STATISTICAL INFERENCE RELATED TO THE INVERSE GAUSSIAN DISTRIBUTION

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CHAPTER I

INTRODUCTION

Tweedie [19] characterized inverse statistical variates by inversion of the logarithm of moment generating functions and discussed examples of inverses to Binomial, Poisson and Normal variates. This has led to the distribution of inverse to the normal variate known as inverse Gaussian. To put it in the simplest form, one may say that the inverse Gaussian is to the normal as (i) the negative Binomial is to the Binomial and (ii) the exponential is to the Poisson. This relationship between any such pair of random variables can also be described in terms of physical phenomena and its study constitutes the subject matter of the theory of stochastic processes. The physical aspect of the inverse Gaussian is inherited in the Brownian motion, and this has been mathematically established in detail, among others, by Cox and Miller [3]. Briefly, it says that the time X for a particle to cover a given distance d along a line in a diffusion (Gaussian) process for the first time is described by the density function

$$f(x) = \frac{d}{\sqrt{2\pi\beta x^3}} e^{-(d-vx)^2/2\beta x}, \quad x > 0$$

where β is a diffusion constant and v is the velocity associated with the particle movement.

While dealing with the same structure but with different situations, Wald [22] and Feller [7] also derived this distribution in an asymptotic way. But the family of such distributions lurked completely until Tweedie [20], [21] published some interesting results on its statistical properties. In fact, the development on sampling distributions given in his papers became the milestone for further statistical inferential investigation on the inverse Gaussian distribution. Recently, Wasan and his associates have elaborated on some analytical and characteristic properties of this distribution in its various forms including limiting cases. Wasan [23] has also formulated an inverse Gaussian process.

In this thesis, our main objective is to investigate some problems of statistical inference related to the inverse Gaussian distribution and to draw attention to some useful areas for its applications. Among its various equivalent forms often found in the literature, we will consider the form which has become almost standard. A random variable X is distributed as inverse Gaussian if its density function is given by

$$f(x;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi x^3}}\; e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}, \quad x > 0 \qquad (1)$$

$$= 0, \qquad \text{otherwise}$$

where μ and λ are assumed to be positive and these constitute a pair of independent parameters. See Chapter II for further properties.

Our interest in the inverse Gaussian distribution derives mainly from its direct relationship with the normal distribution and its skewness property, especially in the case of large variance. We note that this distribution shares with the Gamma and log-normal distributions the asymptotic convergence to normality when the variance gets small. We also demonstrate another common feature of the inverse Gaussian and the log-normal as life-time distributions in Chapter VII.

In Chapter II, we give some properties of this distribution. Chapter III consists of some distributions related to the inverse Gaussian. The problem of testing statistical hypotheses on the parameters μ and λ of (1) is discussed in Chapter IV and optimal test procedures are derived. Chapter V is a follow-up on the interval estimation.

In Chapters VI and VII, we propose to investigate the inverse Gaussian as a life-time distribution as to various aspects of reliability theory. Since this distribution arises as a distribution of first-passage time in a Gaussian process, its applicability to a life-testing or a life-time situation is a natural consequence. Many distributions, including log-normal, have been studied for their use in reliability problems; but no such study seems to be available on the inverse Gaussian as a life-time distribution.

Notations being adopted here are often given in any standard texts on topics of this thesis. As a result of its general nature as to the study on inverse Gaussian distribution and a coverage of different aspects of statistical inference, it was not possible to introduce a topic and discuss it beyond our needs for its applications to the inverse Gaussian distribution. However, we feel that the discussion of the material is self-explanatory to those who are acquainted with the area of statistical inference. Otherwise, a slight refuge in references given at the end is sufficient to overcome the initial difficulty.

CHAPTER II

SOME STATISTICAL PROPERTIES OF THE INVERSE GAUSSIAN DISTRIBUTION

In this chapter we outline briefly some properties of the inverse Gaussian distribution given by (1) in Chapter I. For more details, see Johnson and Kotz [11], Tweedie [20] and Wasan [24].

Tweedie [20] showed that the distribution is unimodal, with its mode at

$$x_m = \mu\left(1 + \frac{9\mu^2}{4\lambda^2}\right)^{\frac{1}{2}} - \frac{3\mu^2}{2\lambda} \ . \tag{1}$$

Next, the characteristic function of an inverse Gaussian random variable X is given by

$$\varphi_X(t) = \exp\left[\frac{\lambda}{\mu}\{1 - (1 - \frac{2i\mu^2 t}{\lambda})^{\frac{1}{2}}\}\right] \tag{2}$$

and all moments exist. The r^{th} moments about 0 is given

$$E[X^r] = \mu^r \sum_{s=0}^{r-1} \frac{(r-1+s)!}{s!\,(r-1-s)!} \left(\frac{\mu}{2\lambda}\right)^s \tag{3}$$

and, in particular,

$$E[X] = \mu \quad \text{and} \quad \mathrm{Var}(X) = \frac{\mu^3}{\lambda} \ . \tag{4}$$

The density curve is positively skewed, and λ is a shape parameter.

Further, we list the following properties that either have been established earlier or can be easily proved.

Property (additive) 1: Let $X_1, X_2, \ldots, X_n$ be n independent inverse Gaussian random variables, and X_i is distributed according to (1) in Chapter I with $\mu = \mu_i$, $\lambda = \lambda_i$ $(i = 1, 2, \ldots, n)$. Then $\sum_1^n X_i$ is an inverse Gaussian distributed random variable if and only if ξ_i are the same for all i, where $\xi_i = \lambda_i/\mu_i^2$ $(i = 1, 2, \ldots, n)$. Letting $\xi_i = \xi$ $(i = 1, 2, \ldots, n)$, the distribution of $\sum_1^n X_i$ is the inverse Gaussian with parameters $\sum_1^n \mu_i$ and $\xi\left(\sum_1^n \mu_i\right)^2$.

Property 2: If $Z = \lambda X/\mu^2$, then Z has inverse Gaussian distribution with parameters ϕ and ϕ^2, where $\phi = \lambda/\mu$. Moreover, $E[Z] = \phi$ and $\mathrm{Var}[Z] = \phi$.

Wasan and Roy [25] have tabulated the percentage points of the distribution of Z for different values of ϕ. Also, Wasan [23] has formulated an inverse Gaussian process on such properties of a random variable. Note that $\sqrt{1/\phi}$ is the coefficient of variation for both random variables X and Z.

Property 3: Let $Y = (Z - \phi)/\sqrt{Z}$. Then (i) Y^2 has a χ^2 distribution with 1 degree of freedom, and (ii) Y has a non-linear weighted normal distribution given by (1.2), Chapter III.

Property 4: The distribution of X is asymptotically normal with mean μ and variance μ^3/λ as $\lambda \to \infty$. (This result is due to Wald [22].)

Property 5: The density function $f(x;\mu,\lambda)$ as in (1) of Chapter I is TP_2 (Total positive of order 2) but it is not PF_2 (Polya frequency function of order 2). (See Karlin [12] for definition of TP_2 and PF_2.)

Wasan [23] establishes only the TP_2 property.

Property 6: The family of inverse Gaussian density functions is complete. (See Wasan [23].)

For a sample $X = (X_1, X_2, \ldots, X_n)$ from an inverse Gaussian population with parameters μ and λ, Tweedie [20] obtained maximum likelihood estimates (MLE's) of μ and λ given by

$$\hat{\mu} = \frac{1}{n}\sum_{1}^{n} X_i \quad \text{and} \quad \hat{\lambda}^{-1} = \frac{1}{n}\sum_{1}^{n}\left(\frac{1}{X_i} - \frac{1}{\overline{X}}\right). \tag{5}$$

He also proved that $\hat{\mu}$ and $\hat{\lambda}^{-1}$ are stochastically independent; and (i) $\hat{\mu}$ has inverse Gaussian distribution with parameters μ and $n\lambda$, and (ii) $n\lambda\,\hat{\lambda}^{-1}$ is distributed as χ^2 with $n-1$ degrees of freedom. $\left(\sum_{1}^{n} X_i, \sum_{1}^{n} X_i^{-1}\right)$ is a minimal sufficient statistic for (μ, λ). Furthermore, Tweedie developed an analogue of the analysis of variance for nested classification.

The distribution given by (1) in Chapter I can equivalently be written as

$$f(x;\mu,\phi) = \sqrt{\frac{\mu\phi}{2\pi x^3}}\; e^{-\phi(x-\mu)^2/2\mu x}, \quad x > 0 \tag{6}$$

where μ and ϕ are positive. For $\mu = 1$, (6) reduces to the form known as the Wald distribution, an alternative name used in Russian literature.

The MLE's of μ and ϕ are

$$\hat{\mu} = \frac{1}{n}\sum_{1}^{n} X_i \quad \text{and} \quad \hat{\phi} = \left[n^{-2}\left(\sum_{1}^{n} X_i\right)\left(\sum_{1}^{n} X_i^{-1}\right) - 1\right]^{-1}. \tag{7}$$

It can be shown that the density function of $\hat{\phi}$ is

$$p(\hat{\phi};\mu,\phi) = \frac{2}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}\left(\frac{n\phi}{2}\right)^{\frac{n}{2}} e^{n\phi}\sqrt{\hat{\phi}^{-1}}\left[\hat{\phi}(1+\hat{\phi})\right]^{-\frac{n}{4}} K_{\frac{n}{2}}\left(n\phi\sqrt{\frac{1+\hat{\phi}}{\hat{\phi}}}\right), \tag{8}$$

where

$$K_{\pm\alpha}(z) = \frac{1}{2}\left(\frac{1}{2}z\right)^{\alpha}\int_0^{\infty} t^{-(\alpha+1)} e^{-(t+\frac{z^2}{4t})}\,dt$$

as given by Watson [26, pg. 183]. Also

$$E[\hat{\phi}^{r}] = \left(\frac{n\phi}{2}\right)^{r} \frac{\Gamma\left(\frac{n-1}{2}-r\right)}{\Gamma\left(\frac{n-1}{2}\right)} \sum_{s=0}^{r} \frac{(r+s)!}{s!\,(r-s)!\,(2n\phi)^{s}} \tag{9}$$

and

$$E[\hat{\phi}^{-r}] = \left(\frac{2}{n\phi}\right)^{r} \frac{\Gamma\left(\frac{n-1}{2}+r\right)}{\Gamma\left(\frac{n-1}{2}\right)} \sum_{s=0}^{r-1} \frac{(r-1+s)!}{s!\,(r-1-s)!\,(2n\phi)^{s}}\,. \tag{10}$$

Consequently

$$E[\hat{\phi}] = \frac{n\phi+1}{n-3} \quad \text{and} \quad \mathrm{Var}(\hat{\phi}) = \frac{2n^2\phi^2 + n(n+1)\phi + 2(n-2)}{(n-3)^2(n-5)} \tag{11}$$

and

$$E[\hat{\phi}^{-1}] = \frac{n-1}{n\phi} \quad \text{and} \quad Var(\hat{\phi}^{-1}) = \frac{(n-1)(2n\phi+n+1)}{n^3\phi^3} . \tag{12}$$

Next, the correlation coefficient between $\hat{\mu}$ and $\hat{\phi}$ is given by

$$\rho = -\sqrt{\frac{n(n-5)\phi}{2n^2\phi^2 + n(n+1)\phi + 2(n-2)}} . \tag{13}$$

For large n, we have approximately

$$E[\hat{\phi}] = \phi , \quad n\,Var(\hat{\phi}) = 2\phi^2 + \phi , \tag{14}$$

$$E[\hat{\phi}^{-1}] = \frac{1}{\phi} , \quad n\,Var(\hat{\phi}^{-1}) = \frac{2}{\phi^2} + \frac{1}{\phi^3} \tag{15}$$

and

$$\rho = -\frac{1}{\sqrt{2\phi+1}} . \tag{16}$$

Results in (14) and (16) are also given by Johnson and Kotz [11]. The discussion in this section is useful toward an investigation of statistical inference associated with the parameter ϕ or a function of ϕ. For example, approximate confidence bounds on the coefficient of variation $\sqrt{1/\phi}$ can be constructed on the basis of results in (14) and (15) when n is large.

CHAPTER III

DISTRIBUTIONS RELATED TO THE INVERSE GAUSSIAN

In this chapter, we will derive the distributions of some functions of the inverse Gaussian distributed random variables. These distributions are slight variants of some well known distributions.

A Non-Linear Weighted Normal Distribution

We first prove the following theorem.

Theorem 1: Let X be an inverse Gaussian random variable with the density function

$$f(x;\mu,\lambda) = \sqrt{\lambda/2\pi x^3}\ \exp\left[-\lambda(x-\mu)^2/2\mu^2 x\right], \quad 0 < x < \infty$$

$$\mu > 0,\ \lambda > 0.$$

$$= 0 \qquad \text{otherwise} \qquad (1.1)$$

Consider $Y = \sqrt{\lambda}\,(X-\mu)/\mu\sqrt{X}$. Then the random variable Y has the following density function.

$$g(y;\mu,\lambda) = \frac{1}{\sqrt{2\pi}}\left[1 - \frac{y}{\sqrt{\frac{4\lambda}{\mu} + y^2}}\right]\exp\left[-\frac{y^2}{2}\right],\ -\infty < y < \infty. \qquad (1.2)$$

Proof: The transformation $y = \sqrt{\lambda}\,(x-\mu)/\mu\sqrt{x}$ is one-to-one and as x varies from 0 to ∞, y varies from $-\infty$ to $+\infty$. Inversely, we find that

$$x = \left[(2\mu\lambda + \mu^2 y^2) \pm \sqrt{4\mu^3 \lambda y^2 + \mu^4 y^4} \right] \Big/ 2\lambda$$

$$= \left[(2\mu\lambda + \mu^2 y) \pm \mu\,|y|\,\sqrt{4\mu\lambda + \mu^2 y^2} \right] \Big/ 2\lambda \, .$$

Since $y \in (-\infty, 0) \Longleftrightarrow x \in (0, \mu)$ and $y \in (0, \infty) \Longleftrightarrow x \in (\mu, \infty)$, it can easily be seen that

$$x = \left[(2\mu\lambda + \mu^2 y) + \mu y \sqrt{4\mu\lambda + \mu^2 y^2} \right] \Big/ 2\lambda \, . \qquad (1.3)$$

Also,

$$\frac{dx}{dy} = \left[\frac{dy}{dx} \right]^{-1} = \frac{2\mu x^{3/2}}{\sqrt{\lambda}\,(x+\mu)}$$

So the density function of Y is given by

$$g(y\,;\mu\,,\lambda) = f(x\,;\mu\,,\lambda)\,\frac{dx}{dy}$$

$$= \frac{4\mu\lambda}{\sqrt{2\pi}}\, e^{-\frac{y^2}{2}} \Big/ \left[(4\mu\lambda + \mu^2 y) + \mu y \sqrt{4\mu\lambda + \mu^2 y^2} \right] .$$

Thus,

$$g(y\,;\mu\,,\lambda) = \frac{1}{\sqrt{2\pi}} \left[1 - \frac{y}{\sqrt{\frac{4\lambda}{\mu} + y^2}} \right] e^{-\frac{y^2}{2}} , \quad -\infty < y < \infty \, .$$

This establishes the theorem.

The result of Theorem 1 is of much significance as it establishes a relationship between a normal density function and an inverse Gaussian density function.

The result of Shuster [18] and Zigangirov [28] for evaluating the inverse Gaussian distribution from the normal can now easily be obtained because

$$F_X(x) = \int_0^x f(t;\mu,\lambda)\,dt = \int_{-\infty}^y g(z;\mu,\lambda)\,dz = F_Y(y)$$

where $y = \sqrt{\lambda}(x-\mu)/\mu\sqrt{x}$ and

$$F_Y(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^y e^{-\frac{z^2}{2}}\,dz - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^y \frac{z}{\sqrt{\frac{4\lambda}{\mu}+z^2}}\,e^{-\frac{z^2}{2}}\,dz\,.$$

To evaluate the second term on the right side, let $u = \sqrt{4\lambda/\mu + z^2}$.

Then

$$\text{2nd term} = \begin{cases} e^{\frac{2\lambda}{\mu}}\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{\sqrt{\frac{4\lambda}{\mu}+y^2}}^{\infty} e^{-\frac{u^2}{2}}\,du\,, & \text{if } y \le 0 \\[2ex] e^{\frac{2\lambda}{\mu}}\dfrac{1}{\sqrt{2\pi}}\left[\displaystyle\int_{\sqrt{\frac{4\lambda}{\mu}}}^{\infty} e^{-\frac{u^2}{2}}\,du - \int_{\sqrt{\frac{4\lambda}{\mu}}}^{\sqrt{\frac{4\lambda}{\mu}+y^2}} e^{-\frac{u^2}{2}}\,du\right], & \text{if } y > 0 \end{cases}$$

$$= e^{\frac{2\lambda}{\mu}}\,\phi\left(\sqrt{\frac{4\lambda}{\mu}+y^2}\right)\,, \quad -\infty < y < \infty\,,$$

where

$$\phi(a) = \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-\frac{u^2}{2}} du .$$

Thus,

$$F_Y(y) = \phi(-y) + e^{\frac{2\lambda}{\mu}} \phi\left(\sqrt{\frac{4\lambda}{\mu} + y^2}\right), \tag{1.4}$$

or

$$F_X(x) = \phi\left(\sqrt{\frac{\lambda}{x}}\left(1 - \frac{x}{\mu}\right)\right) + e^{\frac{2\lambda}{\mu}} \phi\left(\sqrt{\frac{\lambda}{x}}\left(1 + \frac{x}{\mu}\right)\right) , \tag{1.5}$$

the form given in Shuster [18] and Zigangirov [28]. The distribution function of Y given in (1.2) or (1.4) will be called the non-linear weighted normal distribution. This distribution contains a parameter $\frac{\lambda}{\mu}$, which is the inverse of the square of coefficient of variation for the inverse Gaussian random variable. We may write $g(y\,;\frac{\lambda}{\mu})$ for the density function.

The density function $g(y\,;\frac{\lambda}{\mu})$ is monotonically decreasing for $y < 0$ and increasing for $y > 0$ with respect to $\frac{\lambda}{\mu}$; and asymptotically standard normal as $\frac{\lambda}{\mu} \to \infty$. A sketch of its curves for some values of $\frac{\lambda}{\mu}$ is given in Figure 1.

The moments ν_r of this distribution are finite for all r. However, it is not possible to write a simple expression for ν_r. For r even number, $\nu_r = \mu_r$, and for r odd, $-\frac{1}{2}\sqrt{\mu/\lambda}\,\mu_{r+1} \leq \nu_r \leq 0$, where μ_r is the r^{th} moment of the standard normal distribution.

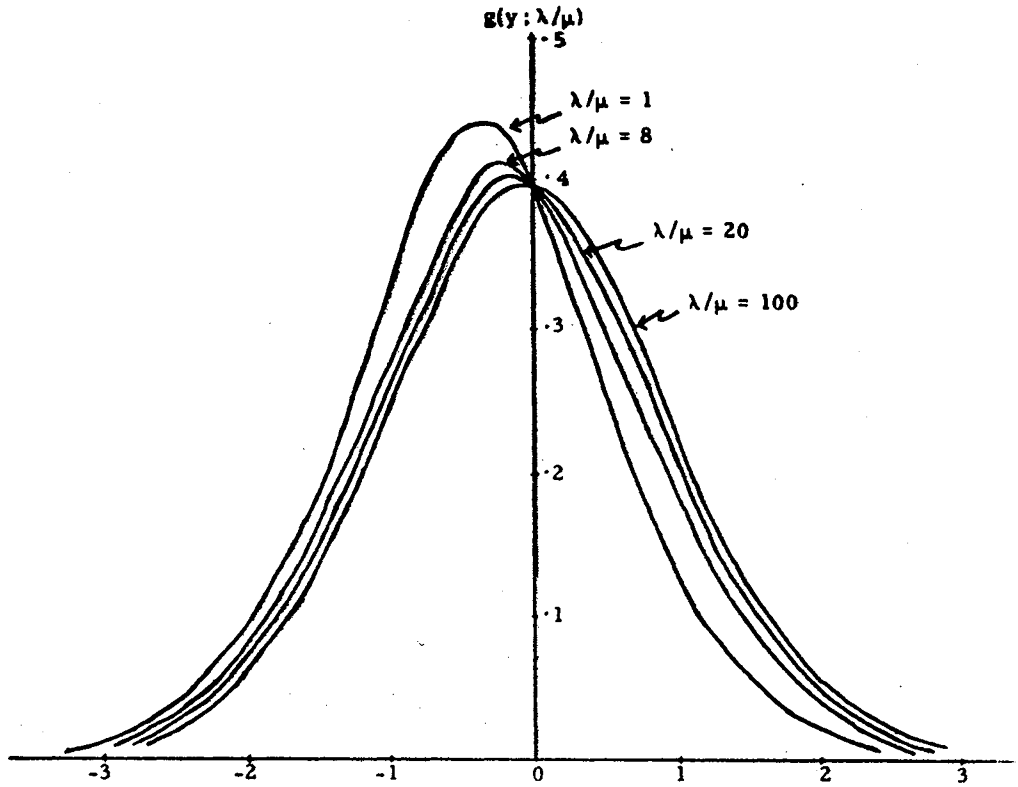

Figure 1. The Non-Linear Weighted Normal Distribution, Density Curve for $\lambda/\mu = 1, 8, 20, 100$

In fact,

$$E(Y) = -\sqrt{\frac{2}{\pi}}\,\frac{\lambda}{\mu}\; e^{\frac{\lambda}{\mu}}\,[K_1(\tfrac{\lambda}{\mu}) - K_0(\tfrac{\lambda}{\mu})]\,,$$

where $K_\alpha(z)$ stands for a modified Bessel function of the second kind (see Chapter II, pg. 7). Since extensive tables are available for $e^z(K_0(z)$ and $e^zK_1(z)$ (see Watson [26]), $E(Y)$ can be found for a given value of $\frac{\lambda}{\mu}$. Observe that the variance of Y is not greater than 1. Furthermore, an exact expression for ν_r can be written in terms of Bessel functions $K_\alpha(z)$.

It can be easily seen that (i) the random variable $|Y|$ has the standard half-normal distribution, and (ii) the random variable Y^2 has the χ^2-distribution with 1 degree of freedom.

Many papers have been written in the past on the linear combination of normal distributions; especially the linear combination of two normal distributions. One of the reasons for doing this was to seek substitutes for a normal distribution for a non-normal situation. However, little attempt has been made on the study of non-linear combination of normal distributions.

Results (i) and (ii) in the above paragraph suggest the usefulness of this distribution for statistical applications. In fact, the basis of statistical inference obtained in the following chapters is the one-to-one relationship which exists between an inverse Gaussian distributed random variable and random variable Y of (1.2).

A Non-Linear Weighted Student's t Distribution

First, we give the following theorem without proof.

Theorem 2: Let $Y, Y_1, \ldots, Y_n$ be $(n+1)$ independent and identically distributed random variables, each distributed according to (1.2). Consider $Z = \sum_1^n Y_i^2$ and $U = \sqrt{n}\, Y \big/ \sqrt{Z}$. Then the random variable U has its density function given by

$$h(u\,;\frac{\lambda}{\mu}) = \frac{1}{\sqrt{n}\,\beta\left(\frac{1}{2}, \frac{n}{2}\right)} \cdot$$

$$\cdot \left\{ \frac{1}{\left[1+\frac{u^2}{n}\right]^{\frac{n+1}{2}}} - \frac{u}{\sqrt{n}\,2^{\frac{n+1}{2}}\,\Gamma\left(\frac{n+1}{2}\right)} \int_0^{\infty} \frac{z^{\frac{n}{2}} e^{-\frac{1}{2}(1+\frac{u^2}{n})z}}{\sqrt{\frac{4\lambda}{\mu}+\frac{u^2}{n}z}}\, dz \right\},$$

$$-\infty < u < \infty \,. \qquad (2.1)$$

It may be noted that the distribution of U is unimodal but unsymmetrical, slightly skewed to the right. This property is a direct consequence of the unsymmetry in the distribution function of Y. Next, the behavior of the distribution function of U is the same as that of Y as to their common parameter $\frac{\lambda}{\mu}$; and $h(u, \frac{\lambda}{\mu})$ is asymptotically the Student's t density function as $\frac{\lambda}{\mu} \to \infty$.

Though it is not possible to write $P[U \le u]$ in any known closed form, it can be "approximately" expressed as a non-linear combination of Student's t distributions with n degrees of freedom. So, the distribution of U with density function in (2.1) will be called the non-linear weighted Sutdent's t distribution.

It is easy to see that all moments of this distribution exist; even moments are the same as that of the Student's t distribution with n degrees of freedom and odd moments are all non-positive. Furthermore, (i) the random variable $|U|$ has the truncated Student's t distribution on the positive real line, and (ii) the random variable U^2 has the F distribution with 1 and n degrees of freedom. Next, it follows that if $Y_1, \ldots, Y_n$, $Z_1, \ldots, Z_m$ are $(n+m)$ independent and identically distributed random variables, each distributed according to

(1.2) and $\xi = \sum_1^n Y_i^2$, $\eta = \sum_1^m Z_i^2$, then (i) $\frac{\xi}{\eta}$ has F distribution with n and m degrees of freedom and (ii) $\frac{\xi}{\xi+\eta}$ has Beta distribution with parameters n and m. See Chapter 18, Cramér [4] for details.

As an application of Theorem 2, consider a random sample $X_1, X_2, \ldots, X_n$ from an inverse Gaussian population with density function in (1.1). Then, it is known that $\overline{X} = \sum_1^n X_i/n$ and $V = \frac{1}{n-1} \sum_1^n \left(\frac{1}{X_i} - \frac{1}{\overline{X}}\right)$ are independently distributed, where $\overline{X}$ has an inverse Gaussian distribution with parameters μ and $n\lambda$, and λV has a χ^2 distribution with $n-1$ degrees of freedom. Since λV can be expressed by

$$\lambda V = \frac{1}{n-1}\left[\sum_1^n \frac{\lambda(X_i-\mu)^2}{\mu^2 X_i} - \frac{n\lambda(\overline{X}-\mu)^2}{\mu^2 \overline{X}}\right],$$

we can apply Theorem 2 to the transformed random variables

$$\frac{\sqrt{n\lambda}\,(\overline{X}-\mu)}{\mu\sqrt{\overline{X}}} \quad \text{and} \quad \frac{\sqrt{\lambda}\,(X_i-\mu)}{\mu\sqrt{X_i}}, \qquad i = 1, 2, \ldots, n.$$

and conclude that the random variable

$$U_1 = \frac{\sqrt{n}\,(\overline{X}-\mu)}{\mu\sqrt{\overline{X}\,V}} \tag{2.2}$$

is distributed according to (2.1) with n replaced by $n-1$ and $n\lambda$ instead of λ.

First, we note that the random variable U_1 is the same as the test statistic given by (2.27), Chapter IV for a test of hypothesis regarding μ when λ is unspecified. Secondly, the distribution of U_1

depends upon nuisance parameter λ, whereas its conditional distribution given $T(X)$, where $T(X) = \sum_1^n (X_i + \frac{1}{X_i})$, is independent of λ (see Lemma 2, Chapter IV). This further suggests that λ is not a scale parameter in the usual sense.

Since $(n-1)s^2 = \sum_1^n (X_i - \overline{X})^2$ can be expressed as

$$(n-1)s^2 = (n-1)\mu^2 \overline{X} V - \sum_1^n \frac{(X_i - \mu)^2}{X_i} (\overline{X} - X_i),$$

we have from (2.2)

$$U_1 = \frac{\sqrt{n}(\overline{X} - \mu)}{\sqrt{s^2 + \frac{1}{n-1} \sum_1^n \frac{(X_i - \mu)^2}{X_i} (\overline{X} - X_i)}} .$$

Letting

$$Y_n = \frac{1}{n-1} \sum_1^n \frac{(X_i - \mu)^2}{X_i} (\overline{X} - X_i),$$

it can be easily shown that $E(Y_n) = 0$ and $Var(Y_n) = O(n^{-1})$, and so, Y_n converges to 0 in probability. Let $t = \sqrt{n}(\overline{X} - \mu)/s$. Then it follows from (x) and (xiv), Rao [15, pages 102-104] that $U_1 - t$ converges to 0 in probability. Hence U_1 is asymptotically equivalent to t in the sense that both have the same limiting distribution.

Next, we have also derived two more distributions, a non-linear weighted non-central χ^2 distribution and a non-linear weighted non-central t distribution. However, these will not be discussed here.

CHAPTER IV

OPTIMUM PROCEDURES OF TESTING STATISTICAL HYPOTHESES ON PARAMETERS

Let $X = (X_1, X_2, \ldots, X_n)$ be a sample from an inverse Gaussian population with parameters μ and λ. Then, the density functions of X

$$f(x;\mu,\lambda) = \left(\sqrt{\lambda/2\pi}\; e^{\lambda/\mu}\right)^n \left(\prod_1^n x_i^{-3/2}\right) \exp\left[-\frac{\lambda}{2\mu^2}\Sigma x_i - \frac{\lambda}{2}\Sigma\frac{1}{x_i}\right] \quad (1)$$

constitute a two-parameter exponential family. We will investigate the problem of testing various hypotheses on parameters μ and λ of (1). Uniformly most powerful (U. M. P.) or uniformly most powerful unbiased tests will be obtained and the test statistics will be derived so that a test can be easily performed for a given size of the test (i. e., level of significance). More emphasis will be placed on the problem of testing hypotheses on μ with both λ known and λ unknown. Presently, we will not discuss the power of these tests. However, the power of tests concerning hypotheses on λ can be obtained explicitly in terms of χ^2-distributions.

Uniformly Most Powerful Test Procedures

Consider the problem of testing a statistical hypothesis on a parameter of (1) when the other parameter is specified. In this case,

(1) reduces to a one-parameter exponential family. Consequently, (1) possesses the monotone likelihood ratio [MLR] property. It then follows that a U. M. P. size α test exists for one-sided hypothesis, say,

$$H_0^{(1)}: \lambda \leq \lambda_0 \quad \text{vs} \quad H_1^{(1)}: \lambda > \lambda_0 \,, \quad \mu \text{ known}$$

or

$$H_0^{(2)}: \mu \leq \mu_0 \quad \text{vs} \quad H_1^{(2)}: \mu > \mu_0 \,, \quad \lambda \text{ known} \,.$$

(i) For testing $H_0^{(1)}$ vs $H_1^{(1)}$, the U. M. P. size α test is

$$\phi(x) = \begin{cases} 1 \,, & \text{if } x \in A \\ 0 \,, & \text{otherwise} \end{cases} \tag{1.1}$$

where

$$A = \left\{ x = (x_1, \ldots, x_n) : \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{x_i} < c \right\} \,,$$

and $c' = \frac{\lambda_0}{\mu^2} c = \chi^2_\alpha(n)$, the α percentile of a χ^2 distribution with n degrees of freedom.

(ii) For testing $H_0^{(2)}$ vs $H_1^{(2)}$, the U. M. P. size α test is

$$\phi(x) = \begin{cases} 1 \,, & \text{if } x \in B \\ 0 \,, & \text{otherwise} \end{cases} \tag{1.2}$$

where $B = \{x = (x_1, \ldots, x_n) : \bar{x} > k\}$, and k depends upon size α of the test and is the solution of the following equation

$$\alpha = \phi\left(\sqrt{\frac{n\lambda}{k}}\left(\frac{k}{\mu_0} - 1\right)\right) - e^{2n\lambda/\mu_0}\,\phi\left(\sqrt{\frac{n\lambda}{k}}\left(\frac{k}{\mu_0} + 1\right)\right) \qquad (1.3)$$

where

$$\phi(a) = \int_a^\infty \frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}}\, dt\ .$$

In fact, if n and λ are not small, k is approximately $(1 - \alpha)$ percentile of the standard normal distribution.

Uniformly Most Powerful Unbiased Test Procedures

Two-Sided Hypotheses on a Parameter When the Other is Known

Let $f(x;\theta)$, $\theta \in \Theta$, be the probability density function of a random variable X, where θ may be a real vector-valued parameter. For testing a hypothesis $H_0: \theta \in \Theta_0$ vs $H_1: \theta \in \Theta_1$ with $\Theta_0 \cup \Theta_1 = \Theta$, a size α test ϕ is said to be unbiased if the power function $E_\theta[\phi(X)] \geq \alpha$, for all $\theta \in \Theta_1$.

As the UMP test does not exist in the case of a two-sided hypothesis on a parameter of (1), we will find UMP unbiased tests instead.

(i) Let $X = (X_1, X_2, \ldots, X_n)$ be a sample from an inverse Gaussian population with parameters μ and λ, where μ is assumed to be known. Then the density function of X is

$$f(x;\lambda) = \left(\frac{\lambda}{2\pi}\right)^{n/2}\left(\prod_1^n x_i^{-3/2}\right) \exp\left[-\frac{\lambda}{2\mu^2}\sum_1^n \frac{(x_i - \mu)^2}{x_i}\right] .$$

Clearly,

$$T(X) = \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{X_i}$$

is sufficient for λ , and is distributed as a $\frac{\mu^2}{\lambda}\ \chi^2$- random variable with n degrees of freedom. It follows from Lehmann [14] that an UMP unbiased test of the hypothesis $H_0: \frac{1}{\lambda} = \frac{1}{\lambda_0}$ vs $H_1: \frac{1}{\lambda} \neq \frac{1}{\lambda_0}$ is

$$\phi(x) = \begin{cases} 1\ , & \text{if } T(x) \leq k_1 \text{ or } T(x) \geq k_2 \\ 0\ , & \text{otherwise} \end{cases} \tag{2.1}$$

where, for a given size α of the test, $c_1 = \left(\frac{\lambda_0}{\mu^2} k_1\right)$ and $c_2 = \left(\frac{\lambda_0}{\mu^2} k_2\right)$ are determined by

$$\int_{c_1}^{c_2} g_n(t)\,dt = 1 - \alpha \quad \text{and} \quad \int_{c_1}^{c_2} t\,g_n(t)\,dt = n(1 - \alpha) \tag{2.2}$$

with

$$g_n(t) = \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)}\ t^{\frac{n-2}{2}}\ e^{-\frac{t}{2}}\ , \quad t > 0\ .$$

The first condition in (2.2) can be written in terms of the χ^2 distribution with n degrees of freedom, $F_{\chi^2(n)}(a)$ as

$$F_{\chi^2(n)}(c_2) - F_{\chi^2(n)}(c_1) = 1 - \alpha\ ,$$

and the second condition as

$$n(\log c_2 - \log c_1) = c_2 - c_1 .$$

This form of the second condition follows due to integration by parts in (2.2) since $t\, g_n(t) = n\, g_{n+2}(t)$. c_1 and c_2 can also be easily obtained from tables of the χ^2-distribution due to the following relationship.

$$F_{\chi^2(n)}(c_2) - F_{\chi^2(n)}(c_1) = F_{\chi^2(n+2)}(c_2) - F_{\chi^2(n+2)}(c_1) ,$$

and each is equal to $1-\alpha$. Further, as it is pointed out in Lehmann [14, p. 130] regarding a similar situation, we may actually conclude that the equal tails test given by

$$\int_0^{c_1} g_n(t)\, dt = \int_{c_2}^{\infty} g_n(t)\, dt = \frac{\alpha}{2}$$

is a good approximation of the above test, provided n is large or λ is not small compared to μ.

(ii) Consider a sample $X = (X_1, \ldots, X_n)$ from the inverse Gaussian with parameter μ and λ (known). The density function of X is

$$f(x;\mu) = \left(\frac{\lambda}{2\pi}\right)^{n/2} \left(\prod_1^n x_i^{-3/2}\right) \exp\left[-\frac{\lambda}{2\mu^2} \Sigma x_i + \frac{n\lambda}{\mu} - \frac{\lambda}{2} \Sigma \frac{1}{x_i}\right] .$$

Clearly, $T(x) = \Sigma X_i$ is a sufficient statistic for μ. Then a UMP unbiased test of the hypothesis $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$ exists and has the form

$$\phi(x) = \begin{cases} 1, & \text{if } \bar{x} < k_1 \text{ or } \bar{x} > k_2 \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

where, for a given size α of the test, the constants k_1 and k_2 are determined by

$$\int_{k_1}^{k_2} g(t)\,dt = 1 - \alpha \quad \text{and} \quad \int_{k_1}^{k_2} t\,g(t)\,dt = (1-\alpha)\mu_0 , \tag{2.4}$$

with

$$g(t) = \sqrt{\frac{n\lambda}{2\pi}} \frac{1}{t^{3/2}} e^{-\frac{n\lambda}{2\mu_0^2} \frac{(t-\mu_0)^2}{t}} , \quad t > 0 .$$

For simplifying conditions in (2.4), let

$$y = \frac{\sqrt{n\lambda}\,(t - \mu_0)}{\mu_0 \sqrt{t}}$$

It is a one-to-one transformation $\ni t \in (0, \infty)$ implies $y \in (-\infty, \infty)$, and inversely,

$$t = \frac{\mu_0}{2n\lambda} \left[2n\lambda + \mu_0 y^2 + y\sqrt{4n\mu_0\lambda + \mu_0^2 y^2} \right]$$

Then, the first and second conditions in (2.4) can respectively be reduced to

$$\frac{1}{\sqrt{2\pi}} \int_{c_1}^{c_2} \left[1 - \frac{y}{\sqrt{\frac{4n\lambda}{\mu_0} + y^2}}\right] e^{-\frac{y^2}{2}} \, dy = 1 - \alpha$$

and

$$\frac{1}{\sqrt{2\pi}} \int_{c_1}^{c_2} \left(1 + \frac{y}{\sqrt{\frac{4n\lambda}{\mu_0} + y^2}}\right) e^{-\frac{y^2}{2}} \, dy = 1 - \alpha$$

where

$$c_i = \frac{\sqrt{n\lambda}\,(k_i - \mu_0)}{\mu_0\sqrt{k_i}}, \quad i = 1, 2$$

and they are different from zero, provided $\alpha > 0$. It follows that $c_1 = -c_2$ and $c_2 = Z_{1-\alpha/2}$, the $(1-\alpha/2)$ percentile of standard normal distribution. Hence, the UMP unbiased size α test of $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$ is

$$\phi(x) = \begin{cases} 1, & \text{if } |y| > c \\ 0, & \text{otherwise} \end{cases} \tag{2.5}$$

where c is $(1-\alpha/2)$ percentile of standard normal distribution and

$$y = \frac{\sqrt{n\lambda}\,(\bar{x} - \mu_0)}{\mu_0\sqrt{\bar{x}}} . \tag{2.6}$$

Testing Hypotheses on a Parameter When the Other is Unknown

In this section we will consider both one-sided and two-sided hypotheses on a parameter of (1) when the other is unspecified and will obtain UMP unbiased test criteria. The theory of hypotheses testing given in sections 3 and 4, Chapter IV, Lehmann [14] will be applied to find such tests. First, we define a few concepts associated with the subject and show that these hold for the family of inverse Gaussian density functions given in (1).

Definition 1: A size α test ϕ of $H_0: \theta \in \Theta_0$ vs $H_1: \theta \in \Theta_1$ is said to be similar on the boundary $\Theta_B = \bar{\Theta}_0 \cap \bar{\Theta}_1$ if $E_\theta[\phi(X)] = \alpha$ for all $\theta \in \Theta_B$.

Definition 2: A test ϕ satisfying $E_\theta[\phi(X)] = \alpha$ for all $\theta \in \Theta_B$, is said to have Neyman structure if $E[\phi(X) | T = t] = \alpha$, a.e. $[\wp^T]$, with T as sufficient statistics for $\theta \in \Theta_B$. ($\wp^T$ is the family of distributions induced by T for $\theta \in \Theta_B$).

Definition 3: A statistic T is complete (boundedly complete) for $\theta \in \Theta_B$ if $E_\theta[f(T)] = 0$ for all $\theta \in \Theta_B$ implies $f(t) = 0$, a.e. $[\wp^T]$, for any real-valued function (bounded function) f.

It follows from Theorem 2, Lehmann [14, p. 134], that the bounded completeness of a sufficient statistic T ensures a similar test with Neyman structure which is easier to construct as compared to an unbiased test. Since the class of similar tests contains the class of unbiased tests, we obtain an UMP unbiased test by constructing an UMP similar test with Neyman structure whenever the power function of each test for a given family of distributions is continuous.

Let $X = (X_1, X_2, \ldots, X_n)$ be a sample from an inverse Gaussian population with parameters μ and λ. Then $\sum_1^n X_i$ and $\sum_1^n\left(\frac{1}{X_i} - \frac{1}{\bar{X}}\right)$

are sufficient statistics for μ and λ with $\sum_1^n X_i$ having the inverse Gaussian distribution and $\lambda \sum_1^n \left(\frac{1}{X_i} - \frac{1}{\overline{X}}\right)$ having the χ^2 distribution with $(n-1)$ degrees of freedom. Because of inverse Gaussian and χ^2 distribution being complete families of distribution functions, independence of $\sum_1^n X_i$ and $\sum_1^n \left(\frac{1}{X_i} - \frac{1}{\overline{X}}\right)$ implies that

$$\left(\sum_1^n X_i \, , \, \sum_1^n \left(\frac{1}{X_i} - \frac{1}{\overline{X}}\right)\right)$$

is a complete sufficient statistic for μ and λ. In fact, it is boundedly complete.

Now we shall discuss the following hypotheses on parameters μ and λ.

(i) $H_0^{(1)}: \lambda \leq \lambda_0$ vs $H_1^{(1)}: \lambda > \lambda_0$, $\quad 0 < \mu < \infty$

(ii) $H_0^{(2)}: \lambda = \lambda_0$ vs $H_1^{(2)}: \lambda \neq \lambda_0$, $\quad 0 < \mu < \infty$

(iii) $H_0^{(3)}: \mu \leq \mu_0$ vs $H_1^{(3)}: \mu > \mu_0$, $\quad 0 < \lambda < \infty$

(iv) $H_0^{(4)}: \mu = \mu_0$ vs $H_1^{(4)}: \mu \neq \mu_0$, $\quad 0 < \lambda < \infty$

For any other hypothesis, the treatment of the problem is similar to one of these four hypotheses, and hence will not be discussed.

(i) By Theorem 3 (Chapter IV) in Lehmann [14], an UMP unbiased size α test of $H_0^{(1)}$ vs $H_1^{(1)}$ exists and the rejection region given by

$$R = \left\{ x = (x_1, \ldots, x_n) : \sum_1^n \frac{1}{x_i} < k(\overline{x}) \right\} ,$$

is determined by $E_{\lambda_0}[I_R(X)|\bar{x}] = \alpha$. Due to stochastic independence of $\bar{X}$ and $\sum_1^n \left(\frac{1}{X_i} - \frac{1}{\bar{X}}\right)$, the rejection region can be written as

$$R = \left\{ x = (x_1, \ldots, x_n) : \sum_1^n \left(\frac{1}{x_i} - \frac{1}{\bar{x}}\right) < c \right\}$$

where c is independent of $\bar{x}$ and is determined by $E_{\lambda_0}[I_R(X)] = \alpha$. Since $\lambda \sum_1^n \left(\frac{1}{X_i} - \frac{1}{\bar{X}}\right)$ has a χ^2 distribution with $(n-1)$ degrees of freedom, it follows that $c_0 = \lambda_0 c$ is the α percentile of the $\chi^2(n-1)$ distribution.

(ii) Similarly, the UMP unbiased test of $H_0^{(2)}$ vs $H_1^{(2)}$ can be shown to have the rejection region

$$A = \left\{ x = (x_1, \ldots, x_n) : c_1 < \Sigma\left(\frac{1}{x_i} - \frac{1}{\bar{x}}\right) < c_2 \right\} \tag{2.7}$$

where c_1 and c_2 are determined by

$$\int_{\lambda_0 c_1}^{\lambda_0 c_2} g(t)\,dt = \frac{1}{n-1} \int_{\lambda_0 c_1}^{\lambda_0 c_2} t\,g(t)\,dt = 1 - \alpha$$

with

$$g(t) = \frac{1}{2^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)}\, t^{\frac{n-3}{2}}\, e^{-\frac{t}{2}}, \quad t > 0 .$$

For further simplification, follow the argument as given in (i) of the previous subsection. Moreover, it has also been discussed by Roy and

Wasan [16].

Tests in (i) and (ii) are identical to those obtained earlier with $\sum_1^n \left(\frac{1}{x_i} - \frac{1}{\bar{x}}\right)$ in place of $\sum_1^n (x_i - \mu)^2/\mu^2 x_i$ and (n-1) degrees of freedom instead of n. The reason is obvious because a UMP unbiased test in (i) or (ii) depends only on the statistic $\sum_1^n \left(\frac{1}{X_i} - \frac{1}{\bar{X}}\right)$ whose distributions are independent of μ, and constitute an exponential family in λ. Hence, the problems are reduced to the corresponding ones for a one-parameter exponential family which have been discussed earlier.

Next, we establish the following two lemmas which will be useful in obtaining conditional distribution functions of test statistics in (iii) and (iv).

Lemma 1: Let $X_1, \ldots, X_n$ be i.i.d. random variables distributed as the inverse Gaussian with parameters μ and λ with $\mu = 1$. Consider the functions U and V of X's given by

$$U = \sum_1^n X_i \quad \text{and} \quad V = \sum_1^n X_i + \sum_1^n \frac{1}{X_i} . \tag{2.8}$$

Then, the conditional density function of U for a given value of V is given by

$$h_{U|V=v}(u) = \frac{n}{\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \frac{1}{\sqrt{u^3(v-2n)}} \left[1 - \frac{(u-n)^2}{u(v-2n)}\right]^{\frac{n-3}{2}}$$

$$-1 < \frac{u-n}{\sqrt{u(v-2n)}} < 1$$

$$= 0, \qquad \text{otherwise} \tag{2.9}$$

Proof: Since U and

$$V_1 = \sum_1^n \left[\frac{1}{X_i} - \frac{n}{\Sigma X_i} \right]$$

are independently distributed, and $V_1 = V - [U + \frac{n^2}{U}]$, the joint density of U and V is

$$f_{U,V}(u,v) = \sqrt{\frac{n^2\lambda}{2\pi}} \frac{1}{u^{3/2}} e^{-\lambda(u-n)^2/2u} \frac{\lambda^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}} \left[v - \left(u + \frac{n^2}{u}\right)\right]^{\frac{n-3}{2}}$$

$$\cdot\; e^{-\frac{\lambda}{2}\left[v - \left(u + \frac{n^2}{u}\right)\right]}$$

$$= \frac{n\,\lambda^{n/2}\,e^{n\lambda}}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right) 2^{n/2}} \frac{1}{u^{3/2}} \left[v - \left(u + \frac{n^2}{u}\right)\right]^{\frac{n-3}{2}} e^{-\lambda v/2} ,$$

$$u > 0 \text{ and } v > u + \frac{n^2}{u} .$$

Next,

$$V = \sum_1^n \frac{(X_i - 1)^2}{X_i} + 2n ,$$

and $\lambda \sum_1^n (X_i - 1)^2 / X_i$ has a χ^2 distribution with n degrees of freedom. So the marginal distribution of V is given by

$$g_V(v) = \frac{\lambda^{n/2}}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} (v - 2n)^{\frac{n-2}{2}} e^{-\lambda(v-2n)/2} , \quad v > 2n .$$

Then

$$h_{U|V=v}(u) = f_{U,V}(u,v)/g_V(v) = \frac{n}{\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \frac{1}{u^{3/2}} \frac{\left[v - (u + \frac{n^2}{u})\right]^{\frac{n-3}{2}}}{[v-2n]^{\frac{n-2}{2}}}$$

$$= \frac{n}{\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \frac{1}{\sqrt{u^3(v-2n)}} \left[1 - \frac{(u-n)^2}{u(v-2n)}\right]^{\frac{n-3}{2}},$$

$$0 < \frac{(u-n)^2}{u(v-2n)} < 1 .$$

This establishes the lemma.

Lemma 2: In Lemma 1, let

$$W = \sqrt{n-1} \frac{U-n}{\sqrt{U(V-2n)}} \Bigg/ \left[1 - \frac{(U-n)^2}{U(V-2n)}\right]^{\frac{1}{2}} . \tag{2.10}$$

Then, the conditional distribution of W, given $V=v$, is given by

$$p_{W|V=v}(w) = \frac{1}{\sqrt{n-1}} \frac{1}{\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \left[1 - \frac{w\sqrt{(v-2n)/n-1}}{\sqrt{4n + (v+2n)\frac{w^2}{n-1}}}\right] \frac{1}{\left[1 + \frac{w^2}{n-1}\right]^{\frac{n}{2}}}$$

$$-\infty < w < \infty . \tag{2.11}$$

Proof: w is a monotone non-decreasing function in u and inversely, we obtain

$$u = n + \left[(v-2n)\frac{w^2}{n-1} + w\sqrt{\frac{v-2n}{n-1}\left[4n + (v+2n)\frac{w^2}{n-1}\right]}\right] \Bigg/ 2\left(1 + \frac{w^2}{n-1}\right)$$

and

$$\frac{du}{dw} = \left[\frac{dw}{du}\right]^{-1} = \frac{1}{\sqrt{n-1}} \quad \frac{2\sqrt{v-2n}\; u^{\frac{3}{2}}}{u+n} \quad \left[1 - \frac{(u-n)^2}{u(v-2n)}\right]^{\frac{3}{2}}$$

where

$$\frac{1}{u+n} = \frac{2(1+\frac{w^2}{n-1})}{4n+(v+2n)\frac{w^2}{n-1} + w\sqrt{\frac{v-2n}{n-1}\left[4n+(v+2n)\frac{w^2}{n-1}\right]}}$$

$$= \frac{1}{2n}\left[1 - \frac{w\sqrt{(v-2n)/n-1}}{\sqrt{4n+(v+2n)\frac{w^2}{n-1}}}\right]$$

and

$$1 - \frac{(u-n)^2}{u(v-2n)} = \left[1 + \frac{w^2}{n-1}\right]^{-1} .$$

By making these substitutions in (2.9), the result in (2.11) follows. Hence, the proof is complete.

Observe that W if expressed in X's, has the simple form

$$W = \sqrt{n(n-1)} \; \frac{\overline{X} - 1}{\sqrt{\overline{X} \sum_1^n \left(\frac{1}{X_i} - \frac{1}{\overline{X}}\right)}} \tag{2.12}$$

where, $\overline{X} = \frac{1}{n}\sum_1^n X_i$.

Now we shall discuss the problem of statistical hypotheses considered in (iii) and (iv). Since $f(x;\mu,\lambda) = \mu^{-1} f(x/\mu; 1, \lambda/\mu)$ in (1), without loss of generality, assume $\mu_0 = 1$ in $H_0^{(3)}$ and $H_0^{(4)}$. Next, we can write (1) in the form

$$f(x;\mu,\lambda) = c(\theta_1, \theta_2) h(x) \exp[\theta_1 T_1(x) + \theta_2 T_2(x)]$$

with

$$\theta_1 = -\frac{\lambda}{2\mu^2}, \qquad \theta_2 = -\frac{\lambda}{2}, \qquad T_1(x) = \sum_1^n x_i, \qquad T_2(x) = \sum_1^n \frac{1}{x_i}$$

Furthermore,

$$f(x;\mu,\lambda) = c(\theta, \theta_2) h(x) \exp[\theta T_1(x) + \theta_2 T(x)]$$

with

$$\theta = (\theta_1 - \theta_2) \quad \text{and} \quad T(x) = \sum_1^n x_i + \sum_1^n \frac{1}{x_i}$$

Then the hypotheses in (iii) and (iv) are equivalent to the following ones.

(iii) $H_0'^{(3)}: \theta \leq 0$ vs $H_1'^{(3)}: \theta > 0$, $-\infty < \theta_2 < 0$

(iv) $H_0'^{(4)}: \theta = 0$ vs $H_1'^{(4)}: \theta \neq 0$, $-\infty < \theta_2 < 0$

(iii) By Theorem 3 (Chapter IV), Lehmann [14], an UMP unbiased test of $H_0'^{(3)}: \theta \leq 0$ vs $H_1'^{(3)}: \theta > 0$ exists and is of the form

$$\phi(t_1, t) = \begin{cases} 1 & \text{if } t_1 > k(t) \\ \gamma(t) & \quad = \\ 0 & \quad < \end{cases}$$

where, for a given size α of the test, $k(t)$ and $\gamma(t)$ are determined by

$$E_{\theta=0}[\phi(T_1, T) \mid T = t] = \alpha \quad \text{for all } t .$$

By Lemma 1 and 2, it follows that the test is equivalently given by the rejection region

$$R = \left\{ w : w > c(t),\ w = \sqrt{n-1}\ \frac{t_1 - n}{\sqrt{t t_1 - t_1^2 - n^2}} \right\} \tag{2.13}$$

which can be explicitly obtained by the condition

$$\alpha = \int_{c(t)}^{\infty} p(w \mid t)\, dw \tag{2.14}$$

where

$$p(w \mid t) = \frac{1}{\sqrt{n-1}\ \beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \left[1 - \frac{w\sqrt{(t-2n)/n-1}}{\sqrt{4n + (t+2n)\frac{w^2}{n-1}}} \right] \frac{1}{\left[1 + \frac{w^2}{n-1}\right]^{\frac{n}{2}}} ,$$

$$-\infty < w < \infty . \tag{2.15}$$

The condition (2.14) can be expressed in terms of student's t distribution

$$F_{t,n-1}(\ell) = \int_{\ell}^{\infty} \frac{1}{\sqrt{n-1}\;\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \frac{dy}{\left[1 + \frac{y^2}{n-1}\right]^{\frac{n}{2}}} .$$

After the change of variable

$$u = \sqrt{4n + (t+2n)\frac{w^2}{n-1}}$$

in the second term of (2. 15), the condition (2. 14) reduces to

$$\alpha = F_{t,n-1}(c) - \left[\frac{t+2n}{t-2n}\right]^{\frac{n}{2}-1} F_{t,n-1}\left(\sqrt{4n+(t+2n)c^2}\right) . \quad (2.16)$$

For a known α and t, one can easily find c from student's t tables by iteration because $F_{t,n-1}(c)$ and $F_{t,n-1}\left(\sqrt{4n+(t+2n)c^2}\right)$ are both monotonically decreasing with respect to c. In fact, (2. 16) can be written in terms of Beta distribution for which extensive tables are available and, hence, iteration can be more accurately carried out.

By further simplification in (2. 13) and (2. 16) we find that the UMP unbiased test of $H_0: \mu \leq 1$ vs $H_1: \mu > 1$ has the rejection region

$$R = \left\{ x = (x_1, x_2, \ldots, x_n) : \sqrt{n}\,\frac{(\bar{x}-1)}{\sqrt{\bar{x}v}} > c \right\} \quad (2.17)$$

where $\bar{x} = \frac{1}{n}\sum_1^n x_i$ and $v = \frac{1}{n-1}\sum_1^n\left(\frac{1}{x_i} - \frac{1}{\bar{x}}\right)$, and c is obtained from

$$\alpha = F_{t,n-1}(c) - \left[\frac{\sum_1^n (x_i+1)^2/x_i}{\sum_1^n (x_i-1)^2/x_i}\right]^{\frac{n-2}{2}} F_{t,n-1}\left(\sqrt{4n + c^2 \sum_1^n \frac{(x_i+1)^2}{x_i}}\right). \quad (2.18)$$

For the general case of testing $H_0 : \mu \leq \mu_0$ vs $H_1 : \mu > \mu_0$, the rejection region corresponds to

$$t'(x) = \frac{\sqrt{n}\,(\bar{x} - \mu_0)}{\mu_0 \sqrt{\bar{x}\, v}} > k \quad (2.19)$$

where k is determined by

$$\alpha = F_{t,n-1}(k) - \left[\frac{\sum_1^n (x_i+\mu_0)^2/x_i}{\sum_1^n (x_i-\mu_0)^2/x_i}\right]^{\frac{n-2}{2}} F_{t,n-1}\left(\sqrt{4n + k^2 \mu_0 \sum_1^n \frac{(x_i+\mu_0)^2}{x_i}}\right). \quad (2.20)$$

Observe that the rejection region in (2.19) is reduced to (2.17) by transforming to the variables X_i/μ_0 $(i = 1, 2, \ldots, n)$.

(iv) For testing $H_0'^{(4)} : \theta = 0$ vs $H_1'^{(4)} : \theta \neq 0$, the existence of an UMP test follows again from Theorem 3, Chapter IV, Lehmann [14] and is of the form

$$\phi(t_1, t) = \begin{cases} 1, & \text{if } t_1 < k_1(t) \text{ or } t_1 > k_2(t) \\ \gamma_i(t), & t_1 = k_i(t), \quad i = 1, 2 \\ 0, & \text{otherwise}, \end{cases}$$

where, for a given size α of the test, constants $k_1(t)$, $k_2(t)$ and $\gamma_1(t)$, $\gamma_2(t)$ are determined by

$$E_{\theta=0}[\phi(T_1, T)\,|\,T=t] = \alpha$$

and

$$E_{\theta=0}[T_1\,\phi(T_1, T)\,|\,T=t] = \alpha\ E_{\theta=0}[T_1\,|\,T=t]\,.$$

By incorporating the results of Lemma 1 and 2 above, the test is given by the rejection region

$$R = \left\{ w : w < c_1(t) \ \text{ or } \ w > c_2(t)\,,\ w = \sqrt{n-1}\ \frac{t_1 - n}{\sqrt{t\,t_1 - t_1^2 - n^2}} \right\} \qquad (2.21)$$

with $c_1(t)$ and $c_2(t)$ are given by conditions

$$\int_{-\infty}^{c_1(t)} p(w\,|\,t)\,dw + \int_{c_2(t)}^{\infty} p(w\,|\,t)\,dw = \alpha \qquad (2.22)$$

and

$$\int_{-\infty}^{c_1(t)} q(w\,|\,t)\,dw + \int_{c_2(t)}^{\infty} q(w\,|\,t)\,dw = \alpha \int_{-\infty}^{\infty} q(w\,|\,t)\,dw \qquad (2.23)$$

where $p(w\,|\,t)$ is given in (2.15) and

$$q(w\,|\,t) = \frac{1}{\sqrt{n-1}\ \beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \left[1 + \frac{w\sqrt{(t-2n)/n-1}}{\sqrt{4n + (t+2n)\,\frac{w^2}{n-1}}} \right] \frac{1}{\left[1 + \frac{w^2}{n-1}\right]^{\frac{n}{2}}}$$

$$-\infty < w < \infty\,. \qquad (2.24)$$

The simplification of the second condition, as in (2.23), can be realized due to transformation of Lemma 2 which in the present notations leads to the relation

$$t_1 h(t_1|t) = q(w|t) .$$

Recognizing that

$$\int_{-\infty}^{\infty} q(w|t)\,dw = 1 ,$$

it follows from (2.22) and (2.23) that $c_1(t)$ and $c_2(t)$ do not depend on t, and $c_1 = -c_2$. Moreover, these are different from zero provided $\alpha > 0$ and c_2 is the $(1-\alpha/2)$ percentile of student's t distribution with $(n-1)$ degrees of freedom.

Thus, it follows from (2.21) that the UMP unbiased size α test of $H_0^{(4)}: \mu = 1$ vs $H_1^{(4)}: \mu \neq 1$ has its rejection region given by

$$\left| \sqrt{n}\, \frac{\bar{x}-1}{\sqrt{\bar{x}\, v}} \right| > c \qquad (2.25)$$

where $\bar{x} = \frac{1}{n}\sum_1^n x_i$, $v = \frac{1}{n-1}\sum_1^n \left(\frac{1}{x_i} - \frac{1}{\bar{x}}\right)$ and $c = t_{1-\alpha/2}$, the $(1-\alpha/2)$ percentile of student's t distribution with $(n-1)$ degrees of freedom.

As mentioned in the case of one-sided hypothesis, similar argument leads to the existence of UMP unbiased size α test of $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$ and its rejection region is given by

$$|t'(x)| > c \qquad (2.26)$$

where

$$t'(x) = \sqrt{n}\ \frac{(\bar{x} - \mu_0)}{\mu_0 \sqrt{\bar{x}\ v}} \qquad (2.27)$$

and $c = t_{1-\alpha/2}$, the $(1 - \alpha/2)$ percentile of student's t distribution with $(n-1)$ degrees of freedom.

As discussed in Chapter III, for statistics $t' = \sqrt{n}\,(\bar{X} - \mu)/\mu\sqrt{\bar{X}\,V}$ and $t = \sqrt{n}\ (\bar{X} - \mu)/s$, where $s^2 = \frac{1}{n-1} \sum_1^n (X_i - \bar{X})^2$, $t' - t$ converges to 0 in probability. Next, it can easily be shown that conditionally or otherwise, t' has asymptotically the standard normal distribution. Since the asymptotic distribution of the student's t is also standard normal, it follows that, for large sample, the test statistic $t'(X)$ corresponding to (2.19) and (2.27) can be replaced by $t = \sqrt{n}\,(\bar{X} - \mu_0)/s$ distributed as student's t with $(n-1)$ degrees of freedom.

CHAPTER V

INTERVAL ESTIMATION

We will examine the problem of confidence intervals for the parameters and the reliability function associated with the inverse Gaussian distribution. However, the discussion on confidence intervals for λ will not be included here because these are easily obtained from the χ^2 distributions. Instead of confidence bounds on the reliability function, we will briefly discuss the tolerance limits.

Confidence Bounds on Parameter μ

Since there is a direct relationship between confidence sets and families of tests of hypotheses, the results of the last chapter can be used to derive the confidence intervals on μ . In order to have this direct relationship between the two, we first formulate the concept.

Definition 1: A family of subsets $\{S(X)\}$ of the parameter space Θ is said to be a family of confidence sets at the confidence level $(1 - \alpha)$ if

$$P_\theta\{\theta \in S(X)\} = \alpha \quad \text{for all} \quad \theta \in \Theta$$

Next, we define an optimal criterion for the choice of a family of confidence sets such that there is a small probability of covering false values of the parameter.

Definition 2: A family of confidence sets at the confidence level $(1 - \alpha)$ is said to be uniformly most accurate (UMA) if

$$P_{\theta'}\{\theta \in S(X)\} = \text{minimum} \quad \text{for all } \theta, \theta' \in \Theta .$$

By Theorem 1, section 5.8, Ferguson [8], a UMP test leads to a UMA family of confidence sets. Due to the non-existence of UMP tests in many situations, we cannot obtain confidence sets with the UMA property in a large number of cases. This suggests restricting attention to confidence sets which are unbiased.

Definition 3: A family $\{S(X)\}$ of confidence sets at the confidence level $(1 - \alpha)$ is said to be unbiased if

$$P_{\theta'}\{\theta \in S(X)\} \leq 1 - \alpha \quad \text{for all } \theta, \theta' \in \Theta$$

By Theorem 2, section 5.8, Ferguson [8], a UMP unbiased test leads to a UMA unbiased family of confidence sets. For greater details, see sections 4 and 5 in Chapter V, Lehmann [14].

With these considerations, we now obtain UMA or UMA unbiased confidence bounds at the confidence level $(1 - \alpha)$ for μ in the following cases:

(i) Two-sided bounds when λ is known.

(ii) Two-sided bounds when λ is unknown.

(iii) Lower bound when λ is known.

(iv) Lower bound when λ is unknown.

The cases for the upper bound can be dealt with in a manner analogous to the treatment in (iii) and (iv). Exact results are given for (i) and (ii), and approximate results are given for (iii) and (iv).

(i) If $X = (X_1, \ldots, X_n)$ is a sample from the inverse Gaussian population with parameters μ and λ (known), the UMP unbiased size α test of $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$ has the acceptance region given by

$$\left| \frac{\sqrt{n\lambda}\,(\bar{x} - \mu_0)}{\mu_0 \sqrt{\bar{x}}} \right| \leq Z_{1-\alpha/2} , \tag{1.1}$$

where $Z_{1-\alpha/2}$ is the $(1 - \alpha/2)$ percentile of the standard normal. [See (2.5) and (2.6) in Chapter IV]. Then the confidence set $S(x)$ consists of all μ's satisfying (1.1) with $\mu = \mu_0$. Accordingly, this set is given by the interval

$$1 - \sqrt{\frac{\bar{x}}{n\lambda}}\, Z_{1-\alpha/2} \leq \frac{\bar{x}}{\mu} \leq 1 + \sqrt{\frac{\bar{x}}{n\lambda}}\, Z_{1-\alpha/2} .$$

Since we are concerned with a population of positive real values, it is appropriate to take $\max\left(1 - \sqrt{\bar{x}/n\lambda}\; Z_{1-\alpha/2} , 0\right)$ for the lower bound. Hence, the UMA unbiased confidence interval for μ at the confidence level $(1 - \alpha)$ is given by

$$\left(\bar{x}\left[1 + \sqrt{\frac{\bar{x}}{n\lambda}}\, Z_{1-\alpha/2}\right]^{-1},\ \bar{x}\left[\max\left(\left[1 - \sqrt{\frac{\bar{x}}{n\lambda}}\, Z_{1-\alpha/2}\right], 0\right)\right]^{-1} \right). \tag{1.2}$$

where $Z_{1-\alpha/2}$ is the $(1 - \alpha/2)$ percentile of standard normal distribution.

(ii) For a sample $X = (X_1, \ldots, X_n)$ from an inverse Gaussian population with unknown parameter λ, the UMP unbiased size α test of $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$ has the acceptance region given by

$$\left| \frac{\sqrt{n}\ (\bar{x} - \mu_0)}{\mu_0 \sqrt{\bar{x}\, v}} \right| \leq t_{1-\alpha/2}\ , \tag{1.3}$$

where $v = \frac{1}{n-1} \sum_1^n \left(\frac{1}{x_i} - \frac{1}{\bar{x}} \right)$ and $t_{1-\alpha/2}$ is the $(1-\alpha/2)$ percentile of the student's t distribution with $(n-1)$ degrees of freedom. [See (2.26) and (2.27) in Chapter IV]. Similar to the case in (i), it can be easily concluded that the UMA unbiased confidence interval for μ at the confidence level $(1-\alpha)$ is given by

$$\left(\bar{x} \left[1 - \sqrt{\frac{\bar{x}\, v}{n}}\ t_{1-\alpha/2} \right]^{-1} ,\ \bar{x} \left[\max \left(\left[1 - \sqrt{\frac{\bar{x}\, v}{n}}\ t_{1-\alpha/2} \right], 0 \right) \right]^{-1} \right) \tag{1.4}$$

where $t_{1-\alpha/2}$ is the $(1-\alpha/2)$ percentile of the student's t distribution with $(n-1)$ degrees of freedom.

(iii) By (1.2) and (1.3) in Chapter IV, the acceptance region of the UMP size α test of $H_0: \mu \leq \mu_0$ vs $H_1: \mu > \mu_0$ is

$$A(\mu_0) = \left\{ x = (x_1, x_2, \ldots, x_n) : \frac{\sqrt{n\lambda}\ (\bar{x} - \mu_0)}{\mu_0 \sqrt{\bar{x}}} \leq c(\mu_0) \right\}, \tag{1.5}$$

where $c(\mu_0)$ is determined by

$$\alpha = \phi(c) - e^{2n\lambda/\mu_0}\ \phi\left(\sqrt{4n\lambda/\mu_0 + c^2} \right) \tag{1.6}$$

with $\phi(a)$ denoting the probability that the standard normal variate exceeds a . It follows that the confidence set S(X) consists of all μ's satisfying the inequality in (1.5) with $\mu = \mu_0$; or, we may write

$$S(x) = \left\{ \mu : \mu \geq \bar{x} \left[1 + c(\mu) \sqrt{\frac{\bar{x}}{n\lambda}} \right]^{-1} \right\} . \tag{1.7}$$

But the determination of $c(\mu)$ depends on the knowledge of both μ and α as can be seen from (1.6). Thus, we fail to find a lower-bound for μ. However $c(\mu)$ can be approximated by

$$\alpha \doteq \phi(c) ,$$

provided n and λ are not small. Then, from (1.7) an approximate lower confidence bound for μ is given by

$$\mu \geq \bar{x} \left[1 + Z_{1-\alpha} \sqrt{\frac{\bar{x}}{n\lambda}} \right]^{-1} , \tag{1.8}$$

where $Z_{1-\alpha}$ is the $(1-\alpha)$ percentile of standard normal.

Next, we give another approximate but slightly more conservative lower confidence bound for μ when the sample size is large. Let $X_1, \ldots, X_n$, n large, be a sample from an inverse Gaussian population, with parameters μ and λ, where λ is assumed to be known. Then $\bar{X}$ has asymptotically the normal distribution with mean μ and variance $\mu^3[n\lambda]^{-1}$. By the convergence theorem, 6 a.2 (i), Rao [15], it follows that $[\bar{X}]^{-1/2}$ has asymptotically a normal distribution with mean $\mu^{-1/2}$ and variance $[4n\lambda]^{-1}$. It implies that $\sqrt{4n\lambda}\left[[\bar{X}]^{-1/2} - [\mu]^{-1/2}\right]$ has the standard normal distribution asymptotically. By this fact, a lower confidence bound at level $(1-\alpha)$ can be obtained as

$$\mu \geq \bar{x} \left[1 + \frac{1}{2} \sqrt{\frac{\bar{x}}{n\lambda}} \, Z_{1-\alpha} \right]^{-2} , \tag{1.9}$$

where $Z_{1-\alpha}$ is the $(1-\alpha)$ percentile of the standard normal distribution.

(iv) To obtain a lower confidence bound for μ, consider the acceptance region

$$A(\mu_0) = \left\{ x = (x_1, \ldots, x_n) : \frac{\sqrt{n}\,(\bar{x} - \mu_0)}{\mu_0 \sqrt{\bar{x}\, v}} \leq c(\mu_0) \right\} \qquad (1.10)$$

obtained for the UMP unbiased size α test of $H_0 : \mu \leq \mu_0$ vs $H_1 : \mu > \mu_0$ in (2.19) and (2.20) of Chapter IV, where,

$$\alpha = F_{t,n-1}(c) - \left[\frac{\sum_1^n (x_i + \mu_0)^2 / x_i}{\sum_1^n (x_i - \mu_0)^2 / x_i} \right]^{\frac{n-2}{2}} F_{t,n-1}\left(\sqrt{4n + c^2 \mu_0 \sum_1^n \frac{(x_i + \mu_0)^2}{x_i}} \right) \qquad (1.11)$$

Here $F_{t,n-1}(a)$ denotes the probability that the student's t, with $(n-1)$ degrees of freedom, variate exceeds a. By a similar argument as, for example in (iii), we do not obtain an exact confidence bound from (1.10). However, an approximate lower confidence bound for μ at level $(1-\alpha)$ is obtained as

$$\mu \geq \bar{x} \left[1 + t_{1-\alpha} \sqrt{\frac{\bar{x}\, v}{n}} \right]^{-1} \qquad (1.12)$$

provided n is not small.

As a result of the last paragraph in Chapter IV, another approximate lower confidence bound for μ in the case of large sample can be obtained by

$$\mu \geq \bar{x} - t_{1-\alpha} \frac{s}{\sqrt{n}} \tag{1.13}$$

In (1.12) and (1.13), $t_{1-\alpha}$ is the $(1-\alpha)$ percentile of the student's t distribution with $(n-1)$ degrees of freedom.

Tolerance Limits

With the results in the previous section, one may attempt to obtain confidence sets for the reliability function $R(x;\theta)$, or to find the tolerance limits, e.g., find $L(x)$ such that $P_\theta[R(L(x);\theta) > p] = \gamma$ for some specified probabilities p and γ, on the basis of confidence bounds given for μ or λ. See the following lemma for the definition of $R(x;\theta)$. An exact solution of such a problem depends on at least two conditions. First, (i) there exists a function $g(\theta)$ of θ such that $R(x;\theta)$ is a monotone function in $g(\theta)$ for all x, and secondly, (ii) exact confidence bounds are available for $g(\theta)$. Since $R(x;\theta)$ does not have a simple closed form, the tolerance bounds are difficult to express in terms of given parameters μ and λ. For example, it can best be obtained by the quantity $\mu + \frac{\mu^2}{2\lambda}\left[Z_p^2 + Z_p\sqrt{Z_p^2 + 4\lambda/\mu}\right]$ for an upper tolerance bound for a proportion p of the inverse Gaussian population with given parameters μ and λ, where Z_p is the p percentile of the standard normal distribution and this bound is not the greatest lower upper bound. This suggests that the condition (i) is hard to be met for a general treatment of the problem for the inverse Gaussian distribution. Consequently, it is not feasible to have an exact solution of the problem unless it is restricted to a rather simple case. Henceforth, our discussion will be limited to the case of one-sided tolerance limits. These limits will have optimal properties analogous to the UMA property of a confidence bound given in the previous section. Refer to Faulkenberry [6] for a general discussion on UMA tolerance limits.

First we prove the following lemma.

Lemma: For any given $x > 0$, let $R(x;\theta) = E[I_{(x,\infty)}(X)]$, where X is the inverse Gaussian random variable with a density function $f(x;\theta)$ and $\theta = (\mu, \lambda)$. Then (a) $R(x;\theta)$ is a monotone non-decreasing function in μ for a fixed λ and (b) $R(x;\theta)$ is a monotone non-increasing function in λ for a fixed μ.

Proof: $R(x;\theta)$ is an analytic function in $\theta = (\mu, \lambda)$ and differentiation can be performed under the integral sign.

(a) Differentiating $R(x;\theta)$ with respect to μ, we have

$$\frac{\partial R(x;\theta)}{\partial \mu} = \frac{\lambda^{\frac{3}{2}}}{\sqrt{2\pi}\,\mu^3} \int_x^\infty \frac{t-\mu}{\sqrt{t^3}} \cdot e^{-\lambda(t-\mu)^2/2\mu^2 t}\,dt$$

$$= \frac{\lambda}{\mu^3}\left[\int_x^\infty t \cdot \frac{f(t,\theta)}{R(x;\theta)}\,dt - \mu\right] R(x;\theta)$$

$$= \frac{\lambda}{\mu^3}\left[E_\theta[X \mid X > x] - E[X]\right] R(x;\theta) .$$

Clearly, the right side is non-negative for all $x > 0$. Thus, the monotonicity of $R(x;\theta)$ in μ follows.

(b) Once again, differentiating $R(x;\theta)$ with respect to λ, we get

$$\frac{\partial R(x;\theta)}{\partial\lambda} = \frac{1}{2\lambda} \int_x^\infty \left[1 - \frac{\lambda(t-\mu)^2}{\mu^2 t}\right] f(t,\theta)\,dt$$

$$= \frac{1}{2\lambda}\,\frac{1}{\sqrt{2\pi}} \int_{\sqrt{\lambda}\,(x-\mu)/\mu\sqrt{x}}^\infty (1-y^2)\left(1 - \frac{y}{\sqrt{\frac{4\lambda}{\mu} + y^2}}\right) e^{-\frac{y^2}{2}}\,dy$$

by the transformation $y = \sqrt{\lambda}\,(t-\mu)/\mu\sqrt{t}$. Also see Theorem 1, Chapter III. Letting

$$u = \sqrt{\lambda}\,(x-\mu)/\mu\sqrt{x} \quad \text{and} \quad F(u) = \int_u^\infty \left[1 - \frac{y}{\sqrt{\frac{4\lambda}{\mu} + y^2}}\right] e^{-\frac{y^2}{2}}\,dy ,$$

$$\frac{\partial R(x;\theta)}{\partial\lambda} = \frac{1}{2\lambda}\left[1 - \frac{1}{\sqrt{2\pi}} \int_u^\infty y^2\, g(y;\lambda/\mu)\Big/F(u)\,dy\right] F(u)$$

$$= \frac{1}{2\lambda}\left[1 - E[U^2 \mid U > u]\right] F(u) ,$$

where $U = \sqrt{\lambda}\,(X-\mu)/\mu\sqrt{X}$ is the transformed random variable with the density function

$$g(u,\lambda/\mu) = \frac{1}{\sqrt{2\pi}}\left(1 - \frac{u}{\sqrt{\frac{4\lambda}{\mu} + u^2}}\right) e^{-\frac{u^2}{2}} , \quad -\infty < u < \infty .$$

Since $E[U^2] = 1$, we have

$$\frac{\partial R(x;\theta)}{\partial \lambda} = \frac{1}{2\lambda}\left[E[U^2] - E[U^2 | U > u]\right] F(u) \leq 0$$

Thus, $R(x;\theta)$ is a monotone non-increasing function in λ, thus completing the proof of (a) and (b).

As a result of this lemma, one-sided confidence bounds at a given confidence level can be easily obtained for $R(x;\theta)$ when (i) $\theta = \mu$, λ is known and (ii) $\theta = \lambda$, μ is known on the basis of one-sided confidence bounds on μ given in the last section and those of λ, respectively. Such confidence bounds will be exact in the case of $R(x;\lambda)$ and will be approximate in the case of $R(x;\mu)$. For one-sided tolerance limits, we first outline the procedure and then give results for cases (i) and (ii).

Suppose $\theta^*(X)$ is a lower confidence bound for θ at confidence level γ, i.e., $P[\theta^*(X) < \theta] = \gamma$. Consider $L(x) \ni R(L(x);\theta^*(x)) = p$. If $R(x;\theta)$ is a monotone non-decreasing function in θ, we can conclude that $R(L(x);\theta) > p \Longleftrightarrow \theta > \theta^*(x)$, so,

$$P[R(L(X);\theta) > p] = \gamma . \tag{2.1}$$

Then, a lower tolerance limit $L(x)$ is obtained by solving the equation

$$R(L(x), \theta^*(x)) = p \tag{2.2}$$

for $L(x)$. However, such a limit would be approximate unless we get an exact solution from (2.2).

Now we consider cases (i) and (ii).

(i) Suppose $\theta = \mu$ and λ is assumed to be known. Then, from (1.8), we have approximately

$$\mu^*(x) = \bar{x}\left[1 + Z_{1-\gamma}\sqrt{\frac{\bar{x}}{n\lambda}}\right]^{-1} \tag{2.3}$$

and so, it follows from (2.2) and the above lemma that

$$p = \phi\left(\sqrt{\frac{\lambda}{L(x)}}\left(\frac{L(x)}{\mu^*(x)} - 1\right)\right) - e^{2\lambda/\mu^*(x)}\,\phi\left(\sqrt{\frac{\lambda}{L(x)}}\left(\frac{L(x)}{\mu^*(x)} + 1\right)\right) \tag{2.4}$$

where

$$\phi(a) = \int_a^\infty \frac{1}{\sqrt{2\pi}}\, e^{-\frac{t^2}{2}}\, dt\,. \tag{2.5}$$

From (2.4), we cannot express $L(x)$ explicitly as a function of other quantities. But for λ large, the second term on the right side of (2.4) is small, since $\Delta = e^{2\lambda/\mu^*(x)}\,\phi\left(2\sqrt{\lambda/\mu^*(x)}\right)$ is small and the second term is not greater than Δ. It can be concluded that the solution of (2.4) for $L(x)$ is bounded by

$$\mu^*(x) + \frac{\mu^{*2}(x)}{2\lambda}\left[Z_p^2 + Z_p\sqrt{Z_p^2 + 4\lambda/\mu^*(x)}\right] \le L(x)$$

$$\le \mu^*(x) + \frac{\mu^{*2}(x)}{2\lambda}\left[Z_{p+\Delta}^2 + Z_{p+\Delta}\sqrt{Z_{p+\Delta}^2 + 4\lambda/\mu^*(x)}\right]. \tag{2.6}$$

Here, Z_p and $Z_{p+\Delta}$ denote the $(1-p)$ and $(1-p-\Delta)$ percentiles of standard normal distribution.

(ii) Suppose $\theta = \lambda$ and μ is assumed to be known. A lower confidence bound for $\frac{1}{\lambda}$ at confidence level γ is easily obtained as

$$\left[\lambda^*(x)\right]^{-1} = \sum_1^n \frac{(x_i-\mu)^2}{x_i} \Big/ \mu^2 \chi^2_\gamma(n) , \tag{2.7}$$

where $\chi^2_\gamma(n)$ is the γ percentile of the χ^2 distribution with n degrees of freedom. From the lemma, $R(x;\theta)$ is a monotone non-decreasing function in $\frac{1}{\lambda}$, and so, due to (2.2), we have $L(x)$ as a solution of the equation

$$p = \phi\left(\sqrt{\frac{\lambda^*(x)}{L(x)}}\left(\frac{L(x)}{\mu} - 1\right)\right) - e^{2\lambda^*(x)/\mu}\,\phi\left(\sqrt{\frac{\lambda^*(x)}{L(x)}}\left(\frac{L(x)}{\mu} + 1\right)\right) , \tag{2.8}$$

where $\phi(a)$ as in (2.5). By a similar argument as above in (i), $L(x)$ is bounded by

$$\mu + \frac{\mu}{2\lambda^*(x)}\left[Z_p^2 + Z_p\sqrt{Z_p^2 + 4\lambda^*(x)/\mu}\right] \le L(x)$$

$$\le \mu + \frac{\mu}{2\lambda^*(x)}\left[Z^2_{p+\Delta'} + Z_{p+\Delta'}\sqrt{Z^2_{p+\Delta'} + 4\lambda^*(x)/\mu}\right] \tag{2.9}$$

where, $\Delta' = e^{2\lambda^*(x)/\mu}\,\phi\left(2\sqrt{\lambda^*(x)/\mu}\right)$.

Though we are unable to express $L(x)$ in (i) and (ii) exactly in terms of known quantities, these exact solutions can directly be obtained from (2.4) and (2.8) by iteration using the standard normal table.

CHAPTER VI

MINIMUM VARIANCE UNBIASED ESTIMATION OF RELIABILITY

Our aim is to obtain the minimum variance unbiased estimate (MVUE) of the reliability function $R(x;\theta) = E[I_{(x,\infty)}(X)]$ when X is an inverse Gaussian random variable. Here, $I_A(\cdot)$ denotes the indicator function of the set A and θ is the set of unknown parameters associated with a distribution function of a random variable. The method of estimation is that given by Kolmogorov [13] and, again, by Basu [2]. It is based on finding the conditional distribution of a sample observation given the sufficient statistic. Then, making use of the Lehmann-Scheffé and Rao-Blackwell theorems, the unique MVUE of the reliability function $R(x;\theta)$ is obtained.

Recently, Sathe and Varde [17] and Eaton and Morris [5] have derived MVUE's of similar parametric functions by considering ancillary statistics independent of complete sufficient statistics. Though their approach is elegant, it falls short as a technique, since it does not provide any method of constructing ancillary statistics whose distributions are utilized for acquiring such estimates.

Let $X = (X_1, X_2, \ldots, X_n)$ be a sample from an inverse Gaussian population with parameter $\theta = (\mu, \lambda)$. Then $I_{[x,\infty)}(X_1)$ is an unbiased estimate of $R(x;\theta)$. If $T(X)$ is a complete sufficient statistic, the

MVUE of $R(x;\theta)$ is given by $E[I_{[x,\infty)}(X_1)|T(X)]$, and it will be denoted by $\hat{R}(x;\theta)$.

We will consider the problem of estimation for all three cases. They are, namely: (1) μ unknown and λ known, (2) μ known and λ unknown, and (3) both μ and λ unknown. Our results will be given in the forms which can be evaluated by using the standard normal table for case (1), and by use of the student's t table for cases (2) and (3). These forms will be obtained by following the technique given in Folks, Pierce and Stewart [9].

MVUE of $R(x;\theta)$ When $\theta = \mu$

The sample mean $\overline{X} = \sum_1^n X_i/n$ is a complete sufficient statistic and its distribution is inverse Gaussian with parameters μ and $n\lambda$, where λ is assumed to be known. To derive the MVUE of $R(x;\mu)$ using the above method, we first find the conditional density function of X_1, given $\overline{X} = \overline{x}$.

The joint density function of random variables X_1 and $\overline{Y} = \sum_2^n X_i/(n-1)$ is

$$f(x_1, \overline{y}) = \frac{\sqrt{n-1}\,\lambda}{2\pi\,(x_1\,\overline{y})^{\frac{3}{2}}} \exp\left[-\frac{\lambda}{2\mu^2}\left\{\frac{(x_1-\mu)^2}{x_1} + \frac{(n-1)(\overline{y}-\mu)^2}{\overline{y}}\right\}\right]$$

With the transformation $\overline{Y} = (n\overline{X} - X_1)/(n-1)$, we obtain the density function of X_1 and $\overline{X}$ as

$$f(x_1, \bar{x}) = \frac{n(n-1)\lambda}{2\pi[x_1(n\bar{x}-x_1)]^{\frac{3}{2}}} \exp\left[-\frac{\lambda}{2\mu^2}\left\{\frac{(x_1-\mu)^2}{x_1} + \frac{[n(\bar{x}-\mu)-(x_1-\mu)]^2}{n\bar{x}-x_1}\right\}\right]$$

$$0 < x_1 < n\bar{x}$$

We already know that the density function of $\bar{X}$ is

$$g(\bar{x}) = \sqrt{n\lambda/2\pi\bar{x}^3} \exp\left[-\frac{n\lambda(\bar{x}-\mu)^2}{2\mu^2\bar{x}}\right], \quad \bar{x} > 0$$

so the conditional density function of X_1, given $\bar{X} = \bar{x}$, is given by

$$h(x_1|\bar{x}) = f(x_1, \bar{x})/g(\bar{x})$$

$$= \sqrt{\frac{n\lambda}{2\pi}} \frac{(n-1)\bar{x}^{\frac{3}{2}}}{[x_1(n\bar{x}-x_1)]^{\frac{3}{2}}} \exp\left[-\frac{n\lambda(x_1-\bar{x})^2}{2x_1\bar{x}(n\bar{x}-x_1)}\right], \quad 0 < x_1 < n\bar{x} \tag{1.1}$$

Hence, we derive the MVUE of $R(x;\mu)$ as

$$\hat{R}(x;\mu) = \int_x^{n\bar{x}} h(x_1|\bar{x})\,dx_1, \quad x > 0 \tag{1.2}$$

with $h(x_1|\bar{x})$ in (1.1).

Next, we will express the right side of (1.2) in the form in which the standard normal table can be used to evaluate $R(x;\mu)$ for all $x > 0$.

Let $w = \sqrt{n\lambda}\,(x_1-\bar{x})\Big/\sqrt{x_1\bar{x}(n\bar{x}-x_1)}$. We have a one-to-one transformation, and w varies from $-\infty$ to ∞ as x_1 varies from 0 to

$n\bar{x}$. Then

$$\frac{dx_1}{dw} = \left[\frac{dw}{dx_1}\right]^{-1} = \frac{2[x_1(n\bar{x} - x_1)]^{\frac{3}{2}}}{\sqrt{n\lambda \ \bar{x}}\ [(n-2)x_1 + n\bar{x}]} \tag{1.3}$$

Inversely, it can be found that

$$x_1 = \frac{\bar{x}\left[(2n\lambda + n\bar{x}\,w^2) \pm \sqrt{n\bar{x}}\ |w|\ [4(n-1)\lambda + n\bar{x}\,w^2]^{\frac{1}{2}}\right]}{2[n\lambda + \bar{x}\,w^2]}$$

Since $w \in (-\infty, 0) \iff x_1 \in (0, \bar{x})$ and $w \in (0, \infty) \iff x_1 \in (\bar{x}, n\bar{x})$, we can write

$$x_1 = \frac{\bar{x}\left[(2n\lambda + n\bar{x}\,w^2) + \sqrt{n\bar{x}}\ w[4(n-1)\lambda + n\bar{x}\,w^2]^{\frac{1}{2}}\right]}{2[n\lambda + \bar{x}\,w^2]}\ .$$

Also,

$$(n-2)\frac{x_1}{\bar{x}} + n = \frac{4n(n-1)\lambda + n^2\bar{x}\,w^2}{2(n\lambda + \bar{x}\,w^2)}\left[1 + \frac{(n-2)\,w\sqrt{\bar{x}}}{\sqrt{4n(n-1)\lambda + n^2\bar{x}\,w^2}}\right]. \tag{1.4}$$

After substitution in (1.2) from (1.1), (1.3) and (1.4),

$$\hat{R}(x\,;\mu) = \sqrt{\frac{2}{\pi}}\int_{w'}^{\infty} \frac{2(n-1)[n\lambda + \bar{x}\,w^2]\ \exp\left[-\frac{w^2}{2}\right]}{\left(4n(n-1)\lambda + n^2\bar{x}\,w^2\right) + (n-2)\,w\sqrt{\bar{x}}\sqrt{4n(n-1)\lambda + n^2\bar{x}\,w^2}}\,dw,$$

where

$$w' = \frac{\sqrt{n\lambda}\ (x - \bar{x})}{\sqrt{x\,\bar{x}\,(n\bar{x} - x)}} . \tag{1.5}$$

After further simplification, we have

$$\hat{R}(x;\mu) = \frac{1}{\sqrt{2\pi}} \int_{w'}^{\infty} \left[1 - \frac{(n-2)\,w\sqrt{\bar{x}}}{\sqrt{4n(n-1)\lambda + n^2\bar{x}\,w^2}} \right] e^{-\frac{w^2}{2}}\, dw \tag{1.6}$$

with w' in (1.5). The integral in (1.6) can be evaluated from the standard normal table for a given value for w'. Separating the integrand, we have

$$\hat{R}(x;\mu) = \frac{1}{\sqrt{2\pi}} \int_{w'}^{\infty} e^{-\frac{w^2}{2}}\, dw - \frac{n-2}{n\sqrt{2\pi}} \int_{w'}^{\infty} \frac{w}{\sqrt{\frac{4(n-1)\lambda}{n\bar{x}} + w^2}}\, e^{-\frac{w^2}{2}}\, dw$$

To eavluate the second term on the right side, let

$$u = \sqrt{\frac{4(n-1)\lambda}{n\bar{x}} + w^2} .$$

Then, by a similar argument as given on page 11, Chapter III for evaluating a similar integral,

$$\text{2nd term} = \frac{n-2}{n}\, e^{2(n-1)\lambda/n\bar{x}}\ \phi\left(\sqrt{\frac{4(n-1)\lambda}{n\bar{x}} + w'^2}\right), \quad -\infty < w' < \infty .$$

where

$$\phi(a) = \frac{1}{\sqrt{2\pi}} \int_a^\infty e^{-\frac{u^2}{2}} du .$$

So,

$$\hat{R}(x;\mu) = \phi(w') - \frac{n-2}{n} e^{2(n-1)\lambda/n\bar{x}} \phi\left(\sqrt{\frac{4(n-1)\lambda}{n\bar{x}} + w'^2}\right), \quad -\infty < w' < \infty .$$

Hence, the MVUE of $R(x;\mu)$ is

$$\hat{R}(x;\mu) = \begin{cases} 0 , & x > n\bar{x} \\ 1 , & x < 0 \\ \phi(w') - \frac{n-2}{n} e^{2(n-1)\lambda/n\bar{x}} \phi(w'') , & \text{otherwise} \end{cases} \tag{1.7}$$

where

$$w' = \frac{\sqrt{n\lambda}\ (x - \bar{x})}{\sqrt{x\bar{x}\ (n\bar{x} - x)}} \quad \text{and} \quad w'' = \frac{\sqrt{\lambda}\ (n\bar{x} + (n-2)x)}{\sqrt{n\bar{x}\ x(n\bar{x} - x)}} .$$

MVUE of $R(x;\theta)$ When $\theta = \lambda$

The statistic $T(X) = \sum_1^n (X_i - \mu)^2/X_i$ is completely sufficient and $\lambda T(X)/\mu^2$ has the χ^2 distribution with n degrees of freedom. First, we find the conditional density function of X_1, given $T(X) = t$.

The joint density function of random variables X_1 and $Y(X) = \sum_2^n (X_i - \mu)^2/X_i$ is

$$f(x;y) = \frac{\lambda^{\frac{n}{2}}}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n}{2}} \mu^{n-1} x_1^{\frac{3}{2}}} \; y^{\frac{n-3}{2}} \; e^{-\frac{\lambda}{2\mu^2}\left[y + \frac{(x_1-\mu)^2}{x_1}\right]} .$$

Since

$$Y(X) = T(X) - \frac{(X_1-\mu)^2}{X_1} ,$$

the joint density function of X_1 and $T(X)$ is

$$f(x_1, t) = \frac{\lambda^{\frac{n}{2}}}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n}{2}} \mu^{n-1} x_1^{\frac{3}{2}}} \left[t - \frac{(x_1-\mu)^2}{x_1}\right]^{\frac{n-3}{2}} e^{-\frac{\lambda t}{2\mu^2}} ,$$

$$0 < \frac{(x_1-\mu)^2}{x_1} < t .$$

The density function of the random variable $T(X)$ is

$$g(t) = \frac{\lambda^{\frac{n}{2}}}{2^{\frac{n}{2}} \mu^n \Gamma\left(\frac{n}{2}\right)} t^{\frac{n}{2}-1} e^{-\frac{\lambda t}{2\mu^2}} , \quad t > 0 .$$

So, the conditional density function of X_1 given $T(X) = t$ is

$$h(x_1|t) = \frac{\mu}{\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \frac{1}{\sqrt{t x_1^3}} \left[1 - \frac{(x_1-\mu)^2}{t x_1}\right]^{\frac{n-3}{2}} , \quad 0 < \frac{(x_1-\mu)^2}{x_1} < t . \tag{2.1}$$

As shown earlier, we can now obtain the MVUE of $R(x;\lambda)$ and it is given by

$$\hat{R}(x;\lambda) = \int_x^{\frac{1}{2}\left[(2\mu+t)+\sqrt{4\mu t+t^2}\right]} h(x_1|t)\,dx_1\,, \quad x > \frac{1}{2}\left[2\mu+t-\sqrt{4\mu t+t^2}\right] \tag{2.2}$$

with $h(x_1|t)$ in (2.1).

Next, we simplify (2.2) so that $\hat{R}(x;\mu)$ can be evaluated using a student's t table. Let

$$w = \frac{\sqrt{n-1}\,(x_1-\mu)}{\sqrt{tx_1}} \Bigg/ \left[1 - \frac{(x_1-\mu)^2}{tx_1}\right]^{\frac{1}{2}} .$$

It is a one-to-one transformation and $-1 < (x_1-\mu)\Big/\sqrt{tx_1} < 1$ implies $w \in (-\infty, \infty)$. Next,

$$\frac{dx_1}{dw} = \left[\frac{dw}{dx_1}\right]^{-1} = \frac{2\sqrt{tx_1^3}}{\sqrt{n-1}\,(x_1+\mu)}\left[1 + \frac{(x_1-\mu)^2}{tx_1}\right]^{\frac{3}{2}}, \tag{2.3}$$

and

$$x_1 = \frac{\left[2\mu\left(1+\frac{w^2}{n-1}\right) + t\,\frac{w^2}{n-1}\right] \pm \frac{|w|}{\sqrt{n-1}}\sqrt{4\mu t\left(1+\frac{w^2}{n-1}\right) + t^2\,\frac{w^2}{n-1}}}{2\left(1+\frac{w^2}{n-1}\right)}$$

or

$$x_1 = \frac{\left[2\mu\left(1+\frac{w^2}{n-1}\right)+t\,\frac{w^2}{n-1}\right]+\frac{w}{\sqrt{n-1}}\sqrt{4\mu t\left(1+\frac{w^2}{n-1}\right)+t^2\,\frac{w^2}{n-1}}}{2\left(1+\frac{w^2}{n-1}\right)}, \qquad (2.4)$$

since $w \in (-\infty, \infty) \iff x_1 \in (L, \mu)$ and $w \in (0, \infty) \iff x_1 \in (\mu, U)$ where

$$L = \frac{1}{2}\left[(2\mu + t) - \sqrt{4\mu t + t^2}\right] \quad \text{and} \quad U = \frac{1}{2}\left[(2\mu + t) + \sqrt{4\mu t + t^2}\right]. \qquad (2.5)$$

By substitution in (2.2) from (2.1), (2.3) and (2.4), we have

$$\hat{R}(x;\lambda) = \frac{4\mu}{\sqrt{n-1}\;\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \cdot$$

$$\cdot \int_{w'}^{\infty} \frac{\left[1+\frac{w^2}{n-1}\right]^{-\frac{n}{2}+1}}{\left[4\mu\left(1+\frac{w^2}{n-1}\right)+t\,\frac{w^2}{n-1}\right]+\frac{w}{\sqrt{n-1}}\sqrt{4\mu t\left(1+\frac{w^2}{n-1}\right)+t^2\,\frac{w^2}{n-1}}}\,dw$$

where

$$w' = \frac{\sqrt{n-1}\,(x-\mu)}{\sqrt{tx-(x-\mu)^2}} \qquad (2.6)$$

The integrand can further be simplified and then

$$\hat{R}(x;\lambda) = \frac{1}{\sqrt{n-1}\;\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \int_{w'}^{\infty} \left[1 - \frac{\frac{w}{\sqrt{n-1}}}{\sqrt{\frac{4\mu}{t}\left(1+\frac{w^2}{n-1}\right)+\frac{w^2}{n-1}}}\right] \frac{dw}{\left[1+\frac{w^2}{n-1}\right]^{\frac{n}{2}}} \qquad (2.7)$$

where w' is given by (2.6). The right side in (2.7) can be expressed in terms of student's t distribution,

$$F_{t,n-1}(c) = \frac{1}{\sqrt{n-1}} \int_c^{\infty} \frac{1}{\beta\left(\frac{1}{2}, \frac{n-1}{2}\right)} \frac{du}{\left[1 + \frac{u^2}{n-1}\right]^{\frac{n}{2}}} .$$

With the change of variable

$$u = \sqrt{n-1} \sqrt{\frac{4\mu}{t}\left(1 + \frac{w^2}{n-1}\right) + \frac{w^2}{n-1}} ,$$

the second term on the right side in (2.7) can be simplified to

$$\left[1 + \frac{4\mu}{t}\right]^{\frac{n}{2}-1} F_{t,n-1}\left(\sqrt{n-1} \sqrt{\frac{4\mu}{t}\left(1 + \frac{w'^2}{n-1}\right) + \frac{w'^2}{n-1}}\right) .$$

Accordingly,

$$\hat{R}(x;\lambda) = F_{t,n-1}(w') - \left[1 + \frac{4\mu}{t}\right]^{\frac{n}{2}-1} F_{t,n-1}\left(\sqrt{n-1} \sqrt{\frac{4\mu}{t}\left(1 + \frac{w'^2}{n-1}\right) + \frac{w'^2}{n-1}}\right),$$

$$-\infty < w' < \infty$$

with w' in (2.6). Hence, the MVUE of $R(x;\lambda)$ is

$$\hat{R}(x;\mu) = \begin{cases} 0 & x > \frac{1}{2}\left[(2\mu + t) + \sqrt{4\mu t + t^2}\right] \\ 1 & x < \frac{1}{2}\left[(2\mu + t) - \sqrt{4\mu t + t^2}\right] \\ F_{t,n-1}(w') - \left[\frac{t + 4\mu}{t}\right]^{\frac{n}{2}-1} F_{t,n-1}(w''), & \text{otherwise} \end{cases} \qquad (2.8)$$

where

$$w' = \frac{\sqrt{n-1}\,(x-\mu)}{\sqrt{tx-(x-\mu)^2}} \quad , \quad w'' = \frac{\sqrt{n-1}\,(x+\mu)}{\sqrt{tx-(x-\mu)^2}}$$

and $t = \sum_1^n (x_i - \mu)^2 / x_i$ is obtained from the sample observations $x_1, x_2, \ldots, x_n$.

MVUE of $R(x;\theta)$ When $\theta = (\mu, \lambda)$

The statistic $T(X) = (\overline{X}, V)$, where

$$\overline{X} = \frac{1}{n}\sum_1^n X_i \quad \text{and} \quad V = \sum_1^n\left(\frac{1}{X_i} - \frac{1}{\overline{X}}\right) ,$$

forms a complete sufficient statistic. Tweedie [20] further showed that $\overline{X}$ and V are stochastically independent; and (a) $\overline{X}$ has the inverse Gaussian distribution with parameters μ and $n\lambda$, and (b) λV has the χ^2 distribution with $(n-1)$ degrees of freedom. To find the MVUE, we first want to find the conditional distribution of X_1 given $T(X)$. Let

$$\overline{Y} = \frac{1}{n-1}\sum_2^n X_i \quad \text{and} \quad V_1 = \sum_2^n\left(\frac{1}{X_i} - \frac{1}{\overline{Y}}\right) .$$

The joint density of random variables X_1, $\overline{Y}$ and V_1 is

$$f(x_1, \overline{y}, v_1) = \frac{\sqrt{n-1}\;\lambda^{\frac{n}{2}}\, v_1^{\frac{n-4}{2}}}{\Gamma\!\left(\frac{n-2}{2}\right)\pi\sqrt{2^n x_1^3\, \overline{y}^{\,3}}}\; e^{-\frac{\lambda}{2\mu^2}\left[\frac{(x_1-\mu)^2}{x_1} + \frac{(n-1)(\overline{y}-\mu)^2}{\overline{y}}\right] - \frac{\lambda v_1}{2}} .$$

Consider the transformation

$$X_1 = X_1$$

$$\overline{X} = [(n-1)\overline{Y} + X_1]\Big/ n$$

$$V = V_1 + \frac{n-1}{\overline{Y}} + \frac{1}{X_1} - \frac{n}{\overline{X}} .$$

Then, inversely,

$$X_1 = X_1$$

$$\overline{Y} = (n\overline{X} - X_1)\Big/ n-1$$

$$V_1 = V - \frac{n(X_1 - \overline{X})^2}{X_1 \overline{X}(n\overline{X} - X_1)}$$

and the Jacobian of the transformation is $\frac{n}{n-1}$. So the joint density function of r.v.'s X_1, $\overline{X}$ and V is

$$f(x_1, \overline{x}, v) = \frac{n(n-1)\,\lambda^{\frac{n}{2}}}{\Gamma\!\left(\frac{n-2}{2}\right)\pi\sqrt{2^n x_1^3 (n\overline{x} - x_1)^3}} \left[v - \frac{n(x_1 - \overline{x})^2}{x_1 \overline{x}(n\overline{x} - x_1)} \right]^{\frac{n-4}{2}} .$$

$$\cdot \exp\left[-\frac{\lambda}{2\mu^2}\left\{ \frac{(x_1 - \mu)^2}{x_1} + \frac{[n(\overline{x} - \mu) - (x_1 - \mu)]^2}{n\overline{x} - x_1} \right\} - \frac{\lambda}{2}\left(v - \frac{n(x_1 - \overline{x})^2}{x_1 \overline{x}(n\overline{x} - x_1)} \right) \right],$$

$$0 < x_1 < n\overline{x} , \quad 0 < \frac{n(x_1 - \overline{x})^2}{x_1 \overline{x}(n\overline{x} - x_1)} < v .$$

The density function of statistics $T(X) = (\overline{X}, V)$ is

$$g(\bar{x}, v) = \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \sqrt{\frac{n\ \lambda^n\ t^{n-3}}{\pi\ 2^n\ \bar{x}^3}}\ e^{-\frac{n\lambda}{2\mu^2}\frac{(\bar{x}-\mu)^2}{\bar{x}} - \frac{\lambda v}{2}} .$$

Hence, the conditional density function of X_1 given $T(X) = (\bar{x}, v)$ is

$$h(x_1 | T(x)) = \frac{\sqrt{n}\ (n-1)}{\beta\left(\frac{1}{2}, \frac{n-2}{2}\right)} \sqrt{\frac{\bar{x}^3}{v x_1^3 (n\bar{x} - x_1)^3}} \left[1 - \frac{n(x_1 - \bar{x})^2}{v x_1 \bar{x}(n\bar{x} - x_1)}\right]^{\frac{n-4}{2}},$$

$$x_1 \in (L, U) \tag{3.1}$$

where

$$L = \bar{x}\ \frac{n[2 + v\bar{x}] - \sqrt{4n(n-1)v\bar{x} + n^2 v^2 \bar{x}^2}}{2[n + v\bar{x}]},$$

$$U = \bar{x}\ \frac{n[2 + v\bar{x}] + \sqrt{4n(n-1)v\bar{x} + n^2 v^2 \bar{x}^2}}{2[n + v\bar{x}]}. \tag{3.2}$$

Accordingly, the MVUE of $R(x;\mu\ \lambda)$ is

$$\hat{R}(x;\mu,\lambda) = \int_x^U h(x_1 | T(x))\, dx_1 \tag{3.3}$$

with $h(x_1 | T(x))$ in (3.1) and U in (3.2).

Next, we simplify (3.3) so that $\hat{R}(x;\mu,\lambda)$ can be evaluated using a student's t table. Let

$$w = \frac{\sqrt{n}\,(x_1 - \bar{x})}{\sqrt{v x_1 \bar{x}(n\bar{x} - x_1)}} \Bigg/ \left[1 - \frac{n(x_1 - \bar{x})}{v x_1 \bar{x}(n\bar{x} - x_1)}\right]^{\frac{1}{2}} ,$$

We have a one-to-one transformation and, inversely

$$x_1 = \bar{x}\,\frac{n[2(1+w^2) + v\bar{x}w^2] + w\sqrt{4n(n-1)v\bar{x}(1+w^2) + n^2v^2\bar{x}^2w^2}}{2[n(1+w^2) + v\bar{x}w^2]}$$

In addition,

$$\frac{dx_1}{dw} = \left[\frac{dw}{dx_1}\right]^{-1} = \frac{2\sqrt{v\bar{x}x_1^3(n\bar{x} - x_1)^3}}{\sqrt{n}\,\bar{x}[(n-2)x_1 + n\bar{x}]}\left[1 - \frac{n(x_1 - \bar{x})^2}{v x_1 \bar{x}(n\bar{x} - x_1)}\right]^{\frac{3}{2}}$$

and $w \in (-\infty, \infty)$ as $x_1 \in (L, U)$. From (3.1) and (3.3), after substitution

$$\hat{R}(x;\mu, \;) = \frac{4(n-1)}{\beta\left(\frac{1}{2}, \frac{n-2}{2}\right)}$$

$$\cdot \int_{w'}^{\infty} \frac{\left[n(1+w^2) + v\bar{x}w^2\right]}{\left(4n(n-1)(1+w^2) + n^2v\bar{x}w^2\right) + (n-2)\,w\sqrt{4n(n-1)v\bar{x}(1+w^2) + n^2v^2\bar{x}^2w^2}}$$

$$\cdot \frac{dw}{\left[1+w^2\right]^{\frac{n-1}{2}}} .$$

Equivalently,

$$\hat{R}(x;\mu,\lambda) = \frac{1}{\beta\left(\frac{1}{2}, \frac{n-2}{2}\right)} \int_{w'}^{\infty} \left[1 - \frac{(n-2)\, w \sqrt{v\bar{x}}}{\sqrt{4n(n-1)(1+w^2) + n^2 v \bar{x} w^2}} \right] \frac{dw}{\left[1+w^2\right]^{\frac{n-1}{2}}}, \tag{3.4}$$

where,

$$w' = \frac{\sqrt{n}\,(x-\bar{x})}{\sqrt{v x \bar{x}(n\bar{x}-x) - n(x-\bar{x})^2}} . \tag{3.5}$$

$\hat{R}(x;\mu,\lambda)$ can further be expressed in terms of student's t distribution,

$$F_{t,\,n-2}(c) = \frac{1}{\sqrt{n-2}\;\beta\left(\frac{1}{2}, \frac{n-2}{2}\right)} \int_{c}^{\infty} \frac{du}{\left[1+\frac{u^2}{n-2}\right]^{\frac{n-1}{2}}} .$$

The first term on the right side in (3.4) is equal to $F_{t,\,n-2}(w')$, and with the change of variable

$$u = \sqrt{\frac{4(n-1)(1+w^2)}{v\bar{x}} + n w^2} ,$$

we obtain the second term equal to

$$\frac{n-2}{n} \left[1 + \frac{4(n-1)}{n v \bar{x}}\right]^{\frac{n-3}{2}} F_{t,\,n-2}\left(\left[\frac{4(n-1)(1+w'^2)}{v\bar{x}} + n w'^2\right]^{\frac{1}{2}}\right) ,$$

where w' is given by (3.5). Hence, the MVUE of $R(x;\mu,\lambda)$ is

$$\hat{R}(x;\mu,\lambda) = \begin{cases} 0, & x > U \\ 1, & x < L \\ F_{t,n-2}(w') - \frac{n-2}{2}\left[1 + \frac{4(n-1)}{nv\bar{x}}\right]^{\frac{n-3}{2}} F_{t,n-2}(w''), & \text{otherwise}, \end{cases}$$

$$(3.6)$$

where

$$w' = \frac{\sqrt{n}\,(x - \bar{x})}{\sqrt{vx\bar{x}(n\bar{x} - x) - n(x - \bar{x})^2}}, \quad w'' = \frac{n\bar{x} + (n-2)x}{\sqrt{vx\bar{x}(n\bar{x} - x) - n(x - \bar{x})^2}},$$

and

$$L = \frac{\bar{x}}{2(n + v\bar{x})}\left[n(2 + v\bar{x}) - \sqrt{4n(n-1)v\bar{x} + n^2 v^2 \bar{x}^2}\right]$$

$$U = \frac{\bar{x}}{2(n + v\bar{x})}\left[n(2 + v\bar{x}) + \sqrt{4n(n-1)v\bar{x} + n^2 v^2 \bar{x}^2}\right].$$

Comparison of these estimates as given in (1.7), (2.8) and (3.6) with those of their counterparts for the Gaussian distribution (see Folks, Pierce and Stewart [9]) shows a remarkable similarity. Not only are their MVUE's commonly expressed in terms of the same distributions, standard normal and student's t, but these estimates are also similar in character as to the form and the respective parameters involved. Of course, the obvious difference that MVUE's obtained for the inverse Gaussian are expressed as the non-linear combination of standard normal distributions or that of student's t distributions is expected since the inverse Gaussian distribution is itself a non-linear weighted normal distribution (see (1.5), Chapter III).

CHAPTER VII

SOME RELIABILITY ASPECTS

Considering a physical phenomenon with some dynamic stress operating on a device or unit under normal use, assume that the device or unit will fail at a given level of stress. If the level of stress obeys the Gaussian law, it can be proven that the life-time τ of the device is described by an inverse Gaussian model. This is a known result in terms of a first passage time random variable for the Gaussian process. It is, therefore, appropriate to consider the inverse Gaussian distribution as a mathematical model representing some life-time distribution and to undertake its investigation as to its failure rate and other aspects of reliability theory.

In Chapter VI, we have already given MVUE's of the reliability function. In the following section, we will show that the inverse Gaussian distribution has a non-monotone failure rate. In addition, a brief description on the mean residual life-time will be given. The present discussion is for the case of a single component device or unit and is only of preliminary nature.

Failure Rate

We define the failure rate of a life-time distribution F with the density function $f(t)$ by

$$r(t) = \lim_{h\to 0} P[\tau \in (t, t+h) \,|\, \tau > t] , \qquad t > 0$$

or equivalently,

$$r(t) = \frac{f(t)}{1 - F(t)} , \qquad t \in (0, \infty) . \tag{1}$$

Let $f(t)$ be unimodal and let t_m be the modal point. Clearly, $r(t)$ is non-decreasing for $t \in (0, t_m]$. It is therefore sufficient to restrict our investigation for $t \in (t_m, \infty)$. Assuming that $r(t)$ is differentiable, it is seen from (1) that

$$\frac{r'(t)}{r(t)} = \frac{f'(t)}{f(t)[1 - F(t)]} \left[[1 - F(t)] + \frac{f^2(t)}{f'(t)} \right] \tag{2}$$

provided $r(t) > 0$. Letting $p(t) = -f'(t)/f(t)$, we have $p(t) > 0$ for $t \in (t_m, \infty)$ and have (2) written as

$$\frac{r'(t)}{r(t)} = \frac{p(t)}{1 - F(t)} \left[\int_t^\infty \frac{f'(x)}{p(x)} dx + \frac{f(t)}{p(t)} \right] . \tag{3}$$

Assume that $p(t)$ is non-decreasing for $t \in (t_m, \infty)$. Then it follows from (3) that

$$\frac{r'(t)}{r(t)} \geq \frac{p(t)}{1 - F(t)} \left[\frac{1}{p(t)} \int_t^\infty f'(x)\,dx + \frac{f(t)}{p(t)} \right]$$

implying $r'(t) \geq 0$ because

$$\int_t^\infty f'(x)dx = -f(t) .$$

Hence, $r(t)$ is monotone non-decreasing for all t. On the other hand, suppose $p'(t) \geq 0$ does not hold for all $t \epsilon (t_m, \infty)$. Then, assuming without loss of generality that $p(t)$ is unimodal, there exists a t_0 such that $p'(t) \geq 0$ for $t \epsilon (t_m, t_0]$ and $p'(t) < 0$ for $t \epsilon (t_0, \infty)$. Accordingly, it can be easily deduced from (3) that $r'(t) \geq 0$, $t \epsilon t_m, t_n)$ and $r'(t) < 0$, $t \epsilon (t_n, \infty)$, where $t_m \leq t_n \leq t_0$. Thus, $r(t)$ is not monotone as it is increasing for $t \epsilon (t_m, t_n)$ and is decreasing for $t \epsilon (t_n, \infty)$. Observing that t_n is a solution of $r'(t) = 0$ only for values of t greater than t_m and $p(t) = -\frac{d}{dt} \log f(t)$, we have $r'(t) \geq 0$ if and only if $\frac{d^2}{dt^2} \log f(t) \leq 0$ for $t \epsilon (t_m, \infty)$. This establishes the following theorem.

Theorem: Let F be a life-time distribution with the unimodal density function $f(t)$, the mode at t_m. Then F is an increasing failure rate (IFR) distribution if and only if $\log f(t)$ is a concave function for $t \epsilon (t_m, \infty)$.

Next, if the mode of $f(t)$ is at 0 or $f(t)$ has infinite value to begin with, then as in the proof of the above theorem it can be shown that F is a decreasing failure rate (DFR) distribution if and only if $\log f(t)$ is a convex function for all t.

It may be noted that the concavity of $\log f(t)$ does not hold if $f(t)$ decreases slowly enough as t gets large. We see such behavior in the density function of many distributions, e.g., log-normal, inverse Gaussian, Pareto, Gamma and Weibull, etc., depending on their parametric values.

Clearly, the inverse Gaussian is not a DFR distribution. We show further that it is not, in general, an IFR distribution as well.

Considering an inverse Gaussian life-time distribution with parameters μ and λ, we can write

$$\log f(t) = k - \frac{3}{2}\log t - \frac{\lambda(t-\mu)^2}{2\mu^2 t}, \qquad k = \text{constant}.$$

Then

$$\frac{d^2}{dt^2}\log f(t) = \frac{(3t-2\lambda)}{2t^3};$$

and so due to $\frac{d^2}{dt^2}\log f(t) > 0$ for $t = \frac{2}{3}\lambda$, a value greater than the mode, the result follows from the above theorem.

The failure rate is given by

$$r(t) = \sqrt{\frac{\lambda}{2\pi t^3}}\; e^{-\frac{\lambda}{2\mu^2 t}(t-\mu)^2} \Bigg/ \phi\left(\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu}-1\right)\right) - e^{\frac{2\lambda}{\mu}}\,\phi\left(\sqrt{\frac{\lambda}{t}}\left(1+\frac{t}{\mu}\right)\right) \quad (4)$$

where $\phi(a)$ denotes the probability that the standard normal variate exceeds a. It can be seen that $r(t)$ is increasing for $t \in (0, t^*)$ and is decreasing for $t \in (t^*, \infty)$, where t^* is the solution of the equation

$$r(t) = \frac{\lambda}{2\mu^2} + \frac{3}{2t} - \frac{\lambda}{2t^2}$$

and $r(t^*)$ is the maximum value $r(t)$ can achieve. Observe that $r(t^*) < \infty$ unless $t^* \to \infty$. But $t^* \to \infty$ only if $\lambda \to \infty$, and then $r(t)$ is monotone non-decreasing for all t. Since $f(t)$ is asymptotically a normal density as $\lambda \to \infty$ and the normal distribution is IFR with the failure rate becoming infinite as $t \to \infty$, the same conclusion is reached.

Bounds on Failure Rate

The failure rate $r(t)$ for an inverse Gaussian distribution can be expressed as

$$\frac{1}{r(t)} = \int_t^\infty \left(\frac{t}{x}\right)^{\frac{3}{2}} e^{-\frac{\lambda}{2\mu^2}\left\{(x-t) - \frac{\mu^2(x-t)}{tx}\right\}} dx .$$

Letting $x - t = z$, it can be simplified to

$$\frac{1}{r(t)} = e^{\frac{\lambda}{2t}} \int_0^\infty \frac{1}{(1+\frac{z}{t})^{\frac{3}{2}}} e^{-\frac{\lambda}{2t}/(1+\frac{z}{t}) - \frac{\lambda}{2\mu^2} z} dz . \qquad (5)$$

Since $e^{-\frac{\lambda}{2\mu^2} z} \leq 1$ for all $z \geq 0$,

$$\frac{1}{r(t)} \leq e^{\frac{\lambda}{2t}} \int_0^\infty \frac{1}{(1+\frac{z}{t})^{\frac{3}{2}}} e^{-\frac{\lambda}{2t}/(1+\frac{z}{t})} dz .$$

Consequently, we can obtain a lower bound of $r(t)$ given by

$$r(t) \geq \sqrt{\frac{\lambda}{2\pi t^3}}\, e^{-\frac{\lambda}{2t}} \Big/ 2h\left(\sqrt{\frac{\lambda}{t}}\right) \qquad (6)$$

where

$$h(a) = \int_0^a \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du .$$

For any fixed $t > 0$,

$$e^{-\frac{\lambda}{2t}} \leq e^{-\frac{\lambda}{2t} \cdot \frac{1}{1+\frac{z}{t}}} \leq 1 . \tag{7}$$

So from (5), we have

$$\frac{1}{r(t)} \geq \int_0^{\infty} \frac{1}{\left(1+\frac{z}{t}\right)^{\frac{3}{2}}} e^{-\frac{\lambda}{2\mu^2} z} dz = t e^{\frac{\lambda t}{2\mu^2}} \int_1^{\infty} y^{-\frac{3}{2}} e^{-\frac{\lambda t}{2\mu^2} y} dy .$$

The last term can be expressed as an infinite series, and approximating it by finite number of terms, we get an upper bound given by

$$r(t) \leq \frac{\lambda}{2\mu^2} \left[1 - \frac{3\mu^2}{\lambda t} + \frac{15\mu^4}{\lambda^2 t^2} - \frac{105\mu^6}{\lambda^3 t^3} + 0(t^{-4}) \right]^{-1} . \tag{8}$$

In fact, another lower bound follows due to the last inequality in (7) and it is given by

$$r(t) \geq \frac{\lambda}{2\mu^2} e^{-\frac{\lambda}{2t}} . \tag{9}$$

This bound is more conservative than the one in (6).

From (8) and (9), it is noted that $r(t) \to 0$ as $t \to 0$ and $r(t) \to \frac{\lambda}{2\mu^2}$ as $t \to \infty$. Hence, the failure rate for the inverse Gaussian distribution first increases up to a certain point depending on parametric values, and then decreases monotonically to some constant value. There is similar behavior with the failure rate of the log-normal distribution except there it finally decreases to 0. Due to this similarity and serious attempts made in representing life-time distributions by the log-normal distribution in the past by many authors, we feel that the inverse Gaussian distribution is a good competitor of the log-normal and, in fact, it could be better as it is more natural to have some failure rate than no failure rate in the end.

Mean Residual Life-Time

The failure rate $r(t)$ of a life-time distribution is closely related with the conditional expectation of the remaining life-time under the assumption that the item did not fail up to a certain instant T. Let s be the random variable for the residual life-time; that is, the period from an instant of time T until the instant of failure under the condition that there was no failure prior to the instant T. Then from (1) it can be seen that the distribution of s is obtained by

$$P[s \leq t] = 1 - \exp\left(-\int_T^{t+T} r(x)\,dx\right).$$

Consequently, the mean residual life-time

$$\mu_T = \int_0^{\infty} \exp\left(-\int_T^{t+T} r(x)\,dx\right) dt \,. \qquad (10)$$

Clearly, μ_T decreases (increases) monotonically for the life-time distribution with IFR (DFR). However, if $r(t)$ first increases and then begins to decrease monotonically at some instant t^*, as the case with the inverse Gaussian, we have $\mu_{T_2} \geq \mu_{T_1}$ for $T_2 > T_1 > t^*$. That is, the mean residual life-time will increase from some instant onward. The increase in the mean residual life-time after the instant t^* is explained by the fact that the items with a high rate of wear can fail and the rate of wear of the surviving items is relatively small resulting in a higher longevity. Based on such considerations, Watson and Wells [27] have examined the log-normal distribution, among others, for the possibility of improving what they call "mean useful life of items" instead of mean residual life-time. A similar investigation of the inverse Gaussian distribution, which also has non-monotonic failure rate, is warranted.

We conclude this chapter with a description showing the appropriateness of incorporating ideas mentioned in the preceding section. In the past, reliability investigators have most often, attempted to represent a life-time phenomenon by a probability distribution with IFR property. This is undoubtedly adequate in a situation which mainly involves the cause of the aging or the wearing out process. But failure of a unit may be due to various causes other than the aging process, such as a technological defect, improper usage; or, say, instantaneous injury. See Gertsbakh and Kordonskiy [10] for details on causes of failure. In order to deal with these unavoidable causes, an appropriate measure as to the model of failure and possible methods of improvement should be adopted.

Consider, for example, a manufacturing process with isolated random flaws which may lead to the occurence of weak spots in an item. The weak spots may remain unnoticed unless items are subjected to a test for a certain period of time. As a result, a fraction of the product will fail and the fraction of the product surviving will, on the average, have higher longevity. Consequently, the probability that an item will fail during $(t, t+\Delta)$, given that it did not fail for time t is less than the probability of its failure during $(0, \Delta)$. For this reason, it is desirable to have a testing period for items of a product before these are released to the market. However, this will lead to the decrease in the life-time of an item because the testing period may not be short since a defective item may not show its weakness immediately. But, if the defective items are eliminated, the residual life-time in the fraction of items that do not fail should, on the average, exceed the average life-time of the initial product so as to compensate for the testing period. This is possible when a life-time model can be represented by a probability distribution that has non-monotonic failure rate. Hence, with the consideration of the inverse Gaussian as a model of failure due to reasons mentioned in the beginning of this chapter, this study on the inverse Gaussian distribution can lead to interesting applications in reliability theory.

CHAPTER VIII

SUMMARY

Our study is devoted to the discussion of the inverse Gaussian distribution with an emphasis on the theory of its statistical inference. Problems of testing statistical hypotheses and interval estimation for its parameters μ and λ are investigated in detail. Further, its development in the area of reliability is suggested and various results are established.

As the inverse Gaussian density functions constitute a two-parameter exponential family, a comprehensive account of optimum test procedures is developed for testing hypotheses on parameters μ and λ by exploiting the theory of testing statistical hypotheses, as given in Lehmann [14]. Test statistics of UMP unbiased tests of $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$ when (i) λ is specified and (ii) λ is unspecified, are shown to have standard half-normal and truncated student's t distributions, with their domain on the positive real line, respectively. We note that similar results are well known in the case of testing hypotheses on the mean of a normal population with two-sided alternatives. Exact optimal test procedures for such one-sided hypotheses are also derived. Next, we have discussed the problem of uniformly most accurate (UMA) confidence bounds on μ. Exact results in the case of desiring two-sided confidence bounds and

approximate results in the case of desiring one-sided confidence bounds are given. A discussion on tolerance limits is appended.

As an application, we have indicated a theoretical reasoning to consider inverse Gaussian as a life-time distribution. Minimum variance unbiased estimates of its reliability function,
$R(x;\mu,\lambda) = E[I_{(x,\infty)}(X)]$ are obtained for all different cases; namely, (1) μ unknown, λ known, (2) μ known, λ unknown and (3) both μ and λ unknown. Interestingly, these estimates and similar such estimates in the case of normal distribution have certain common features and, consequently, it helps visualizing more on the characteristics of parameters μ and λ. As to other concepts of reliability, it was proved that its failure rate is a non-monotone function and, thereby, the mean residual life-time for an inverse Gaussian distribution first decreases and then begins to increase after an instant, depending on the parametric values. On this basis we have suggested an application of the inverse Gaussian as a model of failure. Furthermore, bounds on its failure rate are given and its comparison with the log-normal distribution is made.

For more insight into the inverse Gaussian distribution theory, a relationship between its density function and the normal density function is established. As a consequence, we have been able to derive some related distributions. This has led this author to a better understanding of the silent features of the family of inverse Gaussian distributions and of the statistical inference discussed in this thesis.

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