SOME OSCILLATORY PROPERTIES OF THIRD AND FOURTH ORDER LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

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CHAPTER I

ORDINARY LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

Introduction

This thesis is primarily concerned with some oscillatory properties of third and fourth order linear ordinary homogeneous differential equations. In particular, the dimensions of the oscillatory and nonoscillatory subspaces are investigated.

The third order theory started with Birkhoff's paper in 1911, [1], in which he studied the oscillation and separation theory. Reynolds, [2], extended Birkhoff's work to equations of arbitrarily order n. However, the modern theory did not start until the late 1940's. Significant contributions were made by Gregus, Hanan, Lazer, Ráb, Švec, and Zlamal. It was shown that many of the asymptotic and oscillatory properties of linear third order ordinary homogeneous differential equations with constant coefficients are carried over to those with variable coefficients which do not change sign.

Leighton and Nehari in [3], Howard, Barrett, Lazer, Ahmad were among the first to undertake a systematic development of the fourth order theory. The dimensions of oscillatory and nonoscillatory subspaces were investigated in the case where the coefficients do not change sign.

In this paper, we consider the maximum dimensions of oscillatory and nonoscillatory subspaces for the third order linear differential equation. We will study the case where the maximum dimensions of oscillatory and nonoscillatory subspaces are respectively two and one.

The relations between the dimensions of subspaces of solutions of the differential equation and those of its adjoint are also investigated. This paper contains one example which answers a question raised by Ahmad [5].

Concerning the fourth order ordinary homogeneous differential equations, conditions are given under which all oscillatory solutions are bounded. Also are shown some techniques of getting from the fourth order differential equation

$$y^{IV} = Py'' + Qy' + Ry,$$

where $P > 0$, $Q < 0$, and $R > 0$, a third order differential equation

$$y''' = py'' + qy' + ry,$$

where $p < 0$, $q > 0$, and $r < 0$.

In the remaining sections of this chapter, we shall state propositions which are basic to the development of the succeeding chapters. Since these results are all well-known, we shall not prove any of them.

Third Order Linear Differential Equations

Consider the differential equation

$$z''' + P(x)z'' + Q(x)z' + R(x)z = 0, \tag{1}$$

where $P(x) \varepsilon C^2[a,\infty)$, $Q(x) \varepsilon C^1[a,\infty)$, $R(x) \varepsilon C[a,\infty)$, and a is a real number. The adjoint of (1) is the differential equation given by

$$z''' - (P(x)z)'' + (Q(x)z)' - R(x)z = 0. \tag{1'}$$

The well-known substitution

$$z(x) = y(x) \exp\left[-\frac{1}{3}\int_a^x P(s)\, ds\right]$$

transforms the differential equation (1) into

$$y''' + p(x)y' + q(x)y = 0. \tag{2}$$

The oscillatory properties remain invariant under this transformation. Therefore, (2) is not much less general than (1).

In what follows, by a solution of (2) we shall mean a nontrivial solution. A solution of (2) is said to be oscillatory if its set of zeros is not bounded above. Solutions which are not oscillatory are called nonoscillatory.

Definition 1.1: The differential equation is said to be oscillatory if it has at least one oscillatory solution.

Definition 1.2: A subspace of solutions of the differential equation is said to be nonoscillatory [strongly oscillatory] if and only if none of [all] the solutions in the subspace oscillate.

Definition 1.3: The differential equation (2) is said to have property R on [a,) if it has both oscillatory and nonoscillatory solutions, and further, it has two solutions u_1 and u_2 with $W(u_1,u_2)(x) \neq 0$ for $x \in [a,\infty)$, where $W(u_1,u_2)$ represents the wronskian of u_1 and u_2.

<u>Theorem</u> 1.4: If (2) has solutions u_1 and u_2 such that $u_1(x) \neq 0$ on $[a,\infty)$ and $W(u_1,u_2)(x) \neq 0$ on $[a,\infty)$, then no solution of (2) can have more than two zeros on $[a,\infty)$ (counting multiplicities) (see [4]).

<u>Remark</u>: It follows that if (2) has property R and u_1 and u_2 are the solutions in Definition 1.3, then u_1 and u_2 are both oscillatory. For, suppose $u_1(x) \neq 0$ on some interval $[b,\infty)$. Then $u_1(x) \neq 0$ and $W(u_1,u_2)(x) \neq 0$ on $[b,\infty)$ imply by Theorem 1.4 that no solution of (2) can have more than two zeros.

<u>Definition</u> 1.5: The differential equation (2) is said to have <u>property</u> RO if it has property R and a solution of (2) is oscillatory if and only if it is a nontrivial linear combination of u_1 and u_2, where u_1 and u_2 are the solutions in Definition 1.3. Equation (2) is said to have <u>property</u> RN if it has property R and every nonoscillatory solution of (2) is a constant multiple of a fixed nonoscillatory solution.

The following three theorems have been established in [5].

<u>Theorem</u> 1.6: The differential equation (2) has property R on $[a,\infty)$ if and only if its adjoint has property R on some interval $[b,\infty)$ for some $b \geq a$.

<u>Theorem</u> 1.7: Suppose that equation (2) has solutions u_1, u_2, and v such that $v(x) \neq 0$ for $x \geq a$, and u_1 and u_2 are oscillatory with $W(u_1,u_2)(x) \neq 0$ for $x \geq a$. Then (2) has property RO if and only if

$$\lim_{x \to \infty} \frac{u_1(x)}{v(x)} = \lim_{x \to \infty} \frac{u_2(x)}{v(x)} = 0.$$

<u>Theorem</u> 1.8: Suppose that equation (2) has solutions u_1, u_2, and v such that $v(x) \neq 0$ for $x \geq a$, and u_1 and u_2 are oscillatory with $W(u_1,u_2)(x) \neq 0$ for $x \geq a$. Then (2) has property RN if and only if every nontrivial linear combination of

$$\frac{u_1(x)}{v(x)} \quad \text{and} \quad \frac{u_2(x)}{v(x)}$$

is unbounded above and below.

<u>Theorem</u> 1.9: If equation (2) has property RO on $[a,\infty)$, then its adjoint has property RN on some interval $[b,\infty)$, $b \geq a$.

<u>Proposition</u> 1.10: If $p(x) \leq 0$, $q(x) \leq 0$, and $y(x)$ is a solution of (2) such that $y(x_0) \geq 0$, $y'(x_0) \geq 0$, and $y''(x_0) > 0$ for some $x_0 \in [a,\infty)$, then $y(x) > 0$, $y'(x) > 0$, $y''(x) > 0$ for all $x > x_0$, and

$$\lim_{x \to \infty} y(x) = \lim_{x \to \infty} y'(x) = +\infty$$

(see [3]).

<u>Definition</u> 1.11: Equation (2) is said to be (1.2) disconjugate [(2.1) disconjugate] if and only if every solution $y(x)$ for which

$y(b) = y'(b) = 0$, $y''(b) > 0$ $(a < b < \infty)$ has the property that $y(x) > 0$ in (a,b) [in (b,∞)].

Proposition 1.12: Equation (2) is (1.2) disconjugate [(2.1) disconjugate] if $2q(x) - p'(x) > 0$ [$2q(x) - p'(x) < 0$]. (See [6]).

Fourth Order Linear Differential Equation

Consider the differential equation

$$z^{IV} + P(x)z''' + Q(x)z'' + R(x)z' + L(x)z = 0. \tag{3}$$

If $Q(x)$, $R(x)$, and $L(x) \varepsilon C[a,\infty)$ and $P(x) \varepsilon C^3[a,\infty)$, the substitution

$$z(x) = y(x) \exp\left[-\frac{1}{4}\int_a^x P(s)\, ds\right]$$

transforms (3) into

$$y^{IV} = p(x)y'' + q(x)y' + r(x)y \tag{4}$$

where $p(x)$, $q(x)$, and $r(x)$ are continuous on $[a,\infty)$. The oscillatory properties remain invariant under this transformation.

The techniques in this part were extensively used by Lazer in [7]. Later, Lazer and Hasting [8] applied these methods to a simple self-adjoint differential equation of order four. Ahmad used it also for equations of type (4).

For Chapter IV, we need the following lemma (see [3]).

Lemma 1.13: Let $y(x)$ be a solution of the differential equation (4) where $p(x)$, $q(x)$, and $r(x) \varepsilon C[a,\infty)$, $p(x) > 0$, $q(x) < 0$, and

$r(x) > 0$. If $y(b) \geq 0$, $y'(b) \leq 0$, $y''(b) \geq 0$, $y'''(b) < 0$ for some $b > a$, then $y(x) > 0$, $y'(x) < 0$, $y''(x) > 0$, and $y'''(x) < 0$ for $x \in [a,b)$.

CHAPTER II

SOME OSCILLATION PROPERTIES OF THIRD ORDER LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

Consider the differential equation

$$y''' + p(x)y' + q(x)y = 0 \tag{2}$$

and its adjoint

$$y''' + (p(x)y)' - q(x)y = 0. \tag*{(2)'}$$

Throughout this chapter we shall assume that $p(x) \in C^1[a,\infty)$ and $q(x) \in C[a,\infty)$ for some fixed real number a. We shall study successively the cases:

(i) $p(x) > 0$, $q(x) < 0$;

(ii) $p(x) < 0$, $q(x) < 0$.

We first consider the case when $p(x) > 0$ and $q(x) < 0$. In the case where p and q are constant, a fundamental set of solutions of equation (2) consists of $e^{-bx}\sin cx$, $e^{-bx}\cos cx$, e^{dx} where $b > 0$, $c > 0$ and $d > 0$. Therefore, for the constant case, equation (2) has property RO. The question is whether this is the case when $p(x)$ and $q(x)$ are not necessarily constant. The next part answers partially this question under the supplementary condition that $2q(x) - p'(x) < 0$

for $x \varepsilon [a,\infty)$. The following lemma of Hanan is needed for the proof of our first result.

Lemma 2.1: If $u(x)$ and $v(x)$ are linearly independent solutions of (2) such that $u(b) = v(b) = 0$ for some $b > a$, and equation (2) is (2,1) disconjugate, then the zeros of $u(x)$ and $v(x)$ separate in $[a,b)$ (see [9]).

Theorem 2.2: If $p(x) > 0$, $q(x) < 0$, $2q(x) - p'(x) < 0$ on $[a,\infty)$, and equation (2) has an oscillatory solution $y(x)$, then (2) has another oscillatory solution $z(x)$ such that $y(x)$ and $z(x)$ are linearly independent.

Proof: Let $y(x)$ be an oscillatory solution of (1). Let b be a point at which $y(b) \neq 0$ and let $\{x_i\}_{i=1}^{\infty}$ be an increasing sequence of zeros of $y(x)$ such that $b < x_1$ and $x_i \to \infty$ as $i \to \infty$. Let $z_1(x)$, $z_2(x)$, and $z_3(x)$ be the fundamental set of solutions of (2) such that

$$W(z_1,z_2,z_3)(b) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

For each integer n, let $z_n(x)$ be the solution of (2) defined by the two boundary conditions, $z_n(b) = 0$ and $z_n(x_n) = 0$. The solution $z_n(x)$ can be written as a linear combination of $z_1(x)$, $z_2(x)$, and $z_3(x)$, and since $z_n(b) = 0$, it is a linear combination of $z_2(x)$ and $z_3(x)$ only. Therefore, $z_n(x) = c_{2n}z_2(x) + c_{3n}z_3(x)$ where c_{2n} and c_{3n} are real numbers. We can choose c_{2n} and c_{3n} such that $c_{2n}^2 + c_{3n}^2 = 1$. The sequence $\{(c_{2n},c_{3n})\}_{n=1}^{\infty}$ is a bounded sequence from a compact set. Hence, there exists a sequence of integers $\{n_k\}$ such that $\lim_{n_k\to\infty} c_{2n_k} = c_2$, $\lim c_{3n_k} = c_3$, and $c_2^2 + c_3^2 = 1$. It follows, by continuity, if we let $z(x) = c_2z_2(x) + c_3z_3(x)$, that $\lim_{n_k\to\infty} z_{n_k}^{(j)}(x) = z^{(j)}(x)$, $j = 0,1,2,3$. The solution $z(x)$ is a nontrivial solution of (2) since $c_2^2 + c_3^2 = 1$, and $z_2(x)$ and $z_3(x)$ are linearly independent. The solutions $z_{n_k}(x)$ and $y(x)$ have in common the zero x_{n_k}. Also, they are linearly independent since $z_{n_k}(b) = 0$ and $y(b) \neq 0$. Also, Proposition 1.10 implies that equation (2) is (2.1) disconjugate. Therefore, by Lemma 2.1, the zeros of $y(x)$ and $z_{n_k}(x)$ separate in $[a,x_{n_k}]$.

Next, we want to show that $z(x)$ is oscillatory. Let c and d be consecutive zeros of $y(x)$ such that $c < d$. There exist n_k such that $n_k > d$. It follows that $z_{n_j}(x)$ would have a zero in $[c,d]$ for all $n_j > n_k$. But $z(x) = \lim_{n_k\to\infty} z_{n_k}(x)$; hence, $z(x)$ would have a zero in $[c,d]$.

Therefore, $z(x)$ has a zero between any two consecutive zeros of $y(x)$. Hence, $z(x)$ is also oscillatory since $y(x)$ is. The solutions $z(x)$ and $y(x)$ are linearly independent because $z(b) = 0$ and $y(b) \neq 0$.

Theorem 2.3: Every linear combination of $y(x)$ and $z(x)$ of Theorem 2.2 is oscillatory.

Proof: Let $v(x) = c_1y(x) + c_2z(x)$. If $c_1 = 0$ or $c_2 = 0$, then $v(x)$ is oscillatory. Suppose $c_1c_2 \neq 0$, and assume that $v(x)$ is nonoscillatory. Without loss of generality, we can assume that $v(x) > 0$ for $x \in [b,\infty)$ for some $b > a$. Since equation (2) is (2.1) disconjugate, the zeros of $y(x)$ and $z(x)$ are all simple zeros.

The inequality $v(x) > 0$ for $x \varepsilon [b,\infty)$ reduces to $c_2z(x) > -c_1y(x)$ for $x \varepsilon [b,\infty)$. The zeros of $-c_1y(x)$, being all simple zeros, implies there exist two consecutive zeros x_1 and x_2 of $y(x)$ such that $b < x_1 < x_2$ and $-c_1y(x) > 0$ for $x \varepsilon (x_1,x_2)$. The inequality $c_2z(x) > -c_1y(x)$ implies that $c_2z(x)$ has no zero in $[x_1,x_2]$, which is a contradiction to Theorem 2.2. Hence, $v(x)$ is oscillatory.

We can show further that $W(y(x),z(x)) \neq 0$ for $x \varepsilon [b,\infty)$ for some $b \geq a$. Suppose not, then there exists $x_0 \varepsilon [a,\infty)$ such that $W(y(x_0),z(x_0)) = 0$. It follows that there exists a nontrivial linear combination $w(x)$ of $y(x)$ and $z(x)$ having a double zero or $x = x_0$. From Theorem 2.3 $w(x)$ is oscillatory. But, this is a contradiction to the (2.1) disconjugacy of (2). Hence, $W(y(x),z(x)) \neq 0$ for $x \varepsilon [b,\infty)$ for some $b \geq a$.

We will prove the next lemma in order to show a separation theorem for the oscillatory solution $y(x)$ and its derivative $y'(x)$.

<u>Lemma</u> 2.4: Let $p(x) \geq 0$, $2q(x) - p'(x) < 0$ for $x \varepsilon [a,\infty)$. If $y(x)$ is an oscillatory solution of (1), then $F[y(x)] > 0$ for all $x \varepsilon [a,\infty)$, where $F[y(x)] = [y'(x)]^2 - 2y(x)y''(x) - p(x)y^2(x)$.

Proof: It can be verified by differentiation that

$$F[y(x)] = F[y(a)] + \int_a^x [2q(t) - p'(t)]y^2(t)dt.$$

It follows under the assumption that $2q(x) - p'(x) < 0$ for all $x \varepsilon [a,\infty)$ that $F[y(x)]$ is a strictly decreasing function. Suppose now that $y(x)$ is an oscillatory solution of (2) and let $\{x_i\}_{i=1}^{\infty}$ be a sequence of zeros of $y(x)$ such that $x_k \to \infty$ as $k \to \infty$. Then

$$F[y(x_i)] = [y'(x_i)]^2 - 2y(x_i)y''(x_i) - p(x_i)y^2(x_i)$$

$$F[y(x_i)] = [y'(x_i)]^2 \geq 0.$$

$y'(x_i)^2 > 0$ since (2) is (2.1) disconjugate.

From this, it follows readily that $F[y(x)] > 0$ for all $x \in [a,\infty)$.

Theorem 2.5: If $p(x) > 0$, $2q(x) - p'(x) < 0$, and $y(x)$ is an oscillatory solution of (2), then the zeros of $y(x)$ and $y'(x)$ separate.

Proof: Suppose that $y(x)$ is an oscillatory solution of (2). It follows from Lemma 2.4 that $F[y(x)] > 0$ for all $x \in [a,\infty)$. The function $F(y(x)]$ is strictly decreasing and is nonnegative. Therefore, $\lim_{x \to \infty} F[y(x)]$ exists and is nonnegative.

Let x_k be a zero of $y'(x)$; then

$$F[y(x_k)] = [y'(x_k)]^2 - 2y(x_k)y''(x_k) - p(x_k)y^2(x_k)$$

$$F[y(x_k)] = 2y(x_k)y''(x_k) - p(x_k)y^2(x_k) > 0.$$

Since $p(x_k) > 0$, it follows that

$$y(x_k)y''(x_k) < 0. \tag{5}$$

Let now a_1 and a_2 be two consecutive zeros of $y(x)$. Without loss of generality, we can assume that $y(x) > 0$ for $x \in (a_1,a_2)$. By Rolle's Theorem, there exists $x_1 \in (a_1,a_2)$ such that $y'(x_1) = 0$. We want to show that x_1 is the only zero of $y'(x)$ in $[a_1,a_2]$. First, $y'(a_1) \neq 0$ and $y'(a_2) \neq 0$ follow from the fact that equation (2) is (2.1) disconjugate. The function $y'(x)$ cannot have an infinite number of zeros in the interval $[a_1,a_2]$. For, if it were the case, then there exists a point $x^* \in [a_1,a_2]$ such that $y'(x^*) = y''(x^*) = 0$. At the point x^*, $F[y(x^*)] = -p(x^*)y^2(x^*) < 0$ which is a contradiction since $F[y(x)]$ is a nonnegative function. Suppose now that x_2 is another zero of $y'(x)$ such that $x_2 \in [a_1,a_2]$, and x_1 and x_2 are consecutive zeros. The zeros x_1 and x_2 of $y'(x)$ are simple zeros. For, if not, then again $F[y(x_1)] < 0$ and $F[y(x_2)] < 0$, and this is a contradiction. Therefore, $y''(x_1) \neq y''(x_2)$. This contradicts (5) since $y(x_1)y''(x_1) < 0$, $y(x_2)y''(x_2) < 0$ and $y(x) > 0$ for $x \in [x_1,x_2]$.

Therefore, there is only one zero of $y'(x)$ between any two consecutive zeros of $y(x)$, and consequently, the theorem is proved.

For the case when $p(x) < 0$ and $q(x) < 0$, we will give a sufficient condition for equation (2) to have property RO, and give an example to answer the question raised in [5].

First, we consider the dimension of oscillatory solutions of

$$y''' = p(x)y' + q(x)y. \tag{2)''$$

<u>Lemma</u> 2.6: Let $p,q \in C[a,\infty)$ with $p > 0$ and $q > 0$. If $p \in C^1[a,\infty)$ with $p' \geq 0$, then all oscillatory solutions of (2)", if there are any, are bounded on $[a,\infty)$.

Proof: Let $y(x)$ be any oscillatory solution of (2)", and let x_1 be a fixed zero of $y'(x)$. Let x_2 be any other zero of $y'(x)$ such that $x_2 > x_1$. If

$$\max_{x \in [x_1,x_2]} [y(x)^2 = [y(\bar{x})]^2,$$

$\bar{x} \in [x_1,x_2]$, then $y'(\bar{x}) = 0$.

Define

$$F[y(x)] = [y'(x)]^2 - 2y(x)y''(x) + p(x)y^2(x). \tag{5}$$

By differentiation

$$F[y(\bar{x})] = F[y(x_1)] + \int_{x_1}^{\bar{x}} p'(s)y^2(s)ds - 2\int_{x_1}^{\bar{x}} q(s)y^2(s)ds.$$

If $x = x_1$, then

$$\max_{x \in [x_1,x_2]} [y(x)]^2 = y^2(x_1).$$

If $x_1 < \bar{x}$, then

$$\begin{aligned} F[y(\bar{x})] &= F[y(x_1)] + \int_{x_1}^{\bar{x}} p'(s)y^2(s)ds - 2\int_{x_1}^{\bar{x}} q(s)y^2(s)ds \\ &\leq F[y(x_1)] + y^2(\bar{x}) \cdot \int_{x_1}^{\bar{x}} p'(s)ds \\ &= F[y(x_1)] + y^2(\bar{x})[p(\bar{x}) - p(x_1)]. \qquad (6) \end{aligned}$$

From (5) and the fact that $y'(\bar{x}) = 0$, we have

$$F[y(\bar{x})] = [y'(\bar{x})]^2 - 2y(\bar{x})y''(\bar{x}) + p(\bar{x})y^2(\bar{x})$$

$$= -2y(\bar{x})y''(\bar{x}) + p(\bar{x})y^2(\bar{x}).$$

Therefore,

$$-2y(\bar{x})y''(\bar{x}) + p(\bar{x})y^2(\bar{x}) \leq F[y(x_1)] + p(\bar{x})y^2(\bar{x}) - p(x_1)y^2(\bar{x})$$

or

$$p(x_1)y^2(\bar{x}) - 2y(\bar{x})y''(\bar{x}) \leq F[y(x_1)].$$

Now, by Lemma 2.1 of [3], $y(\bar{x})y''(\bar{x}) < 0$. For $y(\bar{x})y''(\bar{x}) > 0$ and $y'(\bar{x}) = 0$ would imply that y is nonoscillatory. Hence,

$$p(x_1)y^2(\bar{x}) \leq F[y(x_1)],$$

or

$$y^2(\bar{x}) \leq \frac{F[y(x_1)]}{p(x_1)},$$

Consequently,

$$\max_{x \in [x_1,x_2]} [y(x)]^2 = y^2(\bar{x}) \leq y^2(x_1) + \frac{F[y(x_1)]}{p(x_1)},$$

and the lemma is proved.

Theorem 2.7: Assume that the hypothesis of Lemma 2.6 holds. Then, if (2)" is oscillatory, it has property RO.

Proof: Using Lemma 2.1 of [7] and the technique used in the proof of Theorem 2 [10], it follows that (2)" has two linearly independent oscillatory solutions u and v whose linear combinations are also oscillatory. Further, $W(u,v)(x)$ does not vanish anywhere.

Let z be the solution of (2)" defined by the initial conditions $z(a) = z'(a) = 0$, and $z''(a) = 1$. Then z is nonoscillatory. Let $y = c_1u + c_2v + c_3z$ by any solution of (2)". By Lemma 2.6, u and v are bounded. Also, $\lim_{x \to \infty} z(x) = \infty$. Hence, y can not be oscillatory unless $c_3 = 0$. This shows that (2)" has property RO.

An unresolved question raised in [5] was whether the converse of Theorem 1.7 is true. We give a counter example to show that the answer is in the negative. But before doing that, we need to prove the following lemma.

Lemma 2.8: If equation (1) has property RO, then for any three linearly independent solutions U_1', U_2', V' such that U_1' and U_2' are oscillatory, V' is nonoscillatory, and $W(U_1', U_2')(x) \neq 0$ for $x \in [a,\infty)$, it follows that a solution of (1) is oscillatory if and only if it is a linear combination of U_1' and U_2'.

Proof: Suppose that (1) has property RO. Then (1) has three linearly independent solutions U_1, U_2, and V such that U_1 and U_2 are oscillatory, $W(U_1,U_2)(x) \neq 0$ for $x \in [a,\infty)$, V is nonoscillatory and a solution is oscillatory if and only if it is a linear combination of U_1 and U_2.

Let U_1', U_2', V' be solutions of (1) satisfying the hypothesis of the lemma. Since U_1' is oscillatory, it must be a linear combination of U_1 and U_2. Similarly, U_2' is a linear combination of U_1 and U_2. Since V' is nonoscillatory, it is a linear combination of U_1, U_2, and V with coefficient of V with coefficient of V different from zero. Hence,

$$U_1' = c_1U_1 + c_2U_2$$

$$U_2' = k_1U_1 + k_2U_2$$

$$V' = m_1U_1 + m_2U_2 + V. \tag{7}$$

Let $z(x)$ be a linear combination of U_1' and U_2'. Then it is oscillatory since U_1' and U_2' are oscillatory, and $W(U_1',U_2')(x) \neq 0$ for $x \varepsilon [a,\infty)$. Suppose $Z(x)$ is a solution of (1) which is oscillatory. Then $Z(x) = n_1U_1' + n_2U_2' + n_3V'$. By substituting from (7) we have

$$Z(x) = n_1(c_1U_1 + c_2U_2) + n_2(k_1U_1 + k_2U_2) + n_3(m_1U_1 + m_2U_2 + V)$$

$$Z(x) = (n_1c_1 + n_2k_1 + n_3m_1)U_1 + (n_1c_2 + n_2k_2 + n_3m_2)U_2 + n_3V$$

Since (1) has property RO, and $Z(x)$ is oscillatory, it must be a linear combination of U_1 and U_2. Therefore, $n_3 = 0$. Hence, $Z(x)$ is a linear combination of U_1' and U_2'. This completes the proof of the lemma.

Example 2.9: Let $u_1 = \sin x^2$, $u_2 = \cos x^2$, and $v = (2 + 1/x) + (2 - 1/x)\cos 4x$, $x > 0$. It follows that v is nonoscillatory, $W(u_1,u_2)(x) = -2x < 0$ for $x > 0$. Furthermore, calculating $W(u_1,u_2,v)$, it can be verified that $W(u_1,u_2,v)(x) < 0$ on $[a,\infty)$ for a sufficiently large positive number a. Hence, there exists an equation of the form (1) with solutions u_1, u_2 and v. Thus, we may assume that u_1, u_2 and v are solutions of (1).

Consider the adjoint

$$y''' - (py)'' + (qy)' - ry = 0 \tag{1)'}$$

of (1). If

$$F(x) = e^{\int_a^x p(t)dt}$$

Then $U_1 = F(x)W(u_1,v)(x)$, $U_2 = F(x)W(u_2,v)(x)$, and $V = F(x)W(u_1,u_2)(x)$ are solutions of (1)', (see [1]). Clearly, U_1 is oscillatory since u_1 is oscillatory and v is not. Similarly, U_2 is oscillatory and V is nonoscillatory. It is easy to verify that

$$\lim_{x \to \infty} \frac{U_1(x)}{V(x)} \neq 0.$$

Hence, it follows from Theorem 2.7 and Lemma 2.8 that (1)' does not have property RO.

Let $y = c_1 \sin x^2 + c_2 \cos x^2 + (2 + 1/x) + (2 - 1/x)\cos 4x$. In order to show that (1) has property RN, we consider four exhaustive cases.

Case I. Suppose $c_1 \geq 0$, $c_2 \geq 0$, and $c_1^2 + c_2^2 \neq 0$. Then y can be written as

$$y = \sqrt{c_1^2 + c_2^2} \sin (x^2 + \alpha) + (2 + 1/x) + (2 - 1/x)\cos 4x,$$

$0 \leq \alpha \leq \pi/2$. Suppose y is nonoscillatory. Then for some positive number b, $y(x) > 0$ for $x > b$. For, there exist arbitrarily large values of x for which $y(x) > 0$ since $v(x) > 0$ and $x^2 + \alpha = \beta$ has solutions for any number $\beta \geq \alpha$. We note that $v((2n + 1)\pi/4) = 8/(2n + 1)\pi$. Consequently,

$$y((2n+1)\pi/4) = \sqrt{c_1^2 + c_2^2}\,\sin((2n+1)^2\pi^2/16 + \alpha) + \frac{8}{\sqrt{c_1^2 + c_2^2}\,(2n+1)\pi}\,.$$

Let $\epsilon = \sin \pi/16$, and let N be a number such that $N > b$, $(2N + 1)\pi/4 > b$, and

$$\frac{8}{\sqrt{c_1^2 + c_2^2}\,(2n+1)\pi} < \epsilon$$

for all $n \geq N$. In order to obtain a contradiction to the assumption that $y(x) > 0$ for $x > b$, it is sufficient to show that $\sin((2n+1)^2\pi^2/16 + \alpha) < -\epsilon$ for some integer n, $n > N > b$. Thus, it is sufficient to find integers n, $n > N > b$, such that $(2n+1)\pi/4 > b$,

$$\frac{8}{\sqrt{c_1^2 + c_2^2}\,(2n+1)\pi} < \epsilon$$

and $\sin((2n+1)^2\pi^2/16 + \alpha) < -\sin \pi/16$. The latter inequality is satisfied if $17\pi/16 + 2k\pi < (2n+1)^2\pi^2/16 + \alpha < 31\pi/16 + 2k\pi$. Since α is between 0 and $\pi/2$, $\alpha = m\pi/16$ for some real number m, $0 \leq m \leq 8$. Thus, it suffices to show the existence of arbitrarily large integers n satisfying

$$17 + 32k < (2n+1)^2\pi + m < 31 + 32k, \tag{8}$$

where k is an integer.

There exist arbitrarily large integers n such that $8n\pi$ can be written as $8n\pi = 32p + r$, where p is an integer and $0 < r < 1$. This follows since for any integer n divisible by 4, $0 < 8n\pi - 32p < 1$ is equivalent to $0 < q\pi - p < 1/32$, $q = n/4$, which is certainly certified by arbitrarily large integers q and p. Let n be any positive integer such that $8n\pi = 32p + r$, p an integer and $0 < r < 1$. We show that for any choice of m in the prescribed range, one of the numbers $x_i = (2i + 1)^2\pi$, $i = n, n+1, \ldots, n+8$, satisfies (8). Let $x_n = (2n + 1)^2\pi = 32k + r'$, k an integer and $0 < r' < 32$. Then $x_{n+1}((2n + 1) + 2)^2\pi = 32k + r' + 8n\pi + 8\pi$. Similarly, each x_{n+j}, $j = 2, 3, \ldots, 8$, can be written in the form $x_{n+j} = 32k + r' + p_j(8n\pi) + q_j(8\pi)$, p_j and q_j integers. Replacing $8n\pi$ by $32p + r$ and dividing $q_j(8\pi)$ by 32, we have

$$x_{n+1} = 32k_1 + r + r' + 25.13272\ \ldots,$$

k_1 an integer, $0 < r < 1$, $0 < r' < 32$,

$$x_{n+2} = 32k_2 + 2r + r' + 11.39816\ \ldots,$$

$$x_{n+3} = 32k_3 + 3r + r' + 22.79632\ \ldots,$$

$$x_{n+4} = 32k_4 + 4r + r' + 27.3272\ \ldots,$$

$$x_{n+5} = 32k_5 + 5r + r' + 24.9908\ \ldots,$$

$$x_{n+6} = 32k_6 + 6r + r' + 15.78712\ \ldots,$$

$$x_{n+7} = 32k_7 + 7r + r' + 31.7161\ \ldots,$$

$$x_{n+8} = 32k_8 + 8r + r' + 8.7719\ \ldots\ .$$

By dividing up the range of the values of m into subintervals $i \le m \le i + 1$, $i = 0, 1, \ldots, 7$, one can verify that for each value of m, one of the numbers x_i, $i = n, n+1, \ldots, n+8$, satisfies (8). For example, suppose $0 \le m \le 1$. Then if $r' \le 3$, x_{n+1} satisfies (8). If $3 \le r' \le 8$, then x_{n+6} satisfies (8). If $8 \le r' \le 16$, then x_{n+2}

satisfies (8). If $16 \leq r' \leq 17$, then either x_{n+8} or x_{n+7} satisfies (8), depending on whether $r < .5$ or $r \geq .5$. If $17 < r' < 30$, then x_n satisfies (2). If $r' \geq 30$, then x_{n+1} satisfies (2).

<u>Case</u> II. Suppose $c_1 \geq 0$, $c_2 \leq 0$, and $c_1^2 + c_2^2 \neq 0$. Then we can write

$$y = \sqrt{c_1^2 + c_2^2}\cos(x^2 - \alpha) + (2 + 1/x) + (2 - 1/x)\cos 4x,$$

$\pi/2 \leq \alpha \leq \pi$. If we let $-\epsilon = \cos 9\pi/16$, the same reasoning as in Case I reduces our problem to showing the existence of arbitrarily large integers n satisfying

$$9 + 32k < (2n+1)^2\pi - m < 23 + 32k,$$

where k is an integer and m is a number such that $8 \leq m \leq 16$. As in Case I, it can be verified that for each value of m, some x_i, $i = n, n+1, \ldots, n+8$, satisfies the above inequality.

<u>Case</u> III. Suppose $c_1 \leq 0$, $c_2 \leq 0$, and $c_1^2 + c_2^2 \neq 0$. Then we can write

$$y = -\sqrt{c_1^2 + c_2^2}\sin(x^2 + \alpha) + (2 + 1/x) + (2 - 1/x)\cos 4x,$$

$0 \leq \alpha \leq \pi/2$. If we let $\epsilon = \sin \pi/16$, our problem reduces to showing the existence of arbitrarily large integers n satisfying

$$1 + 32k < (2n+1)^2\pi + m < 15 + 32k,$$

where k is an integer and m is a number such that $0 \leq m \leq 8$.

Again, it can be verified that for each value of m, some x_i, $i = n, n+1, \ldots, n+8$, satisfies the above inequality.

Case IV. Suppose $c_1 \leq 0$, $c_2 \geq 0$, and $c_1^2 + c_2^2 \neq 0$. Then y can be written as

$$y = -\sqrt{c_1^2 + c_2^2} \cos (x^2 - \alpha) + (2 + 1/x) + (2 - 1/x)\cos 4x,$$

$\pi/2 \leq \alpha \leq \pi$. Let $\epsilon = \cos 7\pi/16$. It suffices to show that $-\cos(x^2 - \alpha) < -\epsilon$. It suffices to show the existence of arbitrarily large integers n satisfying one of the inequality

$$32k < (2n+1)^2\pi - m < 7 + 32k$$

or

$$25 + 32k < (2n+1)^2\pi - m < 32 + 32k,$$

where k is an integer and m is a number such that $8 \leq m \leq 16$. Again, by considering values of m in subintervals of length one, it can be verified that for each value of m, one of the x_i's satisfies one of the above two inequalities.

CHAPTER III

SOME OSCILLATION PROPERTIES OF FOURTH ORDER ORDINARY LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS

Consider the differential equation

$$y^{IV} = p(x)y'' + q(x)y' + r(x)y, \tag{4}$$

where $p(x)$, $q(x)$, and $r(x)$ are continuous on $[a,\infty)$ for some fixed real number a. We will show, in the following two theorems which insure the boundedness of oscillatory solutions, and use a technique of getting three particular independent solutions of (4) to form a fundamental set for a third linear differential equation. We first prove the following lemma.

<u>Lemma</u> 3.1: Let $p > 0$, $q > 0$, $r > 0$, p, q and $r \varepsilon C^1[a,\infty)$, $p' \leq 0$, $q' \leq 0$, and $r' \geq 0$. Then, if $y(x)$ is any solution of (4) with $y'(b) = y'(c) = 0$ where $a \leq b < c$,

$$\max_{x \varepsilon [b,c]} [y(x)]^2 \leq [y(b)]^2 + \frac{M[y(b)]}{r(b)},$$

where

$$M[y(x)] = r(x)y^2(x) + p(x)y'^2(x) + 2q(x) \int_a^x y'^2(s)ds + y''^2(x) - 2y'(x)y'''(x).$$

Proof: Define

$$M[y(x)] = r(x)y^2(x) + p(x)y'^2(x) + 2q(x) \int_a^x y'^2(s)ds + y''^2(x)$$

$$- 2y'(x)y'''(x). \tag{9}$$

It can be verified by differentiation that

$$M[y(x)] = M[y(a)] + \int_a^x r'(s)y^2(s)ds + \int_a^x p'(s)y'^2(s)ds$$

$$+ \int_a^x \left[2q'(s) \int_a^s y'^2(t)dt \right] ds.$$

If $\max_{x \varepsilon [b,c]} [y(x)]^2 = [y(\overline{x})]^2$, $\overline{x} \varepsilon [b,c]$, then $y'(\overline{x}) = 0$. If $\overline{x} = b$, then

$$\max_{x \varepsilon [b,c]} [y(x)]^2 = y^2(b). \tag{10}$$

If $b < \overline{x}$, then

$$M[y(\overline{x})] = M[y(b)] + \int_b^{\overline{x}} r'(s)y^2(s)ds + \int_b^{\overline{x}} p'(s)y'^2(s)ds$$

$$+ \int_b^{\overline{x}} \left[2q'(s) \int_a^s y'^2(r)dr \right] ds$$

$$\leq M[y(b)] + \int_b^{\overline{x}} r'(s)y^2(s)ds$$

$$\leq M[y(b)] + y^2(\overline{x})[r(\overline{x}) - r(b)].$$

From (9) and the fact that $y'(\overline{x}) = 0$, we have

$$M[y(\bar{x})] = r(\bar{x})y^2(\bar{x}) + 2q(\bar{x}) \int_b^{\bar{x}} y'^2(s)ds + y''^2(\bar{x}).$$

Therefore,

$$r(\bar{x})y^2(\bar{x}) + 2q(\bar{x}) \int_b^{\bar{x}} y'^2(s)ds + y''^2(x)$$

$$\leq M[y(b)] + r(\bar{x})y^2(\bar{x}) - r(b)y^2(\bar{x})$$

or

$$r(b)y^2(\bar{x}) + 2q(\bar{x}) \int_b^{\bar{x}} y'^2(s)ds + y''^2(\bar{x}) \leq M[y(b)].$$

Hence,

$$r(b)y^2(\bar{x}) \leq M[y(b)]$$

since $q(x) \geq 0$ or

$$y^2(\bar{x}) \leq \frac{M[y(b)]}{r(b)} . \tag{11}$$

Therefore, combining (10) and (11)

$$\max_{x \in [b,c]} [y(x)]^2 \leq y^2(b) + \frac{M[y(b)]}{r(b)} .$$

Hence, the lemma is proved.

As a direct consequence of Lemma 3.1, we can state the following theorem.

Theorem 3.2: Assume the hypothesis for Lemma 3.1. Then all oscillatory solutions of (4), if there are any, are bounded.

The next theorem gives also a sufficient condition for all oscillatory solutions of equation (4) to be bounded.

<u>Theorem</u> 3.3: Let $p,q,r,p',r' \in C(a,\infty)$, $p \geq 0$, $r > 0$, $r'(t) \geq 0$, and $p'(t) - 2q(t) < 0$. Then all oscillatory solutions of (4), if there are any, are bounded.

Proof: Let

$$G[y(x)] = r(x)y^2(x) - 2y'(x)y'''(x) + y''^2(x) + p(x)y'(x)^2. \qquad (12)$$

It can be verified by differentiation that

$$G[y(x)] = G[y(a)] + \int_a^x r'(s)y(s)^2 ds$$

$$- \int_a^t (2q(s) - p'(s))y'^2(s)ds. \qquad (13)$$

Let x_1 and x_2 be zeros of $y'(x)$ where $y(x)$ is an oscillatory solution of (4).

If $\max_{x \in [x_1,x_2]} [y^2(x)] = y^2(\bar{x})$, $\bar{x} \in [x_1,x_2]$, then $y'(\bar{x}) = 0$. If $x_1 = \bar{x}$, then $\max_{x \in [x_1,x_2]} [y^2(x)] = y^2(x_1)$. If $x_1 < x$ then using (13),

$$G[y(\bar{x})] = G[y(x_1)] + \int_{x_1}^{\bar{x}} r'(s)y^2(s)ds - \int_{x_1}^{\bar{x}} (2q(s) - p'(s))y'^2(s)ds$$

$$\leq G[y(x_1)] + \int_{x_1}^{\bar{x}} r'(s)y^2(s)ds$$

$$\leq G[y(x_1)] + y^2(\bar{x})[r(\bar{x}) - r(x_1)].$$

From (12) and the fact that $y'(\bar{x}) = 0$

$$r(\bar{x})y^2(\bar{x}) + [y''(\bar{x})]^2 \leq G[y(x_1)] + r(\bar{x})y^2(\bar{x}) - r(x_1)y^2(\bar{x})$$

or

$$r(x_1)y^2(\bar{x}) + y''^2(\bar{x}) \leq G[y(x_1)].$$

Hence,

$$y^2(\bar{x}) \leq \frac{G[y(x_1)]}{r(x_1)} .$$

Therefore, in both cases

$$\max_{x \in [x_1,x_2]} [y^2(x)] = y^2(\bar{x}) \leq y^2(x_1) + \frac{G[y(x_1)]}{r(x_1)} .$$

Hence, it follows that any oscillatory solution is bounded.

We finally illustrate the technique used to obtain a third order differential equation whose fundamental set forms a set of linearly independent solutions of (4) for the case $p > 0$, $q < 0$ and $r > 0$ in $[a,\infty)$.

Consider the fundamental set of solutions of (4), y_0, y_1, y_2, and y_3 such that

$$W(y_0,y_1,y_2,y_3)(a) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

We want to show that y_0, y_1, and y_2 are solutions of a third order differential equation of the form

$$y''' = py'' + qy' + ry, \tag{2)''}$$

where $p < 0$, $q > 0$, and $r < 0$. Hence, any properties about the solutions of equation (2)'' are also properties of equation (4). First, we show that $W(y_0,y_1,y_2)(x) \neq 0$ for $x > a$. Suppose $W(y_0,y_1,y_2)(b) = 0$ where $b > a$. Then, there exist constants c_0, c_1, and c_2, not all zeros, such that

$$c_0y_0(b) + c_1y_1(b) + c_2y_2(b) = 0$$

$$c_0y_0'(b) + c_1y_1'(b) + c_2y_2'(b) = 0$$

$$c_0y_0''(b) + c_1y_1''(b) + c_2y_2''(b) = 0.$$

Let $z(x)$ be the solution of (4) given by

$$z(x) = c_0y_0(x) + c_1y_1(x) + c_2y_2(x).$$

The solution $z(x)$ is a nontrivial solution of (4) since $z(a) = c_0$, $z'(a) = c_1$, $z''(a) = c_2$, and c_0, c_1, and c_2 are not all equal to zero.

We can assume that $z'''(b) < 0$.

Then Lemma 1.13 implies $z''(x) > 0$ for all $x \in [a,b)$. But $z''(a) = 0$, and this is a contradiction. Hence, $W(y_0,y_1,y_2)(x) \neq 0$ for all $x > a$. Since $W(y_0,y_1,y_2)(a) = 1$, then it follows that $W(y_0,y_1,y_2)(x) > 0$ for $x \geq a$. Hence, y_0, y_1, and y_2 are solutions of the third order differential equation given by

$$L[y] = \frac{\begin{vmatrix} y_0 & y_1 & y_2 & y \\ y_0' & y_1' & y_2' & y' \\ y_0'' & y_1'' & y_2'' & y'' \\ y_0''' & y_1''' & y_2''' & y''' \end{vmatrix}}{W(y_0,y_1,y_2)} = 0$$

or after expansion of the determinant

$$L[y] = y''' - \frac{\begin{vmatrix} y_0 & y_1 & y_2 \\ y_0' & y_1' & y_2' \\ y_0''' & y_1''' & y_2''' \end{vmatrix}}{W(y_0,y_1,y_2)} y'' + \frac{\begin{vmatrix} y_0 & y_1 & y_2 \\ y_0'' & y_1'' & y_2'' \\ y_0''' & y_1''' & y_2''' \end{vmatrix}}{W(y_0,y_1,y_2)} y'$$

$$- \frac{\begin{vmatrix} y_0' & y_1' & y_2' \\ y_0'' & y_1'' & y_2'' \\ y_0''' & y_1''' & y_2''' \end{vmatrix}}{W(y_0,y_1,y_2)} y = 0.$$

Let

$$\alpha = \begin{vmatrix} y_0 & y_1 & y_2 \\ y_0' & y_1' & y_2' \\ y_0''' & y_1''' & y_2''' \end{vmatrix}$$

$$\beta = \begin{vmatrix} y_0 & y_1 & y_2 \\ y_0'' & y_1'' & y_2'' \\ y_0''' & y_1''' & y_2''' \end{vmatrix}$$

$$\gamma = \begin{vmatrix} y_0' & y_1' & y_2' \\ y_0'' & y_1'' & y_2'' \\ y_0''' & y_1''' & y_2''' \end{vmatrix}.$$

$$\alpha' = \begin{vmatrix} y_0 & y_1 & y_2 \\ y_0'' & y_1'' & y_2'' \\ y_0''' & y_1''' & y_2''' \end{vmatrix} + \begin{vmatrix} y_0 & y_1 & y_2 \\ y_0' & y_1' & y_2' \\ py_0'' & py_1'' & py_2'' \end{vmatrix} = m + n,$$

where

$$n = \begin{vmatrix} y_0 & y_1 & y_2 \\ y_0' & y_1' & y_2' \\ py_0'' & py_1'' & py_2'' \end{vmatrix} < 0$$

and

$$m = \begin{vmatrix} y_0 & y_1 & y_2 \\ y_0'' & y_1'' & y_2'' \\ y_0''' & y_1''' & y_2''' \end{vmatrix} .$$

$$m' = \begin{vmatrix} y_0' & y_1' & y_2' \\ y_0'' & y_1'' & y_2'' \\ y_0''' & y_1''' & y_2''' \end{vmatrix} + \begin{vmatrix} y_0 & y_1 & y_2 \\ y_0'' & y_1'' & y_2'' \\ qy_0' & qy_1' & qy_2' \end{vmatrix} = m_1 + n_1,$$

where

$$n_1 = \begin{vmatrix} y_0 & y_1 & y_2 \\ y_0'' & y_1'' & y_2'' \\ qy_0' & qy_1' & qy_2' \end{vmatrix} < 0$$

$$m_1 = \begin{vmatrix} y_0' & y_1' & y_2' \\ y_0'' & y_1'' & y_2'' \\ y_0''' & y_1''' & y_2''' \end{vmatrix} .$$

$$m_1' = \begin{vmatrix} y_0' & y_1' & y_2' \\ y_0'' & y_1'' & y_2'' \\ ry_0 & ry_1 & ry_2 \end{vmatrix} = r \begin{vmatrix} y_0 & y_1 & y_2 \\ y_0' & y_1' & y_2' \\ y_0'' & y_1'' & y_2'' \end{vmatrix} < 0.$$

Since $m_1(a) = 0$ then $m_1 < 0$. Hence, $m' < 0$. But $m(a) = 0$. Therefore, $m < 0$. Hence, $\alpha' < 0$. But $\alpha(a) = 0$. Therefore, $\alpha < 0$.

Consider

$$\beta = \begin{vmatrix} y_0 & y_1 & y_2 \\ y_0'' & y_1'' & y_2'' \\ y_0''' & y_1''' & y_2''' \end{vmatrix} .$$

$\beta = m$. By previous argument $m < 0$. Hence, $\beta < 0$. Finally, $\gamma = m_1$, and thus, $m_1 < 0$. Therefore, y_0, y_1, and y_2 are solutions of the differential equation

$$y''' = py'' + qy' + ry$$

with $p < 0$, $q > 0$, and $r < 0$.

CHAPTER IV

SUMMARY AND CONCLUSIONS

The purpose of this paper is the study of some oscillatory properties of third and fourth order ordinary homogeneous differential equations.

Chapter II discusses the third order differential equations. It contains an example of a differential equation having property RN, but its adjoint does not have property RO. One theorem gives a sufficient condition under which the differential equation has property RO. Also, one case where the dimension of the oscillatory subspace is at least two in the oscillatory case is given.

Chapter III discusses the fourth order linear differential equation, and contains two theorems which insure the boundedness of the oscillatory solutions. An example illustrates the case where three particular solutions of a fourth order differential equation form a fundamental set of a third order differential equation,

$$y''' = py'' + qy' + ry,$$

where $p < 0$, $q > 0$, and $r < 0$.

There are questions suggested by this thesis. One might attempt a necessary condition for RN to imply RO. Also, one can examine the case of RO and RN properties of $y''' + py' + qy = 0$ when

$p < 0$ and $q > 0$. Finally, it appears possible to extend the RO and RN properties to the fourth order differential equations.

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