SYMMETRIC EXCHANGE GRAPHS OF POLYMATROIDS

By

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SYMMETRIC EXCHANGE GRAPHS

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Abstract: White has conjectured that the toric ideal of a matroid is generated by quadric binomials corresponding to symmetric basis exchanges. Herzog and Hibi made a similar conjecture for discrete polymatroids. In this paper we study symmetric exchange graphs of polymatroids. We show that for polymatroids on at most seven variables, the degree two part of the associated toric ideal is generated by symmetric exchange binomials.

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CHAPTER 1

Introduction

Polymatroids are generalizations of matroids that allow us to ask the interesting questions from matroid theory in the language of monomials. Specifically, one assigns to each basis B in a polymatroid a variable Y_B in some polynomial ring and each ground element e of the polymatroid is also assigned a variable x_e in a different polynomial ring. Define a map between these polynomial rings by $Y_B \mapsto \prod_{e \in B} x_e$ and then extend this linearly to a homomorphism between these polynomial rings. In *A unique exchange property for bases*, Neil White conjectured that the kernel of this map is generated by symmetric exchange binomials.

Graph theoretic methods have proved useful in studying this conjecture. Both Schweig and Blasiak were able to use graphs to show that the kernel of this map was generated by degree two binomials and that these binomials related to symmetric exchanges in the case of graphic matroids and lattice path polymatroids.

In this paper we study a graph related to symmetric exchanges in polymatroids and establish some connectivity results. Although these connectivity results do not prove White's conjecture, they can reduce the problem to determining if the toric ideal is generated in degree 2. In this paper we show that the degree 2 part of the toric ideal of a polymatroid on at most 7 variables is generated by symmetric exchange binomials. Along the way we develop theory behind the structures that appear in this graph.

CHAPTER 2

Background Material

2.1 Matroids

Matroid theory formalizes the notion of independence in a combinatorial way. The subject was pioneered by Whitney as he was attempting to describe properties of dependence that were common to graphs and Matrices [Oxley]. We recall some of the basic definitions.

Let E be a finite set and I be a collection of subsets of E. Then we say the pair (E, I) is a **matroid** if the following conditions hold:

(I1) $\emptyset \in I$.

(I2) If $A \in I$ and $B \subset A$ then $B \in I$.

(I3) If A and B are members of I and $|B| < |A|$, then there exists $a \in A - B$ such that $B \cup \{a\}$ is a member of I.

The members of I are called **independent sets**. As an initial example of a matroid consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ with column vectors $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $e_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $e_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. There is a natural matroid associated to A where E is taken to be $\{e_1, e_2, e_3, e_4\}$ and I is the collection of all linearly independent subsets of E. Then $M(A) = (E, I)$ is a matroid and the independent sets are the elements of

$$I = \big\{\{e_1, e_2\}, \{e_2, e_3\}, \{e_3, e_1\}, \{e_1\}, \{e_2\}, \{e_3\}, \emptyset\big\}.$$

Notice that maximal members of I, ordered under inclusion, have the same cardinality. Indeed for any matroid its maximal sets have the same cardinality, and we call a maximal member of I a **basis**. This terminology seems suspiciously suggestive but for good reason. In the previous example the sets $\{e_1, e_2\}, \{e_2, e_3\}, \{e_3, e_1\}$ each form a basis for the column space of A.

Property (I2) says that matroids are closed under set inclusion, hence a matroid is completely determined from its bases. Property (I3) captures a familiar phenomenon in linear algebra. Recall that given two sets of linearly independent vectors A and B, such that dim(span(A)) is larger than dim(span(B)) then we can add some vector v from A to B to increase the dimension of span(B). For this reason ($I3$) is often called the exchange axiom. Since E is finite we see immediately that each element of I is contained in a maximal element, or equally each independent set is contained in a basis.

The subsets of E that are not elements of I are called **dependent sets**. So in our earlier example the dependent sets of $M(A)$ are those subsets of E that contain $\{e_4\}$ or have more than 2 elements. The minimally dependent subsets of a matroid are called **circuits** and every dependent set contains a circuit. If B is a basis then for any e which is not in B we know that $B \cup \{e\}$ is dependent since B is maximal. Therefore $B \cup \{e\}$ contains a circuit. Furthermore this circuit is unique, contains e and is usually denoted by $C(B, e)$. The circuit $C(B, e)$ is called the *fundamental circuit of e with respect to B*. Circuits are fundamental structures in a matroid and in fact a matroid is completely determined by its circuits. ([Oxley]) Many examples of matroids arise naturally when considering graphs, which will be discussed in the next section.

Notice that if B_1 and B_2 are distinct bases of a matroid (E, I) then for any e in $B_1 - B_2$ we have $B_1 - \{e\}$ is independent and $|B_1 - \{e\}| < |B_2|$. Therefore property ($I3$) implies the existence of an element f in $B_2 - \big(B_1 - \{e\}\big)$ such that $\big(B_1 - \{e\}\big) \cup \{f\}$ is in I. Evidently $\big(B_1 - \{e\}\big) \cup \{f\}$ is maximal and is therefore a basis for (E, I). This is more precisely stated by the following lemma.

Lemma 2.1.1 *([Oxley], Lemma 1.2.2)*
If B_1 and B_2 are bases of a matroid and $x \in B_1 - B_2$, then there is an element y of $B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\}$ is a basis.

The previous lemma is an example of an exchange property satisfied by the bases of a matroid. The next result shows that the bases of a matroid satisfy a stronger property called *symmetric exchange.*

Theorem 2.1.1 *(Symmetric Exchange, [Brualdi], Theorem 2)*
Let B_1 and B_2 be distinct bases for a matroid. For each e in $B_1 - B_2$ there exists f in $B_2 - B_1$ such that $\big(B_1 - \{e\}\big) \cup \{f\}$ and $\big(B_2 - \{f\}\big) \cup \{e\}$ are both bases.

For fixed bases of a matroid B_1 and B_2 and a given element e in B_1, the possible choices for

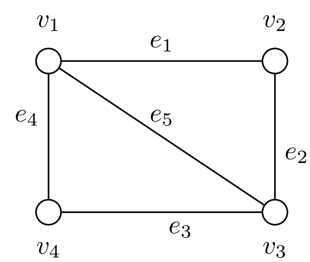
f satisfying the conclusion of the symmetric exchange theorem are known to be the members of $C(B_2, e)$. ([Brualdi])

Lemma 2.1.2 *([Brualdi], Lemma 1)*
Let B be a basis for a matroid and $e \in B^c$. Then $B \cup \{e\}$ contains a unique circuit $C(B, e)$. This circuit contains e , and moreover for $f \in B$, $(B - \{f\}) \cup \{e\}$ is a basis if and only if $f \in C(B, e)$.

The symmetric exchange property allows us to associate to any matroid M a graph $G(M)$ which captures some properties of the toric ideal associated to M. We will discuss the toric ideal in a later section but for now we'll review some graph theory.

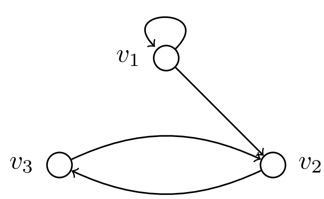
2.2 Graphs

A **graph** is a finite set V of **vertices** and a set E of edges each joining two vertices.



A graph with $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$.

Formally the edges are pairs of vertices. So in the diagram above $e_1 = (v_1, v_2)$, $e_5 = (v_1, v_3)$ and so on. The two ends of an undirected edge may be written in either order. For example $e_1 = (v_2, v_1)$ and (v_1, v_2). Occasionally we would like to create graphs that assign a direction to each edge, in which case we would distinguish between the order of the ends of an edge.



A Graph with $V = \{v_1, v_2, v_3\}$ and
$E = \{(v_1, v_1), (v_1, v_2), (v_2, v_3), (v_3, v_2)\}$

In the above example the edge (v_2, v_3) is identified by the arrow arrow that begins at v_2 and points to v_3. The edge (v_3, v_2) is the arrow that starts at v_3 and points to v_2. When this distinction between the direction of edges is made we say the edges are **directed edges**. In a **directed graph** all edges are directed. Note that in both a directed graph and an undirected graph we allow the same vertex to represent both ends of an edge. Such edges are called **loops**, so in the previous example (v_1, v_1) is a loop.

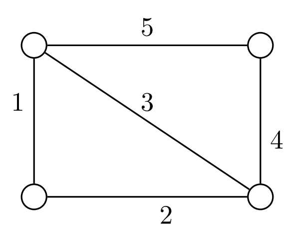
When two vertices are joined by an edge we say these vertices are **adjacent**. So in the example above v_1 and v_2 are adjacent, however v_1 and v_3 are not. If every pair of vertices of a graph G is adjacent then G is called a **complete graph** and is written K_n if it has n vertices. Of interest to us will be sequences of distinct vertices $v_1, v_2, ..., v_n$ such that v_{i+1} is adjacent to v_i . Such sequences of vertices are called **paths** and will be written as $v_1 - v_2 - \cdots - v_n$. If in addition there is an edge (v_n, v_1) the sequence is called a **cycle** and is written as $v_1 - v_2 - \cdots - v_n - v_1$. The number of vertices adjacent to a vertex, v, is called the **degree** of the vertex and is written $\deg(v)$. The length of a path, or a cycle, is the number of edges in the path or the cycle respectively. A **chord** of the cycle $v_1 - v_2 - \cdots - v_n - v_1$ is an edge (v_i, v_j) where $j \neq i$ and $j \neq i \pm 1 \pmod{n}$.

Theorem 2.2.1

In any graph, the sum of the degrees of all vertices is equal to twice the number of edges.

Given any graph G one can form a matroid $M(G) = (E, I)$ by taking E to be the edges of G and I to be the subsets of edges that do not contain a cycle.

As an example of this, in the graph below we would take $E = \{1, 2, 3, 4, 5\}$.



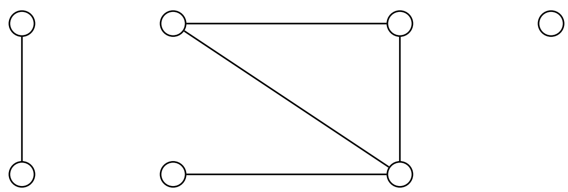
The cycles of this graph are $\{1, 2, 3\}, \{3, 4, 5\}, \{1, 2, 4, 5\}$ and the bases of I will be those maximal subsets of edges that don't contain any of these cycles. So the bases of $M(G)$ are

$$\{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}.$$

Those familiar with graph theory will recognize the bases as the spanning trees of the graph. Since graphs are not always connected, in the general cases the bases of this matroid will be the spanning forests of the graph.

The most important property in graph theory concerning us is *connectivity*. The idea is intuitive enough but the definition differs slightly depending on whether or not the graph is directed. An undirected graph is **connected** if there is a path between every pair of vertices. If in a directed graph we replace each edge with an undirected edge and the resulting graph is connected, we say the original directed graph is **weakly connected**. A directed graph is connected if for any pair of distinct vertices u and v there either exists a directed path from u to v or there exists a directed path from v to u. When discussing directed graphs we will use the phrase connected when we mean weakly connected.

If a graph is not connected then it can be separated into connected pieces that are called **components**.



A graph with 3 components.

Notice that if a graph consists of two components which are each complete graphs, then drawing a new edge between any distinct vertices will form a connected graph.

Lemma 2.2.1 *([Tucker])*

Let G be a graph with n vertices. If the number of edges is greater than $\frac{1}{2}(n-1)(n-2)$ then G is connected.

Lemma 2.2.2

Let G be a graph without loops such that the degree of each vertex is at least k. If G is disconnected then G has at least $2k+2$ vertices.

Proof.

Let x be a vertex of G. Then x is contained in some component C_x of G. The component C_x must contain every vertex that is adjacent to x. Since $\deg(x) \geq k$ and there are no loops incident to x then this component must have at least $k+1$ vertices. Now if G is disconnected there must be a vertex y which is not in C_x. This vertex is itself contained in a component C_y and by similar reasoning we can conclude that C_y contains at least $k+1$ vertices. Hence $|G| \geq |C_x| + |C_y| = 2(k+1)$, where $|G|$ represents the number of vertices in G.

■

Theorem 2.2.2 *([Pach, Agarwal] , Theorem 9.1)*

Let G be a connected graph with $n \geq 3$ vertices such that $\deg(x) + \deg(y) \geq r$ for every pair of nonadjacent vertices x and y.

(i) If $r = n$, then G has a circuit that passes through each of its vertices.

(ii) If $r < n$, then G contains a path of length r and G contains a cycle of length $\lceil (r+2)/2 \rceil$.

CHAPTER 3

Polymatroids

3.1 Definitions and Notation

Let $S = \mathbb{C}[x_1, ..., x_n]$. If m is a monomial in S we define $\deg_{x_i}(m)$ to be the largest non-negative integer α such that $x_i^\alpha \Big| m$. The total degree of a monomial m, written as $\deg(m)$, is defined to be $\sum_i \deg_{x_i}(m)$.

A **discrete polymatroid** Γ is a finite collection of monic monomials in S satisfying the following properties.

1. If $m \in \Gamma$ and $p|m$ then $p \in \Gamma$.
2. If $m, p \in \Gamma$ and $\deg(m) > \deg(p)$ then there exists an index i such that $\deg_{x_i}(m) > \deg_{x_i}(p)$ and $x_i p \in \Gamma$.

Notice that a matroid is just a squarefree polymatroid. Analogous to the case of matroids, the monomials in Γ will be called independent and the **bases** will be the maximal elements of Γ ordered by divisibility.

Lemma 3.1.1 *(Symmetric Exchange)*
Let m and m' be bases of the discrete polymatroid Γ. Let i be an index such that $\deg_{x_i}(m) > \deg_{x_i}(m')$. Then there exists an index j such that $\deg_{x_j}(m') > \deg_{x_j}(m)$ and both $\frac{x_j}{x_i}m$ and $\frac{x_i}{x_j}m'$ are bases in Γ.

Given bases m and n and variables x_i and x_j satisfying Lemma 1, if $m' = \frac{x_j}{x_i}m$ and $n' = \frac{x_i}{x_j}n$ then we will say that the pair (m', n') can be obtained from the pair (m, n) by a symmetric exchange.

3.2 Toric Map and Toric Ideal

Let Γ be a polymatroid with $\{x_1, ..., x_n\}$ as its ground set and let $\mathbb{B}$ be the collection of all bases of Γ. We can use $\mathbb{B}$ to construct a polynomial ring $\mathbb{C}[Y_{B_1}, ..., Y_{B_r}]$ where each basis B_i is assigned a variable. The map $\phi : \mathbb{C}[Y_{B_1}, ..., Y_{B_r}] \longrightarrow \mathbb{C}[x_1, ..., x_n]$ defined by $\phi(Y_{B_i}) = \prod_{x \in B_i} x$ and extended naturally to a ring homomorphism on $\mathbb{C}[Y_{B_1}, ..., Y_{B_r}]$ is called the **toric map** defined by $\mathbb{B}$.

Heuristically the toric map takes a basis and multiplies everything inside the basis together to form a monomial. Extending naturally just means we define products and sums to be preserved by this map.

The kernel of the toric map defined by $\mathbb{B}$ is called the **toric ideal** of Γ. The ring $\mathbb{C}[Y_{B_1}, ..., Y_{B_r}]$ is called the base ring associated to Γ. Given bases A_1, B_1, A_2, and B_2 in Γ, if (A_2, B_2) can be obtained from (A_1, B_1) by a symmetric exchange then we define $Y_{A_1}Y_{B_1} - Y_{A_2}Y_{B_2}$ to be a symmetric exchange binomial.

Suppose $\phi : \mathbb{C}[Y_{B_1}, ..., Y_{B_r}] \to \mathbb{C}[x_1, ..., x_n]$ is the toric map for the toric ideal of Γ. Then for any symmetric exchange binomial, $Y_{A_1}Y_{B_1} - Y_{A_2}Y_{B_2}$, we have that

$$\phi(Y_{A_1}Y_{B_1} - Y_{A_2}Y_{B_2}) = \left(\prod_{x \in A_1} x\right)\left(\prod_{y \in B_1} y\right) - \left(\prod_{x \in A_2} x\right)\left(\prod_{y \in B_2} y\right) = 0$$

since the multiset union of A_1 and B_1 is the same as the multiset union of A_2 and B_2. This shows that every symmetric exchange binomial lies in the toric ideal.

Conjecture 3.2.1 *([Herzog, Hibi])*
The toric ideal of Γ *is generated by symmetric exchange binomials.*

This conjecture can be viewed as two separate conjectures. One that says that the toric ideal is generated by quadratic binomials and the other that says that these binomials are related by symmetric exchanges. The symmetric exchange graph of some polymatroids has been used to show that the degree 2 part of the corresponding toric ideal is generated by symmetric exchange binomials. An example of this can be found in both Blasiak's paper and Schweig's paper. Therefore determining when the symmetric exchange graph of a polymatroid is connected can resolve the second part White's conjecture.

CHAPTER 4

The Symmetric Exchange Graphs of Polymatroids

Let Γ be a discrete polymatroid. We can define a directed graph $G(\Gamma)$ in the following way.

1. The vertex set V is the collection of unordered pairs of bases from Γ.

2. If $A, B, C, D \in \Gamma$ such that (C, D) can be obtained from (A, B) by a symmetric exchange, then we draw a directed edge from (A, B) to (C, D) in the graph.

The graph $G(\Gamma)$ is called the **symmetric exchange graph** of Γ. Let ϕ be the toric map for Γ. A **section** of Γ is defined to be the subgraph of $G(\Gamma)$ induced by the vertex set

$$S = \{(A, B) : A, B \in \mathbb{B} \text{ and } \phi(Y_A Y_B) = m \text{ for some fixed monomial } m\}.$$

Each edge of $G(\Gamma)$ relates to a particular symmetric exchange binomial. For example, an edge between (A, B) and (C, D) corresponds to the symmetric exchange binomial $Y_A Y_B - Y_C Y_D$.

Let M^d represent a degree d monomial. Each $p \in \ker(\phi)$ can be written as a sum of binomials in each degree.

$$p = \sum_{d=0}^{\deg(p)} \sum_{i=0}^{R_d} \left(M_i^d - N_i^d \right)$$

It follows that for each $d \in \{0, ..., \deg(p)\}$ we must have have $\sum_{i=0}^{R_d} \left(M_i^d - N_i^d \right) = 0$. Note that after regrouping we can assume that $M_i^d - N_i^d = 0$ for each i.

If we restrict consideration to the degree 2 part of p then we have a sum of quadratic binomials and one can ask whether these binomials can be rewritten as a sum of symmetric exchange binomials.

For each quadratic binomial satisfying $\phi(Y_A Y_B - Y_C Y_D) = 0$ we have corresponding section of $G(\Gamma)$ defined by $\phi(Y_A Y_B)$. Evidently (C, D) is in the same section of $G(\Gamma)$ as (A, B). If this section is connected then there exists a sequence of vertices (V_i, W_i) starting at (A, B) and ending at (C, D)

such that consecutively indexed vertices are adjacent. In this case we have ,

$$Y_AY_B - Y_CY_D = Y_{V_0}Y_{W_0} + (-Y_{V_1}Y_{W_1} + Y_{V_1}Y_{W_1}) + \cdots + (-Y_{V_{n-1}}Y_{W_{n-1}} + Y_{V_{n-1}}Y_{W_{n-1}}) - Y_{V_n}Y_{W_n}$$
$$Y_AY_B - Y_CY_D = (Y_{V_0}Y_{W_0} - Y_{V_1}Y_{W_1}) + (Y_{V_1}Y_{W_1} - Y_{V_2}Y_{W_2}) + \cdots + (Y_{V_{n-1}}Y_{W_{n-1}} - Y_{V_n}Y_{W_n})$$

In the last equality every binomial in parentheses is a symmetric exchange binomial. Therefore if we could determine conditions which make the sections of $G(\Gamma)$ connected, then the degree two part of the corresponding toric ideal is generated by symmetric exchange binomials.

Observation: The edges in $G(\Gamma)$ had to be defined to be directed.

To see this let Γ be the polymatroid in the ring $\mathbb{C}[a, b, c]$ with bases,

$$\{abc, abd, ac^2, acd, ad^2, b^2c, b^2d, bc^2, bcd, bd^2\}.$$

The pair (abc, bcd) can be derived from the pair (ac^2, b^2d) by the exchange $\frac{d}{c}ac^2$ and $\frac{c}{d}b^2d$. However the only exchange that Lemma 1 allows with the pair (abc, bcd) is, $\frac{d}{a}abc$ and $\frac{a}{d}bcd$ which results in the same pair. Notice that this situation doesn't arise in the matroid case since each basis has degree 1 in each variable dividing that basis.

4.1 Depth

We define a function on the vertices in the symmetric exchange graph of Γ.

Let (A, B) be a basis pair. The depth of (A, B) in the variable x_i is defined to be $F_{x_i}(A, B) = |\deg_{x_i}(A) - \deg_{x_i}(B)|$.

Define the depth function $F : \Gamma^2 \to \mathbb{N}$ by, $F(A, B) = \sum_{x_i} F_{x_i}(A, B)$.

This function makes a lot of properties easier to talk about. For example if $F_{x_i}(A, B) > 0$ then there exists a symmetric exchange between A and B involving x_i.

Lemma 4.1.1

Let (A, B) and (C, D) be vertices in $G(\Gamma)$ such that there is an edge from (A, B) to (C, D). Then *for each variable x_i we have $F_{x_i}(A,B) > F_{x_i}(C,D)$ or $F_{x_i}(A,B) = F_{x_i}(C,D)$. For each x_i the equality case occurs when either $F_{x_i}(A,B) = 0$, $F_{x_i}(A,B) = 1$, or x_i is not involved in the exchange inducing the edge.*

Proof.

Since there is an edge from (A,B) to (C,D) then there exist variables a and b such that $\deg_a(A) > \deg_a(B)$, $\deg_b(B) > \deg_b(A)$ and $\frac{b}{a}A = C$ and $\frac{a}{b}B = D$.

From these equalities we can conclude that if $x_i \neq a$ and $x_i \neq b$ then,

$$F_{x_i}\left(A,B\right) = F_{x_i}\left(\tfrac{b}{a}A, \tfrac{a}{b}B\right) = F_{x_i}(C,D).$$

Now for the variable a we have,

$$F_a(C,D) = F_a\left(\frac{b}{a}A, \frac{a}{b}B\right) = |(\deg_a(A) - 1) - (\deg_a(B) + 1)| = |\deg_a(A) - \deg_a(B) - 2|.$$

Since $\deg_a(A) > \deg_a(B)$ then either $\deg_a(A) \geq \deg_a(B) + 2$, or $\deg_a(A) = \deg_a(B) + 1$.

If $\deg_a(A) = \deg_a(B) + 1$ then,

$$F_a(C,D) = |\deg_a(A) - \deg_a(B) - 2| = |\deg_a(B) + 1 - \deg_a(B) - 2| = |-1| = 1.$$

Since $\deg_a(A) = \deg_a(B) + 1$, we know that $F_a(A,B) = 1$. Therefore $F_a(A,B) = F_a(C,D)$.

Now if $\deg_a(A) \geq \deg_a(B) + 2$ then,

$$F_a(C,D) = |\deg_a(A) - \deg_a(B) - 2| = \deg_a(A) - \deg_a(B) - 2 = F_a(A,B) - 2,$$

which says that $F_a(C,D) < F_a(A,B)$.

By an analogous argument we can prove the same results for $F_b(C,D)$.

□

Corollary 4.1.1

If (A,B) is a vertex in $G(\Gamma)$ and $F_{x_i}(A,B) > 2$ for some variable x_i, then (A,B) is connected to a vertex with smaller depth.

Proof.

Since $F_{x_i}(A,B)$ is positive then there exists a symmetric exchange between (A,B) involving x_i. Hence there is a vertex (C,D) such that x_i induces an edge from (A,B) to (C,D) via the aforementioned exchange. Since $F_{x_i}(A,B) > 2$ then Lemma 4.1.1 implies that $F_{x_i}(A,B) > F_{x_i}(C,D)$.

□

Theorem 4.1.1

The depth function is nonincreasing along directed paths in $G(\Gamma)$.

Proof.

Lemma 4.1.1 implies that F is the sum of nonincreasing functions. □

Theorem 4.1.2

Let Γ be a polymatroid and (A,B) be a vertex in $G(\Gamma)$. Suppose that (C,D) can be derived from (A,B) by the symmetric exchange $C = \frac{y}{x}A$ and $D = \frac{x}{y}B$. Then there is an edge from (C,D) to (A,B) if and only if $\deg_x(A) = \deg_x(B) + 1$ and $\deg_y(B) = \deg_y(A) + 1$.

Proof.

Since there is an edge from (A,B) to (C,D) then Lemma 4.1.1 implies $F_{x_i}(A,B) \geq F_{x_i}(C,D)$ for each variable x_i.

Now suppose there is also an edge from (C,D) to (A,B). Then for each x_i we have $F_{x_i}(C,D) \geq F_{x_i}(A,B)$, and therefore $F_{x_i}(A,B) = F_{x_i}(C,D)$. Since x is involved in the exchange that induces the edge from (A,B) to (C,D) then, and $F_x(A,B) = F_x(C,D)$ then the Lemma 4.1.1 says equality holds when $F_x(A,B) = 1$. Since $\deg_x(A) > \deg_x(B)$ then we can conclude that $\deg_x(A) = \deg_x(B) + 1$. A similar argument will show that $\deg_y(B) = \deg_y(A) + 1$. Thus we have shown one direction of the claim.

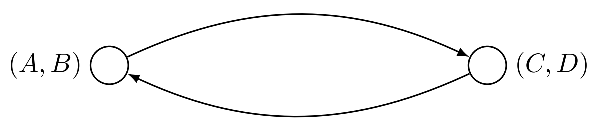
Now for the other direction we suppose $\deg_x(A) = \deg_x(B) + 1$ and $\deg_y(B) = \deg_y(A) + 1$. The equation $\deg_x(A) = \deg_x(B) + 1$ implies $\deg_x(C) + 1 = \deg_x(D)$ and similarly we have $\deg_y(C) = \deg_y(D) + 1$. Note that $\deg_y(C) > \deg_y(D)$ and $\deg_x(D) > \deg_x(C)$ and the exchange

$$\frac{x}{y}C = \frac{x}{y}\frac{y}{x}A = A \text{ and similarly } \frac{y}{x}D = \frac{y}{x}\frac{x}{y}B = B$$

results in a pair of bases in the same section as (C, D). So the exchange is valid and there is an edge from (C, D) to (A, B).

□

The above claim answers the question of when we can move backwards along a path in the exchange graph. So in the graph below we know that the pair (A, B) relates to the pair (C, D) by a symmetric exchange involving two variables x and y. Further we know that the degrees of A and B in these variables differ by 1.



Intuitively it's clear that we can create long paths by repeatedly applying the symmetric exchange lemma to most vertices. Since this lemma only guarantees the existence of an exchange, it may be the case that there are multiple exchange paths from a vertex. It is natural to ask what other types of graph structures may be present in the exchange graph. For instance are there any cycles? If they exist what is the structure of those cycles? etc.

A **bidirected cycle** in a digraph G is a directed cycle $(a_0 \to a_1 \to \cdots \to a_{n-1} \to a_0)$ such that $(a_0 \to a_{n-1} \to \cdots \to a_1 \to a_0)$ is also a directed cycle. A **polygonal cycle** in an exchange graph is a bidirected cycle that has no directed chords.

Theorem 4.1.3

For any polygonal cycle of length n there exists a polymatroid Γ such that $G(\Gamma)$ contains a section whose exchange graph is a length n polygonal cycle.

Before we prove this theorem we need to introduce some terminology.

Consider the family of nonempty sets $S_1, S_2, ..., S_n$. We allow the S_i to have different cardinalities and we allow the same set to appear at multiple indices. Form a multiset V by choosing exactly one element from each S_i. Let $\mathbb{B}$ be the collection of all possible multisets formed in this way. Let $\mathcal{B} = \{\prod_{x \in V} x : V \in \mathbb{B}\}$.

Theorem 4.1.4

$\mathcal{B}$ is the collection of bases for a polymatroid.

The polymatroid described above is called a **transversal polymatroid**. Recall that a bipartite graph is a graph whose vertex set can be partitioned into two disjoint vertex sets V_1 and V_2 where members of the same partition are not allowed to be adjacent. For any bipartite graph G we can form a transversal polymatroid. If one of the vertex sets is $\{v_1, ..., v_r\}$ we take S_i to be the set of vertices adjacent to v_i. This forms a nonempty family of subsets on which we can construct a transversal polymatroid.

Proof. (Theorem 4.1.3)
The proof proceeds by constructing a transversal polymatroid whose exchange graph contains a polygonal cycle of length n. Recall that for a transversal polymatroid we need a family of sets. Let the family of sets be $S_1 = \{x_1, y_1\}, S_2 = \{x_2, y_2\}, ..., S_n = \{x_n, y_n\}$. Let Γ be the transversal polymatroid for this family of sets. Let G be the exchange graph of Γ and consider the section of G corresponding to the multiset union $\{x_1, y_1, x_2, y_2, ..., x_n, y_n\}$.

The vertex $A = (x_1 x_2 \cdots x_n, y_1 y_2 \cdots y_n)$ is in this section of G. The bases are the 2^n size 4 sets containing exactly 1 member from each S_i. Therefore in this section of G a symmetric exchange always exists by interchanging x_i and y_i for a particular index i. So we can perform a sequence of symmetric exchanges by starting with A and interchanging x_1 with y_1, then x_2 with y_2, and so on until we finally interchange x_n, y_n and get back to A. The diagram below describes this particular cycle in this section.

$$
\begin{array}{r}
A = (x_1 x_2 \cdots x_n \ , \ y_1 y_2 \cdots y_n) \\
\updownarrow \\
(y_1 x_2 \cdots x_n \ , \ x_1 y_2 \cdots y_n) \\
\updownarrow \\
(y_1 y_2 \cdots x_n \ , \ x_1 x_2 \cdots y_n) \\
\vdots \\
(y_1 y_2 \cdots y_i x_{i+1} \cdots x_n \ , \ x_1 x_2 \cdots x_i y_{i+1} \cdots y_n) \\
\vdots \\
(y_1 y_2 \cdots y_{n-1} x_n \ , \ x_1 x_2 \cdots x_{n-1} y_n) \\
\updownarrow \\
A = (y_1 y_2 \cdots y_n \ , \ x_1 x_2 \cdots x_n)
\end{array}
$$

This cycle has length n and next we'll show that it contains no directed chords. Label the vertices in this sequence of symmetric exchanges by saying vertex A occurs at the 0^{th} step and the vertex after interchanging x_i with y_i is said to be at the i^{th} step.

Now suppose that there is a chord between a vertex P_i at the i^{th} step and a vertex, P_j, at the j^{th} step. Then the vertices P_i and P_j differ by a single symmetric exchange. Since the edge joining P_i and P_j is a chord we neccessarily have $n > j \geq i + 2$ and $i \geq 0$. The vertex P_i contains a basis that has the variables $\{y_1, ..., y_{i-1}, y_i\}$ and P_j contains a basis that has the variables $\{y_1, ..., y_{i-1}, y_i, y_{i+1}, y_{i+2}\}$ and possibly more. Hence P_i and P_j differ by at least two exchanges involving the y variables. This contradicts the existence of a chord joining P_i and P_j.

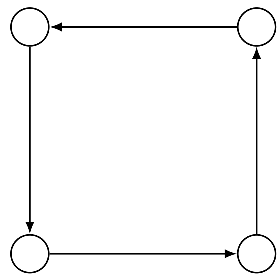
This shows that we can construct polygonal cycles of length n for $n \geq 3$.

$\square$

Theorem 4.1.5

An exchange graph of a polymatroid cannot contain directed cycles unless the directed cycle is a part of a bidirected cycle.

That is to say, structures like the graph below cannot exist.



Proof.

The proof is by contradiction. Suppose there exists a directed cycle of length $N+1$ with initial vertex (A_0, B_0). Then we can describe the cycle in terms of symmetric exchanges.

$$(A_0, B_0) \to (A_1, B_1) \to \cdots \to (A_{N-1}, B_{N-1}) \to (A_N, B_N) = (A_0, B_0)$$

Here $A_{i+1} = \dfrac{b_i}{a_i} A_i$ and $B_{i+1} = \dfrac{a_i}{b_i} B_i$ for each $i \in \{0, 1, ..., N-1\}$ where a_i and b_i are involved in the symmetric exchange for the pair (A_i, B_i) that induces the edge to (A_{i+1}, B_{i+1}). Since the depth function is nonincreasing along paths it follows that $F(A_i, B_i) = F(A_j, B_j)$ for each i and j in $\{0, 1, ..., N-1\}$, in particular for $j = i+1$. Using Lemma 4.1.1, either $F_{a_i}(A_i, B_i) = 0$ or $F_{a_i}(A_i, B_i) = 1$. We note however that $F_{a_i}(A_i, B_i) \neq 0$ since a_i is involved in a symmetric exchange between A_i and B_i, hence $F_{a_i}(A_i, B_i) = 1$. By an analogous argument $F_{b_i}(A_i, B_i) = 1$ as well.

Since both $F_{a_i}(A_i, B_i) = 1$ and $F_{b_i}(A_i, B_i) = 1$, and also $A_{i+1} = \dfrac{b_i}{a_i} A_i$ and $B_{i+1} = \dfrac{a_i}{b_i} B_i$ for each $i \in \{0, 1, ..., N-1\}$ then Theorem 4.1.2 says that there is a path from (A_{i+1}, B_{i+1}) to (A_i, B_i) for each $i \in \{0, 1, ..., N-1\}$. Therefore each edge in the path is bidirected.

□

The depth function has another straightforward property.

Lemma 4.1.2

Let Γ be a polymatroid and let S be a section in $G(\Gamma)$. Then the restriction of the depth function F to S achieves a minimum value.

Proof.

The set $X = \{F(A, B) : (A, B) \in S\}$ is a nonempty subset of $\mathbb{N}$ hence it has a smallest element. □

Often the most interesting properties in a structure occur in the extremal cases. Let S be a section in $G(\Gamma)$. We say the vertex (A, B) is a **presink** of S if the restriction of the depth function to the section S achieves a minimum value at (A, B). The idea is that the presinks of a section are the vertices that have minimum depth in this section. They are in a sense at the lowest level of the graph.

A **descending path** in $G(\Gamma)$ is a directed path in a section, such that the depth function's values are decreasing along consecutive vertices. A descending path **stabilizes** once the depth function achieves its minimum value in that section. We say that a descending path stabilizes at a vertex if it is the first vertex in the path for which the depth function achieves its minimum. Starting at any vertex that is not a presink, we can repeatedly apply Corollary 4.1.1 to construct a descending path.

There are a few situations in which a descending path could stabilize at a vertex (A, B). First there may be no possible exchanges between A and B and so the path terminates. Since there is always an exchange whenever the depth is positive, a descending path can only stabilize in this way if the depth function is 0 which occurs when $A = B$. We define a **terminal vertex** to be a vertex with 0 depth. Another way a path could stabilize at a vertex is if the only out edges for that vertex are loops. A **loop vertex** is a vertex such that a loop is incident at that vertex. It is also possible to possible to keep the depth constant if you follow a cycle or a path. Note that since these would be paths or cycles of presinks then Lemma 4.1.3 implies that the underlying vertices have the form $(x_1 \cdots x_k \cdot m\ , y_1 \cdots y_k \cdot m)$. This form together with Lemma 4.1.1 implies that these paths and cycles are bidirected. A summary of these results is presented after the following lemma.

Lemma 4.1.3

Let Γ be a polymatroid in $\mathbb{C}[x_1, ..., x_n]$ and let S be a section in $G(\Gamma)$. The presinks of S have the same form, $(x_1 \cdots x_k m\ , y_1 \cdots y_k m)$ where $k \leq \frac{n}{2}$.

Proof.

A vertex (A, B) in S can be written as $(x_1^{a_1} \cdots x_n^{a_n}, x_1^{b_1} \cdots x_n^{b_n})$.

Suppose $(x_1^{a_1} \cdots x_n^{a_n}, x_1^{b_1} \cdots x_n^{b_n})$ and $(x_1^{e_1} \cdots x_n^{e_n}, x_1^{f_1} \cdots x_n^{f_n})$ are distinct vertices. Since they are in the same section, then $a_i + b_i = e_i + f_i$. Therefore if $a_i + b_i \equiv 0 \mod 2$ then $e_i + f_i \equiv 0 \mod 2$. Thus for each variable x_i such that $F_{x_i}(A, B) \equiv 0 \mod 2$ for one vertex in the section, we must have that $F_{x_i} \equiv 0 \mod 2$ for all vertices in S.

Without loss of generality assume that the last $n - r$ of the variables $x_1, x_2, ..., x_n$ have the property that $F_{x_i} \equiv 0 \mod 2$ for all vertices in S.

Now let $(A,B) = (x_1^{a_1}\cdots x_n^{a_n}, x_1^{b_1}\cdots x_n^{b_n})$ be a presink in S. Since (A,B) is not connected to any vertices with smaller depth, Corollary 4.1.1 implies that $F_{x_i}(A,B) \le 1$ for each x_i. It follows that $F_{x_i}(A,B) = 0$ for each $i \in \{n-r,...,n-1,n\}$. Therefore (A,B) has the form $(x_1^{a_1}\cdots x_r^{a_r}m\ , x_1^{b_1}\cdots x_r^{b_r}m)$ where $m = x_{n-r}\cdots x_n$.

Suppose there are more indices for which $a_i > b_i$. Then $\sum_i a_i > \sum_i b_i$ which implies that A and B have different total degree, a contradiction. By symmetry of argument we can assume that the number of indices where $a_i > b_i$ is the same as the number of indices where $b_i > a_i$. Therefore $r = 2k$ for some positive integer k. After relabeling we may assume that the first k of the x_i satisfy the inequality that $a_i > b_i$. Then the presink (A,B) has the form $(x_1\cdots x_k m'\ , x_{k+1}\cdots x_{2k}m')$, where $m' = m\left(\prod_{i=1}^{k} x_i^{a_i-1}\right)\left(\prod_{i=k+1}^{2k} x_i^{b_i-1}\right)$.

□

Theorem 4.1.6 *(Structure of presinks)*

Let Γ be a polymatroid and let S be a section in $G(\Gamma)$. Each presink of S is one of the following,

1. *a terminal vertex and is of the form (m,m),*
2. *a loop vertex and is of the form $(x_i m, x_j m)$,*
3. *a vertex in a bidirected cycle or a bidirected path of presinks, and is of the form $(x_1\cdots x_k m\ , y_1\cdots y_n m')$. Here the x_i are distinct, the y_i are distinct and $x_i \neq y_j$ for each i and j. Also these variables x_i and y_j are precisely the variables involved in the symmetric exchanges defining the path or cycle respectively.*

Proof. First suppose (A,B) is a terminal vertex. Then the out degree of (A,B) is 0. If A were not equal to B then a symmetric exchange could be performed with these two bases, which would imply that the pair (A,B) has positive out degree. Since (A,B) does not have positive out degree then A must be equal to B.

Second suppose (A,B) is a loop vertex. Then there exists variables x and y with $x \neq y$ such that $\frac{y}{x}A = B$ and $\frac{x}{y}B = A$, in particular $xB = yA$. Since y divides B and x divides A then $B = ym$ and $A = xn$ for some monomials n and m. So the equation $xB = yA$ implies $xym = yxn$, therefore

$n = m$. This shows $(A, B) = (xm, ym)$.

The third case is a consequence of Lemma 4.1.3.

□

Corollary 4.1.2

Let S be a section in $G(\Gamma)$. Then every vertex in S is connected to a presink in S.

Proof.

For any vertex that is not a presink we may induce a descending path by repeated applications of Corollary 4.1.1. Since the depth is decreasing on such paths and achieves a minimum in S then this induced path must stabilize. □

Theorem 4.1.7

Suppose that S is a section in $G(\Gamma)$ that contains a terminal vertex or a loop vertex. Then S has a unique sink and is therefore weakly connected.

Proof.

It is clear that the statement holds when S has exactly one element. So from here on we'll assume that S has at least two elements.

S contains a presink (A, B) that is either a loop vertex or a terminal vertex.

Suppose (A, B) is a terminal vertex. Then Lemma 4.1.3 and Theorem 4.1.6 imply any other presink is also a terminal vertex. We also know that $B = A$ therefore if (C, C) is another presink we must have that $C^2 = A^2$ since these presinks are in the same section. But then $(A+C)(A-C) = 0$ and since $\mathbb{C}[x_1, ..., x_n]$ is an integral domain and A, C are both monic then $A = C$. Thus if there is a terminal vertex it is the unique presink.

Suppose (A, B) is a loop vertex. Just as before any other presink (E, F) must also be a loop. There exist variables a, b, e, f such that $(A, B) = (am, bm)$ and $(E, F) = (en, fn)$.

Since (E, F) and (A, B) are in the same section then $efn^2 = abm^2$. Also $\deg_a(n^2)$ is even and $\deg_a(abm^2)$ is odd therefore either $a = e$ or $a = f$.

Without loss of generality we may assume that $a = e$ implying $fn^2 = bm^2$. Since $\deg_b(n^2)$ is even and $\deg_b(bm^2)$ is odd then $b = f$. Hence $n^2 = m^2$ and therefore $(A, B) = (E, F)$. Thus if there is a loop vertex, then it is the unique presink.

In each of these cases there was a unique presink and since each vertex is connected to a presink it follows that S actually contains a unique sink and is therefore weakly connected. □

4.2 Subgraph of Presinks

If there was any candidate for a structure that would keep a section of $G(\Gamma)$ connected, it would be that section's subgraph of presinks.

Let Γ be a polymatroid and let S be a section in $G(\Gamma)$. We define the **subgraph of presinks** P_S to be the graph whose vertices are the presinks of S. An edge exists between two vertices in P_S if and only if it exists in S.

Lemma 4.2.1

Let S be a section in $G(\Gamma)$ and let P_S be its subgraph of presinks. Then the maximum number of vertices P_S can have is $\frac{1}{2}\binom{2k}{k}$ where $2k$ is the depth of S.

Proof.

Recall that Lemma 4.1.3 says the presinks of S have the same form, $(x_1 \cdots x_k m \ , y_1 \cdots y_k m)$. An upper bound on the number of possible presinks is to count the number of size k subsets from the set $\{x_1, y_1, x_2, y_2, ..., x_k, y_k\}$. There are $\binom{2k}{k}$ such subsets. Notice that each size k subset V of the set $\{x_1, y_1, x_2, y_2, ..., x_k, y_k\}$ would appear in a vertex of the form $\left(m \prod_{x \in V} x, \ m \prod_{x \in V^c} x\right)$. Therefore each vertex is counted twice, so there are at most $\frac{1}{2}\binom{2k}{k}$. □

CHAPTER 5

Main Results

Theorem 5.0.1

Let Γ be a polymatroid in $\mathbb{C}[x_1, ..., x_n]$ and let S be a section in $G(\Gamma)$. If S contains at most 3 presinks then S is weakly connected.

Proof.

If S has one presink then S is connected since every vertex is connected to a presink.

If S has 2 or 3 presinks then Theorem 4.1.7 implies that none of these presinks is a loop vertex or a terminal vertex.

Therefore they are each connected to a node distinct from themselves. Also presinks can only have out edges to other presinks since the depth must be nonincreasing. So when S has two sinks they must be connected to each other. When S has three sinks then at least one of them is connected to the other two. □

Theorem 5.0.2

Let Γ be a polymatroid in $\mathbb{C}[x_1, ..., x_n]$ and let $n \leq 7$. Then every section in the exchange graph of Γ is weakly connected.

Proof.

Let Γ be a polymatroid in $\mathbb{C}[x_1, ..., x_n]$, let S be a section in the exchange graph of Γ and let P_S be the subgraph of presinks of S. We will show that P_S is connected when $n \leq 7$. Which will imply that S is weakly connected. Theorem 4.1.7 tells us that sections containing loops are weakly connected, we will restrict our consideration to sections S for which P_S does not contain loops.

Let (A, B) be a vertex in P_S. Since (A, B) is a presink then (A, B) has the form $(a_1 \cdots a_k m\ , b_1 \cdots b_k m)$, where a_i and b_i are distinct variables and $k \leq \frac{n}{2}$. Since n is at most 7 then

k can be at most 3.

The bases A and B have at least k exchanges between them, one induced by each of the a_i, hence the degree of (A, B) is at least k. Since (A, B) was an arbitrary vertex we can conclude that each vertex of P_S has degree at least k.

If P_S is disconnected then Lemma 2.2.2 says that P_S must have at least $2k+2$ vertices. Also, Lemma 4.2.1 says that P_S can have at most $\frac{1}{2}\binom{2k}{k}$ vertices. Therefore P_S cannot be disconnected whenever $2k+2 > \frac{1}{2}\binom{2k}{k}$.

By inspection we see that this inequality is true for $k \in \{0, 1, 2\}$ which shows that P_S is connected when $k \in \{0, 1, 2\}$. The inequality fails when $k = 3$, yet we still see that P_S must have at least $2 \cdot 3 + 2 = 8$ vertices if it is disconnected. We will conclude the proof by showing that P_S is connected if $k = 3$ and P_S has at least 8 vertices.

After relabeling we can assume that $(x_1x_2x_3m\ ,\ x_4x_5x_6m)$ is a vertex in P_S. Let (C, D) be a vertex in P_S distinct from $(x_1x_2x_3m\ ,\ x_4x_5x_6m)$. Since $C \neq x_1x_2x_3m$ and $C \neq x_4x_5x_6m$ then $\frac{C}{m}$ must be divisible by two elements from $\{x_1, x_2, x_3\}$ or two elements from $\{x_4, x_5, x_6\}$. A similar argument can be made for $\frac{D}{m}$. Without loss of generality suppose that $\frac{C}{m}$ is divisible by two elements from $\{x_1, x_2, x_3\}$ and $\frac{D}{m}$ is divisible by two elements from $\{x_4, x_5, x_6\}$. Then (C, D) differs from $(x_1x_2x_3m, x_4x_5x_6m)$ by a single exchange. This shows that (C, D) is connected to $(x_1x_2x_3m, x_4x_5x_6m)$. Since (C, D) was arbitrary then P_S is connected.

□

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