that subject with that specific purpose in view."[1] In further discussion on transfer, they note a trend away from Thorndike's identical elements toward Judd's theory of generalization. Regarding teaching for transfer, Butler and Wren felt that the teacher should help the children to recognize similarities between new situations and other situations with which they are familiar and to form the habit of looking for such similarities.[2]

The Commission on Post-War Plans (1944) offered support of the views of the National Committee on Mathematical Requirements (1923).

> Even more important is the fact that the age old debate on transfer is finally "out of the window." In the military activities, countless young men learned something

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[1]Charles Butler and F. Lynwood Wren, The Teaching of Secondary Mathematics (New York: McGraw-Hill Book Company, Inc., 1941), p. 68.

[2]*Ibid.*, p. 159.

> new overnight and over twenty million workers changed jobs last year. The authorities have assumed that a worker could adjust quicker at situation B, because he had lived through situation A. If this assumption had not been sound, it might have taken us ten years to get on with a machine war. It is significant that no writer of insight and prominence has challenged or even questioned the case for disciplinary values, as published in the Report of the National Committee on Mathematical Requirements twenty-one yea s ago. That has stood the test, and it now guarantees that mathematics will be an important part of education of some of our youth for a long time to come.[3]

Some new ideas on transfer of training are seen in the Twenty-first Yearbook of NCTM (1953). The chapter devoted to transfer of training mentions four theories of transfer: formal discipline, identical elements, generalization, and reorganization of experience. Only the last of these has not been covered in preceding chapters. Reorganization of experience, as a theory of transfer, utilizes many gestaltist ideas. When encountering a new problem, the student "attempts to organize his past experiences, to reconstruct from them a way of attack, to see relationships between this new problem and past problems."[4] The restructuring of the problem situation was to provide the student with a better chance of success when encountering a strange problem.

Concerning "teaching for transfer," three

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[3]Commission on Post-War Plans, "First Report of the Commission on Post-War Plans," The Mathematics Teacher, XXXVII (May, 1944), 231-32.

[4]Myron F. Rosskopf, "Transfer of Training," The Learning of Mathematics: Its Theory and Practice, Twenty-first Yearbook of the National Council of Teachers of Mathematics (Washington, D.C.: National Council of Teachers of Mathematics, 1953), p. 216.

recommendations were given.

> 1. Teaching should be for understanding; for developing concepts. This means that the methods of exploration, discovery, and organization should be used.
> 2. After understanding is assured, enough practice is furnished students so that they will have an opportunity to reorganize or reconstruct experiences in terms of the concept involved.
> 3. Those students who progress to higher levels of mathematics study should learn to verbalize principles that are appropriate to their level of progress.[5]

These recommendations placed more emphasis on a discovery approach than those of the thirties. The movement away from Thorndike's connectionism was accompanied by a trend away from his theory of "identical elements" as a basis for transfer of training.

Transfer of training receives considerable attention in the Sixty-ninth Yearbook of NSSE. The opposing views of two principal psychologists, Jerome Bruner and Robert Gagné, are discussed in Shulman's chapter entitled "Psychology and Mathematics Education." In reference to Gagné's position on transfer, there is noted a marked relationship to Thorndike's "identical elements." "Transfer occurs because of the occurrence of specific identical (or highly similar) elements within developmental sequences."[6] Transfer then occurs when an element that is learned is applied in a new situation.

Shulman makes a point of differentiating between Gagné's

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[5] Ibid., pp. 220-21.

[6] Lee Shulman, "Psychology of Mathematics Education," Mathematics Education, Sixty-ninth Yearbook of the National Society for the Study of Education (Chicago: University of Chicago Press, 1970), p. 57.

vertical and lateral transfer. Vertical transfer refers to "the manner in which the learning of a subordinate capability serves to facilitate the mastery of some subsequent learning at a higher level of the same hierarchy."[7] Lateral transfer refers to "the manner in which the learning of a capability in one domain can facilitate the mastery of some parallel capability in another domain."[8] Vertical transfer can be illustrated by the role of multiplication in studying division. An example of lateral transfer would be the "principle of balance" in the trade between nations, a physics laboratory, or balancing equations. Hence, Gagné's emphasis on hierarchies of learning favors vertical transfer.[9]

Bruner, on the other hand, favors broad or lateral transfer of training. "Broad transfer of training occurs when one can identify in the structures of subject matters basic, fundamentally simple concepts--principles or strategies which, if learned well, can be transferred both to other topics within that discipline and to other disciplines as well."[10] Bruner's stress on understanding and discovery of the basic structure and principles of a discipline favors lateral transfer.

As to the question of teaching for transfer, Robert

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[7] Ibid., p. 55.

[8] Ibid.

[9] Ibid., pp. 55-56.

[10] Ibid., p. 56.

Craig, in a review of research on discovery states that "the discovery treatment has been inadequately tested; but, when differences among treatment groups in later ability to infer and use new principles have been found, they favor discovery techniques over the giving of guidance."[11] While the discovery approach seems to be superior in transfer, one must consider the various types of discovery teaching and the degree of guidance used. Furthermore, the material to be transferred should be given due consideration; Craig for instance points out that expository forms of teaching are superior when applications of rules are the criterion.[12]

As of 1970, the question of the best teaching approach to facilitate transfer remains unanswered. However, transfer of training has had considerable influence on mathematics education. The earlier form, faculty psychology or formal discipline, seemed to keep the place of mathematics in the secondary curriculum secure. The twenties, under the connectionist influence of Thorndike and the use of arithmetic as a tool subject, saw the importance of "identical elements." After the mid-thirties the movement was away from the theory of "identical elements" toward a broader view of transfer that was more consistent with Gestalt psychology. The thirties also saw the rise of journal articles concerned with

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[11] Robert C. Craig, "Recent Research on Discovery," Educational Leadership, XXVI (February, 1969), 503.

[12] Ibid.

teaching for transfer. Then the decade of the sixties saw Gagné's emphasis on "identical elements" as opposed to Bruner's broad transfer. Only through carefully controlled experimentation in the classroom will the problem be resolved. Most likely, the solution will be an eclectic one, with various theories being applicable to different problems.

Connectionism: Gagné and Skinner

Connectionism and its relationship to secondary mathematics prior to World War II was considered in Chapter IV. Since World War II, the influence of connectionism has been varied. Startled by the results of the Armed Forces testing programs, some mathematics educators of the post-war period established performance as a goal of learning. With such a goal in mind, the "speed and accuracy" of the 1920's again grew in importance, and with this came a return to connectionist ideas.[13] It should be noted that connectionism did not disappear with the rise of the field theories in the middle 1930's, but it was not the most prominent theory of that period as was seen in Chapter IV. However, two occurrences were to affect the influence of connectionism on the secondary mathematics program. These were the appearance of the so-called modern programs in mathematics in the decade

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[13]Howard F. Fehr, "Theories of Learning Related to the Field of Mathematics," The Learning of Mathematics: Its Theory and Practice, Twenty-first Yearbook of the National Council of Teachers of Mathematics (Washington, D.C.: National Council of Teachers of Mathematics, 1953), pp. 15-16.

of the fifties and the growth of interest in teaching machines and programmed instruction. The modern programs, such as SMSG and UICSM, stressed understanding and structure, and generally favored the field theories of learning to the extent that they espoused any theory at all. Furthermore, UICSM placed additional emphasis on discovery of mathematical ideas. Hence these programs and their related goals did not rely heavily on learning facts and skills. Interest in such programs precipitated a decline in connectionist influence.

Programmed instruction, on the other hand, favored breaking a subject into its basic parts and leading the learner through a learning sequence designed to master a particular skill. Hence, connectionism was the principal theory for those who were in favor of programmed instruction.

A learning theorist closely involved in this area was B. F. Skinner. Skinner's interpretation of learning, consistent with stimulus-response psychology, was related somewhat to Thorndike's law of effect. Responses, according to Skinner, are of two types--elicited and emitted. Elicited responses or respondents are involuntary responses elicited by known stimuli. Examples would include the knee-jerk or eye blink. Emitted responses, or operants, appear spontaneously, rather than as responses to particular stimuli. A large part of human behavior is of the operant variety. While respondent behavior is characterized by being a response to stimuli, operant behavior is emitted by the organism. Operant behavior operates on the environment and no particular stimulus consistently elicits an operant response.[14]

The following example may clarify the problem. Consider a rat in a cage with a bar which operates a feed magazine. If the rat presses the bar, food is delivered to the cage. Here, the food is a reinforcing stimulus to the stimulus (bar) and response (pressing). "If the occurrence of an operant is followed by a presentation of a reinforcing stimulus, the strength is increased."[15]

Thorndike's original law of effect (page 64) and his later reformulation--". . . a stimulus-response sequence could be strengthened by a reward that followed it"[16]--are consistent with Skinner's interpretation.

Hence, programmed materials are easily adapted to Skinner's theory of learning. A Skinnerian type program would have items to which the student would respond, and upon responding, would then be exposed to the correct answer. The student sees a problem (stimulus), determines the answer (response), and it is reinforced by seeing the correct answer.

Skinner's importance in the field of programmed

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[14] Ernest R. Hilgard, Introduction to Psychology (New York: Harcourt, Brace, and World, Inc., 1962), pp. 258-59.

[15] B. F. Skinner, The Behavior of Organisms: An Experimental Analysis (New York: Appleton-Century-Crofts, 1938), p. 38.

[16] Hilgard, Psychology, p. 265.

instruction is perhaps best exhibited by Hilgard in Theories of Learning. Hilgard credits one of Skinner's papers in Science (1958) as catalyzing the interest that had been mounting. Furthermore, the interest in programmed materials in the 1950's was due to Skinner, whose efforts also include several texts, including a programmed text, and numerous articles.

Mention of Skinner or programmed instruction is not found in the Report of the Commission on Mathematics of the CEEB. A possible reason is that the interest in programmed instruction was still growing when the Commission's report was being published. Similarly, the Secondary School Curriculum Committee did not mention programmed instruction. The extent of consideration of programmed instruction in the Cambridge Report is seen in the following. "In this spirit, we have not attempted to discuss the use of Cuisenaire rods, and we have not considered the question of how much (if any) of the mathematics curriculum might lend itself to the methods of programmed learning."[17]

However, SMSG did produce three programmed versions of their first course in algebra. One followed the Skinner paradigm (Form CR--constructed response); the second followed the Crowder Mode (Form MC--multiple choice); and the third version (Form H--hybrid) utilized conventional exposition

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[17] Cambridge Conference on School Mathematics, Goals for School Mathematics (Boston: Houghton Mifflin Company, 1963), p. 3.

and problem sets along with both Skinner's and Crowder's styles of programming.[18] It should be noted that UICSM also produced programmed materials.[19]

Other indications of interest in programmed instruction can be seen by examining the table below which shows the number of articles concerned with programmed instruction in The Mathematics Teacher.

Year	No. Articles
1959	0
1960	0
1961	1
1962	3
1963	2
1964	3
1965	2
1966	4
1967	2
1968	3
1969	3
1970	1

Some dissertation research was also centered on programmed instruction in mathematics. A list of dissertations which dealt with programmed instruction in mathematics can be found in the appendix. Furthermore, Kalin reported in 1966 that more than 100 programmed texts in mathematics were available.[20]

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[18] Kenneth O. May, "Programming and Automation," The Mathematics Teacher, LIX (May, 1966), 444.

[19] Herbert Wills, "The UICSM Programmed Instruction Project," The American Mathematical Monthly, LXIX (October, 1962), 804.

[20] Robert Kalin, "Some Guidelines for Selecting a Programed Text in Mathematics," The Mathematics Teacher, LIX (January, 1966), 14.

By the latter 1960's, interest in programmed instruction had decreased somewhat.

> The pragmatic concern of these materials [programmed], which appeared in many classrooms nearly a decade ago, was to feed material to the learner at a rate consistent with his unique intellectual metabolism. After a brief flurry, stirred up by the belief that a solution had been found, hopes waned considerably while criticism of the process grew.[21]

A 1965 report by the Committee on Education Media of MAA considered the claims of programmed instruction in mathematical education. While asserting that students do learn from programmed materials, the committee offered several conclusions which were not generally favorable toward programmed instruction.

> SCE [Skinner--Crowder--eclectic] -programmed materials are incapable of eliciting the full range of behavior included in the goals of mathematics teaching.[22]
>
> SCE-programmed materials are less adaptable to individual differences than are hybrid, problem and Pressey programs.[23]
>
> No fixed programming pattern can take care of the full range of objectives in mathematical education.[24]
>
> . . . , there is no conclusive evidence that students learn more or with greater efficiency.[25]

Despite reports such as the one mentioned above and the decline in the uses of programmed materials in schools,

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[21] Vere Devault and Thomas Kriewall, "Differentiation of Mathematics Instruction," Mathematics Education, Sixty-ninth Yearbook of the National Society for the Study of Education (Chicago: University of Chicago Press, 1970), p. 410.

[22] May, "Programming," p. 446.

[23] Ibid.

[24] Ibid., p. 447.

[25] Ibid., p. 445.

research continues in the use of programmed materials. Efforts have been made to extend programmed instruction to include teaching creativity. With one of the positive factors for programmed instruction being the ability to vary the rate of the instruction, research is now being directed at varying the path the learner may take.[26]

A second theorist with connectionist leanings is Robert Gagné of Princeton. Gagné, like Bruner, has worked directly with mathematics and has served as a consultant for the UMMaP. Several publications of Gagné have been concerned with learning in the field of mathematics. In The Conditions of Learning, Gagné discusses the following eight different types of learning: signal learning, stimulus-response learning, chaining, verbal-associate learning, multiple discrimination, concept learning, principle learning, and problem solving.[27]

The first two types, signal learning and stimulus-response learning are related to earlier S-R theories. Signal learning, however, differs by the response being of an involuntary nature and usually not under complete control of the organism.[28]

Chaining is simply the combining of two or more

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[26]Devault and Kriewall, "Differentiation of Instruction," p. 411.

[27]Robert Gagné, The Conditions of Learning (New York: Holt, Rinehart and Winston, Inc., 1965), p. 33.

[28]Ibid., pp. 33-39.

previously learned stimulus-response connections into a sequence. For example, unlocking a door consists of identifying the correct key, positioning it properly, rotating it in the right way, and pushing the door open.[29]

Verbal association learning frequently occurs in naming. Upon seeing a parallelogram, the student may notice the parallel sides which he then links to the verbal output--"parallelogram."

Multiple discrimination describes the learning that occurs when a person attempts to distinguish one item from another. This requires previous S-R bonds which differentiate stimuli from each other.[30]

Concept learning refers to learning to respond to stimuli in terms of abstract properties like "color," "shape," "middle," or "number" as opposed to concrete properties. Thus concept learning requires the learner to respond to a particular class of stimuli even though the stimuli themselves are quite dissimilar.[31]

Principle learning is the seventh type derived by Gagné. Basically, a principle is a chain of two or more concepts. Principle learning is then the acquisition of an idea.[32] The distributive property, for example, can be

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[29] Ibid., pp. 39-42.

[30] Ibid., pp. 42-47.

[31] Ibid., pp. 47-51.

[32] Ibid., p. 51.

memorized. $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$. The principle is acquired when the student sees that you may "multiply and then add or add and then multiply." Full understanding would come when the student successfully handles a problem like $(81 \cdot 93) + (81 \cdot 7) = 81(93+7) = 81(100) = 8100$. The eighth type of learning, problem solving, requires the learner to combine previously learned principles into a higher order principle.[33]

Gagné's principal contributions have been concerned with hierarchies of principles. Simply expressed, this means that before attempting to teach a particular principle, one should determine all prerequisite principles and concepts.[34] The following diagram illustrates this idea.

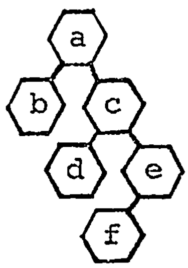

Fig. 6

Mastery of "a" requires prior knowledge of "b" and "c." Understanding "c" in turn requires mastery of "d" and "e." "f" must also be understood before understanding "e." Hence b, c, d, e, and f must be covered before attempting a.

Figures 7 and 8 illustrate the types of hierarchies mentioned by Gagné. Based on research conducted about the

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[33] *Ibid.*, pp. 54-57.

[34] *Ibid.*, p. 151.

hierarchy shown on page 118, Gagné reported, for instance, that of seventy-two pupils who performed correctly on Principle IIA, only one did not perform Principle IIIA correctly. Furthermore, of the eighteen who performed incorrectly on Principle IIA, all did Principle IIIA incorrectly. Similar studies with all possible combinations showed that predictions could be made correctly in 95 per cent or better in all possible cases.[35] Other studies by Gagné also point toward the hierarchial nature of knowledge.

The subject matter of mathematics is such that the idea of mastering prerequisite concepts is essential, and is therefore not directly attributable to Gagné. However, examples consistent with several of the types of learning mentioned in <u>The Conditions of Learning</u> can be found in the texts of both SMSG and SSMCIS. Furthermore, in both principle learning and problem solving, the two groups have insured that prerequisite concepts and principles are covered before attempting some higher order principle.

Principle learning and problem solving, though quite similar in nature, do differ. While both types of learning involve combining previously acquired principles or concepts to arrive at a higher order principle, they differ in the amount of guidance offered by the text or teacher. Thus in the problem solving method of learning, the learner is <u>not</u> told the higher order principle that is the ultimate goal,

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[35] <u>Ibid</u>., p. 153.

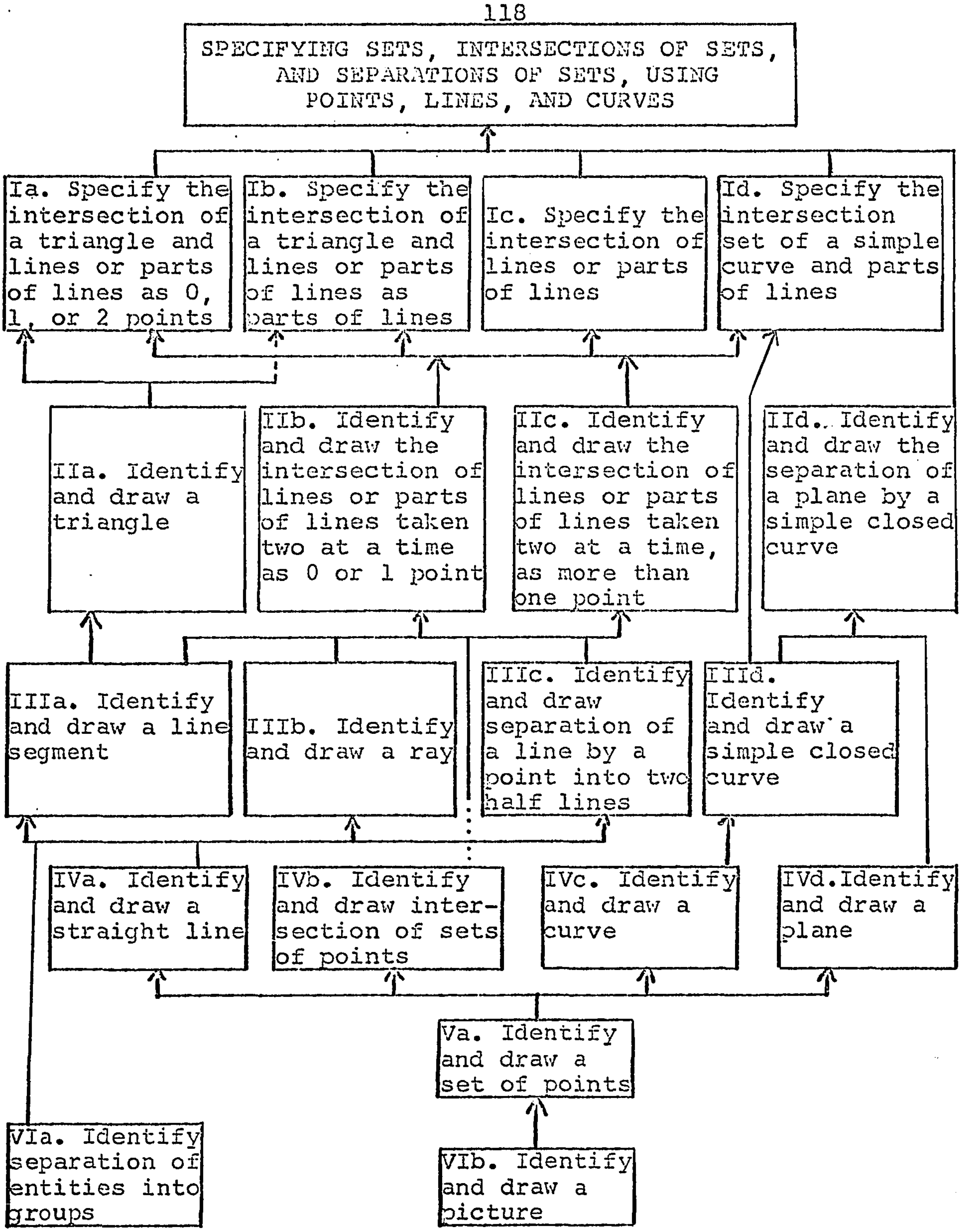

Fig. 7[36]

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[36]Robert M. Gagné, "Learning Proficiency in Mathematics," The Mathematics Teacher, LVI (December, 1963), 623.

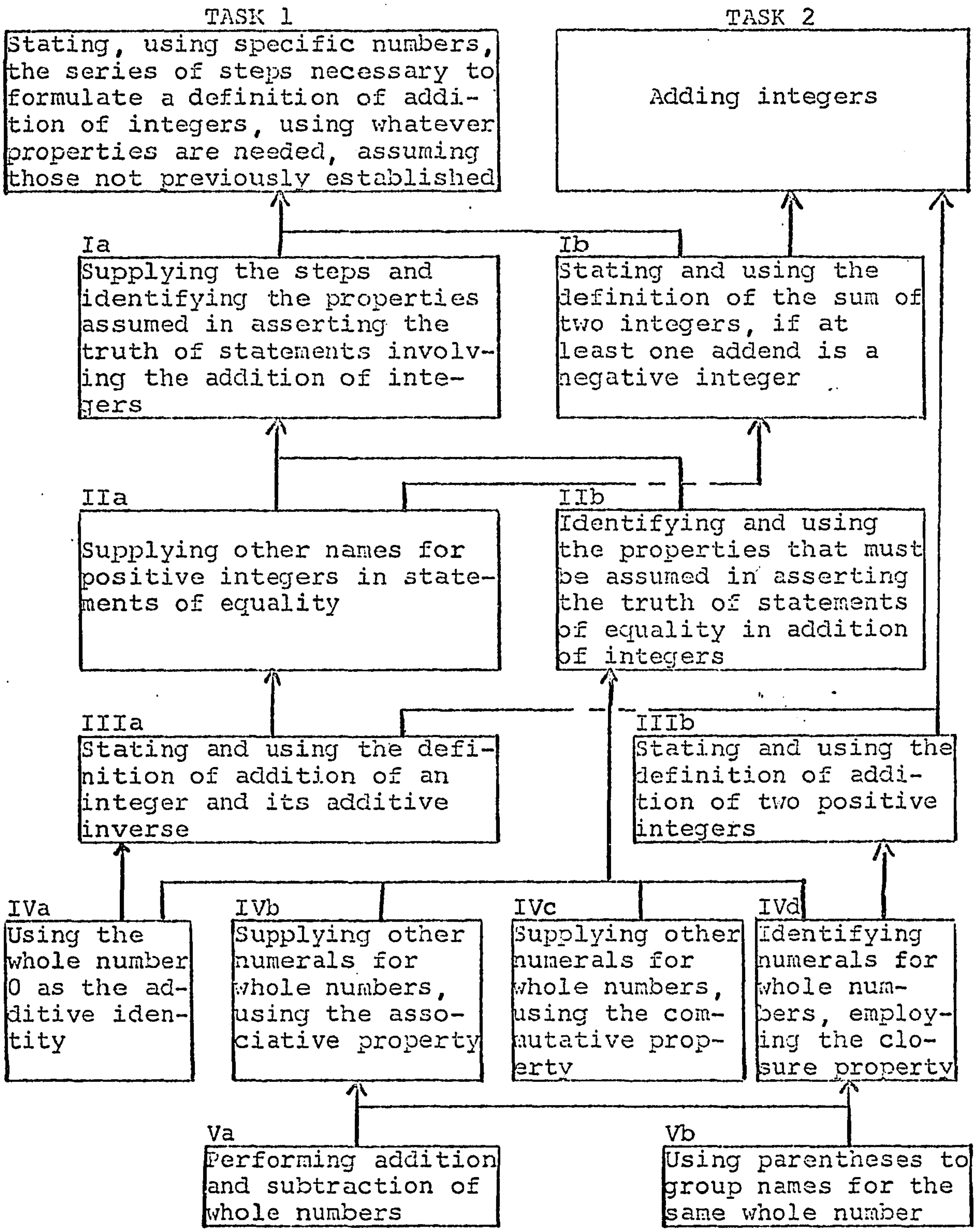

Fig. 8[37]

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[37] Shulman, "Psychology of Mathematics Education," p. 32.

but is to "discover" it. Gagné's problem solving, while being quite close to Bruner's "discovery," does insure that all prerequisite principles have been mastered. Bruner, on the other hand, often gives very little guidance in similar situations. The following problems illustrate the difference between principle learning and problem solving. The following problem is from SSMCIS Course III.

Example 4. In a club with 12 members, how many 5 member subsets are there?

$$\binom{12}{5} = \frac{(12)_5}{5!} = \frac{\frac{12!}{(12-5)!}}{5!}$$

$$= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot (7!)}{7!\ 5!}$$

$$= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 792$$

Notice that in Example 4 each time you selected a subset of 5 elements from the set of 12 elements, there were 7 elements remaining that were not selected. In general, whenever you select a subset of r elements from a set of n elements there are n - r elements remaining that are not selected. This means that there are just as many subsets with n - r elements as there are subsets with r elements.

Example 5. (a) Compute $\binom{7}{5}$ and $\binom{7}{2}$.
(b) Did you get the same number for each of the computations in part (a)?
(c) If the answer to (b) is yes explain why. If not, do your computations again.
(d) Which of the two computations in (a) was easier? Why?

These results may be expressed more generally as:

Theorem 3.

$$\binom{n}{r} = \binom{n}{n-r}$$

The proof is left as an exercise.[38]

The problem below is found in the SMSG eighth grade text.

> Whenever three of the five students are chosen for a special purpose, such as membership on a committee, then the remaining two have also been chosen---chosen, in the sense of <u>not serving</u> on this particular committee. In other words, the selection of a committee, in effect, separates the club members into two sets. One method for selecting the membership of a committee is to decide which club members will not serve. For example, if it is decided that a committee should not include C and D, then we know that the committee is A,B,E .[39]
>
> ⋮
>
> 3. Use the Pascal triangle to find each of the following:
>   (a) $\binom{6}{2}$ and $\binom{6}{4}$
>   (b) $\binom{7}{4}$ and $\binom{7}{3}$
>   (c) $\binom{8}{3}$ and $\binom{8}{5}$
>   (d) $\binom{3}{1}$ and $\binom{3}{2}$
>   (e) $\binom{8}{2}$ and $\binom{8}{6}$
>
> 4. Suppose that a and b are two counting numbers and let S be the sum a + b. What important relationship between $\binom{S}{a}$ and $\binom{S}{b}$ is suggested by Problem 3? (Use some ideas in Section 1 [see above] to convince yourself that this relationship is true in every case.[40]

Note that in the first example, the desired result, that being $\binom{n}{r} = \binom{n}{n-r}$, is in the text. The second does

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[38]SSMCIS, <u>Unified Modern Mathematics, Course III, Part I</u> (New York: Teachers College, Columbia University, 1969), pp. 269-70.

[39]School Mathematics Study Group, <u>Mathematics for Junior High School, Volume II, Part II</u> (New Haven, Connecticut: Yale University Press, 1961), p. 282.

[40]<u>Ibid</u>., pp. 304-05.

not carry the process that far.

Similarly, the quadratic formula is presented in the two textbook series in much the same way. Note however, that the problem from the SSMCIS text now illustrates problem solving.

7. Solve the following quadratic equations if possible.

⋮

(i) $ax^2 + bx + c = 0$ (where a, b, c are real numbers, $a \neq 0$). (The result of Exercise 7(i) is a formula, called the quadratic formula, which can be used to find the real number solutions of any quadratic equation, provided they exist.[41]

The problem above is from the SSMCIS text for the ninth grade. The following one is from the SMSG text in ninth grade algebra.

5. Consider the general quadratic polynomial $Ax^2 + Bx + C$. Show that

(a) $Ax^2 + Bx + C = A\left((x + \frac{B}{2A})^2 - \frac{B^2 - 4AC}{4A^2}\right)$

(b) If $B^2 - 4AC < 0$, then $Ax^2 + Bx + C = 0$ has no real solution.

(c) If $B^2 - 4AC = 0$, then $Ax^2 + Bx + C = 0$ has one real solution, $x = -\frac{B}{2A}$.

(d) If $B^2 - 4AC > 0$, then $Ax^2 + Bx + C = 0$ has two real solutions.

$$x = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \quad x = \frac{-B - \sqrt{B^2 - 4AC}}{2A}.$$

(This latter sentence is called the quadratic formula for finding the solutions of the quadratic equation.)[42]

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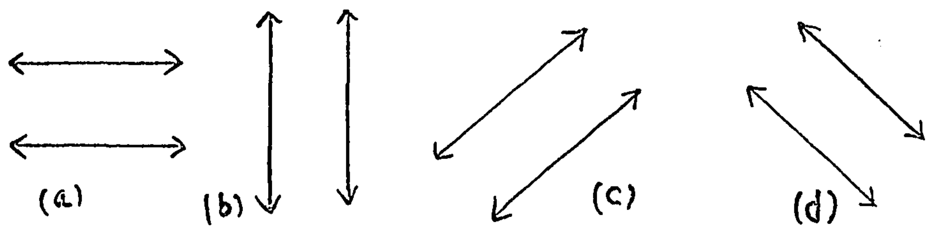
[41] SSMCIS, Unified Modern Mathematics, Course III, Part II (New York: Teachers College, Columbia University, 1969), p. 52.

[42] School Mathematics Study Group, First Course in Algebra, Part II (New Haven, Connecticut: Yale University Press, 1961), p. 545.

Additional examples of both problem solving and principle learning are found in texts of both groups. Below are examples to supplement those already discussed in this section.

6. For each of the sets of parallel lines in the figure below, draw a line perpendicular to one of them. (Do not write in your book. Copy the lines in approximately these positions on a separate piece of paper.) Are the lines which are perpendicular to one of two parallel lines perpendicular to the other also?[43]



*10. Draw any circle and any line tangent to the circle. Draw also the line which joins the center of the circle and the point of tangency.
   (a) Do you believe there is an important relationship between these two lines? If so, what is it?
   (b) How many radii of a circle may be drawn having one point of tangency as an endpoint?[44]

9. In this problem we consider some tests that may be applied to divisibility questions involving base ten. These tests will generally fail when numbers are represented with numerals in bases different from ten.
   Assume the following is true for natural numbers $a, b_1, b_2, \ldots, b_m$:
   If $a \mid b_1, a \mid b_2, \ldots, a \mid b_{m-1}$ and
   if $a \mid (b_1 + b_2 + \ldots b_{m-1} + b_m)$ then $a \mid b_m$.
   Also note that any natural number $\underline{N}$ can be written in the form $N = a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_2 10^2 +$

[43] School Mathematics Study Group, Mathematics for Junior High School, Volume I, Part II (New Haven, Connecticut: Yale University Press, 1961), p. 436.

[44] Ibid., p. 480.

$a_1 \cdot 10 + a_0$, where a_0, a_1, ..., a_n and n are natural numbers.

(a) Prove that a natural number is divisible by 2 if and only if the last digit of its (base ten) numeral is even.

(b) Note $3 \mid (10-1)$, $3 \mid (10^2-1)$, $3 \mid (10^3-1)$, etc. Assume $3 \mid (10^k-1)$ where $\underline{K}$ is any natural number. Prove a natural number is divisible by 3 if and only if the sum of the digits of its (base ten) numeral is divisible by 3. [Hint: $10^k = 10^k - 1 + 1$.][45]

Number of sides of convex polygon	Number of diagonals from A	Number of triangular regions	Sum of measures of angles of the polygon
4	1	2	2 x 180 = 360
5	2		
6	3		
7	4		
8	5		
n			

This exploratory problem leads us to the following important result.

THEOREM 11-1. The sum of the measures of the angles of a convex polygon of n sides is (n - 2) x 180.[46]

In discussing multiple discriminations, Gagné mentions areas in which discrimination is necessary. These are in the position of the subscript, coefficient, and exponent,

[45] SSMCIS, Unified Modern Mathematics, Course I, Part II (New York: Teachers College, Columbia University, 1968), p. 207.

[46] School Mathematics Study Group, Geometry With Coordinates, Part II (New Haven, Connecticut: Yale University Press, 1962), p. 737.

parentheses and brackets, extents of angles, opposite and adjacent angles and sides of figures, and direction of lines. "Whenever potentially confusable symbols or figures are newly introduced in mathematics, multiple discrimination learning must occur."[47] In the first course of SSMCIS there is an effort made to clearly differentiate between two expressions which are frequently confused.

> 23. Consider the following expressions:
>
> $$2(n^3);\ (2n)^3$$
>
> They are <u>not</u> the same. The first one is often written as "$2n^3$," without parentheses.[48]

In the SMSG ninth grade course, several exercises are included which require the student to differentiate between various positions of parentheses.

<u>Problem Set 10-7c</u>

In Problems 1 - 8 simplify (assuming no variable has the value 0) and write answers with positive exponents only.[49]

1. (a) $(3a^3)^2$ (b) $3(a^3)^2$ (c) $(3a^2)^3$ (d) $3a^{(3^2)}$

2. (a) $\dfrac{5x^2}{15xy^2}$ (b) $\dfrac{(5x)^2}{15xy^2}$ (c) $\dfrac{5x^2}{15(xy)^2}$

Following the introduction of the symbol $<$ in <u>First Course in Algebra</u> and the notation for congruence of segments in <u>Geometry With Coordinates</u>, both texts offer the following exercises which are consistent with Gagné's

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[47]Gagné, <u>Conditions</u>, p. 177.

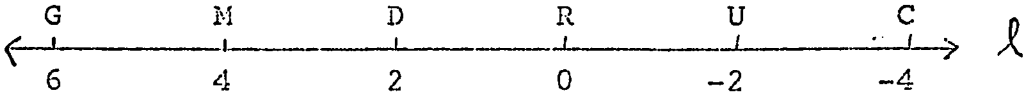
[48]SSMCIS, <u>Unified Modern Mathematics, Course I, Part I</u> (New York: Teachers College, Columbia University, 1968), p. 81.

[49]SMSG, <u>First Course in Algebra, Part II</u>, p. 277.

writings.

6. In which of A, B, C, D, E does the sentence have the same truth set as the sentence "$x \leq 5$"?[50]
   (A) $x > 5$ or $x = 5$
   (B) $x < 5$ and $x = 5$
   (C) $x \ngtr 5$
   (D) $x \nless 5$
   (E) $x \ngeq 5$

3. The diagram below indicates the coordinates which have been assigned to various points on line ℓ by a coordinate system.



   In each of the following, if the statement is meaningful, indicate whether it is true or false. If a statement is not meaningful, write "not meaningful" as your answer.[51]
   (a) $GD = RC$
   (b) $\overline{GM} = MD$
   (c) $\overline{GM} \cong MD$
   (d) $\overline{RM} \cong \overline{RC}$
   (e) $\overline{DG} \cong \overline{RU}$

Thus we find instances in both the SMSG series and the SSMCIS series which are consistent with the writings of Robert Gagné. As was previously mentioned, approaches were <u>not</u> consistent with one group or another, but were dependent upon the approach chosen for the presentation of a particular topic.

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[50]School Mathematics Study Group, <u>First Course in Algebra, Part I</u> (New Haven, Connecticut: Yale University Press, 1961), p. 74.

[51]School Mathematics Study Group, <u>Geometry With Coordinates, Part I</u> (New Haven, Connecticut: Yale University Press, 1962), p. 119.

Jerome Bruner

Many of the ideas of the Gestalt psychologists have been absorbed by the cognitive theorists of the 1960's. One such psychologist, Jerome Bruner, stands out in mathematics education. Bruner, who draws many of his examples from the field of mathematics, stresses the importance of discovery and structure in mathematics. To credit Bruner with introducing discovery teaching in mathematics would be an error as discovery-oriented approaches were utilized by Max Beberman (UICSM) prior to the publication of Bruner's The Process of Education. The previous statement is not meant to imply that Beberman introduced discovery teaching in mathematics. In writing about Bruner in the Sixty-ninth Yearbook of NSSE, Shulman felt

> it is an error to say that Bruner initiated the learning-by-discovery approach. It is far more accurate to say, that more than any one man, he managed to capture its spirit, provide it with a theoretical foundation, and disseminate it. Bruner is not the discoverer of discovery; he is its prophet.[52]

Bruner offers this sentence in The Process of Education. "We begin with the hypothesis that any subject can be taught effectively in some intellectually honest form to any child at any stage of development."[53] Such a statement represents a stand somewhat different from that of Piaget. Bruner offers three levels of representation, not marked

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[52] Shulman, "Psychology of Mathematics Education," p. 29.

[53] Jerome S. Bruner, The Process of Education (Cambridge, Massachusetts: Howard University Press, 1960), p. 33.

with specific age ranges, through which the child progresses in learning. The first level, referred to as enactive, is characterized by active manipulation of materials. Here, there is physical action by the student. Following this level is the second form of representation, iconic, in which the student no longer manipulates materials. Here, he would depend upon visual or other sensory organization of the problem. The last level, symbolic representation, is characterized by the use and manipulation of symbols, with the learner no longer relying on mental images or materials. The symbolic level has two other features, compactibility and a system of rules for manipulation of symbols.[54] Implementing this theory takes the form of the "spiral" curriculum mentioned in Chapter II.

Bruner sees the teaching and learning of structure, as opposed to mastery of facts and techniques, as the center of the classic problem of transfer of training.

> . . . There are many things that go into learning of this kind, not the least of which are supporting habits and skills that make possible the active use of the materials one has come to understand. If earlier learning is to render later learning easier, it must do so by providing a general picture in terms of which the relations between things encountered earlier and later are made as clear as possible.[55]

A similar view on structure is seen in the recommendations of the Commission on Mathematics of the College

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[54] Jerome S. Bruner, Toward a Theory of Instruction (Cambridge, Massachusetts: Belknap Press, 1966), pp. 10-11.

[55] Bruner, Process of Education, p. 12.

Entrance Board.

> . . . The contemporary point of view, while not discounting the manipulative skills necessary for effecient mathematical thought, puts chief emphasis on the structure or pattern of the system and on deductive thinking.[56]
>
> ⋮
>
> . . . Rather instruction should be oriented toward the development and understanding of the properties of a number field.[57]
>
> ⋮
>
> Development of generalized abstract concepts is difficult, too difficult to be a point of departure for the beginner, but nevertheless it is absolutely necessary if any true understanding of mathematics is to be obtained.[58]

Teaching of structure in mathematics takes the form of presenting various mathematical systems such as the group, ring, and field. Then, these ideas are tied in with other areas in mathematics. Examples of this occur in the commission's report. The concepts of group, ring, and field are introduced in the twelfth grade, and are illustrated by the rational numbers or modular systems. Then the idea is extended to the transformation group, illustrating the interrelationship between algebra and geometry. Concepts such as set, relation, and function are utilized throughout the secondary curriculum to aid in unifying mathematics.[59]

Discovery does not receive the coverage devoted to

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[56] College Entrance Examination Board, Comission on Mathematics, Program for College Preparatory Mathematics (New York: College Entrance Examination Board, 1959), p. 2.

[57] Ibid., p. 20.

[58] Ibid.

[59] Ibid., pp. 45-46.

structure. However, a "note" about Elementary Mathematics II reveals the following: "The presentation should encourage creative work on the part of the student, giving him the thrill of discovering as many geometric facts and relations as he can."[60] While not outwardly recommending a discovery approach, most commission members preferred "to see a developmental approach, which would encourage the student to discover as much of the mathematical subject matter for himself as his ability and the time available (for this is a time-consuming method) will permit."[61]

Specific examples in the texts of the School Mathematics Study Group are also consistent with Bruner's thinking. Some discovery exercises are shown below.

> 4. A segment connecting two vertices of a polygon which are not end-points of the same side is a diagonal of the polygon.
>
>   a. How many diagonals has a polygon with 3 sides? 4 sides? 5 sides? 6 sides? 103 sides? n sides?[62]

> In the next three problems, first discover a formula for the sum and then prove, by mathematical induction, that your formula is correct.[63]
>
> 8. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$

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[60] Ibid., p. 39.

[61] Ibid., p. 19.

[62] School Mathematics Study Group, Mathematics for High School: Geometry, Part II (New Haven, Connecticut: Yale University Press, 1961), p. 509.

[63] School Mathematics Study Group, Elementary Functions (Palo Alto, California: Stanford University Press, 1965), p. A-14.

9. $1^3 + 2^3 + 3^3 + \ldots + n^3$. (Hint: Compare the sums you get here with Example 1 in the text, or alternatively, assume that the required result is a polynomial of degree 4.)
10. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n \cdot (n + 1)$. (Hint: Compare this one with Example 8.)

The importance of structure can be seen in the emphasis given the field properties in grades seven through nine. Further stress on structure occurs when directed segment equivalence, congruence for segments, angles, and triangles, and proportionality are all shown to satisfy the properties of an equivalence relation--reflexive, symmetric, and transitive.

<u>THEOREM</u> <u>5-1</u>. Congruence for segments has the following properties.

Reflexive: $\overline{AB} \cong \overline{AB}$.
Symmetric: If $\overline{AB} \cong \overline{CD}$ then $\overline{CD} \cong \overline{AB}$.
Transitive: If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.

<u>THEOREM</u> <u>5-2</u>. Congruence for angles has the following properties.

Reflexive: $\angle A \cong \angle A$.
Symmetric: If $\angle A \cong \angle B$, then $\angle B \cong \angle A$.
Transitive: If $\angle A \cong \angle B$, and $\angle B \cong \angle C$, then $\angle A \cong \angle C$.

<u>THEOREM</u> <u>5-3</u>. Congruence for triangles has the following properties.

Reflexive: $\triangle ABC \cong \triangle ABC$
Symmetric: If $\triangle ABC \cong \triangle DEF$, then $\triangle DEF \cong \triangle ABC$.
Transitive: If $\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle GHI$, then $\triangle ABC \cong \triangle GHI$.[64]

<u>Properties of Directed Segment Equivalence</u>.

1. Directed segment equivalence is reflexive:

$$(\overline{A,B}) \doteq (\overline{A,B})$$

2. Directed segment equivalence is symmetric:

If $(\overline{A,B}) \doteq (\overline{C,D})$ then $(\overline{C,D}) \doteq (\overline{A,B})$

3. Directed segment equivalence is transitive:

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[64] SMSG, <u>Geometry With Coordinates, Part I</u>, pp. 234-35.

If $(\overline{A,B}) \doteq (\overline{C,D})$ and $(\overline{C,D}) \doteq (\overline{E,F})$,
then $(\overline{A,B}) \doteq (\overline{E,F})$.[65]

1. <u>The reflexive property of proportionality</u>

   $$(a, b, c, \ldots) \underset{p}{=} (a, b, c, \ldots)$$

   The proportionality has one as its proportionality constant.

2. <u>The symmetric property of proportionality</u>

   If $(a', b', c', \ldots) \underset{p}{=} (a, b, c, \ldots)$, then
   $$(a, b, c, \ldots) \underset{p}{=} (a', b', c', \ldots).$$

   The hypothesis tells us that $a' = ka$, $b' = kb$, $c' = kc$, $\ldots$. Since $k \neq 0$, it follows that $a = \frac{1}{k} a'$, $b = \frac{1}{k} b'$, $c = \frac{1}{k} c'$, $\ldots$. Therefore, $(a, b, c, \ldots) \underset{p}{=} (a', b', c', \ldots)$ with $\frac{1}{k}$ as the constant of proportionality

3. <u>The transitive property of proportionality</u>

   If $(a, b, c, \ldots) \underset{p}{=} (e, f, g, \ldots)$ and
   $(e, f, g, \ldots) \underset{p}{=} (q, r, s, \ldots)$ then
   $(a, b, c, \ldots) \underset{p}{=} (q, r, s, \ldots)$.

   If h and k are the respective constants of proportionality, then $a = he$ and $e = kq$; therefore $a = hkq$. Similarly $b = hkr$, $c = hks$, $\ldots$.
   Therefore, $(a, b, c, \ldots) \underset{p}{=} (q, r, s, \ldots)$.[66]

The Cambridge Conference on School Mathematics (1963) lists Bruner as director. Hence, one might expect his influence to be present in the report. Whether due to Bruner's presence or not, structure and discovery play a more prominent role here than in earlier conferences. Structure in mathematics is stressed throughout the suggested curriculum.

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[65] SMSG, <u>Geometry With Coordinates, Part II</u>, p. 687.

[66] SMSG, <u>Geometry With Coordinates, Part I</u>, p. 397-98.

> . . . We hope to make each student in the early grades truly familiar with the structure of the real number system and the basic ideas of geometry, both synthetic and analytic.[67]
>
> ⋮
>
> Moreover, we want to make the students familiar with part of the global structure of mathematics.[68]
>
> ⋮
>
> . . . The coordinated development of these two main streams of mathematics algebra and geometry is a central theme of the curriculum proposed here. We believe that, if the relations between arithmetic and geometry are brought out so that arithmetic ideas can be interpreted geometrically and vice versa, this will contribute to the student's understanding of both.[69]

The quote above represents a considerable change from the view presented in 1912 by the American Commissioners (see page 16).

Concerning the discovery method of teaching, the conference felt that "the discovery approach, in which the student is asked to explore a situation in his own way, is invaluable in developing creative and independent thinking in the individual."[70] However, further discussion was concerned with the amount of guidance to be provided by the teacher.

> . . . As a minimum the context and the very statement of the problem, or the equipment given to work with, is a guide to the student--a very important one.
>
> ⋮
>
> We believe that usually one should go farther than

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[67]Cambridge Conference, Goals, p. 8.

[68]Ibid.

[69]Ibid., p. 16.

[70]Ibid., p. 17.

> this in aiding discovery: that the teacher should be prepared to introduce required ideas when they are not forthcoming from the class; that he should bring attention to misleading statements in the way of the discussion, and summarize results as they come forward.[71]

While Bruner favors a "pure" discovery approach, the fact remains that discovery teaching in one form or another remains an important recommendation of the Cambridge Report.

Specific recommendations of the conference included the study of rings, integral domains, and fields in the seventh and eighth grades and vector spaces in grade ten. Group structure was extended to geometry through the study of transformations.[72]

The SSMCIS texts, attempting to implement a first approximation to the Cambridge Report, contain many examples which are consistent with the Cambridge Report and Jerome Bruner. Discovery problems with very little guidance are found in the SSMCIS texts. The example below comes from the seventh grade text.

> 2. Operational Checkers
>
> This game is played by two players on a finite set of lattice points. For example:
>
> | | | |
> |---|---|---|
> | (0,2) | (1,2) | (2,2) |
> | (0,1) | (1,1) | (2,1) |
> | (0,0) | (1,0) | (2,0) |
>
> You will need to use arithmetic of $(Z_3,+)$ so we will

---

[71] Ibid.

[72] Ibid., pp. 33-34.

list the necessary facts: $0 + 0 = 0$; $0 + 1 = 1$; $0 + 2 = 2$; $1 + 1 = 2$; $1 + 2 = 0$; $2 + 2 + 1$; and the commutative property will provide the other basic facts.

(a) One player has red checkers and the other has black checkers. A coin is tossed to determine who starts.

(b) The first player places a checker on any point that he wishes.

(c) The second player may then place a checker on any uncovered point and another point with coordinates obtained by adding the corresponding coordinates of the last two points covered. The addition to be used is that for $(Z_3,+)$.

(d) On each subsequent play, if the player's opponent had just placed a checker on (c,d), then the player may not only cover any uncovered point (a,b) but also $(a + c,\ b + d)$. If this point is already covered by his opponent's checkers, the player replaces it with one of his own. For example, if one player has just covered (2,1), the other player may cover (2,2) and also $(2 + 2,\ 1 + 2)$ which is (1,0).

(e) The game ends when all points are covered. The winner is the player with the most points covered. As you play the game you will see that it involves several interesting strategies.[73]

In this problem, the interesting strategies to be discovered are concerned with the additive identity and additive inverse.

In the following section from the seventh grade text, the student is to discover that in modular systems with prime moduli each non-zero element has a multiplicative inverse while those with composite moduli do not have this property. The student is also asked to discover the multiplication property of zero.

4. Construct multiplication tables for $(Z_6, \cdot)$ and for

---

[73] SSMCIS, Unified Modern Mathematics, Course I, Part I, pp. 320-21.

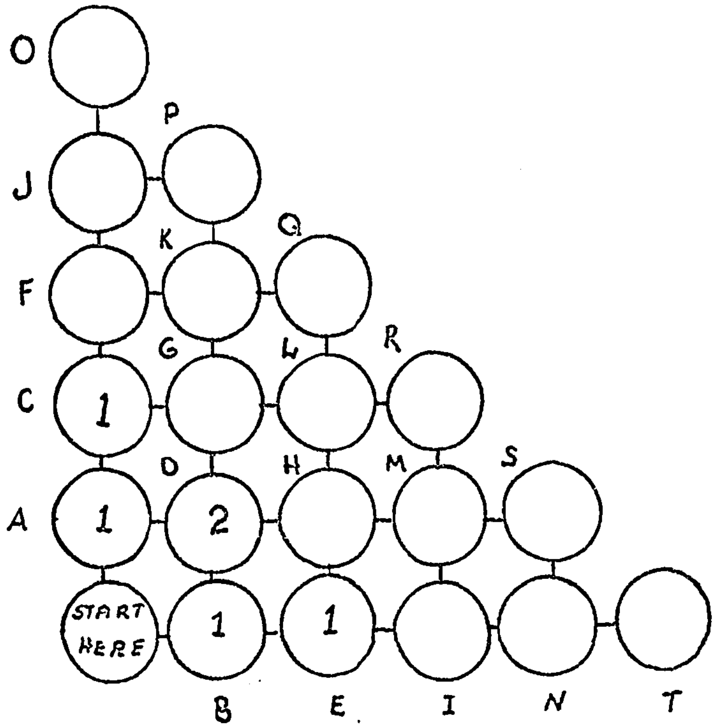
$(Z_7, \cdot)$.

(a) Examine the tables for $(Z_4, \cdot)$ and $(Z_6, \cdot)$. In what ways are these tables similar? In what ways are they different?

(b) Examine the tables for $(Z_6, \cdot)$ and $(Z_7, \cdot)$. What properties do they have in common? Can you find some essential differences between the tables?[74]

The very searching problem below can be seen as part of the spiral toward the binomial theorem and Pascal's rule.

In the diagram below the circles are called <u>states</u> and the routes for legally getting from one state to another are called <u>paths</u>. The numerals in circles A, B, C, D, and E indicate the number of paths from the start to the respective states.



(1) Procedure
  (a) Place a small disk on the lower left state labeled "start here."
  (b) Toss a coin.
  (c) If the coin lands heads-up, move to the next state on the right. If the coin

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[74] <u>Ibid</u>., p. 35.

lands tails-up, move to the next state above. (No moves to the left or down are allowed.)

(2) Experiment
   (a) Toss a coin five times and make the proper moves on each toss. What state did you reach?
   (b) Repeat the five-toss experiment 64 times and each time record your destination.
   (c) What was the relative frequency for each destination?
   (d) What do you notice about the location of your destinations?

(3) Experiment
   (a) Record your destinations for a two-toss experiment with 32 repetitions.
   (b) What was the relative frequency of each destination?
   (c) What do you notice about the location of these destinations?

(4) Counting Paths
   (a) Using the rules of our game, there is only one path to each of A, B, C and E but there are two paths to D. State G would have 3 paths, A-C-G, A-D-G, and B-D-G. Make a copy of the diagram of states and record the number of legal paths to each state inside the corresponding circles in the diagram.
   (b) Except for the border states in the left column and the bottom row, each state has exactly two possible predecessors, the one below and the one to the left. Find a method of computing the number of paths to a state by using the number of paths to each predecessor.
   (c) There are 2 one-toss paths, A and B. There are 4 two-toss paths, A-C, A-D, B-D and B-E. How many three-toss paths are there? Four-toss?
   (d) There are 32 five toss paths and 10 of these go to state Q. What is the probability of arriving at Q in five tosses, if we assume each path to be equally probable?
   (e) Compute the probabilities for each state in the diagram.[75]

Shorter discovery problems similar to those of the

[75]*Ibid.*, pp. 250-53.

SMSG texts are also present.

2. Let A, B, C, D be any 4 points, no three of which are collinear. Draw all the lines you can each containing two points.
   (a) How many did you get?
   (b) Do the same thing for 5 points, no 3 of which are collinear.
   (c) Copy the table below, fill in the blanks, and try to discover a pattern that you feel should continue.

Number of Points	2	3	4	5	6
Number of Lines					

   (d) Try to give an argument to support your generalization.

3.

   (a) Name the line shown in as many ways as you can using the names of the given points. There are 12 possible ways.
   (b) Name all the different rays you can find in the figure. Note $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are all the same ray.
   (c) How many different rays did you find?
   (d) Copy the table and fill in the blanks.

Number of Points on a Line	1	2	3	4	5	6
Number of Rays						

   (e) Try to discover a pattern that you feel ought to continue.
   (f) Try to give an argument to support your generalization.
   (g) Name all the segments formed by points A, B, C, D.
   (h) How many different segments did you get?
   (i) Copy the table and fill in the blanks.

Number of Points on a Line	2	3	4	5	6
Number of Segments					

   (j) Try to discover a pattern that you feel ought

to continue.
(k) Try to give an argument to support your generalizations.[76]

A considerable amount of importance is given to structure in the SSMCIS texts. There are numerous instances in which such mathematical systems as the group, ring, field, and vector space are studied and their properties emphasized. The interrelationships between algebra and geometry are brought out in the following exercises.

2. Prove the set of direct isometries is a group under composition.
3. Does the set of opposite isometries form a group?
4. Is the set of half-turns a subgroup of the group of isometries? (Hint: Consider the composition of two half-turns.)
5. Is the set consisting of half-turns and translations a subgroup of the group of isometries?
6. Does the set of rotations with the same center form a subgroup? In your investigations designate a rotation with center P and measure a by $r(P,a)$. Interpret $r(P,a)$ to be in the counter-clockwise directions when $a>0$, in the clockwise direction when $a<0$.[77]

12. Let T be the set of all translations of a line which are of the form

$$n \longrightarrow n + a,$$

where a is an integer.
(a) If two of these translations are applied, one after the other, is the result another such translation?
(b) If "o" is used to mean composition of translations in T, is (T,o) an operational system?
(c) Is composition of these translations associative? (See Exercises 11i and 11j.)
(d) Is composition of these translations commutative?

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[76] SSMCIS, Unified Modern Mathematics, Course I, Part II, pp. 83-84.

[77] SSMCIS, Unified Modern Mathematics, Course II, Part II (New York: Teachers College, Columbia University, 1968), p. 171.

(e) Is there an identity translation in T?
(f) Does each of the translations in T have an inverse?
(g) Is the system (T,o) a commutative group?[78]

10. Show that all translations having rules of the form $(x,y) \longrightarrow (x + pa, y + qb)$, where _a_ and _b_ are fixed rational numbers, and _p_ and _q_ are integers, form a group with the operation "o" (composition of mappings). (Difficult.)[79]

SSMCIS, in attempting a first approximation to the Cambridge Report, sought to unify secondary mathematics, thus departing from the tradition of teaching algebra and geometry separately. This effort was consistent with the views of the early Gestalt psychologists and cognitive theorists of the sixties. Mathematics is presented as a "whole," with emphasis being placed on the various interrelationships of its parts.

Thus, whether directly attributable to Bruner or not, there is considerable evidence that both programs in question, SMSG and SSMCIS, contain material and exercises which are consistent with his thinking.

Developmental Psychology

Developmental psychology has influenced the mathematics curriculum principally in two decades, the 1930's and the 1960's. The influence in the thirties was through the work of the Committee of Seven on Grade Placement in Arithmetic,

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[78] SSMCIS, _Unified Modern Mathematics, Course I, Part I_, pp. 190-91.

[79] SSMCIS, _Unified Modern Mathematics, Course I, Part II_, p. 356.

while that in the sixties was by Jean Piaget.

The Committee of Seven on Grade Placement in Arithmetic first reported their findings in the Twenty-ninth Yearbook of NSSE (1929). Their research continued throughout the decade of the 1930's, with the Thirty-eighth Yearbook of NSSE reporting their further findings. The work was directed by the Northern Illinois Conference on Supervision, which was formerly the Superintendents' and Principals' Association of Northern Illinois. Although their principal interest was arithmetic, their research is included here because of the later effects of their efforts. Furthermore, their research included material in the junior high school curriculum.

The Committee of Seven tried to answer these questions:

> 1. At what stage of a child's mental growth, as measured by intelligence tests, can he most effectively learn the following phases of arithmetic: addition facts, subtraction facts, subtraction process, multiplication facts simple long division, meaning of fractions, graphs, and Case-I percentage?
> 2. What degree of mastery of more elementary facts and skills is necessary for the effective learning of each of the above topics.[80]

Below is a chart which gives the recommended minimum mental age levels for the various topics. These levels were determined by the mental age range necessary to have three-fourths of the students scoring 80 per cent or better on the various topic tests.

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[80] Carleton W. Washburne, "The Grade Placement of Topics: A 'Committee of Seven' Investigation," Report of the Society's Committee on Arithmetic, Twenty-ninth Yearbook of the National Society for the Study of Education (Bloomington, Illinois: Public School Publishing Company, 1930), p. 641.

Topic	Minimal Mental-Age Level	Minimal Arith. Foundations Test Score (in Percent)
Addition facts		
Sums 10 and under	6 yr. 5 mo.	-
Sums over 10	7 yr. 4 mo.	-
Subtraction Facts		
Easier 50	7 yr. 0 mo.	84
Harder 50	8 yr. 3 mo.	96
Subtraction with Carrying	8 yr. 9 mo.	57
Multiplication Facts	8 yr. 4 mo. or	-
	10 yr. 2 mo.	96
Simple Long Division		
1- and 2- place Quotient	10 yr. 9 mo.	81
Meaning of Fractions		
Non-Grouping	9 yr. 0 mo.	-
Grouping	11 yr. 7 mo.	-
Graphs, Simple Bar	10 yr. 5 mo.	-
Percentage, Case I	12 yr. 4 mo. or	-
	13 yr. 4 mo.	100[81]

The recommendations of the Committee included a general postponement of arithmetic topics. These recommendations fell on very fertile soil. William Brownell, professor of educational psychology at Duke University, received many letters from schools that had adopted the Committee of Seven's placement of topics, asking what they should do next. Brownell's fundamental criticism concerned the mental age standards of the committee, while secondary concern was for the neglect of other fundamental issues in favor of grade placement.[82] Ten years later, harsher comments were

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[81] *Ibid.*, p. 670.

[82] William Brownell, "A Critique of the Committee of Seven's Investigations of the Grade Placement of Arithmetic Topics," *The Elementary School Journal*, XXXVIII (March, 1938), 506-07.

credited to William Betz in his address to the tenth summer meeting of the National Council of Teachers of Mathematics.

> The protagonists of postponement paid no heed to severe criticisms of outstanding students of arithmetic and of organizations like the National Council of Teachers of Mathematics. They rode roughshod over every type of opposition. The children of America became their hopeless victims.
>
> One evil begets another. The consequences of postponement speedily showed themselves. The war years furnished dramatic evidence of the widespread collapse of mathematics as a school subject. Very many of the men taking the selective examinations given by the armed forces lacked even a rudimentary acquaintance with fractions and decimals.[83]

The work of the Committee of Seven was also questioned by the Commission on Post-War Plans. While admitting that the "stepped-up" curriculum had been accepted, the Commission felt that there were some severe limitations.

> (a) There is no magic in birthdays. So far as learning the school subjects is concerned, increase in age makes for readiness only (or at least predominantly) to the extent that extra time gives opportunity for relevant experience. Postponement of arithmetical topics can by itself be only a questionable device for removing learning difficulties. (b) The earlier years are wasted. Children are deprived of ideas and skills which would give them surer control over their environment and their activities. (c) Not only this, but unless we are prepared to shift much of the traditional arithmetic of Grades 7 and 8 into high school, there must inevitably be a jamming of content in Grades 3, 4, 5, and 6, with consequent superficiality in learning. (d) When children are adequately prepared (i.e., when they have been provided with relevant experiences), the traditional placement <u>for mastery</u> is not far from wrong. Admittedly, the more troublesome phases of topics may well be postponed; but a movement of the easier phases to earlier

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[83] William Betz, "Five Decades of Mathematical Reform--Evaluation and Challenge," <u>The Mathematics Teacher</u>, XLIII (December, 1950), 383.

grades may be justified equally as well.[84]

During and immediately after World War II, the work of developmental psychologists did not affect the secondary curriculum. Buswell offered the following in the Fiftieth Yearbook of NSSE in 1951.

> . . . At present the effects of these studies [developmental] are more noticeable in the point of view of teachers than in actual placement of topics or in the distribution of the load of learning, but as the relationships become more clear the implications for the teaching of arithmetic will be more apparent.[85]

The amount of attention paid to developmental ideas was to change with the influence in the United States of the work of Jean Piaget. He lists four periods of development through which the child proceeds from birth to adolescence. These stages of development include the sensori-motor stage, the pre-operational stage, the stage of concrete operations, and the stage of formal operations. The sensori-motor, from birth to approximately two years, sees the child move from a neonatal, reflex level of existence to a mental level compatible with that of the more advanced sub-human animals. His activity changes from body-centered to activity centered. As the child nears two years of age, we find the capacity

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[84]Commission on Post-War Plans, "The Second Report of the Commission on Post-War Plans," The Mathematics Teacher, XXXVIII (May, 1945), 202-03.

[85]G. T. Buswell, "The Psychology of Learning in Relation to the Teaching of Arithmetic," The Teaching of Arithmetic, Fiftieth Yearbook of the National Society for the Study of Education (Chicago: University of Chicago Press, 1951), p. 153.

for symbolization appearing.[86]

The pre-operational stage extends from age two to age seven. This period sees the development of language and the use of symbols. The child is self-centered and is apt to affix his attention to one thing or happening to the exclusion of all others. In the latter part of the period there is the beginning of the idea of conservation. For the young child in this period, two rows of coins with an equal number in each become "unequal" when one row is altered by spacing the coins further apart. Conservation, according to Piaget, is acquired around seven years of age.[87]

The period of concrete operations, extending from age seven to age eleven, is characterized by the first appearances of logical thought. The child can perform operations, such as classifying and ordering, but this is done with concrete examples. Reversibility is acquired during this period.[88] Simply expressed, reversibility refers to the existence, for each operation, of an opposite operation which will cancel it. For example, $3 + 4 = 7$ while $7 - 4 = 3$. Subtraction and addition "undo" each other.

In the stage of formal operations, the child is

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[86] John H. Flavell, The Developmental Psychology of Jean Piaget (New York: D. Van Nostrand Company, Inc., 1963), pp. 85-86.

[87] Irving Adler, "Mental Growth and the Art of Teaching," The Mathematics Teacher, LIX (December, 1966), 708.

[88] D. E. Berlyne "Stages in Intellectual Development According to Jean Piaget," p. 3. (Mimeographed.)

capable of formal mathematical reasoning. He is no longer limited to the manipulation of physical objects, but can carry on operations on abstract ideas. He can reason from hypotheses and test his conclusions against reality. Essentially, this stage could be called the stage of adult reasoning.[89]

Irving Adler discussed significant features of Piaget's theories in The Mathematics Teacher. Those which are applicable at the secondary level are listed below.

1. A child's thinking is more flexible when it is based on reversible operations. For this reason we should teach pairs of inverse operations in arithmetic together.
2. The child in the stage of concrete operations has an incomplete grasp of the relations among the subsets of a set.
3. Physical action is one of the bases of learning. To learn effectively, the child must be a participant in events, not merely a spectator.
4. Since mental growth is associated with the discovery of invariants, we should make more frequent use of a systematic search for those features of a situation that remain unchanged under a particular group of transformations.
5. Piaget points out that topological relations are the first geometric relations that are observed by the child, but they are the last ones that were studied explicitly and formally by mathematicians.[90]

While most of the age ranges mentioned in Piaget's stages of development lie in the realm of elementary arithmetic, the teacher of secondary mathematics encounters students, possibly in the first half of the seventh grade, who are still in the third stage. Practically all secondary

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[89] Adler, "Mental Growth," p. 708.

[90] *Ibid.*, pp. 713-14.

students are in the stage of formal operations.

Did Piagetian-type thinking enter into the recommendations of any of the prominent national committees? This question will be considered by seeking direct references to Piaget or recommendations that favor Piaget's theories in the various committee reports. Further areas of agreement will be sought in the texts of SMSG and SSMCIS, which were intended in part to implement the recommendations of the Commission on Mathematics and the Cambridge Conference, respectively.

An early direct reference to Piaget can be found in the Report of the Secondary School Curriculum Committee in 1959. The committee cited Piaget's studies when they proposed that students of twelve years of age should be able to work with mathematical ideas at a higher level of abstraction than had been required in the traditional mathematics courses. A Wisconsin study was cited in which the basic principles of number relations and operations were successfully taught, with some degree of abstraction, to eighth-grade pupils. It also reported that "if-then" statements, usually taught in tenth-grade geometry, were successfully introduced to eighth-grade students.[91]

The Commission on Mathematics (1959) has several recommendations which are consistent with Piaget's stages of

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[91] Secondary School Curriculum Committee, "The Secondary Mathematics Curriculum," <u>The Mathematics Teacher</u>, LII (May, 1959), 403.

development. With the stage of formal operations beginning at eleven or twelve years of age, the child can then handle "adult thinking." The commission recommended that deductive reasoning should not be limited to geometry alone. Deductive reasoning was to be introduced in the ninth grade course.[92] This recommendation is consistent with Piaget's thinking as the ninth grade student has achieved the stage of formal operations, and is capable of reasoning from hypotheses.

Piaget's article in Scientific American (1953) mentioned that children form some topological concepts before Euclidean concepts. However, the Commission's recommendation of introduction of simple topological concepts to "stimulate the imagination" is not necessarily consistent with this. On the other hand, several examples in the SMSG texts, which implemented the program of the Commission on Mathematics, are consistent with Piaget's writings. The topological concept of the "interior" of a set of points is found in the seventh grade text in conjunction with angles. This is presented prior to the introduction of the metric concepts of the angle.

> From the figure it looks as if the angle ABC separates the plane containing it. It is true that the angle does separate the plane. The two pieces into which the angle separates the plane look somewhat different. They look like:

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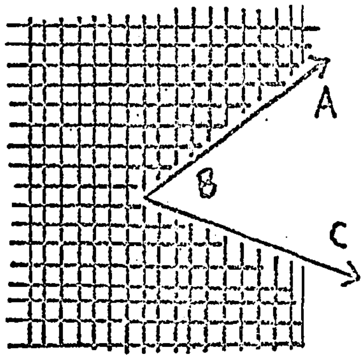
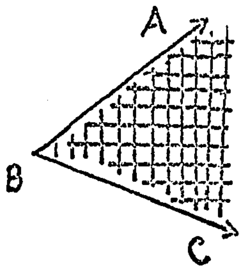
[92]College Entrance Examination Board, Program for College Preparatory Mathematics, p. 22.





We call the piece on the right the interior of the angle and the one on the left the exterior. We can define the interior of the ABC as the intersection of the A-side of the line $\overline{BC}$ and the C-side of the line $\overline{AB}$. It is the intersection of two half planes and does not include the angle. The exterior is the set of all points of the plane not on the angle or in the interior.[93]

The topological concept of an "interior point" is found in Geometry With Coordinates.

DEFINITIONS. The interior of a ray is the set of all points of the new ray except the endpoint.
Each point in the interior of a ray is called an interior point of the ray.
The interior of a segment is the set of all points of the segment except the two endpoints.
Each point in the interior of a segment is called an interior point of the segment.[94]

In general, the modern programs are introducing topological concepts at a level somewhat lower than has been customary in recent years. This action may not be totally consistent with Piaget's discovery that some topological concepts are formed before Euclidean and projective concepts. However, this action might be a result of Piaget's research. Confronted with the notion that topological concepts on a

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[93] School Mathematics Study Group, Mathematics for Junior High School, Volume I. Part I (New Haven, Connecticut: Yale University Press, 1960), pp. 138-39.

[94] SMSG, Geometry With Coordinates, Part I, p. 90.

intuitive level are relatively easy and can be understood by the young child, the mathematics educators are simply introducing these concepts earlier in the mathematics program.

Reversibility, while a characteristic of the stage of concrete operations, is used to bolster concepts at many levels of instruction. Early examples are concerned with the fundamental operations. Appropriate outside examples are also provided.

3-6. Inverse Operations

Often we do something and then we undo it. We open the door; we shut the door. We open the window; we close the window. One operation is the inverse of the other.

The inverse of putting on your coat is taking off your coat. The inverse operation of division is multiplication. The inverse operation of addition is subtraction.

Suppose you have \$220 in the bank and you add \$10 to it. Then you have \$220 + \$10 = \$230. Now undo this by drawing out \$10. The amount that remains is \$230 - \$10 = \$220. The athletic fund at your school might have \$1800 in the bank and after a game have \$300 more. Then the fund has \$1800 + \$300 or \$2100 in it. But the team needs new uniforms which cost \$300 so \$300 is withdrawn to pay for them. The amount left is \$2100 - \$300, or \$1800. These operations undo each other. Subtraction is the inverse of addition.

Of course, we could express this idea in more general terms. Let x represent the number of dollars originally in the bank. If the amount we deposit is b, then $x + b = a$, where a represents the number of dollars we now have in the bank. How shall we undo this operation? From the number of dollars represented by a, we subtract the number of dollars withdrawn, represented by b, and we have the number represented by x. We write $x = a - b$.

You use the idea of inverse operation when you use addition in checking subtraction. For example:

$$\begin{array}{r}203\\-\ 96\\\hline 107\end{array}\qquad\begin{array}{r}a\\-\ b\\\hline x\end{array}\qquad\text{check:}\quad\begin{array}{r}107\\+\ 96\\\hline 203\end{array}\qquad\begin{array}{r}x\\+\ b\\\hline a\end{array}$$

You also use the idea of inverse operation when you use multiplication to check division. For example:

$$\begin{array}{r} 18 \\ 16\overline{)288} \\ \underline{160} \\ 128 \\ \underline{128} \\ 0 \end{array} \qquad \text{check:} \quad \begin{array}{r} 16 \\ \times\ \underline{18} \\ 128 \\ \underline{160} \\ 288 \end{array}$$

or $288 \div 16 = 18$ check: $288 = 18 \times 16$.

Notice that if a and b are whole numbers, and if $a > b$, then there is a whole number x so that $b + x = a$. Examples: If a is 17 and b is 10, then x is the whole number 7 so that $10 + 7 = 17$; if a is 41 and b is 35, then x is the whole number 6 so that $35 + 6 = 41$. When a is greater than b it is always possible to find x so that $a = b + x$. Can you make the same generalization if the above operation $b + x = a$, is changed to multiplication, $b \cdot x = a$? If you substitute 2 for b and 3 for a you will see that there is no whole number that can be substituted for x such that $2 \cdot x = 3$. If one substitutes certain numbers--for example, if $a = 20$ and $b = 4$--then there is a whole number that can be substituted for x such that $4 \cdot x = 20$. In this example x must represent 5, since $4 \cdot 5 = 20$. We get the 5 by dividing 20 by 4. Also:

If b is 6 and a is 24 then x must be 4 since $6 \cdot 4 = 24$.
If b is 5 and a is 40 then x must be 8 since $5 \cdot 8 = 40$.
If b is 3 and a is 30 then x must be 10 since $3 \cdot 10 = 30$.

In each example the number for x is found by dividing the number represented by a by the number represented by b. In general, if there is a counting number x that can be multiplied by a counting number b to get counting number a, then this number x can be found by dividing a by b. We write this as $b \cdot x = a$. We multiply x by b to obtain a. To undo the operation we must perform the inverse operation which means that we must divide a by b to obtain x: $b\overset{x}{\overline{)a}}$. The inverse operation of multiplying by b is dividing by b.[95]

Exercises 3-6

1. Select the words or phrases that describe operations that have an inverse. An operation followed by its inverse returns to the original situation.
   a. Picking up the pencil. (Remember, "not picking up the pencil" is not an inverse operation. "Not

[95] SMSG, Mathematics for Junior High School, Volume I, Part I, pp. 88-90.

picking up the pencil" does not undo the operation of picking up the pencil.)

b. Put on your hat.
c. Getting into a car.
d. Extend your hand.
e. Multiply.
f. Build.
g. Smell the rose.
h. Step forward.
i. Jump from a flying airplane.
j. Addition.
k. Cutting off a dog's tail.
l. Subtraction.
m. Looking at the stars.
n. Talking.
o. Taking a tire off a car.

2. Write the inverse operation to each of those operations selected in Exercise 1.[96]

With our new symbol for division we write,

$\frac{12}{3} = 4$ because $3 \cdot 4 = 12$,

$\frac{6}{2} = 3$ because $2 \cdot 3 = 6$,

$\frac{10}{2} = 5$ because $2 \cdot 5 = 10$,

$\frac{63}{9} = 7$ because $9 \cdot 7 = 63$.[97]

<u>Class Discussion Problems</u>.

1. (a) If $x = \frac{10}{2}$, $2 \cdot x$ is what number?
   (b) If $x = \frac{63}{9}$, $9 \cdot x$ is what number?
   (c) If $x = \frac{5}{2}$, $2 \cdot x$ is what number?
   (d) If $x = \frac{5}{3}$, $3 \cdot x$ is what number?
   (e) If $x = \frac{4}{9}$, $9 \cdot x$ is what number?[98]

Later, in the senior course, the reversibility nature of inverse functions is stressed. This is especially true in the presentation of the exponential and logarithmic

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[96] <u>Ibid</u>., p. 90.

[97] <u>Ibid</u>., p. 191.

[98] <u>Ibid</u>., p. 192.

functions.

14. Express in exponential form
    a) $\log_{10}35 = y$ c) $\log_e d = b$
    b) $\log_2 25 = x$ d) $2 \log_{10}5 = x$

16. Express each of the following in logarithmic form.
    a) $\sqrt[3]{125} = 5$ c) $27^{4/3} = 81$
    b) $10^{-2} = 0.01$ d) $0.04^{3/2} = 0.008$ [99]

As a slightly more complicated example we may take

$$f: x \longrightarrow 2x - 3 \text{ and } g: x \longrightarrow \frac{x + 3}{2}.$$

Here f says "Take a number, double it, and then subtract 3." To reverse this, we must add three and divide by 2. This is the effect of the function g. In symbols,

$$(gf)(x) = g(f(x)) = g(2x - 3) = \frac{(2x - 3) + 3}{2} = x$$

Similarly,

$$(fg)(x) = f(g(x)) = f(\frac{x + 3}{2}) = 2\frac{x + 3}{2} - 3 = x.$$

In terms of our representation of a function as a machine, the g machine in each of these examples is equivalent to the f machine running backwards; each machine then undoes what the other does, and if we hook up the two machines in tandem, every element that gets through both will come out just the same as it originally went in.

We now generalize these two examples in the following definition of inverse functions.

Definition 1-8. If f and g are functions so related that $(fg)(x) = x$ for every element x in the domain of g and $(gf)(y) = y$ for every element y in the domain of f, then f and g are said to be inverses of each other. In this case both f and g are said to have an inverse, and each is said to be an inverse of the other.[100]

Some seventh grade students, approximately twelve years old, may not have progressed into the stage of formal operations. This child may have difficulty in considering the relations between the various subsets of a set. It thus

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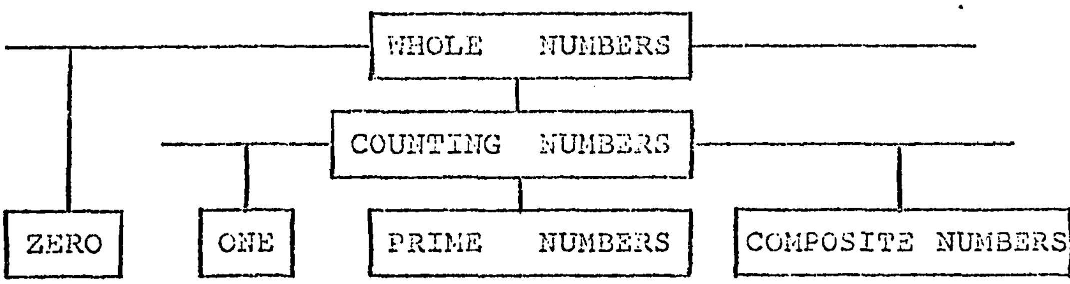
[99] School Mathematics Study Group, Elementary Functions, pp. 204-05.

[100] Ibid., pp. 30-31.

seems to be the relationship of inclusion that is the stumbling block for these children.[101] Related to the ideas of conservation and reversibility, the problem lies in considering the whole and a part simultaneously. The younger child, when considering a part simply forgets the whole and vice versa. A prerequisite for complete understanding of inclusion is the idea of reversibility, which is achieved in the stage of concrete operations.[102]

The very first volume of the junior high program offers the following illustration in which the set of whole numbers and its various subsets are represented in a diagram. For instance, the idea stressed below is that prime numbers are counting numbers <u>and</u> whole numbers.



[103]

Fig. 9

The programmed course in algebra also illustrates the structure of the real numbers, exhibiting the various inclusions. However, since this course is intended for the ninth

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[101] Jean Piaget, <u>The Child's Conception of Number</u> (New York: W. W. Norton and Company, 1965), p. 171.

[102] <u>Ibid</u>.

[103] SMSG, <u>Mathematics for Junior High School, Volume I, Part I</u>, p. 184.

grade student, the illustration was probably intended for review and general understanding rather than for clarification of inclusion.

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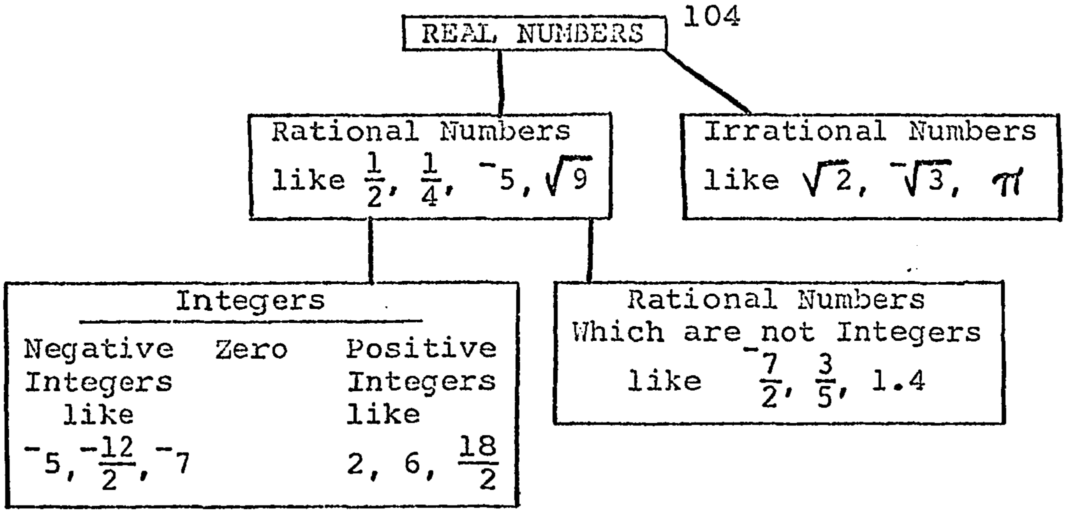

Fig. 10

The Cambridge Report (1963) mentions Piaget directly when explaining the task of the conference.

> We made no attempt to take account of recent researches in cognitive psychology. It has been argued by Piaget and others that certain ideas and degrees of abstraction cannot be learned until certain ages. We regard this question as open, partly because there are cognitive psychologists on both sides of it, and partly because the investigations of Piaget, taken at face value, do not justify any conclusion relevant to our task.[105]

Comments pertinent to accelerating the developmental stages of Piaget will be offered at the end of this section.

It can be noted that while the Cambridge Report does

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[104]School Mathematics Study Group, Programmed First Course in Algebra, Part 2 (Palo Alto, California: Stanford University Press, 1963), p. 179.

[105]Cambridge Conference for School Mathematics, Goals for School Mathematics, p. 3.

recommend proof in advance of Piaget's stage of formal operations, the proofs would be few in number and quite simple.[106]

Reversibility, the topological concept of neighborhood, and manipulation of physical objects are among the recommendations for the K-6 program. The secondary level includes deductive reasoning in the seventh grade, where the child has achieved the stage of formal operations. In the tenth grade course, continuity is to be defined using inverse images of neighborhoods instead of the usual ϵ, δ definition. Other topological concepts include the closed interval and the compact set. Reversibility, stressed in earlier courses in terms of fundamental operations, now appears in work with functions and matrices.[107]

The importance of invariants in Piaget's thinking extends throughout the report. In the K-2 program, importance is given to "symmetry and other transformations leaving geometrical figures invariant."[108] Invariance of concurrence under affine transformations is mentioned in the ninth grade geometry course. Invariance of subspaces under a transformation is included in the linear algebra portion of the tenth grade program.[109]

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[106] *Ibid.*, p. 39.

[107] *Ibid.*, pp. 31-66.

[108] *Ibid.*, p. 33.

[109] *Ibid.*, pp. 56-58.

The SSMCIS texts offer several exercises which are in line with Piaget's experiments. While no examples are listed below, the SSMCIS program requires formal proofs as early as the seventh grade. The examples of reversibility shown below are from the sections on subtraction of integers and clock arithmetic.

Example 1: $3 - (-14) = 17$, since $17 + (-14) = 3$
Example 2: $28 - 13 = 15$, since $15 + 13 = 28$
Example 3: $-17 - 5 = -22$ since $-22 + 5 = -17$
Example 4: $-33 - (-15) = -18$ since $-18 + (-15) = -33$[110]

Example 1. How do we evaluate $\frac{4}{3}$ in Z_5? In order to evaluate $\frac{4}{3}$ in Z_5 we proceed as follows. We ask, "Does there exist one and only one number in Z_5 which when multiplied by 3 in $(Z_5, \cdot)$ yields 4?" $3 \cdot 3 = 4$ in $(Z_5, \cdot)$, and no other number in Z_5 has this property. Thus we conclude that $\frac{4}{3} = 3$ in Z_5.[111]

The following example comes from a chapter that precedes the formal introduction to the inverse of a matrix. Hence, it is of an introductory nature.

1.7 Matrices and Coded Messages

A simple way to code a message is to substitute for each letter in the message a numeral, as given, for instance, in Figure 1.16.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

Figure 1.16

Thus the message GOOD LUCK would be sent as 7-15-15-4

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[110] SSMCIS, Unified Modern Mathematics, Course I, Part I, p. 193.

[111] Ibid., pp. 41-42.

12-21-3-11. The recipient of the message then decodes using the inverse substitution in Figure 1.16. An outsider can easily decode a message of this type by noting the frequency of numerals. One would expect, in general, the most frequent numeral to correspond to E, the next frequent numeral to T, and so on. To make it more difficult for an outsider to decode a message one can use a <u>coding matrix</u> in conjunction with the substitution transformation described above. After using the substitution determined by Figure 1.16 the numerals are arranged in 2x2 matrices. For GOOD LUCK this gives

$$\begin{bmatrix} 7 & 15 \\ 15 & 4 \end{bmatrix} \qquad \begin{bmatrix} 12 & 21 \\ 3 & 11 \end{bmatrix}$$

Then we introduce a coding matrix, say C = $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, multiplying (on the right) each matrix in the message by C.

$$\begin{bmatrix} 7 & 15 \\ 15 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 + 15 & 21 + 30 \\ 30 + 4 & 45 + 8 \end{bmatrix} = \begin{bmatrix} 29 & 51 \\ 34 & 53 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 21 \\ 3 & 11 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 24 + 21 & 36 + 42 \\ 6 + 11 & 9 + 22 \end{bmatrix} = \begin{bmatrix} 45 & 78 \\ 17 & 31 \end{bmatrix}$$

The coded message is 29-51-34-53 45-78-17-31. The recipient of this message has the problem of decoding it. First he restores the matrices and then multiplies each restored matrix by a decoding matrix, which in this case is D = $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

The process then is the following:

$$\begin{bmatrix} 29 & 51 \\ 34 & 53 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 58 + (-51) & (-87) + 102 \\ 68 + (-53) & (-102) + 106 \end{bmatrix} = \begin{bmatrix} 7 & 15 \\ 15 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 45 & 78 \\ 17 & 31 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 90 + (-78) & (-135) + 156 \\ 34 + (-31) & (-51) + 62 \end{bmatrix} = \begin{bmatrix} 12 & 21 \\ 3 & 11 \end{bmatrix}$$

Finally the inverse substitution, according to Figure 1.16 reveals the message GOOD LUCK.

The choice of coding and decoding matrices involves some mathematics that we will consider in Chapter 2.[112]

The topological concept "open" is found in the seventh grade course in relation to separation.

Our first statement about lines is obvious. It is called the <u>Line Separation Principle</u> and it expresses in

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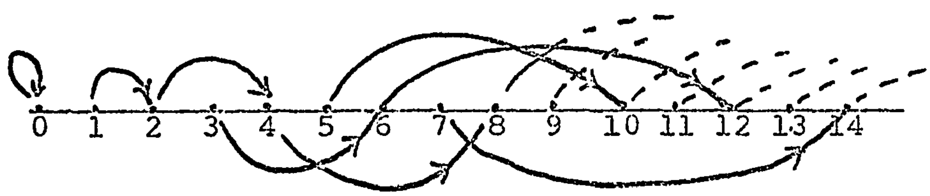
[112]SSMCIS, <u>Unified Modern Mathematics, Course III, Part I</u> (New York: Teachers College, Columbia University, 1969), pp. 19-21.

> a precise way the following idea: If we imagine one single point P removed from a line ℓ, the rest of the line "falls apart" into two distinct portions (subsets). Each of these portions is called an <u>open</u> <u>halfline</u>.[113]

The term "interior point of an angle" is also introduced in the seventh grade course.

> . . . All points of the angle, not in endrays, are called <u>interior</u> points of the angle and the set of interior points is called the interior of the angle.[114]

Invariance is illustrated pictorially in several mappings from the seventh grade course.[115]



The idea of a fixed point is not stressed in the section from which the above example is taken. However, in the same course, the section on dilations and reflections makes mention of a fixed point.[116]

Hence, we see that ideas consistent with the thinking of Jean Piaget are present in two modern programs in secondary mathematics. Further acknowledgement of his influence is seen in the following.

> . . . Today it is literally impossible to discuss the

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[113] SSMCIS, <u>Unified Modern Mathematics, Course I, Part II</u>, pp. 115-16.

[114] <u>Ibid</u>., p. 150.

[115] SSMCIS, <u>Unified Modern Mathematics, Course I, Part I</u>, p. 135.

[116] <u>Ibid</u>., pp. 289-93.

> psychology of instruction in mathematics without placing his [Piaget] contributions at center stage. His influence is not limited to the psychology of instruction. Many psychologists are seriously suggesting that his stature will eventually equal that of Freud as a pioneering giant in the behavioral sciences.[117]

With the constant movement of higher level material into the elementary and secondary curriculum, the older approach of omitting topics cannot suffice to handle the influx. Hence, research, especially in the lower elementary grades, is being conducted to determine the levels at which children can handle certain concepts. Much of the work that has been done is related to accelerating the progress through Piaget's various stages of development. Piaget seems to prefer not to attempt to accelerate the intellectual development of children. While not denying the possibility, he has questioned the possible outcomes of such acceleration.[118]

An example of such efforts is found in the work of Leslie Steffe. After finding in a study of conservation of numerousness that first grade children who are conservers handle certain problem solving tasks better than non-conservers, Steffe sought to test the hypothesis that certain experiences could enhance the ability of kindergarten children to conserve numerousness. The result was that he was able to effect the learning of conservation of numerousness with children whose mean age was five years eight

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[117]Shulman, "Psychology of Mathematics Education," p. 40.

[118]*Ibid.*, p. 45.

months, considerably lower than previous results and much lower than Piaget's estimate of seven-eight years.[119]

Wallach, in his research with conservation, found that reversibility was a key factor in inducing conservation in children. He added that "Experiences with such consequences--experiences with reversibility, in our sense of the term--thus may be what leads to the development of number conservation, not only in the present experiment, but in normal life."[120]

A later study by Wallach and others found that "in order for a child to conserve, he must both recognize reversibility and not rely on inappropriate clues."[121] An example would be an arrangement where one row of objects was arranged so that one object was left completely away from the others.

*   *   *   *   *   *   *

\#   \#   \#   \#   \#   \#   \#

This is what Wallach calls a misleading perceptual clue.[122]

Considerable dissertation research was directed toward

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[119]Leslie P. Steffe, E. Harold Harper, and Henry Van Engen, "An Evaluation of Teaching Conservation of Numerousness," School Science and Mathematics, LXIX (April, 1969), 287-93.

[120]Lise Wallach and R. I. Spratt, "Inducing Number Conservation in Children," Child Development, XXXV (March, 1964), 1069.

[121]Lise Wallach, A. Jack Wall, and Lorna Anderson, "Number Conservation: Its Roles of Reversibility, Addition-Subtraction and Misleading Perceptual Clues," Child Development, XXXVIII (June, 1967), 441.

[122]Ibid., p. 440.

investigations testing Piaget's theories. Concerning his stages of development, one study found that similar stages of development were found in the children of Nigeria.[123] Piaget's stages also occur in the educable mentally retarded. However, as expected, there was an age lag in the stages.[124]

Conservation was a frequently investigated topic as shown in the annual listing of research in The Arithmetic Teacher. Consistent with the work mentioned above by Wallach, Roll found that reversibility training precipitated a significant increase in conservation.[125] Concerning two methods used to teach conservation--using materials and visualizing materials--both methods were successful. The latter was better-suited for girls.[126] Following demonstrations involving conservation at the kindergarten level, one study found that the verbal statement of rules immediately after the demonstration was more successful than giving visual cues or no help at all.[127] Training also proved

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[123] Elizabeth Eme Samson Etuk, "The Development of Number Concepts: An Examination of Piaget's Theory with Yoruba-Speaking Nigerian Children," Dissertation Abstracts, XXVIII (October, 1967), 1295A.

[124] Alton David Quick, "Number and Related Concepts for Arithmetic for the Educable Mentally Retarded," Dissertation Abstracts, XXVII (March-April, 1967), 2953-54A.

[125] Marilyn N. Suydam, "Research on Mathematics Education, Grades K-8, for 1969," The Arithmetic Teacher, XVII (October, 1970), 525.

[126] Ibid., p. 526.

[127] Ibid., p. 524.

successful in attaining conservation with educable mentally retarded students.[128] Another study found that the attainment of stable and generalized levels of conservation were related to verbal competence.[129]

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[128] *Ibid.*, p. 525.

[129] *Ibid.*

CHAPTER VII

CONCLUSION

Throughout the period under consideration, the mathematics curriculum has reflected the changing ideas in psychology. Concerning the classical learning theories considered here, the prevailing theory was the one mentioned in the curriculum reports and journal articles of the time. This is especially true from 1893 until World War II. The "popular" theory at the time was also the principal theory involved in the teaching of mathematics.

The latter part of the nineteenth century and the early part of the twentieth century saw faculty psychology providing a positive argument for retaining mathematics in the secondary curriculum. Secondary mathematics in general, and geometry in particular, was good for the reasoning faculty of the mind. The importance of this should not be overlooked as the high school enrollments were increasing, and many students were seeking "practical" training in the high schools. However, faculty psychology soon ceased being the prominent theory. The first two decades of the twentieth century saw the work of Thorndike and others bring into question the entire theory of faculty psychology, thus providing an opportunity for connectionism to arise as "the theory." The climate was receptive to connectionism for several reasons. First, a theory was needed to replace faculty psychology. Second, connectionism fitted in well with the "tool" role mathematics was to take. And third, the nature of the subject itself permitted subdivision into small units. Connectionism's hold on the curriculum was strengthened by the fact that E. L. Thorndike, the principal contributor to the theory, was active in the field of mathematics itself. Thorndike wrote two texts directly related to mathematics teaching, The Psychology of Algebra and The Psychology of Arithmetic. These texts, coupled with his contributions to the report of the National Committee on Mathematical Requirements in 1923, and his journal articles, helped promote his image, and therefore his theory of learning. His enthusiastic students and followers frequently overworked his theories; and their efforts, more so than his own, were probably the cause of much of the subsequent criticism.

The introduction of Gestalt psychology to America and the attack on Thorndike by the Gestaltists weakened the hold of connectionism during the 1930's. Gestalt psychology contributed to a trend away from the atomistic approach of the connectionists to one which utilized the structure and interrelationships of mathematics. Once again, as with the connectionists, the fundamental nature of mathematics itself was such that many principles of Gestalt psychology were easily applied.

Thus prior to World War II, there was usually a discernible theory of learning in mathematics as evidenced by the chosen sources. However, the emergency of the war somewhat muddled the picture. The post-war years have seen no real theory in complete domination. Shortly after World War II, connectionism was again an important theory. However, thinking consistent with Gestalt psychology, now referred to as "cognitive theories," also rose in popularity.

Since the introduction of the modern programs in mathematics, there has been no clear-cut evidence of one predominant theory. On the contrary, both connectionism and Gestalt psychology can be identified as forerunners of the psychological theories active in mathematics education today. Bruner's theories of today incorporate ideas consistent with earlier Gestalt psychology. The work of Gagné and Skinner has its roots in the early stimulus-response psychologies. Evidence of thinking consistent with the ideas of these three men was shown in Chapter VI, thus reinforcing the contention that no one theory dominated the others.

To complement the influence of learning theories since 1930, the effects of developmental psychology were considered. Since the unfortunate results of the investigations of the Committee of Seven, developmental psychology, under the leadership of Jean Piaget, has come into prominence in mathematics education. Under fire in the forties and inactive in much of the fifties, developmental psychology has grown with the influence of Piaget on American education. Mathematics education, on the whole, has accepted many of Piaget's theories. Conservation, reversibility, and the various stages of intellectual growth now play an active role in modern curriculum planning.

Psychology also has had certain negative effects on secondary mathematics. The early part of the century was under the dying influence of faculty psychology, and saw difficult problems in mathematics being used for training of the various faculties of the mind. Recommendations contrary to this were seen in the committee reports of the latter part of the nineteenth century.

Other evidence of negative effects can be seen during the decade of the twenties. During the twenties, connectionism, with its S-R bonds, was the prevailing learning theory. Mathematics, in conjunction with this theory, was fragmentized. With little appeal to the structure of the number system, mathematics teachers stressed the "fixing of bonds." Evidence of this was seen in the lower grades in the number of addition facts to be memorized (100) as compared to the forty-five required today. At higher levels, the emphasis on single bonds also existed. Even the most complex tasks were decomposed into a seriation of single bonds to be mastered. Such practices led to a stress on "speed and accuracy." Goals such as these often precluded important goals such as understanding. Success in school mathematics could be attained without thorough understanding. However, with no basic foundation or structure to build on, the myriad of bonds soon weakened.

Thorndike, because of his role in the "testing movement," can be linked to The Committee of Seven. Relying strongly on their own tests, the Committee caused many appropriate topics in arithmetic to be postponed until the junior high school years. Criticism of this Committee was quite profound in the Reports of the Commission on Post-War Plans and in a 1950 address by William Betz.

One trend concerning stimulus-response psychologies is evident in this study. When the education community wishes to use mathematics as a "tool" subject or whenever an "emergency" exists, there is a turn toward some form of a stimulus-response psychology. For example, consider the 1920's and early 1930's (before Gestalt psychology was really active). Here, mathematics was important for its role in solving problems encountered in everyday life. When the occasion arose for a particular topic in mathematics, then it became necessary to teach that topic. To this end, connectionism was the answer. The World War II testing program and Sputnik sent educators scrambling for a "way to catch up." Here too, stimulus-response psychology had a crucial role, although its evidence is not as strong in the second instance.

The influence of psychological theories on the mathematics curriculum of the future may be more marked than that of the past decades. While several distinct theories played important roles in the sixties, the fact that many of the modern programs are utilizing psychologists to aid in producing materials could strengthen the influence. Furthermore, the School Mathematics Study Group, whose materials have been widely used, has included research in learning theory in mathematics as part of their future plans.

Suggestions for Further Research

This study could be complemented in any of several ways. A study similar in design to this one could be made by examining representative texts of the twentieth century instead of the principal committee reports. A second study might try to determine the influence of a single psychologist on one or more of the modern programs.

Studies of a similar nature might be attempted in other subject matter areas to see if psychological theories have influenced these curricula. Such studies, along with this one, might determine that the basic nature of mathematics makes it more responsive to psychological influence.

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Appendix I

COMMITTEE REPORTS

1. The Committee of Ten on Secondary School Subjects, NEA, 1893.
2. The Committee of Fifteen, NEA, 1893.
3. The Committee on College Entrance Requirements, NEA, 1899.
4. National Committee of Fifteen on Geometry Syllabus, American Federation of Teachers of the Mathematical and Natural Sciences and the National Education Association, 1909.
5. The International Commission on the Teaching of Mathematics, 1912.
6. National Committee on Mathematical Requirements, MAA, 1923.
7. The Committee of Seven on Grade Placement in Arithmetic, Northern Illinois Conference on Supervision, 1930.
8. The Committee on Geometry, NCTM, 1932.
9. The Committee on the Function of Mathematics in General Education, Progressive Education Association, 1940.
10. The Joint Commission to Study the Place of Mathematics in Secondary Education, MAA and NCTM, 1940.
11. War Preparedness Committee, AMS and MAA, 1940.
12. Committee on Essential Mathematics for Minimum Army Needs, NCTM, 1943.
13. Committee on Pre-Induction Courses in Mathematics, NCTM, 1943.
14. War Policy Committee, AMS and MAA, 1943.
15. The American Association for the Advancement of Science Co-operative Committee on the Teaching of Science, American Association for the Advancement of Science, 1944.
16. The Commission on Post-War Plans, NCTM, 1945.
17. The Commission on Mathematics, College Entrance Examination Board, 1955.
18. Science Teaching Improvement Program, American Association for the Advancement of Science, 1955.
19. Secondary School Curriculum Committee, NCTM, 1959.
20. Cambridge Conference on School Mathematics, 1963.

Appendix II

SECONDARY CURRICULUM GROUPS

1. Boston College Mathematics Institute (BCMI)
2. Comprehensive School Mathematics Project (CSMP)
3. Des Moines Public Schools Experimental Project in General Mathematics
4. Developmental Project in Secondary Mathematics At Southern Illinois University
5. Greater Cleveland Mathematics Program (GCMP)
6. Ontario Mathematics Commission (OMC)
7. School Mathematics Study Group (SMSG)
8. Secondary School Mathematics Curriculum Improvement Study (SSMCIS)
9. Stanford Program in Computer-Assisted Instruction
10. Syracuse University-Webster College Madison Project
11. University of Maryland Mathematics Project (UMMaP)
12. University of Illinois Committee on School Mathematics (UICSM)

Appendix III

DISSERTATION RESEARCH CENTERED ON PROGRAMMED INSTRUCTION

1. Alton, E. V. "An Experiment Using Programed Material in Teaching a Non-credit Algebra Course at the College Level (with) Supplement"

2. Austin, G. R. "A Study of Programed Instruction Response Styles and Reinforcement Schedules for Teaching Multiplication of Fractions"

3. Biddle, J. C. "Effectiveness of Two Methods of Instruction of High School Geometry on Achievement, Retention and Problem Solving Ability"

4. Bobier, D. T. "The Effectiveness of the Independent Use of Programed Textbooks in Aiding Students to Overcome Skill Weaknesses in English, Mechanics, and Arithmetic"

5. Eldredge, G. M. "Expository and Discovery Learning in Programed Instruction"

6. Farberm I. J. "A Study of the Use of Programed Instruction in a Group Situation"

7. Johnson, D. C. "Programed Learning: A Comparison of the School Mathematics Study Group Programed and Conventional Textbooks in Elementary Algebra"

8. Kellems, R. L. "A Comparative Analysis of the Effect of the Use of a Programed Text on Achievement and Efficiency in College Algebra"

9. Key, J. F. "An Investigation in the Teaching of Quadratic Equations Using Programed Instruction"

10. Krauser, A. W. "An Investigation of the Development of Abstract Thinking in Children Through Programed Instruction"

11. Little, C. E. "An Experimental Study of Programed Instruction in College Algebra at Colorado State College"

12. MacPherson, E. D. "Some Correlates of Anxiety in Learning Programed Mathematics"

13. Meadowcraft, B. A. "An Experiment with Programed Materials in Seventh Grade Arithmetic"

14. Neuhauser, D. L. "A Comparison of Three Methods of Teaching Programed Unit on Exponents to Eighth Grade Students"

15. Pagano, A. V. "A Study of the Effectiveness of a Programed Course in Contemporary Algebra Adapted for Presentation via Closed-Circuit Television"

16. Rafig, R. "A Scientific Evaluation of the Unit 'Relations and Functions' in Three Algebra Programed Textbooks in Terms of Educational Objectives"

17. Riggs, C. W. "The Construction and Evaluation of a Programed Text on the Interpretation of Graphs for Grade Five"

18. Robson, A. M. "A Comparative Study of the Teaching of First Year Algebra"

19. Tanner, G. L. "A Comparative Study of the Efficacy of Programed Instruction with Seventh Grade Low Achievers in Arithmetic"

Appendix IV

JOURNAL ARTICLES CONCERNING CONNECTIONISM

1. "The Nature of Algebraic Abilities"
2. "The Psychology of Errors in Algebra"
3. "The Psychology of the Equation"
4. "The Psychology of Problem Solving"
5. "The Strength of the Mental Connections Formed in Algebra"
6. "The Constitution of Algebraic Abilities"
7. "Systematic Procedure in the Solution of Algebraic Problems"
8. "The Thorndike Philosophy of Teaching the Processes and Principles of Arithmetic"
9. "Habit in the Education Process"
10. "A Study of Errors Made in a Ninth Year Algebra Class"
11. "An Analysis of the Learning-Units in N Processes in Algebra"