ADDITIVE NUMBER-THEORETIC FUNCTIONS

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CHAPTER I

INTRODUCTION

An integral part of mathematics is the study of functions. This study begins early in the student's mathematical career and continues throughout his association with mathematics. Some classes of functions are peculiar to a particular branch of mathematics, whereas other classes of functions cross over the sometimes hazy lines which separate the branches, and are associated with several of the branches.

Number-Theoretic Functions

The functions to be considered in this dissertation form a subclass of the class of functions known as number-theoretic functions.

Definition 1.1. A real-valued or complex-valued function whose domain is the set $N = \{1, 2, 3, \ldots\}$ of positive integers is a *number-theoretic* (or *arithmetic*) *function*.

Let $\mathfrak{F}$ denote the class of all number-theoretic functions. Then it can be shown [1, p. 237] that $(\mathfrak{F}, +, *)$, where $+$ denotes pointwise addition and $*$ denotes convolution product, is an abelian ring with unity.

A well-known subclass of $\mathfrak{F}$ is the class of multiplicative functions.

Definition 1.2. Let $f \in \mathfrak{F}$ and let $m, n \in N$. If $f(mn) = f(m)f(n)$ for $(m, n) = 1$, then f is a multiplicative function. If $f(mn) = f(m)f(n)$ for any pair of positive integers, then f is completely multiplicative.

The function z which is zero for all $n \in N$ (i.e., $z(n) = 0$) is multiplicative. The multiplicative functions which are of interest are those different from the function z. Let $\mathfrak{M}$ denote this particular class of functions. The properties of these functions can be found in any introductory number theory text and will not be developed here. Three of the more familiar multiplicative functions which will be used in this dissertation are defined below:

1. The function τ:

   $\tau(n)$ = the number of positive divisors of n.

2. The function φ (Euler's function):

   $\varphi(n)$ = the number of positive integers less than or equal to n, and relatively prime to n.

3. The function μ (Möbius function):

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ -1, & \text{if } n \text{ is a prime}, \\ 0, & \text{if } n \text{ is divisible by a power of a prime with exponent greater than } 1. \end{cases}$$

This paper will explore the subclass of $\mathfrak{F}$ known as additive functions. The identifying property of this class is the property $f(mn) = f(m) + f(n)$, whenever $(m, n) = 1$. This subclass forms a subgroup of the additive group $(\mathfrak{F}, +)$. It should be noted that if $f \in \mathfrak{M}$ is everywhere positive, then $\log f(n)$ is defined and is additive.

Conversely, given h additive, the function $e^{h(n)}$ is defined and is multiplicative. Furthermore, $e^{h(n)}$ has positive values. The base of the logarithm and of the exponential need not be restricted to the number e. The foregoing discussion holds as well for any base k, where k is positive and different from 1.

Most of the classical functions of number theory are additive or multiplicative, and many classical problems of arithmetic are closely connected with the behavior of these functions. Thus, the study of such functions occupies a significant place in the problems of number theory. Some of the literature in relation to additive functions deals with the existence of a distribution function. The earliest result is credited to I. J. Schoenberg [10, p. 46] who, in 1928, proved that $\frac{\varphi(n)}{n}$ has a distribution function. Considerable attention has been given to distribution functions by such authors as P. Erdős, M. Kac, J. Kubilius, C. Ryavec and others, in survey articles as well as in books. This particular area is not included in this dissertation.

Purpose

The purpose of this dissertation is to provide an introduction to additive functions, to develop their fundamental properties, and to indicate some of the principal areas of study related to these functions.

Chapter II is concerned with the fundamental concepts and properties of additive functions. Many examples are considered in connection with these. Included in the development are the properties: completely additive, strongly additive and prime-independent. Developed also, is the idea of an average of an additive function and, stemming from this, an inversion formula. Two isomorphisms related to additive and multiplicative functions exist. One is the connection stated earlier between a subgroup of the multiplicative functions with respect to pointwise multiplication and the additive functions with respect to addition. The other isomorphism results from defining an operator L_1 on multiplicative functions, and is an isomorphism between the two subclasses themselves, with the operations convolution product and addition, respectively. A list of additive functions which are used in this dissertation appears in the Appendix.

The sum $\frac{T(x)}{x}$, where $T(x) = \sum_{n \leq x} \Omega_k(n)$, is the arithmetic average value of functional values of $\Omega_k(n)$ over the first n integers. In Chapter III, a formula for approximating the value of $T(x)$ is developed. After considering the case for $k = 0$, a general form is then derived.

There are some theorems which show that if an additive function is in some sense "smooth", then it must be a very special type. Chapter IV deals with this type of function. One result associated with this "smoothness" is that the logarithmic function is essentially the only nondecreasing additive function. With this in mind, conditions on additive, and completely additive, functions to ensure they will be constant multiples of the logarithmic function are discussed.

A rather extensive bibliography concerning additive functions is included. Articles related to distribution functions have been included, even though this area is not developed here. This bibliography was compiled in order to have available a ready reference of articles on additive functions, and distribution functions of additive functions.

The material in this dissertation assumes the knowledge obtained from an introductory number theory course, as well as some knowledge of abstract algebra and advanced calculus, all obtained on the undergraduate level. Little more than basic number theory is required to read and understand the material in Chapter II. For Chapter III, the reader will have to understand some of the concepts of advanced calculus. A greater maturity is necessary for Chapter IV, in that comprehension of limit superior and limit inferior, and other basic concepts from analysis is essential. If the reader wishes to go further than the scope of this dissertation, it will be necessary for him to have more than a basic knowledge of analysis and probability.

Notation

A word about notation! Throughout this paper, the letters p and q (with and without subscripts) will denote primes; unless stated otherwise, x and y will represent real numbers; and other Roman letters will denote positive integers. Also, if the base of a logarithm is not given, the logarithm is a natural logarithm. That is, $\log n = \log_e n$.

Used, also, will be the convenient notations O, o, and $\sim$. A discussion of these notations, which were introduced by E. Landau, can be found in either LeVeque [36, pp. 92-95] or Hardy and Wright [17, pp. 7-8]. To define these, let $f(x)$ be any function defined on some unbounded set S of positive numbers, and let $g(x)$ be defined and positive for all positive x.

Definition 1.3. If there is a number M such that $|f(x)| < Mg(x)$ for all sufficiently large $x \in S$, then $f(x) = O(g(x))$.

Thus, $10x = O(x)$ because there is an M such that $|10x| < Mx$; in particular, M can be any number larger than 10. Again, $\sin x = O(1)$, since $|\sin x| < M \cdot 1$ for any $M > 1$. When the statement $f(x) = O(1)$ is used, this means that $f(x)$ remains bounded as x increases. An immediate consequence of the definition is that $O(1) \pm O(1) = O(1)$.

Another useful notation is the o-notation.

Definition 1.4. If

$$\lim_{\substack{x \to \infty \\ x \in S}} \frac{f(x)}{g(x)} = 0 ,$$

then $f(x) = o(g(x))$.

This implies that the function g grows faster than does the function f. For example, $x = o(x^2)$, since $\lim \frac{x^2}{x} = \lim \frac{1}{x} = 0$. If the notation $f(x) = o(1)$ is used, this means $f(x) \to 0$ as $x \to \infty$.

Some of the properties of these two notations which will be used in this paper are proved below.

Lemma 1.1. $O(O(g(x)) = O(g(x))$.

Proof: Suppose $f(x) = O(g(x))$ and $h(x) = O(f(x))$, i.e., suppose $f(x)$ and $g(x)$ are positive, $|f(x)| < M'g(x)$ and $|h(x)| < M''f(x)$. Then $|h(x)| < M'M''g(x)$. Hence, the lemma. △

Lemma 1.2. $O(g(x)) \pm O(g(x)) = O(g(x))$.

Proof: Let $f(x) = O(g(x))$ and let $h(x) = O(g(x))$. Then $|f(x)| < M'g(x)$ and $|h(x)| < M''g(x)$. Since

$$
\begin{aligned}
|f(x) \pm h(x)| &\le |f(x)| + |h(x)| \\
&< M'g(x) + M''g(x) \\
&= M\,g(x),
\end{aligned}
$$

where $M = M' + M''$, the lemma follows. $\triangle$

Lemmas 1.1 and 1.2 imply that $O(f(x)) \pm O(g(x)) = O(\max\{f(x), g(x)\})$.

<u>Lemma 1.3.</u> If $f(x) < g(x)$, then $O(f(x)) = O(g(x))$.

Proof: Let $h(x) = O(f(x))$. Then $|h(x)| < M f(x) < M g(x)$. Thus, $h(x) = O(g(x))$. Hence, the lemma follows. $\triangle$

Note that here symmetry of equality does not follow. For example, $O(x) = O(x^2)$, but $O(x^2) \neq O(x)$. Also, symmetry between o and O does not hold. As an example, if $f(x) = o(1)$, then $f(x) \to 0$, which implies that $|f(x)| < M \cdot 1$ for all sufficiently large x. Hence $f(x) = O(1)$, or $o(1) = O(1)$. But the reverse is not true, since $f(x) = O(1)$ implies $f(x)$ remains bounded, which does not necessarily imply that $f(x) \to 0$.

<u>Lemma 1.4.</u> If for all $x \in S$ $f(x) > 0$, then $f(x)\,O(g(x)) = O(f(x)\,g(x))$.

Proof: Let $h(x) = O(g(x))$. Then $|h(x)| < M g(x)$. Since $f(x)$ is positive, $f(x)|h(x)| = |f(x)\,h(x)| < M f(x)\,g(x)$. Therefore, the lemma follows. $\triangle$

<u>Lemma 1.5.</u> $o(1)(A + o(1)\,A) = o(1)A$.

Proof: Let $g(x) = o(1)$ and $h(x) = o(1)$. Then $g(x) \to 0$, $h(x) \to 0$, and

$$\begin{aligned} o(1)(A + o(1)A) &= g(x)(A + h(x)A) \\ &= g(x)A + g(x)\,h(x)A \\ &= g(x) + gh(x)A \to 0 \ . \end{aligned}$$

Thus, the lemma. △

Lemma 1.6. $o(1)a = o(1)$, for a constant.

Proof: Suppose $g(x) = o(1)a$. Then $\frac{g(x)}{a} \to 0$ and therefore, $g(x) \to 0$. Therefore, the lemma follows. △

Finally, $\sim$ is considered.

Definition 1.5. If

$$\lim_{\substack{x\to\infty \\ x \in S}} \frac{f(x)}{g(x)} = 1 ,$$

then $f(x)$ is said to be asymptotically equal to $g(x)$, written $f(x) \sim g(x)$.

As an example, $1 + x \sim x$ since $\frac{1+x}{x} \to 1$ as $x \to \infty$. The expression $f(x) \sim g(x)$ is equivalent to the equation $f(x) = g(x) + o(g(x))$.

In arguments concerning the behavior of functions as x becomes infinite, these notations are useful, in that a complicated expression can be replaced by its principal term plus an additional term, whose possible size is indicated.

Definitions and notation from number theory which are necessary in order to read the material in the remainder of the paper have been included here. Other notation will be described as needed. In the next chapter, the reader is introduced to additive functions and is given a chance to become familiar with them through the many examples which are provided.

CHAPTER II

CLASSIFICATION OF ADDITIVE FUNCTIONS

Familiar to the first-year algebra or trigonometry student is the logarithmic function. One of the properties of logarithms is that the logarithm of a product is the sum of the logarithms of the factors, i.e., $\log mn = \log m + \log n$. It is this property that is the defining property of additive functions.

Definition 2.1. For $f \in \mathfrak{F}$, f is an additive function if

$$f(mn) = f(m) + f(n) , \tag{1}$$

whenever $(m, n) = 1$. If (1) is true for any pair of natural numbers, then f is a completely additive function.

Some segments of literature refer to additive functions as "restrictedly additive," and to completely additive functions as "totally additive," but throughout this paper, the nomenclature used will be additive and completely additive. It should be noted that a completely additive function is always additive, but not conversely. Since $\log mn = \log m + \log n$ for any values of m and n, the log function is completely additive.

First, some examples are considered in order to become familiar with the definition. In the way of notation, the canonical representation of a natural number n is given either by

$$\prod_{p^a \| n} p^a, \text{ where } p^a \| n \text{ means } p^a | n \text{ but } p^{a+1} \nmid n,$$

or by

$$\prod_{i=1}^{r} p_i^{a_i}, \text{ where } p_i^{a_i} \| n.$$

Example 2.1. Let $z(n) = 0$ for all n. Then $z(mn) = 0 = 0+0 = z(m) + z(n)$. Thus, z is completely additive and hence, additive. The function z so defined is the identity element for addition in $\mathfrak{F}$.

Example 2.2. Let $\omega(n)$ be the number of distinct prime divisors of n. If $\prod_{i=1}^{r} p_i^{a_i} = n$, then $\omega(n) = \sum_{i=1}^{r} 1$. Note $\omega(1) = 0$. The function ω is additive since, for $(m, n) = 1$, $\omega(mn)$ is the number of distinct prime divisors of m plus the number of distinct prime divisors of n, i.e., $\omega(mn) = \omega(m) + \omega(n)$. Or, if $m = \prod_{j=1}^{s} q_j^{b_j}$, where $p_i \neq q_j$ for any i and j,

$$\begin{aligned} \omega(m) + \omega(n) &= \sum_{j=1}^{s} 1 + \sum_{i=1}^{r} 1 \\ &= s + r = \sum_{i=1}^{s+r} 1 \\ &= \omega(mn). \end{aligned}$$

To show that ω is not completely additive, note that $\omega(12) = 2$, since 2 and 3 are the only distinct primes that divide 12; $\omega(9) = 1$, since 3 is the only distinct prime divisor of 9. But $\omega(12 \cdot 9) = 2$ and $\omega(12) + \omega(9) = 2+1 = 3$.

Example 2.3. Let $\Omega(n)$ be the number of all prime factors of n. As an example, $\Omega(12) = \Omega(2^2 3) = 2 + 1 = 3$. Then $\Omega(n) = \sum_{i=1}^{r} a_i$, where $n = \prod_{i=1}^{r} p_i^{a_i}$. Further, let $m = \prod_{i=1}^{r} p_i^{b_i}$, where $a_i \geq 0$ and $b_i \geq 0$. Then n and m are said to be in comparable form. Thus $nm = \prod_{i=1}^{r} p_i^{a_i+b_i}$, and $\Omega(nm) = \sum_{i=1}^{r} (a_i + b_i) = \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i = \Omega(n) + \Omega(m)$. Hence Ω is completely additive.

Example 2.4. A generalization of the additive function in the previous example is $\Omega_k(n) = \sum_{i=1}^{r} a_i^k$, where k is a nonnegative integer. It should be noted that $\Omega_0(n) = \omega(n)$ (Example 2.2), while $\Omega_1(n) = \Omega(n)$ (Example 2.3). The function Ω_k is additive. Suppose $(n, m) = 1$ and n and m are in comparable form. Then $a_i > 0$ implies $b_i = 0$, and conversely, so that $a_i + b_i = a_i$ or b_i for each i. Then

$$\sum_{i=1}^{r} (a_i + b_i)^k = \sum_{i=1}^{r} a_i^k + \sum_{i=1}^{r} b_i^k .$$

However, if $(n, m) \neq 1$, there is some i such that $a_i \neq 0$ and $b_i \neq 0$, so that $(a_i + b_i)^k \neq a_i^k + b_i^k$ (unless $k = 1$). Therefore, Ω_k is not completely additive.

Example 2.5. Define the function γ by $\gamma(1) = 0$, $\gamma(n) = \sum_{i=1}^{r} a_i p_i$, $n \geq 2$. This is known as Chawla's function [2]. Given m and n in comparable form then $nm = \prod_{i=1}^{r} p_i^{a_i+b_i}$ and

$$\gamma(nm) = \sum_{i=1}^{r} (a_i + b_i) p_i$$

$$= \sum_{i=1}^{r} a_i p_i + \sum_{i=1}^{r} b_i p_i = \gamma(n) + \gamma(m) .$$

Thus γ is completely additive, so additive. Also, $\gamma(p) = p$. Chawla proves that the function γ is uniquely determined by the three conditions $\gamma(1) = 0$, $\gamma(nm) = \gamma(n)+\gamma(m)$, and $\gamma(p) = p$.

Example 2.6. An extension of Chawla's function is the function defined by

$$\gamma_k(n) = \sum_{i=1}^{r} a_i p_i^k ,$$

where k is a nonnegative integer and n is in canonical form. If $(n,m) = 1$, n and m in comparable form, and $nm = \prod_{i=1}^{r} p_i^{a_i+b_i}$, then

$$\gamma_k(nm) = \sum_{i=1}^{r} (a_i + b_i)\, p_i^k$$

$$= \sum_{i=1}^{r} a_i p_i^k + \sum_{i=1}^{r} b_i p_i^k$$

$$= \gamma_k(n) + \gamma_k(m) .$$

Hence γ_k is completely additive.

In Example 2.4, the function Ω_k which is defined on the exponents of the prime power factors of a number was discussed. In the next example, consider the prime factors themselves. Again, k is a nonnegative integer.

Example 2.7. Define the function $s_k(n)$ by

$$s_k(n) = \sum_{p \mid n} p^k .$$

If $(n, m) = 1$, $p|n$ and $q|m$, then for r prime,

$$s_k(nm) = \sum_{r|nm} r^k .$$

Since r is prime, either $r = p$ or $r = q$, so

$$\sum_{r|nm} r^k = \sum_{p|n} p^k + \sum_{q|m} q^k$$

$$= s_k(n) + s_k(m).$$

Hence s_k is additive.

That s_k is not completely additive can be seen by considering $n = 12$ and $m = 15$: $s_k(12) = s_k(2^2 3) = 2^k + 3^k$; $s_k(15) = s_k(3 \cdot 5) = 3^k + 5^k$; $s_k(12 \cdot 15) = s_k(2^2 3^2 5) = 2^k + 3^k + 5^k$.

Throughout the remainder of the paper, $\mathfrak{a}$ will denote the class of additive number-theoretic functions. It is seen from the examples that $\mathfrak{a} \neq \emptyset$. Then with addition of functions as defined on $\mathfrak{F}$, $\mathfrak{a}$ is an abelian subgroup of the additive group of the ring $(\mathfrak{F}, +, *)$.

<u>Theorem 2.1.</u> $(\mathfrak{a}, +)$ is an abelian subgroup of $(\mathfrak{F}, +)$.

Proof: Recall from modern algebra [39], that to prove $(\mathfrak{a}, +)$ is a subgroup of $(\mathfrak{F}, +)$, it is sufficient to prove for $f, g \in \mathfrak{a}$, that $(f - g) \in \mathfrak{a}$.

Since $f, g \in \mathfrak{a}$ and $\mathfrak{a} \subset \mathfrak{F}$, then $f, g \in \mathfrak{F}$. For functions in $\mathfrak{F}$, and for $(m, n) = 1$,

$$(f - g)(mn) = f(mn) - g(mn)$$

$$= (f(m) + f(n)) - (g(m) + g(n))$$

$$= (f(m) - g(m)) + (f(n) - g(n))$$

$$= (f - g)(m) + (f - g)(n) .$$

But this implies $(f - g) \in \mathfrak{A}$. Hence $(\mathfrak{A}, +)$ is a subgroup of $(\mathfrak{F}, +)$. Since addition is commutative in $\mathfrak{F}$, addition is commutative in $\mathfrak{A}$. Hence, the theorem. △

In fact, more can be said about $(\mathfrak{A}, +)$. Let α be a real number and let $f \in \mathfrak{A}$. Then $\alpha f \in \mathfrak{A}$, for if $(m, n) = 1$,

$$\alpha f(mn) = \alpha[f(m) + f(n)]$$

$$= \alpha f(m) + \alpha f(n) .$$

Thus, since $(\mathfrak{A}, +)$ is an abelian group, it follows that the group $(\mathfrak{A}, +)$ is a vector space over the reals.

It would be desirable to be able to identify a set of generating elements for the group $(\mathfrak{A}, +)$. This has not yet been done. However, a subset of $\mathfrak{A}$ which is generated by the logarithm function is considered in a later chapter.

The additive property (1) yields a way to express the value of an additive function $f(n)$ in terms of its value at the prime power factors of n.

<u>Theorem 2.2</u>. If $f \in \mathfrak{A}$ and n_i, $1 \leq i \leq k$, are relatively prime in pairs, then $f(1) = 0$ and

$$f\left(\prod_{i=1}^{r} n_i\right) = \sum_{i=1}^{r} f(n_i) . \tag{2}$$

In particular, if n is in canonical form, then

(3) $$f(n) = \sum_{p^a \| n} f(p^a) .$$

Proof: Since $(n, 1) = 1$, for any n, $f(n) = f(n \cdot 1) = f(n) + f(1)$. Thus, $f(1) = 0$.

The proof of (2) is by induction on k. If $k = 1$, then $f\left(\prod_{i=1}^{1} n_i\right) = f(n_1) = \sum_{i=1}^{1} f(n_i)$. So the theorem is true for 1. The statement becomes the defining property of additive functions if $k = 2$. Suppose, then, the property is true for some fixed k. Consider the integer $n = n_1 n_2 \cdots n_k n_{k+1}$. Now $(n_i, n_{k+1}) = 1$, for $1 \leq i \leq k$, which implies $\left(\prod_{i=1}^{k} n_i, n_{k+1}\right) = 1$. Since f is additive,

$$f\left(\prod_{i=1}^{k+1} n_i\right) = f\left(\left(\prod_{i=1}^{k} n_i\right) n_{k+1}\right) = f\left(\prod_{i=1}^{k} n_i\right) + f(n_{k+1}) .$$

But by the induction hypothesis, $f\left(\prod_{i=1}^{k} n_i\right) = \sum_{i=1}^{k} f(n_i)$. Hence, $f\left(\prod_{i=1}^{k+1} n_i\right) = \sum_{i=1}^{k} f(n_i) + f(n_{k+1}) = \sum_{i=1}^{k+1} f(n_i)$. Therefore, (2) is true for any natural number k.

Since the primes $p_1, p_2, \ldots, p_k$ are all distinct, $(p_i^a, p_j^b) = 1$, if $i \neq j$. Then if $n = \prod_{p^a \| n} p^a$ is a product of k factors which are relatively prime in pairs, (3) follows from (2) by the substitution $n_i = p_i^{a_i}$. △

The beauty of this theorem is that it reduces the problem of deriving a formula for $f(n)$ to the much easier job of deriving a formula for $f(p^a)$. In order to use this formula, though, it is necessary to know that the function f is in $\mathfrak{G}$.

Special Types of Additive Functions

In general, if $f \in \mathcal{G}$, then $f(p^a) \neq af(p)$. To see this, consider the function Ω_k as defined in Example 2.4. If $p^a = 2^5$, then $\Omega_k(2^5) = 5^k$, while $5\Omega_k(2) = 5(1^k)$. The reason for the inequality is that Ω_k is not completely additive. That the two expressions $f(p^a)$ and $af(p)$ are equal for completely additive functions follows directly from Theorem 2.2, since a completely additive function has the additive property (1) for any pair of natural numbers.

Corollary 2.2.1. If $f \in \mathcal{G}$, then f is completely additive if and only if $f(p^a) = af(p)$.

Corollary 2.2.2. If $f \in \mathcal{G}$, then f is completely additive if and only if $f(n^a) = af(n)$.

Consider the function $\log \frac{\varphi(n)}{n}$. The function $\varphi(n)$ is multiplicative, but not completely multiplicative [1, p. 82]. Let $(m, n) = 1$. Then $\varphi(mn) = \varphi(m)\varphi(n)$. Thus,

$$\log \frac{\varphi(mn)}{mn} = \log \frac{\varphi(m)\varphi(n)}{m \cdot n} = \log \frac{\varphi(m)}{\varphi m} + \log \frac{\varphi(n)}{\varphi n},$$

and therefore, $\log \frac{\varphi(n)}{n}$ is additive. This function is not completely additive. This can be shown by letting $m = 2$ and $n = 6$:

$$\log \frac{\varphi(2)}{2} = \log \frac{1}{2} = -\log 2 ;$$

$$\log \frac{\varphi(6)}{6} = \log \frac{2}{6} = \log \frac{1}{3} = -\log 3 ;$$

$$\log \frac{\varphi(12)}{12} = \log \frac{4}{12} = \log \frac{1}{3} = -\log 3 .$$

Now, a formula for $\varphi(n)$ is given by $\prod_{p^a \| n} (p^a - p^{a-1})$, so that

$$\log \frac{\varphi(p^a)}{p^a} = \log \frac{(p^a - p^{a-1})}{p^a} = \log \frac{(p-1)}{p} .$$

But $\log \frac{\varphi(p)}{p} = \log \frac{(p-1)}{p}$. That is, this function is dependent only on the prime p and not on the exponent of p. This property is called strongly additive.

Definition 2.2. If $f \in \mathfrak{a}$, then f is strongly additive if $f(p^a) = f(p)$, for $a \geq 2$.

Other functions which are strongly additive are $\omega(n)$ and $s_k(n)$.

Since strongly additive functions depend only on p and not on the exponent of the power of p, one might ask if there are functions for which the function value depends only on the exponent and not on the prime p. Consider the additive function $\Omega(n)$ as defined in Example 2.3. Note that $\Omega(p_1^2 p_2^3) = \Omega(p_3^3 p_4 p_5) = 5$, where p_1, p_2, p_3, p_4 and p_5 are all distinct primes. This property is called prime-independent, and is not peculiar to additive functions, since $\tau(n) = \prod_{p^a \| n} (a+1)$ depends only on the exponents and $\tau(n)$ is a multiplicative function.

Definition 2.3. Let $f \in \mathfrak{F}$. The function f is prime-independent if $f(p^a)$ depends only on a.

The functions $\Omega_k(n)$ and $\log \tau(n)$ are other examples of additive functions which are prime-independent.

An Average Function

The question might arise as to whether an additive function can generate an additive function, and if so, how. Let h be an additive function. Consider the function f such that $f(n)$ is the sum consisting of a term $h(d)$ for each divisor d of n. This, if $n = 6$ and $h(n) = \omega(n)$, then $f(6) = \omega(1) + \omega(2) + \omega(3) + \omega(6) = 1 + 1 + 2 = 4$, while $f(2) + f(3) = 2$. It is seen, then, that f is not additive. Some adjustment is necessary before it is at all possible for f to be additive. As is seem in the next theorem, such an adjustment can be made so that the resulting sum is an additive function. This new function can be thought of as an average, since the sum of the functional values at the divisors of a given n is divided by the number of divisors of n.

<u>Theorem 2.3</u>. Let $h \in \mathcal{A}$. Define $H(n) = \frac{1}{\tau(n)} \sum_{d|n} h(d)$. Then $H \in \mathcal{A}$.

Proof: From the definition, $H(mn) = \frac{1}{\tau(mn)} \sum_{d|mn} h(d)$. If $(m,n) = 1$ and $d|mn$, then $d = d'd''$, where $d'|m$, $d''|n$ and $(d', d'') = 1$. Thus,

$$
\begin{aligned}
H(mn) &= \frac{1}{\tau(m)\tau(n)} \sum_{\substack{d'|m \\ d''|n}} h(d'd'') \\
&= \frac{1}{\tau(m)\tau(n)} \sum_{\substack{d'|m \\ d''|n}} (h(d') + h(d'')) \\
&= \frac{1}{\tau(m)\tau(n)} \left[\sum_{d''|n} \sum_{d'|m} h(d') + \sum_{d'|m} \sum_{d''|n} h(d'') \right] \\
&= \frac{1}{\tau(m)\tau(n)} \left[\tau(n) \sum_{d'|m} h(d') + \tau(m) \sum_{d''|n} h(d'') \right]
\end{aligned}
$$

$$= \frac{1}{\tau(m)} \sum_{d'|m} h(d') + \frac{1}{\tau(n)} \sum_{d''|n} h(d'')$$

$$= H(m) + H(n).$$

Therefore, $H \in \mathcal{G}$. △

Since $H \in \mathcal{G}$, in order to obtain $H(n)$ for some $h \in \mathcal{G}$, by Theorem 2.2, it is necessary to evaluate H only at powers of a prime. Recall that $\tau(p^a) = a+1$.

Example 2.8. By Example 2.3, $\Omega(p^a) = a$, so that

$$H(p^a) = (a+1)^{-1} \sum_{d|p^a} \Omega(d)$$

$$= (a+1)^{-1}(\Omega(1) + \Omega(p) + \ldots + \Omega(p^a))$$

$$= (a+1)^{-1}(0+1+\ldots+a)$$

$$= \frac{1}{a+1} \frac{a(a+1)}{2} = \frac{a}{2}$$

$$= \frac{1}{2}\Omega(p^a).$$

Hence,

$$H(n) = \sum_{p^a \| n} H(p^a) = \sum_{p^a \| n} \frac{a}{2} = \frac{1}{2} \sum_{p^a \| n} a = \frac{1}{2}\Omega(n).$$

Example 2.9. Recall that $\Omega_k(p^a) = a^k$. So that

$$H(p^a) = (a+1)^{-1} \sum_{d|p^a} \Omega_k(d)$$

$$= (a+1)^{-1}\left[\Omega_k(1) + \Omega_k(p) + \ldots + \Omega_k(p^a)\right]$$

$$= (a+1)^{-1}\left[1^k + 2^k + \dots + a^k\right]$$

$$= (a+1)^{-1} \sum_{i=1}^{a} i^k .$$

Now $\sum_{i=1}^{a} i^k = (a+1)\,g(a)$, where $g(a)$ is a polynomial in a of degree k, with rational coefficients in absolute value less than 1. Thus,

$$H(p^a) = (a+1)^{-1}(a+1)\,g(a)$$

$$= g(a) = b_0 + b_1 a + \dots + b_k a^k$$

$$= \sum_{i=1}^{k} b_i \Omega_i(p^a) ,$$

where $|b_i| < 1$ and $b_k \neq 0$. Therefore,

$$H(n) = \sum_{p^a \| n} \sum_{i=1}^{k} b_i \Omega_i(p^a) .$$

As an illustration, consider the cases for $k = 2$ and $k = 3$. For $k = 2$,

$$\sum_{i=1}^{a} i^2 = \frac{a(a+1)(2a+1)}{6} .$$

Therefore,

$$H(p^a) = \frac{(a+1)^{-1}a(a+1)(2a+1)}{6}$$

$$= \frac{2a^2 + a}{6}$$

$$= \frac{1}{3}a^2 + \frac{1}{6}a ,$$

and hence,

$$H(n) = \sum_{p^a \| n} \left[\frac{1}{3} \Omega_2(p^a) + \frac{1}{6} \Omega_1(p^a) \right] .$$

For the case where $k = 3$, $\sum_{i=1}^{a} i^3 = \frac{a^2(a+1)^2}{4}$, which implies

$$H(p^a) = \frac{(a+1)^{-1} a^2 (a+1)^2}{4} = \frac{a^3 + a^2}{4}$$

This implies

$$H(n) = \sum_{p^a \| n} \left[\frac{1}{4} \Omega_3(p^a) + \frac{1}{4} \Omega_2(p^a) \right] .$$

Let h be strongly additive. Then $h(p^a) = h(p)$ and

$$H(p^a) = \frac{1}{a+1} \sum_{d \| p^a} h(d)$$

$$= \frac{1}{a+1} [h(1) + h(p) + \dots + h(p^a)]$$

$$= \frac{a}{a+1} h(p) .$$

But since $H(p) = \frac{1}{2} [h(1) + h(p)] = \frac{h(p)}{2}$, it follows that H is not strongly additive whenever h is. This leads to the question whether H retains the property of being completely additive or prime-independetn when h has these properties. Since Ω is completely additive, it follows that $H(n) = \frac{1}{2} \Omega(n)$, in Example 2.8, is completely additive. So here is an example where h completely additive implies H completely additive. In fact, this is always true as is seen in the next theorem.

Theorem 2.4. If $f \in \mathcal{G}$ is completely additive, then H is completely additive. In particular, $H(n) = \frac{f(n)}{2}$.

Proof: By Corollary 2.2.1, it suffices to show $H(p^a) = aH(p)$. Thus

$$\begin{aligned} H(p^a) &= (a+1)^{-1} \sum_{d \mid p^a} f(d) \\ &= (a+1)^{-1}[f(1) + f(p) + f(p^2) + \dots + f(p^a)] \\ &= (a+1)^{-1}[f(p) + 2f(p) + \dots + af(p)] \\ &= (a+1)^{-1}(1 + 2 + \dots + a)\, f(p) \\ &= (a+1)^{-1}(\tfrac{1}{2})\, a(a+1)\, f(p) \\ &= \frac{af(p)}{2} = \frac{f(p^a)}{2}, \end{aligned}$$

since f is completely additive. Also,

$$aH(p) = a\{\tfrac{1}{2}(f(1) + f(p))\} = \frac{af(p)}{2} = \frac{f(p^a)}{2},$$

which implies H is completely additive. △

As to the question of prime-independence of H, suppose f is prime-independent, i.e., suppose $f(p^a)$ depends only on the exponent a. Let $f(p^a) = g(a)$, a function of a only. Then

$$\begin{aligned} H(p^a) &= (a+1)^{-1} \sum_{d \mid p^a} f(d) \\ &= (a+1)^{-1}[f(1) + f(p) + f(p^2) + \dots + f(p^a)] \\ &= (a+1)^{-1}[g(1) + g(2) + \dots + g(a)], \end{aligned}$$

which is an expression that depends only on a. This proves the following theorem.

Theorem 2.5. If $f \in \mathfrak{a}$ is prime-independent, then H is prime-independent.

A natural question to ask concerning the function H is whether the additive function h can be recovered if H is given. That it can be is a consequence of the Möbius inversion formula [1, p. 88]. This formula states that if $f \in \mathfrak{F}$ and $F(n) = \sum_{d|n} f(d)$, then $f(n) = \sum_{d|n} \mu(d) F(\frac{n}{d})$, where μ is the Möbius function.

Theorem 2.6. If $H(n) = \frac{1}{\tau(n)} \sum_{d|n} h(d)$ is additive, then h is additive and is given by

$$h(n) = \sum_{d|n} \mu(d)\, \tau(\tfrac{n}{d})\, H(\tfrac{n}{d}) . \tag{4}$$

Proof: Applying the Möbius inversion formula to $\tau(n) H(n) = \sum_{d|n} h(d)$ yields (4).

To show $h \in \mathfrak{a}$, let $(m,n) = 1$, then

$$h(mn) = \sum_{d'd''|mn} \mu(d'd'')\, \tau\left(\frac{m}{d'}\,\frac{n}{d''}\right) H\left(\frac{m}{d'}\,\frac{n}{d''}\right),$$

where $d = d'd''$, $d'|m$, $d''|n$, and $(d',d'') = 1$. Therefore, since μ and τ are multiplicative, and H is additive,

$$h(mn) = \sum_{\substack{d'|m \\ d''|n}} \mu(d')\mu(d'')\, \tau\left(\frac{m}{d'}\right) \tau\left(\frac{n}{d''}\right) \left[H\left(\frac{m}{d'}\right) + H\left(\frac{n}{d''}\right)\right]$$

$$= \sum_{d''|n} \mu(d'')\,\tau\!\left(\frac{n}{d''}\right) \sum_{d'|m} \mu(d')\,\tau\!\left(\frac{m}{d'}\right) H\!\left(\frac{m}{d'}\right)$$

$$+ \sum_{d'|m} \mu(d')\,\tau\!\left(\frac{m}{d'}\right) \sum_{d''|n} \mu(d'')\,\tau\!\left(\frac{n}{d''}\right) H\!\left(\frac{n}{d''}\right) .$$

Now $\sum_{d|n} \mu(d)\,\tau(\frac{n}{d}) = (\mu * \tau)(n)$, and $\mu * \tau$ is multiplicative, since both μ and τ are. So to evaluate $\mu * \tau$, it is necessary to evaluate this function at prime powers. Since $\mu(p^a) = 0$, for $a > 1$,

$$(\mu * \tau)(p^a) = \mu(1)\,\tau(p^a) + \mu(p)\,\tau(p^{a-1}) + 0$$

$$= a - (a-1) = 1 .$$

Hence, $(\mu * \tau)(n) = \prod_{p^a \| n} (\mu * \tau)(p^a) = \prod 1 = 1$, and therefore,

$$h(mn) = \sum_{d'|m} \mu(d')\,\tau\!\left(\frac{m}{d'}\right) H\!\left(\frac{m}{d'}\right) + \sum_{d''|n} \mu(d'')\,\tau\!\left(\frac{n}{d''}\right) H\!\left(\frac{n}{d''}\right)$$

$$= h(m) + h(n) .$$

Thus, $h \in \mathcal{Q}$ when $H \in \mathcal{Q}$. △

Example 2.10. Let $H(n) = \log \tau(n)$. Then by (4), and since $\mu(p^a) = 0$ for $a > 1$,

$$h(p^a) = \sum_{d|p^a} \mu(d)\,\tau\!\left(\frac{p^a}{d}\right) \log \tau\!\left(\frac{p^a}{d}\right)$$

$$= \mu(1)\,\tau(p^a) \log \tau(p^a) + \mu(p)\,\tau(p^{a-1}) \log \tau(p^{a-1})$$

$$= (a+1) \log (a+1) - a \log a$$

$$= \log \frac{(a+1)^{a+1}}{a^a} .$$

Hence,

$$h(n) = \sum_{p^a \| n} h(p^a) = \sum_{p^a \| n} \log \frac{(a+1)^{a+1}}{a^a} .$$

The average function gives a way of generating an additive function given any additive function h. It is clear from Theorem 2.6 that if h is addi6ive, $\mu * \tau h$, where τh is the pointwise product of τ and h, is also additive.

Note that the property of μ and τ required in the proof of Theorem 2.6 is that $\mu * \tau = u_0$, where $u_0(n) = 1$ for all n. This suggests that given any pair of multiplicative functions, f and g, such that $f * g = u_0$ the convolution product $f * gh$ is additive if h is additive.

Theorem 2.7. If $f, g \in \mathfrak{M}$ such that $f * g = u_0$, and $h \in \mathfrak{A}$, then $f * gh$ is additive.

Proof: Let $(m, n) = 1$. If $d | mn$, then $d = d'd''$, $d' | m$, $d'' | n$, and $(d', d'') = 1$. Thus,

$$\begin{aligned}(f * gh)(mn) &= \sum_{d | mn} f(d)\, gh\left(\frac{mn}{d}\right) \\ &= \sum_{\substack{d' | m \\ d'' | n}} f(d'd'')\, g\left(\frac{m}{d'} \frac{n}{d''}\right) h\left(\frac{m}{d'} \frac{n}{d''}\right) \\ &= \sum_{d'' | n} f(d'')\, g\left(\frac{n}{d''}\right) \sum_{d' | m} f(d')\, g\left(\frac{m}{d'}\right) h\left(\frac{m}{d'}\right) \\ &\quad + \sum_{d' | m} f(d')\, g\left(\frac{m}{d'}\right) \sum_{d'' | n} f(d'')\, g\left(\frac{n}{d''}\right) h\left(\frac{n}{d''}\right) .\end{aligned}$$

Since $\sum_{d|n} f(d)\, g(\frac{n}{d}) = (f * g)(n) = u_0(n) = 1$,

$$(f * gh)(mn) = \sum_{d'|m} f(d')\, g\left(\frac{m}{d'}\right) h\left(\frac{m}{d'}\right) + \sum_{d''|n} f(d'')\, g\left(\frac{n}{d''}\right) h\left(\frac{n}{d''}\right)$$

$$= (f * gh)(m) + (f * gh)(n) . \qquad \triangle$$

Even though $\mu * \tau = \tau * \mu = u_0$, it is interesting to note that $\mu * \tau h \neq \tau * \mu h$, for h additive. This can be seen in the following example. Hence, in general, it follows that $f * gh \neq g * fh$ when $f * g = g * h = u_0$.

Example 2.11. Let h be additive. Then

$$(\tau * \mu h)(p^a) = \tau(p^a)\, \mu h(1) + \tau(p^{a-1})\, \mu h(p) + 0 ,$$

since $\mu(p^a) = 0$ for $a > 1$. Thus

$$(\tau * \mu h)(p^a) = (a+1)\, \mu(1)\, h(1) + a\mu(p)\, h(p) = -ah(p) ,$$

because $h(1) = 0$. But,

$$(\mu * \tau h)(p^a) = \mu(1)\, \tau h(p^a) + \mu(p)\, \tau h(p^{a-1})$$

$$= (a+1)\, h(p^a) - ah(p^{a-1}) .$$

Identification of Completely Additive Functions

A somewhat different way to generate a completely additive function is given by the next theorem. It gives a more sophisticated way of defining an additive function than has been illustrated previously. This result is used several times later in the paper.

Theorem 2.8. Let $f \in \mathcal{a}$. If

$$g(n) = \lim_{t\to\infty} \frac{f(n^t)}{t} \tag{5}$$

then g is completely additive.

Proof: Let $(m, n) = 1$, then

$$g(mn) = \lim_{t\to\infty} \frac{f(m^t n^t)}{t} = \lim_{t\to\infty} \frac{f(m^t) + f(n^t)}{t}$$

$$= \lim_{t\to\infty} \frac{f(m^t)}{t} + \lim_{t\to\infty} \frac{f(n^t)}{t}$$

$$= g(m) + g(n) ,$$

which implies $g \in \mathcal{a}$.

By Corollary 2.2.2, it suffices to show $g(n^r) = rg(n)$, for any n. Thus

$$g(n^r) = \lim_{t\to\infty} \frac{f((n^r)^t)}{t} = r \lim_{t\to\infty} \frac{f(n^{rt})}{rt}$$

$$= r \lim_{s\to\infty} \frac{f(r^s)}{s}$$

$$= rg(n) . \qquad \triangle$$

Chapter IV is concerned with additive functions which have a "smoothness" about them. One condition which implies smoothness is the condition

$$\frac{1}{x} \sum_{n \leq x} |f(n+1) - f(n)| \to 0 , \quad \text{as} \quad x \to \infty . \tag{6}$$

The next theorem shows that if an additive function satisfies the condition in (6), then it must be completely additive.

Theorem 2.9. If $f \in G$ and f satisfies (6), then f is completely additive.

Proof: Let p^a be an arbitrary prime power. For a fixed A let

(7) $$\Delta_A(n) = \max\{|f(n+k) - f(n)|\}, \quad k = 1, 2, \ldots, A.$$

Now

$$|f(n+k) - f(n)| \leq \sum_{i=1}^{k} |f(n+i) - f(n+i-1)|,$$

so that

$$\Delta_A(n) \leq \sum_{i=1}^{A} |f(n+i) - f(n+i-1)|.$$

But

$$\sum_{n \leq x} \sum_{i=1}^{A} |f(n+i) - f(n+i-1)| \leq A \sum_{n \leq x+A} |f(n+1) - f(n)| = o(x),$$

by the limit in (6). Hence, for A fixed,

(8) $$\sum_{n \leq x} \Delta_A(n) = o(x).$$

Let N be an arbitrary positive integer such that $(N, p) = 1$. Then, since f is additive, $f(p^a N) - f(N) = f(p^a)$, so that

(9) $$f(p^a) - af(p) = f(p^a N) - af(p) - f(N).$$

If it can be shown that the left hand member of (9) is zero, then Corollary 2.2.1 implies that f is completely additive. To this end, consider

$$f(p^aN+1) - af(p) - f(N+1) = f(p^aN+1) - f(p) - f(p^{a-1}N+1) + f(p^{a-1}N+1)$$
$$- (a-1)f(p) - f(N+1)$$
$$= \{f(p^aN+1) - f(p) - f(p^{a-1}N+1)\}$$
$$+ \{f(p^{a-1}N+1) - f(p) - f(p^{a-2}N+1)\}$$
$$+ f(p^{a-2}N+1) - (a-2)f(p) - f(N+1) .$$

Repeating this process a total of a times results in

$$f(p^aN+1) - af(p) - f(N+1) = \{f(p^aN+1) - f(p) - f(p^{a-1}N+1)\}$$
$$+ \{f(p^{a-1}N+1) - f(p) - f(p^{a-2}N+1)\}$$
$$+ \dots + \{f(pN+1) - f(p) - f(p^0N+1)\}$$
$$= \sum_{i=1}^{a} \{f(p^iN+1) - f(p) - f(p^{i-1}N+1)\} .$$

Thus, by the additivity of f, $f(p) + f(p^{i-1}N+1) = f(p^iN+p)$, so that

$$(10) \qquad f(p^aN+1) - af(p) - f(N+1) = \sum_{i=1}^{a} \{f(p^iN+1) - f(p^iN+p)\} .$$

Now, by (7), for $n = p^iN+1$,

$$|f(p^iN+1) - f(p^iN+p)| \le \Delta_p(p^iN+1) .$$

Substitution into (10) produces

$$(11) \qquad |f(p^aN+1) - af(p) - f(N+1)| \le \sum_{i=1}^{a} \Delta_p(p^iN+1) .$$

But

$$|f(p^a N) - af(p) - f(N)| = |f(p^a N+1) - f(p^a N) - f(p^a N+1) - af(p) - f(N+1) + f(N+1) - f(N)|$$

$$\leq |f(p^a N+1) - f(p^a N)| + |f(N+1) - f(N)| + |f(p^a N+1) - af(p) - f(N+1)| .$$

Therefore, by (9) and (11),

$$|f(p^a) - af(p)| \leq |f(p^a N+1) - f(p^a N)| + |f(N+1) - f(N)| + \sum_{i=1}^{a} \Delta_p(p^i N+1) .$$

Now take the sum of this inequality over all $N \leq x$ such that $(N, p) = 1$. Then,

$$\sum_{N \leq x} |f(p^a) - af(p)| \leq \sum_{N \leq x} |f(p^a N+1) - f(p^a N)| + \sum_{N \leq x} |f(N+1) - f(N)| + \sum_{i=1}^{a} \sum_{N \leq x} \Delta_p(p^i N+1) .$$

There exists a positive constant c such that $\sum_{\substack{N \leq x \\ (N,p)=1}} 1 > cx$. Since

$$\sum_{N \leq x} |f(p^a) - af(p)| = |f(p^a) - af(p)| \sum_{N \leq x} 1 ,$$

$$(12)\quad |f(p^a) - af(p)| \le \frac{1}{c}\left[\frac{\sum_{N\le x} |f(p^aN+1) - f(p^aN)|}{x} + \frac{\sum_{N\le x} |f(N+1) - f(N)|}{x} + \frac{\sum_{i=1}^{a} \sum_{N\le x} \Delta_p(p^iN+p)}{x}\right].$$

Since

$$\sum_{N\le x} |f(p^aN+1) - f(p^aN)| \le \sum_{n\le p^a x} |f(n+1) - f(n)| = o(p^a x) = o(x),$$

the first term in the right hand member of (12) approaches 0. Also, by (6), the middle term approaches 0, and the last term approaches 0, by (8). Since the left hand member does not depend on x and it is less than or equal to some value which approaches 0, it follows that $f(p^a) - af(p) = 0$. In other words, f is completely additive. Δ

In a 1969 article, Ryavec [42] establishes, using different means, a more general form of this theorem, by using the weaker condition

$$(13)\qquad \liminf_{x\to\infty} \frac{1}{x} \sum_{n\le x} |f(n+1) - f(n)| = 0.$$

This condition is seen again in Chapter IV in formula (16). At that point it is used in the determination of conditions on additive functions for them to be constant multiples of $\log n$.

Double Sequences and the L-Operator

Additive functions can be considered in a more analytical vein. An additive function f is a sequence $f(1), f(2), \ldots$ in which f obeys the additive property (1). Let p_k denote the kth prime ($p_1 = 2$, $p_2 = 3$, $p_3 = 5, \ldots$) and let $f^{(k)}$ be the function defined by

$$(14) \qquad f^{(k)}(n) = \begin{cases} 0 & , \text{ if } n \not\equiv 0 \bmod p_k , \\ f(p_k^{\,a}) , & \text{if } p_k^{\,a} \| n . \end{cases}$$

Thus, the function defined in (14) is in $\mathfrak{G}$, since $f \in \mathfrak{G}$.

Define, for k fixed, the function

$$(15) \qquad f_k(n) = \sum_{j=1}^{k} f^{(j)}(n) .$$

Since $\mathfrak{G}$ is a group with respect to addition (Theorem 2.1), it follows that the function f_k is in $\mathfrak{G}$. Thus, formulas (14) and (15) imply that

$$(16) \qquad f(n) = \sum_{k=1}^{\infty} f^{(k)}(n) = \lim_{k\to\infty} f_k(n) ,$$

where, for a fixed n, all but a finite number of terms of the infinite series are zero. Therefore, associated with the additive function f is a double sequence of numbers $\{\{f(p_k^{\,a})\}\}$.

Let $\{\{f(p_k^{\,a})\}\}$ be a double sequence of numbers, and let $(m, n) = 1$. If m and n both are not congruent to $0 \bmod p_k$, then $mn \not\equiv 0 \bmod p_k$. Hence $f^{(k)}(mn) = 0 = 0 + 0 = f^{(k)}(m) + f^{(k)}(n)$. If $p_k^{\,a} \| mn$, then $p_k^{\,a} \| m$ or $p_k^{\,a} \| n$, but not both since m and n are relatively prime. Without loss of generality, assume $p_k^{\,a} \| m$. Then $n \not\equiv 0 \bmod p_k$ and

$$f^{(k)}(mn) = f(p_k^a) = f(p_k^a) + 0 = f^{(k)}(m) + f^{(k)}(n) .$$

Since m and n are relatively prime, these are the only two cases that need to be considered. Thus, $f^{(k)}$ is an additive function. Also, as before, f_k is an additive function. Thus, f as defined in (16) is an additive function. Hence, if $f(p_k^a) = c_{ak}$, given the double sequence $\{\{c_{ak}\}\}$, formulas (14), (15) and (16) define the additive functions $f^{(k)}$, f_k and f, respectively.

Example 2.12. Let $n = p_1^2 p_2^4$. Then $f^{(1)}(n) = f(p_1^2)$, $f^{(2)}(n) = f(p_2^4)$, and $f^{(j)}(n) = 0$ for $j \geq 3$. Also,

$$f_1(n) = \sum_{j=1}^{1} f^{(j)}(n) = f^{(1)}(n) = f(p_1^2) ,$$

$$f_2(n) = \sum_{j=1}^{2} f^{(j)}(n) = f(p_1^2) + f(p_2^4) ,$$

and

$$f_j(n) = f_2(n) \quad \text{for } j \geq 3 .$$

Therefore, by (16), $f(n) = f(p_1^2) + f(p_2^4)$.

If $\mathfrak{P}$ is the subset of $\mathfrak{F}$ such that f is real-valued and $f(1) > 0$, then consider the operator L defined by

$$\text{(17)} \qquad L f(n) = \begin{cases} \log f(1) , & \text{for } n = 1 , \\ \sum_{d \mid n} f(d) f^{-1}(\frac{n}{d}) \log d , & \text{for } n > 1 . \end{cases}$$

It is known [1, p. 259] that $f \in \mathfrak{F}$ is multiplicative if and only if

$Lf(n) = 0$, whenever n is not a power of a prime. Given f multiplicative, then, Lf is a function h such that

$$h(n) = 0, \quad \text{if} \quad n \neq p^a,$$

$$h(p_k^a) = c_{ak}, \quad \text{for } p_k \text{ the kth prime}.$$

This function is not additive; e.g., let $h(p_k^a) = 1$. Then $h(12) = 0$, $h(3) = 1$, $h(4) = 1$, and therefore, $h(12) \neq h(3) + h(4)$.

Let n be in canonical form. Define the function h_1 by

$$(18) \qquad h_1(n) = \sum_{i=1}^{r} h(p_i^{a_i}).$$

The function h_1 is additive. Suppose $(m, n) = 1$. Then $m = \prod_{i=1}^{r} p_i^{a_i}$ and $n = \prod_{i=1}^{s} q_i^{b_i}$, where $p_i \neq q_j$, and $mn = \prod_{i=1}^{r} p_i^{a_i} \prod_{i=1}^{s} q_i^{b_i}$. From the definition, it is immediate that $h_1(mn) = h_1(m) + h_1(n)$.

Given any sequence c_{ak}, the function h_1 is uniquely determined by this sequence. Let $L_1f(n)$ be the function $h_1(n)$ defined when $h(p_k^a) = Lf(p_k^a)$. In other words,

$$(19) \qquad L_1f(n) = \sum_{p^a \| n} Lf(p^a).$$

Let f be given. Then Lf is uniquely determined at the prime powers. Hence L_1f, is uniquely determined. Conversely, given L_1f, by (19) Lf is determined by $Lf(p^a) = L_1f(p^a)$, $Lf(n) = 0$ if n is not a prime power. It is known [1, p. 257] that given h there is a unique $f \in \mathcal{P}$ such that $Lf = h$. Thus, f is uniquely determined.

Therefore, the operator L_1 establishes a one-to-one correspondence between $\mathfrak{M}$ and $\mathfrak{A}$.

It is known, also, that (1) if f and g are multiplicative, then so is $f * g$ [1, p. 247], and (2) $L(f * g)(n) = L f(n) + L g(n)$ [1, p. 255]. Thus,

$$\begin{aligned} L_1(f * g)(n) &= \sum_{p^a \| n} L(f * g)(p^a) \\ &= \sum_{p^a \| n} \{L f(p^a) + L g(p^a)\} \\ &= \sum_{p^a \| n} L f(p^a) + \sum_{p^a \| n} L g(p^a) \\ &= L_1 f(n) + L_1 g(n) . \end{aligned}$$

Therefore, L_1 is an isomorphism between the multiplicative group $(\mathfrak{M}, *)$ and the additive group $(\mathfrak{A}, +)$.

It has been pointed out earlier that if $f \in \mathfrak{M}$ and $f(n) > 0$ for every n, then $\log f(n)$ is defined and is an additive function. Conversely, given any additive function $h(n)$ the function $e^{h(n)}$ is defined, is multiplicative, and has nonzero values. Thus the log function maps the functions from $\mathfrak{M}^+$ to $\mathfrak{A}$, where $\mathfrak{M}^+$ denotes the set of multiplicative functions which have only positive functional values. Thus, two isomorphisms have been established: one between $(\mathfrak{M}, *)$ and $(\mathfrak{A}, +)$ by the L_1-operator and one between $(\mathfrak{M}^+, \cdot)$ and $(\mathfrak{A}, +)$ by the log-operator.

If f is completely multiplicative, in general $L_1 f$ is not completely additive. For example, $u_1(n) = n$ is completely multiplicative. Then $L_1 u_1(n) = \sum_{p^a \| n} L u_1(p^a)$. Since

$$L u_p(p^a) = \sum_{d \mid p^a} u_1(d) u_1^{-1}(\frac{p^a}{d}) \log d$$

$$= u_1(p^a) u_1^{-1}(1) \log p^a + u_1(p^{a-1}) u_1^{-1}(p) \log p^{a-1} + 0 ,$$

because $\log 1 = 0$ and $u_1^{-1}(p^a) = 0$ for $a > 1$ [1, p. 250]. But $u_1^{-1}(p) = -p$, so,

$$L u_1(p^a) = p^a \log p^a + p^{a-1}(-p) \log p^{a-1}$$

$$= p^a \log \frac{p^a}{p^{a-1}}$$

$$= p^a \log p .$$

Thus, $L_1 u_1(n) = \sum_{p^a \| n} p^a \log p$. Then, for $n = 24 = 2^3 3$,

$$L_1 u_1(24) = 8 \log 2 + 3 \log 3 ,$$

but

$$L_1 u_1(4) + L_1 u_1(6) = 4 \log 2 + 2 \log 2 + 3 \log 3$$

$$= 6 \log 2 + 3 \log 3 ,$$

hence, $L_1 u_1$ is not completely additive.

This material concludes the development of this chapter, but this is not all that can be said about the L_1-operator. The formula for determining Lf, formula (17), and hence, the formula for determining $L_1 f$, formula (19), can be simplified for the case where f is completely multiplicative, by using results about completely multiplicative functions. For example, the expression for $L_1 u_1$ in the above example can be obtained in this manner without having to know the value of u_1^{-1}.

CHAPTER III

GROWTH PROBLEMS RELATED TO THE FUNCTION Ω_k

Both the functions $\omega(n)$ and $\Omega(n)$ are concerned with the number of prime divisors of n. In Example 2.9, the function

$$H(n) = (\tau(n))^{-1} \sum_{d|n} h(d) ,$$

where $h(n) = \Omega_k(n)$, was introduced. The function H represents a type of average of the functional values of h based on divisors. Another type of average, the ordinary arithmetic average, is given by $\frac{1}{x} \sum_{n \leq x} h(n)$. In the present chapter the average of the function Ω_k is considered.

Recall that if n is written in canonical form, then $\Omega_k(n) = \sum_{i=1}^{r} a_i^k$. But

$$a_i^k = [a_i^k - (a_i - 1)^k] + [(a_i - 1)^k - (a_i - 2)^k] + \ldots + [1^k - 0^k] .$$

Hence, if $p_i^{a_i} \| n$,

$$a_i^k = \sum_{p_i^a | n} \{a^k - (a - 1)^k\} ,$$

where a takes on the values from 1 to a_i inclusive. Thus

$$\Omega_k(n) = \sum_{i=1}^{r} \sum_{p_i^a | n} \{a^k - (a-1)^k\} ,$$

or

(1) $$\Omega_k(n) = \sum_{p^a | n} \{a^k - (a-1)^k\} .$$

The functions $\omega(n) = \Omega_0(n)$ and $\Omega(n) = \Omega_1(n)$ are both related to the distribution of primes, and have irregular behavior for large values of n. They are both 1 when n is a prime, while $\Omega(n) = \frac{\log n}{\log p}$ when n is a power of p. If p_r is the rth prime and n is the product of the first r primes, then $\omega(n) = r = \pi(p_r)$, where π counts the primes less than or equal to a given number. So the sums of these functions are increasing functions, but they exhibit a somewhat erratic growth pattern which reflects the erratic distribution of the primes.

An Asymptotic Formula for $\omega(n)$

Let

(2) $$S(x) = \sum_{n \le x} \omega(n) .$$

Then $\frac{S(x)}{x}$ is the arithmetic average of ω over the first n integers, and hence related to the distribution of prime powers throughout this sequence. If f is a function such that $\frac{S(x)}{x} \sim f(x)$, then f is called the average order of ω. In the remainder of the chapter, an approximation, or an asymptotic formula, for the sum $\sum_{n \le x} \Omega_k(n)$ is established. Once this is accomplished, it is easy to establish that the average order of $\omega(n)$ is $\log\log n$, and that the average order of $\Omega_k(n)$, in general, is $\log\log n$.

A constant used throughout this chapter is Euler's constant, γ, which is defined by $\lim_{n\to\infty}(1+\frac{1}{2}+\ldots+\frac{1}{n}-\log n)$. Used also, is Abel's partial summation formula [17, p. 346] which is stated in the following theorem.

Theorem 3.1. Suppose that $\{c_i\}$ is a sequence of numbers, that

$$C(t) = \sum_{n\leq t} c_n ,$$

and that $f(t)$ is any function of t. Then

$$(3) \qquad \sum_{n\leq x} c_n f(n) = \sum_{n\leq x-1} C(n)\{f(n) - f(n+1)\} + C(x) f([x]) .$$

If, in addition, $c_j = 0$ for $j < n_1$ and $f(t)$ has a continuous derivative for $t \geq n_1$, then

$$(4) \qquad \sum_{n\leq x} c_n f(n) = C(x) f(x) - \int_{n_1}^{x} C(t) f'(t)\, dt .$$

If, in (4), $c_p = \frac{\log p}{p}$, $c_n = 0$ for n not a prime, and $f(t) = \frac{1}{\log t}$, it can be shown that

$$(5) \qquad \sum_{p\leq x} \frac{1}{p} = \log\log x + A_1 + O\left(\frac{1}{\log x}\right) ,$$

where A_1 is the value given by

$$(6) \qquad A_1 = \gamma + \Sigma\{\log(1-p^{-1}) + p^{-1}\} .$$

The proofs of these particular results can be found in Hardy and Wright [17, pp. 350-353].

In addition, the estimation given by the Prime Number Theorem [17, p. 9] is used. This states that

$$(7) \qquad \pi(x) \sim \frac{x}{\log x} .$$

A result concerning $S(x)$, the sum in (2), will be established first.

Theorem 3.2. The average order of $\omega(n)$ is $\log\log n$. In particular,

$$(8) \qquad S(x) = \sum_{n \leq x} \omega(n) = x \log\log x + A_1 x + O\left(\frac{x}{\log x}\right) ,$$

where A_1 is defined in (6).

Proof: Since there are just $[\frac{x}{p}]$ values of $n \leq x$ which are multiples of p,

$$S(x) = \sum_{n \leq x} \sum_{p|n} 1 = \sum_{p \leq x} [\frac{x}{p}] .$$

Also, $\frac{x}{p} = [\frac{x}{p}] + r$, $0 \leq r < 1$, implies that

$$\sum_{p \leq x} [\frac{x}{p}] = \sum_{p \leq x} (\frac{x}{p} - r) = \sum_{p \leq x} \frac{x}{p} - \sum_{p \leq x} r .$$

Because $|- \sum_{p \leq x} r| < \sum_{p \leq x} 1 = \pi(x)$, it follows that

$$- \sum_{p \leq x} r = O(\pi(x)) = O\left(\frac{x}{\log x}\right) ,$$

by (7). Thus,

$$\sum_{p \le x} \left[\frac{x}{p}\right] = \sum_{p \le x} \frac{x}{p} + O\left(\frac{x}{\log x}\right) = x \sum_{p \le x} \frac{1}{p} + O\left(\frac{x}{\log x}\right),$$

since x is fixed. By (5), it follows that

$$\sum_{p \le x} \left[\frac{x}{p}\right] = x\left\{\log\log x + A_1 + O\left(\frac{1}{\log x}\right)\right\} + O\left(\frac{x}{\log x}\right)$$

$$= x \log\log x + A_1 x + x\,O\left(\frac{1}{\log x}\right) + O\left(\frac{x}{\log x}\right).$$

Applying Lemmas 1.4 and 1.2 to the O-terms yields the required results. △

An Asymptotic Formula for $\Omega_k(n)$

In 1962, Duncan [5] extended the development of asymptotic formulas for $\Sigma\,\omega(n)$ and $\Sigma\,\Omega(n)$ (which was already known) to include a general formula for $\Sigma\,\Omega_k(n)$, for all nonnegative values of k. Before this generalization is considered, the bounds on some sums need to be established. They are presented here as lemmas so as not to interrupt the flow of the argument in the proof of the generalization.

<u>Lemma 3.1.</u> $\displaystyle\sum_{p^a \le x} \{a^k - (a-1)^k\} = O\left(\frac{x}{\log x}\right).$

Proof: Let q be a fixed prime and let a be the largest exponent such that $q^a \le x$. Then

$$\sum_{q^b \le x} \{b^k - (b-1)^k\} = (1^k - 0^k) + (2^k - 1^k) + \ldots + (a^k - (a-1)^k)$$

$$= a^k.$$

Since a is the largest exponent under the conditions stated (i.e., the largest $a \leq \frac{\log x}{\log q}$), it follows that $a = \left[\frac{\log x}{\log q}\right]$. Thus,

$$\sum_{q^b \leq x} \{b^k - (b-1)^k\} = \left[\frac{\log x}{\log q}\right]^k .$$

Therefore, if all prime powers less than or equal to x are taken into account,

$$\sum_{p^a \leq x} \{a^k - (a-1)^k\} = \sum_{p \leq x} \left[\frac{\log x}{\log p}\right]^k$$

$$\leq \log^k x \sum_{p \leq x} \frac{1}{\log^k p} .$$

Consider the difference $\pi(n) - \pi(n-1)$. This difference will be zero unless n is a prime; in that case, the difference will be 1. Therefore, the summation

$$\sum_{n \leq x} \frac{\pi(n) - \pi(n-1)}{\log^k n} = \sum_{p \leq x} \frac{1}{\log^k p} ,$$

and

$$\sum_{p^a \leq x} \{a^k - (a-1)^k\} = \log^k x \sum_{n \leq x} \frac{\pi(n) - \pi(n-1)}{\log^k n} .$$

By the Prime Number Theorem (7) and partial summation (5),

$$\sum_{n \leq x} \frac{\pi(n) - \pi(n-1)}{\log^k n} = \frac{\pi(x)}{\log^k x} + k \int_2^x \frac{\pi(t)\, dt}{t \log^{k+1} t}$$

$$= O\left(\frac{x}{\log^{k+1} x}\right) + O\left(\int_2^x \frac{dt}{\log^{k+2} t}\right)$$

$$= O\left(\frac{x}{\log^{k+1} x}\right) .$$

Hence,

$$\sum_{p^a \leq x} \{a^k - (a-1)^k\} = \log^k x \; O\left(\frac{x}{\log^{k+1} x}\right)$$

$$= O\left(\frac{x}{\log x}\right) ,$$

by Lemma 1.3. △

Lemma 3.2. $\sum_{\substack{p^a > x \\ a \geq 2}} \{a^k - (a-1)^k\} p^{-a} = O\left(\frac{x}{\log x}\right)$.

Proof: It is clear that, for $a \geq 2$,

(9) $$\sum_{p^a > x} \{a^k - (a-1)^k\} p^{-a} \leq \sum_{p^a > x} a^k p^{-a} .$$

Since $p^a > x$, it follows that $a \log p > \log x$ or $a > \frac{\log x}{\log p} = y$. Then

(10) $$\sum_{\substack{p^a > x \\ a \geq 2}} a^k p^{-a} \leq \sum_{p > \log x} \sum_{a=2}^{\infty} a^k p^{-a} + \sum_{p \leq \log x} \sum_{a > y}^{\infty} a^k p^{-a} .$$

Each of the two double summations in the right member of the inequality in (10) is examined individually. The first double summation simplies as follows:

$$\sum_{a=2}^{\infty} \sum_{p > \log x} a^k p^{-a} \leq \sum_{a=2}^{\infty} \sum_{n > \log x} a^k n^{-a}$$

$$\leq \sum_{a=2}^{\infty} a^k \sum_{n > \log x} n^{-a} .$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^a}$ converges, an approximation of $\sum_{n > \log x} \frac{1}{n^a}$ can be obtained by

$$\int_{\log x}^{\infty} \frac{1}{t^a}\,dt = \frac{t^{-a+1}}{-a+1}\Big|_{\log x}^{\infty} = \frac{1}{a-1}(\log x)^{-a+1}$$

$$= \frac{1}{a-1}\left(\frac{1}{\log x}\right)^{a-1} .$$

Therefore,

$$\sum_{a=2}^{\infty} \sum_{p > \log x} a^k p^{-a} = O\left[\sum_{a=2}^{\infty} a^k \left(\frac{1}{\log x}\right)^{a-1}\right] .$$

By shifting the index, this yields

$$(11) \quad \sum_{a=2}^{\infty} \sum_{p > \log x} a^k p^{-a} = O\left[\frac{1}{\log x} \sum_{a=0}^{\infty} (a+2)^k \left(\frac{1}{\log x}\right)^a\right] = O\left[\frac{1}{\log x}\right],$$

because the series $\sum_{a \geq 0} (a+2)^k \left(\frac{1}{\log x}\right)^a$ is a convergent series.

For the second double summation in (10), let $\delta = \frac{\log x}{\log \log x}$. Since $p \leq \log x$ implies $\log p \leq \log \log x$, it follows that $\frac{\log x}{\log p} \geq \frac{\log x}{\log \log x}$ (i.e., $y \geq \delta$) and thus,

$$\sum_{p \leq \log x} \sum_{a > y} a^k p^{-a} \leq \sum_{p \leq \log x} \sum_{a > \delta} a^k 2^{-a} ,$$

since $2 \leq p$ implies that $2^{-a} \geq p^{-a}$. But

$$\sum_{p \leq \log x} \sum_{a > \delta} a^k 2^{-a} = \left(\sum_{a > \delta} a^k 2^{-a}\right) \sum_{p \leq \log x} 1 = \left(\sum_{a > \delta} a^k 2^{-a}\right) \pi(\log x) .$$

By the Prime Number Theorem (7), it follows that

$$\pi(\log x) \sim \frac{\log x}{\log\log x} = \delta .$$

Thus, there exists an M such that $M\delta > \pi(\log x)$, so that

$$\Sigma\,\Sigma\, a^k 2^{-a} = O\left(\delta \sum_{a>\delta} a^k 2^{-a}\right).$$

Now for any a, $a^k 2^{-a} = e^{k\log a} e^{-a\log 2} = e^{k\log a - a\log 2}$.
Let $k\log a - a\log 2 = -a(\log 2 - \frac{k}{a}\log a) = -a\alpha$. As $a \to \infty$, $\frac{k}{a}\log a \to 0$. Thus $\alpha > 0$, and since $0 \leq e^{-\alpha} < 1$, $\sum_{a\geq 0} (e^{-\alpha})^a$ converges. Therefore,

$$O\left(\delta \sum_{a>\delta} a^k 2^{-a}\right) = O\left(\delta \sum_{a>\delta} e^{-\alpha a}\right),$$

by Lemma 1.3.

Since $\delta = \frac{\log x}{\log\log x}$, $0 < e^{-\alpha\delta} < 1$. But for $a > \delta$, $0 < e^{-\alpha a} < e^{-\alpha\delta} < 1$. Also, $\sum_{a\geq\delta} e^{-\alpha a}$ is a geometric series with $r = e^{-\alpha a} < 1$. Then

$$\sum_{a>\delta} e^{-\alpha a} < \sum_{a\geq\delta} e^{-\alpha a} = \frac{e^{-\alpha\delta}}{1 - e^{-\alpha a}} < M e^{-\alpha\delta}.$$

Therefore, $O\left(\delta \sum_{a>\delta} e^{-\alpha a}\right) = O(\delta e^{-\alpha\delta})$.

Because $(\log\log x)^2 < \log x = \frac{\log^2 x}{\log x}$, it follows that $\frac{(\log\log x)^2}{\log^2 x} < \frac{1}{\log x}$; that is, $\frac{1}{\delta^2} < \frac{1}{\log x}$. Also, since $e^{-\alpha\delta} < 1$, $\delta e^{-\alpha\delta} < \delta$. Now $\delta > 1$, so that $\frac{1}{\delta^2} < \delta$. By the Archimedean property, there is an M such that $\delta < \frac{M}{\delta^2}$. Hence,

$$\delta e^{-\alpha\delta} = O\left(\frac{1}{\delta^2}\right) = O\left(\frac{1}{\log x}\right).$$

Therefore,

$$(12) \qquad \sum_{p \leq \log x} \sum_{a > y} a^k p^{-a} = O\left(\frac{1}{\log x}\right) .$$

Hence, from (9), (10), (11) and (12),

$$\sum_{\substack{p^a > x \\ a \geq 2}} \{a^k - (a-1)^k\} p^{-a} = O\left(\frac{1}{\log x}\right) + O\left(\frac{1}{\log x}\right)$$

$$= O\left(\frac{1}{\log x}\right) ,$$

by Lemma 1.2. △

Let θ be the differential operator $x\left(\frac{d}{dx}\right)$ and define $G_k(x)$ to be $\theta^k\left(\frac{x}{1-x}\right)$. Since $\frac{x}{1-x} = \sum_{n \geq 1} x^n$, it follows that

$$G_1(x) = x\left(\frac{d}{dx}\right) \sum_{n \geq 1} x^n = x \sum_{n \geq 1} n x^{n-1} = \sum_{n \geq 1} n x^n ,$$

$$G_2(x) = x\left(\frac{d}{dx}\right) G_1(x) = x \sum_{n > 1} n(n x^{n-1}) = \sum_{n \geq 1} n^2 x^n ,$$

and in general,

$$(13) \qquad G_k(x) = x\left(\frac{d}{dx}\right) G_{k-1}(x) = x \sum_{n \geq 1} n^{k-1}(n x^{n-1}) = \sum_{n \geq 1} n^k x^n .$$

It is necessary to define θ^0 to be the identity operator in order for the generalization to be consistent. That is, $\theta^0\{\frac{x}{1-x}\} = \frac{x}{1-x}$. In the discussion to follow, let $R = a^k - (a-1)^k$.

<u>Lemma 3.3.</u> For p a fixed prime

$$\sum_{a=2}^{\infty} \{a^k - (a-1)^k\} p^{-a} = (1-p^{-1}) G_k(p^{-1}) - p^{-1} .$$

Proof: Expanding the left member results in

$$\sum_{a=2}^{\infty} R p^{-a} = (2^k - 1^k)p^{-2} + (3^k - 2^k)p^{-3} + \dots .$$

Regrouping the terms in the right member produces

$$\sum_{a=2}^{\infty} R p^{-a} = -(1^k)p^{-1} + 1^k p^{-1} - 1^k p^{-2} + 2^k(p^{-2} - p^{-3})$$

$$+ 3^k(p^{-3} - p^{-4}) + \dots$$

$$= -p^{-1} + 1^k(p^{-1} - p^{-2}) + 2^k(p^{-2} - p^{-3}) + \dots .$$

This can be written as

$$\sum_{a=2}^{\infty} R p^{-a} = -p^{-1} + \sum_{a=1}^{\infty} a^k(p^{-a} - p^{-a-1})$$

$$= -p^{-1} + \sum_{a=1}^{\infty} a^k p^{-a}(1 - p^{-1})$$

$$= -p^{-1} + (1 - p^{-1}) \sum_{a=1}^{\infty} a^k p^{-a} .$$

Then by (13),

$$\sum_{a=2}^{\infty} \{a^k - (a-1)^k\} p^{-a} = (1-p^{-1}) G_k(p^{-1}) - p^{-1} . \qquad \triangle$$

Lemma 3.4. For $a \geq 2$,

$$\sum_{p^a \leq x} \{a^k - (a-1)^k\} p^{-a} = \sum_p \{a^k - (a-1)^k\} p^{-a} + O\left(\frac{1}{\log x}\right).$$

Proof: Adding and subtracting the same amount yields

$$\sum_{p^a \leq x} R p^{-a} = \sum_{p^a \leq x} R p^{-a} + \sum_{p^a > x} R p^{-a} - \sum_{p^a > x} R p^{-a}$$

$$= \sum_p R p^{-a} - \sum_{p^a > x} R p^{-a}.$$

By Lemma 3.2, for $a \geq 2$, $-\sum_{p^a > x} R p^{-a} = O\left(\frac{1}{\log x}\right)$. Thus, the lemma follows. △

It is now possible to evaluate $\sum_{n \leq x} \Omega_k(n)$ to within an error of $O\left(\frac{x}{\log x}\right)$.

Theorem 3.3. $\sum_{n \leq x} \Omega_k(n) = x \log \log x + B_k x + O\left(\frac{x}{\log x}\right)$, where

$$B_k = \gamma + \sum_p \{(1 - p^{-1}) G_k(p^{-1}) + \log(1 - p^{-1})\}$$

and γ is Euler's constant.

Proof: Let $T(x) = \sum_{n \leq x} \Omega_k(n)$. Using the result in equation (1),

$$T(x) = \sum_{n \leq x} \sum_{p^a | n} \{a^k - (a-1)^k\},$$

which can be regrouped as follows:

$$T(x) = \sum_{p^a \leq n} \{a^k - (a-1)^k\} \left[\frac{x}{p^a}\right],$$

since there are $\left[\frac{x}{p^a}\right]$ values of $n \leq x$ which are multiples of p^a. Separation of this sum into two sums, one for $a = 1$ and the other for $a \geq 2$, results in

$$T(x) = \sum_{p \leq x} \left[\frac{x}{p}\right] + \sum_{\substack{p^a \leq x \\ a \geq 2}} \{a^k - (a-1)^k\} \left[\frac{x}{p^a}\right]$$

$$= \sum_{p \leq x} \left[\frac{x}{p}\right] + \sum_{\substack{p^a \leq x \\ a \geq 2}} \{a^k - (a-1)^k\} \left(\frac{x}{p^a} - r\right),$$

where $\frac{x}{p^a} = \left[\frac{x}{p^a}\right] + r$ and $0 \leq r < 1$. Using the notation in Lemma 3.3, this last expression, then, is equal to

$$\sum_{p \leq x} \left[\frac{x}{p}\right] + x \sum_{\substack{p^a \leq x \\ a \geq 2}} R p^{-a} - r \sum_{\substack{p^a \leq x \\ a \geq 2}} R,$$

so that

$$(14) \qquad T(x) = \sum_{p \leq x} \left[\frac{x}{p}\right] + x \sum_{\substack{p^a \leq x \\ a \geq 2}} R p^{-a} + O\left(\sum_{p^a \leq x} R\right).$$

The first term in the right member can be replaced by $x \log\log x + A_1 x + O\left(\frac{x}{\log x}\right)$, by Theorem 3.2. The last term can be replaced by $O\left(\frac{x}{\log x}\right)$, by Lemma 3.1. By Lemma 3.4, the summation in the middle term can be replaced by $\sum_{\substack{p \\ a \geq 2}} R p^{-a} + O\left(\frac{1}{\log x}\right)$. Therefore,

$$T(x) = x \log\log x + A_1 + O\left(\frac{x}{\log x}\right) + x\left\{\sum_{\substack{p \\ a \geq 2}} R\,p^{-a} + O\left(\frac{1}{\log x}\right)\right\} + O\left(\frac{x}{\log x}\right).$$

Now A_1 is equal to B_0, as defined in the statement of the theorem, since

$$B_0 = \gamma + \sum_p \{(1-p^{-1})\,G_0(p^{-1}) + \log(1-p^{-1})\}$$

$$= \gamma + \sum_p \{(1-p^{-1})\left(\frac{p^{-1}}{1-p^{-1}}\right) + \log(1-p^{-1})\}$$

(15) $$= \gamma + \sum_p \{p^{-1} + \log(1-p^{-1})\}\ .$$

Thus,

$$T(x) = x \log\log x + x\left\{B_0 + \sum_{\substack{p \\ a \geq 2}} R\,p^{-a}\right\} + O\left(\frac{x}{\log x}\right),$$

using Lemmas 1.4 and 1.2. By Lemma 3.3 and (15),

$$B_0 + \sum_p \sum_{a \geq 2} R\,p^{-a} = \gamma + \sum_p \{p^{-1} + \log(1-p^{-1})\}$$

$$+ \sum_p \{(1-p^{-1})\,G_k(p^{-1}) - p^{-1}\}$$

$$= \gamma + \sum_p \{(1-p^{-1})\,G_k(p^{-1}) + \log(1-p^{-1})\}$$

$$= B_k\ .$$

Therefore, $T(x) = x \log\log x + B_k x + O\left(\frac{x}{\log x}\right)$. △

The Average Based on Divisors

Let $n = \prod_{k=1}^{r} p_i^{a_i}$. If $d | n$, then $d = \prod_{i=1}^{r} p_i^{b_i}$, where $0 \leq b_i \leq a_i$. Then

$$\sum_{d|n} \Omega_k(d) = \sum_{b_i} \Omega_k\left(\prod_{i=1}^{r} p_i^{b_i}\right) = \sum_{b_i} (b_1^k + \ldots + b_r^k) .$$

For each i, there are $a_i + 1$ divisors of n which have some power of p_i as a factor. Then in the summation $\Sigma (b_1^k + \ldots + b_r^k)$ there are $\frac{\tau(n)}{(a_i + 1)}$ expressions $1^k + 2^k + \ldots + a_i^k$. Thus,

$$\sum_{d|n} \Omega_k(n) = \frac{\tau(n)(1^k + \ldots + a_1^k)}{a_1 + 1} + \ldots + \frac{\tau(n)(1^k + \ldots + a_r^k)}{a_r + 1}$$

$$(16) \qquad = \tau(n) \sum_{i=1}^{r} \frac{(1^k + \ldots + a_i^k)}{a_i + 1} .$$

For example, let $n = 2 \cdot 3^2$, whence $a_1 = 1$ and $a_2 = 2$. The divisors of n, other than 1, are 2, $2 \cdot 3$, $2 \cdot 3^2$, 3 and 3^2, and $\tau(n) = (1+1)(2+1) = 6$. Then

$$\sum_{d|n} \Omega_k(d) = \Omega_k(1) + \Omega_k(2) + \Omega_k(2 \cdot 3) + \Omega_k(2 \cdot 3^2) + \Omega_k(3) + \Omega_k(3^2)$$

$$= 0^k + (1^k) + (1^k + 1^k) + (1^k + 2^k) + (1^k) + (2^k) .$$

There are $\frac{\tau(n)}{(a_1 + 1)} = \frac{6}{2} = 3$ of 1^k associated with 2, and there are $\frac{\tau(n)}{(a_2 + 1)} = \frac{6}{2} = 2$ of $1^k + 2^k$ associated with 3.

Let $\alpha_k(n) = H(n)$, for $h(n) = \Omega_k(n)$. Thus, by the definition of $\alpha_k(n)$,

$$\alpha_k(n) = \sum_{i=1}^{r} \frac{1^k + \ldots + a_i^k}{a_i + 1} ,$$

and hence, since $a_i + 1 \geq 2$, $\alpha_k(n) \leq \frac{1}{2}\Omega_{k+1}(n)$, for $k \geq 0$. But for any $a > 0$, $a^{k-1} \leq a^k$. Thus $a^{k-1} \leq 2(1^k + 2^k + \ldots + (a-1)^k) + a^k$, so, $a^k + a^{k-1} \leq 2(1^k + 2^k + \ldots + a^k)$ or $a^{k-1}(a+1) \leq 2(1^k + 2^k + \ldots + a^k)$. Therefore,

$$\frac{a^{k-1}}{2} \leq \frac{1^k + 2^k + \ldots + a^k}{a+1} .$$

Hence,

$$\sum_{i=1}^{r} \frac{a_i^{k-1}}{2} < \sum_{i=1}^{r} \frac{1^k + 2^k + \ldots + a_i^k}{a_i + 1} ,$$

which implies $\frac{1}{2}\Omega_{k-1}(n) \leq \alpha_k(n)$, for $k > 1$. Now

$$\alpha_0(n) = \sum_{i=1}^{r} \frac{(1 + \ldots + 1)}{(a_i + 1)} = \sum_{i=1}^{r} \frac{a_i}{(a_i + 1)}$$

and

$$\frac{1}{2}\Omega_0(n) = \frac{1}{2}\sum_{i=1}^{r} 1 = \sum_{i=1}^{r} \frac{1}{2} .$$

For each i, $\frac{a_i}{(a_i+1)} \geq \frac{1}{2}$, hence $\sum_{i=1}^{r} \frac{a_i}{(a_i+1)} \geq \sum_{i=1}^{r} \frac{1}{2}$. That is, $\alpha_0(n) \geq \frac{1}{2}\Omega_0(n)$. Then

$$\frac{1}{2}\Sigma\,\Omega_{k-1}(n) \leq \sum_{n \leq x} \alpha_k(n) \leq \frac{1}{2}\Sigma\,\Omega_{k+1}(n) ,$$

and therefore, $\sum_{n \leq x} \alpha_k(n) \sim \frac{1}{2} x \log\log x$; that is, the average order of $\alpha_k(n)$ is $\frac{1}{2} \log\log n$, for all k.

With the previous theorem, it is easy to obtain an exact expression for B_k as a sum over the primes, providing the value of k is small. The procedure is cumbersome though, if k is large. In a later article, Duncan [6] develops an asymptotic formula for B_k.

Theorem 3.4. Let $G_k(x)$ and B_k be as in Theorem 3.3. Then

$$B_k \sim \frac{k!}{2 \log^{k+1} 2} \quad . \tag{17}$$

Proof: By definition,

$$\begin{aligned} B_k &= \gamma + \sum_p \{(1 - p^{-1}) G_k(p^{-1}) + \log(1 - p^{-1})\} \\ &= \gamma + (1 - 2^{-1}) G_k(2^{-1}) + \log(1 - 2^{-1}) \\ &\quad + \sum_{p>2} \{(1 - p^{-1}) G_k(p^{-1}) + \log(1 - p^{-1})\} \\ &= \gamma + \frac{1}{2} G_k(2^{-1}) + \log \frac{1}{2} + \sum_{p>2} \{(1 - p^{-1}) G_k(p^{-1}) + \log(1 - p^{-1})\} \ . \end{aligned}$$

By (13), $G_k(2^{-1}) = \sum_{n=1}^{\infty} n^k 2^{-1}$, thus

$$B_k = \gamma - \log 2 + \frac{1}{2} \sum_{n=1}^{\infty} n^k 2^{-n} + \sum_{p>2} \{(1 - p^{-1}) G_k(p^{-1}) + \log(1 - p^{-1})\} \ .$$

Now

$$\sum_{n=1}^{\infty} n^k 2^{-n} \sim \int_0^{\infty} x^k 2^{-x} dx = \int_0^{\infty} x^k e^{-x \log 2} dx .$$

It is known that

$$(18) \qquad \int_0^{\infty} \frac{a^p}{\Gamma(p)} x^{p-1} e^{-ax} dx = 1 ,$$

where Γ is the gamma function, and a and p are greater than zero. Then, if $a = \log 2$ and $p = k+1$,

$$\int_0^{\infty} x^k e^{-x \log 2} dx = \frac{\Gamma(k+1)}{\log^{k+1} 2} \int_0^{\infty} \frac{\log^{k+1} 2}{\Gamma(k+1)} x^k e^{-x \log 2} dx$$

$$= \frac{\Gamma(k+1)}{\log^{k+1} 2} = \frac{k!}{\log^{k+1} 2} .$$

Since $\frac{1}{2} \leq 1 - p^{-1} < 1$, it follows that $-\log 2 \leq \log (1 - p^{-1}) < 0 < p^{-1}$. Also, $(1 - p^{-1}) G_k(p^{-1}) < G_k(p^{-1})$. These inequalities make it possible to write

$$\sum_{p>2} \{(1 - p^{-1}) G_k(p^{-1}) + \log (1 - p^{-1})\} \leq \sum_{p>2} \{G_k(p^{-1}) - p^{-1}\} .$$

Replace G_k by (13) so that the right hand side becomes

$$\sum_{p>2} \left\{ \sum_{n=1}^{\infty} n^k (p^{-1})^n - p^{-1} \right\} = \sum_{p>2} \left\{ \sum_{n=2}^{\infty} n^k p^{-n} - p^{-1} + 1^k p^{-1} \right\}$$

$$= \sum_{p \geq 2} \sum_{n=2}^{\infty} n^k p^{-n}$$

$$= \sum_{p>2} \{2^k p^{-2} + 3^k p^{-3} + 4^k p^{-4} + \dots\}$$

$$= \sum_{p>2} p^{-2} \{2^k + 3^k p^{-1} + 4^k p^{-2} + \ldots\}$$

$$= \left(\sum_{p>2} p^{-2}\right)\left(\sum_{n=0}^{\infty} (n+2)^k p^{-n}\right) .$$

Since $p > 2$ implies $\frac{1}{p} \leq \frac{1}{3} < \frac{1}{2}$,

$$\sum_{p>2} p^{-2} \sum_{n=0}^{\infty} (n+2)^k p^{-n} \leq \left(\sum_{p>2} p^{-2}\right)\left(\sum_{n=0}^{\infty} (n+2)^k 3^{-n}\right).$$

Now

$$\sum_{n=0}^{\infty} (n+2)^k 3^{-n} = 2^k 3^0 + 3^k 3^{-1} + 4^k 3^{-2} + \ldots$$

$$= 3^2(2^k 3^{-2} + 3^k 3^{-3} + 4^k 3^{-4} + \ldots)$$

$$= 9 \sum_{n=2}^{\infty} n^k 3^{-n} .$$

As a result of this reduction,

$$\sum_{p>2} \{(1-p^{-1}) G_k(p^{-1}) + \log(1-p^{-1})\} \leq 9\left(\sum_{p>2} p^{-2}\right)\left(\sum_{n=2}^{\infty} n^k 3^{-n}\right) .$$

Because $\sum_{p>2} p^{-2}$ converges, and because

$$\sum_{n=2}^{\infty} n^k 3^{-n} \leq \int_0^{\infty} x^k 3^{-x} dx ,$$

it follows that

$$9\left(\sum_{p>2} p^{-2}\right)\left(\sum_{n=2}^{\infty} n^k 3^{-n}\right) = O\left(\int_0^{\infty} x^k 3^{-x} dx\right) .$$

From (18), with $a = \log 3$ and $p = k+1$, it follows that

$$\int_0^\infty x^k 3^{-x} dx = \frac{\Gamma(k+1)}{\log^{k+1} 3} = \frac{k!}{\log^{k+1} 3} ,$$

and

$$\sum_{p>2} \{(1-p^{-1}) G_k(p^{-1}) + \log(1-p^{-1})\} = O\left(\frac{k!}{\log^{k+1} 3}\right) .$$

Thus,

$$B_k = \gamma - \log 2 + \frac{k!}{2 \log^{k+1} 2} + O\left(\frac{k!}{\log^{k+1} 3}\right)$$

or

$$B_k \sim \frac{k!}{2 \log^{k+1} 2} . \qquad \triangle$$

The results in this chapter have involved considerable manipulative detail. The early work in this area is largely due to G. H. Hardy and the generalizations are mostly due to R. L. Duncan. The investigation of additive functions from a completely different point of view is presented in the chapter to follow.

CHAPTER IV

SMOOTH ADDITIVE FUNCTIONS

Since the function $c \log n$ is a familiar example of a monotone completely additive function, it is natural to ask whether there are other such functions. In general one might ask under what conditions is an additive function closely related to $\log n$. The class $\mathfrak{B}$ of functions is defined to be the set of functions of the form $f(n) = c \log n$, i.e., $f(n)$ is a constant multiple of $\log n$. One segment of the literature about additive functions deals with the problem of determining the conditions on the additive function f under which f is also in $\mathfrak{B}$.

In his initial article, Erdős [9] set forth some conditions under which this might be true. Some he proved, others he had to leave as conjectures. Since that time some of his conjectures have been proved, whereas others are still open. Also, in the intervening time, other conditions have been set forth - some proved, some not. In this chapter, the author will try to coalesce this information. Unless stated otherwise, the additive functions to be considered will be real-valued.

Conditions on Additive Functions

The first theorem to be considered shows that essentially the only nondecreasing additive function is the logarithmic function. The proof presented here is a simplification of the proof in Erdős' article and is attributed to Moser and Lambek [40].

Theorem 4.1. Assume that $f \in \mathfrak{G}$ and $f(n+1) \geq f(n)$ for every n. Then $f \in \mathfrak{B}$.

Proof: Let $g(n) = e^{f(n)}$. Then g is multiplicative and $g(n+1) \geq g(n)$, since f is nondecreasing. It is shown first that, since g is a nondecreasing multiplicative function, $g(n) = n^k$. Then $e^{f(n)} = n^k$ and hence, the conclusion of the theorem follows.

Let a be a fixed integer greater than 1. For t a positive integer, let $R(t)$ and $S(t)$ be defined by:

(1) $$R(t) = a^t + a^{t-1} + \ldots + a + 1,$$

and

(2) $$S(t) = a^t - a^{t-1} - \ldots - a - 1.$$

Note that $R(t) - 1 = aR(t-1)$, so that the multiplicative property of g and the fact that $(a, R(t)) = 1$ for any t, imply that

$$g(R(t) - 1) = g(a)\,g(R(t-1)).$$

This, together with g nondecreasing yields

$$g(R(t)) \geq g(R(t) - 1) = g(a)\,g(R(t-1)) \geq \ldots \geq (g(a))^t.$$

Similarly, $(a, S(t)) = 1$ and

$$g(S(t)) \leq g(S(t) + 1) = g(a)\,g(S(t+1)) \leq \ldots \leq (g(a))^t.$$

Let n be given and let r be the integer determined by

(3) $$a^r < n \leq a^{r+1},$$

so that $r < \log_a n \leq r+1$. Equations (1), (2), and (3) imply, then, that for $a > 1$, $R(r-1) < n$, and for $a > 2$,

$$S(r+2) > a^{r+2} - 2a^{r+1} > a^{r+1} \geq n .$$

From these two inequalities, it is seen that for $a > 1$,

$$g(n) \geq g(R(r-1)) \geq (g(a))^{r-1} \geq (g(a))^{\log_a n - 2} , \tag{4}$$

and for $a > 2$,

$$g(n) \leq g(S(r+2)) \leq (g(a))^{r+2} \leq (g(a))^{\log_a n + 2} . \tag{5}$$

Now both (4) and (5) hold for all $a > 2$, so in the following discussion let $a, b > 2$, and let $\log_a n + 2 = \alpha \log n$ and $\log_b n - 2 = \beta \log n$. Then (4) and (5) imply

$$(g(a))^{\alpha} \geq g(n)^{1/\log n} \geq (g(b))^{\beta} .$$

Since $\log_a n = \log n / \log a$, $\alpha \log n = (\log n / \log a) - 2$ and $\alpha \log a = 1 - 2 \log a / \log n$. Thus $\alpha \log a \leq 1$ and $\alpha \leq 1/\log a$. A similar reduction yields $\beta \geq 1/\log b$. So, if α is replaced by something larger and β is replaced by something smaller, the inequality remains the same, i.e.,

$$(g(a))^{1/\log a} \geq (g(b))^{1/\log b} .$$

Now a and b are interchangeable, which implies these two quantities are equal. Thus, for $n > 2$, $(g(n))^{1/\log n}$ is a constant c.

If $c = 0$, then $g(n) \equiv 0$ for all n, which contradicts $g \not\equiv z$. Thus $c = e^k$ and so $g(n) = n^k$, for $n > 2$. Also, since g is multiplicative, $g(1) = 1$. In addition, since $g(6) = g(2)g(3)$, $g(2) = 2^k$. Therefore, $g(n) = n^k$ for all n. Since $f(n) = \log g(n)$, it follows that $f(n) = k \log n$, i.e., $f(n)$ is a constant multiple of $\log n$. △

A weaker condition was conjectured by Erdös [9, p. 3] and later proved by Kátai [23, p. 411] in which the conclusion follows if the monotonicity condition holds for almost all n, i.e., for all n except for a sequence of density 0. By the density of a sequence S of natural numbers $n_i < n_j$, for $i < j$, is meant

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\substack{n_i \in S \\ n_i < n}} 1 ,$$

if this limit exists. A sequence has density 0 if

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{\substack{n_i \in S \\ n_i < n}} 1 = 0 .$$

If a sequence of natural numbers has density 0, then the complement of that sequence with respect to the natural numbers has density 1.

Theorem 4.2. Suppose, for $f \in \mathfrak{A}$ and for an increasing sequence of natural numbers $A = \{n_i\}$ having density 1, $f(n_{i+1}) \geq f(n_i)$ holds. Then $f \in \mathfrak{B}$.

Proof: For n a natural number, consider the equation

$$(n+1)x - ny = 1 . \tag{6}$$

All the integral solutions of (6) are given by $x = 1 + nk$ and $y = 1 + (n+1)k$. Consider the congruences $nk \equiv a - 1 \bmod (n+1)$, where $(a, n+1) = 1$, and $(n+1) \equiv b - 1 \bmod n$, where $(b, n) = 1$. Since $(n, n+1) = 1$, each of these congruences has a solution. Thus, for some m' and m'',

$$k \equiv m' \bmod (n+1)$$
$$k \equiv m'' \bmod n .$$

By the Chinese Remainder Theorem [1, p. 110], there is an infinite set of values of k which are congruent $\bmod n(n+1)$ and which satisfy these simultaneous congruences.

For any m' and m'', the density of the set of k's is $\frac{1}{n(n+1)}$ [43, p. 6] and is, therefore, positive. Since the density of S is 1, there exist infinitely many solutions x and y for which all of x, y, $(n+1)x$, and ny are in S. From (6), $(n+1)x \geq ny$, and hence $f((n+1)x) \geq f(ny)$. Because of the additivity of f, this implies

$$f(n+1) - f(n) \geq f(y) - f(x) \geq 0 ,$$

since $y, x \in S$ and $y \geq x$. Hence $f(n)$ is nondecreasing on the whole set of natural numbers. Theorem 4.1 implies, then, that $f(n)$ is a constant multiple of $\log n$. △

If, instead of the monotonicity condition, the condition

$$\lim_{n\to\infty} f(n+1) - f(n) = 0 \tag{7}$$

is used, f is again assured of being in $\mathfrak{B}$. The conclusion holds as well for complex-valued functions under this condition.

Theorem 4.3. If $f \in \mathfrak{A}$ and f satisfies (7), then $f \in \mathfrak{B}$.

Proof: Consider the sequence $\{P_i\}$, where P_i is either a prime or a power of a prime, and $P_i < P_j$ for $i < j$. Let

$$\limsup_{i\to\infty} \frac{f(P_i)}{\log P_i} = c . \tag{8}$$

Let $\{p_i\}$ be the sequence of primes, where $p_i < p_j$ for $i < j$. Now, c is either finite or infinite, so consider the following cases.

Case I: Let c be finite and assume for infinitely many primes p_i there exists an a_i such that

$$\frac{f(p_i^{a_i})}{\log p_i^{a_i}} > c .$$

Let $p_i^{a_i} = Q_i$, and order the Q_i by

$$\frac{f(Q_1)}{\log Q_1} \geq \frac{f(Q_2)}{\log Q_2} \geq \dots > c ,$$

so that $\lim f(Q_i)/\log Q_i = c$. Thus there exists a j such that

$$\frac{f(Q_1)}{\log Q_1} = \dots = \frac{f(Q_{j-1})}{\log Q_{j-1}} > \frac{f(Q_j)}{\log Q_j} \geq \frac{f(Q_{j+1})}{\log Q_{j+1}} \geq \dots$$

Let $N_k = Q_1 Q_2 \cdots Q_k$, for $k > j$; let $c_1 = f(Q_1)/\log Q_1$ and $c_k = f(Q_k)/\log Q_k$, where $c < c_k < c_1$ and $k \geq j$. Hence $c_1 = c_2 = \dots = c_{j-1} > c_j \geq c_{j+1} \geq \dots$. Now $(Q_i, Q_j) = 1$ for $i \neq j$, and since f is additive,

$$
\begin{aligned}
f(N_k) &= f(Q_1) + f(Q_2) + \dots + f(Q_k) \\
&= c_1 \log Q_1 + c_2 \log Q_2 + \dots + c_k \log Q_k \\
&= (c_1 - c_j) \log Q_1 + c_j \log Q_1 + c_2 \log Q_2 + \dots + c_k \log Q_k \\
&> (c_1 - c_j) \log Q_1 + c_k(\log Q_1 + \log Q_2 + \dots + \log Q_k) \\
&= (c_1 - c_j) \log Q_1 + c_k \log N_k \,.
\end{aligned}
$$

Now $Q_j \nmid (N_k - 1)$, for $j \leq k$. Since $c_k < c_1$ for $k \geq j$ and $f(N_k) > c_k \log N_k$, c_k is the largest value such that $f(N_k - 1) < c_k \log (N_k - 1)$. Then

$$f(N_k) - f(N_k - 1) > c_k \log N_k + (c_1 - c_j) \log Q_1 - c_k \log (N_k - 1) \,.$$

Since $c_k \log N_k - c_k \log (N_k - 1) = c_k \log N_k/(N_k - 1) > 0$, and $(c_1 - c_j) \log Q_1 = \delta > 0$,

$$f(N_k) - f(N_k - 1) > \delta \,,$$

which contradicts $f(n+1) - f(n) \to 0$.

Case II: Assume next for c finite there are a finite number of p_i such that

$$\frac{f(p_i^{a_i})}{\log p_i^{a_i}} > c \,.$$

Let $p_1, \dots, p_j$ be these primes and let $p_i^{a_i} = Q_i$ for $1 \leq i \leq j$.

Define $N_j = Q_1 Q_2 \cdots Q_j$. Now $\frac{f(N_j)}{\log N_j} = c_0 > c$. For all other primes, $f(p_i^{a_i})/\log p_i^{a_i} \leq c$; and given any n such that $(n, N_j) = 1$, $\frac{f(n)}{\log n} \leq c$.

Let n_1 be the least n such that

$$\frac{f(m)}{\log m} \leq \frac{f(n_1)}{\log n_1} ,$$

for $m < n_1$ and $(m, N_j) = 1$. Let n_2 be the least integer greater than or equal to n_1 such that

$$\frac{f(n_2)}{\log n_2} \geq \frac{f(n_1)}{\log n_1} .$$

Then $m < n_1$ and $\frac{f(m)}{\log m} \leq \frac{f(n_2)}{\log n_2}$. Continue in this fashion to obtain a sequence $\{n_i\}$ such that $n_i < n_j$ for $i < j$,

$$\frac{f(m)}{\log m} \leq \frac{f(n_i)}{\log n_i} ,$$

and $(m, N_j) = (n_i, N_j) = 1$.

Let n_i be large and let r be the least prime which is greater than N_j. An integer u is chosen in such a way that $u < 2r$ and

$$n_i N_j - u \equiv 0 \bmod r$$

but

$$n_i N_j - u \not\equiv 0 \bmod r^2 ,$$

This implies $n_i N_j - u = rx$, for some x. Now $(r, x) = 1$ because

$r \mid (n_i N_j - u)$ and $r^2 \nmid (n_i N_j - u)$. Also, since $rx < n_i N_j$ and $r > N_j$, it follows that $x < n_i$.

Let $\frac{f(n_i)}{\log n_i} = c_i \leq c$. Also, since $r \neq Q_i$, $1 \leq i \leq j$, $\frac{f(r)}{\log r} \leq c$. Then, since f is additive,

$$f(rx) = f(r) + f(x) \leq c \log r + c_i \log x .$$

Thus,

$$\begin{aligned} f(n_i N_j) - f(n_i N_j - u) &= f(n_i N_j) - f(rx) \\ &= f(n_i) + f(N_j) - (f(r) + f(x)) \\ &\geq c_i \log n_i + c_0 \log N_j - c \log r - c_i \log x \\ &\quad - c \log r - c_i \log x \\ &= (c_0 - c) \log N_j + c_i \log\left(\frac{n_i}{x}\right) + c \log\left(\frac{N_j}{r}\right) \\ &\geq (c_0 - c) \log N_j + c_i \log\left(\frac{n_i N_j}{xr}\right) , \end{aligned}$$

since $c > c_i$. Let $(c_0 - c) \log N_i = \delta > 0$. Then

$$f(n_i N_j) - f(n_i N_j - u) \geq \delta$$

for each n_i. And since

$$\begin{aligned} f(n_i N_j) - f(n_i N_j - u) &= f(n_i N_j) - f(n_i N_j - 1) + f(n_i N_j - 1) \\ &\quad - f(n_i N_j - 2) + \ldots + f(n_i N_j - u + 1) \\ &\quad - f(n_i N_j - u) \geq \delta , \end{aligned}$$

there is a contradiction that $f(n+1) - f(n) \to 0$.

Case III: Assume, now, for all Q_i, $\frac{f(Q_i)}{\log Q_i} < c$. In this case, $c = +\infty$ is allowed.

The construction as in Case II, using all the integers, gives a sequence $\{n_i\}$ where

$$\frac{f(m)}{\log m} \leq \frac{f(n_i)}{\log n_i},$$

for $m \leq n_i$. It follows from (8) that $\limsup \frac{f(n_i)}{\log n_i} = c$. Also, assume there exists a p_i^a such that

$$\frac{f(p_i^a)}{\log p_i^a} < c_1 < c.$$

Let $n_i > 3p_i^a$ and choose $u < 2p_i^a$ such that

$$n_i - u \equiv 0 \bmod p_i^a$$

but

$$n_i - u \not\equiv 0 \bmod p_i^{a+1}.$$

Then $n_i - u = p_i^a x$ and $x < n_i$. Let $\frac{f(n_i)}{\log n_i} = c_i$. Then $\frac{f(x)}{\log x} < c_i$. Also

$$\begin{aligned} f(n_i) - f(n_i - u) &= f(n_i) - f(p_i^a x) \\ &= f(n_i) - f(p_i^a) - f(x) \\ &\geq c_i \log n_i - c_1 \log p_i^a - c_i \log x . \end{aligned}$$

Since $c_1 < c$, pick n_i such that $c - c_i < \frac{1}{2}(c - c_1)$. Then

$$f(n_i) - f(n_i - u) \geq c_i \log n_i - c_i \log p_i^a + (c_i - c_1) \log p_i^a - c_i \log x$$

$$= c_i \log\left(\frac{n_i}{p_i^a x}\right) + (c_i - c_1) \log p_i^a .$$

Let $(c_i - c_1) \log p_i^a = \delta > 0$. Because $c_i \log(n_i/p_i^a x) \to 0$, it follows that $f(n_i) - f(n_i - u) > \delta$, a contradiction that $f(n+1) - f(n) \to 0$.

Thus, for all Q_i, $f(Q_i)/\log Q_i = c$ -- the only possibility left -- or $f(n)$ is a constant multiple of $\log n$. △

In the year prior to Rényi's article, Erdös [10, p. 48] stated a generalization which includes both Theorems 4.1 and 4.3. The generalization states that if lin inf $f(n+1) - f(n) \geq 0$, for f additive, then f is a constant multiple of $\log n$. This generalization means that if $f(n)$ is not a constant multiple of $\log n$, then $f(n+1) - f(n)$ has both positive and negative limit points. These limit points may be either finite or infinite. Both Kátai [21], [24] and Atilla Máté [37] have established proofs of this theorem, with Máté's proof presented here. But before the generalization is proved, it is necessary to establish some inequalities. Recall, also, from Theorem 2.8 that if f is additive and $g(n) = \lim \frac{f(n^t)}{t}$, as $t \to \infty$, then g is completely additive.

In the next two theorems, let ε be an arbitrary positive quantity, and $c(n, \varepsilon)$ be a quantity which depends only on n and ε. Let $H(\varepsilon) = \{n : f(n+1) - f(n) < -\varepsilon\}$. Note that the set $H(\varepsilon)$ is finite if $\liminf f(n+1) - f(n) \geq 0$. Define $c(\varepsilon) = -\Sigma[f(n+1) - f(n)]$, where the sum is over all n in $H(\varepsilon)$. Now for an arbitrary set S of natural numbers with r elements, by the definition of $c(\varepsilon)$,

(9) $$\sum_{n \in S} f(n+1) - f(n) \geq -r\varepsilon - c(\varepsilon) .$$

Theorem 4.4. If $f \in \mathcal{G}$, then

(10) $$f(n^k) - kf(n) \geq c_1(n,\varepsilon) - kn\varepsilon ,$$

(11) $$f(n^k) - kf(n) \leq c_2(n,\varepsilon) + kn\varepsilon .$$

Proof: Proof of both the inequalities depends on (9). To prove the first inequality, consider $f(n^k) - kf(n)$ and employ the familiar trick of adding and subtracting the same quantities:

$$f(n^k) - kf(n) = [f(n^k) - f(n^k - 1)] + \sum_{r=2}^{k} [f(n^r - 1) - f(n^{r-1} - 1) - f(n)]$$

$$+ [f(n-1) - f(n)]$$

$$= [f(n^k) - f(n^k - 1)] + \sum_{r=2}^{k} [f(n^r - 1) - f(n^r - n)] + [f(n-1) - f(n)]$$

$$= [f(n^k) - f(n^k - 1)] + \sum_{i=1}^{n-1} [f(n^2 - i) - f(n^2 - i - 1)]$$

$$+ \sum_{i=1}^{n-1} [f(n^3 - i) - f(n^3 - i - 1)] + \dots$$

$$+ \sum_{i=1}^{n-1} [f(n^k - i) - f(n^k - i - 1)] + [f(n-1) - f(n)]$$

$$= [f(n^k) - f(n^k - 1)] + \sum_{r=2}^{k} \sum_{i=1}^{n-1} [f(n^r - i) - f(n^r - i - 1)]$$

$$+ [f(n-1) - f(n)] .$$

In the first two terms of this last expression there are $(k-1)(n-1)+1$ natural numbers of the form $f(m+1) - f(m)$ so that, by (9),

$$f(n^k) - k f(n) \geq -(kn - n - k)\varepsilon - c(\varepsilon) + f(n+1) - f(n)$$

$$\geq -kn\varepsilon - c(\varepsilon) + f(n+1) - f(n) = c_1(n, \varepsilon) - kn\varepsilon .$$

The proof of (11) is similar if

$$f(n^k) - k f(n) = [f(n^k) - f(n^k + 1)] + \sum_{r=2}^{k} [f(n^r + 1) - f(n^{r-1} + 1) - f(n)] + [f(n+1) - f(n)]$$

is reduced in the same manner as above. △

As a result, it can be seen that $f(n^t) = t f(n)$, since Theorem 4.4 implies that $|f(n^k) - k f(n)| \leq c_3(n, \varepsilon) + kn\varepsilon$. That is,

$$\limsup_{k\to\infty} \left| \frac{f(n^k)}{k} - f(n) \right| \leq n\varepsilon$$

for every $\varepsilon > 0$, or

$$\lim_{k\to\infty} \left| \frac{f(n^k)}{k} - f(n) \right| = 0 ,$$

and hence,

$$\lim_{k\to\infty} \frac{f(n^k)}{k} = f(n) .$$

Then, by Theorem 2.8, f is completely additive, a fact which will be needed later.

<u>Theorem 4.5.</u> Let $f \in \mathfrak{a}$, and let $2^k \leq n < 2^{k+1}$. Then

(12) $$f(n) \geq -k\varepsilon - c(\varepsilon) + k f(2) ,$$

(13) $$f(n) \leq (k+1)\varepsilon + c(\varepsilon) + (k+1) f(2) .$$

Proof: Define a finite sequence of integers $\{x_k\}$, where $x_0 = n$ and $x_{r+1} = \frac{1}{2} x_r'$, for x_r' the even one of x_r and $x_r - 1$. Now since $2^k \leq x_0 < 2^{k+1}$, it follows that $2^{k-1} \leq x_1 < 2^k, \ldots, 2^{k-k} \leq x_k < 2^{k-(k-1)}$. Since x_k is an integer such that $1 \leq x_k < 2$, x_k must be equal to 1, and since f is completely additive, $f(x_k) = f(1) = 0$. Then

$$\begin{aligned} f(n) &= f(x_0) - f(x_k) \\ &= \sum_{r=0}^{k-1} [f(x_r) - f(x_r')] + \sum_{r=0}^{k-1} [f(x_r') - f(x_{r+1})] \\ &= \Sigma [f(x_r) - f(x_r')] + \Sigma [f(2x_{r+1}) - f(x_{r+1})] \\ &= \Sigma [f(x_r) - f(x_r')] + \Sigma f(2) , \end{aligned}$$

by the additive property of f. Hence, $f(n) \geq -k\varepsilon - c(\varepsilon) + k f(2)$, by (9).

Again, define a finite sequence of integers $\{y_{k+1}\}$, where $y_0 = n$ and $y_{r+1} = \frac{1}{2} y_r'$, for y_r' the even one of y_r and $y_r + 1$. By reasoning similar to above, $y_{k+1} = 1$ and $f(y_{k+1}) = 0$. Thus, since f is completely additive, (9) implies

$$f(n) \leq (k+1)\varepsilon + c(\varepsilon) + (k+1) f(2) . \qquad \triangle$$

Combining the inequalities in (12) and (13) yields

$$|f(n) - kf(2)| \leq 2k\varepsilon + c(\varepsilon) + |f(2)| .$$

But since $k \leq \log_2 n < k+1$,

(14) $$|f(n) - f(2)\log_2 n| < 2\varepsilon \log_2 n + c(\varepsilon) + 2|f(2)| .$$

With these results it is now possible to establish the following condition for a function to belong to $\mathfrak{B}$.

<u>Theorem 4.6.</u> Let $f \in \mathfrak{A}$. If $\liminf f(n+1) - f(n) \geq 0$, then $f \in \mathfrak{B}$.

Proof: Replacing n, in (14), by n^t results in

$$|f(n^t) - f(2)\log_2 n^t| < 2\varepsilon \log_2 n^t + c(\varepsilon) + 2|f(2)| .$$

Since f is completely additive, $f(n^t) = t f(n)$; dividing by t produces

$$|f(n) - f(2)\log_2 n| < 2\varepsilon \log_2 n + \frac{1}{t}\{c(\varepsilon) + 2|f(2)|\} .$$

If $t \to \infty$, then

$$|f(n) - f(2)\log_2 n| \leq 2\varepsilon \log_2 n ,$$

for every $\varepsilon > 0$, which shows

$$f(n) = f(2)\log_2 n = \frac{f(2)}{\log 2} \log n..$$

Hence $f(n)$ is a constant multiple of $\log n$. △

It was also conjectured by Erdős that if the condition in Theorem 4.3 is replaced by the condition

(15) $$\lim_{x\to\infty} \frac{1}{x} \sum_{n \le x} |f(n+1) - f(n)| = 0 ,$$

the conclusion follows. The proof of this theorem was published by Kátai [29] in 1970. A conjecture, which is still open, made by Ryavec [45] is that f is in $\mathfrak{B}$, if $f \in \mathfrak{a}$ and

(16) $$\liminf_{x\to\infty} \frac{1}{x} \sum_{n \le x} |f(n+1) - f(n)| = 0 .$$

In the same paper, Ryavec proves a weaker version of this conjecture, in which f is in $\mathfrak{B}$ if, for $f \in \mathfrak{a}$, f satisfies (16) and $f(n) = O(\log n)$. Another condition which assures that f is in $\mathfrak{B}$ is the following: $f(n) \ge 0$ and

(17) $$\liminf_{x\to\infty} \frac{1}{\log x} \sum_{n \le x} \left| \frac{f(n+1)}{n+1} - \frac{f(n)}{n} \right| = 0 .$$

This is also proved by Ryavec.

Conditions on Completely Additive Functions

The functions in $\mathfrak{B}$ are all completely additive, since they are constant multiples of the completely additive function $\log n$. Up to now attention has been focused on conditions on additive functions to assure they are in $\mathfrak{B}$. Since not every completely additive function is in $\mathfrak{B}$, what about conditions on these functions so they will be in $\mathfrak{B}$? Wirsing [49], in the process of proving a theorem conjectured by

Erdős, provides such a condition. Wirsing's theorem is stated here without proof, and the reader is referred to his article for the proof.

Theorem 4.7. If f is completely additive and fulfills $f(n+1) \leq f(n) + K$, for all n, with K a constant, then $f \in \mathfrak{B}$.

Another condition which assures that a completely additive function is in $\mathfrak{B}$ was set forth, and proved, by Kátai [25].

Theorem 4.8. If $f \in \mathfrak{C}$, such that f is completely additive, and

(18) $$f(2n+1) - f(n) \to K,$$

with K a constant, then $f \in \mathfrak{B}$. Further, for $f(n) = c \log n$, $c = \frac{K}{\log 2}$.

Proof: Since f and the log function are completely additive, then so is the function $g(n) = f(n) - \frac{K \log n}{\log 2}$. If $f(2n+1) - f(n) \to K$, then $g(2n+1) - g(n) \to 0$. Thus, the problem can be restated as: f completely additive and $f(2n+1) - f(n) \to 0$ implies $f(n) \equiv 0$.

Given f completely additive, $K = 0$ and (18) implies

$$f(2n+1) - f(2n) = f(2n+1) - f(n) - f(2) \to -f(2),$$

which implies

(19) $$f(2n+1) - f(2n) = -f(2) + o(1).$$

Let N be a large number of the form

$$N = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_k},$$

where $a_1 > a_2 > \ldots > a_k$, and let $\lambda(N) = k$, the length of N. Then

$$N = 2^{a_k}(2^{a_1-a_k} + 2^{a_2-a_k} + \dots + 1)$$

and

$$f(N) = f(2^{a_k}) + f(2^{a_1-a_k} + 2^{a_2-a_k} + \dots + 1)$$

$$= f(2^{a_k}) + f(2^{a_1-a_k} + \dots + 2^{a_{k-1}-a_k} + 1)$$

$$- f(2^{a_1-a_k} + \dots + 2^{a_{k-1}-a_k})$$

$$+ f(2^{a_1-a_k} + \dots + 2^{a_{k-1}-a_k}) .$$

Since f is completely additive, formula (19) makes this expression become

$$f(N) = a_k f(2) - f(2) + f(2^{a_1-a_k} + \dots + 2^{a_{k-1}-a_k}) + o(1)$$

$$= a_k f(2) - f(2) + f\left(2^{a_{k-1}-a_k}(2^{a_1-a_k-a_{k-1}+a_k} + \dots + 1)\right) + o(1) .$$

The additivity of f implies

$$f(N) = a_k f(2) - f(2) + o(1) + f(2^{a_{k-1}-a_k}) + f(2^{a_1-a_{k-1}} + \dots + 1)$$

$$- f(2^{a_1-a_{k-1}} + \dots + 2^{a_{k-2}-a_{k-1}})$$

$$+ f(2^{a_1-a_{k-1}} + \dots + 2^{a_{k-2}-a_{k-1}}) .$$

Again the complete additivity of f and (19) produces

$$f(N) = a_k f(2) - f(2) + o(1) + (a_{k-1} - a_k) f(2) - f(2) + o(1)$$

$$+ f(2^{a_1-a_{k-1}} + \dots + 2^{a_{k-2}-a_{k-1}}) .$$

Simplification yields

$$f(N) = (a_k + a_{k-1} - a_k)\, f(2) - 2f(2) + 2o(1) + f(2^{a_1 - a_{k-1}} + \ldots + 2^{a_{k-2} - a_{k-1}})$$

$$= a_{k-1}\, f(2) - 2f(2) + 2o(1) + f(2^{a_1 - a_{k-1}} + \ldots + 2^{a_{k-2} - a_{k-1}}) .$$

This process is repeated a total of k times, the length of N, so that after the last but one reduction,

$$f(N) = a_2 f(2) - (k-1)\, f(2) + (k-1)\, o(1) + f(2^{a_1 - a_2} + 1) .$$

Applying the reduction one last time produces

$$f(N) = a_2 f(2) - (k-1)\, f(2) + (k-1)\, o(1) + f(2^{a_1 - a_2} + 1)$$

$$- f(2^{a_1 - a_2}) + f(2^{a_1 - a_2})$$

$$= a_2 f(2) - (k-1)\, f(2) + (k-1)\, o(1) - f(2) + o(1) + (a_1 - a_2)\, f(2) .$$

Therefore,

(20) $$f(N) = a_1 f(2) - k\, f(2) + k\, o(1) .$$

By the definition of N, $2^{a_1} \leq N < 2^{a_1 + 1}$; Thus

$$a_1 \log 2 \leq \log N < a_1 \log 2 + \log 2 ,$$

and

$$\frac{a_1 \log 2}{\log N} \leq 1 < \frac{a_1 \log 2}{\log N} + \frac{\log 2}{\log N} .$$

Hence,

$$0 \leq 1 - \frac{a_1 \log 2}{\log N} < \frac{\log 2}{\log N} .$$

From this, it follows that

$$\frac{a_1 \log 2}{\log N} = 1 - \frac{\log 2}{\log N} + o(1) .$$

So, $a_1 = \frac{\log N}{\log 2} - 1 + \frac{o(1) \log N}{\log 2}$.

To show $f(2) = 0$, let $N_t = 2 + 2^3 + \ldots + 2^{2t+1}$, for $t \geq 0$. Then

$$\begin{aligned} 3N_t &= (2+1)(2+2^3 + \ldots + 2^{2t+1}) \\ &= 2 \cdot 2 + 2 \cdot 2^3 + \ldots + 2 \cdot 2^{2t+1} + 2 + 2^3 + \ldots + 2^{2t+1} \\ &= 2 + 2^2 + 2^3 + \ldots + 2^{2t+1} + 2^{2t+2} . \end{aligned}$$

It follows that $\lambda(N_t) = \frac{1}{2}(2t + 1 + 1) = t+1$, and $\lambda(3N_t) = 2t + 2 = 2\lambda(N_t)$. From the fact that $2t + 1 = a_1$, with a_1 given above, it follows that

$$2t = \frac{\log N_t}{\log 2} - 2 + \frac{o(1) \log N_t}{\log 2}$$

and

$$t = \frac{\log N_t}{2 \log 2} - 1 + \frac{o(1) \log N_t}{2 \log 2} \quad \text{or} \quad t + 1 = \frac{\log N_t}{2 \log 2} + \frac{o(1) \log N_t}{2 \log 2} .$$

Because $(3, N_t) = 1$ and f is additive, $f(3 N_t) = f(3) + f(N_t)$, and therefore,

$$f(3) = f(3N_t) - f(N_t)$$

$$= (2t+2)f(2) - (2t+2)f(2) + (2t+2)o(1) - (2t+1)f(2)$$

$$+ (t+1)f(2) - (t+1)o(1) ,$$

after substituting into (20). Thus,

$$f(3) = -t\,f(2) + (t+1)\,o(1)$$

$$= -\left(\frac{\log N_t}{2\log 2} - 1 + \frac{o(1)\log N_t}{2\log 2}\right) f(2) + \left(\frac{\log N_t}{2\log 2} + \frac{o(1)\log N_t}{2\log 2}\right) o(1)$$

$$= \frac{-f(2)\log N_t}{2\log 2}\left(1 + o(1)\right) + f(2) + o(1) ,$$

by Lemmas 1.5 and 1.6. This last expression approaches ∞ if $f(2) \neq 0$, a contradiction that $f(n+1) - f(n) \to \infty$. Thus, $f(2) = 0$. Then

(21)
$$\lim_{N\to\infty} \frac{f(N)}{\log N} = 0 .$$

Since f is completely additive, (21) implies

$$\frac{f(N)}{\log N} = \lim_{N\to\infty} \frac{f(N^k)}{\log N^k} = 0 .$$

Hence $f(N) = 0$. △

Decomposition of Additive Functions

So far interest has been centered on the conditions under which a real-valued additive function will be in $\mathfrak{B}$. It is of interest now to focus attention on the matter of decomposing an additive function into a sum of two additive functions, one of which is a completely additive function (in particular, a constant multiple of $\log n$) and the other, a bounded function.

Erdős [9, p. 3] conjectured that if an additive function f is such that $f(n+1) - f(n)$ is bounded, then f decomposes into the sum of a constant multiple of $\log n$ and a bounded function.

<u>Theorem 4.9.</u> Assume for $f \in \mathcal{G}$, $f(n+1) - f(n) < c_1$, for all n. Then

$$f(n) = c \log n + g(n) \tag{22}$$

where $|g(n)| < c_2$, for all n.

The proof of this theorem was published in 1970 by Wirsing [49].

The converse of this theorem is also true. For if $f \in \mathcal{G}$ and g is a bounded function such that (22) is true, then

$$\begin{aligned} f(n+1) - f(n) &= c \log(n+1) + g(n+1) - c \log n - g(n) \\ &\leq c(\log(n+1) - \log n) + 2c_2 \\ &= c \log \frac{n+1}{n} + 2c_2 \\ &< c \log 2 + 2c_2 . \end{aligned}$$

Thus the set of differences $f(n+1) - f(n)$ will be bounded.

Eörs Máté [38] proves a more general, and somewhat weaker, form of this conjecture. Instead of a constant multiple of $\log n$, his decomposition involves a completely additive function, and the decomposition is unique. Assuming the existence of such a decomposition, the completely additive function is $h(n) = \lim \frac{f(n^t)}{t}$ (see Theorem 2.8) and the bounded function is $g(n) = f(n) - h(n)$.

<u>Theorem 4.10</u>. If $f \in \mathcal{G}$, $|f(n+1) - f(n)| \leq M$, for every n and $(n-1, s) = 1$, then $|f(n^s) - s f(n)| \leq 2sM$.

Proof: Consider

$$\begin{aligned} |f(n^s) - sf(n)| &= |f(n^s) - f(n^s - 1) + f(n^s - 1) - sf(n)| \\ &\leq |f(n^s - 1) - sf(n)| + M \\ &= \left| f\left((n-1) \sum_{i=0}^{s-1} n^i\right) - sf(n)\right| + M . \end{aligned}$$

Since

$$s = \sum_{i=0}^{s-1} 1 \equiv \sum_{i=0}^{s-1} n^i \bmod (n-1), \left(\sum_{i=0}^{s-1} n^i, n-1\right) = 1 ,$$

and since f is additive,

$$\begin{aligned} |f(n^s) - sf(n)| &\leq M + |f(n-1) - f(n)| + \left| f\left(\sum_{i=0}^{s-1} n^i\right) - (s-1) f(n)\right| \\ &\leq 2M + \left| f\left(\sum_{i=0}^{s-1} n^i\right) - f\left(\sum_{i=0}^{s-2} n^i\right) - f(n) + f\left(\sum_{i=0}^{s-2} n^i\right) - (s-2) f(n)\right| \\ &= 2M + \left| \left\{ f\left(\sum_{i=0}^{s-1} n^i\right) - f\left(\sum_{i=0}^{s-2} n^i\right) - f(n)\right\} \right. \\ &\quad \left. + \left\{ f\left(\sum_{i=0}^{s-2} n^i\right) - f\left(\sum_{i=0}^{s-3} n^i\right) - f(n)\right\} + f\left(\sum_{i=0}^{s-3} n^i\right) - (s-3) f(n)\right| . \end{aligned}$$

This process is repeated until $f\left(\sum_{i=0}^{s-(s-1)} n^i\right) - (s-s+1) f(n)$ is reached. This then can be replaced by

$$f\left(\sum_{i=0}^{1} n^i\right) - f\left(\sum_{i=0}^{0} n^i\right) - f(n) ,$$

since $f\left(\sum_{i=0}^{0} n^i\right)$ is equal to $f(1) = 0$. Therefore

$$|f(n^s) - sf(n)| \leq 2M + \left| \sum_{j=1}^{s-1} \left\{ f\left(\sum_{i=0}^{j} n^i\right) - f\left(\sum_{i=0}^{j-1} n^i\right) - f(n) \right\} \right| .$$

Because $(n, \Sigma n^i) = 1$, the additive property of f implies

$$f\left(\sum_{i=0}^{j-1} n^i\right) + f(n) = f\left(n \cdot \sum_{i=0}^{j-1} n^i\right) = f\left(\left(\sum_{i=0}^{j} n^i\right) - 1\right) .$$

With this substitution,

$$|f(n^s) - sf(n)| \leq 2M + \sum_{j=1}^{s-1} \left| f\left(\sum_{i=0}^{j} n^i\right) - f\left(\left(\sum_{i=0}^{j} n^i\right) - 1\right) \right|$$

$$\leq 2M + (s-1)M = (s+1)M \leq 2sM . \qquad \triangle$$

<u>Theorem 4.11</u>. If $f \in \mathfrak{a}$ and $|f(n+1) - f(n)| \leq M$, for each n, then for any two integers $k \geq 0$ and $n > 0$,

$$|f(n^{2^k}) - 2^k f(n)| \leq 4(2^k) M .$$

Proof: If n is even, then $(2^k, n-1) = 1$ so that Theorem 4.10 can be used with $2(2^k)M < 4(2^k)M$. Using this result for n odd,

$$\left|f(n^{2^k}) - 2^k f(n)\right| = \left|\left(f(n^{2^k}) + f(2^{2^k})\right) - f(2^{2^k}) - \left(2^k f(n) + 2^k f(2)\right) + 2^k f(2)\right|$$

$$= \left| f\left((2n)^{2^k}\right) - f(2^{2^k}) - 2^k f(2n) + 2^k f(2) \right| ,$$

since $(2, n) = 1$ and f is additive. Hence,

$$f(n^{2^k}) - 2^k f(n) \le \left| f\left((2n)^{2^k}\right) - 2^k f(2n)\right| + \left| f(2^{2^k}) - 2^k f(2)\right|$$

$$\le 2(2^k)M + 2(2^k)M$$

$$= 4(2^k)M \,. \qquad \triangle$$

Theorem 4.12. If $f \in \mathfrak{a}$ and $|f(n+1) - f(n)| \le M$, for each n, then for any two positive integers s and t,

$$|f(n^t) - f(n^s)| \le 4|t - s|nM \,.$$

Proof: There is no loss in generality in assuming $t > s$; then

$$|f(n^t) - f(n^s)| = |f(n^t) - f(n^t - 1) + f(n^t - 1) - f(n^s) + f(n^s - 1) - f(n^s - 1)|$$

$$\le 2M + |f(n^t - 1) - f(n^s - 1)| \,,$$

since $|f(n+1) - f(n)| \le M$ for any n. Because $t > s$, add and subtract the quantity $f(n^{i-1} - 1) - f(n)$ for i strictly between s and t. This results in

$$|f(n^t) - f(n^s)| = 2M + \left| \sum_{i=s+1}^{t} \{f(n^i - 1) - f(n^{i-1} - 1) - f(n)\} + (t-s)f(n)\right|$$

$$\le 2M + \sum_{i=s+1}^{t} |f(n^i - 1) - f(n^{i-1} - 1) - f(n)| + (t-s)|f(n)|$$

$$= 2M + \sum_{i=s+1}^{t} |f(n^i - 1) - f(n^i - n)| + (t-s)\,|f(n)| \,,$$

by the additivity of f. The summation $\sum_{i=s+1}^{t} |f(n^i - 1) - f(n^i - n)|$ is equal to

$$|f(n^{s+1} - 1) - f(n^{s+1} - n) + f(n^{s+2} - 1) - f(n^{s+2} - n) + \ldots + f(n^t - 1) - f(n^t - n)|$$

which is less than or equal to

$$|f(n^{s+1} - 1) - f(n^{s+1} - 2)| + |f(n^{s+1} - 2) - f(n^{s+1} - 3)| + \ldots + |f(n^{s+1} - (n-1)) - f(n^{s+1} - n)| + \ldots + |f(n^t - 1) - f(n^t - 2)| + \ldots + |f(n^t - (n-1)) - f(n^t - n)| .$$

For i in the given range there were $(n - 1)$ terms added and then subtracted. Also there are $(t - s)$ values of i. Thus, given $|f(n+1) - f(n)|$ bounded by M,

$$\sum_{i=s+1}^{t} |f(n^i - 1) - f(n^i - n)| \leq (n-1)(t-s)M .$$

In addition,

$$f(n) = [f(n-1+1) - f(n-1)] + [f(n-1) - f(n-2)] + \ldots + [f(3) - f(2)] + [f(2) - f(1)] ,$$

since $f(1) = 0$. With $|f(n+1) - f(n)|$ bounded, it follows that

$$|f(n)| \leq \sum_{i=1}^{n-1} |f(i+1) - f(i)| \leq (n-1)M .$$

Using these results yields

$$|f(n^t) - f(n^s)| \leq 2M + 2(t-s)(n-1)M \leq 4(t-s)nM . \qquad \triangle$$

With Theorems 4.10 - 4.12, which yield some significant bounds in their own right, the theorem which guarantees the decomposition of an additive function can now be proved.

Theorem 4.13. Suppose that for $f \in \mathcal{C}$, the difference $f(n+1) - f(n)$ is bounded. Then there exists a decomposition $h(n) + g(n)$ of $f(n)$, where h is completely additive and g is bounded.

Proof: Consider the sequence defined by $\frac{f(n^t)}{t}$. If t runs over numbers of the form 2^k, then for any k and h,

$$\left| \frac{f(n^{2^k})}{2^k} - \frac{f(n^{2^h})}{2^h} \right| = \frac{1}{2^k 2^h} \left| 2^h f(n^{2^k}) - 2^k f(n^{2^h}) \right|$$

$$\leq \frac{1}{2^k 2^h} \left\{ \left| 2^h f(n^{2^k}) - f(n^{2^{k+h}}) \right| + \left| f(n^{2^{k+h}}) - 2^k f(n^{2^h}) \right| \right\}.$$

Since $n^{2^{k+h}} = n^{2^k 2^h}$, $f(n^{2^{k+h}})$ is equal to either $f((n^{2^k})^{2^h})$ or $f((n^{2^h})^{2^k})$, so that Theorem 4.11 is applicable. The last expression of the above inequality, then, is less than or equal to

$$\frac{1}{2^{k+h}} [4(2^h)M + 4(2^k)M].$$

Thus,

$$\left| \frac{f(n^{2^k})}{2^k} - \frac{f(n^{2^h})}{2^h} \right| \leq 4M\left(\frac{1}{2^k} + \frac{1}{2^h}\right).$$

As k and h approach ∞, this last expression approaches 0. Hence the sequence, for t of the form 2^k, is Cauchy. Thus,

$$h(n) = \lim_{k\to\infty} \frac{f(n^{2^k})}{2^k}$$

exists, and in particular,

$$\left| \frac{f(n^{2^k})}{2^k} - h(n) \right| \leq \frac{4M}{2^k} \quad . \tag{23}$$

Let k be large fixed integer. If s runs over the primes less than or equal to $n^{2^k} - 1$, then

$$\left| \frac{f(n^s)}{s} - h(n) \right| = \left| \frac{f(n^s)}{s} - \frac{1}{s2^k} f(n^{s2^k}) + \frac{1}{s2^k} f(n^{s2^k}) - \frac{f(n^{2^k})}{2^k} + \frac{f(n^{2^k})}{2^k} - h(n) \right|$$

$$\leq \frac{1}{s2^k} \left| 2^k f(n^s) - f(n^{s2^k}) \right| + \frac{1}{s2^k} \left| f(n^{s2^k}) - sf(n^{2^k}) \right|$$

$$+ \left| \frac{f(n^{2^k})}{2^k} - h(n) \right| .$$

By Theorem 4.11, the first term is less than or equal to

$$\frac{1}{s2^k}\left(4(2^k)M\right) = \frac{4M}{s} \quad . \tag{24}$$

By Theorem 4.10, since $(n^{2^k} - 1, s) = 1$, the second term is less than or equal to

$$\frac{1}{s2^k}\left(2sM\right) = \frac{2M}{2^k} \quad . \tag{25}$$

By (23), the last term is less than or equal to $\frac{4M}{2^k}$. These yield

$$\left| \frac{f(n^s)}{s} - h(n) \right| \leq \frac{4M}{s} + \frac{2M}{2^k} + \frac{4M}{2^k} .$$

Thus, for $s \leq n^{2^k} - 1$ and s prime,

$$\text{(26)} \qquad \limsup_{s\to\infty} \left| \frac{f(n^s)}{s} - h(n) \right| \leq \frac{6M}{2^k} .$$

Consider $s \leq t < s + n^{2^k}$. Then the difference

$$\left| \frac{f(n)^t}{t} - \frac{s}{t} \frac{f(n^s)}{s} \right| = \frac{1}{t} \left| f(n^t) - f(n^s) \right| \leq \frac{1}{t} \cdot 4(t-s)nM ,$$

by Theorem 4.12. For any t, then,

$$\left| \frac{f(n^t)}{t} - h(n) \right| = \left| \frac{f(n^t)}{t} - \frac{s}{t} \frac{f(n^s)}{s} + \frac{s}{t} \frac{f(n^s)}{s} - h(n) \right|$$

$$\leq \frac{1}{t} \cdot 4(t-s)nM + \frac{6M}{2^k} .$$

Let $t \to \infty$. Then $s \to \infty$, $k \to \infty$, $\frac{s}{t} \to 1$ and

$$\limsup_{t\to\infty} \left| \frac{f(n^t)}{t} - h(n) \right| \leq \frac{6M}{2^k} .$$

The left member does not depend on k, so by making $k \to \infty$,

$$\limsup_{t\to\infty} \left| \frac{f(n^t)}{t} - h(n) \right| = 0 .$$

Therefore,

$$\lim_{t\to\infty} \frac{f(n^t)}{t} = h(n)$$

exists, and by Theorem 2.8, h is completely additive.

Let $g(n) = f(n) - h(n)$. If, in (23), $k = 0$, then $|f(n) - h(n)| \leq 4M$. Hence, $|g(n)| \leq 4M$ and g is bounded. Therefore, the theorem follows. △

The material presented here is by no means all inclusive. Additional conditions are set forth by Kátai, Erdös, and others. But the proofs of these are beyond the scope intended for this paper. It is intended only to give some insight into the types of investigation which can be carried out in this area, and an introduction to the methods by which these results can be established.

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APPENDIX

A list of additive functions which appear in the dissertation is included here for the convenience of the reader. Let

$$n = \prod_{i=1}^{r} p_i^{a_i} = \prod_{p^a \| n} p^a .$$

1. $z(n) = 0$, for all n.

2. $\omega(n) = \sum_{i=1}^{r} 1$ = number of distinct prime divisors of n.

3. $\Omega(n) = \sum_{i=1}^{r} a_i$ = number of prime divisors of n.

4. $\Omega_k(n) = \sum_{i=1}^{r} a_i^k$, $k \geq 0$.

5. $\gamma(n) = \sum_{i=1}^{r} a_i p_i$ (Chawla's function).

6. $\gamma_k(n) = \sum_{i=1}^{r} a_i p_i^k$, $k \geq 0$ (generalization of Chawla's function).

7. $s(n) = \sum_{i=1}^{r} p_i$.

8. $s_k(n) = \sum_{i=1}^{r} p_i^k$, $k \geq 0$.

9. $\log \frac{\varphi(n)}{n} = \sum_{i=1}^{r} \log(1 - p_i^{-1})$, where φ is Euler's function.

10. $H(n) = \frac{1}{\tau(n)} \sum_{d|n} h(n)$, where h is additive (average based on divisors).

11. $h(n) = \sum_{d|n} \mu(d)\tau H(\frac{n}{d})$, where H is additive.

12. $h(n) = \sum_{p^a \| n} \log \frac{(a+1)^{a+1}}{a^a}$, where $H(n) = \log \tau(n)$.

13. $(f * gh)(n) = \sum_{d|n} f(d)\, g(\frac{n}{d})\, h(\frac{n}{d})$, where f and g are multiplicative, h is additive, $u_0(n) = 1$ and $f * g = u_0$.

14. $g(n) = \lim_{t\to\infty} \frac{f(n^t)}{t}$, where f is additive.

15. $L_1 f(n) = \sum_{p^a \| n} L f(p^a)$, where f is multiplicative, $L f(1) = \log f(1)$ and $L f(n) = \sum_{d|n} f(d) f^{-1}(\frac{n}{d}) \log d$, for $n > 1$.

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