EQUILIBRIUM PROBLEMS IN POTENTIAL THEORY

By

MYKHAILO BILOGLIADOV

Bachelor of Science in Applied Physics
Donetsk National University
Donetsk, Ukraine
2004

Master of Science in Radiophysics and Electronics
Donetsk National University
Donetsk, Ukraine
2005

Master of Arts in Mathematics
Indiana University
Bloomington, Indiana
2010

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Dissertation Approved:

Dr. Igor E. Pritsker

Dissertation Advisor

Dr. Alan V. Noell

Dr. Roger C. Zierau

Dr. Jacques H.H. Perk

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Name: Mykhailo Bilogliadov

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Abstract: We study two types of equilibrium problems arising in potential theory. The first type is concerned with a special case of Maxwell's problem on the number of equilibrium points of the Riesz potential of positive unit point charges placed at the vertices of a regular polygon, given by

$$U_s(z) := \sum_{j=1}^{n} \frac{1}{|z-\zeta_n^j|^{2s}}, \quad z \in \mathbb{C}, \quad n \geq 3, \quad s > 0,$$

where $\zeta_n = e^{2\pi i/n}$ is the n-th primitive root of unity. We study the location and asymptotic behavior of the equilibrium points. In addition, we obtain precise results on the number of equilibrium points for certain values of the Riesz parameter s.

The second type is concerned with minimizing the weighted energy

$$I_Q(\mu) := \iint \frac{1}{|x-y|^s} \, d\mu(x) d\mu(y) + 2 \int Q(x) d\mu(x),$$

over the class $\mathcal{M}(E)$ of unit positive Borel measures supported on a compact subset E of $\mathbb{R}^d$, where $d \geq 3$. It is assumed that the set E is immersed into a smooth rotationally invariant external field Q.

We then restrict ourselves by considering the minimum energy problem on the unit sphere $\mathbb{S}^{d-1}$, assuming the charges interact according to Newtonian potential. We also consider the minimum Riesz s-energy problem on the hyperdisk $\mathbb{D} \subset \mathbb{R}^d$, $d \geq 3$, where the charges are assumed to interact via the Riesz potential $1/r^s$, with $d-3 < s < d-1$, with r denoting the Euclidean distance. The problems are solved by finding the support of the extremal measure, and obtaining an explicit expression for the density. We then consider applications to the external field generated by a point charge of positive magnitude and to the external fields of polynomial type. Our results take especially explicit and precise form in the case of Coulomb potential in $\mathbb{R}^3$. Finally, we study the problem of recovering of the extremal measure when the support is no longer the entire disk, but rather a ring, again paying special attention to the case of classical Coulomb potential in $\mathbb{R}^3$.

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LIST OF SYMBOLS

$\mathbb{N}$	Set of natural numbers: $1, 2, 3, \dots$
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{R}^d$	d-dimensional real vector space
$\mathbb{S}^{d-1}$	unit sphere in $\mathbb{R}^d$
$\mathbb{D}$	unit disk in $\mathbb{R}^d$
$\mathbb{D}_R$	disk of radius R in $\mathbb{R}^d$
$\mathscr{R}(a, b)$	ring of inner radius a and outer radius b in $\mathbb{R}^d$
$\mathcal{M}(E)$	class of unit positive Borel measures supported on E

CHAPTER 1

Equilibrium Problems in Potential Theory

1.1 Maxwell's problem

Consider a system of n positive unit point charges in $\mathbb{R}^3$ located at points x_i, $i = 1, 2, \ldots, n$. They produce an electric field

$$E(x) = \sum_{i=1}^{n} \frac{x - x_i}{|x - x_i|^3}, \quad x_i,\, x \in \mathbb{R}^3.$$

In 1873 Maxwell [55] raised a question about the number of equilibrium points of the system of n point charges, which is the number of points where $E(x)$ vanishes. This is the same as the number of critical points of the Coulomb potential $U(x)$ associated with this electrostatic field $E(x)$. Maxwell conjectured that, assuming that considered configuration of charges has only non-degenerate critical points, the number of equilibrium points for the system of n point charges has an upper bound $(n-1)^2$. It was only in 2007 that Gabrielov, Novikov, and Shapiro [30] were able to show the existence of an upper bound of $4^{n^2}(3n)^{2n}$ for any non-degenerate configuration of charges in $\mathbb{R}^3$. In 2009, Killian [43] improved the result obtained in [30] in the case when all charges lie in a plane. Namely, he established that for n charges in the plane there is an upper bound of $(2^{n-1}(3n-2))^2$ on the number of equilibrium points. He also showed that in the case when unit point charges are located at the vertices of an equilateral triangle, there are exactly four equilibrium points. Shortly after that Peretz [59] proved that in the case of three positive point charges on a plane, there must be at least two equilibrium points, and generically an even number of them, i.e. 2, 4, 6, 8, 10 or 12. Also Tsai [71] considered the case of three point charges of arbitrary signs placed at the vertices of isosceles and equilateral triangle, and obtained an upper bound of 4 on the number of equilibrium points.

We also observe that there is a direct connection between electrostatics and the theory of many-body problem in celestial mechanics, in particular due to the fact that the potentials which appear in both are the same. It is often of an interest in celestial mechanics to consider a many-body problem where the bodies are subjected to their mutual attractions, governed by a central potential of the type $1/r^{2s}$, with $s > 0$, where r is the Euclidean distance, see [29], [6]. The Coulomb potential is then just a special case when $s = 1/2$. One often is interested in the equilibrium configurations in the context of many-body problem, which amounts to nothing else but the study of the critical points of a potential resulting from an arrangement of celestial bodies [6].

Furthermore, in 1859 Maxwell wrote a paper on the stability of Saturn's rings [56]. Since at that time the structure of the rings was unknown, Maxwell considered an approximation by studying the case of n equal mass bodies orbiting Saturn at a common radius and uniformly distributed about a circle of this radius. Since then this model gained a widespread recognition and was studied immensely. However, the question on the number of equilibrium points in such a model remained open. The first step in studying equilibria in a regular polygon arrangements of points by means of a special integral representation of the gravitational potential was initiated in 1899 by Tisserand [69] and later on appeared in a modern form in work of Lindow [53]. These results were extended in 2003 by Bang and Elmabsout [5] to the general central-type potentials of the form $1/r^{2s}$, with $s > 0$. Bang and Elmabsout used their aforementioned work [5] to study the existence and the linear stability of equilibrium positions of a zero-mass particle, submitted to the gravitational field generated by n bodies of equal mass placed at the vertices of a regular n-gon, and rotating rigidly around an additional mass at its center with a constant angular velocity [7].

In our paper [8] the case when n positive unit point charges are placed at the vertices of a regular n-gon was investigated. We showed that the Riesz potential $1/r^{2s}$ (where r is the Euclidean distance) generated by this system, has critical points located on the perpendicular bisectors to the sides of the polygon, as well as at the origin. We also studied the asymptotic behavior of the critical points with respect to n and s. Namely, for a fixed s and large n we prove that all the non-trivial critical points become equidistributed on the unit circumference. When n is fixed and s is approaching zero, it was shown that the non-trivial critical points slide to the origin, while for s infinitely large we show that the non-trivial critical points slide to the sides of a regular polygon. Finally, using the techniques of the theory of stable mappings we demonstrated that for values of the parameter s in a small left hand-side neighborhood of $s = 1$, the Riesz potential $1/r^{2s}$ has only one equilibrium point different from the origin on each perpendicular bisector, and one equilibrium point at the origin.

1.2 Minimum energy problem on the hypersphere and other manifolds

Let $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ be the unit sphere in $\mathbb{R}^d$, with $d \geq 3$, and where $|\cdot|$ is the Euclidean distance. Given a compact set $E \subset \mathbb{S}^{d-1}$, consider the class $\mathcal{M}(E)$ of unit positive Borel measures supported on E. For $0 < s < d$, the Riesz s-potential and Riesz s-energy of a measure $\mu \in \mathcal{M}(E)$ are defined respectively as

$$U_s^\mu(x) := \int \frac{1}{|x-y|^s}\, d\mu(y), \quad I_s(\mu) := \iint \frac{1}{|x-y|^s}\, d\mu(x) d\mu(y).$$

Let

$$W_s(E) := \inf\{I_s(\mu) : \mu \in \mathcal{M}(E)\}.$$

Define the Riesz s-capacity of E as $\text{cap}_s(E) := 1/W_s(E)$. We say that a property holds quasi-everywhere (q.e.), if the exceptional set has a Riesz s-capacity zero. When $\text{cap}_s(E) > 0$, there is a unique μ_E such that $I_s(\mu_E) = W_s(E)$. Such μ_E is called the

Riesz s-equilibrium measure for E.

An external field is defined as a non-negative lower-semicontinuous function $Q : E \to [0, \infty]$, such that $Q(x) < \infty$ on a set of positive Lebesgue surface measure. The weighted energy associated with $Q(x)$ is then defined by

$$I_Q(\mu) := I_s(\mu) + 2\int Q(x)d\mu(x).$$

The energy problem on $\mathbb{S}^{d-1}$ in the presence of the external field $Q(x)$ refers to the minimal quantity

$$V_Q := \inf\{I_Q(\mu) : \mu \in \mathcal{M}(E)\}.$$

A measure $\mu_Q \in \mathcal{M}(E)$ such that $I_Q(\mu_Q) = V_Q$ is called the s-extremal (or positive Riesz s-equilibrium) measure associated with $Q(x)$.

We next state a Frostman-type theorem which guarantees the existence and uniqueness of the measure μ_Q, while also characterizing the measure μ_Q in terms of its potential. The proof of this theorem can be extracted from more general results of Zorii [77], and follows closely the proof of [65, Theorem I.1.3]. The proof can also be found in the forthcoming book [13], also see Theorem 10.3 in [57] or Proposition 3 in [18].

Theorem 1.2.1 *Let $0 < s < d-1$. For the minimum energy problem on $\mathbb{S}^{d-1}$ with external field $Q(x)$ the following properties hold:*

(a) *V_Q is finite.*

(b) *There exists a unique s-extremal measure $\mu_Q \in \mathcal{M}(E)$ associated with $Q(x)$. Moreover, the support $S_Q := \operatorname{supp}(\mu_Q)$ of this measure is contained in a compact set $E_M := \{x \in E : Q(x) \leq M\}$ for some $M > 0$.*

(c) *The measure μ_Q satisfies the Gauss variational inequalities*

$$U_s^{\mu_Q}(x) + Q(x) \geq F_Q \quad q.e. \;\; on \;\; E, \tag{1.2.1}$$

$$U_s^{\mu_Q}(x) + Q(x) \leq F_Q \quad for\ all \;\; x \in S_Q, \tag{1.2.2}$$

where

$$F_Q := V_Q - \int Q(x)\, d\mu_Q(x).$$

(d) *Inequalities* (1.2.1) *and* (1.2.2) *completely characterize the equilibrium measure* μ_Q *in the sense that if* $\nu \in \mathcal{M}(E)$ *is a measure with finite s-energy such that for some constant* C

$$U_s^{\nu_Q}(x) + Q(x) \geq C \quad q.e. \;\; on \;\; E, \tag{1.2.3}$$

$$U_s^{\nu_Q}(x) + Q(x) \leq C \quad for\ all \;\; x \in S_Q, \tag{1.2.4}$$

then $\nu = \mu_Q$ *and* $C = F_Q$.

We remark that for continuous external fields, the inequality in (1.2.1) holds everywhere, which implies that equality holds in (1.2.2).

We will also be studying the minimum energy problem for the Riesz potentials on the hyperdisk $\mathbb{D} := \{(x_1, \ldots, x_n) \in \mathbb{R}^d : x_1 = 0, x_2^2 + x_3^2 + \ldots + x_d^2 \leq 1\}$. A Frostman-type theorem similar to the one mentioned above can be stated in this case as well, with the sphere $\mathbb{S}^{d-1}$ being replaced with the hyperdisk $\mathbb{D}$.

The minimum energy problems with external fields on the sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, $d \geq 3$ were a subject of investigation by a group of Brauchart, Dragnev and Saff. In 2007, Dragnev and Saff, working on separation results for the minimum energy points on the sphere $\mathbb{S}^{d-1}$, solved the minimum energy problem under the presence of an external field produced by a point charge placed at the Northern Pole of the sphere [28]. In 2009, Brauchart, Dragnev and Saff solved the minimum energy problem on the sphere $\mathbb{S}^{d-1}$ immersed in a field of a point charge located outside of the unit sphere [17]. Their group, together with some other researchers, further extended the obtained results to include the fields produced by a negative point charge [19], as well as charges located inside the sphere [20].

We also note that the minimum energy problems on the manifolds other the the unit sphere, were also considered. In particular, a minimum energy problem for logarithmic potential on the sets of revolution in $\mathbb{R}^3$ was considered by Hardin, Saff and Stahl in [38]. The same problem for the Riesz s-potentials with $0 < s < 1$, was treated by Brauchart, Hardin and Saff in papers [14] and [15].

In our paper [10] we provided a solution to the weighted energy problem on the sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, $d \geq 3$, immersed in a general external field, possessing rotational symmetry with respect to the polar axis, when support is a spherical cap. It was assumed that the charges interact according to the classical Newtonian potential $1/r^{d-2}$, where r denotes the Euclidean distance. We obtained an equation that described the support of the extremal measure, and also gave an explicit expression for the equilibrium density, thus extending an approach previously developed in our paper [9] for the case $d = 3$. As an application of our results, we considered an external field produced by a point charge of positive magnitude, placed at the North Pole of the sphere, and explicitly computed support and density of the corresponding extremal measure. As another application of the developed theory, we considered the case when the sphere is immersed into a quadratic external field, and also explicitly found the support and density of the extremal measure.

The treatments of minimum energy problem on the unit disk in $\mathbb{R}^3$ for the case of Newtonian potential can be traced back to the beginning of 20th century. They were studied, however, mostly by physicists. The first record of mathematically rigorous treatment of this problem is due to Copson [26]. He developed an integral representation for the kernel of Newtonian potential, which enabled him to reduce the problem to solving an Abel-type integral equation, thus obtaining a closed form solution. His approach turned out to be fruitful. Using a generalized Copson-type representation, we solve the minimum energy problem on the unit hyperdisk $\mathbb{D}$, immersed into a general rotationally invariant external field by finding the extremal support and then obtaining explicit expressions for the extremal measure. It is assumed that the charges interact according to the Riesz s-potential $1/r^s$, with $d-3 < s < d-1$, where r is the Euclidean distance. We then consider the applications of the obtained results. The first application of our results is concerned with a situation when the disk $\mathbb{D}$ is immersed into a monomial external field. We find the extremal measure corresponding to such a field, while also explicitly finding the extremal support. Our second application is concerned with finding the extremal measure when an external field is generated by a positive point charge, placed on the polar axis at a certain distance above the disk $\mathbb{D}$. A similar problem for the sphere $\mathbb{S}^2$ for the Coulomb potential in $\mathbb{R}^3$ was raised by Gonchar [47], and solved in [17] for general Riesz potentials on the sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, $d \geq 3$. The extensions and further results on Gonchar's problem are contained in works [19], [20]. The problem of finding a signed measure representing the charge distribution on the disk $\mathbb{D}$ in $\mathbb{R}^3$ under the influence of a positive point charge placed on the polar axis above the disk $\mathbb{D}$, for the case of Coulomb potential, was first considered in the classical work of Thomson [70], and later solved by a different method by Gallop [31]. Below we solve this problem for the case of higher dimensions and general Riesz s-potentials by finding the extremal measure representing the positive charge distribution on the disk $\mathbb{D}$. For the case of higher dimensions and general Riesz s-potentials we give an explicit estimate on the height and magnitude of the point charge, which guarantees for the extremal support to occupy the whole disk $\mathbb{D}$. Moreover, in the case of classical Coulomb potential in $\mathbb{R}^3$ and a positive unit point charge, we are able to precisely determine the height of the point charge so that the extremal support occupies the entire disk $\mathbb{D}$.

Moreover, we investigate what happens to the support of the extremal measure μ_Q if one moves a point charge closer to the disk, beyond the aforementioned height threshold. It is demonstrated that under some mild restrictions on a general external field, the support $\operatorname{supp} \mu_Q$ will be a ring, contained in the disk $\mathbb{D}$.

1.3 Minimum energy problem for Coulomb potential on a ring in $\mathbb{R}^3$

We consider a special case when $d = 3$ and the charges are assumed to interact according to the Newtonian potential, while the support of the extremal measure is a ring, contained in disk $\mathbb{D}$. There are essentially two different ways of attacking this problem.

The first way consists of solving a Dirichlet problem for the Laplace equation on the potential U^{μ_Q} of the extremal measure μ_Q, with boundary data $F_Q - Q$ on the ring. The work in this direction was initiated by Wangerin [73, 74] in 1870s. The theory developed by Wangerin was applied specifically to the study of the Dirichlet problem for the Laplace question in the ring by Poole in [61]–[62]. An approach closely related to that of Poole's was considered by Lebedev [51].

Another way to recover the extremal measure supported on a ring consists in solving an integral equation arising from the equilibrium relations (1.2.1)-(1.2.2) on the support $\operatorname{supp}\mu_Q$ of the extremal measure μ_Q. The first attempt in solving such an integral equation was undertaken by Gubenko and Mossakovksii [35]. Their work was further advanced by Cooke [24] and Clements and Love [22]-[23].

CHAPTER 2

Equilibria of Riesz Potentials of Point Charges at the Roots of Unity

This chapter is based on work [9]. Let n positive unit point charges be placed at the vertices of a regular n-gon inscribed in a unit circumference. The Riesz potential of this system is

$$U_s(r,\theta) = \sum_{j=1}^{n} \frac{1}{|z-\zeta_n^j|^{2s}}, \quad n \geq 3,\ s > 0, \tag{2.0.1}$$

where $\zeta_n = e^{2\pi i/n}$ is the n-th primitive root of unity and $z = re^{i\theta}$, with $r \in [0,1)$ and $\theta \in [0, 2\pi)$. We can rewrite the potential (2.0.1) as

$$U_s(r,\theta) = \sum_{j=1}^{n} \frac{1}{(1+r^2-2r\cos(2\pi j/n-\theta))^s}. \tag{2.0.2}$$

Note that for $s = 1/2$, expression (2.0.1) reduces to the usual Coulomb potential

$$U(r,\theta) = \sum_{j=1}^{n} \frac{1}{(1+r^2-2r\cos(2\pi j/n-\theta))^{\frac{1}{2}}}. \tag{2.0.3}$$

The following useful observation is due to Bang and Elmabsout [5]:

Proposition 2.0.1 *Let $\alpha = q + \mu$, where q is positive integer and $\mu \in (0,1)$. Then, for every $n \geq 3$, real θ and real r such that $0 \leq r < 1$*

$$\sum_{j=1}^{n} \frac{1}{(1+r^2-2r\cos(2\pi j/n-\theta))^\alpha} = \frac{n}{\Gamma(\alpha)\Gamma(1-\mu)} \times \int_0^1 \frac{t^{\alpha-1}}{(1-t)^\mu} \frac{\partial^q}{\partial t^q} \left\{ \frac{t^q}{(1-tr^2)^\alpha} \frac{1-(tr)^{2n}}{1+(tr)^{2n}-2(tr)^n\cos(n\theta)} \right\} dt. \tag{2.0.4}$$

In the special case $\alpha = q + 1$ we obtain that

$$\sum_{j=1}^{n} \frac{1}{(1+r^2-2r\cos(2\pi j/n-\theta))^\alpha} = \left\{ \frac{n}{q!} \frac{\partial^q}{\partial t^q} \left\{ \frac{t^q}{(1-tr^2)^\alpha} \frac{1-(tr)^{2n}}{1+(tr)^{2n}-2(tr)^n\cos(n\theta)} \right\} \right\}_{t=1}. \tag{2.0.5}$$

As a corollary we obtain an important integral representation of the Riesz potential (2.0.1):

Corollary 2.0.1 *For $s \in (0,1)$, $r \in [0,1)$ and $0 \le \theta < 2\pi$, the potential $U_s(r,\theta)$ can be written as*

$$U_s(r,\theta) = \frac{n\sin(\pi s)}{\pi} \int_0^1 \frac{t^{s-1}(1-t)^{-s}}{(1-r^2t)^s} \frac{1-(rt)^{2n}}{1+(rt)^{2n}-2(rt)^n\cos(n\theta)}\, dt. \tag{2.0.6}$$

Corollary 2.0.1 implies that in the case of Coulomb potential ($s = 1/2$) we have

$$U(r,\theta) = \frac{n}{\pi} \int_0^1 \frac{t^{-1/2}(1-t)^{-1/2}}{(1-r^2t)^{1/2}} \frac{1-(rt)^{2n}}{1+(rt)^{2n}-2(rt)^n\cos(n\theta)}\, dt.$$

2.1 Critical points of the Riesz potential

Now we can use the integral representation of the Riesz potential $U_s(r,\theta)$, when $s \in (0,1)$, given in Corollary 2.0.1 to determine the location of the critical points of $U_s(r,\theta)$. It follows that all the non-trivial critical points of the potential $U_s(r,\theta)$ are located on the perpendicular bisectors to the sides of a polygon, corresponding to the values of θ given by

$$\theta = \pi k/n, \quad k = 1, 3, 5 \ldots, 2n-1. \tag{2.1.1}$$

More precisely we have

Theorem 2.1.1 *The potential $U_s(r,\theta)$, with $s \in (0,1)$, has all its non-trivial critical points on perpendicular bisectors to the sides of the regular polygon, and a critical point at the origin. Furthermore, each non-trivial critical point (r,θ) satisfies*

$$r \in (r_l(n,s), r_u(n)),$$

where

$$r_l(n,s) = \left(\frac{s}{s+n}\right)^{\frac{1}{n-2}}, \quad r_u(n) = \cos(\pi/n).$$

Example 2.1.1 *($n = 3$, $s = 1/2$) Let $n = 3$ and $s = 1/2$. In this case, the charges are located at the vertices of an equilateral triangle inscribed in a unit circle centered at the origin. The Coulomb potential restricted to the bisectors is $u_{1/2}(r) = 2(r^2 - r + 1)^{-1/2} + (1 + r)^{-1}$. It is easy to see that the critical points are in the interval $[0, 1/2]$. Direct computation shows that the critical points are among the non-negative roots of the polynomial $r(r^5 + 5r^4 + r^3 + r^2 - 4r + 1)$. Applying the Descarte's Rule of Signs to the polynomial $r^5 + 5r^4 + r^3 + r^2 - 4r + 1$ we infer that there are two positive roots, one of which must be discarded as being outside of $[0, 1/2]$. Thus there is a unique critical point for $u_{1/2}(r)$ different from the origin, and therefore the potential $U(r, \theta)$ has exactly four critical points.*

Example 2.1.2 *($n = 4$, $s = 1/2$) Let $n = 4$ and $s = 1/2$. Now the charges are placed at the vertices of a square inscribed in a unit circumference centered at the origin. For the Coulomb potential on the bisectors we obtain that $u_{1/2}(r) = 2(r^2 - \sqrt{2}r + 1)^{-1/2} + 2(r^2 + \sqrt{2}r + 1)^{-1/2}$. It follows that the critical points for $u_{1/2}(r)$ must be in the interval $[0, 1/\sqrt{2}]$. It is not hard to see that the critical points are among the non-negative roots of the polynomial $r(4r^6 + r^5 - 4r^3 + 1)$ and again applying the Descarte's Rule of Signs to the polynomial $4r^6 + r^5 - 4r^3 + 1$ we conclude that it has two positive zeros one of which must be dropped as being outside of $[0, 1/\sqrt{2}]$. Thus $u_{1/2}(r)$ has exactly two critical points, namely 0 and some $r_0 \neq 0$. This shows that $U(r, \theta)$ has exactly five critical points in the case of the square.*

2.2 Asymptotic behavior of critical points

It is not hard to see that the potential $U_s(r, \theta)$ has the same number of the critical points on each perpendicular bisector to the sides of a polygon. Indeed, as it follows from (2.0.6) and (2.1.1) the potential on the perpendicular bisectors is independent of θ. We also remark that, if $k(n)$ is the number of non-trivial critical points on each perpendicular bisector, then if Maxwell's conjecture is true, $k(n) \leq n - 2$.

We also observe that the radial coordinate r corresponding to an equilibrium position on each perpendicular bisector naturally depends on n and s. We will show that when s is fixed and the number of charges n increases to infinity, $r(n,s)$ approaches 1. That is, we have the following:

Theorem 2.2.1 *Let $r(n,s) \in (0,1)$ be a radial coordinate of a non-trivial critical point on a perpendicular bisector to the sides. Then for a fixed $s \in (0,1)$ we have*

$$\lim_{n\to\infty} r(n,s) = 1. \tag{2.2.1}$$

Recalling that all the non-trivial critical points are located on perpendicular bisectors to the sides of a polygon, Theorem 2.2.1 shows that the critical points (except the one located at the origin) cluster on the unit circumference for large n. We also see from (2.1.1) that for each fixed $n \geq 3$ all the non-trivial critical points consist of n-tuple subsets equidistributed on circumference with radius $r(n,s)$. We deduce that when n becomes infinitely large all the non-trivial critical points become uniformly distributed on the unit circumference.

Now we investigate the dependence of $r(n,s)$ on the parameter s when n is fixed. We will consider the asymptotic behavior of critical points when $s \to 0+$ and $s \to +\infty$. First we treat the case $s \to 0+$.

Theorem 2.2.2 *Let $n \geq 3$ be fixed. Let $r(n,s)$ be a radius corresponding to an equilibrium position different from the origin. Then*

$$\lim_{s\to 0+} r(n,s) = 0. \tag{2.2.2}$$

We turn to the case $s \to +\infty$ now. First, we remark that since all the charges are positive, it is a direct consequence of the generalized Lucas theorem, proved in [27] (see also [32]), that all the critical points lie in a convex hall of the point charges. Therefore, it is clear that an equilibrium position $r(n,s)$ can not exceed (the length of) an apothem of a regular polygon, for all non-negative s and all $n \geq 3$. In our case we have a regular polygon inscribed in a unit circumference, therefore its apothem is just $\cos(\pi/n)$, and thus

$$r(n,s) \le \cos\left(\frac{\pi}{n}\right). \tag{2.2.3}$$

Furthermore, the following result shows that equality in (2.2.3) is attained in the limit $s \to +\infty$.

Theorem 2.2.3 *Assume $n \ge 3$ is fixed, and let $r(n,s)$ be a radius corresponding to an equilibrium position different from the origin. Under these assumptions we have*

$$\lim_{s\to+\infty} r(n,s) = \cos\left(\frac{\pi}{n}\right). \tag{2.2.4}$$

2.3 Number of critical points of the Riesz potential $U_s(z)$ in the left hand-side neighborhood of $s = 1$

In the case $s = 1$ the Riesz potential (2.0.2) takes the form

$$U_1(r,\theta) = \sum_{j=1}^{n} \frac{1}{1 + r^2 - 2r\cos(2\pi j/n - \theta)}. \tag{2.3.1}$$

What is rather surprising is that the potential $U_1(r,\theta)$ admits the following simple closed representation.

Proposition 2.3.1 *For $r \in [0,1)$ and $0 \le \theta < 2\pi$*

$$U_1(r,\theta) = \frac{n}{1-r^2}\frac{1-r^{2n}}{1+r^{2n}-2r^n\cos(n\theta)}. \tag{2.3.2}$$

Now it is not hard to count the number of critical points of the potential $U_1(r,\theta)$.

Theorem 2.3.1 *The potential $U_1(r,\theta)$ has exactly $n+1$ critical points. Namely, it has exactly one critical point on each perpendicular bisector to the sides of the polygon, and a critical point at the origin.*

Next, we observe that the restriction of the potential $U_s(r,\theta)$ to a perpendicular bisector to the sides is an analytic function in s, r. This fact, coupled with the theory of stable mappings shows that the property of the potential $U_s(r,\theta)$ to have exactly $n+1$ critical points is preserved when s varies in a small left hand-side neighborhood of $s = 1$.

Theorem 2.3.2 *There exists a number $s_l \in (0,1)$ such that for each $s \in (s_l, 1]$ the potential $U_s(r,\theta)$ has $n+1$ critical points. In particular, for each $s \in (s_l, 1]$ the potential $U_s(r,\theta)$ has only one critical point different from the origin on each perpendicular bisector to the sides of the polygon, and a critical point at the origin.*

In fact, as the next example clearly demonstrates, Maxwell's conjecture for the Riesz potentials holds for three unit positive point charges placed at the vertices of an equilateral triangle *for all* $s \in (0,1)$.

Example 2.3.1 ($n = 3$, $s \in (0,1)$) *Let $n = 3$ and $s \in (0,1)$. Then the Riesz potential $U_s(r,\theta)$ has exactly four critical points. Namely, for all $s \in (0,1)$ there is one critical point on each perpendicular bisector to the sides of a triangle, and a critical point at the intersection of the perpendicular bisectors. The proof of this statement is given at the Proofs section.*

2.4 Proofs

Proof of Corollary 2.0.1. Proof of Corollary 2.0.1 can be found in [29] or [5], where it arises in connection with a polygonal many-body problem. There is similarity between the ideas arising in the theory of many-body problem and electrostatics, in particular due to the fact that the potentials which appear in both are identical. The idea behind Corollary 2.0.1 has its origin in the book of Tisserand [69] on celestial mechanics, and later it appears more or less explicitly in the paper of Lindow [53]. For the sake of self-containment we will reproduce the proof from the paper of Ferrario and Portaluri [29].

First we observe that potential (2.0.1) can be rewritten in the following manner:

$$U_s(r,\theta) = \sum_{j=1}^{n} \frac{1}{|1 - z\zeta_n^j|^{2s}}, \quad n \geq 3, \quad s \in (0,1). \tag{2.4.1}$$

This follows from a fact that if ζ_n is the n-th primitive root of unity, we have that $\zeta_n^{-1} = 1/\zeta_n$ is an n-th root of unity, and an observation $|1 - z\zeta_n^j| = |z - \zeta_n^{-j}|$. In order to transform the right-hand side of (2.4.1), we expand $|1 - r\xi|^{-2s}$ in a double power series as follows:

Lemma 2.4.1 *For each $r \in [0,1)$, $\xi = e^{i\theta}$, $0 \leq \theta < 2\pi$, and $s \in (0,1)$, we have the expansion*

$$\frac{1}{|1 - r\xi|^{2s}} = \sum_{m=-\infty}^{+\infty} b_m \xi^m,$$

where

$$b_m = b_{-m} = \frac{\sin(s\pi)}{\pi} r^m \int_0^1 (1-t)^{-s} t^{s-1} t^m (1 - tr^2)^{-s}\, dt, \quad m \geq 0.$$

Proof. We have

$$\begin{aligned}
|1 - r\xi|^{-2s} &= (1 - r\xi)^{-s}(1 - r\xi^{-1})^{-s} \\
&= \left(\sum_{k=0}^{\infty} \binom{-s}{k} (-r\xi)^k\right) \cdot \left(\sum_{h=0}^{\infty} \binom{-s}{h} (-r\xi^{-1})^h\right) \\
&= \sum_{h,k=0}^{\infty} \binom{-s}{k}\binom{-s}{h} (-r)^{k+h} \xi^{k-h} \\
&= \sum_{m=-\infty}^{\infty} \left((-1)^m \sum_{\substack{k-h=m \\ k,h\geq 0}} \binom{-s}{k}\binom{-s}{h} r^{k+h} \right) \xi^m.
\end{aligned}$$

Let

$$b_m = (-1)^m \sum_{\substack{k-h=m \\ k,h\geq 0}} \binom{-s}{k}\binom{-s}{h} r^{k+h}.$$

Next, recall that for each $s > 0$ and $N \in \mathbb{N}$ we have

$$\begin{aligned}
\binom{-s}{N} &= (-1)^N \binom{N+s-1}{N} \\
&= (-1)^N \frac{\Gamma(N+s)\Gamma(1-s)}{\Gamma(N+1)\Gamma(s)\Gamma(1-s)} \\
&= \frac{(-1)^N}{\Gamma(s)\Gamma(1-s)} \frac{\Gamma(N+s)\Gamma(1-s)}{\Gamma(N+1)} \\
&= \frac{(-1)^N \sin(\pi s)}{\pi} B(1-s, N+s),
\end{aligned}$$

where $B(x, y)$ is the Beta function and $\Gamma(x)$ is the Gamma function. Here we have used the following identities [2],

$$\begin{aligned}
\Gamma(s)\Gamma(1-s) &= \frac{\pi}{\sin(s\pi)}, \\
\binom{-s}{N} &= (-1)^N \frac{\Gamma(N+s)}{\Gamma(N+1)\Gamma(s)}, \\
\binom{s}{N} &= \frac{\Gamma(1+s)}{\Gamma(N+1)\Gamma(s-N+1)}.
\end{aligned}$$

Using the integral representation for the Beta function [2, p. 4], we obtain

$$\binom{-s}{N} = (-1)^N \frac{\sin(s\pi)}{\pi} \int_0^1 (1-t)^{-s} t^{s-1} t^N \, dt.$$

This implies that for $N = h + m$,

$$\begin{aligned}
b_m &= (-1)^m \sum_{h=0}^{\infty} \left((-1)^{m+h} \frac{\sin(s\pi)}{\pi} \int_0^1 (1-t)^{-s} t^{s-1} t^{m+h} \, dt \binom{-s}{h} r^{m+2h} \right) \\
&= \frac{\sin(s\pi)}{\pi} r^m \sum_{h=0}^{\infty} \left((-1)^h \int_0^1 (1-t)^{-s} t^{s-1} t^{m+h} \, dt \binom{-s}{h} r^{2h} \right) \\
&= \frac{\sin(s\pi)}{\pi} r^m \int_0^1 (1-t)^{-s} t^{s-1} t^m \left(\sum_{h=0}^{\infty} (-1)^h t^h \binom{-s}{h} r^{2h} \right) dt \\
&= \frac{\sin(s\pi)}{\pi} r^m \int_0^1 (1-t)^{-s} t^{s-1} t^m (1-tr^2)^{-s} \, dt.
\end{aligned}$$

Here, we used the fact that

$$\sum_{h=0}^{\infty} (-tr^2)^h \binom{-s}{h} = (1-tr^2)^{-s}.$$

■

Now observe that for each $k \in \mathbb{Z}$

$$\frac{1}{n}\sum_{y:y^n=\xi} y^k = \begin{cases} \xi^{k/n}, & \text{if } k \equiv 0 \mod n, \\ 0, & \text{if } k \not\equiv 0 \mod n, \end{cases}$$

so for every $s \in (0,1)$, $r \in [0,1)$ and integer $n \geq 2$, we obtain that

$$U_s(r,\theta) = n \sum_{k=-\infty}^{\infty} b_{nk}\xi^k.$$

Next, we have

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} b_{nk}\xi^k &= \sum_{k=0}^{\infty} b_{nk}\xi^k + \sum_{k=1}^{\infty} b_{nk}\xi^{-k} \\
&= \sum_{k=0}^{\infty}\left(\frac{\sin(s\pi)}{\pi} r^{nk}\int_0^1 (1-t)^{-s}t^{s-1}t^{nk}(1-tr^2)^{-s}dt\right)\xi^k \\
&+ \sum_{k=1}^{\infty}\left(\frac{\sin(s\pi)}{\pi} r^{nk}\int_0^1 (1-t)^{-s}t^{s-1}t^{nk}(1-tr^2)^{-s}dt\right)\xi^{-k} \\
&= \frac{\sin(s\pi)}{\pi}\int_0^1 \frac{(1-t)^{-s}t^{s-1}}{(1-tr^2)^s}\left[\sum_{k=0}^{\infty}(tr)^{nk}\xi^k + \sum_{k=1}^{\infty}(tr)^{nk}\xi^{-k}\right] dt.
\end{aligned}$$

Since

$$\sum_{k=0}^{\infty}(tr)^{nk}\xi^k + \sum_{k=1}^{\infty}(tr)^{nk}\xi^{-k} = \frac{1-(tr)^{2n}}{|1-(tr)^n\xi|^2},$$

we arrive at (2.0.6):

$$U_s(r,\theta) = \frac{n\sin(s\pi)}{\pi}\int_0^1 \frac{(1-t)^{-s}t^{s-1}}{(1-tr^2)^s}\,\frac{1-(tr)^{2n}}{|1-(tr)^n\xi|^2}\,dt.$$

■

Proof of Theorem 2.1.1. First we find the critical values of θ by using the integral representation (2.0.6) given in Corollary 2.0.1. The derivative with respect to θ is

$$\begin{aligned}
\frac{\partial U_s(r,\theta)}{\partial \theta} = &-\sin(n\theta)\frac{2n^2r^n\sin(\pi s)}{\pi}\times \\
&\int_0^1 \frac{t^{n+s-1}(1-t)^{-s}}{(1-r^2t)^s}\,\frac{1-(rt)^{2n}}{(1+(rt)^{2n}-2(rt)^n\cos(n\theta))^2}\,dt.
\end{aligned}$$

Then if θ is a critical value of the potential, we have $\partial U_s(r,\theta)/\partial\theta = 0$. This implies that $\sin(n\theta) = 0$, so that $n\theta = \pi k, k \in \mathbb{Z}$, or

$$\theta = \pi k/n, \quad k \in \mathbb{Z}, \tag{2.4.2}$$

because the above integral is positive.

Consider the case $k = 0$ in (2.4.2). We see that there are possible critical points on a ray coming from the origin where our regular n-gon is centered, and going through a vertex which sits on the x-axis. Now letting $\theta = 0$ in (2.0.6), we see the potential on that ray is given by

$$U_s(r, 0) = \frac{n \sin(\pi s)}{\pi} \int_0^1 \frac{t^{s-1}(1-t)^{-s}}{(1-r^2 t)^s} \frac{1 + (rt)^n}{1 - (rt)^n} \, dt. \tag{2.4.3}$$

The critical points of $U_s(r, 0)$ are found from

$$\frac{\partial U_s(r, 0)}{\partial r} = 0. \tag{2.4.4}$$

Since the integrand on the right-hand side of (2.4.3) and its derivative are continuous with respect to $(r, t) \in [0, x_0) \times [0, 1]$, for any $x_0 \in [0, 1)$, we can differentiate under the sign of the integral [64, p. 236]. Then for the derivative of $U_s(r, 0)$ we obtain

$$\frac{\partial U_s(r, 0)}{\partial r} = \frac{n \sin(\pi s)}{\pi} \times \tag{2.4.5}$$
$$\int_0^1 t^{s-1}(1-t)^{-s} \left\{ \frac{2srt}{(1-r^2 t)^{s+1}} \frac{1 + (rt)^n}{1 - (rt)^n} + \frac{1}{(1-r^2 t)^s} \frac{2nr^{n-1}t^n}{(1-(rt)^n)^2} \right\} dt.$$

For $r \in (0, 1)$ and $t \in [0, 1]$, we see that

$$\frac{1}{1-t} \geq 1, \quad \frac{1}{1-tr^2} \geq 1, \quad \frac{1}{1-r^n t^n} \geq 1, \quad \frac{1+r^n t^n}{1-r^n t^n} \geq 1.$$

Therefore

$$\begin{aligned} \frac{\partial U_s(r, 0)}{\partial r} &\geq \frac{n \sin(\pi s)}{\pi} \int_0^1 t^{s-1} \left(2srt + 2nr^{n-1}t^n \right) dt \\ &= \frac{n \sin(\pi s)}{\pi} \left\{ \frac{2sr}{s+1} + \frac{2nr^{n-1}}{n+s} \right\} > 0. \end{aligned}$$

Hence $\partial U_s(r, 0)/\partial r$ is a strictly positive function for $r \in (0, 1)$. Now looking at the expression (2.4.5) for $\partial U_s(r, 0)/\partial r$, it is clear that it can be zero if and only if $r = 0$. Thus for $r \in [0, 1)$ the potential on that ray has only one critical point, namely at the origin. This also shows that the potential $u_s(r)$ does not have the critical points different from the origin for the values of $\theta = \pi k/n$, corresponding to the even values of $k \in \mathbb{Z}$.

So far we have showed that all the critical points of the Riesz potential (2.0.1) (if they exist) lie on the rays $\theta = \pi k/n$, where $k = 1, 3, 5, \ldots, 2n-1$. We now want to obtain further information on the critical points on those rays. For that we let $\theta = \pi k/n, k = 1, 3, 5, \ldots, 2n-1$ in (2.0.2) and consider the case $k = 1$, which is enough for our purposes. We therefore obtain

$$u_s(r) := U_s(r, \pi/n) = \frac{n \sin(\pi s)}{\pi} J_n^s(r), \tag{2.4.6}$$

where

$$J_n^s(r) := \int_0^1 \frac{t^{s-1}(1-t)^{-s}}{(1-r^2t)^s} \frac{1-(rt)^n}{1+(rt)^n} \, dt. \tag{2.4.7}$$

We are interested in the critical points of $J_n^s(r)$,

$$\frac{dJ_n^s(r)}{dr} = 0, \quad n \geq 3.$$

Clearly the integrand in (2.4.7) and its derivative are continuous with respect to $(r, t) \in [0, x_0) \times [0, 1]$, for any $x_0 \in [0, 1)$. Hence we can differentiate under the sign of the integral in (2.4.7) [64, p. 236].

Now let us show that $u_s'(r) > 0$ for r from a small right-hand side neighborhood of the origin, for all $n \geq 3$. It is clear that

$$\frac{dJ_n^s(r)}{dr} = 2r \int_0^1 \frac{t^s(1-t)^{-s}}{(1-r^2t)^{s+1}} \frac{g_n(r,t)}{(1+(rt)^n)^2} \, dt, \quad n \geq 3, \tag{2.4.8}$$

where

$$g_n(r,t) = s(1-(rt)^{2n}) - nr^{n-2}t^{n-1}(1-r^2t). \tag{2.4.9}$$

We can estimate $g_n(r,t)$ from below in the following manner,

$$\begin{aligned}
g_n(r,t) &= s(1-(rt)^{2n}) - nr^{n-2}t^{n-1}(1-r^2t) \\
&= s - sr^{2n}t^{2n} - nr^{n-2}t^{n-1} + nr^nt^n \\
&\geq s - sr^{2n} - nr^{n-2} \\
&\geq s - sr^{n-2} - nr^{n-2} \\
&= s - r^{n-2}(s+n),
\end{aligned}$$

for all $t \in [0,1]$, all $n \geq 3$ and all $s \in (0,1)$. Thus if we take

$$r < r_l(n,s) := \left(\frac{s}{s+n}\right)^{\frac{1}{n-2}} \tag{2.4.10}$$

it follows that $g_n(r,t) > 0$ for all $t \in [0,1]$ and all $n \geq 3$.

Next, we note that the function $t^s(1-t)^{-s}(1-r^2t)^{-(s+1)}$ is continuous and non-negative for all $0 \leq t \leq 1$. Therefore by the second Mean Value Theorem for integrals there exists $t^* \in [0,1]$ such that

$$\begin{aligned}
&\int_0^1 \frac{t^s(1-t)^{-s}}{(1-r^2t)^{s+1}} \frac{g_n(r,t)}{(1+(rt)^n)^2}\,dt \\
&\qquad = \frac{g_n(r,t^*)}{(1+(rt^*)^n)^2} \int_0^1 \frac{t^s(1-t)^{-s}}{(1-r^2t)^{s+1}}\,dt \\
&\qquad = \frac{g_n(r,t^*)}{(1+(rt^*)^n)^2} B(1+s,1-s)\,{}_2F_1(1+s,1+s;2;r^2),
\end{aligned}$$

where ${}_2F_1$ is the Gauss hypergeometric function [2]. Therefore, we see that if $r \in (0, r_l(n,s))$, then $u_s'(r) > 0$ for all $n \geq 3$. One can also check that $u_s'(r_l(n,s)) \neq 0$. Thus we proved the following

Lemma 2.4.2 *Potential $u_s(r)$ has no critical points on $(0, r_l(n,s))$.*

To obtain an upper estimate on the location of a critical point on a bisector we recall that the Riesz potential on a bisector for $0 \leq r < 1$ and $n \geq 3$ is given by

$$u_s(r) = \sum_{j=1}^{n} \frac{1}{(1+r^2-2r\cos(\theta_j))^s}, \tag{2.4.11}$$

where $\theta_j = \pi(2j-1)/n$, $j = 1,2,\ldots,n$.

For the derivative $u_s'(r)$ we find

$$u_s'(r) = -2s\sum_{j=1}^{n} \frac{r-\cos(\theta_j)}{(1+r^2-2r\cos(\theta_j))^{s+1}}. \tag{2.4.12}$$

so that

$$u_s'(\cos(\pi/n)) = -2s\sum_{j=1}^{n} \frac{\cos(\pi/n)-\cos(\theta_j)}{(1+\cos^2(\pi/n)-2\cos(\pi/n)\cos(\theta_j))^{s+1}}.$$

It is now obvious that $u_s'(\cos(\pi/n)) < 0$ for all $n \geq 3$. Let $r_u(n) := \cos(\pi/n)$.

Since $u_s'(r)$ is continuous and changes sign on $(0, r_u(n))$, we infer that there exists $r_0 \in (0, r_u(n))$ such that $u_s'(r_0) = 0$. Also from (2.4.12) it is clear that $u_s'(r) < 0$ for $r \in [r_u(n), 1)$. Hence we proved the following

Lemma 2.4.3 *Potential $u_s(r)$ does not have critical points on $[r_u(n), 1)$.*

Combining Lemma 2.4.2 and Lemma 2.4.3 we obtain

Lemma 2.4.4 *Potential $u_s(r)$ has all its critical points in the interval $(r_l(n,s),\, r_u(n))$.*

Since the potential $u_s(r)$ is a restriction of 2D potential $U_s(r,\theta)$ to the perpendicular bisectors, the proof of the theorem follows. ■

Remark 2.4.1 *We note that for the potential $u_s(r)$ a brief calculation reveals that 0 is a critical point and $u_s''(0) = 2s^2 n$. Therefore 0 is a local minimum for the potential $u_s(r)$.*

Proof of Theorem 2.2.1. We know that $r(n,s) \in (r_l(n,s), r_u(n))$ for all $s \in (0,1)$ and all $n \geq 3$. Note that

$$\lim_{n\to\infty} r_l(n,s) = \lim_{n\to\infty} \left(\frac{s}{s+n}\right)^{\frac{1}{n-2}} = 1,$$

$$\lim_{n\to\infty} r_u(n) = \lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = 1.$$

Therefore

$$\lim_{n\to\infty} r(n,s) = 1.$$

■

Proof of Theorem 2.2.2. Without loss of generality we may assume that $s \in (0, 1/2)$. Recall that according to (2.4.6) the potential $u_s(r)$ has the form

$$u_s(r) = \frac{n\sin(\pi s)}{\pi} \int_0^1 \frac{t^{s-1}(1-t)^{-s}}{(1-r^2t)^s} \frac{1-(rt)^n}{1+(rt)^n} dt,$$

and so

$$u_s'(r) = \frac{2nr\sin(\pi s)}{\pi}\Big\{ s\int_0^1 \frac{t^s(1-t)^{-s}}{(1-r^2t)^{s+1}}\frac{1-(rt)^n}{1+(rt)^n}\,dt \\ -nr^{n-2}\int_0^1 \frac{t^{n+s-1}(1-t)^{-s}}{(1-r^2t)^s}\frac{1}{(1+(rt)^n)^2}\,dt\Big\}. \tag{2.4.13}$$

We want to estimate the integrals on the right-hand side of (2.4.13). For that a few simple inequalities will prove handy.

Observe that $1-(rt)^n = (1-(rt))(1+rt+(rt)^2+\ldots+(rt)^{n-1})$. Hence, as $r \in (0,1)$ and $t \in [0,1]$ we deduce that $1-(rt)^n = (1-(rt))(1+rt+(rt)^2+\ldots+(rt)^{n-1}) \le (1-r^2t)(1+1+\ldots+1) = n(1-r^2t)$. In addition, $(1-r^2t)^{-s} \le (1-t)^{-s}$ and, trivially $(1+r^nt^n)^{-1} \le 1$. Therefore

$$\begin{aligned}\int_0^1 \frac{t^s(1-t)^{-s}}{(1-r^2t)^{s+1}}\frac{1-(rt)^n}{1+(rt)^n}\,dt &= \int_0^1 \frac{t^s(1-t)^{-s}}{(1-r^2t)(1-r^2t)^s}\frac{1-(rt)^n}{1+(rt)^n}\,dt \\ &\le \int_0^1 \frac{t^s(1-t)^{-s}}{(1-r^2t)(1-t)^s}\frac{n(1-r^2t)}{1+(rt)^n}\,dt \\ &\le n\int_0^1 t^s(1-t)^{-2s}\,dt \\ &= n\,B(1+s,1-2s),\end{aligned} \tag{2.4.14}$$

where $B(x,y)$ is the beta-function.

Also, the second integral on the right-hand side of (2.4.13) can be estimated as follows,

$$\begin{aligned}\int_0^1 \frac{t^{n+s-1}(1-t)^{-s}}{(1-r^2t)^s}\frac{1}{(1+(rt)^n)^2}\,dt &\ge \frac{1}{4}\int_0^1 t^{n+s-1}\,dt \\ &= \frac{1}{4(n+s)}.\end{aligned} \tag{2.4.15}$$

Inequalities (2.4.14) and (2.4.15) imply that

$$u_s'(r) \le \frac{2n^2r\sin(\pi s)}{\pi}\left\{ sB(1+s,1-2s) - \frac{r^{n-2}}{4(n+s)}\right\}. \tag{2.4.16}$$

Let

$$r_u(n,s) := \{4s(n+s)B(1+s,1-2s)\}^{\frac{1}{n-2}}. \tag{2.4.17}$$

Then, as it follows from (2.4.16), if $r > r_u(n,s)$ we have $u_s'(r) < 0$. This shows that there are no critical points on the right from $r_u(n,s)$. Hence all the critical points belong to the interval $(r_l(n,s), r_u(n,s))$.

Note that ($n \geq 3$ is fixed)

$$\lim_{s\to 0+} r_u(n,s) = \lim_{s\to 0+} \{4s(n+s)B(1+s,1-2s)\}^{\frac{1}{n-2}} = 0,$$

and also

$$\lim_{s\to 0+} r_l(n,s) = \lim_{s\to 0+} \left(\frac{s}{s+n}\right)^{\frac{1}{n-2}} = 0.$$

Therefore, since $r_l(n,s) < r(n,s) < r_u(n,s)$ clearly

$$\lim_{s\to 0+} r(n,s) = 0,$$

as desired. ∎

Proof of Theorem 2.2.3. The idea of the following proof was suggested by Prof. Boris Shapiro.

According to Theorem 1.7 of [30], we know that for a generic configuration of point charges of the same sign, and for s sufficiently large, the critical points of the potential are in one-to-one correspondence with the effective cells in the Voronoi diagram of the considered configuration. Therefore, when positive unit point charges are placed at the vertices of a regular n-gon, it follows that the equilibrium points of the potential $u_s(r)$ are within the distance $\mathcal{O}(1/s)$ from the point of intersection of a perpendicular bisector and a side of a regular polygon. In our case the distance from the origin to the point of intersection is just the length of an apothem of a regular n-gon inscribed in a unit circumference, that is $\cos(\pi/n)$. Hence, we see that in the limit $s \to +\infty$, we have

$$\lim_{s\to +\infty} r(n,s) = \cos\left(\frac{\pi}{n}\right),$$

which proves the desired result. ∎

Proof of Proposition 2.3.1. Setting $q = 0$ in the second statement of Proposition 2.0.1, we easily obtain that

$$U_1(r,\theta) = \sum_{j=1}^{n} \frac{1}{1 + r^2 - 2r\cos(2\pi j/n - \theta)}$$
$$= \frac{n}{1-r^2}\frac{1-r^{2n}}{1+r^{2n}-2r^n\cos(n\theta)},$$

as desired. ■

Proof of Theorem 2.3.1. It is easy to see that the critical values of the angle θ are given by $\theta = \pi k/n$, where $k \in \mathbb{Z}$. Next, we sort out even and odd values of k. First, we have the following fact.

Lemma 2.4.5 *For $r \in [0,1)$, the potential $U_1(r,\theta)$ on a ray corresponding to even k has only one critical point, namely the origin.*

Proof. It is sufficient to consider the case $k = 0$. We readily find

$$U_1(r,0) = \frac{n}{1-r^2}\frac{1+r^n}{1-r^n}.$$

Differentiating, we obtain

$$\frac{\partial U_1(r,0)}{\partial r} = n\left\{\frac{2r}{(1-r^2)^2}\frac{1+r^n}{1-r^n} + \frac{2n}{1-r^2}\frac{r^{n-1}}{(1-r^n)^2}\right\}.$$

Now trivial estimates show that $\partial U_1(r,0)/\partial r > 0$ for $r \in (0,1)$. Therefore the potential $U_1(r,0)$ has only one critical point on the rays corresponding to even k, namely the origin. ■

Lemma 2.4.5 implies that all the critical points lie on the rays stemming from the origin and bisecting the edges of the regular polygon, i.e. for $\theta = \pi k/n, k \in \mathbb{Z}$, with odd k.

Let us consider a restriction of the potential $U_1(r,\theta)$ to the perpendicular bisectors of the sides. Without loss of generality we can assume $k = 1$ and set $v(r) := U_1(r,\pi/n)$. Then

$$v(r) = \frac{n}{1-r^2}\frac{1-r^n}{1+r^n}. \tag{2.4.18}$$

To find the critical points of $v(r)$, we differentiate (2.4.18),

$$v'(r) = 2n\frac{r(1 - r^{2n} - nr^{n-2} + nr^n)}{(1 - r^2)^2(1 + r^n)^2}. \tag{2.4.19}$$

We clearly see that there is a critical point of the potential $v(r)$ located at the origin. Now consider the polynomial $p_n(r) = -r^{2n} + nr^n - nr^{n-2} + 1$. Then $p_n'(r) = -2nr^{2n-1} + n^2r^{n-1} - n(n-2)r^{n-3}$, and $p_n''(r) = -2n(2n-1)r^{2n-2} + n^2(n-1)r^{n-2} - n(n-2)(n-3)r^{n-4}$. Note that $p_n(1) = p_n'(1) = 0$ and $p_n''(1) = -4n \neq 0$ for all $n \geq 3$. This shows that $r = 1$ is a zero of multiplicity 2 of the polynomial $p_n(r)$. Now observe that by Decartes's rule polynomial $p_n(r)$ has exactly three positive roots. Next, we will need the following lemma.

Lemma 2.4.6 *For all $r > 1$ and for all $n \geq 3$ $p_n(r) < 0$.*

Proof. Let $r > 1$ and $n \geq 3$. First we show that $p_n'(r) < 0$. As $p_n'(r) = -2nr^{2n-1} + n^2r^{n-1} - n(n-2)r^{n-3}$, we need to show that $2nr^{2n-1} - n^2r^{n-1} + n(n-2)r^{n-3} > 0$. That is the same as $n - 2 > nr^2 - 2r^{n+2}$. Consider the function $\psi(r) = nr^2 - 2r^{n+2}$. Then $\psi'(r) = 2nr - 2(n+2)r^{n+1} = 2r(n - (n+2)r^n) < 0$ for $r > 1$, so that the function $\psi(r)$ is strictly decreasing. Hence $\psi(1) > \psi(r)$ for $r > 1$. But $\psi(1) = n - 2$, so $n - 2 > nr^2 - 2r^{n+2}$, as claimed. This tells us that $p_n'(r) < 0$ for $r > 1$, as desired. Thus we see that $p_n(r)$ is a strictly decreasing function of r for $r > 1$. Therefore $p_n(1) > p_n(r)$ for $r > 1$, which implies that $p_n(r) < 0$. This concludes the proof. ■

Since by the Lemma 2.4.6 $p_n(r) < 0$ for all $r > 1$ and all $n \geq 3$, we conclude that $p_n(r)$ has exactly one simple zero on $(0, 1)$. Thus the potential $v(r)$ has exactly one critical point on $(0, 1)$. This completes the proof of the Theorem 2.3.1. ■

At this point we note that in the case $s = 1$ it is quite easy to show that the potential is in fact a Morse function. Indeed, we have the following

Lemma 2.4.7 *The potential $v(r)$ is a Morse function, that is its critical points are non-degenerate.*

Proof. We compute the first and the second derivatives as follows:

$$v'(r) = 2nrp_n(r)(1-r^2)^{-2}(1+r^n)^{-2},$$

$$v''(r) = 2n(1-r^2)^{-4}(1+r^n)^{-4}$$
$$\times \{(p_n(r)+rp_n'(r))(1-r^2)^2(1+r^n)^2 + rp_n(r)((1-r^2)^2(1+r^n)^2)'\}.$$

Suppose that $r_0 \neq 0$ is a critical point of $v(r)$, that is $v'(r_0) = 0$. Then $p_n(r_0) = 0$ for all $n \geq 3$, and

$$v''(r_0) = 2n(1-r_0^2)^{-4}(1+r_0^n)^{-4}\times$$
$$\{(p_n(r_0)+r_0p_n'(r_0))(1-r_0^2)^2(1+r_0^n)^2 + r_0p_n(r_0)((1-r^2)^2(1+r^n)^2)'(r_0)\}.$$

so that

$$v''(r_0) = \frac{2nr_0p_n'(r_0)}{(1-r_0^2)^2(1+r_0^n)^2} \neq 0.$$

It is clear from the expressions for $v'(r)$ and $v''(r)$ that $r = 0$ is a critical point for $v(r)$ and that it is not degenerate. Therefore all critical points of $v(r)$ are non-degenerate, which means that $v(r)$ is Morse on $[0, 1)$. ■

Proof of Theorem 2.3.2. First we will need a few facts from the theory of real analytic functions [46]. In particular, we will need

Proposition 2.4.1 (Real Analytic Identity Theorem) *Let $D \subset \mathbb{R}^m$ be an open connected set and let f be a real-analytic function on D. If there is a non-empty open set $U \subset D$ such that $f(x) = 0$ for all $x \in U$, then $f \equiv 0$ on D. If D is an open interval in $\mathbb{R}$, and if there is $U \subset D$ with an accumulation point in D, such that $f(x) = 0$ for all $x \in U$, then $f \equiv 0$ on D.*

We will also be making use of

Proposition 2.4.2 (Real Analytic Implicit Function Theorem) *Suppose $F : \mathbb{R}^2 \to \mathbb{R}$ is real analytic in a neghborhood of (x_0, y_0) for some $x_0 \in \mathbb{R}$ and some $y_0 \in \mathbb{R}$. If $F(x_0, y_0) = 0$ and*

$$\frac{\partial F(x_0, y_0)}{\partial y} \neq 0,$$

then there exists a function $h : \mathbb{R} \to \mathbb{R}$ which is real analytic in a neighborhood of x_0 and such that

$$F(x, h(x)) = 0$$

holds in a neighborhood of x_0.

Let us now briefly mention a few standard facts from the theory of stable mappings that will be used in the content of the discussion below.

Let X and Y be smooth manifolds. Denote by $C^\infty(X, Y)$ the set of smooth mappings from X to Y. The set $C^\infty(X, Y)$ equipped with Whitney topology becomes a topological space (for details see [33]).

Let φ and ψ be elements of $C^\infty(X, Y)$. We will call φ and ψ *isotopic* if there exist diffeomophisms $f : X \to X$ and $g : Y \to Y$, each homotopic to the identity on their respective space, such that $\varphi = g \circ \psi \circ f$. An element φ of $C^\infty(X, Y)$ will be called *stable* if there is a neighborhood W_φ of φ in $C^\infty(X, Y)$ such that each ψ in W_φ is isotopic to φ. Note that the stable mappings in $C^\infty(X, Y)$ always form an open subset.

Now employ the following observation, which can be found, for example, in [40]. Assume that two stable mappings in $C^\infty(X, Y)$ are connected by a path γ consisting of stable mappings. Then we can cover γ by a finite collection of open sets such that any two mappings in each set are isotopic. Using an induction argument it follows that any two mappings in γ are isotopic. Recalling that any two isotopic mappings have the same number of critical points, it follows that any two mappings in γ will have the same number of critical points.

Suppose X is a compact manifold and let φ be an element of $C^\infty(X, \mathbb{R})$. Then φ is stable if and only if φ is a Morse function whose critical values are pairwise distinct. In particular, it follows that if we have two Morse functions with distinct critical values connected by a path γ in $C^\infty(X, \mathbb{R})$ consisting of Morse functions with distinct critical values, any two functions in γ will have the same number of the critical points.

We also note [36, Ex. 19, p. 47] that if f is a Morse function with not necessarily distinct critical values, we can find a function $\tilde{f}$ that has the same critical points as f and is arbitrarily close to f in the C^2 topology. Moreover, the critical values of $\tilde{f}$ are distinct.

Now prove the following easy but important fact:

Lemma 2.4.8 *The potential $u_s(r)$ is a real analytic function in s, r.*

Proof. Note that $(1 + r^2 - 2r\cos(\theta_j)) > 0$ is an analytic function of r. Hence $\log(1 + r^2 - 2r\cos(\theta_j))$ is also analytic in r. Next, observe that $(1 + r^2 - 2r\cos(\theta_j))^{-s} = \exp(-s\log(1 + r^2 - 2r\cos(\theta_j))$, which is an analytic function in s, r. ■

We are now ready to state our extension result for the potential $u_s(r)$.

Lemma 2.4.9 (Extension Theorem for $u_s(r)$) *There exist real numbers s_l and s_r such that $1 \in (s_l, s_r)$ so that for each $s \in (s_l, s_r)$ the potential $u_s(r)$ has a unique critical point different from the origin, as well as a critical point at the origin.*

Remark 2.4.2 *Lemma 2.4.9 guarantees uniqueness for a non-trivial critical point of $u_s(r)$, with $s \in (s_l, s_r)$, where $1 \in (s_l, s_r)$. However, when utilizing this result to make appropriate claims for the 2D potential $U_s(r, \theta)$, we are only permitted to use the left hand-side neighborhood of $s = 1$, since a fact guaranteeing that all the non-trivial critical points of the Riesz potential $U_s(r, \theta)$ being located on the perpendicular bisectors to the sides, was proved for $s \in (0, 1]$.*

Proof. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined as $g(s, r) := u_s'(r)$. By Lemma 2.4.8 the potential $u_s(r)$ is analytic in s, r, so its derivatives with respect to s and r are also analytic in s, r. This implies that the function $g(s, r)$ is analytic in s, r. Let r_0 be a non-zero critical point of the potential $u_{s_0}(r)$ for $s_0 = 1$. By Lemma 2.4.7 we know that r_0 is a non-degenerate critical point, that is $u_{s_0}''(r_0) \neq 0$. Therefore

$$g(s_0, r_0) = 0, \quad \frac{\partial g(s_0, r_0)}{\partial r} \neq 0.$$

By the Real Analytic Implicit Function Theorem there exist a neighborhood $U_0 = (a,b) \times (c,d)$ of (s_0, r_0) and an analytic function $h : \mathbb{R} \to \mathbb{R}$ such that $\{(s,r) \in U_0 : g(s,r) = 0\} = \{(s, h(s)) : s \in (a,b)\}$. That is, $u_s'(h(s)) = 0$ in (a,b), and $h(s)$ is the only such a root of $u_s'(r) = 0$.

We conclude that for each s in a neighborhood of $s_0 = 1$ there exists a unique critical point of $u_s(r)$, call it r_s, and r_s depends analytically on s.

Now observe that from the Real Analytic Identity Theorem it follows that an analytic function on a closed interval can only have a finite number of zeros. Therefore $\partial g(s,r)/\partial r$ has a finite number of zeros. One may shrink U_0 such that $\partial g(s,r)/\partial r \neq 0$ on U_0 (since $\partial g(s,r)/\partial r \neq 0$ at (s_0, r_0)). Denote an interval of such s's as (s_l, s_r). Then all critical points r_s are non-degenerate for $s \in (s_l, s_r)$. Note that $1 \in (s_l, s_r)$.

After these preliminary remarks let us make the following observation. Since we know that the potential $u_s(r)$ has no critical points on $(\cos(\pi/n), 1)$, we can assume that $r \in [0, \cos(\pi/n)]$ and $s \in (s_l, s_r)$. It is clear that $u_s \in C^\infty([0, \cos(\pi/n)], \mathbb{R})$ for all $s \in (s_l, s_r)$.

Next, note that for each $s \in (s_l, s_r)$ the potential $u_s(r)$ is a Morse function. Hence u_s defines a path in $C^\infty([0, \cos(\pi/n)], \mathbb{R})$. Therefore by above considerations any two functions in the path defined by u_s have the same number of critical points. As we have shown that $u_s(r)$ for $s = 1$ has exactly one critical point (excluding the origin), and $1 \in (s_l, s_r)$, it follows that for any $s \in (s_l, s_r)$ the potential $u_s(r)$ also has exactly one critical point (excluding origin). The proof of the lemma is complete. ■

Recalling that the potential $u_s(r)$ is a restriction of the 2D potential $U_s(r,\theta)$ to the perpendicular bisectors to the sides, and utilizing the fact that all the non-trivial critical points of the potential $U_s(r,\theta)$ for $s \in (0,1]$ are located on perpendicular bisectors to the sides, the statement of the theorem follows. ■

Proof of Example 2.3.1. In this case the Riesz potential on the bisectors is of the form

$$u_s(r) = \frac{1}{(1+r)^{2s}} + \frac{2}{(1+r^2-r)^s}, \quad r \in (0,1).$$

Its derivative is

$$\begin{aligned} u_s'(r) &= \frac{-2s}{(1+r)^{2s+1}} + \frac{-2s(2r-1)}{(1+r^2-r)^{s+1}} \\ &= \frac{-2s}{(1+r)^{2s+1}(1+r^2-r)^{s+1}}((2r-1)(1+r)^{2s+1} + (1+r^2-r)^{s+1}) \\ &= -2s(1+r)^{-(2s+1)}(1+r^2-r)^{-(s+1)} f_s(r), \end{aligned}$$

where $f_s(r) := (1+r)^{2s+1}(2r-1) + (1+r^2-r)^{s+1}$. Differentiating, we obtain

$$f_s'(r) = (2s+1)(1+r)^{2s}(2r-1) + 2(1+r)^{2s+1} + (s+1)(1+r^2-r)^s(2r-1).$$

For the second derivative we have

$$\begin{aligned} f_s''(r) &= (2s+1)(2s(1+r)^{2s-1}(2r-1) + 2(1+r)^{2s}) \\ &\quad + 2(2s+1)(1+r)^{2s} \\ &\quad + (s+1)(s(1+r^2-r)^{s-1}(2r-1)^2 + 2(1+r^2-r)^s) \\ &= 2(2s+1)(1+r)^{2s-1}(s(2r-1) + (1+r)) \\ &\quad + 2(2s+1)(1+r)^{2s} \\ &\quad + (s+1)(s(1+r^2-r)^{s-1}(2r-1)^2 + 2(1+r^2-r)^s) \\ &= 2(2s+1)(1+r)^{2s-1}((2s+1)r + (1-s)) \\ &\quad + 2(2s+1)(1+r)^{2s} \\ &\quad + (s+1)(s(1+r^2-r)^{s-1}(2r-1)^2 + 2(1+r^2-r)^s). \end{aligned}$$

As $s \in (0,1)$, it follows that $1-s > 0$, and we see from the above that $f_s'' > 0$ for all $r \in (0,1)$ and all $s \in (0,1)$. That says that f_s is strictly convex on $(0,1)$ for all $s \in (0,1)$.

Observe that the roots of $f_s(r)$ are exactly the critical points of the potential $u_s(r)$. Applying Theorem 2.1.1 with $n = 3$, we see that $f_s(r)$ has a positive root on $(0, 1/2)$, for all $s \in (0,1)$. Note that $f_s(0) = -1+1 = 0$, and $f_s(1/2) = (3/4)^{s+1} > 0$. The strict convexity of $f_s(r)$ implies that this positive root is unique.

We have shown that the potential u_s has a unique non-trivial critical point for all $s \in (0,1)$. Recalling that the potential $u_s(r)$ is a restriction of the 2D Riesz potential $U_s(r,\theta)$ to the perpendicular bisectors, we conclude that $U_s(r,\theta)$ has exactly four critical points in the case of an equilateral triangle for *all* $s \in (0,1)$, as claimed. ■

CHAPTER 3

Minimum Energy Problem on the Hypersphere

3.1 Introduction and main results

This section is based on work [10]. We introduce hyperspherical polar coordinates $r, \theta_1, \theta_2, \ldots, \theta_{d-2}, \varphi$, defined by

$$\begin{aligned}
x_1 &= r\cos\theta_1,\\
x_2 &= r\sin\theta_1\cos\theta_2,\\
x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3,\\
&\vdots\\
x_{d-2} &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{d-3}\cos\theta_{d-2},\\
x_{d-1} &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{d-2}\cos\varphi,\\
x_d &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{d-2}\sin\varphi,
\end{aligned}$$

where $0 \le r$, $0 \le \theta_j \le \pi$, $j = 1, 2, \ldots, d-2$, and $0 \le \varphi \le 2\pi$, see [3]. The surface area element of the unit sphere $\mathbb{S}^{d-1}$, written in hyperspherical coordinates, is given by

$$d\sigma_d = \sin^{d-2}\theta_1\,\sin^{d-3}\theta_2\ldots\sin\theta_{d-2}\,d\theta_1\,d\theta_2\ldots d\theta_{d-2}\,d\varphi.$$

The total surface area of the sphere $\mathbb{S}^{d-1}$ is given by

$$\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

A spherical cap on the sphere $\mathbb{S}^{d-1}$, centered at the North Pole, is defined via an angle α, $0 < \alpha \leq \pi$, as

$$C_{N,\alpha} := \{(r, \theta_1, \ldots, \theta_{d-2}, \varphi) : r = 1, \quad 0 \leq \theta_1 \leq \alpha, \quad 0 \leq \theta_j \leq \pi, \\ j = 2, \ldots, d-2, \quad 0 \leq \varphi \leq 2\pi\}.$$

Similarly, a spherical cap centered at the South Pole, is defined in terms of an angle α, $0 < \alpha \leq \pi$, as

$$C_{S,\alpha} := \{(r, \theta_1, \ldots, \theta_{d-2}, \varphi) : r = 1, \quad \alpha \leq \theta_1 \leq \pi, \quad 0 \leq \theta_j \leq \pi, \\ j = 2, \ldots, d-2, \quad 0 \leq \varphi \leq 2\pi\}.$$

In what follows, we will need to use certain special functions, for which we fix the notation here. The incomplete Beta function $\mathrm{B}(z; a, b)$ is defined as

$$\mathrm{B}(z; a, b) := \int_0^z t^{a-1}(1-t)^{b-1}\, dt. \tag{3.1.1}$$

The Gauss hypergeometric function ${}_2F_1(a, b; c, z)$ is defined via series

$${}_2F_1(a, b; c, z) := \sum_{n=0}^{\infty} \frac{(a)_n\, (b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z| < 1, \tag{3.1.2}$$

where $(a)_0 := 1$ and $(a)_n := a(a+1)\ldots(a+n-1)$ for $n \geq 1$ is the Pochhammer symbol.

We begin by recording sufficient conditions on an external field Q, that guarantee that the extremal support S_Q of the equilibrium measure μ_Q is a spherical cap $C_{S,\alpha}$, centered at the South Pole. The following proposition is a consequence of a more general statement, proved in [17].

Proposition 3.1.1 *Let a non-negative external field Q be rotationally invariant with respect to rotations about the polar axis x_1, that is $Q(x) = Q(x_1)$, where $x = (x_1, x_2, \ldots, x_d) \in \mathbb{S}^{d-1}$. Suppose that $Q(x_1)$ is an increasing convex function on $[-1, 1]$. Then the support of the extremal measure μ_Q is a spherical cap centered at the South Pole, that is $S_Q = C_{S,\alpha}$.*

The following result is an important step towards the recovery of the equilibrium measure.

Theorem 3.1.1 *Suppose that an external field Q is rotationally invariant with respect to rotations about the polar axis, and is such that $Q \in C^2(N)$, where N is an open neighborhood of $S_Q = C_{S,\alpha}$ on the sphere $\mathbb{S}^{d-1}$. Then the equilibrium measure μ_Q is absolutely continuous with respect to the Lebesgue surface measure, with a locally bounded density, that is $d\mu_Q = f(\theta_1)\, d\sigma_d$, where $f \in L^\infty([\alpha, \pi])$.*

The support S_Q is a main ingredient in determining the equilibrium measure μ_Q itself. Indeed, if S_Q is known, then the equilibrium measure μ_Q can be recovered by solving the singular integral equation

$$\int \frac{1}{|x-y|^{d-2}}\, d\mu(y) + Q(x) = F_Q, \quad x \in S_Q, \tag{3.1.3}$$

where F_Q is a constant (see (1.2.2)).

We solve this equation and obtain the following two theorems, that describe explicitly the equilibrium density when support S_Q is either a spherical cap $C_{N,\alpha}$, centered at the North Pole, or $C_{S,\alpha}$, a spherical cap centered at the South Pole. This extends the corresponding results stated in Theorem 2 and Theorem 3 in [9], for the case $d = 3$.

Theorem 3.1.2 *Suppose that an external field Q is rotationally invariant with respect to rotations about the polar axis, and is such that $Q \in C^2(N)$, where N is an open neighborhood of S_Q in $\mathbb{S}^{d-1}$. Assume that S_Q is a spherical cap $C_{N,\alpha}$ centered at the North Pole, with $0 < \alpha \leq \pi$. Let*

$$F(\eta) = \frac{\Gamma((d-2)/2)}{2\,\pi^{(d+2)/2}}\, \frac{1}{\sin\eta}\, \sec^{d-3}\left(\frac{\eta}{2}\right) \frac{d}{d\eta} \int_\eta^\alpha \frac{g(\zeta)\sin\zeta\, d\zeta}{\sqrt{\cos\eta - \cos\zeta}}, \tag{3.1.4}$$

with $0 \leq \eta \leq \alpha$, where

$$g(\zeta) = \cot^{d-3}\left(\frac{\zeta}{2}\right) \frac{d}{d\zeta} \int_0^\zeta \frac{Q(\theta)\, \sin^{d-3}(\theta/2) \sin\theta\, d\theta}{\sqrt{\cos\theta - \cos\zeta}}, \quad 0 \leq \zeta \leq \alpha. \tag{3.1.5}$$

Then the density f of the equilibrium measure μ_Q is given by

$$f(\eta) = C_Q \left(\frac{1+\cos\alpha}{1+\cos\eta} \right)^{\frac{d-1}{2}} \left(\frac{1+\cos\alpha}{\cos\eta - \cos\alpha} \right)^{\frac{1}{2}} \times \\ {}_2F_1\left(1, \frac{d-1}{2}; \frac{1}{2}; \frac{\cos\eta - \cos\alpha}{1+\cos\eta}\right) + F(\eta), \quad 0 \le \eta \le \alpha. \tag{3.1.6}$$

The constant C_Q is uniquely defined by

$$C_Q = \frac{\Gamma(d/2-1)}{2^{d-2}\sqrt{\pi}\,\Gamma((d-1)/2)} \left(\mathrm{B}\left(\sin^2\left(\frac{\alpha}{2}\right); \frac{d-2}{2}, \frac{d}{2} \right) \right)^{-1} \times \\ \left\{ \frac{\Gamma((d-1)/2)}{2\pi^{(d-1)/2}} - \int_0^\alpha F(\eta)\, \sin^{d-2}\eta\, d\eta \right\}. \tag{3.1.7}$$

An analogous statement for the support being a spherical cap $C_{S,\alpha}$ centered at the South Pole, is of the following nature.

Theorem 3.1.3 *Suppose that an external field Q is rotationally invariant with respect to rotations about the polar axis, and is such that $Q \in C^2(N)$, where N is an open neighborhood of S_Q in $\mathbb{S}^{d-1}$. Assume that S_Q is a spherical cap $C_{S,\alpha}$, centered at the South Pole, with $0 < \alpha \le \pi$. Let*

$$F(\eta) = \frac{\Gamma((d-2)/2)}{2\,\pi^{(d+2)/2}} \frac{1}{\sin\eta} \csc^{d-3}\left(\frac{\eta}{2}\right) \frac{d}{d\eta} \int_\alpha^\eta \frac{g(\zeta)\sin\zeta\, d\zeta}{\sqrt{\cos\zeta - \cos\eta}}, \tag{3.1.8}$$

with $\alpha \le \eta \le \pi$, where

$$g(\zeta) = \tan^{d-3}\left(\frac{\zeta}{2}\right) \frac{d}{d\zeta} \int_\zeta^\pi \frac{Q(\theta)\, \cos^{d-3}(\theta/2) \sin\theta\, d\theta}{\sqrt{\cos\zeta - \cos\theta}}, \quad \alpha \le \zeta \le \pi. \tag{3.1.9}$$

Then the density f of the equilibrium measure μ_Q is given by

$$f(\eta) = C_Q \left(\frac{1-\cos\alpha}{1-\cos\eta} \right)^{\frac{d-1}{2}} \left(\frac{1-\cos\alpha}{\cos\alpha - \cos\eta} \right)^{\frac{1}{2}} \times \\ {}_2F_1\left(1, \frac{d-1}{2}; \frac{1}{2}; \frac{\cos\alpha - \cos\eta}{1-\cos\eta}\right) + F(\eta), \quad \alpha \le \eta \le \pi. \tag{3.1.10}$$

The constant C_Q is uniquely determined by

$$C_Q = \frac{\Gamma(d/2-1)}{2^{d-2}\sqrt{\pi}\,\Gamma((d-1)/2)} \left(\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right); \frac{d-2}{2}, \frac{d}{2} \right) \right)^{-1} \times \\ \left\{ \frac{\Gamma((d-1)/2)}{2\pi^{(d-1)/2}} - \int_\alpha^\pi F(\eta)\, \sin^{d-2}\eta\, d\eta \right\}. \tag{3.1.11}$$

3.2 Applications to the external field of a point charge and a quadratic external field

We first consider the case of no external field, when the support is a spherical cap centered at the South Pole, that is $S_Q = C_{S,\alpha}$, $Q = 0$. The equilibrium measure for the spherical cap centered at the South Pole $C_{S,\alpha}$, for the case of general Riesz potentials, was first found in [28] (see also [17]).

Proposition 3.2.1 *The density of the equilibrium measure of a spherical cap $C_{S,\alpha}$ with no external field is*

$$f(\eta) = \frac{\Gamma(d/2-1)}{2^{d-1}\,\pi^{d/2}} \left(\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right); \frac{d-2}{2}, \frac{d}{2} \right) \right)^{-1} \times \tag{3.2.1}$$
$$\left(\frac{1-\cos\alpha}{1-\cos\eta}\right)^{\frac{d-1}{2}} \left(\frac{1-\cos\alpha}{\cos\alpha-\cos\eta}\right)^{\frac{1}{2}} {}_2F_1\left(1, \frac{d-1}{2}; \frac{1}{2}; \frac{\cos\alpha-\cos\eta}{1-\cos\eta}\right),$$

where $\alpha \leq \eta \leq \pi$. The capacity of $C_{S,\alpha}$ is given by

$$\mathrm{cap}(C_{S,\alpha}) = \frac{2^{d-2}\,\Gamma((d-1)/2)}{\sqrt{\pi}\,\Gamma(d/2-1)}\,\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right); \frac{d-2}{2}, \frac{d}{2} \right). \tag{3.2.2}$$

Suppose now that the sphere $\mathbb{S}^{d-1}$ is immersed in an external field Q, that satisfies the conditions of Proposition 3.1.1. Then the support S_Q of the weighted equilibrium measure μ_Q will be a spherical cap $C_{S,\alpha}$, centered at the South Pole. The angle α, defining the extremal support $C_{S,\alpha}$, can be found via the Newtonian analog of the Mhaskar-Saff $\mathcal{F}$-functional, which is defined as follows.

Definition 3.2.1 *The $\mathcal{F}$-functional of a compact subset $E \subset \mathbb{S}^{d-1}$ of positive (Newtonian) capacity is defined as*

$$\mathcal{F}(E) := W(E) + \int Q(x)\,d\mu_E(x), \tag{3.2.3}$$

where $W(E)$ is the Newtonian energy of the compact E and μ_E is the equilibrium measure (with no external field) on E.

The main objective of introducing the $\mathcal{F}$-functional is its following extremal property, proved in [17] for the general Riesz potentials.

Proposition 3.2.2 *Let Q be an external field on $\mathbb{S}^{d-1}$. Then $\mathcal{F}$-functional is minimized for $S_Q = \operatorname{supp}(\mu_Q)$.*

If $E = C_{S,\alpha}$, taking into account that $W(C_{S,\alpha}) = 1/\operatorname{cap}(C_{S,\alpha})$, and inserting (3.2.2) and (3.2.1) into (3.2.3), we find that $\mathcal{F}$-functional is given by

$$\begin{aligned}\mathcal{F}(C_{S,\alpha}) = \frac{\sqrt{\pi}\,\Gamma(d/2-1)}{2^{d-2}\,\Gamma((d-1)/2)} &\left(\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right); \frac{d-2}{2}, \frac{d}{2}\right)\right)^{-1} \times \\ &\left\{1 + \frac{1}{\pi}\int_\alpha^\pi Q(\eta)\left(\frac{1-\cos\alpha}{1-\cos\eta}\right)^{\frac{d-1}{2}}\left(\frac{1-\cos\alpha}{\cos\alpha-\cos\eta}\right)^{\frac{1}{2}} \times\right. \\ &\left. {}_2F_1\left(1, \frac{d-1}{2}; \frac{1}{2}; \frac{\cos\alpha-\cos\eta}{1-\cos\eta}\right)\sin^{d-2}\eta\, d\eta\right\}. \qquad (3.2.4)\end{aligned}$$

As a first applications of our results, we consider the situation when the sphere $\mathbb{S}^{d-1}$ is immersed in an external field generated by a positive point charge of magnitude q placed at the North Pole of the sphere, namely

$$Q(x) = q\,(1-x_1)^{-(d-2)/2}, \quad q > 0, \quad x \in \mathbb{S}^{d-1}. \qquad (3.2.5)$$

Note that the extremal measure for such a field was first obtained in [28], for general Riesz potentials. For $d = 3$ and when the charges are assumed to interact according to the Newtonian potential, the extremal density and Mhaskar-Saff functional, along with its critical points, were computed in [9] (see Proposition 4 and Theorems 4 and 5 there). We also remark that it is possible to extend the results of [17] to cover such a case as well.

It is clear that external field Q in (3.2.5) is invariant with respect to the rotations about the polar axis. Also, $Q(x_1)$ is a non-negative increasing convex function on $[-1, 1]$. From Proposition 3.1.1 it then follows that the support of the corresponding equilibrium measure μ_Q will be a spherical cap $C_{S,\alpha}$, centered at the South Pole. The crucial step towards the recovery of the equilibrium measure for this external field is to determine the support of the equilibrium measure. For that we first compute the Mhaskar-Saff $\mathcal{F}$-functional, by inserting expression (3.2.5) for the external field Q into (3.2.4).

Theorem 3.2.1 *The $\mathcal{F}$-functional for the spherical cap $C_{S,\alpha}$ when the external field is produced by a positive point charge of magnitude q, placed at the North Pole, is given by*

$$\begin{aligned}\mathcal{F}(C_{S,\alpha}) =&\frac{\sqrt{\pi}\,\Gamma(d/2-1)}{2^{d-2}\,\Gamma((d-1)/2)}\left(\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{d}{2}\right)\right)^{-1}\times \\ &\left\{1+\frac{q\,2^{(d-2)/2}\,\Gamma((d-1)/2)}{\sqrt{\pi}\,\Gamma(d/2-1)}\,\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{1}{2}\right)\right\}.\end{aligned} \tag{3.2.6}$$

Applying Theorem 3.1.3, we compute the density of the equilibrium measure, corresponding to this external field.

Theorem 3.2.2 *For the external field Q given by* (3.2.5)*, the support S_Q is a spherical cap $C_{S,\alpha}$ centered at the South Pole, with $\alpha=\alpha_0\in(0,\pi)$. The angle α_0 is defined as a unique solution of the equation*

$$\begin{aligned}\csc^{d-1}\left(\frac{\alpha}{2}\right)\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{d}{2}\right)-\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{1}{2}\right)& \\ =\frac{\sqrt{\pi}\,\Gamma(d/2-1)}{q\,2^{(d-2)/2}\,\Gamma((d-1)/2)}&.\end{aligned} \tag{3.2.7}$$

Let

$$\begin{aligned}C_Q =&\frac{\Gamma(d/2-1)}{2^{d-1}\,\pi^{d/2}}\left(\mathrm{B}\left(\cos^2\left(\frac{\alpha_0}{2}\right);\frac{d-2}{2},\frac{d}{2}\right)\right)^{-1}\times \\ &\left\{1+\frac{q\,2^{(d-2)/2}\,\Gamma((d-1)/2)}{\sqrt{\pi}\,\Gamma(d/2-1)}\,\mathrm{B}\left(\cos^2\left(\frac{\alpha_0}{2}\right);\frac{d-2}{2},\frac{1}{2}\right)\right\}.\end{aligned} \tag{3.2.8}$$

The density of the equilibrium measure μ_Q is given by

$$\begin{aligned}f(\eta) = C_Q&\left(\frac{1-\cos\alpha_0}{1-\cos\eta}\right)^{(d-1)/2}\times \\ &\sqrt{\frac{1-\cos\alpha_0}{\cos\alpha_0-\cos\eta}}\,{}_2F_1\left(1,\frac{d-1}{2};\frac{1}{2};\frac{\cos\alpha_0-\cos\eta}{1-\cos\eta}\right) \\ &-\frac{q\,\Gamma((d-1)/2)}{\sqrt{2}\,\pi^{(d+1)/2}}\,\frac{1}{(1-\cos\eta)^{(d-1)/2}}\sqrt{\frac{1-\cos\alpha_0}{\cos\alpha_0-\cos\eta}},\quad \alpha_0\le\eta\le\pi.\end{aligned} \tag{3.2.9}$$

As a second application, we consider the case when the external field Q is a quadratic polynomial of the form

$$Q(x) = (1 + x_1)^2, \quad x \in \mathbb{S}^{d-1}, \quad d \geq 3. \tag{3.2.10}$$

An external field given by a quadratic polynomial was first considered in [9] for the case $d = 3$ and Newton potential, see Proposition 5 and Theorems 7 and 8 there. Below we extend the corresponding statements from [9] to the arbitrary dimension $d \geq 3$, for the monic quadratic polynomial of the form appearing on the right hand side of (3.2.10).

It is a straightforward calculation to verify that $Q(x)$ is a nonnegative convex increasing function on $[-1, 1]$, also possessing rotational symmetry with respect to rotations about the polar axis. Therefore, by Proposition 3.1.1, the support of the equilibrium measure μ_Q for this external field will be a spherical cap $C_{S,\alpha}$. Following the established procedure, we first compute the Mhaskar-Saff $\mathcal{F}$-functional, by substituting expression for the external field (3.2.10) into (3.2.4).

Theorem 3.2.3 *In the case of the rational external field* (3.2.10)*, the Mhaskar-Saff $\mathcal{F}$-functional for the spherical cap $C_{S,\alpha}$ is of the form*

$$\begin{aligned}\mathcal{F}(C_{S,\alpha}) =& \frac{\sqrt{\pi}\,\Gamma(d/2-1)}{2^{d-2}\,\Gamma((d-1)/2)} \left(\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right); \frac{d}{2} - 1, \frac{d}{2} \right) \right)^{-1} \times \\ & \left\{ 1 + \frac{2^d\,\Gamma((d+3)/2)}{\sqrt{\pi}\,\Gamma(d/2+1)} \mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right); \frac{d}{2} + 1, \frac{d}{2} \right) \right\}.\end{aligned} \tag{3.2.11}$$

The density of the corresponding equilibrium measure is found by applying Theorem 3.1.3.

Theorem 3.2.4 *If the external field Q defined via* (3.2.5)*, the support S_Q is a spherical cap $C_{S,\alpha}$ centered at the South Pole, with $\alpha = \alpha_0 \in (0, \pi)$. The angle α_0 is defined as a unique solution of the equation*

$$\begin{aligned}\cos^4\left(\frac{\alpha}{2}\right) \mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right); \frac{d}{2} - 1, \frac{d}{2} \right) - \mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right); \frac{d}{2} + 1, \frac{d}{2} \right)& \\ = \frac{\sqrt{\pi} d(d-2)\,\Gamma(d/2-1)}{2^d(d^2-1)\Gamma((d-1)/2)}&.\end{aligned} \tag{3.2.12}$$

Let

$$F(\eta) = -\frac{2\,\Gamma((d+3)/2)}{d(d-2)\pi^{(d+1)/2}}\times \tag{3.2.13}$$
$$\Bigg\{ \left(\frac{1-\cos\alpha_0}{1-\cos\eta}\right)^{d/2} \sqrt{\frac{1-\cos\eta}{\cos\alpha_0-\cos\eta}}\,(1+\cos\alpha_0)^2$$
$$+\,2(d-1)\,\mathrm{B}\Bigg(\frac{\cos\alpha_0-\cos\eta}{1-\cos\eta};\frac{1}{2},\frac{d}{2}\Bigg)$$
$$-\,2(d+1)\,(1-\cos\eta)\,\mathrm{B}\Bigg(\frac{\cos\alpha_0-\cos\eta}{1-\cos\eta};\frac{1}{2},\frac{d}{2}+1\Bigg)$$
$$+\,\frac{d+3}{2}\,(1-\cos\eta)^2\,\mathrm{B}\Bigg(\frac{\cos\alpha_0-\cos\eta}{1-\cos\eta};\frac{1}{2},\frac{d}{2}+2\Bigg)\Bigg\},$$

where $\alpha_0 \le \eta \le \pi$, *and*

$$C_Q = \frac{\Gamma(d/2-1)}{2^{d-1}\,\pi^{d/2}}\left(\mathrm{B}\left(\cos^2\left(\frac{\alpha_0}{2}\right);\frac{d-2}{2},\frac{d}{2}\right)\right)^{-1}\times \tag{3.2.14}$$
$$\left\{1+\frac{2^d\,\Gamma((d+3)/2)}{\sqrt{\pi}\,\Gamma(d/2+1)}\mathrm{B}\left(\cos^2\left(\frac{\alpha_0}{2}\right);\frac{d}{2}+1,\frac{d}{2}\right)\right\}.$$

The density of the equilibrium measure μ_Q *is given by*

$$f(\eta) = C_Q\left(\frac{1-\cos\alpha_0}{1-\cos\eta}\right)^{(d-1)/2}\sqrt{\frac{1-\cos\alpha_0}{\cos\alpha_0-\cos\eta}}\times \tag{3.2.15}$$
$${}_2F_1\left(1,\frac{d-1}{2};\frac{1}{2};\frac{\cos\alpha_0-\cos\eta}{1-\cos\eta}\right)+F(\eta), \quad \alpha_0\le\eta\le\pi.$$

3.3 Proofs

Proof of Theorem 3.1.1. The idea of the proof is to show that the equilibrium potential U^{μ_Q} is Lipschitz continuous on $C_{S,\alpha}$. If that is established, it will imply that the normal derivatives of U^{μ_Q} exist a.e. on $C_{S,\alpha}$. Then the measure μ_Q can be recovered from its potential by the formula

$$d\mu_Q = -\frac{1}{(d-2)\,\omega_d}\left(\frac{\partial U^{\mu_Q}}{\partial n_+}+\frac{\partial U^{\mu_Q}}{\partial n_-}\right)\,d\sigma := f(\theta_1)\,d\sigma, \tag{3.3.1}$$

where $d\sigma$ is the Lebesgue surface measure on $\mathrm{supp}(\mu_Q)$, n_+ and n_- are the inner and the outer normals to the cap $C_{S,\alpha}$. It is clear that the normal derivatives of U^{μ_Q} are bounded a.e. by the Lipschitz constant, and hence we obtain $f\in L^\infty_{\mathrm{loc}}([\alpha,\pi])$.

Our first step is to construct an extension of the external field Q to $\mathbb{S}^{d-1}$ in such a way that the extremal measures for the cap $C_{S,\alpha}$ and the sphere $\mathbb{S}^{d-1}$ are the same. Recall that the external field Q is a C^2 function on an open neighborhood N of S_Q in $\mathbb{S}^{d-1}$. We can adjust Q in such a way that for the new external field $\widetilde{Q}$ one has

$$U^{\mu_Q}(x) + \widetilde{Q}(x) = F_Q, \quad x \in S_Q,$$

$$U^{\mu_Q}(x) + \widetilde{Q}(x) \geq F_Q, \quad x \in \mathbb{S}^{d-1},$$

and also $\widetilde{Q} \in C^2(\mathbb{S}^{d-1})$. To show that it is indeed possible, we first remark that the external field Q is rotationally symmetric with respect to the rotations about the polar axis. Therefore, Q is a function of the polar angle θ_1 only, that is $Q = Q(\theta_1)$. This symmetry is also inherited by potential, so that on the sphere $\mathbb{S}^{d-1}$ we have $U^{\mu_Q}(x) = U^{\mu_Q}(\theta_1)$, $x = (r, \theta_1, \ldots, \theta_{d-2}, \varphi) \in \mathbb{S}^{d-1}$. Next, note that $N = \{(r, \theta_1, \ldots, \theta_{d-2}, \varphi) : r = 1, \quad \gamma < \theta_1 \leq \pi, \quad 0 \leq \theta_j \leq \pi, j = 2, \ldots, d-2, \quad 0 \leq \varphi \leq 2\pi\}$, with some $\gamma \in (0, \alpha)$. Pick a number ϵ such that $\gamma < \epsilon < \alpha$. We define a new external field $\widetilde{Q}$ as follows: set $\widetilde{Q}(\theta_1) = Q(\theta_1)$, for $\epsilon < \theta_1 \leq \pi$, while on $[0, \epsilon]$ we tweak Q to $\widetilde{Q}$ in such a way that

$$U^{\mu_Q}(\theta_1) + \widetilde{Q}(\theta_1) \geq F_Q,$$

and $\widetilde{Q} \in C^2(\mathbb{S}^{d-1})$. Applying Theorem 4.2.14 from [13], we infer that $\mu_{\widetilde{Q}} = \mu_Q$ and $F_{\widetilde{Q}} = F_Q$.

Let u and v denote the equilibrium potentials for the minimum energy problem on $C_{S,\alpha}$ and $\mathbb{S}^{d-1}$, respectively. Since the equilibrium measure is the same for those two sets, it immediately follows that $u = v$. Now observe that the spherical cap $C_{S,\alpha}$ is a part of the sphere $\mathbb{S}^{d-1}$, which is a closed smooth surface. Thus we can invoke the result of Götz [34] to conclude that v is Lipschitz continuous in an open neighborhood $\mathcal{U}$ of $\mathbb{S}^{d-1}$. Hence U^{μ_Q} is Lipschitz continuous on $C_{S,\alpha}$.

We finish the proof by justifying formula (3.3.1).

Lemma 3.3.1 *Let U^μ be the potential of a measure μ in a domain $G \subset \mathbb{R}^d$. Suppose that the intersection of* $\operatorname{supp}(\mu)$ *and the domain G is a connected C^1 hypersurface Ω. Suppose also that the potential U^μ is Lipschitz continuous on an open neighborhood of Ω. Then on Ω the measure μ is locally absolutely continuous with respect to the Lebesgue surface measure $d\sigma$, and we have the representation*

$$d\mu = -\frac{1}{(d-2)\,\omega_d}\left(\frac{\partial U^\mu}{\partial n_+} + \frac{\partial U^\mu}{\partial n_-}\right) d\sigma, \tag{3.3.2}$$

where n_+ and n_- are the inner and the outer normals to Ω.

Proof. For the case $d = 3$ formula (3.3.2) is proved in [44, p. 164]. Also, when the measure μ is supported on a hyperplane, expression (3.3.2) was derived in [76, Lemma 3.1, p. 48].

We begin by observing that there is a neighborhood of Ω where Ω separates G into two pieces. We will be denoting the intersection of G with that neighborhood again by G. We next pick an interior point $x \in \Omega$ and consider a small ball $B(r, x)$ centered at x, where $r > 0$ is chosen such that $\overline{B(r, x)} \subset G$. We then choose a positive side of Ω and denote the normal in that direction by n_+, while the normal to a negative side of Ω will be denoted by n_-. We will also use the subscripts $+$ and $-$ to distinguish the subsets of G and $B(r, x)$ that lie on positive and negative sides of Ω.

Let $u = U^\mu$ and $v(y) = 1/|x-y|^{d-2}$. Observe that when x is fixed, the function $v(y)$ is harmonic for $y \neq x$. In particular, it is harmonic on a neighborhood of $G_+ \backslash B_+(r, x)$. We also know that u is harmonic in $G \setminus \Omega$. Therefore, there exists a compact set with a neighborhood where u and v are both harmonic. Let $V_\varepsilon := \{x \in G : \operatorname{dist}(x, \Omega) < \varepsilon\}$ be a small open neighborhood of Ω, and set $K_\varepsilon := (G_+ \setminus B_+(r, x)) \setminus V_\varepsilon$. The Green's identity [39, p. 22] states that

$$\int_{K_\varepsilon} (u\Delta v - v\Delta u)\, dy = \int_{\partial K_\varepsilon} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\sigma, \tag{3.3.3}$$

where $\partial/\partial n$ denotes the inward normal derivative on $G \setminus B(r, x)$. As both u and v

are harmonic in a neighborhood of K_ε, the left hand side of (3.3.3) is zero. Therefore,

$$\int_{\partial K_\varepsilon} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) d\sigma = 0. \tag{3.3.4}$$

Since the potential u is Lipschitz continuous, its normal derivative is bounded a.e. by a Lipschitz constant. Therefore, passing to the limit $\varepsilon \to 0+$ in (3.3.4), and applying the Dominated Convergence Theorem, we deduce that

$$\int_{\partial(G_+ \setminus B_+(r,x))} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) d\sigma = 0. \tag{3.3.5}$$

We proceed by splitting the domain of integration in (3.3.5) into a component that lies on Ω, and the two components that do not. On the positive side of Ω relation (3.3.5) reads

$$\begin{aligned} &\int_{\partial(G_+ \setminus B_+(r,x))\setminus\Omega} \left(u\frac{\partial v}{\partial n_+} - v\frac{\partial u}{\partial n_+} \right) d\sigma \\ &\qquad + \int_{\partial(G_+ \setminus B_+(r,x))\cap\Omega} \left(u\frac{\partial v}{\partial n_+} - v\frac{\partial u}{\partial n_+} \right) d\sigma = 0. \end{aligned} \tag{3.3.6}$$

By similar considerations, working with the negative side of Ω, we obtain

$$\begin{aligned} &\int_{\partial(G_- \setminus B_-(r,x))\setminus\Omega} \left(u\frac{\partial v}{\partial n_-} - v\frac{\partial u}{\partial n_-} \right) d\sigma \\ &\qquad + \int_{\partial(G_- \setminus B_-(r,x))\cap\Omega} \left(u\frac{\partial v}{\partial n_-} - v\frac{\partial u}{\partial n_-} \right) d\sigma = 0. \end{aligned} \tag{3.3.7}$$

Adding the right hand sides of (3.3.6) and (3.3.7), and observing that the normal derivatives of v on $\Omega \setminus B(r,x)$ are of opposite values, we infer

$$\begin{aligned} \int_{(G\setminus B(r,x))\cap\Omega} v \left(\frac{\partial u}{\partial n_+} + \frac{\partial u}{\partial n_-} \right) d\sigma = &\int_{\partial G} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) d\sigma \\ &+ \int_{\partial B(r,x)} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) d\sigma. \end{aligned} \tag{3.3.8}$$

We now deal with the first integral on the right hand side of (3.3.8). Observe that in a neighborhood of ∂G, the potential u does not depend on the choice of x, and the function v is clearly harmonic as a function of x. It then follows that the first integral represents a function of x, which is harmonic in a neighborhood of ∂G, and which will be denoted by $g(x)$.

We now turn to the second integral on the right hand side of (3.3.8). First note that

$$\frac{\partial v}{\partial n} = -(d-2)\,\frac{1}{r^{d-1}}, \quad y \in \partial B(r,x). \tag{3.3.9}$$

Using (3.3.9) and the continuity of u, we obtain

$$\begin{aligned}\int_{\partial B(r,x)} u\frac{\partial v}{\partial n} d\sigma &= -\frac{d-2}{r^{d-1}} \int_{\partial B(r,x)} u(y)\, d\sigma \\ &= -(d-2)\,\omega_d\, u(x) + o(1).\end{aligned} \tag{3.3.10}$$

Now recall that the potential u is assumed to be Lipschitz continuous, with the Lipschitz constant which we will denote by L. Then it follows that the normal derivative of u will be bounded a.e. by L. With that in hand, we have the following estimate

$$\begin{aligned}\left|\int_{\partial B(r,x)} v\frac{\partial u}{\partial n}\, d\sigma\right| &= \left|\int_{\partial B(r,x)} \frac{1}{r^{d-2}}\frac{\partial u}{\partial n}\, d\sigma\right| \\ &\leq \frac{1}{r^{d-2}} \int_{\partial B(r,x)} \left|\frac{\partial u}{\partial n}\right|\, d\sigma \\ &\leq \frac{L}{r^{d-2}} \int_{\partial B(r,x)} d\sigma \\ &= \omega_d\, L\, r.\end{aligned} \tag{3.3.11}$$

Estimate (3.3.11) shows that

$$\lim_{r\to 0+} \int_{\partial B(r,x)} v\frac{\partial u}{\partial n}\, d\sigma = 0. \tag{3.3.12}$$

Passing to the limit $r \to 0+$ in left hand side of (3.3.8), and noting that $\chi_{B(r,x)} \to 0$ a.e. as $r \to 0+$, by the Dominated Convergence Theorem we obtain

$$\begin{aligned}&\lim_{r\to 0+} \int_{(G\setminus B(r,x))\cap\Omega} v\left(\frac{\partial u}{\partial n_+} + \frac{\partial u}{\partial n_-}\right) d\sigma \\ &\quad= \lim_{r\to 0+} \int \chi_{G\cap\Omega}\,(1-\chi_{B(r,x)})\, v\left(\frac{\partial u}{\partial n_+} + \frac{\partial u}{\partial n_-}\right) d\sigma \\ &\quad= \int \chi_{G\cap\Omega}\, v\left(\frac{\partial u}{\partial n_+} + \frac{\partial u}{\partial n_-}\right) d\sigma \\ &\quad= \int_{G\cap\Omega} v\left(\frac{\partial u}{\partial n_+} + \frac{\partial u}{\partial n_-}\right) d\sigma.\end{aligned} \tag{3.3.13}$$

Collecting (3.3.13), (3.3.10) and (3.3.12), we obtain

$$\int_{G\cap\Omega}\left(\frac{\partial u}{\partial n_+}+\frac{\partial u}{\partial n_-}\right)\frac{d\sigma}{|x-y|^{d-2}}=-(d-2)\,\omega_d\,u(x)+g(x).$$

The uniqueness part of the Riesz Decomposition Theorem [50, Theorem 1.22′, p. 104] then yields that on Ω the measure μ is given by the expression

$$d\mu=-\frac{1}{(d-2)\,\omega_d}\left(\frac{\partial U^\mu}{\partial n_+}+\frac{\partial U^\mu}{\partial n_-}\right)d\sigma,$$

as desired. ■

The proof of the theorem is now complete.

■

Proof of Theorem 3.1.2. Let the support of the extremal measure μ_Q be a spherical cap centered at the North Pole, that is $S_Q = C_{N,\alpha}$. From Theorem 3.1.1 we know that $d\mu_Q = f(\theta_1)\,d\sigma_d$, where $f \in L^\infty_{\text{loc}}([0,\alpha])$.

In what follows, we will need an expression for the distance between a point on a sphere $\mathbb{S}^{d-1}$ and another point in the space $\mathbb{R}^d$, which is not on the surface of the sphere $\mathbb{S}^{d-1}$. Let

$$x=(x_1,x_2,x_3,\ldots,x_d)=(r\cos\theta_1, r\sin\theta_1\cos\theta_2, r\sin\theta_1\sin\theta_2\cos\theta_3,\ldots,$$
$$r\sin\theta_1\sin\theta_2\ldots\sin\theta_{d-2}\sin\varphi)\in\mathbb{R}^d,$$

and

$$y=(y_1,y_2,y_3,\ldots,x_d)=(\cos\eta_1,\sin\eta_1\cos\eta_2,\sin\eta_1\sin\eta_2\cos\eta_3,\ldots,$$
$$\sin\eta_1\sin\eta_2\ldots\sin\eta_{d-2}\sin\psi)\in\mathbb{S}^{d-1},$$

be such two points, written in hyperspherical coordinates. Then, for the inner product of x and y, we obtain

$$\begin{aligned}\langle x,y\rangle &=\sum_{j=1}^{d}x_jy_j=x_1y_1+\sum_{j=2}^{d}x_jy_j\\ &=r\cos\theta_1\cos\eta_1+r\sin\theta_1\sin\eta_1\langle\bar{x},\bar{y}\rangle,\end{aligned}$$

where

$$\bar{x} =(\cos\theta_2, \sin\theta_2\cos\theta_3, \dots, \sin\theta_2\dots$$
$$\sin\theta_{d-3}\cos\theta_{d-2}\cos\varphi, \sin\theta_2\dots\sin\theta_{d-3}\cos\theta_{d-2}\sin\varphi) \in \mathbb{S}^{d-2},$$
$$\bar{y} =(\cos\eta_2, \sin\eta_2\cos\eta_3, \dots, \sin\eta_2\dots$$
$$\sin\eta_{d-3}\cos\eta_{d-2}\cos\psi, \sin\eta_2\dots\sin\eta_{d-3}\cos\eta_{d-2}\sin\psi) \in \mathbb{S}^{d-2}.$$

Therefore, for the distance $|x-y|$, we obtain

$$\begin{aligned}|x-y|^2 &= |x|^2 + |y|^2 - 2\langle x, y\rangle\\ &= r^2 + 1 - 2r(\cos\theta_1\cos\eta_1 + \sin\theta_1\sin\eta_1\langle\bar{x},\bar{y}\rangle)\\ &= r^2 + 1 - 2r\lambda,\end{aligned}$$

where $\lambda = \cos\theta_1\cos\eta_1 + \sin\theta_1\sin\eta_1\,\langle\bar{x},\bar{y}\rangle$. Thus, the potential U^{μ_Q} becomes

$$\begin{aligned}U^{\mu_Q}(x) &= \int_{C_{N,\alpha}} \frac{d\mu_Q(y)}{|x-y|^{d-2}}\\ &= \int_0^\alpha f(\eta_1)\,\sin^{d-2}\eta_1\,d\eta_1 \int_{\mathbb{S}^{d-2}} \frac{d\sigma_{d-1}(\bar{y})}{(r^2+1-2r\lambda)^{(d-2)/2}}.\end{aligned}$$

On the surface of the sphere $\mathbb{S}^{d-1}$ we have $r=1$, so that

$$U^{\mu_Q}(x) = \int_0^\alpha f(\eta_1)\,\sin^{d-2}\eta_1\,d\eta_1 \int_{\mathbb{S}^{d-2}} \frac{d\sigma_{d-1}(\bar{y})}{(2-2\lambda)^{(d-2)/2}}, \quad x \in C_{N,\alpha}. \tag{3.3.14}$$

We will need the following proposition, which is a special case of the Funk-Hecke theorem [3, p. 247].

Proposition 3.3.1 *If f is integrable on $[-1,1]$ with respect to the weight $(1-t^2)^{(d-3)/2}$, and y is an arbitrary fixed point on the sphere S^{d-1}, then*

$$\int_{\mathbb{S}^{d-1}} f(\langle x,y\rangle)\,d\sigma_d(x) = \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)} \int_{-1}^1 f(t)\,(1-t^2)^{(d-3)/2}\,dt. \tag{3.3.15}$$

Applying Proposition 3.3.1 to the inner integral in (3.3.14), we thus obtain

$$\begin{aligned}&\int_{\mathbb{S}^{d-2}} \frac{d\sigma_{d-1}(\bar{y})}{(2-2\lambda)^{(d-2)/2}}\\ &\quad= \int_{\mathbb{S}^{d-2}} \frac{d\sigma_{d-1}(\bar{y})}{(2-2(\cos\theta_1\cos\eta_1+\sin\theta_1\sin\eta_1\langle\bar{x},\bar{y}\rangle))^{(d-2)/2}}\\ &\quad= \frac{2\pi^{(d-2)/2}}{\Gamma((d-2)/2)} \int_0^\pi \frac{\sin^{d-3}\xi\,d\xi}{(2-2(\cos\theta_1\cos\eta_1+\sin\theta_1\sin\eta_1\cos\xi))^{(d-2)/2}}.\end{aligned}$$

Hence, for the potential U^{μ_Q} on the spherical cap $C_{N,\alpha}$, we finally obtain

$$
\begin{aligned}
U^{\mu_Q}(\theta_1) &= \frac{2\pi^{(d-2)/2}}{\Gamma((d-2)/2)} \int_0^{\alpha} f(\eta_1)\, \sin^{d-2}\eta_1\, d\eta_1 \\
&\quad \times \int_0^{\pi} \frac{\sin^{d-3}\xi\, d\xi}{(2-2(\cos\theta_1\cos\eta_1+\sin\theta_1\sin\eta_1\cos\xi))^{(d-2)/2}} \\
&= \frac{2\pi^{(d-2)/2}}{\Gamma((d-2)/2)} \int_0^{\alpha} f(\eta_1)\, \sin^{d-2}\eta_1\, d\eta_1 \int_0^{\pi} \frac{\sin^{d-3}\xi\, d\xi}{(2-2\gamma)^{(d-2)/2}},
\end{aligned}
$$

with $0 \le \theta_1 \le \alpha$, where γ is defined as

$$\gamma = \cos\theta_1\cos\eta_1 + \sin\theta_1\sin\eta_1\cos\xi. \tag{3.3.16}$$

Therefore, integral equation (3.1.3) now assumes the form

$$\frac{2\pi^{(d-2)/2}}{\Gamma((d-2)/2)} \int_0^{\alpha} f(\eta_1)\, \sin^{d-2}\eta_1\, d\eta_1 \int_0^{\pi} \frac{\sin^{d-3}\xi\, d\xi}{(2-2\gamma)^{(d-2)/2}} = F_Q - Q(\theta_1), \tag{3.3.17}$$

where $0 \le \theta_1 \le \alpha$.

Integral equation (3.3.17), in the case $d = 3$, was first obtained and solved by Collins [25]. A generalization of the results of [25] to the case of d dimensions, $d \ge 3$, was considered by Shail [66]. However, some of the arguments employed in [66], in particular, those used in derivation of a special case of integral equation (3.3.17), are not mathematically rigorous.

Letting

$$
\begin{aligned}
a &:= 2\sin\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\eta_1}{2}\right), \\
b &:= 2\sin\left(\frac{\eta_1}{2}\right)\cos\left(\frac{\theta_1}{2}\right),
\end{aligned}
$$

using elementary trigonometric identities, we can easily see that

$$
\begin{aligned}
2-2\gamma &= 2-2(\cos\theta_1\cos\eta_1+\sin\theta_1\sin\eta_1\cos\xi) \\
&= a^2+b^2-2ab\cos\xi.
\end{aligned}
$$

Observe that as $0 < \alpha \le \pi$, it is clear that $a \ge 0$ and $b \ge 0$ for all and $0 \le \theta_1 \le \alpha$ and $0 \le \eta_1 \le \alpha$. The next step is to further transform the kernel of the integral equation (3.3.17). Namely, it is facilitated via the following

Lemma 3.3.2 *If a and b are positive numbers, $a \neq b$, and $q \geq 0$, then*

$$\int_0^{\pi} \frac{\sin^{2q}\xi \, d\xi}{(a^2+b^2-2ab\cos\xi)^{q+\frac{1}{2}}} = \frac{2}{a^{2q}\, b^{2q}} \int_0^{\min(a,b)} \frac{t^{2q}\, dt}{\sqrt{a^2-t^2}\sqrt{b^2-t^2}}. \tag{3.3.18}$$

We remark that Lemma 3.3.2 is generalization of a result, obtained by Copson [26] when $q = 0$. Lemma 3.3.2 is implicitly mentioned in [66], although with an incorrect numerical coefficient. In [66], the author derives (3.3.18) using identities for the Bessel functions. However, the form of the result suggests that its proof is independent of any special function. We present such a proof below.

Proof. The proof hinges on the following identity, obtained by Kahane [42].

Proposition 3.3.2 *Let a and b be positive numbers such that $a \neq b$, $\tau \in (0,1)$, $\upsilon \in \mathbb{C}$ with $Re(\upsilon) \geq 0$, and u a real number with $|u| \leq 1$. Then*

$$\frac{(ab)^{\upsilon}}{(a^2+b^2-2abu)^{\tau+\upsilon}} = \frac{\Gamma(\tau)\Gamma(\upsilon+1)}{\Gamma(\tau+\upsilon)} \frac{2\sin(\tau\pi)}{\pi} \times \tag{3.3.19}$$
$$\int_0^{\min(a,b)} \frac{1-(t^2/ab)^2}{(1+(t^2/ab)^2-2(t^2/ab)u)^{\upsilon+1}} \left(\frac{t^2}{ab}\right)^{\upsilon} \frac{t^{2\tau-1}\, dt}{(a^2-t^2)^{\tau}(b^2-t^2)^{\tau}}.$$

Setting $\tau = 1/2$, $\upsilon = q \geq 0$ in Proposition 3.3.2, and using Fubini's theorem, we rewrite the left hand side of (3.3.18) as

$$\int_0^{\pi} \frac{\sin^{2q}\xi \, d\xi}{(a^2+b^2-2ab\cos\xi)^{q+\frac{1}{2}}} = \frac{2}{a^{2q}\, b^{2q}} \frac{\Gamma(q+1)}{\sqrt{\pi}\Gamma(q+1/2)} \times \tag{3.3.20}$$
$$\int_0^{\min(a,b)} \frac{(1-(t^2/ab)^2)\, t^{2q}\, dt}{\sqrt{a^2-t^2}\sqrt{b^2-t^2}} \int_0^{\pi} \frac{\sin^{2q}\xi \, d\xi}{(1+(t^2/ab)^2-2(t^2/ab)\cos\xi)^{q+1}}.$$

Next, we show that

$$\int_0^{\pi} \frac{\sin^{2q}\xi \, d\xi}{(1+(t^2/ab)^2-2(t^2/ab)\cos\xi)^{q+1}} = \frac{1}{1-(t^2/ab)^2} \frac{\sqrt{\pi}\,\Gamma(q+1/2)}{\Gamma(q+1)}. \tag{3.3.21}$$

The integral of a type appearing on the left hand side of (3.3.21) was previously considered in [50, p. 400]. It was shown that

$$\int_0^{\pi} \frac{\sin^{p-2}\xi \, d\xi}{(1+\rho^2-2\rho\cos\xi)^{p/2}} = \frac{1}{\rho^{p-2}(\rho^2-1)} \int_0^{\pi} \sin^{p-2}\xi \, d\xi, \tag{3.3.22}$$

where $\rho \geq 1$, and $p \geq 3$ was assumed to be an integer. A careful analysis of the evaluation of integral (3.3.22) in [50, p. 400] shows that, in fact, (3.3.22) holds true for any $p \geq 2$. We hence transform the left hand side of (3.3.21) as follows,

$$\begin{aligned}\int_0^\pi \frac{\sin^{2q}\xi\, d\xi}{(1+(t^2/ab)^2 - 2(t^2/ab)\cos\xi)^{q+1}} &= \frac{1}{1-(t^2/ab)^2}\int_0^\pi \sin^{2q}\xi\, d\xi \\ &= \frac{1}{1-(t^2/ab)^2} 2^{2q}\,\mathrm{B}(q+1/2, q+1/2) \\ &= \frac{1}{1-(t^2/ab)^2}\frac{\sqrt{\pi}\,\Gamma(q+1/2)}{\Gamma(q+1)},\end{aligned}$$

as claimed. Therefore, upon inserting (3.3.21) into (3.3.20), we obtain desired representation (3.3.18). ■

Setting $q = (d-3)/2$ in Lemma 3.3.2, integral equation (3.3.17) is transformed into

$$\begin{aligned}\frac{4\,\pi^{(d-2)/2}}{\Gamma((d-2)/2)}\int_0^\alpha f(\eta_1)\sin^{d-2}\eta_1\, d\eta_1\, \frac{1}{a^{d-3}\, b^{d-3}}\int_0^{\min(a,b)} \frac{t^{d-3}\, dt}{\sqrt{a^2-t^2}\sqrt{b^2-t^2}} \\ = F_Q - Q(\theta_1), \quad 0 \leq \theta_1 \leq \alpha.\end{aligned}$$

To simplify notation, we will use η and θ instead of η_1 and θ_1, respectively. Then, the last equation reads

$$\begin{aligned}\frac{4\,\pi^{(d-2)/2}}{\Gamma((d-2)/2)}\int_0^\alpha f(\eta)\,\sin^{d-2}\eta\, d\eta \,\times \\ \frac{1}{a^{d-3}\, b^{d-3}}\int_0^{\min(a,b)} \frac{t^{d-3}\, dt}{\sqrt{a^2-t^2}\sqrt{b^2-t^2}} = F_Q - Q(\theta), \quad 0 \leq \theta \leq \alpha,\end{aligned} \tag{3.3.23}$$

where

$$a = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\eta}{2}\right),$$
$$b = 2\sin\left(\frac{\eta}{2}\right)\cos\left(\frac{\theta}{2}\right).$$

One can easily check that $a < b$ for $\theta < \eta$, while for $\theta > \eta$, we have $a > b$. Splitting the interval of integration of the outer integral in the left hand side of equation (3.3.23), we rewrite equation (3.3.23) as follows,

$$
\begin{aligned}
&\int_0^{\theta} f(\eta)\sin^{d-2}\eta\, d\eta\, \frac{1}{a^{d-3}\, b^{d-3}} \int_0^{b} \frac{t^{d-3}\, dt}{\sqrt{a^2-t^2}\sqrt{b^2-t^2}} \\
&+\int_{\theta}^{\alpha} f(\eta)\sin^{d-2}\eta\, d\eta\, \frac{1}{a^{d-3}\, b^{d-3}} \int_0^{a} \frac{t^{d-3}\, dt}{\sqrt{a^2-t^2}\sqrt{b^2-t^2}} \\
&=\frac{\Gamma((d-2)/2)}{4\,\pi^{(d-2)/2}}\,(F_Q-Q(\theta)), \quad 0\le\theta\le\alpha.
\end{aligned}
$$

Introducing the substitution

$$
t = 2\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\eta}{2}\right)\tan\left(\frac{\zeta}{2}\right),
$$

we can further transform the last integral equation into

$$
\begin{aligned}
&\int_0^{\theta} f(\eta)\sin\eta\,\cos^{d-3}(\eta/2)\, d\eta \int_0^{\eta} \frac{\tan^{d-3}(\zeta/2)\, d\zeta}{\sqrt{\cos\zeta-\cos\theta}\sqrt{\cos\zeta-\cos\eta}} \\
&+\int_{\theta}^{\alpha} f(\eta)\sin\eta\,\cos^{d-3}(\eta/2)\, d\eta \int_0^{\theta} \frac{\tan^{d-3}(\zeta/2)\, d\zeta}{\sqrt{\cos\zeta-\cos\theta}\sqrt{\cos\zeta-\cos\eta}} \\
&=\frac{\Gamma((d-2)/2)}{2\,\pi^{(d-2)/2}}\,\sin^{d-3}\left(\frac{\theta}{2}\right)(F_Q-Q(\theta)), \quad 0\le\theta\le\alpha.
\end{aligned} \tag{3.3.24}
$$

Inverting the order of integration in the first integral in the left hand side of (3.3.24), we recast equation (3.3.24) into

$$
\begin{aligned}
&\int_0^{\theta} \frac{\tan^{d-3}(\zeta/2)\, d\zeta}{\sqrt{\cos\zeta-\cos\theta}} \int_{\zeta}^{\alpha} \frac{f(\eta)\sin\eta\,\cos^{d-3}(\eta/2)\, d\eta}{\sqrt{\cos\zeta-\cos\eta}} \\
&\quad=\frac{\Gamma((d-2)/2)}{2\,\pi^{(d-2)/2}}\,\sin^{d-3}\left(\frac{\theta}{2}\right)(F_Q-Q(\theta)), \quad 0\le\theta\le\alpha.
\end{aligned} \tag{3.3.25}
$$

Let

$$
S(\zeta) = \int_{\zeta}^{\alpha} \frac{f(\eta)\sin\eta\,\cos^{d-3}(\eta/2)\, d\eta}{\sqrt{\cos\zeta-\cos\eta}}, \quad 0\le\zeta\le\alpha. \tag{3.3.26}
$$

Equation (3.3.25) then becomes

$$
\int_0^{\theta} \frac{S(\zeta)\,\tan^{d-3}(\zeta/2)\, d\zeta}{\sqrt{\cos\zeta-\cos\theta}} = \frac{\Gamma((d-2)/2)}{2\,\pi^{(d-2)/2}}\,\sin^{d-3}\left(\frac{\theta}{2}\right)(F_Q-Q(\theta)), \tag{3.3.27}
$$

where $0\le\theta\le\alpha$.

Equation (3.3.27) is an Abel type integral equation with respect to $S(\zeta)\tan^{d-3}(\zeta/2)$. Since $Q\in C^2$, the solution of (3.3.27) is [60, p. 50, # 23]

$$
S(\zeta) = \frac{\Gamma((d-2)/2)}{2\,\pi^{d/2}}\,\cot^{d-3}\left(\frac{\zeta}{2}\right)\frac{d}{d\zeta}\int_0^{\zeta} \frac{(F_Q-Q(\theta))\,\sin^{d-3}(\theta/2)\sin\theta\, d\theta}{\sqrt{\cos\theta-\cos\zeta}},
$$

with $0 \le \zeta \le \alpha$. Observing that equation (3.3.26) is again an Abel type integral equation with respect to $f(\eta)\sin\eta\,\cos^{d-3}(\eta/2)$, we solve it and obtain

$$f(\eta) = -\frac{1}{\pi}\frac{1}{\sin\eta}\,\sec^{d-3}\left(\frac{\eta}{2}\right)\frac{d}{d\eta}\int_\eta^\alpha \frac{S(\zeta)\sin\zeta\,d\zeta}{\sqrt{\cos\eta-\cos\zeta}}, \quad 0\le\eta\le\alpha. \tag{3.3.28}$$

Denote

$$g(\zeta) = \cot^{d-3}\left(\frac{\zeta}{2}\right)\frac{d}{d\zeta}\int_0^\zeta \frac{Q(\theta)\,\sin^{d-3}(\theta/2)\sin\theta\,d\theta}{\sqrt{\cos\theta-\cos\zeta}}, \quad 0\le\zeta\le\alpha. \tag{3.3.29}$$

and let

$$F(\eta) = \frac{\Gamma((d-2)/2)}{2\,\pi^{(d+2)/2}}\,\frac{1}{\sin\eta}\,\sec^{d-3}\left(\frac{\eta}{2}\right)\frac{d}{d\eta}\int_\eta^\alpha \frac{g(\zeta)\sin\zeta\,d\zeta}{\sqrt{\cos\eta-\cos\zeta}}, \tag{3.3.30}$$

where $0\le\eta\le\alpha$. In view of (3.3.29) and (3.3.30), expression for the density (3.3.28) takes the form

$$\begin{aligned} f(\eta) = &-\frac{F_Q\,\Gamma((d-2)/2)}{2\,\pi^{(d+2)/2}}\,\frac{\sec^{d-3}(\eta/2)}{\sin\eta}\times \\ &\frac{d}{d\eta}\int_\eta^\alpha \frac{\sin\zeta\,\cot^{d-3}(\zeta/2)}{\sqrt{\cos\eta-\cos\zeta}}\left\{\frac{d}{d\zeta}\int_0^\zeta \frac{\sin^{d-3}(\theta/2)\,\sin\theta\,d\theta}{\sqrt{\cos\theta-\cos\zeta}}\right\}d\zeta \\ &+F(\eta), \quad 0\le\eta\le\alpha. \end{aligned} \tag{3.3.31}$$

It is not hard to see that

$$\int_0^\zeta \frac{\sin^{d-3}(\theta/2)\,\sin\theta\,d\theta}{\sqrt{\cos\theta-\cos\zeta}} = \sqrt{2}\,\mathrm{B}\left(\frac{d-1}{2},\frac{1}{2}\right)\sin^{d-2}\left(\frac{\zeta}{2}\right).$$

Upon differentiating last expression with respect to ζ, we find that

$$\frac{d}{d\zeta}\int_0^\zeta \frac{\sin^{d-3}(\theta/2)\,\sin\theta\,d\theta}{\sqrt{\cos\theta-\cos\zeta}} = \frac{d-2}{\sqrt{2}}\,\mathrm{B}\left(\frac{d-1}{2},\frac{1}{2}\right)\sin^{d-3}\left(\frac{\zeta}{2}\right)\cos\left(\frac{\zeta}{2}\right).$$

Using the elementary transformations, one can show that

$$\begin{aligned} &\int_\eta^\alpha \frac{\sin\zeta\,\cot^{d-3}(\zeta/2)\,\sin^{d-3}(\zeta/2)\,\cos(\zeta/2)}{\sqrt{\cos\eta-\cos\zeta}}\,d\zeta \\ &\quad = \sqrt{2}\,\cos^{d-1}\left(\frac{\eta}{2}\right)\,\mathrm{B}\left(\frac{\cos\eta-\cos\alpha}{1+\cos\eta};\frac{1}{2},\frac{d}{2}\right). \end{aligned}$$

Hence, we can conclude that

$$\int_{\eta}^{\alpha} \frac{\sin\zeta\ \cot^{d-3}(\zeta/2)}{\sqrt{\cos\eta-\cos\zeta}}\left\{\frac{d}{d\zeta}\int_{0}^{\zeta}\frac{\sin^{d-3}(\theta/2)\ \sin\theta\, d\theta}{\sqrt{\cos\theta-\cos\zeta}}\right\} d\zeta$$
$$= (d-2)\,\mathrm{B}\left(\frac{d-1}{2},\frac{1}{2}\right)\ \cos^{d-1}\left(\frac{\eta}{2}\right)\ \mathrm{B}\left(\frac{\cos\eta-\cos\alpha}{1+\cos\eta};\frac{1}{2},\frac{d}{2}\right).$$

Differentiating the latter, and simplifying, we see that

$$\frac{d}{d\eta}\int_{\eta}^{\alpha} \frac{\sin\zeta\ \cot^{d-3}(\zeta/2)}{\sqrt{\cos\eta-\cos\zeta}}\left\{\frac{d}{d\zeta}\int_{0}^{\zeta}\frac{\sin^{d-3}(\theta/2)\ \sin\theta\, d\theta}{\sqrt{\cos\theta-\cos\zeta}}\right\} d\zeta \tag{3.3.32}$$
$$= -\frac{d-2}{2}\,\mathrm{B}\left(\frac{d-1}{2},\frac{1}{2}\right)\ \sin\eta\left\{\frac{d-1}{2}\ \cos^{d-3}\left(\frac{\eta}{2}\right)\ \times\right.$$
$$\left.\mathrm{B}\left(\frac{\cos\eta-\cos\alpha}{1+\cos\eta};\frac{1}{2},\frac{d}{2}\right)+\frac{\cos^{d-1}(\alpha/2)}{\cos^{2}(\eta/2)}\sqrt{\frac{1+\cos\alpha}{\cos\eta-\cos\alpha}}\right\}.$$

Inserting (3.3.32) into (3.3.31), after some algebra, we eventually find that

$$f(\eta)=\frac{F_Q\,\Gamma((d-1)/2)}{2\,\pi^{(d+1)/2}}\left\{\frac{d-1}{2}\,\mathrm{B}\left(\frac{\cos\eta-\cos\alpha}{1+\cos\eta};\frac{1}{2},\frac{d}{2}\right)\right. \tag{3.3.33}$$
$$\left.+\left(\frac{1+\cos\alpha}{1+\cos\eta}\right)^{\frac{d-1}{2}}\sqrt{\frac{1+\cos\alpha}{\cos\eta-\cos\alpha}}\right\}+F(\eta),\quad 0\le\eta\le\alpha.$$

The expression in braces on the right hand side of (3.3.33) represents (save for a normalizing constant) the equilibrium density of the spherical cap centered at the North Pole, for the case of no external field. Our goal at this stage will be to transform this expression into a form first obtained in [17, p. 780, expression (44)], for the case of general Riesz potential. This will prove useful in our further considerations.

Recalling that

$${}_2F_1(1,a+b;a+1;z)=\frac{a}{z^a\,(1-z)^b}\,\mathrm{B}(z;a,b), \tag{3.3.34}$$

and denoting for brevity $t:=\cos\alpha$, $u:=\cos\eta$, we see that

$$\mathrm{B}\left(\frac{\cos\eta-\cos\alpha}{1+\cos\eta};\frac{1}{2},\frac{d}{2}\right)$$
$$=\mathrm{B}\left(\frac{u-t}{1+u};\frac{1}{2},\frac{d}{2}\right)$$
$$=\frac{\Gamma(1/2)}{\Gamma(3/2)}\left(\frac{u-t}{1+u}\right)^{1/2}\left(\frac{1+t}{1+u}\right)^{d/2}{}_2F_1\left(1,\frac{d+1}{2};\frac{3}{2};\frac{u-t}{1+u}\right)$$
$$=2\left(\frac{u-t}{1+u}\right)^{1/2}\left(\frac{1+t}{1+u}\right)^{d/2}{}_2F_1\left(1,\frac{d+1}{2};\frac{3}{2};\frac{u-t}{1+u}\right).$$

Therefore,

$$\begin{aligned}
&\frac{d-1}{2}\,\mathrm{B}\left(\frac{\cos\eta-\cos\alpha}{1+\cos\eta};\frac{1}{2},\frac{d}{2}\right)+\left(\frac{1+\cos\alpha}{1+\cos\eta}\right)^{(d-1)/2}\sqrt{\frac{1+\cos\alpha}{\cos\eta-\cos\alpha}}\\
&=\frac{d-1}{2}\,\mathrm{B}\left(\frac{u-t}{1+u};\frac{1}{2},\frac{d}{2}\right)+\left(\frac{1+t}{1+u}\right)^{(d-1)/2}\left(\frac{1+t}{u-t}\right)^{1/2}\\
&=\left(\frac{1+t}{1+u}\right)^{(d-1)/2}\left(\frac{1+t}{u-t}\right)^{1/2}\left\{1+(d-1)\,\frac{u-t}{1+u}\,{}_2F_1\left(1,\frac{d+1}{2};\frac{3}{2};\frac{u-t}{1+u}\right)\right\}.
\end{aligned}$$

For brevity, let $z:=(u-t)/(1+u)$. We will be working with the term $z\,{}_2F_1(1,(d+1)/2;3/2;z)=z\,{}_2F_1(1,(d-1)/2+1;1/2+1;z)$, appearing in the right hand side of the last expression. According to [1, p. 558, # 15.2.20],

$$c\,(1-z)\,{}_2F_1(a,b;c;z)-c\,{}_2F_1(a-1,b;c;z)+(c-b)z\,{}_2F_1(a,b;c+1;z)=0,$$

which, in cojunction with the fact ${}_2F_1(0,b;c;z)=1$, entails

$$z\,{}_2F_1\left(1,\frac{d-1}{2}+1;\frac{1}{2}+1;z\right)=\frac{1}{d}\left\{(1-z)\,{}_2F_1\left(1,\frac{d-1}{2}+1;\frac{1}{2};z\right)-1\right\}. \tag{3.3.35}$$

Furthermore, [1, p. 558, # 15.2.14] states that

$$(b-a)\,{}_2F_1(a,b;c;z)+a\,{}_2F_1(a+1,b;c;z)-b\,{}_2F_1(a,b+1;c;z)=0,$$

which in our circumstances is equivalent to

$$\begin{aligned}
&{}_2F_1\left(1,\frac{d-1}{2}+1;\frac{1}{2};z\right)\\
&\quad=\frac{1}{d-1}\left\{(d-3)\,{}_2F_1\left(1,\frac{d-1}{2};\frac{1}{2};z\right)+2\,{}_2F_1\left(2,\frac{d-1}{2};\frac{1}{2};z\right)\right\}.
\end{aligned} \tag{3.3.36}$$

Inserting (3.3.36) into (3.3.35), we easily obtain that

$$\begin{aligned}
&z\,{}_2F_1\left(1,\frac{d-1}{2}+1;\frac{1}{2}+1;z\right)\\
&\qquad=\frac{1}{d(d-1)}\left\{(d-3)(1-z)\,{}_2F_1\left(1,\frac{d-1}{2};\frac{1}{2};z\right)\right.\\
&\qquad\left.+2(1-z)\,{}_2F_1\left(2,\frac{d-1}{2};\frac{1}{2};z\right)-(d-1)\right\}.
\end{aligned} \tag{3.3.37}$$

It is a direct consequence of [1, p. 558, # 15.2.10] that

$$ {}_2F_1(2,b;c;z) = \frac{1}{z-1}\Big\{(c-2+z(1-b))\,{}_2F_1(1,b;c;z)+1-c\Big\}. \tag{3.3.38} $$

Substituting (3.3.38) into (3.3.37) and simplifying, we deduce that

$$ z\,{}_2F_1\left(1,\frac{d-1}{2}+1;\frac{1}{2}+1;z\right) = \frac{1}{d-1}\left\{{}_2F_1\left(1,\frac{d-1}{2};\frac{1}{2};z\right)-1\right\}. $$

We thus demonstrated that

$$ \begin{aligned} &\frac{d-1}{2}\,\mathrm{B}\left(\frac{u-t}{1+u};\frac{1}{2},\frac{d}{2}\right)+\left(\frac{1+t}{1+u}\right)^{(d-1)/2}\left(\frac{1+t}{u-t}\right)^{1/2}\\ &=\left(\frac{1+t}{1+u}\right)^{(d-1)/2}\left(\frac{1+t}{u-t}\right)^{1/2}{}_2F_1\left(1,\frac{d-1}{2};\frac{1}{2};\frac{u-t}{1+u}\right)\\ &=\left(\frac{1+\cos\alpha}{1+\cos\eta}\right)^{(d-1)/2}\left(\frac{1+\cos\alpha}{\cos\eta-\cos\alpha}\right)^{1/2}{}_2F_1\left(1,\frac{d-1}{2};\frac{1}{2};\frac{\cos\eta-\cos\alpha}{1+\cos\eta}\right) \end{aligned} $$

Expression (3.3.33) can now be written as

$$ \begin{aligned} f(\eta) =& \frac{F_Q\,\Gamma((d-1)/2)}{2\,\pi^{(d+1)/2}}\left(\frac{1+\cos\alpha}{1+\cos\eta}\right)^{\frac{d-1}{2}}\left(\frac{1+\cos\alpha}{\cos\eta-\cos\alpha}\right)^{\frac{1}{2}}\times\\ &{}_2F_1\left(1,\frac{d-1}{2};\frac{1}{2};\frac{\cos\eta-\cos\alpha}{1+\cos\eta}\right)+F(\eta),\quad 0\le\eta\le\alpha. \end{aligned} \tag{3.3.39} $$

Next, we compute the Robin constant F_Q. Recall that μ_Q is a probability measure, so that its mass is 1. Therefore,

$$ 1=\int d\mu_Q=\int_0^\alpha\int_{\mathbb{S}^{d-2}} f(\eta)\,\sin^{d-2}\eta\,d\sigma_{d-1}\,d\eta. \tag{3.3.40} $$

Inserting expression (3.3.39) into (3.3.40), we obtain

$$ \begin{aligned} &\frac{\Gamma((d-1)/2)}{2\pi^{(d-1)/2}}=\frac{1}{\omega_{d-1}}=\int_0^\alpha f(\eta)\,\sin^{d-2}\eta\,d\eta\\ &=\int_0^\alpha F(\eta)\,\sin^{d-2}\eta\,d\eta\\ &+\frac{F_Q\,\Gamma((d-1)/2)}{2\,\pi^{(d+1)/2}}\int_0^\alpha\left(\frac{1+\cos\alpha}{1+\cos\eta}\right)^{\frac{d-1}{2}}\left(\frac{1+\cos\alpha}{\cos\eta-\cos\alpha}\right)^{\frac{1}{2}}\times\\ &{}_2F_1\left(1,\frac{d-1}{2};\frac{1}{2};\frac{\cos\eta-\cos\alpha}{1+\cos\eta}\right)\sin^{d-2}\eta\,d\eta. \end{aligned} \tag{3.3.41} $$

To evaluate the second integral in the right hand side of the last expression, we will be making use of the following result [16, Lemma A.1, p. 40].

Lemma 3.3.3 *Assume $-1 \le a < b < c \le 1$ and $|y| \le 1$. Set $x := (b-a)/(c-a)$. Then for all $\alpha, \beta, \gamma > 0$ such that $\beta + \gamma > \alpha$, one has*

$$\int_a^b (u-a)^{\beta-1}\,(b-u)^{\gamma-1}\,(c-u)^{-\alpha}\,{}_2F_1\!\left(\alpha,\beta;\gamma;y\,\frac{b-u}{c-u}\right)du$$
$$= \frac{\Gamma(\beta)\,\Gamma(\gamma)}{\Gamma(\beta+\gamma-\alpha)\,\Gamma(\alpha)}\,(b-a)^{\beta+\gamma-1}\,(c-a)^{-\gamma}\,(c-b)^{\gamma-\alpha}\,(1-xy)^{-\beta}$$
$$\times \int_0^1 t^{\beta+\gamma-\alpha-1}\,(1-t)^{\alpha-1}\,(1-xt)^{\beta-\gamma}\left(1-\frac{x(1-y)}{1-xy}\,t\right)^{-\beta} dt.$$

We first bring the second integral in the right hand side of (3.3.41) into a form that can be handled by invoking Lemma 3.3.3. Using trivial substitutions, we write

$$\int_0^\alpha \left(\frac{1+\cos\alpha}{1+\cos\eta}\right)^{\frac{d-1}{2}} \left(\frac{1+\cos\alpha}{\cos\eta-\cos\alpha}\right)^{\frac{1}{2}} {}_2F_1\!\left(1,\frac{d-1}{2};\frac{1}{2};\frac{\cos\eta-\cos\alpha}{1+\cos\eta}\right)\times$$
$$\sin^{d-2}\eta\,d\eta = (1+t^*)^{d/2}\times$$
$$\int_{-1}^{t^*} (t^*-u)^{-1/2}\,(1-u)^{-1}\,(1+u)^{(d-3)/2}\,{}_2F_1\!\left(1,\frac{d-1}{2};\frac{1}{2};\frac{t^*-u}{1-u}\right)du,$$

where we set $t^* := -\cos\alpha$.

Applying Lemma 3.3.3 with $a = -1, b = t^*, c = 1, y = 1$ and $\alpha = 1, \beta = (d-1)/2, \gamma = 1/2$, while reasoning along the lines of the proof of Lemma 30 in [17, p. 782], we find that

$$(1+t)^{d/2}\int_{-1}^{t^*}(t^*-u)^{-1/2}(1-u)^{-1}\,(1+u)^{(d-3)/2}{}_2F_1\!\left(1,\frac{d-1}{2};\frac{1}{2};\frac{t^*-u}{1-u}\right)du$$
$$= \frac{\sqrt{\pi}\,2^{d-2}\,\Gamma((d-1)/2)}{\Gamma(d/2-1)}\,\mathrm{B}\!\left(\sin^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{d}{2}\right).$$

Thus we conclude that

$$\int_0^\alpha \left(\frac{1+\cos\alpha}{1+\cos\eta}\right)^{\frac{d-1}{2}} \left(\frac{1+\cos\alpha}{\cos\eta-\cos\alpha}\right)^{\frac{1}{2}} {}_2F_1\!\left(1,\frac{d-1}{2};\frac{1}{2};\frac{\cos\eta-\cos\alpha}{1+\cos\eta}\right)\times$$
$$\sin^{d-2}\eta\,d\eta = \frac{\sqrt{\pi}\,2^{d-2}\,\Gamma((d-1)/2)}{\Gamma(d/2-1)}\,\mathrm{B}\!\left(\sin^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{d}{2}\right).$$

Substituting this into (3.3.41), we find that

$$F_Q = \frac{\pi^{d/2}\,\Gamma(d/2-1)}{2^{d-3}\,(\Gamma((d-1)/2))^2}\left(\mathrm{B}\!\left(\sin^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{d}{2}\right)\right)^{-1}\times \tag{3.3.42}$$
$$\left\{\frac{\Gamma((d-1)/2)}{2\pi^{(d-1)/2}} - \int_0^\alpha F(\eta)\,\sin^{d-2}\eta\,d\eta\right\}.$$

In light of (3.3.42), the expression for the equilibrium density (3.3.39) can be written as

$$f(\eta) = C_Q \left(\frac{1+\cos\alpha}{1+\cos\eta} \right)^{\frac{d-1}{2}} \left(\frac{1+\cos\alpha}{\cos\eta - \cos\alpha} \right)^{\frac{1}{2}} {}_2F_1\left(1, \frac{d-1}{2}; \frac{1}{2}; \frac{\cos\eta - \cos\alpha}{1+\cos\eta}\right)$$
$$+ F(\eta), \quad 0 \le \eta \le \alpha,$$

with the constant C_Q given by

$$C_Q = \frac{\Gamma(d/2-1)}{2^{d-2}\sqrt{\pi}\,\Gamma((d-1)/2)} \left(\mathrm{B}\left(\sin^2\left(\frac{\alpha}{2}\right); \frac{d-2}{2}, \frac{d}{2} \right) \right)^{-1} \times$$
$$\left\{ \frac{\Gamma((d-1)/2)}{2\pi^{(d-1)/2}} - \int_0^\alpha F(\eta)\, \sin^{d-2}\eta\, d\eta \right\}.$$

■

Proof of Theorem 3.1.3. If $S_Q = C_{S,\alpha}$, equation (3.1.3) assumes the form

$$\frac{2\pi^{(d-2)/2}}{\Gamma((d-2)/2)} \int_\alpha^\pi f(\eta) \sin^{d-2}\eta\, d\eta \int_0^\pi \frac{\sin^{d-3}\xi\, d\xi}{(2-2\gamma)^{(d-2)/2}} = F_Q - Q(\theta), \qquad (3.3.43)$$

where $\alpha \le \theta \le \pi$ and $\gamma = \cos\theta\cos\eta + \sin\theta\sin\eta\cos\xi$. Via the change of variables $\widetilde{\theta} = \pi - \theta$, we transform (3.3.43) into

$$\frac{2\pi^{(d-2)/2}}{\Gamma((d-2)/2)} \int_0^\beta f_0(\eta) \sin^{d-2}\eta\, d\eta \int_0^\pi \frac{\sin^{d-3}\xi\, d\xi}{(2-2\widetilde{\gamma})^{(d-2)/2}} = F_Q - Q_0(\widetilde{\theta}), \qquad (3.3.44)$$

with $0 \le \widetilde{\theta} \le \beta$ and $\beta = \pi - \alpha$, $f_0(\widetilde{\eta}) = f(\pi - \widetilde{\eta})$, $Q_0(\widetilde{\theta}) = Q(\pi - \widetilde{\theta})$, and $\widetilde{\gamma} = \cos\widetilde{\theta}\cos\eta + \sin\widetilde{\theta}\sin\eta\cos\xi$.

The integral equation (3.3.44) is of the form (3.3.17). Hence Theorem 3.1.2 applies, and we obtain

$$F_0(\widetilde{\eta}) = \frac{\Gamma((d-2)/2)}{2\,\pi^{(d+2)/2}} \frac{1}{\sin\widetilde{\eta}} \sec^{d-3}\left(\frac{\widetilde{\eta}}{2}\right) \frac{d}{d\widetilde{\eta}} \int_{\widetilde{\eta}}^\beta \frac{g_0(\zeta)\sin\zeta\, d\zeta}{\sqrt{\cos\widetilde{\eta} - \cos\zeta}}, \quad 0 \le \widetilde{\eta} \le \beta,$$

where

$$g_0(\zeta) = \cot^{d-3}\left(\frac{\zeta}{2}\right) \frac{d}{d\zeta} \int_0^\zeta \frac{Q_0(\theta)\, \sin^{d-3}(\theta/2) \sin\theta\, d\theta}{\sqrt{\cos\theta - \cos\zeta}}, \quad 0 \le \zeta \le \beta.$$

The density f_0 of the equilibrium measure μ_{Q_0} is

$$f_0(\widetilde{\eta}) = C_Q \left(\frac{1+\cos\beta}{1+\cos\widetilde{\eta}} \right)^{\frac{d-1}{2}} \left(\frac{1+\cos\beta}{\cos\widetilde{\eta}-\cos\beta} \right)^{\frac{1}{2}} {}_2F_1\left(1, \frac{d-1}{2}; \frac{1}{2}; \frac{\cos\widetilde{\eta}-\cos\beta}{1+\cos\widetilde{\eta}}\right)$$
$$+ F_0(\widetilde{\eta}), \quad 0 \le \widetilde{\eta} \le \beta,$$

where the constant C_Q is given by

$$C_Q = \frac{\Gamma(d/2-1)}{2^{d-2}\sqrt{\pi}\,\Gamma((d-1)/2)} \left(\mathrm{B}\left(\sin^2\left(\frac{\beta}{2}\right); \frac{d-2}{2}, \frac{d}{2} \right) \right)^{-1} \times$$
$$\left\{ \frac{\Gamma((d-1)/2)}{2\pi^{(d-1)/2}} - \int_0^\beta F_0(\widetilde{\eta}) \sin^{d-2}\widetilde{\eta}\, d\widetilde{\eta} \right\}.$$

Going back to the η variable via $\eta = \pi - \widetilde{\eta}$, after some algebra, we find

$$g(\zeta) = \tan^{d-3}\left(\frac{\zeta}{2}\right) \frac{d}{d\zeta} \int_\zeta^\pi \frac{Q(\theta)\, \cos^{d-3}(\theta/2) \sin\theta\, d\theta}{\sqrt{\cos\zeta - \cos\theta}}, \quad \alpha \le \zeta \le \pi,$$

so that

$$F(\eta) = \frac{\Gamma((d-2)/2)}{2\pi^{(d+2)/2}} \frac{1}{\sin\eta} \csc^{d-3}\left(\frac{\eta}{2}\right) \frac{d}{d\eta} \int_\alpha^\eta \frac{g(\zeta)\sin\zeta\, d\zeta}{\sqrt{\cos\zeta - \cos\eta}}, \quad \alpha \le \eta \le \pi.$$

The constant C_Q has the form

$$C_Q = \frac{\Gamma(d/2-1)}{2^{d-2}\sqrt{\pi}\,\Gamma((d-1)/2)} \left(\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right); \frac{d-2}{2}, \frac{d}{2} \right) \right)^{-1} \times$$
$$\left\{ \frac{\Gamma((d-1)/2)}{2\pi^{(d-1)/2}} - \int_\alpha^\pi F(\eta) \sin^{d-2}\eta\, d\eta \right\}.$$

We thus conclude that the equilibrium density, when support is a spherical cap centered at the South Pole, is given by

$$f(\eta) = C_Q \left(\frac{1-\cos\alpha}{1-\cos\eta} \right)^{\frac{d-1}{2}} \left(\frac{1-\cos\alpha}{\cos\alpha-\cos\eta} \right)^{\frac{1}{2}} {}_2F_1\left(1, \frac{d-1}{2}; \frac{1}{2}; \frac{\cos\alpha-\cos\eta}{1-\cos\eta}\right)$$
$$+ F(\eta), \quad \alpha \le \eta \le \pi.$$

■

Proof of Theorem 3.2.1. Recall that the external field Q in question is given by (3.2.5), that is

$$Q(\eta) = q\,(1-\cos\eta)^{-(d-2)/2}, \quad q > 0, \quad 0 \le \eta \le \pi.$$

Substituting this expression into formula (3.2.4) for the $\mathcal{F}$-functional, we are lead to the following integral

$$\int_{\alpha}^{\pi}\left(\frac{1-\cos\alpha}{1-\cos\eta}\right)^{\frac{d-1}{2}}\left(\frac{1-\cos\alpha}{\cos\alpha-\cos\eta}\right)^{\frac{1}{2}}{}_2F_1\left(1,\frac{d-1}{2};\frac{1}{2};\frac{\cos\alpha-\cos\eta}{1-\cos\eta}\right)\times$$
$$(1-\cos\eta)^{-\frac{(d-2)}{2}}\sin^{d-2}\eta\,d\eta.$$

Letting $t := \cos\alpha$, $u := \cos\eta$, after some simple algebra, we obtain that

$$\int_{\alpha}^{\pi}\left(\frac{1-\cos\alpha}{1-\cos\eta}\right)^{\frac{d-1}{2}}\left(\frac{1-\cos\alpha}{\cos\alpha-\cos\eta}\right)^{\frac{1}{2}}{}_2F_1\left(1,\frac{d-1}{2};\frac{1}{2};\frac{\cos\alpha-\cos\eta}{1-\cos\eta}\right) \tag{3.3.45}$$
$$\times(1-\cos\eta)^{-\frac{(d-2)}{2}}\sin^{d-2}\eta\,d\eta=(1-t)^{d/2}\times$$
$$\int_{-1}^{t}(1-u)^{-d/2}(1+u)^{(d-3)/2}(t-u)^{-1/2}\,{}_2F_1\left(1,\frac{d-1}{2};\frac{1}{2};\frac{t-u}{1-u}\right)du.$$

Hence, our original integral further reduces to

$$\int_{-1}^{t}(1-u)^{-d/2}(1+u)^{(d-3)/2}(t-u)^{-1/2}\,{}_2F_1\left(1,\frac{d-1}{2};\frac{1}{2};\frac{t-u}{1-u}\right)du. \tag{3.3.46}$$

The integral in (3.3.46) closely resembles the integral appearing in Lemma 3.3.3. To evaluate integral (3.3.46), we will develop an argument similar to the proof of Lemma A.1 in [16]. Set $a=-1$, $b=t$, $c=1$, $\gamma=1/2$ and $\beta=(d-1)/2$. Also, we let $x=(b-a)/(c-a)$, so that $0<x<1$. Introducing the substitution $(b-u)/(c-u)=xv$, after simplifications we deduce that

$$\int_{-1}^{t}(1-u)^{-d/2}(1+u)^{(d-3)/2}(t-u)^{-1/2}\,{}_2F_1\left(1,\frac{d-1}{2};\frac{1}{2};\frac{t-u}{1-u}\right)du \tag{3.3.47}$$
$$=(b-a)^{\gamma+\beta-1}(c-a)^{-\gamma}(c-b)^{\gamma-d/2}\times$$
$$\int_0^1 v^{\gamma-1}(1-v)^{\beta-1}\,{}_2F_1(1,\beta;\gamma;xv)\,dv$$
$$=(b-a)^{\gamma+\beta-1}(c-a)^{-\gamma}(c-b)^{\gamma-d/2}\,I,$$

where

$$I:=\int_0^1 v^{\gamma-1}(1-v)^{\beta-1}\,{}_2F_1(1,\beta;\gamma;xv)\,dv. \tag{3.3.48}$$

Substituting the series expansion for the Gauss hypergeometric function (3.1.2) into the above integral and integrating term-by-term, we find

$$\begin{aligned}
I &= \int_0^1 v^{\gamma-1}\,(1-v)^{\beta-1}\,{}_2F_1(1,\beta;\gamma;xv)\,dv \\
&= \int_0^1 v^{\gamma-1}\,(1-v)^{\beta-1}\,dv\,\sum_{n=0}^{\infty}\frac{(1)_n\,(\beta)_n}{(\gamma)_n\,n!}\,x^n\,v^n \\
&= \sum_{n=0}^{\infty}\frac{(\beta)_n}{(\gamma)_n}\,x^n\int_0^1 v^{n+\gamma-1}\,(1-v)^{\beta-1}\,dv \\
&= \sum_{n=0}^{\infty}\frac{(\beta)_n}{(\gamma)_n}\,x^n\,\mathrm{B}(n+\gamma,\beta) \\
&= \sum_{n=0}^{\infty}\frac{(\beta)_n}{(\gamma)_n}\frac{\Gamma(n+\gamma)\,\Gamma(\beta)}{\Gamma(n+\gamma+\beta)}\,x^n.
\end{aligned}$$

Taking into account that $(a)_n = \Gamma(a+n)/\Gamma(a)$, we further obtain

$$\begin{aligned}
I &= \sum_{n=0}^{\infty}\frac{(\beta)_n}{(\gamma)_n}\frac{\Gamma(n+\gamma)\,\Gamma(\beta)}{\Gamma(n+\gamma+\beta)}\,x^n \qquad (3.3.49)\\
&= \Gamma(\gamma)\sum_{n=0}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\beta+\gamma+n)}\,x^n \\
&= \Gamma(1/2)\sum_{n=0}^{\infty}\frac{\Gamma(n+(d-1)/2)}{\Gamma(n+d/2)}\,x^n \\
&= \frac{\sqrt{\pi}\,\Gamma((d-1)/2)}{\Gamma(d/2)}\,{}_2F_1\!\left(1,\frac{d-1}{2};\frac{d}{2};x\right).
\end{aligned}$$

Substituting (3.3.49) into (3.3.47), and simplifying, we find that

$$\begin{aligned}
&\int_{-1}^{t}(1-u)^{-d/2}\,(1+u)^{(d-3)/2}\,(t-u)^{-1/2}\,{}_2F_1\!\left(1,\frac{d-1}{2};\frac{1}{2};\frac{t-u}{1-u}\right)du \\
&= \frac{\sqrt{\pi}\,\Gamma((d-1)/2)}{\sqrt{2}\,\Gamma(d/2)}\times \\
&(1-\cos\alpha)^{-(d-1)/2}\,(1+\cos\alpha)^{(d-2)/2}\,{}_2F_1\!\left(1,\frac{d-1}{2};\frac{d}{2};\frac{1+\cos\alpha}{2}\right).
\end{aligned}$$

Inserting the last integral into (3.3.45), we finally infer that

$$\begin{aligned}
&\int_{\alpha}^{\pi} Q(\eta)\left(\frac{1-\cos\alpha}{1-\cos\eta}\right)^{\frac{d-1}{2}}\left(\frac{1-\cos\alpha}{\cos\alpha-\cos\eta}\right)^{\frac{1}{2}}\times \qquad (3.3.50)\\
&{}_2F_1\!\left(1,\frac{d-1}{2};\frac{1}{2};\frac{\cos\alpha-\cos\eta}{1-\cos\eta}\right)\sin^{d-2}\eta\,d\eta = \frac{q\,\sqrt{\pi}\,\Gamma((d-1)/2)}{\sqrt{2}\,\Gamma(d/2)}\times \\
&(1-\cos\alpha)^{1/2}\,(1+\cos\alpha)^{(d-2)/2}\,{}_2F_1\!\left(1,\frac{d-1}{2};\frac{d}{2};\frac{1+\cos\alpha}{2}\right).
\end{aligned}$$

Furthermore, from (3.3.34) it follows that

$$(1-\cos\alpha)^{1/2}\,(1+\cos\alpha)^{(d-2)/2}\,{}_2F_1\left(1,\frac{d-1}{2};\frac{d}{2};\frac{1+\cos\alpha}{2}\right) \\ = \frac{2^{(d-1)/2}\,\Gamma(d/2)}{\Gamma((d-2)/2)}\,\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{1}{2}\right), \tag{3.3.51}$$

thus reducing expression (3.3.50) to

$$\int_\alpha^\pi Q(\eta)\left(\frac{1-\cos\alpha}{1-\cos\eta}\right)^{\frac{d-1}{2}}\left(\frac{1-\cos\alpha}{\cos\alpha-\cos\eta}\right)^{\frac{1}{2}}\times \\ {}_2F_1\left(1,\frac{d-1}{2};\frac{1}{2};\frac{\cos\alpha-\cos\eta}{1-\cos\eta}\right)\sin^{d-2}\eta\,d\eta \\ = \frac{q\sqrt{\pi}\,2^{(d-2)/2}\,\Gamma((d-1)/2)}{\Gamma((d-2)/2)}\,\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{1}{2}\right). \tag{3.3.52}$$

Substituting (3.3.52) into (3.2.4), we finally obtain the desired expression (3.2.6) for the $\mathcal{F}$-functional,

$$\mathcal{F}(C_{S,\alpha}) = \frac{\sqrt{\pi}\,\Gamma(d/2-1)}{2^{d-2}\,\Gamma((d-1)/2)}\left(\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{d}{2}\right)\right)^{-1}\times \\ \left\{1+\frac{q\,2^{(d-2)/2}\,\Gamma((d-1)/2)}{\sqrt{\pi}\,\Gamma(d/2-1)}\,\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{1}{2}\right)\right\}. \tag{3.3.53}$$

■

Proof of Theorem 3.2.2. Assume that the support is a spherical cap $C_{S,\alpha}$, and an external field Q on the sphere $\mathbb{S}^{d-1}$ is given by (3.2.5), that is

$$Q(\theta) = q\,(1-\cos\theta)^{-(d-2)/2}, \quad q>0, \quad \alpha\le\theta\le\pi.$$

The $\mathcal{F}$-functional for this external field is given by expression (3.3.53). Taking into account that

$$\frac{d}{dz}\,\mathrm{B}(z;a,b) = (1-z)^{b-1}\,z^{a-1}, \tag{3.3.54}$$

and differentiating (3.3.53) with respect to α, one can show that

$$\mathcal{F}'(C_{S,\alpha}) \\ = \frac{\sqrt{\pi}\Gamma(d/2-1)}{2^{d-2}\Gamma((d-1)/2)}\left(\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{d}{2}\right)\right)^{-2}\cos^{d-3}\left(\frac{\alpha}{2}\right)\sin^{d-1}\left(\frac{\alpha}{2}\right) \\ \times\left\{1-\frac{q\,2^{(d-2)/2}\,\Gamma((d-1)/2)}{\sqrt{\pi}\,\Gamma(d/2-1)}\left[\csc^{d-1}\left(\frac{\alpha}{2}\right)\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{d}{2}\right) \right.\right. \\ \left.\left. -\,\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{1}{2}\right)\right]\right\}.$$

This shows that the critical points of $\mathcal{F}(C_{S,\alpha})$ satisfy

$$\csc^{d-1}\left(\frac{\alpha}{2}\right) \mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right); \frac{d-2}{2}, \frac{d}{2}\right) - \mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right); \frac{d-2}{2}, \frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(d/2-1)}{q\,2^{(d-2)/2}\,\Gamma((d-1)/2)}. \tag{3.3.55}$$

The proof of existence and uniqueness of a zero for the function defined by expression (3.3.55) is exactly the same in [17, Theorem 13, p. 773]. We therefore obtained an equation for finding an angle α that defines the support $C_{S,\alpha}$ of the extremal measure μ_Q, when the external field is produced by a positive point charge of magnitude q, placed at the North Pole of the sphere $\mathbb{S}^{d-1}$.

We next obtain the expression for the equilibrium density, corresponding to the external field under consideration. Applying Theorem 3.1.3, we first compute the auxiliary function $g(\zeta)$, according to (3.1.9). We are thus led to the following integral, appearing in right hand side of (3.1.9),

$$\int_\zeta^\pi \frac{Q(\theta)\, \cos^{d-3}(\theta/2)\, \sin\theta\, d\theta}{\sqrt{\cos\zeta - \cos\theta}} = q\, 2^{-(d-3)/2} \int_\zeta^\pi \frac{(1-\cos\theta)^{-(d-2)/2}\,(1+\cos\theta)^{(d-3)/2}\, \sin\theta\, d\theta}{\sqrt{\cos\zeta - \cos\theta}}$$

Using the substitution $1+\cos\theta = (1+\cos\zeta)\,t$, after a number of easy manipulations, we infer that

$$\begin{aligned}\int_\zeta^\pi & \frac{(1-\cos\theta)^{-(d-2)/2}\,(1+\cos\theta)^{(d-3)/2}\, \sin\theta\, d\theta}{\sqrt{\cos\zeta - \cos\theta}} \\ &= \cos^{d-2}(\zeta/2) \int_0^1 t^{(d-3)/2}\,(1-t)^{-1/2}\,(1-\cos^2(\zeta/2)t)^{-(d-2)/2}\, dt \\ &= \frac{\sqrt{\pi}\,\Gamma((d-1)/2)}{\Gamma(d/2)} \cos^{d-2}\left(\frac{\zeta}{2}\right) {}_2F_1\left(\frac{d-2}{2}, \frac{d-1}{2}; \frac{d}{2}; \cos^2\left(\frac{\zeta}{2}\right)\right),\end{aligned}$$

where we used the integral representation of the hypergeometric function [1, p. 558, # 15.3.1]. Therefore,

$$\int_\zeta^\pi \frac{Q(\theta)\, \cos^{d-3}(\theta/2)\, \sin\theta\, d\theta}{\sqrt{\cos\zeta - \cos\theta}} = \frac{q\sqrt{\pi}\,\Gamma((d-1)/2)}{2^{(d-3)/2}\,\Gamma(d/2)} \cos^{d-2}\left(\frac{\zeta}{2}\right) {}_2F_1\left(\frac{d-2}{2}, \frac{d-1}{2}; \frac{d}{2}; \cos^2\left(\frac{\zeta}{2}\right)\right).$$

Differentiating the last expression, and taking into account the fact [1, p. 556, #15.2.1]

$$\frac{d}{dz}\,{}_2F_1(a,b;c;z) = \frac{ab}{c}\,{}_2F_1(a+1,b+1;c+1;z),$$

we find, upon inserting the result of the differentiation into (3.1.9),

$$\begin{aligned} g(\zeta) = &-\frac{q\sqrt{\pi}\,(d-2)\,\Gamma((d-1)/2)}{2^{(d-1)/2}\,\Gamma(d/2)}\,\sin^{d-2}\left(\frac{\zeta}{2}\right)\times \\ &\left\{{}_2F_1\left(\frac{d-2}{2},\frac{d-1}{2};\frac{d}{2};\cos^2\left(\frac{\zeta}{2}\right)\right)\right. \\ &\left.+\frac{d-1}{d}\cos^2\left(\frac{\zeta}{2}\right){}_2F_1\left(\frac{d}{2},\frac{d+1}{2};\frac{d}{2}+1;\cos^2\left(\frac{\zeta}{2}\right)\right)\right\},\qquad \alpha_0\le\zeta\le\pi. \end{aligned} \tag{3.3.56}$$

It turns out that the expression in braces on the right hand side of (3.3.56) can be simplified rather dramatically. Indeed, according to the linear transformation formula for the hypergeometric function [1, p. 559, #15.3.3], we have

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z).$$

With this in mind, it is a straightforward calculation to see that

$$\begin{aligned} &{}_2F_1\left(\frac{d-2}{2},\frac{d-1}{2};\frac{d}{2};\cos^2\left(\frac{\zeta}{2}\right)\right) \\ &\qquad = \sin^{-(d-3)}\left(\frac{\zeta}{2}\right){}_2F_1\left(1,\frac{1}{2};\frac{d}{2};\cos^2\left(\frac{\zeta}{2}\right)\right). \end{aligned} \tag{3.3.57}$$

Similarly,

$$\begin{aligned} &{}_2F_1\left(\frac{d}{2},\frac{d+1}{2};\frac{d}{2}+1;\cos^2\left(\frac{\zeta}{2}\right)\right) \\ &\qquad = \sin^{-(d-1)}\left(\frac{\zeta}{2}\right){}_2F_1\left(1,\frac{1}{2};\frac{d}{2}+1;\cos^2\left(\frac{\zeta}{2}\right)\right). \end{aligned} \tag{3.3.58}$$

In light of (3.3.57) and (3.3.58), we see that

$$\begin{aligned} &{}_2F_1\left(\frac{d-2}{2},\frac{d-1}{2};\frac{d}{2};\cos^2\left(\frac{\zeta}{2}\right)\right) \\ &\qquad +\frac{d-1}{d}\cos^2\left(\frac{\zeta}{2}\right){}_2F_1\left(\frac{d}{2},\frac{d+1}{2};\frac{d}{2}+1;\cos^2\left(\frac{\zeta}{2}\right)\right) \\ &\qquad = \sin^{-(d-3)}\left(\frac{\zeta}{2}\right)\left\{{}_2F_1\left(1,\frac{1}{2};\frac{d}{2};\cos^2\left(\frac{\zeta}{2}\right)\right)\right. \\ &\qquad \left.+\frac{d-1}{d}\,\frac{\cos^2(\zeta/2)}{1-\cos^2(\zeta/2)}\,{}_2F_1\left(1,\frac{1}{2};\frac{d}{2}+1;\cos^2\left(\frac{\zeta}{2}\right)\right)\right\}. \end{aligned} \tag{3.3.59}$$

Using again the relation [1, p. 558, # 15.2.20],

$$c\,(1-z)\,{}_2F_1(a,b;c;z) - c\,{}_2F_1(a-1,b;c;z) + (c-b)z\,{}_2F_1(a,b;c+1;z) = 0,$$

it follows that

$${}_2F_1\left(1,\frac{1}{2};\frac{d}{2};\cos^2\left(\frac{\zeta}{2}\right)\right) + \frac{d-1}{d}\frac{\cos^2(\zeta/2)}{1-\cos^2(\zeta/2)}\,{}_2F_1\left(1,\frac{1}{2};\frac{d}{2}+1;\cos^2\left(\frac{\zeta}{2}\right)\right) = \sin^{-2}\left(\frac{\zeta}{2}\right),$$

which, in conjunction with (3.3.59), allows us to conclude

$${}_2F_1\left(\frac{d-2}{2},\frac{d-1}{2};\frac{d}{2};\cos^2\left(\frac{\zeta}{2}\right)\right) + \frac{d-1}{d}\cos^2\left(\frac{\zeta}{2}\right){}_2F_1\left(\frac{d}{2},\frac{d+1}{2};\frac{d}{2}+1;\cos^2\left(\frac{\zeta}{2}\right)\right) = \sin^{-(d-1)}\left(\frac{\zeta}{2}\right).$$

Inserting the latter into (3.3.56), we obtain the desired simplified expression for $g(\zeta)$,

$$g(\zeta) = -\frac{q\sqrt{\pi}\,(d-2)\,\Gamma((d-1)/2)}{2^{(d-1)/2}\,\Gamma(d/2)}\csc\left(\frac{\zeta}{2}\right), \quad \alpha_0 \le \zeta \le \pi. \tag{3.3.60}$$

Having $g(\zeta)$ computed, we now proceed to evaluating the term $F(\eta)$ given by (3.1.8), which describes the contribution of the external field Q in the expression (3.1.10) for the equilibrium measure μ_Q. Upon inserting (3.3.60) into the right hand side of (3.1.8), we are presented with evaluation of the following integral,

$$\int_{\alpha_0}^{\eta}\frac{g(\zeta)\,\sin\zeta\,d\zeta}{\sqrt{\cos\zeta-\cos\eta}} = -\frac{q\sqrt{\pi}\,(d-2)\,\Gamma((d-1)/2)}{2^{(d-2)/2}\,\Gamma(d/2)}\int_{\alpha_0}^{\eta}\frac{\sin\zeta\,d\zeta}{\sqrt{1-\cos\zeta}\,\sqrt{\cos\zeta-\cos\eta}}.$$

The integral on the right hand side of the last expression can be easily evaluated using the substitution $t^2 = \cos\zeta - \cos\eta$, so that

$$\int_{\alpha_0}^{\eta}\frac{\sin\zeta\,d\zeta}{\sqrt{1-\cos\zeta}\,\sqrt{\cos\zeta-\cos\eta}} = 2\,\sin^{-1}\sqrt{\frac{\cos\alpha_0-\cos\eta}{1-\cos\eta}}.$$

Now it is not hard to see that

$$\frac{d}{d\eta}\int_{\alpha_0}^{\eta}\frac{g(\zeta)\,\sin\zeta\,d\zeta}{\sqrt{\cos\zeta-\cos\eta}} = -\frac{q\sqrt{\pi}(d-2)\Gamma((d-1)/2)}{2^{(d-2)/2}\Gamma(d/2)}\frac{\sin\eta}{1-\cos\eta}\sqrt{\frac{1-\cos\alpha_0}{\cos\alpha_0-\cos\eta}}, \quad \alpha_0 \le \eta \le \pi. \tag{3.3.61}$$

Inserting (3.3.61) into (3.1.8) and simplifying, we find

$$F(\eta) = -\frac{q\,\Gamma((d-1)/2)}{\sqrt{2}\,\pi^{(d+1)/2}}\,\frac{1}{(1-\cos\eta)^{(d-1)/2}}\sqrt{\frac{1-\cos\alpha_0}{\cos\alpha_0-\cos\eta}}, \quad \alpha_0 \le \eta \le \pi.$$

Having $F(\eta)$ at hand, the constant C_Q is found from (3.1.11), thus leading us to the integral

$$\begin{aligned}\int_{\alpha_0}^{\pi} & F(\eta)\,\sin^{d-2}\eta\,d\eta \\ &= -\frac{q\,\Gamma((d-1)/2)}{\sqrt{2}\,\pi^{(d+1)/2}}\sqrt{1-\cos\alpha_0}\int_{\alpha_0}^{\pi}\frac{(1+\cos\eta)^{(d-3)/2}\,\sin\eta\,d\eta}{(1-\cos\eta)\,\sqrt{\cos\alpha_0-\cos\eta}}.\end{aligned} \tag{3.3.62}$$

Using the standard substitution $1+\cos\eta = 2t$, it follows that

$$\begin{aligned}&\int_{\alpha_0}^{\pi}\frac{(1+\cos\eta)^{(d-3)/2}\,\sin\eta\,d\eta}{(1-\cos\eta)\,\sqrt{\cos\alpha_0-\cos\eta}} \\ &= 2^{d/2-2}\left(\frac{1+\cos\alpha_0}{2}\right)^{(d-2)/2}\frac{\Gamma((d-1)/2)\sqrt{\pi}}{\Gamma(d/2)}\,{}_2F_1\left(1,\frac{d-1}{2};\frac{d}{2};\cos^2\left(\frac{\alpha_0}{2}\right)\right) \\ &= \frac{\sqrt{\pi}\,2^{(d-3)/2}\,\Gamma((d-1)/2)}{\Gamma(d/2-1)}\,\frac{1}{\sqrt{1-\cos\alpha_0}}\,\mathrm{B}\left(\cos^2\left(\frac{\alpha_0}{2}\right);\frac{d-2}{2},\frac{1}{2}\right),\end{aligned}$$

where we used the integral representation for the hypergeometric function [1, p. 558, # 15.3.1], as well as relation (3.3.51). Substituting the value of the last integral into (3.3.62), we find that

$$\begin{aligned}\int_{\alpha_0}^{\pi} & F(\eta)\,\sin^{d-2}\eta\,d\eta \\ &= -\frac{q\,2^{(d-4)/2}\,(\Gamma((d-1)/2))^2}{\pi^{d/2}\,\Gamma(d/2-1)}\,\mathrm{B}\left(\cos^2\left(\frac{\alpha_0}{2}\right);\frac{d-2}{2},\frac{1}{2}\right).\end{aligned} \tag{3.3.63}$$

Hence, after substituting (3.3.63) into (3.1.11) and some simple algebra, we infer that the constant C_Q is given by

$$\begin{aligned}C_Q = \frac{\Gamma(d/2-1)}{2^{d-1}\,\pi^{d/2}}&\left(\mathrm{B}\left(\cos^2\left(\frac{\alpha_0}{2}\right);\frac{d-2}{2},\frac{d}{2}\right)\right)^{-1}\times \\ &\left\{1+\frac{q\,2^{(d-2)/2}\,\Gamma((d-1)/2)}{\sqrt{\pi}\,\Gamma(d/2-1)}\,\mathrm{B}\left(\cos^2\left(\frac{\alpha_0}{2}\right);\frac{d-2}{2},\frac{1}{2}\right)\right\}.\end{aligned}$$

■

Proof of Theorem 3.2.3. Substituting expression (3.2.10) into (3.2.4), we are presented with the following integral

$$\int_{\alpha}^{\pi} Q(\eta) \left(\frac{1-\cos\alpha}{1-\cos\eta}\right)^{\frac{d-1}{2}} \left(\frac{1-\cos\alpha}{\cos\alpha-\cos\eta}\right)^{\frac{1}{2}} {}_2F_1\left(1, \frac{d-1}{2}; \frac{1}{2}; \frac{\cos\alpha-\cos\eta}{1-\cos\eta}\right) \times$$
$$\sin^{d-2}\eta\, d\eta = (1-t)^{d/2} \times$$
$$\int_{-1}^{t} (1-u)^{-1}\,(1+u)^{(d+1)/2}\,(t-u)^{-1/2}\,{}_2F_1\left(1, \frac{d-1}{2}; \frac{1}{2}; \frac{t-u}{1-u}\right) du,$$

where we set $t := \cos\alpha$ and $u := \cos\eta$. We thus need to evaluate

$$J = \int_{-1}^{t} (1-u)^{-1}\,(1+u)^{(d+1)/2}\,(t-u)^{-1/2}\,{}_2F_1\left(1, \frac{d-1}{2}; \frac{1}{2}; \frac{t-u}{1-u}\right) du.$$

This integral will be evaluated using the same approach we used when evaluating similar integral (3.3.48). Letting $a = -1$, $b = t$, $c = 1$, $\gamma = 1/2$, $\beta = (d-1)/2$, $x = (b-a)/(c-a)$, and again using the substitution $(b-u)/(c-u) = xv$, we find

$$J = \int_{-1}^{t} (1-u)^{-1}\,(1+u)^{(d+1)/2}\,(t-u)^{-1/2}\,{}_2F_1\left(1, \frac{d-1}{2}; \frac{1}{2}; \frac{t-u}{1-u}\right) du \tag{3.3.64}$$
$$= (c-b)^{\gamma-1}\,(b-a)^{\beta+1}\,x^{\gamma} \times$$
$$\int_{0}^{1} v^{\gamma-1}\,(1-v)^{\beta+1}\,(1-xv)^{-(\beta+\gamma+1)}\,{}_2F_1(1,\beta;\gamma;xv)\,dv$$
$$= (c-b)^{\gamma-1}\,(b-a)^{\beta+1}\,x^{\gamma}\,I,$$

where

$$I := \int_{0}^{1} v^{\gamma-1}\,(1-v)^{\beta+1}\,(1-xv)^{-(\beta+\gamma+1)}\,{}_2F_1(1,\beta;\gamma;xv)\,dv.$$

Using the series representation for the Gauss hypergeometric function ${}_2F_1$ [1, p. 556, # 15.1.1] and integrating term-by-term, we further write

$$I = \int_{0}^{1} v^{\gamma-1}\,(1-v)^{\beta+1}\,(1-xv)^{-(\beta+\gamma+1)}\,{}_2F_1(1,\beta;\gamma;xv)\,dv \tag{3.3.65}$$
$$= \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\gamma)_n}\,x^n \int_{0}^{1} v^{n+\gamma-1}\,(1-v)^{\beta+1}\,(1-xv)^{-(\beta+\gamma+1)}\,dv$$
$$= \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\gamma)_n}\,x^n\,I_n,$$

where

$$I_n := \int_0^1 v^{n+\gamma-1}\,(1-v)^{\beta+1}\,(1-xv)^{-(\beta+\gamma+1)}\,dv.$$

According to the integral representation of the function ${}_2F_1$ [1, p. 558, # 15.3.1], we further have

$$I_n = \frac{\Gamma(n+\gamma)\,\Gamma(\beta+2)}{\Gamma(n+\gamma+\beta+2)}\,{}_2F_1(\beta+\gamma+1, n+\gamma; n+\gamma+\beta+2; x).$$

Recalling that the function ${}_2F_1$ is symmetric with respect to switching the first two parameters [1, p. 556, # 15.1.1] , that is ${}_2F_1(\beta+\gamma+1, n+\gamma; n+\gamma+\beta+2; x) = {}_2F_1(n+\gamma, \beta+\gamma+1; n+\gamma+\beta+2; x)$, and using the fact that $n+\gamma+\beta+2 > \beta+\gamma+1$ for all integer $n \geq 0$, we continue by using again the integral representation of ${}_2F_1$ [1, p. 558, # 15.3.1] as follows,

$$\begin{aligned} I_n &= \frac{\Gamma(n+\gamma)\,\Gamma(\beta+2)}{\Gamma(n+\gamma+\beta+2)}\,{}_2F_1(\beta+\gamma+1, n+\gamma; n+\gamma+\beta+2; x) \\ &= \frac{\Gamma(n+\gamma)\,\Gamma(\beta+2)}{\Gamma(n+\gamma+\beta+2)}\,{}_2F_1(n+\gamma, \beta+\gamma+1; n+\gamma+\beta+2; x) \\ &= \frac{\Gamma(n+\gamma)\,\Gamma(\beta+2)}{\Gamma(\beta+\gamma+1)\,\Gamma(n+1)} \int_0^1 v^{\beta+\gamma}\,(1-v)^n\,(1-xv)^{-(n+\gamma)}\,dv. \end{aligned}$$

Inserting the last expression into (3.3.65), and switching the order of integration and summation, which is justified by the uniform convergence of the series as $v \in [0,1]$, we have

$$\begin{aligned} I &= \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\gamma)_n}\,x^n\,I_n \\ &= \frac{\Gamma(\beta+2)}{\Gamma(\beta+\gamma+1)} \int_0^1 v^{\beta+\gamma}\,(1-xv)^{-\gamma}\,dv \sum_{n=0}^{\infty} \frac{(\beta)_n\,\Gamma(n+\gamma)}{\Gamma(n+1)\,(\gamma)_n} \left(x\,\frac{1-v}{1-xv}\right)^n \end{aligned}$$

It is not difficult to see that

$$\sum_{n=0}^{\infty} \frac{(\beta)_n\,\Gamma(n+\gamma)}{\Gamma(n+1)\,(\gamma)_n}\,z^n = \Gamma(\gamma)\,(1-z)^{-\beta}, \quad |z| < 1.$$

We thus eventually find

$$\begin{aligned} I &= \frac{\Gamma(\beta+2)\,\Gamma(\gamma)}{\Gamma(\beta+\gamma+1)}\,(1-x)^{-\beta} \int_0^1 v^{\beta+\gamma}\,(1-xv)^{-(\gamma-\beta)}\,dv \\ &= \frac{\sqrt{\pi}\,\Gamma(\beta+2)}{\Gamma(\beta+\gamma+2)}\,(1-x)^{-\beta}\,{}_2F_1(\gamma-\beta, \beta+\gamma+1; \beta+\gamma+2; x), \end{aligned}$$

where we again used the integral representation of the hypergeometric function ${}_2F_1$ [1, p. 558, # 15.3.1]. Inserting the latter into (3.3.64) and simplifying, we deduce that

$$\int_\alpha^\pi Q(\eta)\left(\frac{1-\cos\alpha}{1-\cos\eta}\right)^{\frac{d-1}{2}}\left(\frac{1-\cos\alpha}{\cos\alpha-\cos\eta}\right)^{\frac{1}{2}}\times$$
$${}_2F_1\left(1,\frac{d-1}{2};\frac{1}{2};\frac{\cos\alpha-\cos\eta}{1-\cos\eta}\right)\sin^{d-2}\eta\,d\eta=\frac{2^{d/2-1}\sqrt{\pi}\,\Gamma((d+3)/2)}{\Gamma(d/2+2)}\times$$
$$(1+\cos\alpha)^{d/2+1}\,{}_2F_1\left(1-\frac{d}{2},\frac{d}{2}+1;\frac{d}{2}+2;\cos^2\left(\frac{\alpha}{2}\right)\right).$$

The last expression can be simplified even further. Indeed, using a linear transformation formula for the hypergeometric function [1, p. 559, #15.3.3] and (3.3.34), we obtain the neat formula

$$\int_\alpha^\pi Q(\eta)\left(\frac{1-\cos\alpha}{1-\cos\eta}\right)^{\frac{d-1}{2}}\left(\frac{1-\cos\alpha}{\cos\alpha-\cos\eta}\right)^{\frac{1}{2}}\times$$
$${}_2F_1\left(1,\frac{d-1}{2};\frac{1}{2};\frac{\cos\alpha-\cos\eta}{1-\cos\eta}\right)\sin^{d-2}\eta\,d\eta$$
$$=\frac{2^{d/2}\sqrt{\pi}\,\Gamma((d+3)/2)}{\Gamma(d/2+1)}\,\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right);\frac{d}{2}+1,\frac{d}{2}\right).$$

Inserting the last integral into (3.2.4), we obtain the desired expression (3.2.11).

■

Proof of Theorem 3.2.4. Let

$$w(\alpha)=\frac{\sqrt{\pi}\,\Gamma(d/2-1)}{2^{d-2}\,\Gamma((d-1)/2)}\left(\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{d}{2}\right)\right)^{-1}$$
$$\left\{1+\frac{2^{d/2}\,\Gamma((d+3)/2)}{\sqrt{\pi}\,\Gamma(d/2+1)}\,\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right);\frac{d}{2}+1,\frac{d}{2}\right)\right\}. \tag{3.3.66}$$

Differentiating $w(\alpha)$, we find

$$w'(\alpha)=\sin^{d-1}\left(\frac{\alpha}{2}\right)\cos^{d-3}\left(\frac{\alpha}{2}\right)\left(\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{d}{2}\right)\right)^{-1}\omega(\alpha), \tag{3.3.67}$$

where

$$\omega(\alpha)=w(\alpha)-\frac{4(d^2-1)}{d(d-2)}\cos^4\left(\frac{\alpha}{2}\right). \tag{3.3.68}$$

We therefore see that the critical points of $w(\alpha)$ are given by solutions of the equation

$$w(\alpha) = \frac{4(d^2-1)}{d(d-2)} \cos^4\left(\frac{\alpha}{2}\right). \tag{3.3.69}$$

Rearranging the latter, we obtain (3.2.12).

We continue by showing the existence and uniqueness of a critical point of $w(\alpha)$. Our approach will be based on an argument developed in [17]. First, observe that from (3.3.66) and (3.3.68) it follows that

$$\lim_{\alpha\to\pi-} \omega(\alpha) = +\infty.$$

Hence, there is a smallest $\alpha_0 \in [0,\pi)$ such that $\omega(\alpha) > 0$ for $\alpha \in (\alpha_0,\pi)$. If $\alpha_0 = 0$, then $w(\alpha)$ is strictly increasing on $(0,\pi)$, and attains minimum at $\alpha = 0$. If $\alpha_0 > 0$, we have that $\omega(\alpha) > 0$ for $\alpha \in (\alpha_0,\pi)$. Taking into account the continuity of $\omega(\alpha)$, by passing to the limit $\alpha \to \alpha_0+$ in the latter inequality, we infer that $\omega(\alpha_0) \geq 0$. Since α_0 was the smallest α such that $\omega(\alpha) > 0$ on (α_0,π), we deduce that $\omega(\alpha_0) = 0$.

From expression (3.3.67) it is clear that the sign of $w'(\alpha)$ is determined by the sign of $\omega(\alpha)$. This shows that $w'(\alpha) > 0$ on (α_0,π), and $w'(\alpha_0) = 0$. Next, suppose that $\xi \in (0,\pi)$ is a critical point of $w(\alpha)$, that is $w'(\xi) = 0$. Using expression (3.3.67), we readily find that

$$\begin{aligned} w''(\alpha) = {} & \\ & \left[\sin^{d-1}\left(\frac{\alpha}{2}\right)\cos^{d-3}\left(\frac{\alpha}{2}\right)\left(\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{d}{2}\right)\right)^{-1}\right]'\omega(\alpha) \\ & +\sin^{d-1}\left(\frac{\alpha}{2}\right)\cos^{d-3}\left(\frac{\alpha}{2}\right)\left(\mathrm{B}\left(\cos^2\left(\frac{\alpha}{2}\right);\frac{d-2}{2},\frac{d}{2}\right)\right)^{-1}\omega'(\alpha), \end{aligned} \tag{3.3.70}$$

where

$$\omega'(\alpha) = w'(\alpha) + \frac{8(d^2-1)}{d(d-2)}\cos^3\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right). \tag{3.3.71}$$

We want to show that $w''(\xi) > 0$. This readily follows from (3.3.70), since for $0 < \xi < \pi$,

$$w''(\xi) = \frac{8(d^2-1)}{d(d-2)}\cos^3\left(\frac{\xi}{2}\right)\sin\left(\frac{\xi}{2}\right) > 0.$$

This means that $w(\alpha)$ has exactly one global minimum on $[0,\pi)$, which is either a unique solution $\alpha_0 \in (0,\pi)$ of equation (3.3.69), if it exists, or $\alpha_0 = 0$, if such a solution does not exist.

We finish by computing the equilibrium density. Substituting expression (3.2.10) into (3.1.9), we are led to the following integral

$$\int_\zeta^\pi \frac{Q(\theta)\,\cos^{d-3}(\theta/2)\,\sin\theta\,d\theta}{\sqrt{\cos\zeta-\cos\theta}} = \frac{1}{2^{(d-3)/2}}\int_\zeta^\pi (1+\cos\theta)^{(d+1)/2}\,\frac{\sin\theta\,d\theta}{\sqrt{\cos\zeta-\cos\theta}}.$$

Making the change of variables $1+\cos\theta = (1+\cos\zeta)t$, we further write

$$\begin{aligned} 2^{-(d-3)/2}&\int_\zeta^\pi (1+\cos\theta)^{(d+1)/2}\,\frac{\sin\theta\,d\theta}{\sqrt{\cos\zeta-\cos\theta}} \\ &= 2^{-(d-3)/2}\,(1+\cos\zeta)^{(d+2)/2}\int_0^1 t^{(d+1)/2}\,(1-t)^{-1/2}\,dt \\ &= 2^{-(d-3)/2}\,(1+\cos\zeta)^{(d+2)/2}\,\mathrm{B}\left(\frac{d+3}{2},\frac{1}{2}\right) \\ &= \frac{2^{5/2}\,\sqrt{\pi}\,\Gamma((d+3)/2)}{\Gamma(d/2+2)}\,\cos^{d+2}\left(\frac{\zeta}{2}\right). \end{aligned}$$

We thus conclude

$$\int_\zeta^\pi \frac{Q(\theta)\,\cos^{d-3}(\theta/2)\,\sin\theta\,d\theta}{\sqrt{\cos\zeta-\cos\theta}} = \frac{2^{5/2}\,\sqrt{\pi}\,\Gamma((d+3)/2)}{\Gamma(d/2+2)}\,\cos^{d+2}\left(\frac{\zeta}{2}\right). \tag{3.3.72}$$

Differentiating (3.3.72) with respect to ζ and inserting the result into (3.1.9), we find

$$g(\zeta) = -\frac{2^{5/2}\,\sqrt{\pi}\,\Gamma((d+3)/2)}{\Gamma(d/2+1)}\,\sin^{d-2}\left(\frac{\zeta}{2}\right)\,\cos^4\left(\frac{\zeta}{2}\right), \quad \alpha_0 \le \zeta \le \pi. \tag{3.3.73}$$

Substituting (3.3.73) into the right hand side of (3.1.8), we arrive to the integral

$$\int_{\alpha_0}^\eta \sin^{d-2}\left(\frac{\zeta}{2}\right)\,\cos^4\left(\frac{\zeta}{2}\right)\,\frac{\sin\zeta\,d\zeta}{\sqrt{\cos\zeta-\cos\eta}}.$$

Making the change of variables $\zeta = \pi - y$, and setting $\widetilde{\alpha_0} = \pi - \alpha_0$, $\widetilde{\eta} = \pi - \eta$, we recast the latter integral as

$$\begin{aligned} &\int_{\alpha_0}^\eta \sin^{d-2}\left(\frac{\zeta}{2}\right)\,\cos^4\left(\frac{\zeta}{2}\right)\,\frac{\sin\zeta\,d\zeta}{\sqrt{\cos\zeta-\cos\eta}} \\ &\quad= \int_{\widetilde{\eta}}^{\widetilde{\alpha_0}} \cos^{d-2}\left(\frac{y}{2}\right)\,\sin^4\left(\frac{y}{2}\right)\,\frac{\sin y\,dy}{\sqrt{\cos y-\cos\widetilde{\eta}}} \\ &\quad= 2^{-(d/2+1)}\int_{\widetilde{\eta}}^{\widetilde{\alpha_0}} (1+\cos y)^{(d-2)/2}\,(1-\cos y)^2\,\frac{\sin y\,dy}{\sqrt{\cos y-\cos\widetilde{\eta}}}. \end{aligned}$$

Making a simple observation that $1 - \cos y = (1 + \cos y) - 2$, allows us to continue the above string of integrals as

$$\begin{aligned}
&\int_{\widetilde{\eta}}^{\widetilde{\alpha_0}} (1+\cos y)^{(d-2)/2} \, (1-\cos y)^2 \, \frac{\sin y \, dy}{\sqrt{\cos y - \cos \widetilde{\eta}}} \\
&\quad = \int_{\widetilde{\eta}}^{\widetilde{\alpha_0}} (1+\cos y)^{(d-2)/2} \, ((1+\cos y) - 2)^2 \, \frac{\sin y \, dy}{\sqrt{\cos y - \cos \widetilde{\eta}}} \\
&\quad = \int_{\widetilde{\eta}}^{\widetilde{\alpha_0}} (1+\cos y)^{(d-2)/2} \, (1+\cos y)^2 \, \frac{\sin y \, dy}{\sqrt{\cos y - \cos \widetilde{\eta}}} \\
&\quad - 4 \int_{\widetilde{\eta}}^{\widetilde{\alpha_0}} (1+\cos y)^{(d-2)/2} \, (1+\cos y) \, \frac{\sin y \, dy}{\sqrt{\cos y - \cos \widetilde{\eta}}} \\
&\quad + 4 \int_{\widetilde{\eta}}^{\widetilde{\alpha_0}} (1+\cos y)^{(d-2)/2} \, \frac{\sin y \, dy}{\sqrt{\cos y - \cos \widetilde{\eta}}}.
\end{aligned}$$

The three integrals on the right hand side of the last expression are evaluated using the change of variables $1 + \cos y = (1 + \cos \widetilde{\eta})t$. Performing these straightforward but tedious evaluations, and reverting back to α_0 and η, we eventually obtain

$$\begin{aligned}
&\int_{\alpha_0}^{\eta} \sin^{d-2}\left(\frac{\zeta}{2}\right) \cos^4\left(\frac{\zeta}{2}\right) \frac{\sin \zeta \, d\zeta}{\sqrt{\cos \zeta - \cos \eta}} \\
&\qquad = 2^{-(d+2)/2} \Bigg\{ (1-\cos\eta)^{(d+3)/2} \, \mathrm{B}\left(\frac{\cos\alpha_0 - \cos\eta}{1-\cos\eta}; \frac{1}{2}, \frac{d}{2}+2\right) \\
&\qquad\qquad -4 \, (1-\cos\eta)^{(d+1)/2} \, \mathrm{B}\left(\frac{\cos\alpha_0 - \cos\eta}{1-\cos\eta}; \frac{1}{2}, \frac{d}{2}+1\right) \\
&\qquad\qquad +4 \, (1-\cos\eta)^{(d-1)/2} \, \mathrm{B}\left(\frac{\cos\alpha_0 - \cos\eta}{1-\cos\eta}; \frac{1}{2}, \frac{d}{2}\right) \Bigg\}.
\end{aligned}$$

Differentiating the above expression with respect to η, and inserting the result into (3.1.8), after simplifications we derive

$$\begin{aligned}
F(\eta) = &-\frac{2\Gamma((d+3)/2)}{d(d-2)\pi^{(d+1)/2}} \times \qquad (3.3.74) \\
&\Bigg\{ \left(\frac{1-\cos\alpha_0}{1-\cos\eta}\right)^{d/2} \sqrt{\frac{1-\cos\eta}{\cos\alpha_0 - \cos\eta}} \, (1+\cos\alpha_0)^2 \\
&+ 2(d-1) \, \mathrm{B}\left(\frac{\cos\alpha_0 - \cos\eta}{1-\cos\eta}; \frac{1}{2}, \frac{d}{2}\right) \\
&- 2(d+1) \, (1-\cos\eta) \, \mathrm{B}\left(\frac{\cos\alpha_0 - \cos\eta}{1-\cos\eta}; \frac{1}{2}, \frac{d}{2}+1\right) \\
&+ \frac{d+3}{2} \, (1-\cos\eta)^2 \, \mathrm{B}\left(\frac{\cos\alpha_0 - \cos\eta}{1-\cos\eta}; \frac{1}{2}, \frac{d}{2}+2\right) \Bigg\},
\end{aligned}$$

where $\alpha_0 \le \eta \le \pi$.

The value of the Robin constant can now be found from (3.1.11). However, going via this standard route with the function $F(\eta)$ of the type (3.3.74) usually involves laborious calculations. Luckily, there is an alternative to that. Indeed, one observes that from the variational inequalities (1.2.1)-(1.2.2) and Proposition 3.2.2 it follows that $\mathcal{F}(C_{S,\alpha_0}) = F_Q$. Therefore, using (3.2.11), we deduce that

$$F_Q = \frac{\sqrt{\pi}\,\Gamma(d/2-1)}{2^{d-2}\,\Gamma((d-1)/2)} \left(\mathrm{B}\left(\cos^2\left(\frac{\alpha_0}{2}\right); \frac{d-2}{2}, \frac{d}{2} \right) \right)^{-1} \left\{ 1 + \frac{2^d\,\Gamma((d+3)/2)}{\sqrt{\pi}\,\Gamma(d/2+1)} \mathrm{B}\left(\cos^2\left(\frac{\alpha_0}{2}\right); \frac{d}{2}+1, \frac{d}{2} \right) \right\},$$

which in turn implies that

$$C_Q = \frac{\Gamma(d/2-1)}{2^{d-1}\,\pi^{d/2}} \left(\mathrm{B}\left(\cos^2\left(\frac{\alpha_0}{2}\right); \frac{d-2}{2}, \frac{d}{2} \right) \right)^{-1} \left\{ 1 + \frac{2^d\,\Gamma((d+3)/2)}{\sqrt{\pi}\,\Gamma(d/2+1)} \mathrm{B}\left(\cos^2\left(\frac{\alpha_0}{2}\right); \frac{d}{2}+1, \frac{d}{2} \right) \right\}.$$

This completes the proof of the theorem. ■

CHAPTER 4

Minimum Riesz Energy Problem on the Hyperdisk

4.1 Introduction and main results

This chapter is based on work [11]. Let $\mathbb{D}_R := \{(x_1,\ldots,x_d) \in \mathbb{R}^d : x_1 = 0, x_2^2 + x_3^2 + \ldots + x_d^2 \le R^2\}$ be the disk of radius R in $\mathbb{R}^d$, with $d \ge 3$, and where $|\cdot|$ is the Euclidean distance. The ring $\mathscr{R}(a,b)$ in $\mathbb{R}^d$ is defined as $\mathscr{R}(a,b) := \{(0, r\bar{x}) \in \mathbb{R}^d : a \le r \le b, \bar{x} \in \mathbb{S}^{d-2}\}$, and the unit disk in $\mathbb{R}^d$ will be denoted by $\mathbb{D}$. We start by taking advantage of the rotationally symmetry of $\mathbb{D}$ using the cylindrical coordinates $z, r, \theta_1, \theta_2, \ldots, \theta_{d-3}, \varphi$, defined as

$$
\begin{aligned}
x_1 &= z,\\
x_2 &= r\cos\theta_1,\\
x_3 &= r\sin\theta_1\cos\theta_2,\\
x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3,\\
&\vdots\\
x_{d-2} &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{d-4}\cos\theta_{d-3},\\
x_{d-1} &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{d-3}\cos\varphi,\\
x_d &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{d-3}\sin\varphi,
\end{aligned}
$$

where $r \ge 0$, $0 \le \theta_j \le \pi$, $j = 1, 2, \ldots, d-3$, and $0 \le \varphi \le 2\pi$. The surface area element on a surface of constant height z, written in cylindrical coordinates, is given by

$$dS = r^{d-2}\,\sin^{d-3}\theta_1\,\sin^{d-4}\theta_2\ldots\sin\theta_{d-3}\,dr\,d\theta_1\,d\theta_2\ldots d\theta_{d-3}\,d\varphi = r^{d-2}\,dr\,d\sigma_{d-1},$$

where σ_d is the surface area element of the unit sphere $\mathbb{S}^{d-1}$. The total surface area of the sphere $\mathbb{S}^{d-1}$ is given by

$$\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

In what follows, we will need to use certain special functions, for which we fix the notation here. The incomplete Beta function $\mathrm{B}(z; a, b)$ is defined as

$$\mathrm{B}(z; a, b) := \int_0^z t^{a-1}(1-t)^{b-1}\, dt, \tag{4.1.1}$$

and the Beta function $\mathrm{B}(a, b) := \mathrm{B}(1; a, b)$. The Gauss hypergeometric function ${}_2F_1(a, b; c, z)$ is defined via series

$${}_2F_1(a, b; c, z) := \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z| < 1, \tag{4.1.2}$$

where $(a)_0 := 1$ and $(a)_n := a(a+1)\dots(a+n-1)$ for $n \geq 1$ is the Pochhammer symbol.

We commence by recording the sufficient conditions on an external field Q that guarantee that the support of the extremal measure μ_Q is a ring or a disk.

Theorem 4.1.1 *Let $s = (d-3)+2\lambda$, with $0 < \lambda < 1$. Assume that the external field $Q : \mathbb{D} \to [0, \infty]$ is invariant with respect to the rotations about the polar axis, that is $Q(x) = Q(r)$, where $x = (0, r\bar{x}) \in \mathbb{D}$, $\bar{x} \in \mathbb{S}^{d-2}$, $0 \leq r \leq 1$. Further suppose that Q is a convex function, that is $Q(r)$ is convex on $[0, 1]$. Then the support of the extremal measure μ_Q is a ring $\mathscr{R}(a, b)$, contained in the disk $\mathbb{D}$. In other words, there exist real numbers a and b such that $0 \leq a < b \leq 1$, so that $\operatorname{supp}\mu_Q = \mathscr{R}(a, b)$.*

Furthermore, if $Q(r)$ is, in addition, an increasing function, then $a = 0$, which implies that the support of the extremal measure μ_Q is a disk of radius $b \leq 1$, centered at the origin.

On the other hand, if $Q(r)$ is a decreasing function, then $b = 1$, that is the support of the extremal measure μ_Q will be a ring with outer radius 1.

The support S_Q is a main ingredient in determining the extremal measure μ_Q itself. Indeed, if S_Q is known, the equilibrium measure μ_Q can be recovered by solving the singular integral equation

$$\int \frac{1}{|x-y|^s}\, d\mu(y) + Q(x) = F_Q, \quad x \in S_Q, \tag{4.1.3}$$

where F_Q is a constant (see (1.2.2)).

We solve this equation and obtain the following theorem, which explicitly gives the density of the extremal measure when the support S_Q is the disk $\mathbb{D}_R$. Our results extend the original work of Copson [26], which dealt with classical Coulomb potential in $\mathbb{R}^3$.

Theorem 4.1.2 *Suppose that the support of the extremal measure μ_Q is the disk $\mathbb{D}_R$, and the external field Q is invariant with respect to rotations about the polar axis, that is $Q(x) = Q(r)$, where $x = (0, r\bar{x}) \in \mathbb{D}_R$, $\bar{x} \in \mathbb{S}^{d-2}$, $0 \le r \le R$. Also assume that $Q \in C^2(\mathbb{D}_R)$. Let $s = (d-3) + 2\lambda$, with $0 < \lambda < 1$, and let*

$$F(t) = \frac{\sin(\lambda\pi)\,\Gamma((d-3)/2+\lambda)}{\pi^{(d+1)/2}\,\Gamma(\lambda)}\,\frac{1}{t}\,\frac{d}{dt}\int_t^R \frac{g(r)\,r\,dr}{(r^2-t^2)^{1-\lambda}}, \quad 0 \le t \le R, \tag{4.1.4}$$

with

$$g(r) = \frac{1}{r^{d+2\lambda-4}}\,\frac{d}{dr}\int_0^r \frac{Q(u)\,u^{d-2}\,du}{(r^2-u^2)^{1-\lambda}}, \quad 0 \le r \le R. \tag{4.1.5}$$

Then for the extremal measure μ_Q we have

$$d\mu_Q(x) = f(r)\,r^{d-2}\,dr\,d\sigma_{d-1}(\bar{x}), \quad x = (0, r\bar{x}) \in \mathbb{D}_R, \quad \bar{x} \in \mathbb{S}^{d-2}, \quad 0 \le r \le R, \tag{4.1.6}$$

where the density f is explicitly given by

$$f(r) = C_Q\,(R^2 - r^2)^{\lambda-1} + F(r), \quad 0 \le r \le R, \tag{4.1.7}$$

with the constant C_Q uniquely defined by

$$C_Q = \frac{2\,\Gamma((d-1)/2+\lambda)}{\Gamma(\lambda)\,\Gamma((d-1)/2)}\,\frac{1}{R^{d+2\lambda-3}}\left\{\frac{\Gamma((d-1)/2)}{2\,\pi^{(d-1)/2}} - \int_0^R F(t)\,t^{d-2}\,dt\right\}. \tag{4.1.8}$$

4.2 Equation for the support and some applications

In what follows we will need to know the equilibrium measure and capacity of a disk $\mathbb{D}_R$ of radius $R > 0$. The following theorem extends the corresponding result of Copson [26], which dealt with the Coulomb potential in $\mathbb{R}^3$.

Theorem 4.2.1 *Let $s = (d-3) + 2\lambda$, with $0 < \lambda < 1$. The equilibrium measure $\mu_{\mathbb{D}_R}$ of the disk $\mathbb{D}_R$ of radius R is given by*

$$d\mu_{\mathbb{D}_R}(x) = f(r)\, r^{d-2}\, dr\, d\sigma_{d-1}(\bar{x}), \quad x = (0, r\bar{x}) \in \mathbb{D}_R, \quad \bar{x} \in \mathbb{S}^{d-2}, \quad 0 \le r \le R, \tag{4.2.1}$$

where the density f is

$$f(r) = \frac{\Gamma((d+2\lambda-1)/2)}{\pi^{(d-1)/2}\,\Gamma(\lambda)} \frac{1}{R^{d+2\lambda-3}} (R^2 - r^2)^{\lambda-1}, \quad 0 \le r \le R. \tag{4.2.2}$$

The capacity of the disk $\mathbb{D}_R$ is given by

$$\mathrm{cap}_s(\mathbb{D}_R) = \frac{\sin(\lambda\pi)\Gamma(\lambda)\Gamma((d-1)/2)}{\pi\Gamma((d+2\lambda-1)/2)} R^{d+2\lambda-3}. \tag{4.2.3}$$

Assume that the disk $\mathbb{D}$ is immersed into a general rotationally invariant external field Q, satisfying the conditions of the second statement of Theorem 4.1.1. It then follows that the support of the extremal measure μ_Q will be a disk $\mathbb{D}_R$ of some radius $R \le 1$. The (presently) unknown radius R can be found by minimizing the Mhaskar-Saff functional, which is defined as follows [17].

Definition 4.2.1 *The $\mathcal{F}$-functional of a compact subset $E \subset \mathbb{D}$ of positive Riesz s-capacity is defined as*

$$\mathcal{F}_s(E) := W_s(E) + \int Q(x)\, d\mu_E(x), \tag{4.2.4}$$

where $W_s(E)$ is the Riesz s-energy of the compact E and μ_E is the equilibrium measure (with no external field) on E.

The main objective of introducing the $\mathcal{F}_s$-functional is its following extremal property, which originally was proved in [17] for the general Riesz potentials on the sphere $\mathbb{S}^{d-1}$, $d \geq 3$.

Proposition 4.2.1 *Let Q be an external field on $\mathbb{D}$. Then $\mathcal{F}_s$-functional is minimized for $S_Q = \text{supp}(\mu_Q)$.*

Utilizing Proposition 4.2.1, we can now explicitly determine the support of the extremal measure provided the external field satisfies some mild restrictions.

Theorem 4.2.2 *Let $s = (d-3) + 2\lambda$, with $0 < \lambda < 1$. Assume that the external field $Q : \mathbb{D} \to [0, \infty]$ is invariant with respect to the rotations about the polar axis, that is $Q(x) = Q(r)$, where $x = (0, r\bar{x}) \in \mathbb{D}$, $\bar{x} \in \mathbb{S}^{d-2}$, $0 \leq r \leq 1$. Further suppose that Q is a convex increasing function, that is $Q(r)$ is convex increasing on $[0, 1]$. Then the support of the extremal measure μ_Q will be a disk of radius $R \leq 1$, centered at the origin. The radius R of this disk is either the unique solution of the equation*

$$\frac{2\sin(\lambda\pi)}{\pi(d + 2\lambda - 3)} \int_0^R Q'(r)\,(R^2 - t^2)^{\lambda-1}\, t^{d-1}\, dt = 1, \tag{4.2.5}$$

on the interval $(0, 1]$ if it exists, or $R = 1$ when such a solution fails to exist.

As the first applications of our results, we consider the situation when the disk $\mathbb{D}$ is immersed into an external field given by a monomial, namely

$$Q(x) = qr^\alpha, \quad q > 0, \quad \alpha \geq 1, \quad x = (0, r\bar{x}) \in \mathbb{D}, \quad \bar{x} \in \mathbb{S}^{d-2}, \quad 0 \leq r \leq 1. \tag{4.2.6}$$

It is clear that external field Q in (4.2.6) is invariant with respect to the rotations about the polar axis. Also, $Q(r)$ is a non-negative increasing convex function on $[0, 1]$. From Theorem 4.1.1 it then follows that the support of the corresponding extremal measure μ_Q will be a disk $\mathbb{D}_R$, with some $R \leq 1$. First invoking Theorem 4.2.2 we compute the extremal support, and then with that knowledge at hand we use Theorem 4.1.2 to find a closed-form expression for the extremal measure.

Theorem 4.2.3 *Let $s = (d-3) + 2\lambda$, with $0 < \lambda < 1$. The extremal measure μ_Q, corresponding to the monomial external field* (4.2.6)*, is supported on the disk $\mathbb{D}_{R_*}$, where R_* is defined as*

$$R_* = \left(\frac{(d+2\lambda-3)\pi\Gamma((d+\alpha+2\lambda-1)/2)}{q\alpha\sin(\lambda\pi)\Gamma(\lambda)\Gamma((d+\alpha-1)/2)} \right)^{1/(d+\alpha+2\lambda-3)}. \tag{4.2.7}$$

For the extremal measure μ_Q we have

$$d\mu_Q(x) = f(r)\, r^{d-2}\, dr\, d\sigma_{d-1}(\bar{x}), \quad x = (0, r\bar{x}) \in \mathbb{D}_{R_*}, \qquad \bar{x} \in \mathbb{S}^{d-2}, \quad 0 \le r \le R_*, \tag{4.2.8}$$

with the density $f(r)$ is given by

$$f(r) = C_Q\, (R^2 - r^2)^{\lambda-1} + F(r), \quad 0 \le r \le R_*, \tag{4.2.9}$$

where

$$\begin{aligned} F(r) = {} & \frac{q\,\sin(\lambda\pi)\,\Gamma((d+\alpha-1)/2)\,\Gamma((d+2\lambda-3)/2)}{\pi^{(d+1)/2}\,\Gamma((d+\alpha+2\lambda-3)/2)}\, R_*^{\alpha}\, (R_*^2 - r^2)^{\lambda-1} \times \\ & \Bigg\{ -\, {}_2F_1\left(-\frac{\alpha}{2}, 1; \lambda+1; 1 - \left(\frac{r}{R_*}\right)^2 \right) \\ & + \frac{\alpha}{2\lambda(\lambda+1)} \left(1 - \left(\frac{r}{R_*}\right)^2 \right) {}_2F_1\left(1 - \frac{\alpha}{2}, 2; \lambda+2; 1 - \left(\frac{r}{R_*}\right)^2 \right) \Bigg\}, \end{aligned} \tag{4.2.10}$$

for $0 \le r \le R_*$, and the constant C_Q is defined as

$$C_Q = \frac{\Gamma((d+2\lambda-1)/2)}{\pi^{(d-1)/2}\,\Gamma(\lambda)} \left\{ \frac{1}{R_*^{d+2\lambda-3}} + \frac{q\sin(\lambda\pi)\,\Gamma((d+\alpha-1)/2)\,\Gamma(\lambda)}{\pi\,\Gamma((d+\alpha+2\lambda-1)/2)}\, R_*^{\alpha} \right\}. \tag{4.2.11}$$

Another application is concerned with finding the extremal measure μ_Q in the case of the Riesz s-potential generated by a positive point charge. We assume that the external field Q is produced by a positive point charge of magnitude q, placed on the positive polar semi-axis at some distance $h > 0$ above the disk $\mathbb{D}$. This external field is given by

$$Q(x) = \frac{q}{(r^2 + h^2)^{s/2}}, \quad h > 0, \quad q > 0, \quad s > 0, \qquad x = (0, r\bar{x}) \in \mathbb{D}, \quad \bar{x} \in \mathbb{S}^{d-2}, \quad 0 \le r \le 1. \tag{4.2.12}$$

This problem is similar to the celebrated Gonchar's problem, which was solved in [17] for the case of classical Newtonian potential in $\mathbb{R}^d$. The Gonchar's problem is concerned with a situation when a positive unit point charge is approaching the insulated unit sphere, carrying a total charge 1, eventually causing a spherical cap free of charge to appear. Gonchar raised a question about finding the smallest distance from the point charge to the sphere such that the whole of the sphere still being positively charged. In a slightly more general setting, the solution of the Gonchar's problem means that if a point charge of a non-negative magnitude q, located on the positive polar semi-axis, is too far from the surface of the sphere $\mathbb{S}^2$, or the magnitude q of the point charge is too small, the electrostatic field, created by this point charge, is too weak to force the equilibrium charge distribution from occupying the whole surface of the sphere $\mathbb{S}^2$. It is known that in the case of the positive unit point charge, the critical height is precisely $1+\rho$, where ρ is the golden ratio $(1+\sqrt{5})/2 \approx 1.6180339887$.

The question about the charge distribution on the surface of the disk $\mathbb{D}$ in $\mathbb{R}^3$, influenced by a positive point charge placed above the disk on the polar axis, was first considered by Thompson [70]. By a different method, the problem was treated by Gallop [31].

Theorem 4.2.4 *Let $s = (d-3)+2\lambda$, with $0 < \lambda < 1$. Assume that the external field Q is given by (4.2.12), with $h > \max\{h_-, h_+\}$, where*

$$h_- := \left(\frac{q\left((1-\lambda)(d-2\lambda+1)+1\right)^2 \sin(\lambda\pi)}{8\pi(d+2\lambda-1)(1-\lambda)} \mathrm{B}\left(\lambda, \frac{d-1}{2}\right)\right)^{1/(d+2\lambda-3)}, \qquad (4.2.13)$$

and h_+ is the largest positive root of the function

$$\begin{aligned} p(h) &= \frac{2\,\Gamma((d-1)/2+\lambda)}{\Gamma(\lambda)\,\Gamma((d-1)/2)} \left\{ \frac{\Gamma((d-1)/2)}{2\,\pi^{(d-1)/2}} + q\,c_{d,\lambda} \right\} \\ &- \frac{q\,\sin(\lambda\pi)\,\Gamma((d-1)/2)}{\pi^{(d+1)/2}} \left\{ \frac{d-2\lambda-1}{2} \frac{1}{h^{d-1}} \mathrm{B}\left(\frac{1}{1+h^2}; \lambda, \frac{d-2\lambda-1}{2}\right) \right. \\ &\left. + \frac{1}{h^{2\lambda}(1+h^2)^{(d-3)/2}} \right\}, \end{aligned} \qquad (4.2.14)$$

with

$$c_{d,\lambda} := \frac{\sin(\lambda\pi)\,\Gamma((d-1)/2)}{\pi^{(d+1)/2}}\,h^{2(1-\lambda)}\times \tag{4.2.15}$$
$$\left\{\frac{d-2\lambda-1}{2}\int_0^1 \frac{t^{d-2}}{(h^2+t^2)^{(d-2\lambda+1)/2}}\,\mathrm{B}\left(\frac{1-t^2}{1+h^2};\lambda,\frac{d-2\lambda-1}{2}\right)dt\right.$$
$$\left.+\frac{\Gamma(\lambda)\,\Gamma((d-1)/2)}{2\,\Gamma((d-1)/2+\lambda)}\,\frac{1}{(1+h^2)^{(d-1)/2}}\,{}_2F_1\left(1,\lambda;\frac{d+2\lambda-1}{2};\frac{1}{1+h^2}\right)\right\}.$$

The extremal measure μ_Q, corresponding to the external field of a point charge (4.2.12)*, is given by*

$$d\mu_Q(x) = f(r)\,r^{d-2}\,dr\,d\sigma_{d-1}(\bar{x}),\quad x=(0,r\bar{x})\in\mathbb{D},\quad \bar{x}\in\mathbb{S}^{d-2},\quad 0\le r\le 1, \tag{4.2.16}$$

where

$$f(r) = C_Q\,(1-r^2)^{\lambda-1} + F(r),\quad 0\le r\le 1. \tag{4.2.17}$$

Here

$$F(r) = -\frac{q\,\sin(\lambda\pi)\,\Gamma((d-1)/2)}{\pi^{(d+1)/2}}\,h^{2(1-\lambda)}\times \tag{4.2.18}$$
$$\left\{\frac{d-2\lambda-1}{2}\,\frac{1}{(h^2+r^2)^{(d-2\lambda+1)/2}}\,\mathrm{B}\left(\frac{1-r^2}{1+h^2};\lambda,\frac{d-2\lambda-1}{2}\right)\right.$$
$$\left.+\frac{(1-r^2)^{\lambda-1}}{(1+h^2)^{(d-3)/2}\,(h^2+r^2)}\right\},\quad 0\le r\le 1,$$

and where the positive constant C_Q is given by

$$C_Q = \frac{2\,\Gamma((d-1)/2+\lambda)}{\Gamma(\lambda)\,\Gamma((d-1)/2)}\left\{\frac{\Gamma((d-1)/2)}{2\,\pi^{(d-1)/2}} + q\,c_{d,\lambda}\right\}, \tag{4.2.19}$$

In the important special case of Newtonian potential in the even dimensions starting with 8, from the above Theorem it follows that the extremal measure can be written in a simplified form.

Corollary 4.2.1 *Let $d=2m+4$ and $s=d-2=2(m+1)$, where $m\ge 2$. Assume that the external field Q is given by* (4.2.12)*, with $h>\max\{h_-,h_+\}$, with*

$$h_- := \left(\frac{q\,(m+2)\,\Gamma(m+3/2)}{8\sqrt{\pi}(m+1)!}\right)^{1/2(m+1)}, \tag{4.2.20}$$

and h_+ is the largest positive root of the function

$$p(h) = \frac{2\,(m+1)!}{\sqrt{\pi}\,\Gamma(m+3/2)}\left\{\frac{\Gamma(m+3/2)}{2\,\pi^{m+3/2}}+qc\right\} \tag{4.2.21}$$
$$-\frac{q\,\Gamma(m+3/2)}{\pi^{m+5/2}}\left\{2(m+1)\,h^{-(2m+3)}\sum_{n=0}^{m}\frac{(-m)_n}{(2n+1)n!}\,(1+h^2)^{-(n+1/2)}\right.$$
$$\left.+\frac{1}{h(1+h^2)^{m+1/2}}\right\},$$

with

$$c := \frac{(\Gamma(m+3/2))^2}{\pi^{m+5/2}}\,h\,(1+h^2)^{-(m+1)}\left\{\Gamma\left(m+\frac{3}{2}\right)\,(1+h^2)^{-(m+5/2)}\times\right. \tag{4.2.22}$$
$$\sum_{n=0}^{m}\frac{(-m)_n}{(2n+1)n!}\,(1+h^2)^{-n}\sum_{l=0}^{m-2}\frac{(2-m)_l\,\Gamma(n+l+3/2)}{(n+m+l+2)!}\,(1+h^2)^{-l}$$
$$\left.+\frac{\sqrt{1+h^2}}{h+\sqrt{1+h^2}}\sum_{n=0}^{m}\frac{(-m)_n}{(m+n+1)!}\,\left(\frac{\sqrt{1+h^2}-h}{\sqrt{1+h^2}+h}\right)^n\right\}.$$

Then the extremal measure μ_Q, corresponding to the external field of a point charge (4.2.12)*, is given by*

$$d\mu_Q(x) = f(r)\,r^{d-2}\,dr\,d\sigma_{d-1}(\bar{x}), \quad x=(0,r\bar{x})\in\mathbb{D}, \tag{4.2.23}$$
$$\bar{x}\in\mathbb{S}^{d-2}, \quad 0\le r\le 1,$$

where

$$f(r) = C_Q\,\frac{1}{\sqrt{1-r^2}} + F(r), \quad 0\le r\le 1. \tag{4.2.24}$$

Here

$$F(r) = -\,qh\frac{\Gamma(m+3/2)}{\pi^{m+5/2}}\left\{\frac{2(m+1)}{(h^2+r^2)^{m+2}}\sum_{n=0}^{m}\frac{(-m)_n}{(2n+1)n!}\,\left(\frac{1-r^2}{1+h^2}\right)^{n+1/2}\right. \tag{4.2.25}$$
$$\left.+\frac{1}{\sqrt{1-r^2}}\,\frac{1}{(h^2+r^2)\,(1+h^2)^{m+1/2}}\right\}, \quad 0\le r\le 1.$$

and where the positive constant C_Q is given by

$$C_Q = \frac{2(m+1)!}{\sqrt{\pi}\Gamma(m+3/2)}\left\{\frac{\Gamma(m+3/2)}{2\pi^{m+3/2}}+qc\right\}. \tag{4.2.26}$$

The case of the three-dimensional Euclidean space $\mathbb{R}^3$ and Coulomb potential, corresponding to $d = 3$ and $\lambda = 1/2$ in the context of Theorem 4.2.4, deserves special attention. Assuming that the disk $\mathbb{D}$ is immersed into the external field generated by a positive unit point charge, in this physically important case we are able to precisely determine the height of the point charge that guarantees the extremal support μ_Q to occupy the whole disk $\mathbb{D}$. This is an improvement of Theorem 4.2.4, where we can only provide an estimate of such a height.

Corollary 4.2.2 *Suppose the external field Q is given by (4.2.12), with $d = 3$ and $s = 1$, and where h is chosen such that $h \geq h_+$, where h_+ is the unique positive root of the function*

$$p(h) = \frac{1}{2\pi}\left(1 + \frac{2h\tan^{-1}(1/h)}{\pi\sqrt{1+h^2}}\right) - \frac{1}{\pi^2 h} - \frac{1}{\pi^2 h^2}\tan^{-1}(1/h).$$

Then, under these assumptions $S_Q = \mathbb{D}$, and the extremal measure μ_Q is given by

$$d\mu_Q(x) = f(r)\, r\, dr\, d\sigma_2(\bar{x}), \quad x = (0, r\bar{x}) \in \mathbb{D}, \quad \bar{x} \in \mathbb{S}^1, \quad 0 \leq r \leq 1, \tag{4.2.27}$$

where the density $f(r)$ is

$$\begin{aligned} f(r) =& \frac{1}{2\pi}\left(1 + \frac{2h\tan^{-1}(1/h)}{\pi\sqrt{1+h^2}}\right)\frac{1}{\sqrt{1-r^2}} - \frac{h}{\pi^2(h^2+r^2)}\frac{1}{\sqrt{1-r^2}} \\ &- \frac{h}{\pi^2}\frac{1}{(h^2+r^2)^{3/2}}\tan^{-1}\sqrt{\frac{1-r^2}{h^2+r^2}}, \quad 0 \leq r \leq 1. \end{aligned} \tag{4.2.28}$$

If the height of the point charge is chosen such that $h < h_+$, then the support of the extremal measure μ_Q will no longer be the entire disk $\mathbb{D}$, as there will be an opening around the origin.

Note that from Corollary 4.2.2 it follows that if the charge is moved closer to the disk past the critical height h_+, then support of the extremal measure will no longer be the entire disk. This means that when $h < h_+$ the point charge clears out an opening in the disk $\mathbb{D}$ at the origin, which will be free of charge and is likely to have a ring structure.

4.3 Proofs

Proof of Theorem 4.1.1. From the rotational invariance of Q it follows that the support of the extremal measure μ_Q is also rotationally invariant. Therefore, there exists a compact set $K \subset [0,1]$ and a non-negative real-valued function $f \in L^1(K)$ such that

$$d\mu_Q(x) = f(r)\, r^{d-2}\, dr\, d\sigma_{d-1}(\bar{x}), \quad x = (0, r\bar{x}), \quad \bar{x} \in \mathbb{S}^{d-2},$$
$$\operatorname{supp} \mu_Q = \{(0, r\bar{x}) \in \mathbb{D} : r \in K, \bar{x} \in \mathbb{S}^{d-2}\}.$$

We will first prove that the set K is connected. We will follow the argument given in [17]. Assume to the contrary that K is not connected. Then there is an interval $[r_1, r_2] \subset [0,1]$ such that $K \cap [r_1, r_2] = \{r_1, r_2\}$. We further denote $K_+ := K \cap [r_2, 1]$ and $K_- := K \cap [0, r_1]$. Then for

$$x = (0, r\bar{x}), \quad r \in (r_1, r_2), \quad \bar{x} \in \mathbb{S}^{d-2},$$
$$y = (0, \rho\bar{y}), \quad \rho \in K_- \cup K_+, \quad \bar{y} \in \mathbb{S}^{d-2},$$

the s-potential of μ_Q can be written as

$$\begin{aligned}
U_s^{\mu_Q}(x) &= \int \frac{1}{|x-y|^s}\, d\mu(y) \\
&= \int_K f(\rho)\, \rho^{d-2}\, d\rho \int_{\mathbb{S}^{d-2}} \frac{d\sigma_{d-1}(\bar{y})}{(r^2 + \rho^2 - 2r\rho\langle \bar{x}, \bar{y}\rangle)^{s/2}} \\
&= \frac{2\pi^{(d-2)/2}}{\Gamma(d/2-1)} \int_K f(\rho)\, \rho^{d-2}\, d\rho \int_0^\pi \frac{\sin^{d-3}\xi\, d\xi}{(r^2+\rho^2-2r\rho\cos\xi)^{s/2}} \\
&= \int_{K_-} f(\rho)\, k(r,\rho)\, d\rho + \int_{K_+} f(\rho)\, k(r,\rho)\, d\rho,
\end{aligned}$$

where $k(r,\rho)$ is given by

$$k(r,\rho) = \frac{2\pi^{(d-2)/2}}{\Gamma(d/2-1)} \int_0^\pi \frac{\sin^{d-3}\xi\, d\xi}{(r^2+\rho^2-2r\rho\cos\xi)^{s/2}}. \tag{4.3.1}$$

The result #3.665 of [37] states that for $\mathrm{Re}(\nu) > 0$ and $|x| < 1$,

$$\int_0^\pi \frac{\sin^{2\nu-1}\xi\, d\xi}{(1+x^2+2x\cos\xi)^s} = \frac{\Gamma(\nu)\sqrt{\pi}}{\Gamma(\nu+1/2)}\, {}_2F_1(s, s-\nu+1/2; \nu+1/2; x^2). \tag{4.3.2}$$

Using (4.3.2), for the case $r > \rho$ ($\rho \in K_-$) we can further transform (4.3.1) as follows,

$$\begin{aligned} k(r,\rho) &= \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)} \frac{\rho^{d-2}}{r^s} \, {}_2F_1\left(\frac{s}{2}, \lambda; \frac{d-1}{2}; \left(\frac{\rho}{r}\right)^2\right) \\ &= \frac{2\pi^{(d-1)/2}\,\rho^{d-2}}{\Gamma((d-1)/2)} \sum_{n=0}^{\infty} \frac{(s/2)_n\,(\lambda)_n\,\rho^{2n}}{((d-1)/2)_n\, n!} \frac{1}{r^{2n+s}}. \end{aligned}$$

Hence, for $r > \rho$ we have

$$k(r,\rho) = \frac{2\pi^{(d-1)/2}\,\rho^{d-2}}{\Gamma((d-1)/2)} \sum_{n=0}^{\infty} \frac{(s/2)_n\,(\lambda)_n\,\rho^{2n}}{((d-1)/2)_n\, n!} \frac{1}{r^{2n+s}}. \tag{4.3.3}$$

It is clear that the functions $r^{-(2n+s)}$, $n = 0, 1, 2, \dots$ are strictly convex for $r \in (0, 1)$. Therefore, taking into account the positivity of all coefficients of the series in the right hand side of (4.3.3), and the fact that this series is uniformly convergent in r on compact subsets of (r_1, r_2), the convexity of the right hand side of (4.3.3) follows by differentiation with respect to r.

Exactly the same approach with $\rho > r$ ($\rho \in K_+$) leads to the following series representation for $k(r,\rho)$,

$$k(r,\rho) = \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)\,\rho^{2\lambda-1}} \sum_{n=0}^{\infty} \frac{(s/2)_n\,(\lambda)_n}{((d-1)/2)_n\, n!\, \rho^{2n}}\, r^{2n}. \tag{4.3.4}$$

Obviously the functions r^{2n}, $n = 0, 1, 2 \dots$ are convex on $(0, 1)$. Hence, we similarly derive the convexity of the right hand side of (4.3.4).

From (4.3.3) and (4.3.4) we infer that the function k is a strictly convex function of r on (r_1, r_2), for any fixed $\rho \in K_- \cup K_+$. Using the convexity of $Q(r)$, we deduce that $U_s^{\mu_Q}(r)+Q(r)$ is a strictly convex function on (r_1, r_2). Furthermore, by (1.2.2), for the weighted potential $U_s^{\mu_Q}(r) + Q(r)$ we have $U_s^{\mu_Q}(r_1) + Q(r_1) = F_Q = U_s^{\mu_Q}(r_2) + Q(r_2)$. Then the strict convexity of $U_s^{\mu_Q}(r) + Q(r)$ implies that $U_s^{\mu_Q}(r) + Q(r) < F_Q$, for $r_1 < r < r_2$. But this is an obvious contradiction with inequality (1.2.1), which is valid for all $0 \le r \le 1$.

We now prove the second part of the statement of Theorem 4.1.1. Assume that $Q(r)$, in addition to being convex, is also an increasing function. Suppose that $a > 0$.

In this case the kernel $k(r,\rho)$ is calculated according to (4.3.4), which shows that $U_s^{\mu_Q}(r)$ is an increasing function on $(0,a]$. This implies that the weighted potential $U_s^{\mu_Q}(r)+Q(r)$ is a strictly increasing function on $(0,a]$. Therefore for any $r'\in(0,a)$ we have $U_s^{\mu_Q}(a)+Q(a)>U_s^{\mu_Q}(r')+Q(r')$. On the other hand, since $a\in S_Q$, from (1.2.2) it follows that $U_s^{\mu_Q}(a)+Q(a)=F_Q$. We thus find that $U_s^{\mu_Q}(r')+Q(r')<F_Q$, which clearly violates inequality (1.2.1).

The proof of the remaining part of the statement of Theorem 4.1.1 follows the same logic as in the last paragraph. Indeed, assume that $Q(r)$ besides being convex, is also a decreasing function of r. Suppose that $b<1$. In this case kernel $k(r,\rho)$ is calculated according to (4.3.3), which shows that $U_s^{\mu_Q}(r)$ is a decreasing function on $[b,1)$. Hence the weighted potential $U_s^{\mu_Q}(r)+Q(r)$ is a strictly decreasing function on $[b,1)$. Thus for any $r'\in(b,1)$ we have $U_s^{\mu_Q}(b)+Q(b)>U_s^{\mu_Q}(r')+Q(r')$. Observing that $b\in S_Q$, from (1.2.2) it follows that $U_s^{\mu_Q}(b)+Q(b)=F_Q$. We thus see that $U_s^{\mu_Q}(r')+Q(r')<F_Q$, which again violates inequality (1.2.1).

■

Proof of Theorem 4.1.2. Assume that the support of the extremal measure μ_Q is the disk $\mathbb{D}_R$, that is $S_Q=\mathbb{D}_R$. Then there exists a non-negative real-valued function $f\in L^1([0,R])$ such that for $x=(0,r\bar{x})\in\mathbb{D}_R$, $\bar{x}\in\mathbb{S}^{d-2}$, $0\le r\le R$,

$$d\mu_Q(x)=f(r)\,r^{d-2}\,dr\,d\sigma_{d-1}(\bar{x}).$$

Let x and y be two points in $\mathbb{D}$ with $|x|=r$ and $|y|=\rho$. Note that $\bar{x}:=x/r\in\mathbb{S}^{d-2}$, and similarly $\bar{y}:=y/\rho\in\mathbb{S}^{d-2}$. For the distance $|x-y|$ we then obtain

$$\begin{aligned}|x-y|^2&=|x|^2+|y|^2-2\langle x,y\rangle\\&=r^2+\rho^2-2r\rho\langle\bar{x},\bar{y}\rangle.\end{aligned}$$

We immediately notice that the rotational invariance of an external field Q is passed on to the Riesz s-potential $U_s^{\mu_Q}$, thanks to the uniqueness of the extremal measure.

Therefore for $x \in \operatorname{supp} \mu_Q$ we have $U_s^{\mu_Q}(x) = U_s^{\mu_Q}(|x|)$. We will be using this fact from now on without mentioning it explicitly on each separate occasion.

The Riesz s-potential $U_s^{\mu_Q}(x)$, with $x = (0, r\bar{x}) \in \mathbb{D}_R$, $\bar{x} \in \mathbb{S}^{d-2}$, $0 \le r \le R$, can be written as

$$\begin{aligned} U_s^{\mu_Q}(x) &= \int_{\mathbb{D}_R} \frac{1}{|x-y|^s}\, d\mu_Q(y) \\ &= \int_0^R f(\rho)\, \rho^{d-2} d\rho \int_{\mathbb{S}^{d-2}} \frac{d\sigma_{d-1}(\bar{y})}{(r^2+\rho^2-2r\rho\langle \bar{x}, \bar{y}\rangle)^{s/2}}. \end{aligned} \tag{4.3.5}$$

We will need the following proposition, which is a special case of the Funk-Hecke theorem [3, p. 247].

Proposition 4.3.1 *If f is integrable on $[-1,1]$ with respect to the weight $(1-t^2)^{(d-3)/2}$, and y is an arbitrary fixed point on the sphere S^{d-1}, then*

$$\int_{\mathbb{S}^{d-1}} f(\langle x, y\rangle)\, d\sigma_d(x) = \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)} \int_{-1}^1 f(t)\,(1-t^2)^{(d-3)/2}\, dt. \tag{4.3.6}$$

Applying Proposition 4.3.1 to the inner integral on the right hand side in (4.3.5), we derive

$$\int_{\mathbb{S}^{d-2}} \frac{d\sigma_{d-1}(\bar{y})}{(r^2+\rho^2-2r\rho\langle \bar{x}, \bar{y}\rangle)^{s/2}} = \frac{2\pi^{(d-2)/2}}{\Gamma((d-2)/2)} \int_0^\pi \frac{\sin^{d-3}\xi\, d\xi}{(r^2+\rho^2-2r\rho\cos\xi)^{s/2}}.$$

Hence the potential $U_s^{\mu_Q}$ in (4.3.5) assumes the form

$$U_s^{\mu_Q}(r) = \frac{2\pi^{(d-2)/2}}{\Gamma((d-2)/2)} \int_0^R f(\rho)\, \rho^{d-2} d\rho \int_0^\pi \frac{\sin^{d-3}\xi\, d\xi}{(r^2+\rho^2-2r\rho\cos\xi)^{s/2}}, \quad 0 \le r \le R.$$

The integral equation (4.1.3) can now be written as

$$\int_0^R f(\rho)\, \rho^{d-2} d\rho \int_0^\pi \frac{\sin^{d-3}\xi\, d\xi}{(r^2+\rho^2-2r\rho\cos\xi)^{s/2}} = \frac{\Gamma((d-2)/2)}{2\pi^{(d-2)/2}}\,(F_Q - Q(r)), \tag{4.3.7}$$

where $0 \le r \le R$.

Our next step is to further transform the inner integral on the left hand side of the integral equation (4.3.7). This is achieved via the following fact.

Lemma 4.3.1 *If a and b are positive numbers, $a \neq b$, $q \geq 0$ and $0 < \lambda < 1$, then*

$$\int_0^{\pi} \frac{\sin^{2q}\xi\, d\xi}{(a^2+b^2-2ab\cos\xi)^{q+\lambda}} = \frac{2}{a^{2q}\,b^{2q}} \frac{\sin(\lambda\pi)\,\Gamma(\lambda)\,\Gamma(q+1/2)}{\sqrt{\pi}\,\Gamma(q+\lambda)} \times \int_0^{\min(a,b)} \frac{t^{2(q+\lambda)-1}\,dt}{(a^2-t^2)^{\lambda}\,(b^2-t^2)^{\lambda}}. \tag{4.3.8}$$

Note that Lemma 4.3.1 in the case when $q = 0$ and $\lambda = 1/2$ was obtained by Copson [26]. Also, Lemma 4.3.1 when $q \geq 0$ and $\lambda = 1/2$, is implicitly mentioned in [66], although with an incorrect numerical coefficient. The correct version of the latter, along with its proof, is given in [9, p. 8].

Proof. The proof is based on the following identity, obtained by Kahane [42].

Proposition 4.3.2 *Let a and b be positive numbers such that $a \neq b$, $\lambda \in (0,1)$, $q \in \mathbb{C}$ with $\mathrm{Re}(q) \geq 0$, and u a real number with $|u| \leq 1$. Then*

$$\frac{(ab)^q}{(a^2+b^2-2abu)^{\lambda+q}} = \frac{\Gamma(\lambda)\Gamma(q+1)}{\Gamma(\lambda+q)} \frac{2\sin(\lambda\pi)}{\pi} \times \int_0^{\min(a,b)} \frac{1-(t^2/ab)^2}{(1+(t^2/ab)^2-2(t^2/ab)u)^{q+1}} \left(\frac{t^2}{ab}\right)^q \frac{t^{2\lambda-1}\,dt}{(a^2-t^2)^{\lambda}(b^2-t^2)^{\lambda}}. \tag{4.3.9}$$

Taking q to be a non-negative real number in Proposition 4.3.2, and applying Fubini's theorem, we rewrite the left hand side of (4.3.8) as

$$\int_0^{\pi} \frac{\sin^{2q}\xi\, d\xi}{(a^2+b^2-2ab\cos\xi)^{q+\lambda}} = \frac{1}{a^{2q}\,b^{2q}} \frac{2\sin(\lambda\pi)\,\Gamma(\lambda)\,\Gamma(q+1)}{\pi\,\Gamma(q+\lambda)} \times \int_0^{\min(a,b)} \frac{(1-(t^2/ab)^2)\,t^{2(q+\lambda)-1}\,dt}{(a^2-t^2)^{\lambda}\,(b^2-t^2)^{\lambda}} \int_0^{\pi} \frac{\sin^{2q}\xi\,d\xi}{(1+(t^2/ab)^2-2(t^2/ab)\cos\xi)^{q+1}}. \tag{4.3.10}$$

We will show that

$$\int_0^{\pi} \frac{\sin^{2q}\xi\,d\xi}{(1+(t^2/ab)^2-2(t^2/ab)\cos\xi)^{q+1}} = \frac{1}{1-(t^2/ab)^2} \frac{\sqrt{\pi}\,\Gamma(q+1/2)}{\Gamma(q+1)}. \tag{4.3.11}$$

Indeed, the integral of a type appearing on the left hand side of (4.3.11) was previously considered in [50, p. 400]. It was shown that

$$\int_0^{\pi} \frac{\sin^{p-2}\xi\,d\xi}{(1+\rho^2-2\rho\cos\xi)^{p/2}} = \frac{1}{\rho^{p-2}(\rho^2-1)} \int_0^{\pi} \sin^{p-2}\xi\,d\xi, \tag{4.3.12}$$

where $\rho \geq 1$, and $p \geq 3$ was assumed to be an integer. A careful analysis of the evaluation of integral (4.3.12) in [50, p. 400] shows that, in fact, (4.3.12) holds true for any $p \geq 2$. We hence transform the left hand side of (4.3.11) as follows,

$$\begin{aligned}\int_0^\pi \frac{\sin^{2q}\xi\, d\xi}{(1+(t^2/ab)^2 - 2(t^2/ab)\cos\xi)^{q+1}} &= \frac{1}{1-(t^2/ab)^2}\int_0^\pi \sin^{2q}\xi\, d\xi \\ &= \frac{1}{1-(t^2/ab)^2} 2^{2q}\,\mathrm{B}(q+1/2, q+1/2) \\ &= \frac{1}{1-(t^2/ab)^2}\frac{\sqrt{\pi}\,\Gamma(q+1/2)}{\Gamma(q+1)},\end{aligned}$$

which is the right hand side of (4.3.11). Substituting (4.3.11) into (4.3.10), we obtain the desired representation (4.3.8). ■

Setting $q = (d-3)/2$ and observing that $s = 2q + 2\lambda$, by letting $a = r$ and $b = \rho$ in Lemma 4.3.8, the inner integral on the left hand side of integral equation (4.3.7) can be written in the following form,

$$\begin{aligned}\int_0^\pi \frac{\sin^{d-3}\xi\, d\xi}{(r^2+\rho^2-2r\rho\cos\xi)^{s/2}} =& \frac{1}{r^{d-3}\,\rho^{d-3}}\,\frac{2\sin(\lambda\pi)\,\Gamma(\lambda)\,\Gamma(d/2-1)}{\sqrt{\pi}\,\Gamma((d-3)/2+\lambda)}\times \\ & \int_0^{\min(r,\rho)} \frac{t^{d+2\lambda-4}\, dt}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda}.\end{aligned} \tag{4.3.13}$$

Using (4.3.13), we recast integral equation (4.3.7) as

$$\begin{aligned}\int_0^R f(\rho)\,\rho\, d\rho \int_0^{\min(r,\rho)} &\frac{t^{d+2\lambda-4}\, dt}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} \\ &= \frac{\Gamma((d-3)/2+\lambda)}{4\sin(\lambda\pi)\,\pi^{(d-3)/2}\,\Gamma(\lambda)}\, r^{d-3}\,(F_Q - Q(r)), \quad 0 \leq r \leq R.\end{aligned} \tag{4.3.14}$$

We now work with the integral on the left hand side of (4.3.14). Splitting the range of integration, and changing the order of integration in the first integral, we derive

$$\begin{aligned}\int_0^R f(\rho)\,\rho\,d\rho \int_0^{\min(r,\rho)} \frac{t^{d+2\lambda-4}\,dt}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} &= \int_0^r f(\rho)\,\rho\,d\rho \int_0^\rho \frac{t^{d+2\lambda-4}\,dt}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} \\ &+ \int_r^R f(\rho)\,\rho\,d\rho \int_0^r \frac{t^{d+2\lambda-4}\,dt}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} \\ &= \int_0^r \frac{t^{d+2\lambda-4}\,dt}{(r^2-t^2)^\lambda} \int_t^r \frac{f(\rho)\,\rho\,d\rho}{(\rho^2-t^2)^\lambda} \\ &+ \int_0^r \frac{t^{d+2\lambda-4}\,dt}{(r^2-t^2)^\lambda} \int_r^R \frac{f(\rho)\,\rho\,d\rho}{(\rho^2-t^2)^\lambda} \\ &= \int_0^r \frac{t^{d+2\lambda-4}\,dt}{(r^2-t^2)^\lambda} \int_t^R \frac{f(\rho)\,\rho\,d\rho}{(\rho^2-t^2)^\lambda}.\end{aligned}$$

We can thus re-write integral equation (4.3.14) as

$$\int_0^r \frac{t^{d+2\lambda-4}\,dt}{(r^2-t^2)^\lambda} \int_t^R \frac{f(\rho)\,\rho\,d\rho}{(\rho^2-t^2)^\lambda} = \frac{\Gamma((d-3)/2+\lambda)}{4\sin(\lambda\pi)\,\pi^{(d-3)/2}\,\Gamma(\lambda)}\, r^{d-3}\,(F_Q - Q(r)), \quad (4.3.15)$$

where $0 \le r \le R$.

Let

$$S(t) = \int_t^R \frac{f(\rho)\,\rho\,d\rho}{(\rho^2-t^2)^\lambda}. \quad (4.3.16)$$

Then (4.3.15) reads

$$\int_0^r \frac{S(t)\,t^{d+2\lambda-4}\,dt}{(r^2-t^2)^\lambda} = \frac{\Gamma((d-3)/2+\lambda)}{4\sin(\lambda\pi)\,\pi^{(d-3)/2}\,\Gamma(\lambda)}\, r^{d-3}\,(F_Q - Q(r)), \quad 0 \le r \le R. \quad (4.3.17)$$

Integral equation (4.3.17) is an Abel-type integral equation with respect to $S(t)\,t^{d+2\lambda-4}$. As $Q \in C^2([0,1])$, applying [60, # 44, p. 122], we solve this equation and find

$$S(r) = \frac{\Gamma((d-3)/2+\lambda)}{2\,\pi^{(d-1)/2}\,\Gamma(\lambda)} \frac{1}{r^{d+2\lambda-4}} \frac{d}{dr} \int_0^r \frac{(F_Q - Q(t))\,t^{d-2}\,dt}{(r^2-t^2)^{1-\lambda}}. \quad (4.3.18)$$

Now observe that (4.3.16) is also an Abel-type integral equation with respect to $f(\rho)\,\rho$. Solving it in a similar fashion, we derive

$$f(t) = -\frac{2\sin(\lambda\pi)}{\pi} \frac{1}{t} \frac{d}{dt} \int_t^R \frac{S(\rho)\,\rho\,d\rho}{(\rho^2-t^2)^{1-\lambda}}, \quad 0 \le t \le R. \quad (4.3.19)$$

Let

$$F(t) = \frac{\sin(\lambda\pi)\,\Gamma((d-3)/2+\lambda)}{\pi^{(d+1)/2}\,\Gamma(\lambda)} \frac{1}{t} \frac{d}{dt} \int_t^R \frac{g(r)\,r\,dr}{(r^2-t^2)^{1-\lambda}}, \quad 0 \le t \le R,$$

where

$$g(r)=\frac{1}{r^{d+2\lambda-4}}\,\frac{d}{dr}\int_0^r \frac{Q(u)\,u^{d-2}\,du}{(r^2-u^2)^{1-\lambda}},\qquad 0\le r\le R.$$

Then expression (4.3.19) can be written as

$$\begin{aligned} f(t)=-F_Q\,&\frac{\sin(\lambda\pi)\,\Gamma((d-3)/2+\lambda)}{\pi^{(d+1)/2}\,\Gamma(\lambda)}\,\frac{1}{t}\times\\ &\frac{d}{dt}\int_t^R\left\{\frac{1}{r^{d+2\lambda-4}}\,\frac{d}{dr}\int_0^r\frac{u^{d-2}\,du}{(r^2-u^2)^{1-\lambda}}\right\}\frac{r\,dr}{(r^2-t^2)^{1-\lambda}}+F(t),\quad 0\le t\le R. \end{aligned}$$

Performing rather straightforward integrations and differentiations appearing on the right hand side of the last expression, we deduce

$$f(t)=F_Q\,\frac{\sin(\lambda\pi)\,\Gamma((d-1)/2)}{\pi^{(d+1)/2}}\,(R^2-t^2)^{\lambda-1}+F(t),\quad 0\le t\le R. \tag{4.3.20}$$

We complete the proof by evaluating the Robin constant F_Q. Recall that μ_Q is a probability measure, so that it has mass one. Therefore,

$$\begin{aligned} 1=\int d\mu_Q&=\int_0^R f(t)\,t^{d-2}\,dt\int_{\mathbb{S}^{d-2}}d\sigma_{d-1}\\ &=\omega_{d-1}\int_0^R f(t)\,t^{d-2}\,dt. \end{aligned}$$

We therefore obtain

$$\begin{aligned} \frac{\Gamma((d-1)/2)}{2\pi^{(d-1)/2}}=\frac{1}{\omega_{d-1}}&=\int_0^R f(t)\,t^{d-2}\,dt\\ &=F_Q\,\frac{\sin(\lambda\pi)\,\Gamma((d-1)/2)}{\pi^{(d+1)/2}}\int_0^R(R^2-t^2)^{\lambda-1}\,t^{d-2}\,dt\\ &+\int_0^R F(t)\,t^{d-2}\,dt. \end{aligned}$$

It is an elementary calculation to see that

$$\int_0^R(R^2-t^2)^{\lambda-1}\,t^{d-2}\,dt=\frac{\Gamma((d-1)/2)\,\Gamma(\lambda)}{2\,\Gamma((d-1)/2+\lambda)}\,R^{d+2\lambda-3}. \tag{4.3.21}$$

Combining the last two expressions, we eventually find

$$\begin{aligned} F_Q=\frac{2\,\pi^{(d+1)/2}\,\Gamma((d-1)/2+\lambda)}{\sin(\lambda\pi)\,(\Gamma((d-1)/2))^2\,\Gamma(\lambda)}\,&\frac{1}{R^{d+2\lambda-3}}\times\\ &\left\{\frac{\Gamma((d-1)/2)}{2\,\pi^{(d-1)/2}}-\int_0^R F(t)\,t^{d-2}\,dt\right\}. \end{aligned} \tag{4.3.22}$$

Inserting expression (4.3.22) into the formula for the extremal density (4.3.20), we derive the following simple formula,

$$f(t) = C_Q\,(R^2 - t^2)^{\lambda-1} + F(t), \quad 0 \le t \le R,$$

where the constant C_Q is given by

$$C_Q = \frac{2\,\Gamma((d-1)/2+\lambda)}{\Gamma(\lambda)\,\Gamma((d-1)/2)}\,\frac{1}{R^{d+2\lambda-3}}\left\{\frac{\Gamma((d-1)/2)}{2\,\pi^{(d-1)/2}} - \int_0^R F(t)\,t^{d-2}\,dt\right\}.$$

■

Proof of Theorem 4.2.1. The expression for the equilibrium measure on the disk $\mathbb{D}_R$ when there is no external field present, follows from Theorem 4.1.2 upon setting $Q = 0$. To obtain expression for the capacity of the disk $\mathbb{D}_R$, we first notice that $F_Q = W_s(\mathbb{D}_R)$ when $Q = 0$. From formula (4.3.22) we find that when $Q = 0$,

$$F_Q = \frac{\pi\,\Gamma((d+2\lambda-1)/2)}{\sin(\lambda\pi)\,\Gamma(\lambda)\,(\Gamma((d-1)/2))^2}\,\frac{1}{R^{d+2\lambda-3}}.$$

Recalling that $\mathrm{cap}_s(\mathbb{D}_R) = 1/W_s(\mathbb{D}_R)$, we obtain desired expression (4.2.3).

■

Proof of Proposition 4.2.1. Let E be any compact subset of $\mathbb{D}$ with positive Riesz s-capacity. For the range of the Riesz s-parameter satisfying $d-3 < s < d-1$, the potential of the equilibrium measure μ_E (with no external field) satisfies the following inequalities [50, p. 136],

$$U_s^{\mu_E}(x) = W_s(E), \quad \text{q. e. on } E, \tag{4.3.23}$$

$$U_s^{\mu_E}(x) \le W_s(E), \quad \text{on } \mathbb{D}. \tag{4.3.24}$$

We first observe that variational inequalities (1.2.1)–(1.2.2) imply

$$\begin{aligned}\mathcal{F}_s(S_Q) &= W_s(S_Q) + \int Q(x)\,d\mu_Q(x)\\ &= I_s(\mu_Q) + \int Q(x)\,d\mu_Q(x)\\ &= F_Q.\end{aligned}$$

We now show that for any compact set $E \subset \mathbb{D}$ with positive Riesz s-capacity we have $\mathcal{F}_s(E) \geq \mathcal{F}_s(S_Q)$. Indeed, integrating inequality (1.2.1) with respect to μ_E, we obtain

$$\int U_s^{\mu_Q}(x)\, d\mu_E(x) + \int Q(x)\, d\mu_E(x) \geq F_Q, \tag{4.3.25}$$

where the inequality holds μ_E-a.e. as μ_E has finite Riesz s-energy. With (4.3.25) and (4.3.23)–(4.3.24) in mind, we write down the following chain of inequalities,

$$\begin{aligned}
W_s(E) &= \int W_s(E)\, d\mu_Q(x) \\
&\geq \int U_s^{\mu_E}(x)\, d\mu_Q(x) \\
&= \int \left(\int \frac{1}{|x-y|^s}\, d\mu_E(y) \right) d\mu_Q(x) \\
&= \int \left(\int \frac{1}{|x-y|^s}\, d\mu_Q(x) \right) d\mu_E(y) \\
&= \int U_s^{\mu_Q}(y)\, d\mu_E(y) \\
&= \int U_s^{\mu_Q}(x)\, d\mu_E(x) \\
&\geq F_Q - \int Q(x)\, d\mu_E(x).
\end{aligned}$$

We now see that

$$\mathcal{F}_s(E) = W_s(E) + \int Q(x)\, d\mu_E(x) \geq F_Q = \mathcal{F}_s(S_Q),$$

so that $\mathcal{F}_s(E) \geq \mathcal{F}_s(S_Q)$, as claimed.

■

Proof of Theorem 4.2.2. If $E = \mathbb{D}_R$, taking into account that $W_s(\mathbb{D}_R) = 1/\operatorname{cap}_s(\mathbb{D}_R)$, and inserting (4.2.3) and (4.2.1) into (4.2.4), we find that $\mathcal{F}_s$-functional is given by

$$\begin{aligned}
\mathcal{F}_s(\mathbb{D}_R) = &\frac{\pi\, \Gamma((d+2\lambda-1)/2)}{\sin(\lambda\pi)\, \Gamma(\lambda)\, \Gamma((d-1)/2)}\, \frac{1}{R^{d+2\lambda-3}} \times \\
&\left\{ 1 + \frac{2\sin(\lambda\pi)}{\pi} \int_0^R Q(r)\, (R^2 - r^2)^{\lambda-1}\, r^{d-2}\, dr \right\}.
\end{aligned} \tag{4.3.26}$$

Using the substitution $R - r = Ru$, we transform the integral on the right hand side of (4.3.26) as follows,

$$\int_0^R Q(r)(R^2 - r^2)^{\lambda-1}\, r^{d-2}\, dr$$
$$= 2^{\lambda-1}\, R^{d+2\lambda-3} \int_0^1 Q((1-u)R)\, u^{\lambda-1}\, (1-u)^{d-2}\, (1-u/2)^{\lambda-1}\, du.$$

This allows us to write expression (4.3.26) as

$$\mathcal{F}_s(\mathbb{D}_R) = c(d,\lambda)\times \left\{ \frac{1}{R^{d+2\lambda-3}} + \frac{2^\lambda \sin(\lambda\pi)}{\pi} \int_0^1 Q((1-u)R)\, u^{\lambda-1}\, (1-u)^{d-2}\, (1-u/2)^{\lambda-1}\, du \right\},$$

where for brevity we set

$$c(d,\lambda) := \frac{\pi\Gamma((d+2\lambda-1)/2)}{\sin(\lambda\pi)\Gamma(\lambda)\Gamma((d-1)/2)}.$$

Differentiating the last expression with respect to R, we derive

$$\begin{aligned} \mathcal{F}_s'(\mathbb{D}_R) &= c(d,\lambda)\Big\{ -\frac{d+2\lambda-3}{R^{d+2\lambda-2}} \\ &+ \frac{2^\lambda \sin(\lambda\pi)}{\pi} \int_0^1 Q'((1-u)R)\, u^{\lambda-1}\, (1-u)^{d-1}\, (1-u/2)^{\lambda-1}\, du \Big\} \\ &= c(d,\lambda)\Big\{ -\frac{d+2\lambda-3}{R^{d+2\lambda-2}} \\ &+ \frac{2\sin(\lambda\pi)}{\pi} \frac{1}{R^{d+2\lambda-2}} \int_0^R Q'(r)\, (R^2-r^2)^{\lambda-1}\, r^{d-1}\, dr \Big\} \\ &= -\frac{2\sin(\lambda\pi) c(d,\lambda)}{\pi R^{d+2\lambda-2}}\, \Delta(R), \end{aligned} \tag{4.3.27}$$

with

$$\Delta(R) := \frac{\pi(d+2\lambda-3)}{2\sin(\lambda\pi)} - \int_0^R Q'(r)\, (R^2-r^2)^{\lambda-1}\, r^{d-1}\, dr,$$

and where the differentiation under the integral sign is justified by invoking the Dominated Convergence Theorem. Since the $\mathcal{F}_s$-functional is minimized on the support of the extremal measure, we obtain by setting $\mathcal{F}_s'(\mathbb{D}_R)$ to zero that the radius R must satisfy the equation

$$\int_0^R Q'(r)\, (R^2-r^2)^{\lambda-1}\, r^{d-1}\, dr = \frac{\pi(d+2\lambda-3)}{2\sin(\lambda\pi)}, \tag{4.3.28}$$

which by a simple rearrangement can be brought into the required form (4.2.5).

We now discuss the existence and uniqueness of a solution to equation (4.3.28). First we note that the left hand side can be written as

$$\begin{aligned} w(R) :&= \int_0^R Q'(r)\,(R^2 - r^2)^{\lambda-1}\, r^{d-1}\, dr \\ &= 2^{\lambda-1}\, R^{d+2\lambda-2} \int_0^1 Q'((1-u)R)\, u^{\lambda-1}\,(1-u)^{d-1}\,(1-u/2)^{\lambda-1}\, du. \end{aligned}$$

From the convexity of Q it follows that $Q'(R) \geq 0$ for $0 \leq R \leq 1$. Using the Dominated Convergence Theorem we can show that the integral on the right hand side of the last expression is an increasing function of R for $0 \leq R \leq 1$. As $w(R)$ is a product of two non-negative increasing functions on $[0,1]$, the function $w(R)$ itself is an increasing function of R for $0 \leq R \leq 1$. Moreover, a simple calculation shows that $w(0) = 0$. These considerations make it clear that in a situation when (4.3.28) does not have a solution on the interval $[0,1]$, it must be the case that

$$\begin{aligned} \Delta(R) &= \frac{\pi(d+2\lambda-3)}{2\sin(\lambda\pi)} - \int_0^R Q'(r)\,(R^2-r^2)^{\lambda-1}\, r^{d-1}\, dr \\ &= \frac{\pi(d+2\lambda-3)}{2\sin(\lambda\pi)} - g(R) > 0, \quad 0 \leq R \leq 1. \end{aligned}$$

From (4.3.27) it then follows that $\mathcal{F}_s'(\mathbb{D}_R) < 0$ for $0 \leq R \leq 1$. Hence $\mathcal{F}_s(\mathbb{D}_R)$ is strictly decreasing on $[0,1]$ and attains its global minimum at $R = 1$.

Suppose now that equation (4.3.28) does have a solution on the interval $[0,1]$, i.e. $\mathcal{F}_s(\mathbb{D}_R)$ has a critical point on $[0,1]$. From (4.3.27) we deduce that

$$\begin{aligned} \mathcal{F}_s''(\mathbb{D}_R) &= c(d,\lambda)\left\{\frac{(d+2\lambda-3)(d+2\lambda-2)}{R^{d+2\lambda-1}}\right. \\ &\quad\left. + \frac{2^{\lambda}\sin(\lambda\pi)}{\pi}\int_0^1 Q''((1-u)R)\, u^{\lambda-1}\,(1-u)^{d}\,(1-u/2)^{\lambda-1}\, du\right\} \\ &= c(d,\lambda)\left\{\frac{(d+2\lambda-3)(d+2\lambda-2)}{R^{d+2\lambda-1}}\right. \\ &\quad\left. + \frac{2\sin(\lambda\pi)}{\pi}\,\frac{1}{R^{d+2\lambda-1}}\int_0^R Q''(r)\,(R^2-r^2)^{\lambda-1}\, r^{d}\, dr\right\} \\ &= \frac{c(d,\lambda)2\sin(\lambda\pi)}{\pi R^{d+2\lambda-1}}\left\{\frac{\pi(d+2\lambda-3)(d+2\lambda-2)}{2\sin(\lambda\pi)} + \int_0^R Q''(r)\,(R^2-r^2)^{\lambda-1}\, r^{d}\, dr\right\}. \end{aligned}$$

Recalling that $Q(r)$ is convex on $[0,1]$, from last expression it transpires that $\mathcal{F}_s''(\mathbb{D}_R) > 0$ for all $0 \le R \le 1$. This means that if R_* is a critical point of $\mathcal{F}_s(\mathbb{D}_R)$, then it will be its global minimum.

We also observe that $R = 0$ cannot satisfy equation (4.3.28), which is evident from (4.3.28) itself. We therefore conclude that $\mathcal{F}_s(\mathbb{D}_R)$ has exactly one global minimum on $(0,1]$, which is either the unique solution $R_* \in (0,1]$ of equation (4.3.28) if it exists, or $R_* = 1$.

■

Proof of Theorem 4.2.3. Inserting (4.2.6) into the integral in equation (4.2.5), we find, using formula (4.3.21),

$$\int_0^R Q'(r)\,(R^2 - r^2)^{\lambda-1}\, r^{d-2}\, dr = \frac{q\alpha\Gamma((d+\alpha-1)/2)\,\Gamma(\lambda)}{2\Gamma((d+\alpha+2\lambda-1)/2)}\, R^{d+\alpha+2\lambda-3}.$$

Inserting the above result back to equation (4.2.5), after simple algebra we derive desired formula (4.2.7).

We now proceed with evaluating the density of the extremal measure μ_Q, as outlined in (4.1.5) and (4.1.4). Inserting expression for the external field (4.2.6) into (4.1.5), we arrive at the integral

$$\int_0^r \frac{Q(u)\, u^{d-2}\, du}{(r^2-u^2)^{1-\lambda}} = \frac{q(\alpha+d-3)}{4\lambda}\, \mathrm{B}\left(1+\lambda, \frac{d+\alpha-3}{2}\right) r^{d+2\lambda-3+\alpha},$$

which is evaluated using the substitution $tr^2 = r^2 - u^2$. We therefore easily find that

$$g(r) = \frac{q(\alpha+d-3)(d+2\lambda+\alpha-3)}{4\lambda}\, \mathrm{B}\left(1+\lambda, \frac{d+\alpha-3}{2}\right) r^{\alpha}.$$

Inserting this result into (4.1.4), we arrive at another integral

$$\int_t^{R_*} \frac{r^\alpha\, r\, dr}{(r^2-t^2)^{1-\lambda}} = \frac{R_*^\alpha}{2\lambda}\,(R_*^2 - t^2)^\lambda\, {}_2F_1\left(-\frac{\alpha}{2}, 1; \lambda+1; 1 - \left(\frac{t}{R_*}\right)^2\right),$$

which is handled by substituting $(R_*^2 - t^2)u = r^2 - t^2$ and then recalling the integral representation [1, #15.3.1, p. 558] of the hypergeometric function ${}_2F_1(a,b;c;z)$. Now inserting the above result into formula (4.1.4), we eventually deduce that

$$F(t) = \frac{q\,\sin(\lambda\pi)\,\Gamma((\alpha+d-1)/2)\,\Gamma((d+2\lambda-3)/2)}{\pi^{(d+1)/2}\,\Gamma((d+\alpha+2\lambda-3)/2)}\,R_*^{\alpha}\,(R_*^2-t^2)^{\lambda-1}$$
$$\Bigg\{-\,{}_2F_1\left(-\frac{\alpha}{2},1;\lambda+1;1-\left(\frac{t}{R_*}\right)^2\right)$$
$$+\frac{\alpha}{2\lambda(\lambda+1)}\left(1-\left(\frac{t}{R_*}\right)^2\right)\,{}_2F_1\left(1-\frac{\alpha}{2},2;\lambda+2;1-\left(\frac{t}{R_*}\right)^2\right)\Bigg\},\qquad 0\le t\le R_*.$$

It remains to find the constant C_Q, which is computed according to (4.1.8). However, to avoid the tedious calculations encountered while evaluating the integral appearing on the right hand side of (4.1.8), we recall that the constant C_Q is related to the Robin constant F_Q by

$$C_Q = \frac{\sin(\lambda\pi)\,\Gamma((d-1)/2)}{\pi^{(d+1)/2}}\,F_Q.$$

Furthermore, from the proof of Proposition 4.2.1 it follows that $F_Q = \mathcal{F}_s(\mathbb{D}_{R_*})$, where R_* is given by (4.2.7). We now easily derive

$$C_Q = \frac{\Gamma((d+2\lambda-1)/2)}{\pi^{(d-1)/2}\,\Gamma(\lambda)}\left\{\frac{1}{R_*^{d+2\lambda-3}}+\frac{q\sin(\lambda\pi)\,\Gamma((d+\alpha-1)/2)\,\Gamma(\lambda)}{\pi\,\Gamma((d+\alpha+2\lambda-1)/2)}\,R_*^{\alpha}\right\}.$$

■

Proof of Theorem 4.2.4. We commence by deriving expression (4.2.16) for the density of the extremal measure μ_Q. For that upon inserting (4.2.12) into (4.1.5), we arrive at the following integral

$$\int_0^r \frac{u^{d-2}\,du}{(u^2+h^2)^{s/2}\,(r^2-u^2)^{1-\lambda}}$$
$$= \frac{r^{d+2\lambda-3}}{2(r^2+h^2)^{s/2}}\int_0^1 z^{\lambda-1}\,(1-z)^{(d-3)/2}\,(1-(r^2/(r^2+h^2))z)^{-s/2}\,dz$$
$$= \frac{\Gamma((d-1)/2)\Gamma(\lambda)}{2\Gamma((d+2\lambda-1)/2)}\,\frac{r^{d+2\lambda-3}}{(r^2+h^2)^{s/2}}\,{}_2F_1\left(\frac{s}{2},\lambda;\frac{d+2\lambda-1}{2};\frac{r^2}{r^2+h^2}\right)$$
$$= \frac{\Gamma((d-1)/2)\Gamma(\lambda)}{2\Gamma((d+2\lambda-1)/2)}\,\frac{r^{d+2\lambda-3}}{(r^2+h^2)^{s/2}}\,{}_2F_1\left(\frac{s}{2},\lambda;\frac{s}{2}+1;\frac{r^2}{r^2+h^2}\right),$$

where we used the substitution $r^2-u^2=r^2z$ and the integral representation of the hypergeometric function [1, #15.3.1, p. 558]. Further recalling the known relation between the hypergeometric function ${}_2F_1(a,b;c;z)$ and the incomplete Beta function $\mathrm{B}(z;a,b)$,

$$\mathrm{B}(z;a,b) = \frac{z^a}{a}\,{}_2F_1(a,1-b;a+1;z), \tag{4.3.29}$$

we eventually find that

$$\int_0^r \frac{u^{d-2}\,du}{(u^2+h^2)^{s/2}\,(r^2-u^2)^{1-\lambda}} \tag{4.3.30}$$
$$= \frac{(d+2\lambda-3)\,\Gamma(\lambda)\,\Gamma((d-1)/2)}{4\Gamma((d+2\lambda-1)/2)}\,\mathrm{B}\left(\frac{r^2}{r^2+h^2};\frac{d+2\lambda-3}{2},1-\lambda\right).$$

Differentiating both sides of (4.3.30) with respect to r, while keeping in mind the fact

$$\frac{d}{dz}\,\mathrm{B}(z;a,b) = (1-z)^{b-1}\,z^{a-1}, \tag{4.3.31}$$

we deduce a concise expression for the function g,

$$g(r) = \frac{q(d+2\lambda-3)\,\Gamma(\lambda)\,\Gamma((d-1)/2)}{2\,\Gamma((d+2\lambda-1)/2)}\,\frac{h^{2(1-\lambda)}}{(r^2+h^2)^{(d-1)/2}}, \quad 0\le r\le 1. \tag{4.3.32}$$

Having expression for the function g obtained, from formula (4.1.4) it follows that we need to evaluate the integral

$$\begin{aligned}\int_t^1 \frac{r\,dr}{(r^2+h^2)^{(d-1)/2}\,(r^2-t^2)^{1-\lambda}} &= \frac{(1-t^2)^\lambda}{2}\int_0^1 \frac{z^{\lambda-1}\,dz}{(t^2+h^2+(1-t^2)z)^{(d-1)/2}}\\ &= \frac{(1-t^2)^\lambda}{2\lambda\,(t^2+h^2)^{(d-1)/2}}\,{}_2F_1\left(\lambda,\frac{d-1}{2};\lambda+1;-\frac{1-t^2}{h^2+t^2}\right)\\ &= \frac{(1-t^2)^\lambda}{2\lambda\,(1+h^2)^{(d-1)/2}}\,{}_2F_1\left(1,\frac{d-1}{2};\lambda+1;\frac{1-t^2}{1+h^2}\right),\end{aligned}$$

where in the last two steps we used the integral representation of the hypergeometric function [1, #15.3.1, p. 558] and the linear transformation formula [1, #15.3.5, p. 559]. Using another relation between the hypergeometric function ${}_2F_1(a,b;c;z)$ and the incomplete Beta function $\mathrm{B}(z;a,b)$,

$$\mathrm{B}(z;a,b) = \frac{z^a\,(1-z)^b}{a}\,{}_2F_1(1,a+b;a+1;z), \tag{4.3.33}$$

we eventually obtain, after some trivial simplifications,

$$\int_t^1 \frac{r\,dr}{(r^2+h^2)^{(d-1)/2}\,(r^2-t^2)^{1-\lambda}} \tag{4.3.34}$$
$$= \frac{1}{2\,(h^2+t^2)^{(d-2\lambda-1)/2}}\,\mathrm{B}\left(\frac{1-t^2}{1+h^2};\lambda,\frac{d-2\lambda-1}{2}\right).$$

Therefore,

$$\int_t^1 \frac{g(r)rdr}{(r^2-t^2)^{1-\lambda}} = \frac{q(d+2\lambda-3)\,\Gamma(\lambda)\,\Gamma((d-1)/2)}{4\,\Gamma((d+2\lambda-1)/2)}\times \frac{h^{2(1-\lambda)}}{(h^2+t^2)^{(d-2\lambda-1)/2}}\,\mathrm{B}\left(\frac{1-t^2}{1+h^2};\lambda,\frac{d-2\lambda-1}{2}\right). \tag{4.3.35}$$

Differentiating both sides of (4.3.35) with respect to t, and substituting the result into (4.1.4), after some simple algebra we find

$$F(t) = -\frac{q\,\sin(\lambda\pi)\,\Gamma((d-1)/2)}{\pi^{(d+1)/2}}\,h^{2(1-\lambda)}\times \left\{\frac{d-2\lambda-1}{2}\,\frac{1}{(h^2+t^2)^{(d-2\lambda+1)/2}}\,\mathrm{B}\left(\frac{1-t^2}{1+h^2};\lambda,\frac{d-2\lambda-1}{2}\right) + \frac{(1-t^2)^{\lambda-1}}{(1+h^2)^{(d-3)/2}\,(h^2+t^2)}\right\},\quad 0\le t\le 1. \tag{4.3.36}$$

Our next step is to evaluate the constant C_Q, following the recipe contained in formula (4.1.8). Inserting expression (4.3.36) into (4.1.8), we find that one of the integrals we need to evaluate is

$$\begin{aligned}\int_0^1 \frac{(1-t^2)^{\lambda-1}}{h^2+t^2}\,t^{d-2}\,dt &= \frac{1}{2h^2}\int_0^1 u^{(d-3)/2}\,(1-u)^{\lambda-1}\,(1-(-h^{-2})u)^{-1}\,du\\ &= \frac{1}{2h^2}\,\frac{\Gamma(\lambda)\,\Gamma((d-1)/2)}{\Gamma((d+2\lambda-1)/2)}\,{}_2F_1\left(1,\frac{d-1}{2};\frac{d+2\lambda-1}{2};-\frac{1}{h^2}\right)\\ &= \frac{\Gamma(\lambda)\,\Gamma((d-1)/2)}{2\,\Gamma((d+2\lambda-1)/2)}\,\frac{1}{1+h^2}\,{}_2F_1\left(1,\lambda;\frac{d+2\lambda-1}{2};\frac{1}{1+h^2}\right),\end{aligned}$$

where the last two expressions were obtained using the integral representation of the hypergeometric function [1, #15.3.1, p. 558] and the linear transformation formula [1, #15.3.4, p. 559]. We subsequently obtain

$$\int_0^1 \frac{(1-t^2)^{\lambda-1}}{h^2+t^2}\,t^{d-2}\,dt = \frac{\Gamma(\lambda)\,\Gamma((d-1)/2)}{2\,\Gamma((d+2\lambda-1)/2)}\,\frac{1}{1+h^2}\,{}_2F_1\left(1,\lambda;\frac{d+2\lambda-1}{2};\frac{1}{1+h^2}\right). \tag{4.3.37}$$

In light of (4.3.37), we finally derive that

$$C_Q = \frac{2\,\Gamma((d-1)/2+\lambda)}{\Gamma(\lambda)\,\Gamma((d-1)/2)}\left\{\frac{\Gamma((d-1)/2)}{2\,\pi^{(d-1)/2}} + q\,c_{d,\lambda}\right\},$$

where

$$
\begin{aligned}
c_{d,\lambda} := {} & \frac{\sin(\lambda\pi)\,\Gamma((d-1)/2)}{\pi^{(d+1)/2}}\, h^{2(1-\lambda)} \times \\
& \left\{ \frac{d-2\lambda-1}{2} \int_0^1 \frac{t^{d-2}}{(h^2+t^2)^{(d-2\lambda+1)/2}}\, \mathrm{B}\left(\frac{1-t^2}{1+h^2}; \lambda, \frac{d-2\lambda-1}{2}\right) dt \right. \\
& \left. + \frac{\Gamma(\lambda)\,\Gamma((d-1)/2)}{2\,\Gamma((d-1)/2+\lambda)} \frac{1}{(1+h^2)^{(d-1)/2}}\, {}_2F_1\left(1, \lambda; \frac{d+2\lambda-1}{2}; \frac{1}{1+h^2}\right) \right\}.
\end{aligned}
$$

Now we show that the obtained extremal measure is indeed a positive measure, provided the height of the point charge satisfies certain restrictions. The first step toward that goal is to demonstrate that the density $f(r)$ of the extremal measure μ_Q is an increasing function of r. Differentiating its expression (4.2.17) and dropping a positive term involving the incomplete Beta function, we see that

$$
\begin{aligned}
f'(r) \geq {} & C_Q \frac{2r(1-\lambda)}{(1-r^2)^{2-\lambda}} + q\frac{\sin(\lambda\pi)\Gamma((d-1)/2)}{\pi^{(d+1)/2}} \frac{h^{2(1-\lambda)}\, r\,(d-2\lambda+1)}{(h^2+r^2)^2\,(1-r^2)^{1-\lambda}\,(1+h^2)^{(d-3)/2}} \\
& - q\frac{\sin(\lambda\pi)\Gamma((d-1)/2)}{\pi^{(d+1)/2}} \frac{h^{2(1-\lambda)}\, 2r\,(1-\lambda)}{(h^2+r^2)\,(1-r^2)^{2-\lambda}\,(1+h^2)^{(d-3)/2}} \\
\geq {} & \frac{\Gamma((d+2\lambda-1)/2)}{\pi^{(d-1)/2}\,\Gamma(\lambda)} \frac{2r(1-\lambda)}{(1-r^2)^{2-\lambda}} \\
& + q\frac{\sin(\lambda\pi)\Gamma((d-1)/2)}{\pi^{(d+1)/2}} \frac{h^{2(1-\lambda)}\, r\,(d-2\lambda+1)}{(h^2+r^2)^2\,(1-r^2)^{1-\lambda}\,(1+h^2)^{(d-3)/2}} \\
& - q\frac{\sin(\lambda\pi)\Gamma((d-1)/2)}{\pi^{(d+1)/2}} \frac{h^{2(1-\lambda)}\, 2r\,(1-\lambda)}{(h^2+r^2)\,(1-r^2)^{2-\lambda}\,(1+h^2)^{(d-3)/2}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& f'(r)\,\pi^{(d+1)/2}\,(1-\lambda)^{-1}\,(h^2+r^2)^2\,r^{-1}\,(1-r^2)^{2-\lambda}\,(1+h^2)^{(d-3)/2} \\
& \geq \frac{2\pi\,\Gamma((d+2\lambda-1)/2)}{\Gamma(\lambda)}\,(h^2+r^2)^2\,(1+h^2)^{(d-3)/2} \\
& + q(d-2\lambda+1)(1-\lambda)\sin(\lambda\pi)\,\Gamma((d-1)/2)\,h^{2(1-\lambda)}\,(1-r^2) \\
& - q\sin(\lambda\pi)\Gamma((d-1)/2)\,h^{2(1-\lambda)}\,(h^2+r^2) \\
& \geq \frac{2h^{d-3}\,\Gamma((d+2\lambda-1)/2)}{\Gamma(\lambda)} \left\{ \pi\,(h^2+r^2)^2 \right. \\
& + q\,\frac{(d-2\lambda+1)\,(1-\lambda)\,\sin(\lambda\pi)}{2}\,\mathrm{B}\left(\lambda, \frac{d-1}{2}\right)\,h^{5-2\lambda-d}\,(1-r^2) \\
& \left. - q\,\frac{\sin(\lambda\pi)}{2}\,\mathrm{B}\left(\lambda, \frac{d-1}{2}\right)\,h^{5-2\lambda-d}\,(h^2+r^2) \right\}.
\end{aligned}
$$

Let $t = r^2 + h^2$. Then the expression in the braces in the last line above becomes

$$\begin{aligned}
&\pi\,(h^2+r^2)^2 + q\,\frac{(d-2\lambda+1)\,(1-\lambda)\,\sin(\lambda\pi)}{2}\,\mathrm{B}\left(\lambda,\frac{d-1}{2}\right)\,h^{5-2\lambda-d}\,(1-r^2)\\
&\qquad - q\,\frac{(1-\lambda)\,\sin(\lambda\pi)}{2}\,\mathrm{B}\left(\lambda,\frac{d-1}{2}\right)\,h^{5-2\lambda-d}\,(h^2+r^2)\\
&\qquad = \pi t^2 - q\,\frac{((1-\lambda)(d-2\lambda+1)+1)\sin(\lambda\pi)}{2}\mathrm{B}\left(\lambda,\frac{d-1}{2}\right)\,h^{5-2\lambda-d}\,t\\
&\qquad + q\,\frac{(d-2\lambda+1)\,(1-\lambda)\,\sin(\lambda\pi)}{2}\,\mathrm{B}\left(\lambda,\frac{d-1}{2}\right)\,h^{5-2\lambda-d}\,(1+h^2).
\end{aligned}$$

The quadratic polynomial

$$\begin{aligned}
m(t) := \pi t^2 &- q\,\frac{((1-\lambda)(d-2\lambda+1)+1)\sin(\lambda\pi)}{2}\mathrm{B}\left(\lambda,\frac{d-1}{2}\right)\,h^{5-2\lambda-d}\,t\\
&+ q\,\frac{(d-2\lambda+1)\,(1-\lambda)\,\sin(\lambda\pi)}{2}\,\mathrm{B}\left(\lambda,\frac{d-1}{2}\right)\,h^{5-2\lambda-d}\,(1+h^2)
\end{aligned}$$

has the discriminant

$$\begin{aligned}
D = -&\,\frac{q\sin(\lambda\pi)h^{2(5-2\lambda-d)}}{4}\,\mathrm{B}\left(\lambda,\frac{d-1}{2}\right)\,\Big\{8\pi(d+2\lambda-1)(1-\lambda)\,(1+h^2)\,h^{d+2\lambda-5}\\
&-q((1-\lambda)(d-2\lambda+1)+1)^2\sin(\lambda\pi)\,\mathrm{B}\left(\lambda,\frac{d-1}{2}\right)\Big\}.
\end{aligned}$$

For the term in braces in the last expression we have the following trivial estimate

$$\begin{aligned}
&8\pi(d+2\lambda-1)(1-\lambda)\,(1+h^2)\,h^{d+2\lambda-5}\\
&-q((1-\lambda)(d-2\lambda+1)+1)^2\sin(\lambda\pi)\,\mathrm{B}\left(\lambda,\frac{d-1}{2}\right)\\
&\geq 8\pi(d+2\lambda-1)(1-\lambda)\,h^{d+2\lambda-3}\\
&-q((1-\lambda)(d-2\lambda+1)+1)^2\sin(\lambda\pi)\,\mathrm{B}\left(\lambda,\frac{d-1}{2}\right)\\
&> 0,
\end{aligned}$$

if h is chosen so that $h > h_-$, where

$$h_- := \left(\frac{q\,((1-\lambda)(d-2\lambda+1)+1)^2\sin(\lambda\pi)}{8\pi(d+2\lambda-1)(1-\lambda)}\,\mathrm{B}\left(\lambda,\frac{d-1}{2}\right)\right)^{1/(d+2\lambda-3)}$$

With such a choice of h it is clear that the discriminant D is strictly negative, and hence the polynomial $m(t)$ does not have real roots. Therefore, $m(t) > 0$, which in turn shows that $f'(r) > 0$. Thus, $f(r) > f(0)$, for all $0 < r < 1$ and if h chosen such that $h > h_-$.

Set $p(h) := f(0)$. We then obtain

$$\begin{aligned} p(h) &= \frac{2\,\Gamma((d-1)/2+\lambda)}{\Gamma(\lambda)\,\Gamma((d-1)/2)} \left\{ \frac{\Gamma((d-1)/2)}{2\,\pi^{(d-1)/2}} + q\, c_{d,\lambda} \right\} \\ &\quad - \frac{q\,\sin(\lambda\pi)\,\Gamma((d-1)/2)}{\pi^{(d+1)/2}} \left\{ \frac{d-2\lambda-1}{2}\, \frac{1}{h^{d-1}}\, \mathrm{B}\left(\frac{1}{1+h^2}; \lambda, \frac{d-2\lambda-1}{2} \right) \right. \\ &\quad \left. + \frac{1}{h^{2\lambda}(1+h^2)^{(d-3)/2}} \right\}. \end{aligned}$$

It is not hard to see that

$$\lim_{h\to 0+} p(h) = -\infty, \qquad \lim_{h\to\infty} p(h) = \frac{\Gamma((d-1)/2+\lambda)}{\pi^{(d-1)/2}\,\Gamma(\lambda)} > 0.$$

As $p(h)$ is a continuous function, it has at least one positive root. Denote the largest such root by h_+. It then follows that $f(r) > 0$ provided $h > \max\{h_-, h_+\}$, and therefore μ_Q is a positive measure, as required.

■

Proof of Corollary 4.2.1. We begin by evaluating the integral appearing in the right hand side of the constant $c_{d,q}$,

$$\begin{aligned} &\int_0^1 \frac{t^{d-2}}{(h^2+t^2)^{(d-2\lambda+1)/2}}\, \mathrm{B}\left(\frac{1-t^2}{1+h^2}; \lambda, \frac{d-2\lambda-1}{2} \right) dt \\ &= \frac{1}{2\lambda}\, \xi^{(d+1)/2} \int_0^1 u^{\lambda}\,(1-u)^{(d-3)/2}\,(1-u\xi)^{-1}\, {}_2F_1\left(1, \frac{d-1}{2}; \lambda+1; u\xi\right), \end{aligned} \tag{4.3.38}$$

where we put $\xi = 1/(1+h^2)$ for brevity. Recall that in the case of Newtonian potential we have $s = d-2$, where $d = 2m+4$, with $m \geq 2$ is a natural number. Using a well-known fact [1, #15.4.1, p. 561] that the hypergeometric function ${}_2F_1(a,b;c;z)$ reduces to a polynomial if either one of its first two parameters is a negative integer, we easily find

$${}_2F_1\left(1, \frac{d-1}{2}; \frac{3}{2}; u\xi\right) = (1-u\xi)^{-(m+1)} \sum_{n=0}^{m} \frac{(-m)_n}{(2n+1)n!}\, u^n \xi^n. \tag{4.3.39}$$

Inserting expression (4.3.39) into the integral on the right hand side of (4.3.38) and again using [1, #15.4.1, p. 561], we eventually find that

$$\int_0^1 \frac{t^{d-2}}{(h^2+t^2)^{(d-2\lambda+1)/2}} \mathrm{B}\left(\frac{1-t^2}{1+h^2}; \lambda, \frac{d-2\lambda-1}{2}\right) dt \tag{4.3.40}$$
$$= \Gamma\left(m+\frac{3}{2}\right) \xi^{m+5/2} \sum_{n=0}^{m} \frac{(-m)_n}{(2n+1)n!} \xi^n \sum_{l=0}^{m-2} \frac{(2-m)_l\, \Gamma(n+l+3/2)}{(n+m+l+2)!} \xi^l$$
$$= \Gamma\left(m+\frac{3}{2}\right) (1+h^2)^{-(m+5/2)} \times$$
$$\sum_{n=0}^{m} \frac{(-m)_n}{(2n+1)n!} (1+h^2)^{-n} \sum_{l=0}^{m-2} \frac{(2-m)_l\, \Gamma(n+l+3/2)}{(n+m+l+2)!} (1+h^2)^{-l}.$$

We also note that the hypergeometric function appearing in the definition of the constant $c_{d,\lambda}$ in this special case reduces to a finite sum involving elementary functions only. Indeed, using [1, #15.3.19, p. 560], with some work we can show that

$${}_2F_1\left(1, \lambda; \frac{d+2\lambda-1}{2}; \frac{1}{1+h^2}\right) \tag{4.3.41}$$
$$= 2(m+1)! \frac{\sqrt{1+h^2}}{h+\sqrt{1+h^2}} \sum_{n=0}^{m} \frac{(-m)_n}{(m+n+1)!} \left(\frac{\sqrt{1+h^2}-h}{\sqrt{1+h^2}+h}\right)^n .$$

We also reduce the Beta-function term, appearing in expression (4.2.18), to a finite sum of expressions in elementary functions only. For that we first recall the relation

$$\mathrm{B}\left(\frac{1-r^2}{1+h^2}; \lambda, \frac{d-2\lambda-1}{2}\right)$$
$$= \frac{1}{\lambda} \frac{(1-r^2)^\lambda}{(1+h^2)^{(d-1)/2}} (h^2+r^2)^{(d-2\lambda-1)/2} {}_2F_1\left(1, \frac{d-1}{2}; \lambda+1; \frac{1-r^2}{1+h^2}\right),$$

which, after taking into account that $\lambda = 1/2$ and $d = 2m+4$, $m \geq 2$, and using the fact [1, #15.4.1, p. 561], gives us

$$\frac{d-2\lambda-1}{2} \frac{1}{(h^2+r^2)^{(d-2\lambda+1)/2}} \mathrm{B}\left(\frac{1-r^2}{1+h^2}; \lambda, \frac{d-2\lambda-1}{2}\right) \tag{4.3.42}$$
$$= \frac{2(m+1)}{(h^2+r^2)^{m+2}} \sum_{n=0}^{m} \frac{(-m)_n}{(2n+1)n!} \left(\frac{1-r^2}{1+h^2}\right)^{n+1/2} .$$

With the help of (4.3.42), expression (4.2.18) is reduced to

$$F(r) = -qh\frac{\Gamma(m+3/2)}{\pi^{m+5/2}}\left\{\frac{2(m+1)}{(h^2+r^2)^{m+2}}\sum_{n=0}^{m}\frac{(-m)_n}{(2n+1)n!}\left(\frac{1-r^2}{1+h^2}\right)^{n+1/2}\right.$$
$$\left.+\frac{1}{\sqrt{1-r^2}}\frac{1}{(h^2+r^2)\,(1+h^2)^{m+1/2}}\right\}, \quad 0 \le r \le 1.$$

We similarly derive that in the case $\lambda = 1/2$ and $d = 2m+4$, $m \ge 2$,

$$c_{d,\lambda} = \frac{(\Gamma(m+3/2))^2}{\pi^{m+5/2}}\,h\,(1+h^2)^{-(m+1)}$$
$$\left\{\Gamma\left(m+\frac{3}{2}\right)(1+h^2)^{-(m+5/2)}\times\right.$$
$$\sum_{n=0}^{m}\frac{(-m)_n}{(2n+1)n!}(1+h^2)^{-n}\sum_{l=0}^{m-2}\frac{(2-m)_l\,\Gamma(n+l+3/2)}{(n+m+l+2)!}(1+h^2)^{-l}$$
$$\left.+\frac{\sqrt{1+h^2}}{h+\sqrt{1+h^2}}\sum_{n=0}^{m}\frac{(-m)_n}{(m+n+1)!}\left(\frac{\sqrt{1+h^2}-h}{\sqrt{1+h^2}+h}\right)^n\right\},$$

and also

$$C_Q = \frac{2(m+1)!}{\sqrt{\pi}\Gamma(m+3/2)}\left\{\frac{\Gamma(m+3/2)}{2\pi^{m+3/2}} + q\,c_{d,\lambda}\right\}.$$

■

Proof of Corollary 4.2.2. The expression (4.2.28) for the density $f(r)$ of the extremal measure μ_Q can be derived either using the recipe given in Theorem 4.1.2, or alternatively from corresponding parts of Theorem 4.2.3.

After obtaining the required expression for the density $f(r)$, the next step is to show sure that the obtained solution is, in fact, represents a positive measure. Below we will show that this is indeed the case, provided a point charge is located sufficiently far from the surface of the disk $\mathbb{D}$. In the course of the proof we precisely determine the critical hight of location of the point charge, guaranteeing the positivity of the extremal measure.

We begin by observing that the density $f(r)$ is a strictly increasing function of r.

Differentiating expression (4.2.28), after simplifications, one finds that

$$\begin{aligned} f'(r) =& \frac{c}{\pi^2}\frac{r}{(1-r^2)^{3/2}} + \frac{3h}{\pi^2}\frac{1}{\sqrt{1-r^2}}\frac{r}{(h^2+r^2)^2} \\ &- \frac{h}{\pi^2}\frac{1}{(1-r^2)^{3/2}}\frac{r}{h^2+r^2} + \frac{3h}{\pi^2}\frac{r}{(h^2+r^2)^{5/2}}\tan^{-1}\sqrt{\frac{1-r^2}{h^2+r^2}}, \end{aligned}$$

with

$$c = \frac{\pi}{2}\left(1 + \frac{2h\tan^{-1}(1/h)}{\pi\sqrt{1+h^2}}\right).$$

Using trivial estimates, we further obtain

$$\begin{aligned} \pi^2 f'(r)(1-r^2)^{3/2}(h^2+r^2)^2 r^{-1} &\geq c(h^2+r^2)^2 + 3h(1-r^2) - h(h^2+r^2) \\ &\geq \frac{\pi}{2}(h^2+r^2)^2 + 3h(1-r^2) - h(h^2+r^2) \\ &= \frac{1}{2}(\pi t^2 - 8ht + 6h(1+h^2)), \end{aligned}$$

with $t = r^2 + h^2$. The quadratic polynomial $m(t) = \pi t^2 - 8ht + 6h(1+h^2)$ has discriminant $D = -8h(3\pi(1+h^2) - 8h) < 0$. Hence $m(t)$ does not have real roots, which implies that $m(t) > 0$. This in turn shows that $f'(r) > 0$, as required, and so $f(r) > f(0)$ for all $0 < r < 1$.

Denoting $p(h) := f(0)$, we readily find

$$p(h) = \frac{1}{2\pi}\left(1 + \frac{2h\tan^{-1}(1/h)}{\pi\sqrt{1+h^2}}\right) - \frac{1}{\pi^2 h} - \frac{1}{\pi^2 h^2}\tan^{-1}(1/h).$$

As observed during the course of the proof of Theorem 4.2.3, we have

$$\lim_{h\to 0+} p(h) = -\infty, \quad \lim_{h\to\infty} p(h) = \frac{1}{2\pi} > 0,$$

and so as $p(h)$ is a continuous function, it has at least one positive root. Denote the largest such a root by h_+. We will show that h_+ is the unique root of $p(h)$ on $[0,\infty)$. This follows from the fact that $p(h)$ is a strictly increasing function on $[0,\infty)$, which we now demonstrate. Indeed, the derivative of $p(h)$ can be easily computed to be

$$\begin{aligned} p'(h) =& \frac{\tan^{-1}(1/h)}{\pi^2\sqrt{1+h^2}} - \frac{h}{\pi^2(1+h^2)^{3/2}} - \frac{h^2\tan^{-1}(1/h)}{\pi^2(1+h^2)^{3/2}} \\ &+ \frac{1}{\pi^2 h^2} + \frac{2\tan^{-1}(1/h)}{\pi^2 h^3} + \frac{1}{\pi^2 h^2(1+h^2)}. \end{aligned} \tag{4.3.43}$$

From the Mean Value Theorem for the derivative it follows that for $x > 0$,

$$\frac{x}{1+x^2} < \tan^{-1} x < x. \tag{4.3.44}$$

Using inequality (4.3.44) in conjunction with (4.3.43), we write down an estimate of $p'(h)$ from below,

$$\begin{aligned}
p'(h) &\geq \frac{h}{\pi^2(1+h^2)^{3/2}} - \frac{h}{\pi^2(1+h^2)^{3/2}} - \frac{h}{\pi^2(1+h^2)^{3/2}} \\
&\quad + \frac{1}{\pi^2 h^2} + \frac{2}{\pi^2 h^2(1+h^2)} + \frac{1}{\pi^2 h^2(1+h^2)} \\
&= \frac{3}{\pi^2 h(1+h^2)} + \frac{1}{\pi^2 h^2} - \frac{h}{\pi^2(1+h^2)^{3/2}} \\
&\geq \frac{3}{\pi^2 h^2(1+h^2)} + \frac{1}{\pi^2 h^2} - \frac{1}{\pi^2 h^2} \\
&= \frac{3}{\pi^2 h^2(1+h^2)} > 0, \quad h > 0.
\end{aligned}$$

Clearly, $p(h) \geq 0$ when $h \geq h_+$ and $p(h) < 0$ when $h < h_+$. This implies that $h = h_+$ is the critical height of the point charge, and so $f(r) \geq 0$ for all $0 \leq r \leq 1$, provided $h \geq h_+$, which means that μ_Q is a positive measure.

If $h < h_+$ then support of the extremal measure will no longer be the entire disk, but rather will have an opening around the origin. Indeed, if this is not so, from the fact that $p(h) < 0$ when $h < h_+$, we see that in this case the density $f(r)$ will be negative for all $0 \leq r \leq 1$. This implies that the measure μ_Q in (4.2.27) will no longer be a positive measure, and thus cannot be the extremal measure. This contradiction shows that when $h < h_+$ the point charge will clear out an opening in the disk $\mathbb{D}$ at the origin.

■

CHAPTER 5

Minimum Energy Problem on a Ring in $\mathbb{R}^3$

5.1 Introduction

Recall that in the case of the three-dimensional space $d = 3$, the hyperdisk is $\mathbb{D} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, z = 0\}$. Let $\mathscr{R}(a, b) := \{(x, y, z) \in \mathbb{R}^3 : 0 \leq a \leq x^2 + y^2 \leq b \leq 1, z = 0\} \subset \mathbb{D}$ be a ring of inner radius a and outer radius b in the xy–plane in $\mathbb{R}^3$. We will consider the minimum energy problem on $\mathscr{R}(a, b)$, assuming that the charges interact according to the Newtonian potential $1/r$, where r denotes Euclidean distance. We will also assume that the ring $\mathscr{R}(a, b)$ is immersed into a smooth rotationally invariant external field Q.

There are essentially two different ways of attacking this problem.

The first way consists of solving a Dirichlet problem for the Laplace equation for the potential U^{μ_Q} of the extremal measure μ_Q, with boundary data $F_Q - Q$ on the ring $\mathscr{R}(a, b)$. The work in this direction was originally initiated by Wangerin [73, 74] in 1870s. His idea was to describe the most general system of curvilinear coordinates, in which the Laplace equation becomes R-separable, see [74]. An approach essentially replicating the original work of Wangerin [74] was developed by Lagrange in 1939, see [49] and a later work [48]. The theory developed by Wangerin was applied specifically to the study of the Dirichlet problem for the Laplace question in the ring $\mathscr{R}(a, b)$ by Poole in 1929. In a series of papers [61]–[62] he obtained an expression for the equilibrium potential (without an external field), in terms of a series of the so-called periodic Lamé functions. An approach closely related to that of Poole's, was considered by Lebedev in 1937. In paper [51] he also obtained an expression for the equilibrium potential (without an external field), in term of a series of some special functions, that were not previously tabulated. In addition, using the obtained expression for the equilibrium potential, Lebedev calculated the density of the equilibrium measure. An important consequence of Lebedev's results was that in the case of no external field, one should expect that the equilibrium density being unbounded at the rims of ring $\mathscr{R}(a,b)$, that is

$$f(r) \sim \frac{1}{\sqrt{r^2-a^2}}, \quad r \to a+, \tag{5.1.1}$$
$$f(r) \sim \frac{1}{\sqrt{b^2-r^2}}, \quad r \to b-\,.$$

We also remark that Nicholson [58] in 1922 considered the aforementioned Dirichlet problem for the ring as well. However, his approach did not fall under the umbrella of Wangerin's ideas, and was more of a physical nature. An important contribution of his work was to obtain an approximate expression for the capacity of a ring, assuming that it was thin enough.

The heart of the second way to recover the extremal measure supported on a ring lies in solving an integral equation, arising from the equilibrium relations (1.2.1)-(1.2.2). Indeed, in the case of a smooth rotationally invariant external field Q, from relations (1.2.1)-(1.2.2) it follows that

$$\int \frac{1}{|x-y|}\, d\mu_Q(y) + Q(x) = F_Q, \quad x \in S_Q, \tag{5.1.2}$$

where F_Q is a constant, and S_Q is the ring $\mathscr{R}(a,b)$. Furthermore, as we already seen before, for a rotationally invariant external field Q, the extremal measure μ_Q will be absolutely continuous with respect to the Lebesgue area measure, with an integrable density, that is $d\mu_Q(r,\varphi) = f(r)\, r\, d\varphi\, dr$, where $f \in L_1([0,1])$, $\varphi \in [0,2\pi)$. Therefore, integral equation (5.1.2) can be written as

$$\int_a^b f(t)\, t\, dt \int_0^{2\pi} \frac{d\varphi}{\sqrt{r^2+t^2-2rt\cos\varphi}} = F_Q - Q(r), \quad a \le r \le b. \tag{5.1.3}$$

Apparently the first attempt in solving integral equation (5.1.3) was undertaken by Gubenko and Mossakovksii in 1960. In paper [35] they considered the problem of calculating the pressure that a rigid die, having a form of a circular concentric ring, exerts on an elastic half-space. Having reduced that mechanics problem to an equation of type (5.1.3), they used certain approximation techniques to arrive to an approximate solution to (5.1.3), with a prescribed degree of accuracy, acceptable for their needs. The next major step was undertaken by Cooke in 1963, while seeking to obtain a closed form solution to equation (5.1.3). In his paper [24], he found a way to reduce equation (5.1.3) to a Fredholm integral equation of the second kind of the form

$$G(r) - \frac{2}{\pi^2} \int_a^b K(u,r)\, G(u)\, du = \frac{F_Q}{2\pi} \frac{r}{\sqrt{r^2 - a^2}} - F(r), \quad a \le r \le b, \qquad (5.1.4)$$

where the function F is computed based on the external field Q, while the unknown function G is used to recover the density f of the extremal measure μ_Q. The approach of Cooke was based on exploiting certain representations of the inner integral in the left hand side of (5.1.3), based on some intricate relations involving Bessel functions. Williams [75] noticed that Cooke's approach is in fact independent of any special functions. However, due to the complexity of the kernel K in (5.1.4), mainly iterative numerical procedures were used to obtain solutions of equation (5.1.4). A further step in investigating equation (5.1.3) was taken by Clements and Love in 1970s. In their papers [22] and [23], instead of reducing (5.1.3) to integral equation (5.1.4) with a complicated kernel K, they reduced equation (5.1.3) to two Fredholm integral equations of the second kind with a simple kernel. Namely, they showed how to convert (5.1.3) to a pair of uncoupled integral equations

$$f_+(x) + \frac{2}{\pi} \int_0^k \frac{xt}{1 - x^2 t^2}\, f_+(t)\, dt = g_+(x), \quad \text{for} \;\; 0 < x < k,$$
$$f_-(x) - \frac{2}{\pi} \int_0^k \frac{xt}{1 - x^2 t^2}\, f_-(t)\, dt = g_-(x), \quad \text{for} \;\; 0 < x < k,$$

where $f_\pm$ are unknown functions, which are used to recover the density f of the extremal measure μ_Q, while $g_\pm$ are known functions calculated based on the external field Q, and $k = \sqrt{a/b}$. Based on these results, Love in [54] obtained a formula for the capacity of a ring, as an infinite series in terms of the kernel of the above pair of uncoupled integral equations. His expression for capacity is useful in obtaining estimates on the capacity, as well as its various approximations. We also remark that paper [23] rigorously established the aforementioned asymptotics (5.1.1). Comparison of the above two approaches for calculating the capacity of a thin ring was done in a fairly recent paper by Lebedev and Skalskaya [52].

5.2 Recovery of the extremal measure via an integral equation on the support

The support of the extremal measure is a main ingredient in determining the extremal measure μ_Q itself. If S_Q is known, the extremal measure μ_Q can be recovered by solving the integral equation

$$\int \frac{1}{|x-y|}\, d\mu_Q(y) + Q(x) = F_Q, \quad x \in S_Q, \tag{5.2.1}$$

where F_Q is a constant.

Suppose that the support of the extremal measure μ_Q is a ring $\mathscr{R}(a,b)$, that is $S_Q = \mathscr{R}(a,b)$.

We remark it it is possible to consider this problem in more general setting of higher dimensions and general Reisz s-potentials. Our next statement addresses such a problem and generalizes the known results to the case of higher dimensions and general Riesz s-potentials.

Theorem 5.2.1 *Let $s = (d-3)+2\lambda$, with $0 < \lambda < 1$. Suppose that an external field Q is invariant with respect to rotations about the polar axis, that is $Q(x) = Q(r)$, where $x = (0, r\bar{x}) \in \mathbb{D}$, $\bar{x} \in \mathbb{S}^{d-2}$, $0 \le r \le 1$, and is such that $Q \in C^2(\mathbb{D})$. Assume that* $\operatorname{supp} \mu_Q$ *is a ring $\mathscr{R}(a,b)$. Let*

$$F(r) = \frac{\Gamma((d+2\lambda-3)/2)\Gamma(3-2\lambda)}{2\,\Gamma((d-2\lambda+3)/2)} \frac{d}{dr} \int_a^r \frac{Q(t)\, t^{d-2}\, dt}{(r^2-t^2)^{1-\lambda}}, \quad a \le r \le b, \tag{5.2.2}$$

and further put

$$K(u,r) = \frac{a^{d-2\lambda+1}}{u^2-r^2}\left\{\frac{1}{r^2}\,{}_2F_1\left(1, \frac{d+2\lambda-3}{2}; \frac{d-2\lambda+3}{2}; \left(\frac{a}{r}\right)^2\right)\right. \\ \left. - \frac{1}{u^2}\,{}_2F_1\left(1, \frac{d+2\lambda-3}{2}; \frac{d-2\lambda+3}{2}; \left(\frac{a}{u}\right)^2\right)\right\}. \tag{5.2.3}$$

Let f be the density of the extremal measure μ_Q, that is $d\mu_Q(x) = f(r)\, r^{d-2}\, dr\, d\sigma_{d-1}(\bar{x})$, $x = (0, r\bar{x}) \in \mathscr{R}(a,b)$, $\bar{x} \in \mathbb{S}^{d-2}$, $a \le r \le b$. We set

$$G(r) = \int_r^b \frac{f(t)\, t\, dt}{(t^2-r^2)^\lambda}, \quad a \le r \le b. \tag{5.2.4}$$

Then the function G can be recovered by solving the Fredholm integral equation of the second kind,

$$G(r) - \frac{\Gamma((d+2\lambda-3)/2)\Gamma(3-2\lambda)}{2\,\Gamma((d-2\lambda+3)/2)} \int_a^b G(u)\, K(u,r)\, du \tag{5.2.5}$$
$$= F_Q \frac{\Gamma((d+2\lambda-3)/2)}{2\pi^{(d-1)/2}\,\Gamma(\lambda)} \times$$
$$\left\{\frac{d+2\lambda-3}{2}\, r^{d+2\lambda-4}\, \mathrm{B}\left(1-\left(\frac{a}{r}\right)^2; \lambda, \frac{d-1}{2}\right) + \frac{a^{d-1}}{r}\,(r^2-a^2)^{\lambda-1}\right\}$$
$$- F(r), \quad a \le r \le b.$$

The constant F_Q is uniquely determined by the relation

$$\int_a^b f(t)\, t^{d-2}\, dt = \frac{\Gamma((d-1)/2)}{2\pi^{(d-1)/2}}. \tag{5.2.6}$$

We note that equation (5.2.5) is a Fredholm integral equation of the second kind. The kernel K, given by (5.2.3), is symmetric, that is $K(u,r) = K(r,u)$, which directly follows from its expression (5.2.3). A closed-form exact solution of integral equations of type (5.2.5) is presently unknown. However, various numerical methods and methods for finding approximate solutions to Fredholm integral equations of the second kind are well-established. These results, among many other facts on virtually all known types of integral equations can be found in book [60].

In the case of the classical Coulomb potential in $\mathbb{R}^3$, correspodnding to $d = 3$ and $s = 1$ ($\lambda = 1/2$), the above Theorem assumes the following simplified form.

Theorem 5.2.2 *Suppose that an external field Q is rotationally invariant with respect to rotations about the polar axis, and is such that $Q \in C^2(\mathbb{D})$. Assume that S_Q is a ring $\mathscr{R}(a,b)$. Let*

$$F(r) = \frac{1}{2\pi}\frac{d}{dr}\int_a^r \frac{Q(\rho)\,\rho\,d\rho}{\sqrt{r^2-\rho^2}}, \quad a \le r \le b, \tag{5.2.7}$$

and let

$$\begin{aligned} K(u,r) =& \frac{ur}{\sqrt{r^2-a^2}\sqrt{u^2-a^2}}\frac{1}{u^2-r^2} \\ &\times \left\{\frac{u^2-a^2}{u}\log\left(\frac{u+a}{u-a}\right) - \frac{r^2-a^2}{r}\log\left(\frac{r+a}{r-a}\right)\right\}. \end{aligned} \tag{5.2.8}$$

Also, if f is the density of the equilibrium measure μ_Q, we set

$$G(r) = \int_r^b \frac{f(u)\,u\,du}{\sqrt{u^2-r^2}}, \quad a \le r \le b. \tag{5.2.9}$$

Then the function G can be recovered by solving a Fredholm integral equation of the second kind,

$$G(r) - \frac{2}{\pi^2}\int_a^b K(u,r)\,G(u)\,du = \frac{F_Q}{2\pi}\frac{r}{\sqrt{r^2-a^2}} - F(r), \quad a \le r \le b. \tag{5.2.10}$$

The constant F_Q is uniquely determined by the relation

$$\int_a^b f(t)\,t\,dt = \frac{1}{2\pi}. \tag{5.2.11}$$

Now we concern ourselves with a different kind of task. For the case of $\mathbb{R}^3$ and the classical Coulomb potential, we want to find an external field producing a prescribed extremal measure, supported on a ring $\mathscr{R}(a,b)$.

Theorem 5.2.3 *Let the external field Q be given by*

$$Q(r) = C - \frac{4}{\pi(b^2-a^2)}\left\{E\left(\frac{r}{b}\right) - rE\left(\frac{a}{r}\right) + \frac{r^2-a^2}{r}K\left(\frac{a}{r}\right)\right\}, \quad a \le r \le b, \tag{5.2.12}$$

where C is any real constant, and where

$$K(k) := \int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \quad 0 < k < 1,$$

is the complete elliptic integral of the first kind, and

$$E(k) := \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \varphi}\, d\varphi, \quad 0 < k < 1,$$

is the complete elliptic integral of the second kind.

Then the extremal measure μ_Q is given by $d\mu_Q = f(r)\, r dr d\varphi$, where

$$f(r) = \frac{1}{\pi(b^2 - a^2)}, \quad a \le r \le b. \tag{5.2.13}$$

5.3 Alternative way of recovering the extremal measure

It is known that the equilibrium potential U^{μ_Q} is a harmonic function outside of the support S_Q, while on the support S_Q it assumes the values $F_Q - Q$. Hence, the equilibrium potential U^{μ_Q} satisfies the following Dirichlet problem,

$$\Delta U^{\mu_Q}(x) = 0, \quad x \in \mathbb{R}^3 \setminus \mathscr{R}(a, b), \tag{5.3.1}$$

$$U^{\mu_Q}(x) = F_Q - Q(x), \quad x \in \mathscr{R}(a, b), \tag{5.3.2}$$

$$U^{\mu_Q}(x) = \mathrm{O}\big(|x|^{-1}\big), \quad |x| \to \infty. \tag{5.3.3}$$

If one can solve Dirichlet problem (5.3.1)–(5.3.3), the extremal measure μ_Q can be recovered from its potential by the formula

$$d\mu_Q(r, \varphi) = -\frac{1}{4\pi}\left(\frac{\partial U^{\mu_Q}}{\partial n_+} + \frac{\partial U^{\mu_Q}}{\partial n_-}\right) dS := f(r) r\, d\varphi\, dr, \tag{5.3.4}$$

where $dS = r\, d\varphi\, dr$ is the Lebesgue surface area measure on $\operatorname{supp} \mu_Q$, and $n_\pm$ are the inner and the outer normals to the ring $\mathscr{R}(a, b)$, see [44, pp. 164–165].

5.3.1 Coordinates of confocal cyclides of revolution

The geometry of the problem suggests a choice of a special coordinate system, which can be described as follows.

The Laplace equation $\Delta\varphi = 0$ in cylindrical coordinates (ρ, z, ψ) takes the form

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \varphi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \varphi}{\partial \psi^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0. \tag{5.3.5}$$

Let u, v be new coordinates in the meridian plane such that $z = z(u, v)$, $\rho = \rho(u, v)$. Wangerin [74] first described the most general systems of orthogonal coordinates u, v in which the Laplace equation is R-separable, that is it admits solutions of the form

$$\varphi = \omega(u, v) M(u) V(v) \Psi(\psi), \tag{5.3.6}$$

where $\omega(u, v)$ is a fixed function, and the functions M, V and Ψ are solutions of second order ordinary differential equations.

In particular, Wangerin in his paper [74] showed that if u, v are orthogonal coordinates such that the Laplace equation written in those coordinates has solutions of the form (5.3.6), then $\omega(u, v) = \rho^{-1/2}$, and the coordinates u, v may be chosen in such a way that the mapping of the (z, ρ)-plane into the (u, v)-plane is conformal. Thus we accordingly set

$$z + i\rho = f(u + iv), \tag{5.3.7}$$

where f is a holomorphic function with $f' \neq 0$. We also set

$$\varphi = \rho^{-1/2}\, \Phi(u, v)\, e^{\pm i m \psi}, \tag{5.3.8}$$

and insert this representation into the Laplace equation (5.3.5), thus obtaining

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{\partial^2 \Phi}{\partial z^2} - \left(m^2 - \frac{1}{4}\right) \frac{1}{\rho^2} \Phi = 0. \tag{5.3.9}$$

Using the fact that f is holomorphic, one deduces that $\rho = \mathrm{Im}(f)$ and $z = \mathrm{Re}(f)$ satisfy the Cauchy-Riemann equations,

$$\frac{\partial z}{\partial u} = \frac{\partial \rho}{\partial v}, \quad \frac{\partial z}{\partial v} = -\frac{\partial \rho}{\partial u}. \tag{5.3.10}$$

After some laborious calculations, this allows us to conclude that

$$\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} = |f'|^2 \left(\frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Phi}{\partial \rho^2} \right).$$

Hence, it is easy to see now that equation (5.3.9) translates into the following partial differential equation on function Φ:

$$\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} - \left(m^2 - \frac{1}{4}\right) F(u, v) = 0, \tag{5.3.11}$$

where

$$F(u,v) = \frac{|f'(u+iv)|^2}{(\mathrm{Im} f(u+iv))^2} = \frac{|f'|^2}{\rho^2}. \tag{5.3.12}$$

Clearly, equation (5.3.11) will have solutions of the form $M(u)V(v)$ if the function F can be factored as a sum of a function of u and a function of v only,

$$F(u,v) = F_1(u) + F_2(v). \tag{5.3.13}$$

If this is the case, the functions $M(u)$ and $N(v)$ will satisfy the ordinary differential equations of the second order,

$$M''(u) + \left(h - \left(m^2 - \frac{1}{4}\right) F_1(u)\right) M(u) = 0, \tag{5.3.14}$$

$$N''(u) + \left(h - \left(m^2 - \frac{1}{4}\right) F_2(u)\right) N(u) = 0, \tag{5.3.15}$$

where h is a separation constant. It can be proved [74] that factorization (5.3.13) is possible if and only if the function f is a solution of the following ordinary differential equation of the second order,

$$(f'(z))^2 = a_0 + a_1 f(z) + a_2 f^2(z) + a_3 f^3(z) + a_4 f^4(z) =: P_4(f), \tag{5.3.16}$$

with $a_0, \ldots, a_4$ being some real constants. Therefore, f is either an elementary function or an elliptic function. Furthermore, the differential equation (5.3.16) is invariant under the Möbius transformations

$$f \mapsto \frac{Af + B}{Cf + D},$$

where A, B, C, D are real constants, chosen such that $AD - BC \neq 0$.

We will assume that P_4 has four distinct zeros, so that P_4 can be reduced to a one of its canonical forms [3, p. 304]. The further classification distinguishes the cases of P_4 having all roots real, all roots complex, and two real and two complex roots. The canonical forms of f in these three cases are

$$\mathrm{sn}(u+iv,k), \quad i\,\mathrm{sn}(u+iv,k), \quad \mathrm{cn}(u+iv,k), \tag{5.3.17}$$

where k is an elliptic modulus.

From this point on we will be presenting a slight variation of a solution given by Poole in [61, 62]. An approach closed to that of Poole's can also be found in a paper of Lagrange [48]. The paper of Lebedev [51] explores the same idea as well, using, however, a different substitution.

The geometry of the problem at hand suggests the choice of a canonical form of f as follows,

$$f = i\,a\,\mathrm{sn}(u + iv, k), \tag{5.3.18}$$

where the elliptic modulus k is defined as $k = a/b$, so that $0 < k < 1$. The complimentary elliptic modulus k' is defined via $k^2 + k'^2 = 1$. Also, let K and K' denote the complete elliptic integrals with moduli k and k', respectively,

$$K = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2(\varphi)}}, \quad 0 < k < 1, \tag{5.3.19}$$

$$K' = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k'^2 \sin^2(\varphi)}}, \quad 0 < k' < 1. \tag{5.3.20}$$

Thus,

$$z + i\rho = i\,a\,\mathrm{sn}(u + iv, k). \tag{5.3.21}$$

Note that the function $\mathrm{sn}(u+iv, k)$ gives the mapping of the quadrant $\{z < 0, \rho > 0\}$ onto the rectangle with vertices at $(0, 0)$, $(K, 0)$, (K, K'), $(0, K')$, in the (u, v)-plane. To complete the mapping we reflect in the (z, ρ)-plane along $z = 0$, and in the (u, v)-plane either along $v = 0$, or along $u = K$. The choice of f, as in (5.3.18), becomes now transparent, since the curves $u = \text{const}$, $v = \text{const}$ in the (z, ρ)-plane are confocal bicircular quadrics with real foci at $z = 0$, $\rho = a$ and $\rho = a/k = b$.

Using the complex argument formulas for elliptic functions [21, p. 24], one can show that the coordinates u, v, ψ are related to the Descartes' coordinates (x, y, z) in the following manner,

$$x = \frac{a}{1-\mathrm{dn}^2(u,k)\,\mathrm{sn}^2(v,k')}\,\mathrm{sn}(u,k)\,\mathrm{dn}(v,k')\cos(\psi), \tag{5.3.22}$$
$$y = \frac{a}{1-\mathrm{dn}^2(u,k)\,\mathrm{sn}^2(v,k')}\,\mathrm{sn}(u,k)\,\mathrm{dn}(v,k')\sin(\psi),$$
$$z = -\frac{a}{1-\mathrm{dn}^2(u,k)\,\mathrm{sn}^2(v,k')}\,\mathrm{cn}(u,k)\,\mathrm{dn}(u,k)\,\mathrm{sn}(v,k')\,\mathrm{cn}(v,k'),$$

Also,

$$\rho(u,v) = \frac{a\,\mathrm{sn}(u,k)\,\mathrm{dn}(v,k')}{1-\mathrm{dn}^2(u,k)\,\mathrm{sn}^2(v,k')} \tag{5.3.23}$$

Theorem 5.3.1 *The equilibrium potential U^{μ_Q} is given by*

$$U^{\mu_Q}(u,v) = \rho^{-1/2}\sum_{n=0}^{\infty} a_n\,\mathrm{Ec}^{2n}_{-1/2}(i(1+k)(u-K),k^2)\,\mathrm{Ec}^{2n}_{-1/2}((1-k)v,k^2), \tag{5.3.24}$$

where $\mathrm{Ec}^{2n}_m(z,k^2)$ are periodic Lamé functions [4, pp. 63–75], orthogonal in $L^2((a,b),w)$, with $[a,b]=[0,K]$ or $[a,b]=[-K',K']$. The positive weight w is given by

$$w = (1+k)^2 \tag{5.3.25}$$

In particular, for $l,n = 0,1,2,\ldots$,

$$\int_{-K'}^{K'} \mathrm{Ec}^{2l}_{m-1/2}((1-k)v,k^2)\,\mathrm{Ec}^{2n}_{m-1/2}((1-k)v,k^2)\,(1+k)^2\,dv = \delta_{ln}. \tag{5.3.26}$$

The convergence of the series in (5.3.24) is understood in the sense of L^2_w. The coefficients a_n are determined from the boundary values of the potential U^{μ_Q}, given in (5.3.2). In other words, they can be readily found from the external field Q, using orthogonality property (5.3.26).

5.3.2 Equilibrium density in the (u,v) coordinates

Recall, that according to (5.3.4), the density $f(r)$ of the extremal measure μ_Q is recovered from the equilibrium potential U^{μ_Q} via

$$f(r) = -\frac{1}{4\pi}\left(\frac{\partial U^{\mu_Q}}{\partial n_+} + \frac{\partial U^{\mu_Q}}{\partial n_-}\right), \tag{5.3.27}$$

where $n_\pm$ are the inner and the outer normals to the ring $\mathscr{R}(a,b)$. Obviously, equation (5.3.27) can be written as

$$f(r) = \frac{1}{2\pi}\left(\frac{\partial U^{\mu_Q}}{\partial z}\right)_{z=0}. \tag{5.3.28}$$

Since the equilibrium potential U^{μ_Q} was computed above in (5.3.24) as a function of the coordinates (u,v), we need to recast formula (5.3.28) into the coordinates (u,v) as well.

Proposition 5.3.1 *The density f of the extremal measure μ_Q in coordinates (u,v) is of the following form,*

$$f(v) = \frac{b}{2\pi a}\frac{\sqrt{r(v)}}{\sqrt{b^2-r^2(v)}\sqrt{r^2(v)-a^2}}\left(\frac{\partial g(u,v)}{\partial u}\right)_{u=K}, \quad -K' \le v \le K', \tag{5.3.29}$$

where $r(v) := \rho(K,v) = a/\mathrm{dn}(v,k')$, and where

$$g(u,v) = \sum_{n=0}^{\infty} a_n \,\mathrm{Ec}_{-1/2}^{2n}(i(1+k)(u-K),k^2)\,\mathrm{Ec}_{-1/2}^{2n}((1-k)v,k^2), \tag{5.3.30}$$

with a_n defined through the external field Q via the orthogonality relation (5.3.26)*, as explained in Theorem 5.3.1 above.*

5.4 Proofs

Proof of Theorem 5.2.1. Recalling that the extremal measure is absolutely continuous with respect to the Lebesgue surface area measure, and invoking Lemma 4.3.8 of Chapter 4, for the Riesz s-potential $U_s^{\mu_Q}$ we have the following representation, for $x=(0,r\bar{x})\in\mathbb{D}$, with $\bar{x}\in\mathbb{S}^{d-2}$ and $0\le a\le r\le b\le 1$,

$$\begin{aligned}
U_s^{\mu_Q}(x) &= \int \frac{1}{|x-y|^s}\,d\mu(y)\\
&= \frac{2\pi^{(d-2)/2}}{\Gamma(d/2-1)}\int_a^b f(\rho)\,\rho^{d-2}\,d\rho\int_0^\pi \frac{\sin^{d-3}\xi\,d\xi}{(r^2+\rho^2-2r\rho\cos\xi)^{s/2}}\\
&= \frac{4\sin(\lambda\pi)\,\pi^{(d-3)/2}\,\Gamma(\lambda)}{\Gamma((d+2\lambda-3)/2)}\,r^{3-d}\int_a^b f(\rho)\,\rho\,d\rho\int_0^{\min(r,\rho)}\frac{t^{d+2\lambda-4}\,dt}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda}
\end{aligned}$$

This allows us to write integral equation (4.1.3) in the following form,

$$\int_a^b \int_0^{\min(\rho,r)} \frac{f(\rho)\,\rho\,t^{d+2\lambda-4}\,dt\,d\rho}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} = \frac{\Gamma((d+2\lambda-3)/2)}{4\sin(\lambda\pi)\,\pi^{(d-3)/2}\,\Gamma(\lambda)}\,r^{d-3}\,(F_Q - Q(r))\,, \quad a \le r \le b. \tag{5.4.1}$$

We continue by working with the left hand side of equation (5.4.1). Splitting the domain of integration for the variable ρ, we write

$$\int_a^b \int_0^{\min(\rho,r)} \frac{f(\rho)\,\rho\,t^{d+2\lambda-4}\,dt\,d\rho}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} = \int_a^r \int_0^\rho \frac{f(\rho)\,\rho\,t^{d+2\lambda-4}\,dt\,d\rho}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} + \int_r^b \int_0^r \frac{f(\rho)\,\rho\,t^{d+2\lambda-4}\,dt\,d\rho}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda}. \tag{5.4.2}$$

In the first integral on the right hand side of the above expression, we further split the domain of integration as follows,

$$\int_a^r \int_0^\rho \frac{f(\rho)\,\rho\,t^{d+2\lambda-4}\,dt\,d\rho}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} = \int_a^r \int_0^a \frac{f(\rho)\,\rho\,t^{d+2\lambda-4}\,dt\,d\rho}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} + \int_a^r \int_a^\rho \frac{f(\rho)\,\rho\,t^{d+2\lambda-4}\,dt\,d\rho}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda}. \tag{5.4.3}$$

Similarly, the second integral is split in the following way,

$$\int_r^b \int_0^r \frac{f(\rho)\,\rho\,t^{d+2\lambda-4}\,dt\,d\rho}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} = \int_r^b \int_0^a \frac{f(\rho)\,\rho\,t^{d+2\lambda-4}\,dt\,d\rho}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} + \int_r^b \int_a^r \frac{f(\rho)\,\rho\,t^{d+2\lambda-4}\,dt\,d\rho}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda}. \tag{5.4.4}$$

We then change the order of integration in the second integral on the right hand side of (5.4.3) as follows,

$$\int_a^r \int_a^\rho \frac{f(\rho)\,\rho\,t^{d+2\lambda-4}\,dt\,d\rho}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} = \int_a^r \int_t^r \frac{f(\rho)\,\rho\,t^{d+2\lambda-4}\,dt\,d\rho}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda}. \tag{5.4.5}$$

Combining the first integral on the right hand side of (5.4.3) with the first integral on the right hand side of (5.4.4), we obtain

$$\int_a^r \int_0^a \frac{f(\rho)\,\rho\,t^{d+2\lambda-4}\,dt\,d\rho}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} + \int_r^b \int_0^a \frac{f(\rho)\,\rho\,t^{d+2\lambda-4}\,dt\,d\rho}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} = \int_a^b f(\rho)\,\rho\,d\rho \int_0^a \frac{t^{d+2\lambda-4}\,dt}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda}, \tag{5.4.6}$$

while similarly combining the integral on the right hand side of (5.4.5) with the second integral on the right hand side of (5.4.4) yields

$$\int_a^r \int_t^\rho \frac{f(\rho)\,\rho\, t^{d+2\lambda-4}\,dt\,d\rho}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} + \int_r^b \int_a^r \frac{f(\rho)\,\rho\, t^{d+2\lambda-4}\,dt\,d\rho}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} = \int_a^r \frac{t^{d+2\lambda-4}\,dt}{(r^2-t^2)^\lambda} \int_t^b \frac{f(\rho)\,\rho\,d\rho}{(\rho^2-t^2)^\lambda}. \tag{5.4.7}$$

Collecting the above calculations, we conclude that integral equation (5.4.1) is transformed into

$$\int_a^r \frac{t^{d+2\lambda-4}\,dt}{(r^2-t^2)^\lambda} \int_t^b \frac{f(\rho)\,\rho\,d\rho}{(\rho^2-t^2)^\lambda} + \int_a^b f(\rho)\,\rho\,d\rho \int_0^a \frac{t^{d+2\lambda-4}\,dt}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} = \frac{\Gamma((d+2\lambda-3)/2)}{4\sin(\lambda\pi)\,\pi^{(d-3)/2}\,\Gamma(\lambda)}\, r^{d-3}\,(F_Q - Q(r)), \quad a \le r \le b. \tag{5.4.8}$$

Our goal now is to further transform the second term on the left hand side of equation (5.4.8). For that we introduce the function $g(u,t)$ as a solution of the following Abel-type integral equation,

$$\int_a^\rho \frac{g(u,t)\,du}{(\rho^2-u^2)^\lambda} = \frac{1}{(\rho^2-t^2)^\lambda}, \quad a \le \rho \le b, \tag{5.4.9}$$

where the variable t, such that $0 \le t \le a$, is fixed. We thus obtain

$$\begin{aligned} &\int_a^b f(\rho)\,\rho\,d\rho \int_0^a \frac{t^{d+2\lambda-4}\,dt}{(r^2-t^2)^\lambda\,(\rho^2-t^2)^\lambda} \\ &= \int_a^b f(\rho)\,\rho\,d\rho \int_0^a \frac{t^{d+2\lambda-4}\,dt}{(r^2-t^2)^\lambda} \int_a^\rho \frac{g(u,t)\,du}{(\rho^2-t^2)^\lambda} \\ &= \int_a^b f(\rho)\,\rho\,d\rho \int_0^a t^{d+2\lambda-4}\,dt \int_a^r \frac{g(s,t)\,ds}{(r^2-s^2)^\lambda} \int_a^\rho \frac{g(u,t)\,du}{(\rho^2-u^2)^\lambda} \\ &= \int_a^r \frac{ds}{(r^2-s^2)^\lambda} \left\{ \int_a^b f(\rho)\,\rho\,d\rho \int_a^\rho \frac{du}{(\rho^2-u^2)^\lambda} \int_0^a g(s,t)\,g(u,t)\,t^{d+2\lambda-4}\,dt \right\} \\ &= \int_a^r \frac{ds}{(r^2-s^2)^\lambda} \left\{ \int_a^b du \int_u^b \frac{f(\rho)\,\rho\,d\rho}{(\rho^2-u^2)^\lambda} \left\{ \int_0^a g(s,t)\,g(u,t)\,t^{d+2\lambda-4}\,dt \right\} \right\}. \end{aligned} \tag{5.4.10}$$

Let

$$G(s) = \int_s^b \frac{f(u)\,u\,du}{(u^2-s^2)^\lambda}. \tag{5.4.11}$$

Combining (5.4.10) and (5.4.11), we recast equation (5.4.8) into

$$\int_a^r \frac{G(t)\,dt}{(r^2-t^2)^\lambda} + \int_a^r \frac{ds}{(r^2-s^2)^\lambda}\left\{\int_a^b G(u)\,du\left\{\int_0^a g(s,t)\,g(u,t)\,t^{d+2\lambda-4}\,dt\right\}\right\}$$
$$= \frac{\Gamma((d+2\lambda-3)/2)}{4\sin(\lambda\pi)\,\pi^{(d-3)/2}\,\Gamma(\lambda)}\,r^{d-3}\left(F_Q - Q(r)\right), \quad a \le r \le b.$$

or,

$$\int_a^r \frac{ds}{(r^2-s^2)^\lambda}\left\{G(s) + \int_a^b G(u)\,du\left\{\int_0^a g(s,t)\,g(u,t)\,t^{d+2\lambda-4}\,dt\right\}\right\} \tag{5.4.12}$$
$$= \frac{\Gamma((d+2\lambda-3)/2)}{4\sin(\lambda\pi)\,\pi^{(d-3)/2}\,\Gamma(\lambda)}\,r^{d-3}\left(F_Q - Q(r)\right), \quad a \le r \le b.$$

Equation (5.4.12) is an Abel-type integral equation with respect to the function

$$G(s) + \int_a^b G(u)\,du\left\{\int_0^a g(s,t)\,g(u,t)\,t^{d+2\lambda-4}\,dt\right\}.$$

Solving this equation [60, # 44, p. 122], we obtain the following integral equation,

$$G(r) + \int_a^b G(u)\,du\left\{\int_0^a g(r,t)\,g(u,t)\,t^{d+2\lambda-4}\,dt\right\} \tag{5.4.13}$$
$$= \frac{\Gamma((d+2\lambda-3)/2)}{2\pi^{(d-1)/2}\,\Gamma(\lambda)}\,\frac{d}{dr}\int_a^r \frac{(F_Q - Q(\rho))\,\rho^{d-3}\,\rho\,d\rho}{(r^2-\rho^2)^{1-\lambda}}, \quad a \le r \le b.$$

We now turn our attention to evaluating the following expression, present on the right hand side of last expression,

$$\frac{d}{dr}\int_a^r \frac{\rho^{d-3}\,\rho\,d\rho}{(r^2-\rho^2)^{1-\lambda}}.$$

Using the substitution $r^2 - t^2 = r^2 z$, after some elementary calculations we find that

$$\int_a^r \frac{\rho^{d-3}\,\rho\,d\rho}{(r^2-\rho^2)^{1-\lambda}} = \frac{1}{2}\,r^{d+2\lambda-3}\,\mathrm{B}\left(1-\left(\frac{a}{r}\right)^2;\lambda,\frac{d-1}{2}\right),$$

where $\mathrm{B}(z;a,b)$ is the incomplete Beta function defined in (4.1.1).

Now it is easy to see that

$$\frac{d}{dr}\int_a^r \frac{\rho^{d-3}\,\rho\,d\rho}{(r^2-\rho^2)^{1-\lambda}} = \frac{d+2\lambda-3}{2}\,r^{d+2\lambda-4}\,\mathrm{B}\left(1-\left(\frac{a}{r}\right)^2;\lambda,\frac{d-1}{2}\right) \tag{5.4.14}$$
$$+ \frac{a^{d-1}}{r}\,(r^2-a^2)^{\lambda-1}.$$

Finally we deal with the inner integral on the left hand side of equation (5.4.13). First we recover the function g from integral equation (5.4.9). Applying [60, #41, p. 11], we find that the function g is given by

$$g(\rho,t) = \frac{2\sin(\lambda\pi)}{\pi}\frac{d}{d\rho}\int_a^\rho \frac{u\,du}{(\rho^2-u^2)^{1-\lambda}\,(u^2-t^2)^\lambda}, \quad 0 \le t \le \rho \le b. \tag{5.4.15}$$

Using the substitution $\rho^2 - u^2 = z$, after some work we find that

$$\int_a^\rho \frac{u\,du}{(\rho^2-u^2)^{1-\lambda}\,(u^2-t^2)^\lambda} = \frac{1}{2\lambda}\left(\frac{\rho^2-a^2}{\rho^2-t^2}\right)^\lambda {}_2F_1\left(\lambda,\lambda;\lambda+1;\frac{\rho^2-a^2}{\rho^2-t^2}\right).$$

Taking derivative of the last expression with respect to ρ, and inserting the result into (5.4.15) produces a remarkably simple expression for the function g,

$$g(\rho,t) = \frac{2\sin(\lambda\pi)}{\pi}\frac{\rho}{\rho^2-t^2}\left(\frac{\rho^2-a^2}{a^2-t^2}\right)^{\lambda-1}, \quad 0 \le t \le a \le \rho \le b. \tag{5.4.16}$$

We therefore obtain

$$\begin{aligned}&\int_0^a g(r,t)g(u,t)\,t^{d+2\lambda-4}\,dt \\ &\quad = \left(\frac{2\sin(\lambda\pi)}{\pi}\right)^2 ru\,(r^2-a^2)^{\lambda-1}(u^2-a^2)^{\lambda-1}\int_0^a \frac{(a^2-t^2)^{2(1-\lambda)}\,t^{d+2\lambda-4}\,dt}{(r^2-t^2)\,(u^2-t^2)}.\end{aligned}$$

After simple but fairly laborious calculations one finds that

$$\begin{aligned}&\int_0^a \frac{(a^2-t^2)^{2(1-\lambda)}\,t^{d+2\lambda-4}\,dt}{(r^2-t^2)\,(u^2-t^2)} = \frac{\Gamma((d+2\lambda-3)/2)\Gamma(3-2\lambda)}{2\,\Gamma((d-2\lambda+3)/2)}\,\frac{a^{d-2\lambda+1}}{u^2-r^2} \\ &\times\left\{\frac{1}{r^2}\,{}_2F_1\left(1,\frac{d+2\lambda-3}{2};\frac{d-2\lambda+3}{2};\left(\frac{a}{r}\right)^2\right)\right. \\ &\left.-\frac{1}{u^2}\,{}_2F_1\left(1,\frac{d+2\lambda-3}{2};\frac{d-2\lambda+3}{2};\left(\frac{a}{u}\right)^2\right)\right\}.\end{aligned}$$

Denote

$$\begin{aligned}K(u,r) = \frac{a^{d-2\lambda+1}}{u^2-r^2}&\left\{\frac{1}{r^2}\,{}_2F_1\left(1,\frac{d+2\lambda-3}{2};\frac{d-2\lambda+3}{2};\left(\frac{a}{r}\right)^2\right)\right. \\ &\left.-\frac{1}{u^2}\,{}_2F_1\left(1,\frac{d+2\lambda-3}{2};\frac{d-2\lambda+3}{2};\left(\frac{a}{u}\right)^2\right)\right\},\end{aligned} \tag{5.4.17}$$

and let

$$F(r) = \frac{\Gamma((d+2\lambda-3)/2)\Gamma(3-2\lambda)}{2\,\Gamma((d-2\lambda+3)/2)}\,\frac{d}{dr}\int_a^r \frac{Q(t)\,t^{d-2}\,dt}{(r^2-t^2)^{1-\lambda}}, \quad a \le r \le b.$$

We can now rewrite integral equation (5.4.13) as follows,

$$
\begin{aligned}
G(r) - \frac{\Gamma((d+2\lambda-3)/2)\Gamma(3-2\lambda)}{2\,\Gamma((d-2\lambda+3)/2)} \int_a^b G(u)\,K(u,r)\,du \qquad (5.4.18)\\
= F_Q \frac{\Gamma((d+2\lambda-3)/2)}{2\pi^{(d-1)/2}\,\Gamma(\lambda)} \left\{ \frac{d+2\lambda-3}{2}\, r^{d+2\lambda-4}\,\mathrm{B}\left(1-\left(\frac{a}{r}\right)^2; \lambda, \frac{d-1}{2}\right)\right.\\
\left. + \frac{a^{d-1}}{r}\,(r^2-a^2)^{\lambda-1}\right\} - F(r), \quad a \le r \le b.
\end{aligned}
$$

Integral equation (5.4.18) is a Fredholm integral equation of the second kind. We remark that its kernel $K(u,r)$ is symmetric, that is $K(u,r) = K(r,u)$, which can be easily seen from expression (5.4.17).

It remains to mention that the constant F_Q is determined using the fact that μ_Q is a probability measure, that is its mass is one. We therefore find that

$$
\int_a^b f(t)\,t^{d-2}\,dt = \frac{\Gamma((d-1)/2)}{2\pi^{(d-1)/2}}.
$$

■

Proof of Theorem 5.2.3. Recalling that the extremal measure μ_Q is absolutely continuous with respect to Lebesgue area measure, with a locally bounded density, that is $d\mu_Q = f(r)\,r dr d\varphi$, with $f \in L_1([0,1])$, integral equation (5.2.1) in the case $S_Q = \mathscr{R}(a,b)$ becomes

$$
\int_a^b f(t)\,t\,dt \int_0^{2\pi} \frac{d\varphi}{\sqrt{r^2+t^2-2rt\cos\varphi}} = F_Q - Q(r), \quad a \le r \le b. \qquad (5.4.19)
$$

First will will transform the right hand side of (5.4.19) to make it more suitable to our goals at this moment. We have the following string of simple transformations,

$$\begin{aligned}
&\int_a^b f(t)\,t\,dt \int_0^{2\pi} \frac{d\varphi}{\sqrt{r^2+t^2-2rt\cos\varphi}} \qquad (5.4.20)\\
&\quad = \int_a^b f(t)\,t\,dt \int_0^{2\pi} \frac{d\varphi}{\sqrt{(r+t)^2-4rt\cos^2(\varphi/2)}}\\
&\quad = \int_a^b \frac{2t}{r+t} f(t)\,dt \int_0^{\pi} \frac{d\varphi}{\sqrt{1-(2\sqrt{rt}/(r+t))^2\cos^2\varphi}}\\
&\quad = \int_a^b \frac{4t}{r+t} f(t)\,dt \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-(2\sqrt{rt}/(r+t))^2\sin^2\varphi}}\\
&\quad = \int_a^b \frac{4t}{r+t} K\left(\frac{2\sqrt{rt}}{r+t}\right) f(t)\,dt,
\end{aligned}$$

where

$$K(k) := \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \quad 0<k<1,$$

is the complete elliptic integral of the first kind.

Now observe that according to [37, p. 864],

$$K(k) = \frac{1}{1+k} K\left(\frac{2\sqrt{k}}{1+k}\right), \quad 0<k<1. \qquad (5.4.21)$$

With (5.4.21) in hand, we continue transforming the last integral in (5.4.20) as

$$\begin{aligned}
&4\int_a^b \frac{1}{(r/t)+1} K\left(\frac{2\sqrt{r/t}}{(r/t)+1}\right) f(t)\,dt \qquad (5.4.22)\\
&\quad = 4\left\{ \int_a^r \frac{1}{(r/t)+1} K\left(\frac{2\sqrt{r/t}}{(r/t)+1}\right) f(t)\,dt \right.\\
&\qquad \left. + \int_r^b \frac{1}{(r/t)+1} K\left(\frac{2\sqrt{r/t}}{(r/t)+1}\right) f(t)\,dt \right\}\\
&\quad = 4\left\{ \int_a^r \frac{t}{r}\frac{1}{(t/r)+1} K\left(\frac{2\sqrt{t/r}}{(t/r)+1}\right) f(t)\,dt \right.\\
&\qquad \left. + \int_r^b \frac{1}{(r/t)+1} K\left(\frac{2\sqrt{r/t}}{(r/t)+1}\right) f(t)\,dt \right\}\\
&\quad = 4\left\{ \int_a^r \frac{t}{r} K\left(\frac{t}{r}\right) f(t)\,dt + \int_r^b K\left(\frac{r}{t}\right) f(t)\,dt \right\}.
\end{aligned}$$

We can now rewrite integral equation (5.4.19) as follows,

$$\int_a^r \frac{t}{r} K\left(\frac{t}{r}\right) f(t)\,dt + \int_r^b K\left(\frac{r}{t}\right) f(t)\,dt = \frac{1}{4}F_Q - \frac{1}{4}Q(r), \tag{5.4.23}$$

where $a \le r \le b$.

Now suppose we are given a prescribed density f of the extremal measure μ_Q, which is of the form

$$f(r) = c, \quad a \le r \le b, \tag{5.4.24}$$

where c is a constant that will be evaluated later. Our task at this point will be to evaluate an external field which generates this prescribed density.

Inserting expression (5.4.24) into integral equation (5.4.23) we derive, after some easy algebra, that

$$Q(r) = F_Q - 4rc\left\{\int_{a/r}^1 uK(u)du - \int_1^{r/b} \frac{K(u)}{u^2}du\right\}, \quad a \le r \le b. \tag{5.4.25}$$

According to [21, 610.01, p. 272], up to an additive constant,

$$\int uK(u)du = E(u) - (1-u^2)K(u), \tag{5.4.26}$$

while, again up to an additive constant, by [21, 612.05, p. 273],

$$\int \frac{K(u)}{u^2}du = -\frac{E(u)}{u}, \tag{5.4.27}$$

where

$$E(k) := \int_0^{\pi/2} \sqrt{1-k^2\sin^2\varphi}\,d\varphi, \quad 0 < k < 1,$$

is the complete elliptic integral of the second kind. Upon substituting (5.4.26) and (5.4.27), we deduce that

$$Q(r) = F_Q - 4c\left\{E\left(\frac{r}{b}\right) - rE\left(\frac{a}{r}\right) + \frac{r^2-a^2}{r}K\left(\frac{a}{r}\right)\right\}, \quad a \le r \le b. \tag{5.4.28}$$

We now evaluate a yet unknown constant c. Its value can be easily found from the fact that μ_Q is a probability measure, that is, its mass is one. We thus have

$$\int_a^b \int_0^{2\pi} f(r)\,r\,dr\,d\varphi = 1,$$

and the substitution of (5.4.24) into the latter expression gives us the value of the constant c,

$$c = \frac{1}{\pi(b^2 - a^2)}. \tag{5.4.29}$$

We therefore finally obtain,

$$Q(r) = C - \frac{4}{\pi(b^2 - a^2)} \left\{ E\left(\frac{r}{b}\right) - rE\left(\frac{a}{r}\right) + \frac{r^2 - a^2}{r} K\left(\frac{a}{r}\right) \right\}, \quad a \leq r \leq b, \tag{5.4.30}$$

where C is any real constant.

■

Proof of Theorem 5.3.1. The usefulness of coordinates (u, v) will become transparent in a moment. First, it is not hard to see that the surface $u = K$ is the surface of the ring $\mathscr{R}(a, b)$. Indeed, setting $u = K$ in (5.3.23), and noting that $\operatorname{sn}(K, k) = 1$, $\operatorname{dn}(K, k) = k'$ [21, p. 20], we find that on the surface $u = K$,

$$\begin{aligned} \rho(K, v) = \frac{a \operatorname{dn}(v, k')}{1 - k'^2 \operatorname{sn}^2(v, k')} &= \frac{a \operatorname{dn}(v, k')}{\operatorname{dn}^2(v, k')} \\ &= \frac{a}{\operatorname{dn}(v, k')}. \end{aligned}$$

If $v = 0$, we find, using the facts that $\operatorname{dn}(0, k') = 1$, $\operatorname{sn}(0, k') = 0$ [21, p. 20],

$$\rho(K, 0) = a,$$

which gives the inner rim a of the ring $\mathscr{R}(a, b)$. If $v = K'$, using the fact that $\operatorname{dn}(K', k') = k$ [21, p. 20], we infer

$$\rho(K, K') = \frac{a}{k} = \frac{a\, b}{a} = b,$$

which gives the outer rim b.

We proceed by computing the function $F(u, v)$. Performing the calculations, we find that

$$\begin{aligned} F(u, v) = &- \left\{ (1 - k) \operatorname{sn}\left(i(1 + k)(u - K), \frac{1 - k}{1 + k} \right) \right\}^2 \\ &+ \left\{ (1 - k) \operatorname{sn}\left((1 + k)v, \frac{1 - k}{1 + k} \right) \right\}^2. \end{aligned} \tag{5.4.31}$$

We continue by determining the separation constant m. We observe that the function φ must be a single-valued function of the coordinates x and y. However, the coordinates x and y are clearly periodic functions of ψ, with period 2π, as can be seen from (5.3.22). Hence, to ensure that the solution φ of the Laplace equation is a single-valued function, we must require that the function $e^{im\psi}$ is periodic with period 2π. This requirement translates into the restriction $m \in \mathbb{Z}$. Without loss of generality we may restrict ourselves to non-negative values of m.

Now we are equipped to write down the ordinary differential equations of the second order for functions M and N. Inserting (5.4.31) into (5.3.14) and (5.3.15), we obtain

$$M''(u) + \left(m^2 - \frac{1}{4}\right)\left(\frac{1-k}{1+k}\right)^2 \operatorname{sn}^2\left(i(1+k)(u-K), \frac{1-k}{1+k}\right) M(u) \\ = -h(1+k)^2 M(u), \tag{5.4.32}$$

$$N''(v) - \left(m^2 - \frac{1}{4}\right)\left(\frac{1-k}{1+k}\right)^2 \operatorname{sn}^2\left((1-k)v, \frac{1-k}{1+k}\right) N(v) \\ = h(1+k)^2 N(v). \tag{5.4.33}$$

We now analyze equations (5.4.32) and (5.4.33). The plan is to impose special boundary conditions on functions $M(u)$ and $N(v)$, which will constitute a Sturm-Liouville problem for equations (5.4.32) and (5.4.33). This will allow us to determine the separation constant h, and the corresponding eigenfunctions.

Following Edélyi [4, pp. 46-47, p. 53]), we remark the following. The endpoints of the intervals for u and v represent ∞ and the degenerate surfaces of our coordinate system. The degenerate surfaces act as branch-cuts, and the requirement of continuity of a potential across these branch-cuts translates into the boundary conditions.

For a potential regular inside or outside a surface $u = \text{const}$, we map the half-plane $\rho > 0$ onto the rectangle with vertices at $(0, \pm K')$, $(K, \pm K')$ in the (u, v)-plane. Note that $v = -K'$ and $v = K'$ are both images of the $z = 0$, $\rho > k^{-1}$. It now follows that $N(v)$ must be a periodic solution of the differential equation (5.4.33), with period $2K'$. This condition determines the characteristic values of h, and the corresponding characteristic function $N(v)$. As the external field is assumed to be at least continuous, an analog of Theorem 4.8 in [65, p. 55] shows that the equilibrium potential will be continuous on the ring. For a potential continuous inside the surface $u = \text{const}$, the continuity condition across $u = K$ ($z = 0$, $a < \rho < b$) dictates that $M(u)$ at $u = K$ and $N(v)$ at $v = 0$ have the same parity. For a potential regular outside the surface $u = \text{const}$, $M(u)$ must remain finite at $u = 0$.

Under these boundary conditions, equations (5.4.32) and (5.4.33) possess solutions, called periodic Lamé functions. Under the assumed boundary conditions, equation (5.4.33) will have two types of solutions, an even and an odd. Only the even functions will be required, since they correspond to a potential regular inside the surface $u = K$. We will follow the standard notation introduced in [4, p. 64], and denote these even solutions by $\mathrm{Ec}_m^{2n}(z, k^2)$, and the value of h belonging to $\mathrm{Ec}_m^{n}(z, k^2)$, will be denoted by $a_m^{2n}(k^2)$ [4, p. 64]. Therefore, the unique solution of Sturm-Liouville problem for equation (5.4.33) exists if and only if $h = a_{m-1/2}^{2n}(k^2)$, and has the form

$$N(v) = \mathrm{Ec}_{m-1/2}^{2n}((1-k)v, k^2), \quad -K' \le v \le K', \quad n = 0, 1, 2, \ldots \tag{5.4.34}$$

Setting $h = a_{m-1/2}^{2n}(k^2)$ in equation (5.4.32), we deduce that the only solution of that equation is

$$M(u) = \mathrm{Ec}_{m-1/2}^{2n}(i(1+k)(u-K), k^2), \quad 0 \le u \le K, \quad n = 0, 1, 2, \ldots \tag{5.4.35}$$

From the Sturm-Liouville theory it follows that the periodic Lamé functions, corresponding to different characteristic values, are orthogonal in $L^2((a, b), w)$, where $[a, b] = [0, K]$ or $[a, b] = [-K', K']$, with positive weight w, given by

$$w = (1+k)^2 \tag{5.4.36}$$

In particular, for $l, n = 0, 1, 2, \ldots$,

$$\int_{-K'}^{K'} \mathrm{Ec}_{m-1/2}^{2l}((1-k)v, k^2)\, \mathrm{Ec}_{m-1/2}^{2n}((1-k)v, k^2)\, (1+k)^2 \, dv = \delta_{ln}. \tag{5.4.37}$$

A detailed account on periodic Lamé functions is given in [4, pp. 63–75].

Taking into account the axial symmetry of the potential, we arrive at the following expression for the equilibrium potential U^{μ_Q},

$$U^{\mu_Q}(u,v) = \rho^{-1/2} \sum_{n=0}^{\infty} a_n \, \mathrm{Ec}^{2n}_{-1/2}(i(1+k)(u-K),k^2)\, \mathrm{Ec}^{2n}_{-1/2}((1-k)v,k^2), \quad (5.4.38)$$

where the convergence of the series in (5.4.38) is understood in the sense of L^2_w. The coefficients a_n are determined from the boundary values of the potential U^{μ_Q}, given in (5.3.2).

■

Proof of Proposition 5.3.1. First observe that, as $U^{\mu_Q}(z,\rho) = U^{\mu_Q}(z(u,v),\rho(u,v))$, we have

$$\begin{aligned}
\frac{\partial U^{\mu_Q}}{\partial u} &= \frac{\partial U^{\mu_Q}}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial U^{\mu_Q}}{\partial \rho}\frac{\partial \rho}{\partial u},\\
\frac{\partial U^{\mu_Q}}{\partial v} &= \frac{\partial U^{\mu_Q}}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial U^{\mu_Q}}{\partial \rho}\frac{\partial \rho}{\partial v}.
\end{aligned}$$

This can be written in matrix form as

$$\begin{pmatrix} (U^{\mu_Q})_u \\ (U^{\mu_Q})_v \end{pmatrix} = \begin{pmatrix} z_u & \rho_u \\ z_v & \rho_v \end{pmatrix} \begin{pmatrix} (U^{\mu_Q})_z \\ (U^{\mu_Q})_\rho \end{pmatrix}.$$

Inverting, we obtain

$$\begin{pmatrix} (U^{\mu_Q})_z \\ (U^{\mu_Q})_\rho \end{pmatrix} = \frac{1}{z_u\rho_v - z_v\rho_u} \begin{pmatrix} \rho_v & -\rho_u \\ -z_v & z_u \end{pmatrix} \begin{pmatrix} (U^{\mu_Q})_u \\ (U^{\mu_Q})_v \end{pmatrix}.$$

Hence,

$$(U^{\mu_Q})_z = \frac{(U^{\mu_Q})_u\rho_v - (U^{\mu_Q})_v\rho_u}{z_u\rho_v - z_v\rho_u}.$$

From (5.3.10) it follows that $z_u\rho_v - z_v\rho_u = \rho_u^2 + \rho_v^2 = |f'|^2$, so that

$$\frac{\partial U^{\mu_Q}}{\partial z} = \frac{1}{|f'|^2}\left(\frac{\partial U^{\mu_Q}}{\partial u}\frac{\partial \rho}{\partial v} - \frac{\partial U^{\mu_Q}}{\partial v}\frac{\partial \rho}{\partial u}\right)$$

Note that $|f'| \neq 0$, as the mapping f is conformal.

The surface of the ring in the coordinates (u, v) is given via setting $u = K$. Since

$$\rho(u,v) = \frac{a\,\mathrm{sn}(u,k)\,\mathrm{dn}(v,k')}{1-\mathrm{dn}^2(u,k)\,\mathrm{sn}^2(v,k')},$$

we easily find, taking into account that $\mathrm{cn}(K,k) = 0$,

$$\left(\frac{\partial \rho}{\partial u}\right)_{u=K} = 0.$$

Therefore,

$$\begin{aligned}
\left(\frac{\partial U^{\mu_Q}}{\partial z}\right)_{z=0} &= \left\{\frac{1}{|f'|^2}\left(\frac{\partial U^{\mu_Q}}{\partial u}\frac{\partial \rho}{\partial v} - \frac{\partial U^{\mu_Q}}{\partial v}\frac{\partial \rho}{\partial u}\right)\right\}_{u=K} \\
&= \left\{\frac{1}{\rho_u^2+\rho_v^2}\left(\frac{\partial U^{\mu_Q}}{\partial u}\frac{\partial \rho}{\partial v} - \frac{\partial U^{\mu_Q}}{\partial v}\frac{\partial \rho}{\partial u}\right)\right\}_{u=K} \\
&= \left\{\frac{\partial U^{\mu_Q}}{\partial u}\left(\frac{\partial \rho}{\partial v}\right)^{-1}\right\}_{u=K}.
\end{aligned}$$

The formula for the density f in coordinates (u, v) has the form

$$f(v) = \frac{1}{2\pi}\left\{\frac{\partial U^{\mu_Q}}{\partial u}\left(\frac{\partial \rho}{\partial v}\right)^{-1}\right\}_{u=K}, \quad -K' \le v \le K'. \tag{5.4.39}$$

We continue by evaluating ρ_v,

$$\begin{aligned}
\frac{\partial \rho}{\partial v} &= -k'^2 a\frac{\mathrm{sn}(u,k)\,\mathrm{sn}(v,k')\,\mathrm{cn}(v,k')}{1-\mathrm{dn}^2(u,k)\,\mathrm{sn}^2(v,k')} \\
&\quad + 2a\frac{\mathrm{sn}(u,k)\,\mathrm{dn}^2(u,k)\,\mathrm{sn}(v,k')\,\mathrm{cn}(v,k')\mathrm{dn}^2(v,k')}{(1-\mathrm{dn}^2(u,k)\,\mathrm{sn}^2(v,k'))^2}.
\end{aligned}$$

Taking into account that $\mathrm{sn}(K,k) = 1$, $\mathrm{dn}(K,k) = k'$, and $\mathrm{dn}^2(v,k') = 1-k'^2\mathrm{sn}^2(v,k')$, we have

$$\begin{aligned}
\left(\frac{\partial \rho}{\partial v}\right)_{u=K} &= -k'^2 a\frac{\mathrm{sn}(v,k')\,\mathrm{cn}(v,k')}{1-k'^2\,\mathrm{sn}^2(v,k')} + 2k'^2 a\frac{\mathrm{sn}(v,k')\,\mathrm{cn}(v,k')\mathrm{dn}^2(v,k')}{(1-k'^2\,\mathrm{sn}^2(v,k'))^2} \\
&= -k'^2 a\frac{\mathrm{sn}(v,k')\,\mathrm{cn}(v,k')}{\mathrm{dn}^2(v,k')} + 2k'^2 a\frac{\mathrm{sn}(v,k')\,\mathrm{cn}(v,k')\mathrm{dn}^2(v,k')}{\mathrm{dn}^4(v,k')} \\
&= -k'^2 a\frac{\mathrm{sn}(v,k')\,\mathrm{cn}(v,k')}{\mathrm{dn}^2(v,k')} + 2k'^2 a\frac{\mathrm{sn}(v,k')\,\mathrm{cn}(v,k')}{\mathrm{dn}^2(v,k')} \\
&= k'^2 a\frac{\mathrm{sn}(v,k')\,\mathrm{cn}(v,k')}{\mathrm{dn}^2(v,k')}
\end{aligned}$$

Thus,

$$\left(\frac{\partial \rho}{\partial v}\right)_{u=K} = k'^2 a\frac{\mathrm{sn}(v,k')\,\mathrm{cn}(v,k')}{\mathrm{dn}^2(v,k')}. \tag{5.4.40}$$

Let

$$g(u,v) = \sum_{n=0}^{\infty} a_n \operatorname{Ec}^{2n}_{-1/2}(i(1+k)(u-K), k^2) \operatorname{Ec}^{2n}_{-1/2}((1-k)v, k^2), \tag{5.4.41}$$

with a_n defined through the external field Q, as explained above. The equilibrium potential U^{μ_Q} can now be written as

$$\begin{aligned} U^{\mu_Q}(u,v) &= \rho(u,v)^{-1/2}\, g(u,v) \\ &= \frac{1}{\sqrt{a}} \left(\frac{1 - \operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')}{\operatorname{sn}(u,k)\operatorname{dn}(v,k')} \right)^{1/2} g(u,v). \end{aligned}$$

Furthermore,

$$\begin{aligned} \frac{\partial U^{\mu_Q}}{\partial u} &= \frac{1}{\sqrt{a}}\frac{1}{2} \left(\frac{1 - \operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')}{\operatorname{sn}(u,k)\operatorname{dn}(v,k')} \right)^{-1/2} g(u,v) \\ &\quad \left\{ \frac{-\operatorname{sn}^2(v,k')\, 2\operatorname{dn}(u,k)(-k^2)\operatorname{cn}(u,k)\operatorname{sn}(u,k)}{\operatorname{sn}(u,k)\operatorname{dn}(v,k')} \right. \\ &\quad \left. - \frac{1 - \operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')}{\operatorname{sn}^2(u,k)\operatorname{dn}(v,k')} 2\operatorname{sn}(u,k)\operatorname{cn}(u,k)\operatorname{dn}(u,k) \right\} \\ &\quad + \frac{1}{\sqrt{a}} \left(\frac{1 - \operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')}{\operatorname{sn}(u,k)\operatorname{dn}(v,k')} \right)^{1/2} \frac{\partial g(u,v)}{\partial u}. \end{aligned}$$

Knowing that $\operatorname{cn}(K,k) = 0$, we find

$$\begin{aligned} \left(\frac{\partial U^{\mu_Q}}{\partial u} \right)_{u=K} &= \frac{1}{\sqrt{a}} \left(\frac{1 - k'^2 \operatorname{sn}^2(v,k')}{\operatorname{dn}(v,k')} \right)^{1/2} \left(\frac{\partial g(u,v)}{\partial u} \right)_{u=K} \\ &= \frac{1}{\sqrt{a}} \left(\frac{\operatorname{dn}^2(v,k')}{\operatorname{dn}(v,k')} \right)^{1/2} \left(\frac{\partial g(u,v)}{\partial u} \right)_{u=K} \\ &= \frac{1}{\sqrt{a}} \sqrt{\operatorname{dn}(v,k')} \left(\frac{\partial g(u,v)}{\partial u} \right)_{u=K}. \end{aligned}$$

Thus

$$\left(\frac{\partial U^{\mu_Q}}{\partial u} \right)_{u=K} = \frac{1}{\sqrt{a}} \sqrt{\operatorname{dn}(v,k')} \left(\frac{\partial g(u,v)}{\partial u} \right)_{u=K}. \tag{5.4.42}$$

Finally, inserting (5.4.40) and (5.4.42) into (5.4.39), we obtain

$$\begin{aligned} f(v) &= -\frac{1}{2\pi} \left\{ \frac{\partial U^{\mu_Q}}{\partial u} \left(\frac{\partial \rho}{\partial v} \right)^{-1} \right\}_{u=K} \\ &= -\frac{1}{2\pi} \frac{1}{k'^2 a} \frac{\operatorname{dn}^2(v,k')}{\operatorname{sn}(v,k')\operatorname{cn}(v,k')} \frac{1}{\sqrt{a}} \sqrt{\operatorname{dn}(v,k')} \left(\frac{\partial g(u,v)}{\partial u} \right)_{u=K} \\ &= -\frac{1}{2\pi} \frac{1}{k'^2 a^{3/2}} \frac{\operatorname{dn}^{3/2}(v,k')}{\operatorname{sn}(v,k')\operatorname{cn}(v,k')} \left(\frac{\partial g(u,v)}{\partial u} \right)_{u=K}. \end{aligned}$$

Hence the expression for the density takes the form

$$f(v) = \frac{1}{2\pi}\frac{1}{k'^2a^{3/2}}\frac{\mathrm{dn}^{3/2}(v,k')}{\mathrm{sn}(v,k')\,\mathrm{cn}(v,k')}\left(\frac{\partial g(u,v)}{\partial u}\right)_{u=K}, \quad -K' \le v \le K'. \tag{5.4.43}$$

Note that $f(v)$ becomes infinite on the rims of the ring, for having the factors $\mathrm{sn}(v,k')$ and $\mathrm{cn}(v,k')$ in the denominator in (5.4.43). Indeed, this follows from the fact that the rims of the ring correspond to $v = 0$ and $v = K'$, while $\mathrm{sn}(0,k') = 0$ and $\mathrm{cn}(K',k') = 0$. Furthermore, since $r(v) := \rho(K,v) = a/\mathrm{dn}(v,k')$, it follows that $\mathrm{dn}(v,k') = a/r(v)$. A brief calculation reveals that

$$\mathrm{sn}(v,k') = \frac{1}{k'\,r(v)}\sqrt{r^2(v)-a^2},$$

$$\mathrm{cn}(v,k') = \frac{a}{k'b\,r(v)}\sqrt{b^2-r^2(v)}.$$

Hence,

$$f(v) = \frac{b}{2\pi a}\frac{\sqrt{r(v)}}{\sqrt{b^2-r^2(v)}\sqrt{r^2(v)-a^2}}\left(\frac{\partial g(u,v)}{\partial u}\right)_{u=K}, \quad -K' \le v \le K'. \tag{5.4.44}$$

■

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VITA

Mykhailo Bilogliadov

Candidate for the Degree of

Doctor of Philosophy

Dissertation: EQUILIBRIUM PROBLEMS IN POTENTIAL THEORY

Major Field: Mathematics

Biographical:

Education:

Completed the requirements for the Doctor of Philosophy in Mathematics at Oklahoma State University, Stillwater, Oklahoma in December, 2016.

Completed the requirements for the Master of Arts in Mathematics at Indiana University, Bloomington, Indiana in 2010.

Completed the requirements for the Master of Science in Radiophysics and Electronics at Donetsk National University, Donetsk, Ukraine in 2005.

Completed the requirements for the Bachelor of Science in Applied Physics at Donetsk National University, Donetsk, Ukraine in 2004.