# CAPACITY, FLEXIBILITY AND PRICING DECISIONS FOR SUBSTITUTABLE PRODUCTS UNDER DEMAND AND SUPPLY UNCERTAINTIES

By

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*Dedicated to Dr. Tieming Liu*

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# CHAPTER 1

# INTRODUCTION

## 1.1 Motivation

In today's competitive economy, satisfying customer needs with the right product at the right time at a competitive price is key to orchestrating a successful supply chain. Customers' needs and tastes evolve and vary widely posing forecasting challenges to firms in making effective resource[1] investment decisions. In addition, exogenous factors such as raw-material or inventory availability, plant disruptions etc. add additional complexity to the firm's decision making process.

Firms typically face three types of uncertainties: demand uncertainties, resource capacity uncertainties and supply disruptions. Uncertainties in demand is typically caused due to imperfect demand forecasts or forecasting with insufficient information (Chod and Rudi 2005). Jordan and Graves (1995) note a strategic average forecast error of 40% in the automotive industry. When Mercedes-Benz first introduced its M-class cars in 1997, it forecasted its annual demand to be about 65,000 vehicles. This forecast was, in fact, too low and the capacity was expanded to 80,000 vehicles during 1998-1999, which was still supposedly insufficient to meet the actual demand (Van Mieghem 2007). Peidro et al. (2009) provide a comprehensive summary of literature related to supply chain planning methods under uncertainties.

[1] We use the term "resource" in its broader sense to include units of manufacturing capacity, inventory of products (raw materials, finished goods), hiring/training workers etc. In this context, resource capacity is measured in terms of output production units such as number of cars or number of computer memory chips in a given time horizon.

Temporary production line shutdowns, machine breakdowns or faulty manufacturing processes that result in lower than expected output can result in capacity uncertainties. In the first-quarter of 2004 IBM reported a \$150 million loss in its electronic division due to yield problems at its plant in New York (Krazit 2004). This reduced capacity created a direct mismatch between supply and demand. Hendricks and Singhal (2005) report that such uncertainties have gained momentum in recent years due to their high impacts on profits and shareholder values.

Supply disruptions, caused by natural or man-made disasters, may leave firms crippled due to complete unavailability of resources. The Taiwan earthquake in 1999 caused an industry-wide shortage of memory for personal computers (Tomlin 2006). Other events, such as the Japanese earthquake (2011), the Thailand flood (2011), the west coast port labor strike (2008, 2015), and the 9/11 attacks, also caused severe shortages of supplies. Sheffi (2005) provides comprehensive real-world examples and offers a detailed review of the impact of such disruptions.

Resource flexibility investment and responsive product pricing are two commonly employed hedging strategies against demand uncertainties (Chod and Rudi 2005). *Resource flexibility* entails a firm to invest in a flexible resource that is capable of producing multiple products. As opposed to product-specific or dedicated resources, flexible resources allow capacities to be allocated later between products. With *responsive pricing* a firm can influence the demand by setting prices according to the actual demand conditions and available resource capacity.

While these two strategies are known to mitigate uncertainties in demand (Lus and Muriel 2009), their effectiveness on capacity uncertainties and supply disruptions have not been explored in detail. For example, does responsive pricing mitigate capacity loss? Is investment in flexibility a good hedging strategy when one resource could be completely unavailable in the production stage? How does product substitutability affect capacity investment and profit under supply disruptions? If the demand for the products are correlated, does the investment portfolio change with the type of correlation?

In this research we investigate the capacity, flexibility and pricing decisions of a price-setting firm facing the three different types of uncertainties described above. The firm produces and sells two products that could vary in the degree of their substitutability. In the planning stage (Stage I), under demand uncertainty, the firm decides on the optimal capacity and flexibility levels. In the selling stage (Stage II), once the market potentials for these two products are revealed, the firm decides on optimal production quantities (and hence sets prices) contingent on the capacity and flexibility level investments in Stage I. Two types of uncertainties are considered in Stage II: (1) capacity uncertainty, where only a proportion of the invested capacities in Stage I is available in Stage II, and (2) 0-1 type supply disruptions where plants or machines may be shut down completely and a resource maybe completely unavailable in Stage II. Figure 1.1 shows the timeline of the decision making process.

Gaps also exist between academic studies and industry practice in resource flexibility planning. The *product-mix* flexibility literature (e.g., Fine and Freund 1990) typically assumes that resources are either dedicated or flexible. A dedicated resource can only produce a particular product while a flexible resource can produce multiple products without affecting the output. The *process flexibility* literature (e.g., Jordan and Graves 1995) considers resources with partial (or limited) flexibility. Their notion of partial flexibility is that multiple plants can produce a subset of the products but not all of them. In this research, a completely flexible resource can produce multiple products without any efficiency loss. A partially flexible resource, however, incurs efficiency loss if it produces a product it was not designed for.

In practice very few resources are completely flexible. When a resource designed for one type of product is used to produce another type productivity may decrease resulting in shrunk capacities for the other product. Machine set-ups, prod-

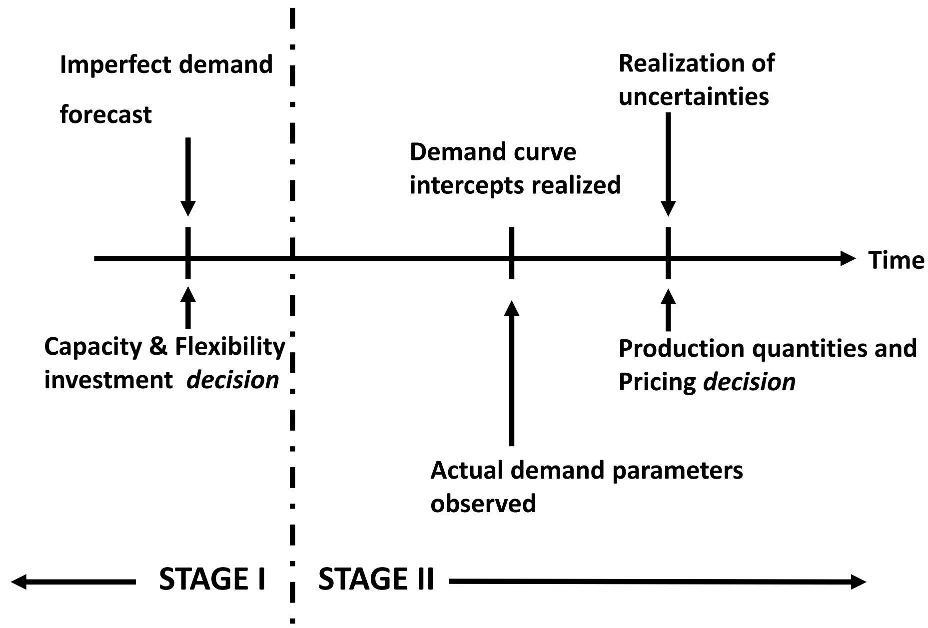


Figure 1.1: Sequence of events

uct changeovers and tooling reconfigurations are some examples where efficiency loss prevails and the process cycle time increases (Coste and Malhotra 1999). Boyer and Leong (1996) defined "changeover costs as the percentage reduction in the available capacity of a manufacturing plant". Hence it is imperative to take efficiency losses into consideration while modeling investment strategies utilizing flexible resources.

*Cross-production*, a term we coined to represent a resource flexibility management strategy, focuses on switching a specialized resource capacity to produce another type of product, with certain efficiency loss incurred. For example, consider a firm producing two products with two types of resources, and each resource is designed for a single product. Because the two products share common features, one production facility could be reconfigured to produce the second product, but this will still incur a certain degree of efficiency loss. Cross-production provides flexibility to manufacturers on the supply side. The lack of cross production abilities may seriously decrease a firm's capacity utilization, profitability, and market share. In 2000, Chrysler put out its PT Cruiser, a new design based on the Neon, and it turned out to be very successful soon after its debut. Its dedicated plant in Toluca, Mexico, however, was not able to satisfy the unexpected high demand. At the same time, the dedicated plant for the original Neon in Belvidere, Illinois, was under-utilized. It has been estimated that Chrysler lost up to $240M pre-tax profit, due to this lack of production flexibility in the first three years (Biller et al. 2006).

Many manufacturers prefer to choose the degree of resource flexibility at the capacity investment stage. An investment in higher degree of flexibility in the planning stage usually leads to lower efficiency loss in the production stage. However, the higher degree of flexibility, the more expensive is the investment. Thus a moderate degree of flexibility that hedges against capacity and demand uncertainties to an acceptable extent might be desirable. A recent industry example of this scenario is of Intel Corp. which announced converting its 22 nm facilities in Chandler, AZ to also produce the next generation 14 nm chips used in processors (Randewich 2014). In fact, the company has postponed opening its new chip factory originally scheduled for 2013 and refocused its efforts on "better capital utilization". Another practical example would be when a firm has to decide how many workers to hire and whether or not to cross-train workers across skill sets, and if yes, to what extent.

### 1.2 Problem Statement

For a firm facing demand uncertainties, literature has established that resource flexibility and responsive product pricing are excellent risk hedging strategies. However, the acquisition cost of a completely flexible resource is usually extremely high to justify the investment (Chou et al. 2010). Very few resources are completely flexible giving rise to efficiency losses during the production stage. Since a majority of the capacity investment decisions made in the planning stage are irreversible, it is imperative to understand the economic impact of these efficiency losses.

Literature has also typically assumed that the capacities invested in planning stage are completely available in the selling stage. In many cases yield problems or supply disruptions cause only a fraction of the invested capacities to be available for production. It then becomes important to investigate the effectiveness of resource flexibility and responsive pricing under different types of capacity uncertainties.

Strategic capacity and flexibility investment decisions feed into tactical and operational decisions affecting multiple periods. Investment in hedging strategies such as the desired degree of flexibility is intimately influenced by factors such as level of product substitutability and correlation of product demands. Further, accounting for the type as well as degree of uncertainty faced by the firm is crucial in making effective resource investment decisions. Hence, a careful and thorough understanding of the various tradeoffs in investment is necessary to make firm-level strategic decisions.

## 1.3 Research Objectives

The objective of this research is to contribute to the operations management knowledge-base on effective resource investment and product pricing decisions under cross-production efficiency losses where firms face demand as well as supply uncertainties.

We study the capacity investment decisions of a firm investing in partially flexible resources to produce two substitutable products under demand uncertainty in the planning stage. Many firms consider retro-fitting their existing dedicated capacities to produce multiple products rather than investing in a separate flexible resource. We model the efficiency loss incurred when switching a resource for one product to produce another product. In the production stage, the firm then sets prices based on the realized demand contingent on the capacity and flexibility investment decisions made earlier. The pricing decision is also influenced by reduced capacity availability due to uncertainties in the yield or complete unavailability of resources due to disruptions.

Using linear demand models that capture the relationship between the demand of a product, its own price as well as price of a substitutable product, we show that investment in complete flexibility is seldom required. We outline conditions under which a moderate degree of flexibility hedges the firm against demand uncertainties, capacity uncertainties, and supply disruptions. These conditions include the degree and type of uncertainty faced by the firm as well as product demand variability, correlation and nature of product substitutability.

The specific objectives of this research are as follows.

**Objective 1**: To explicitly model **efficiency losses** when a product-dedicated resource is switched to produce a different product. We model two different types of efficiency losses: In the Shrinking Capacity model, fewer units of products will be produced if a resource originally specialized for one type of product is switched to produce another type (i.e., cross-produced). In the Additional Cost model, the firm incurs an increase in unit production cost during cross-production. The firm can improve the flexibility of the resource in the planning stage thus lowering this efficiency loss during the production stage. However, flexibility is typically very *expensive* and hence it is imperative to study conditions under which flexibility is desired and if so, how much (investment level). The Shrinking Capacity and Additional Cost models investigate the need for *partially* flexible resources.

**Objective 2**: To model **uncertainties in product demand** in the capacity investment stage. Many investment decisions are made much before the selling season hence it is important to consider demand uncertainties when making capacity and flexibility investment decisions. All the three models studied in this dissertation consider demand uncertainties through the randomness of the demand intercepts in the price-demand relationship. In the planning stage, when only the mean and standard deviation of the product demand intercepts are available, the firm decides on the optimal capacity and flexibility levels. In the selling stage the firm observes the realization of these random variables (a.k.a. market potentials) and subsequently makes the production and pricing decisions. We study the effect of increasing demand uncertainties by increasing the CV of the demand distribution of the intercepts of each product.

**Objective 3**: To extend the efficiency loss models studied under Objectives 1 and 2 to include **uncertainties in capacity and supply disruptions**. The pricing decision in the selling stage is not only affected by *invested* capacities and realized demands, but is also affected by the *realized* capacities. Hence in addition to demand uncertainties, we study two more types of uncertainties viz., capacity uncertainties and 0-1 type disruptions in the selling stage. Capacity uncertainties indicate that only a proportion of the invested capacities are available during the production stage and there is always an average amount of capacity for production. We use Normally distributed random variables in conjunction with installed capacities of the two products to model the variability in the available capacity in selling stage. To study supply disruptions that indicate complete unavailability of one or both resources, we use Bernoulli RVs to model scenarios where the resource is available or not. We increase the CV of these distributions to study the impact of increasing capacity uncertainties and disruptions.

**Objective 4**: To incorporate the effect of **product demand correlation** and **degree of substitutability** between the two products produced by flexible resources. We numerically investigate the impact of correlation between product demand intercepts in both the Shrinking Capacity as well as Additional Cost models. Specifically, for a given level of product substitutability we generate demand scenarios from a multivariate normal distribution whose means are correlated. We vary this level of correlation to study the impact of increasing correlation of demand intercepts.

These objectives lend themselves to be addressed through a two-stage stochastic programming framework. Determining how the optimal capacity and flexibility levels change with exogenous demand and supply parameters is the focus of this research.

The following section presents an overview of the three models in this dissertation that help achieve these objectives.

## 1.4 Dissertation Overview

We study the resource investment and pricing decisions for a profit-maximizing firm producing two substitutable products under three different settings as outlined below:

### 1.4.1 The Contingent Flexible Capacity Model

We investigate the pricing and capacity investment decisions for a monopolistic firm that faces stochastic price-dependent demand for two substitutable products. The firm produces the two products with two dedicated resources and a *separate* contingent flexible resource. The firm makes capacity and pricing decisions in two stages: In Stage I the firm decides the capacity portfolio of dedicated and flexible resources; And in Stage II, after demand information is revealed, the firm decides the production quantities and prices for the products. We examine two demand models, the so-called $\gamma$- and the $b$-demand models, and we give out the closed-form solutions under both models. Both $\gamma$ and $b$ represent the degree of product substitutability but the functional form representing the price-demand relationship under these models are different. Our analysis shows that the $b$-demand model, which is supported by a utility function, better represents the product substitution relationship than the $\gamma$-demand model. Further, using the $b$-demand model, we show that responsive pricing is a more cost-effective strategy than utilizing the contingent flexible capacity.

The model with contingent capacity considers a separate flexible resource. The next two models consider the notion of cross-production where a specialized resource is converted to produce another type of product and hence incurs efficiency losses. They utilize the $b$-demand model that accurately captures the effect of product substitutability.

### 1.4.2 The Shrinking Capacity Model

This model examines the interplay between the cost of investing in flexibility, the efficiency loss due to cross-production as well as the responsive pricing for substitutable products. We consider a firm producing two products with two partially flexible resources and facing three types of uncertainties separately: demand uncertainties, capacity uncertainties, and supply disruptions. The firm can choose the level of resource flexibility in the investment stage. The higher flexibility level, the less efficiency loss will be incurred when switching a resource for one product to produce another product.

We use *shrinking capacity* to study the efficiency loss that occurs when switching a resource for one product to produce another product. Shrinking capacity explicitly captures the fact that fewer units of products will be produced if a resource originally specialized for one type of product is switched to produce another type. In the resource planning stage, the firm determines the degree of flexibility (the percentage of shrinking capacity) in addition to capacity. In the production stage, after the resource capacities and the market potentials are realized, the firm makes capacity allocation and pricing decisions to maximize its profits. In this model if the firm does not invest in any degree of flexibility in Stage I it will not be able to cross-produce in Stage II.

We find that product substitutability, type and severity of the uncertainty play a key role in deciding the optimal investment strategies under cross-production. The ability to use responsive pricing in conjunction with selling substitutable products largely mitigates the need for any flexibility investment in many cases. However, under severe supply disruptions flexibility proves to be extremely beneficial.

### 1.4.3 The Additional Cost Model

Efficiency losses may not be always related to number of units produced during cross-production. In many cases, producing a product using a non-dedicated resource may increase the unit production cost rather than reducing output. For example, firms train workers and this requires extra hours that incurs wages. It is also costly to convert one product to another (e.g. change the color, or downward conversion) as it requires additional effort that directly translates to cost.

We use the *additional cost* model to capture this type of efficiency loss. A higher investment in flexibility in Stage I incurs a lower additional cost in Stage II indicative of lower efficiency loss. In this model, even if the firm does not invest in any flexibility in Stage I it can still cross-produce in Stage II by incurring a higher additional cost. The effect of demand uncertainties, capacity uncertainties and supply disruptions are also investigated.

We find that the firm generally does not invest in flexibility under all three uncertainties. Even under a very high degree of capacity uncertainties or supply disruptions, the firm prefers to simply invest in higher amounts of cheaper dedicated capacities and use cross-production rather than invest in flexibility. This result is true under any level of demand correlation or degree of product substitutability. When cost of dedicated capacities and/or cross-production increases, the firm may still invest in flexibility under specific conditions.

Next, we summarize our key contributions.

## 1.5 Contribution

The contribution of this dissertation to the theory and practice of operations management is stated below:

- First, we explicitly model two different forms of efficiency loss that occurs during cross-production.

- Second, contrary to reviewed literature, we show that flexibility is unnecessary when facing *low* or *moderate* demand and capacity uncertainties. The firm can typically mitigate these uncertainties through responsive product pricing.

- Third, even when facing *high* demand uncertainties, we show that depending on the *type* of efficiency loss, only a *limited* amount of flexibility maybe necessary to mitigate over/under investment costs. Selling substitutable products decreases the need for investing in costly flexibility as demand can be better managed through responsive pricing.

- Fourth, we show that when facing a *high* degree of 0-1 type supply disruptions, flexibility is extremely valuable under *any* demand intercept correlation even for highly substitutable products. This result also depends on the type of efficiency loss that is faced by the firm as well as relative cost of flexibility.

The rest of the document is structured as follows. Chapter 2 presents a review of literature on resource flexibility planning, responsive product pricing as well as impact of disruptions and capacity uncertainties. Chapter 3 contains the Contingent Flexible Capacity model, its analytical solutions and insights. We then investigate the impact of efficiency losses, demand uncertainties, capacity uncertainties and supply disruptions through the Shrinking Capacity model in Chapter 4 and the Additional Cost model in Chapter 5. Chapter 6 summarizes the dissertation and concludes with the scope for future research.

# CHAPTER 2

## LITERATURE REVIEW

Two streams of literature are relevant to this research: flexibility planning and responsive pricing. Due to the large volume of articles in each of these areas we narrow our focus to papers that are closely related to our study.

Investment trade-offs between flexible and dedicated resources to mitigate demand uncertainties have been well studied in literature. Fine and Freund (1990) study the capacity investment decisions for a monopolistic firm producing $n$ products with both flexible and dedicated resources. They use a set of discrete demand-price scenarios, and conclude that the flexible capacity has no value with perfectly positively correlated demands. Van Mieghem (1998) shows that, however, such a conclusion may not hold when the two products have exogenous, different prices. Bish and Wang (2004) investigate the capacity investment decisions with responsive pricing for independent products with both dedicated and fully flexible resources. They show that the flexible resource investment decision follows a threshold policy. Chod and Rudi (2005) model the capacity and pricing decisions for a firm that produces two substitutable products with only one flexible resource. Bish et al. (2009) study the optimal capacity and pricing decisions under additive and multiplicative demand uncertainties for a monopolist investing in a single flexible capacity. These studies focus on the nature of demand uncertainty while supply risk or partial flexibility is not considered. Tomlin and Wang (2008) consider a flexible resource with uncertain yield to produce two *vertically* differentiated products (in contrast to aggregate linear demand models to model horizontally differentiated products as in our research) to investigate the impact of demand and yield uncertainties. Bish and Suwandechochai (2010) explore the impact of product substitution under demand uncertainty for a firm producing two substitutable products with a flexible resource. They find that the firm's cost structure and the degree of product substitution play a key role in determining whether flexibility and pricing are strategic substitutes or complements. Goyal and Netessine (2010) model the trade-off between volume flexibility and product flexibility and investigate the impact of demand correlation in each setting.

In the aforementioned papers, the firm is restricted to investing in a single completely flexible resource to produce both products. Another common assumption in these papers (except Tomlin and Wang 2008) is that the capacities invested in the planning stage are 100% available in the production stage. While demand uncertainties have typically been the dominant focus in the literature on flexibility and pricing, capacity uncertainties and supply disruptions have garnered very little attention. However, these uncertainties may cause capacity shortages in the production stage.

A paper more closely related to ours is Tomlin and Wang (2005), who study the value of mix flexibility and dual sourcing in a newsvendor network setting considering both risk neutral and risk averse firms. They model 0-1 supply disruptions as Bernoulli random variables and investigate the impact of several attributes such as demand correlation, contribution margin, number of products, resource reliabilities and risk tolerance that influence the preference of the network structure. Our research is different from Tomlin and Wang (2005) in that: (1) We study the degree of efficiency loss when switching a dedicated resource to produce a secondary product that differs in it's degree of substitutability with the primary product. Tomlin and Wang (2005) do not consider efficiency loss or product substitutability. (2) In our model, the firm is a price-setter i.e, after demand and capacity uncertainties are resolved the firm decides on both the production levels as well as selling prices for the products. In their model the firm allocates production but does not set prices in the second stage. (3) We model the impact of increasing demand uncertainties by varying the coefficient of variation of its distribution. However, demand variability is not the focus of their study. (4) Finally, one of our major conclusions that flexibility is highly valuable under increasing capacity uncertainties and supply disruptions is a stark contrast to their results on flexible networks. In their model, the preference for flexibility decreases as resource investments become less reliable.

The following three papers consider both dedicated and flexible resources, and the latter two consider capacity uncertainties or supply disruptions. Lus and Muriel (2009) study the optimal resource portfolio (two dedicated and one fully flexible resource) of a price-setting firm selling two substitutable products. They compare the impact of product substitutability and demand correlation for a price-setting firm selling substitutable products, and they show that utilizing the correct demand model (obtained from Singh and Vives 1984) leads to realistic results on optimal prices and profit. Capacity uncertainties or supply disruption risks are not considered.[1] Tomlin (2009) considers a firm that may employ any combination of three strategies: supplier diversification, contingent sourcing, and demand switching. Using a two-product newsvendor setting they model product substitutability by the fraction of customers switching to buy the substitutable product. The firm incurs a switching cost to induce a customer to purchase her non-preferred product. Bish and Chen (2008) study the optimal resource portfolio (again two dedicated and one fully flexible resource) of

[1]We note here that in a related dissertation, Lus (2008) develop analytical solutions for the two different demand models using a state space decomposition approach. Our work on the Contingent Flexible Capacity model is very similar to theirs but was independently developed before this dissertation became available to us through an external review. Our model includes closed form expressions (albeit under the "clearance" assumption) for the optimal capacities that do not rely on any distributional assumptions about product demands. This gives rise to a much finer representation of the state-space of the input-output vector compared to Lus (2008).

a price-setting firm selling two vertically differentiated products using a consumer choice model with both supply and demand uncertainties. They consider capacity disruptions (as modeled by Bernoulli random variables) for the multi-resource case and discuss the impact of yield uncertainties when the firm invests in only a single fully-flexible resource. Because of the complexity of their model the comparative statics on product substitution and demand uncertainty are restricted to the single-resource case.

Related to cross-production, Li et al. (2014) study a two-product two-capacity multi-period planning model in which a newer product replaces an older one at Intel's wafer fabrication, assembly and testing units. Capacity in the form of new production lines can be purchased in addition to converting existing production lines and in both cases "retro-fitted" (i.e., produce the older product). Accounting for inventory carry-over as well as back-orders, they compare an open loop system where both capacity and production decisions are made ex-ante with a closed loop system where production decisions are made ex-post. Due to the nature of the problem, they provide a heuristic solution for the multi-period closed loop system and compare the impact of holding cost and target service level changes. Li et al. (2010) use dynamic programming to model and solve an inventory planning problem for product transitions without replenishment. Allowing for product substitutions they model the optimal planning quantities for both old and new products under deterministic transition start date as well as a stochastic transition start date. In both these papers, the impact of product demand uncertainty, substitutability, correlation, responsive pricing, capacity uncertainties and supply disruptions are not considered.

Moreno and Terwiesch (2015) empirically study the impact of flexibility and endogenous pricing in the U.S. automotive industry. They use manufacturer provided incentives as discounts from the list price to account for the effect of responsive pricing. They find that flexibility is only beneficial to firms that operate under very high demand uncertainty, with highly differentiated (or complementary) products and limited competition. In these situations the flexible firm offers lower discounts and hence avoids substantially marking down the price. They do not study the impact of capacity uncertainties or plant shutdowns.

To summarize, our research differs from existing literature in the following ways:

- We model the investment in partially flexible resources in the planning stage by considering the efficiency loss that may occur when a resource for one product is used to produce other products (i.e., cross-production).
- We study (for the multi-resource case) the different impacts caused by demand uncertainties, capacity uncertainties and supply disruptions on the firm's optimal resource portfolio.
- We also investigate the role of product substitutability, resource investment cost and demand correlation in influencing the capacity, flexibility and pricing decisions.

Our model contributes to the literature on flexibility investment by simultaneously studying the interplay between the cost of flexibility investment and cross-production efficiency loss, as well as responsive pricing for substitutable products. While these effects have been studied in isolation (see for e.g., Bish and Suwandechochai 2010, Lus and Muriel 2009, Tang and Tomlin 2009), our research shows that accounting for them together significantly changes the optimal investment portfolio of a price-setting firm.

# CHAPTER 3

# CAPACITY, FLEXIBILITY AND PRICING DECISIONS: THE CONTINGENT FLEXIBLE CAPACITY MODEL

## 3.1 Introduction

Demand for a product is typically affected by many factors. Two of the most important ones are its own price as well as price of its substitute (Talluri and van Ryzin 2005). In order to make effective pricing decisions, this relationship must be accurately captured. In this chapter, we compare and contrast two commonly used (linear) forms of the price-demand relationship for substitutable products. Specifically, we use both the forms to model and solve a price-setting firm's capacity and flexibility investment problem and derive insights on which form is more appropriate and when.

Fine and Freund (1990) study the capacity investment decisions for a monopolistic firm producing $n$ products with both flexible and dedicated resources. They use a set of discrete demand-price scenarios, and conclude that the flexible capacity has no value with perfectly positively correlated demands. Van Mieghem (1998) shows that, however, such a conclusion may not hold when the two products have exogenous, different prices. Bish and Wang (2004) investigate the capacity investment decisions with responsive pricing for independent products. They show that the contingent flexible resource investment decision follows a threshold policy. Chod and Rudi (2005) model the capacity and pricing decisions for a firm that produces two substitutable products with only one flexible resource. They show that the optimal flexible capacity increases as the products become more substitutable.

The demand models used in the above papers all belong to the so-called $\gamma$-demand model whose functional form is as follows:

$$p_1 = \frac{\mathcal{E}_1 + \gamma \mathcal{E}_2}{1-\gamma^2} - \frac{Q_1 \alpha}{1-\gamma^2} - \frac{\gamma \alpha Q_2}{1-\gamma^2}, \qquad p_2 = \frac{\mathcal{E}_2 + \gamma \mathcal{E}_1}{1-\gamma^2} - \frac{Q_2 \alpha}{1-\gamma^2} - \frac{\gamma \alpha Q_1}{1-\gamma^2}$$

where $p_i$ is the price of product $i$, $\mathcal{E}_i$ is the market potential (also called demand intercept), $Q_i$ is the demand, $\alpha$ represents the demand sensitivity to its own price, and $\gamma \in [0, 1)$ is the degree of product substitutability (also called cross-pricing factor). $\boldsymbol{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2)^T$ are continuous random variables with positive support and finite expectation, and the joint probability density function of $\mathcal{E}_1$ and $\mathcal{E}_2$ is denoted by $\Psi(.,.)$.

The direct demand function is given by:

$$Q_1 = \frac{\mathcal{E}_1 - p_1 + \gamma p_2}{\alpha}, \qquad Q_2 = \frac{\mathcal{E}_2 - p_2 + \gamma p_1}{\alpha}$$

**Demand uncertainties:** $\mathcal{E}_1$ and $\mathcal{E}_2$ are RVs in Stage I (planning stage) and represent the uncertainties in market demand for the two products. At this time, the firm decides the optimal capacity investments so as to maximize its expected profit. In Stage II, uncertainty is resolved indicating that the realization of RV $\mathcal{E}_i$, denoted by $\epsilon_i$ for product $i$, is observed by the firm. The firm then maximizes its revenue through pricing and resource allocation decisions. Prices and production quantity decisions in Stage II are dependent on demand realizations $\epsilon_1$,$\epsilon_2$ as well as Stage I capacity levels $K_1, K_2, K_f$ and parameters such as degree of product substitutability and price sensitivity. The RVs in the inverse and direct demand functions stated above can then be replaced by their respective realizations in Stage II.

Given an output vector $\mathbf{Q}$, the profit vector is determined by the following function,

$$\frac{1}{1-\gamma^2} \begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix} (\boldsymbol{\mathcal{E}} - \alpha \mathbf{Q}) = \mathbf{D}(\boldsymbol{\mathcal{E}} - \alpha \mathbf{Q})$$

While the above linear demand model is widely adopted, and its parameters are easy to obtain through linear-regression techniques, it does come with limitations. An issue that needs justification is the use of $\gamma$ as the substitutability factor. Lus and Muriel (2009) point out that the use of the $\gamma$-demand model has led to unrealistic conclusions such as that optimal prices and profits increase unboundedly as the product substitutability $\gamma$ increases.

A different linear demand function, the so-called $b$-demand model in the current research[1], has recently been used by Goyal and Netessine (2007) and Lus and Muriel (2009) to model the demand and product substitutability relationship. The demand for a product is modeled as a downward sloping function of its own price with the effect of its substitute product. This demand model is derived by maximizing the utility function given by Singh and Vives (1984) as follows:

$$\max_{Q_1,Q_2} U(Q_1,Q_2) = A_1Q_1 + A_2Q_2 - (\vartheta_1 Q_1^2 + 2bQ_1Q_2 + \vartheta_2 Q_2^2)/2$$

where $b \in [0,1)$ is the measure of product substitutability or complementarity, Random Variables (RVs) $A_i$ are the market potentials, for $i = 1,2$, with $\vartheta_{3-i}A_i \geq bA_{3-i}$ (to enforce positive demand realizations). $\boldsymbol{A} = (A_1,\, A_2)$ are continuous RVs with positive support and the mean of their marginal distribution is denoted by $\mu_i$ for $i = 1,2$. Their joint distribution is denoted by $\Psi(A_1, A_2)$ . In Stage II the realizations of $A_1$, $A_2$ is denoted by $a_1$ and $a_2$, respectively.

By maximizing the utility function, we obtain the following linear demand function:

$$Q_1(\mathbf{p},\boldsymbol{a}) = \frac{\vartheta_2 a_1 - b a_2}{\vartheta_1\vartheta_2 - b^2} - \frac{\vartheta_2}{\vartheta_1\vartheta_2 - b^2}p_1 + \frac{b}{\vartheta_1\vartheta_2 - b^2}p_2 \tag{3.1}$$

$$Q_2(\mathbf{p},\boldsymbol{a}) = \frac{\vartheta_1 a_2 - b a_1}{\vartheta_1\vartheta_2 - b^2} - \frac{\vartheta_1}{\vartheta_1\vartheta_2 - b^2}p_2 + \frac{b}{\vartheta_1\vartheta_2 - b^2}p_1 \tag{3.2}$$

[1]We note here that Chapters 4 and 5 that model cross-production efficiency loss through the Shrinking Capacity and Additional Cost models use the $b$-demand model

where $p_i$ is the price for product $i$, where $i = 1, 2$. To simplify expressions in subsequent sections we assume parameters $\vartheta_1$=$\vartheta_2$=1 as they can be easily incorporated into the model without changing any of the insights derived. The inverse of the demand function is as follows,

$$p_1 = a_1 - Q_1 - bQ_2; \qquad p_2 = a_2 - Q_2 - bQ_1. \tag{3.3}$$

Recall that as products become more substitutable (i.e., as $b$ increases), customers become more sensitive to price changes and the overall market size decreases (Talluri and van Ryzin 2005). Further, we need $\vartheta_i > |b|$ to ensure that the demand for a product is more sensitive to it's own price than to price changes of the substitutable product.

The $\gamma$-demand model can be related to the $b$-demand model by letting

$$\epsilon_1 = a_1 - ba_2; \qquad \epsilon_2 = a_2 - ba_1; \qquad \gamma = b; \qquad \alpha = \vartheta_1\vartheta_2 - b^2.$$

The key difference between the $\gamma$- and $b$-demand functions is whether the demand intercepts and sensitivity $\alpha$ change with product substitutability. In the $b$-demand model, they are all dependent on $b$, the product substitutability factor. This confirms the realistic situation that as products become more substitutable (i.e., as $b$ increases), the customers become more sensitive to price changes and the overall market size decreases (Talluri and van Ryzin 2005). However, in the $\gamma$-demand model, they are all independent of $\gamma$.

In this research, we investigate the pricing and contingent capacity investment decisions under both demand models and obtain closed-form analytical solutions for each model. Our results underscore the importance of selecting an appropriate representation of the demand and product substitutability relationship. The results will help avoid over-investment in dedicated and contingent capacities.

Our work differs from the existing literature as follows: Bish and Wang (2004) study the capacity investment problem for a monopolistic firm with independent products (i.e., $\gamma$=0). We extend their model by considering product substitution, and we also analyze the setting with the $b$-demand model. Chod and Rudi (2005) use the $\gamma$-demand model to study the capacity and pricing decisions for a monopolistic firm that produces two substitutable products with only a flexible resource. Goyal and Netessine (2007) use the $b$-demand to investigate the optimal capacity investment decisions for two competing firms. Each firm can invest in either dedicated or flexible capacities but not both. Our work differs from Chod and Rudi (2005) and Goyal and Netessine (2007) by using both the $\gamma$- and the $b$-demand models and considering both dedicated and contingent capacities. The current research is similar to Lus and Muriel (2009), who examine the dedicated and flexible resource investment decisions under both the $\gamma$- and the $b$-demand models. In their model the firm incurs only the unit cost of investment in a flexible resource in the planning stage (Stage I). In our model the firm also incurs an additional cost of using this flexible resource in the production stage (Stage II). This additional cost reflects changes in economic conditions such as market entry of a competitor, currency valuations or labor costs between the planning and selling stages due to long lead times.

Our research reveals contrasting results between the two models. We find that when $\gamma$ is used as the measure of product substitutability (i.e., the $\gamma$-demand model), the firm invests more in contingent flexible capacity as products become closer substitutes. However, when the measure of product substitutability is $b$ (the $b$-demand model), the investment in contingent capacity decreases as products become closer substitutes.

The rest of the chapter is organized as follows. In Section 3.2 we formulate the problem as two-stage optimization problem. In Section 3.3, we solve the Stage II problem analytically under the $\gamma$-demand model. We use these solutions from the second stage to obtain the optimal expected profit in the first stage. In addition, we also obtain necessary and sufficient conditions for the capacity investment (Stage I) problem. In Section 3.4, we solve the Stages I and II problems under the $b$-demand model. In Section 3.5, with the *clearance* assumption, we obtain the closed-form solutions for the optimal contingent and dedicated capacities under both demand models, and compare the different insights under the two demand models. Finally, we summarize our results and conclude in Section 3.6.

## 3.2 Model

We consider a firm that produces two products, indexed by $i = 1, 2$. The firm makes capacity and production decisions in two stages. Before the demand information is revealed, the firm needs to determine the capacities invested in the two dedicated resources ($\bar{K}_1$, $\bar{K}_2$) and a contingent resource ($\bar{K}_f$), that can be used to produce both products. After the demand information becomes available, the firm determines the production quantities ($Q_1$, $Q_2$) and hence prices ($p_1$, $p_2$) to maximize its revenue. We model the decision problem as a two-Stage stochastic programming problem where at the beginning of the planning horizon i.e. Stage I, the firm seeks to determine a capacity investment portfolio, $(\bar{K}_1, \bar{K}_2, \bar{K}_f)$ under demand intercept uncertainty. In Stage II, when the demands are realized (i.e., uncertainty is resolved), the firm determines the capacity allocation and sets the prices to maximize its revenue subject to the capacity investment constraints in Stage I.

In addition to the premium unit investment costs $(g_1, g_2, g_f)$, an additional production cost $c_i$ will be incurred if product $i$ is produced by the contingent resource. We again note here that this additional production cost in Stage II was not part of Lus and Muriel (2009).

Let $\Pi^s$ denote the expected profit in Stage I and $R^s$ denote the revenue in Stage II, where $s \in \{\gamma, b\}$ indicates the underlying demand model. The two-stage decision problem for the $\gamma$-demand model can be formulated as follows:

$$\textit{Stage I}: \qquad \Pi^s = \mathbf{max}\ \boldsymbol{E}[R^s(\bar{K}_1, \bar{K}_2, \bar{K}_f)] - \sum_{i=1,2,f} g_i \bar{K}_i \tag{3.4}$$

$$\textit{subject to}: \ \bar{K}_1, \bar{K}_2, \bar{K}_f \geq 0$$

$$\textit{Stage II}: \qquad R^\gamma = \mathbf{max}\ [\mathbf{Q^T D}(\boldsymbol{\mathcal{E}} - \alpha\mathbf{Q}) - ((\mathbf{Q\text{-}K})^+)^\mathbf{T}\mathbf{c}] \tag{3.5}$$

$$\textit{or} \qquad R^b = \mathbf{max}\ [\mathbf{pQ} - ((\mathbf{Q\text{-}K})^+)^\mathbf{T}\mathbf{c}] \tag{3.6}$$

$$\textit{subject to}: \ Q_1 \leq K_1 + K_f$$

$$Q_2 \leq K_2 + K_f$$

$$Q_1 + Q_2 \leq K_1 + K_2 + K_f$$

$$Q_1, Q_2 \geq 0$$

where $R^\gamma$ and $R^b$ represent the Stage II objective function for the $\gamma$- and $b$-demand functions, respectively.

## 3.3 Solution for $\gamma$-demand Model

We solve the two stage problem by first solving the Stage II formulation subject to capacity constraints. The Stage II decision variables $Q_1$ and $Q_2$ are functions of demand realizations ($a_1$, $a_2$), available capacities ($K_1$,$K_2$,$K_f$), additional production costs $c_1, c_2$ as well as product substitutability $\gamma$. We note here that under only demand uncertainties the firm does not face any resource supply uncertainties and the invested capacities in Stage I ($\bar{K}_1, \bar{K}_2, \bar{K}_f$) are the same as the available capacities $K_1, K_2$ and $K_f$ in Stage II. To simplify the exposition we hence use $K_1$,$K_2$ and $K_f$ as the capacity decision variables in Stage I.

The Stage II objective function could have four possible forms depending on the four feasible regions as shown in Figure 3.1-Right: (1) In Region I, no capacity flexibility is required, and thus no additional production cost is incurred. $Q_1 \leq K_1$ and

$Q_2 \leq K_2$ and $Q_1 \leq K_1 + K_2 + K_f$ (2) In region II, demand realization of product 1 is higher than that of product 2 and contingent capacity is used to produce product 1, and thus an additional production cost $c_1$ is incurred. $Q_1 > K_1$ and $Q_1 \leq K_1 + K_f$ while $Q_2 \leq K_2$. (3) Symmetrically, in region III, contingent capacity is used to produce product 2, and thus an additional production cost $c_2$ is incurred. We have $Q_2 > K_2$ and $Q_2 \leq K_2 + K_f$ while $Q_1 \leq K_1$. (4) In region IV, contingent capacity is used to produce both products as demand for each product exceeds its own dedicated capacity. $Q_1 \geq K_1$, $Q_2 \geq K_2$ and $Q_1 + Q_2 \leq K_1 + K_2 + K_f$. The partition of the feasible region enables us to solve the individual optimization problems and obtain closed form solutions to the production quantities in many regions using KKT conditions. At boundaries we have either $Q_1 = K_1$ or $Q_2 = K_2$ where the objective function is non-differentiable. The Stage II problem in those regions can be solved without using KKT optimality conditions.

The demand space can be partitioned into eleven different regions (Figure 3.1-Left) where each region belongs to one of the scenarios described above. The partition of the feasible region enables us to solve the individual optimization problems and obtain closed form solutions to the production quantities as well as avoid non-differentiability at the boundaries.

### 3.3.1 Stage II Solution

We apply the KKT conditions (cf. see Bazaraa et al. 1993) to solve the Stage II problem under the $\gamma$-demand model in regions where the objective function is differentiable. The constraints can be binding /non-binding in eleven different combinations and hence the state space of $\boldsymbol{\epsilon}$ is partitioned into eleven regions, each corresponding to one of these combinations as shown in Figure 3.1-Left.

The demand regions are defined as follows:

$$\begin{aligned}
\Omega_1: \quad & \{(\epsilon_1,\epsilon_2): 0 \le \epsilon_1 \le 2\alpha K_1 + c_1, 0 \le \epsilon_2 \le 2\alpha K_2 + c_2,\\
& \epsilon_1 + \gamma\epsilon_2 \le 2\alpha(K_1+\gamma K_2) + (1-\gamma^2)c_1, \gamma\epsilon_1 + \epsilon_2 \le 2\alpha(\gamma K_1 + K_2) + (1-\gamma^2)c_2\};\\
\Omega_2: \quad & \{(\epsilon_1,\epsilon_2): 2\alpha K_1 + c_1 \le \epsilon_1 \le 2\alpha(K_1+K_f) + c_1, 0 \le \epsilon_2 \le 2\alpha K_2 - \gamma c_1\};\\
\Omega_3: \quad & \{(\epsilon_1,\epsilon_2): 0 \le \epsilon_1 \le 2\alpha K_1 - \gamma c_2, 2\alpha K_2 + c_2 \le \epsilon_2 \le 2\alpha(K_2+K_f) + c_2\};\\
\Omega_4: \quad & \{(\epsilon_1,\epsilon_2); \epsilon_1 \ge 2\alpha K_1 + c_1 - \gamma c_2, \epsilon_2 \ge 2\alpha K_2 + c_2 - \gamma c_1,\\
& \epsilon_1 + \epsilon_2 \le 2\alpha(K_1+K_2+K_f) + (1-\gamma)(c_1+c_2)\};\\
\Omega_5: \quad & \{(\epsilon_1,\epsilon_2): 2\alpha K_2 - \gamma c_1 \le \epsilon_2 \le 2\alpha K_2 - c_2 - \gamma c_1,\\
& 2\alpha(K_1+\gamma K_2) + (1-\gamma^2)c_1 \le \epsilon_1 + \gamma\epsilon_2 \le 2\alpha(K_1+\gamma K_2+K_f) + (1-\gamma^2)c_1\};\\
\Omega_6: \quad & \{(\epsilon_1,\epsilon_2): 2\alpha K_1 - \gamma c_2 \le \epsilon_1 \le 2\alpha K_1 - c_1 - \gamma c_2,\\
& 2\alpha(K_2+\gamma K_1) + (1-\gamma^2)c_2 \le \epsilon_2 + \gamma\epsilon_1 \le 2\alpha(K_2+\gamma K_1+K_f) + (1-\gamma^2)c_2\};\\
\Omega_7: \quad & \{(\epsilon_1,\epsilon_2): \epsilon_1 \ge 2\alpha(K_1+K_f) + c_1, \epsilon_2 \ge 0, \gamma\epsilon_1 + \epsilon_2 \le 2\alpha\gamma(K_1+K_f) + 2\alpha K_2\};\\
\Omega_8: \quad & \{(\epsilon_1,\epsilon_2): \epsilon_2 \ge 2\alpha(K_2+K_f) + c_2, \epsilon_1 \ge 0, \gamma\epsilon_2 + \epsilon_1 \le 2\alpha\gamma(K_2+K_f) + 2\alpha K_1\};\\
\Omega_9: \quad & \{(\epsilon_1,\epsilon_2): \epsilon_1 + \epsilon_2 \ge 2\alpha(K_1+K_2+K_f) + (1-\gamma)(c_1+c_2),\\
& 2\alpha(K_2-K_1+K_f) - (1+\gamma)(c_1-c_2) \le \epsilon_1 - \epsilon_2 \le 2\alpha(K_1-K_2+K_f)\\
& +(1+\gamma)(c_1-c_2)\};\\
\Omega_{10}: \quad & \{(\epsilon_1,\epsilon_2): \epsilon_2 \ge 0, \gamma\epsilon_1 + \epsilon_2 \ge 2\alpha\gamma(K_1+K_f) + 2\alpha K_2,\\
& \epsilon_1 + \gamma\epsilon_2 \ge 2\alpha(K_1+\gamma K_2+K_f) + (1-\gamma^2)c_1, \epsilon_1 - \epsilon_2 \ge 2\alpha(K_1-K_2+K_f)\\
& +(1+\gamma)(c_1-c_2)\};\\
\Omega_{11}: \quad & \{(\epsilon_1,\epsilon_2): \epsilon_1 \ge 0, \gamma\epsilon_2 + \epsilon_1 \ge 2\alpha\gamma(K_2+K_f) + 2\alpha K_1,\\
& \epsilon_2 + \gamma\epsilon_1 \ge 2\alpha(K_2+\gamma K_1+K_f) + (1-\gamma^2)c_2, \epsilon_1 - \epsilon_2 \le 2\alpha(K_2-K_1+K_f)\\
& -(1+\gamma)(c_1-c_2)\};
\end{aligned}$$

The corresponding optimal production quantities are shown in Figure 3.1-Right and are classified into four regions: In Region I, no contingent capacity is used; In

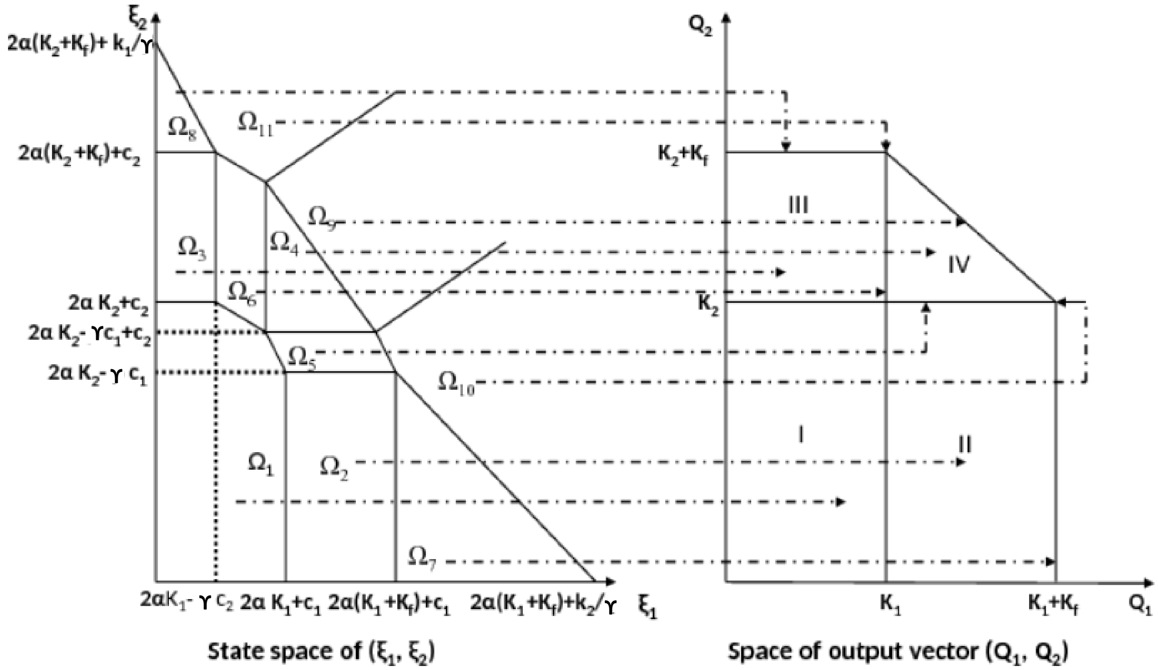


Figure 3.1: Mapping of the state space of $\mathcal{E}$ into the output space

region II, the contingent capacity is used to produce product 1, and thus an additional production cost $c_1$ is incurred; In region III, the contingent capacity is used to produce product 2, and thus an additional production cost $c_2$ is incurred; In region IV, the contingent capacity is used to produce both products.

The objective function of the Stage II problem is jointly concave and all the constraints are linear in $Q_1$ and $Q_2$. The following theorem characterizes the optimal solution of the Stage II problem.

**Theorem 3.1** *Given the realizations of the demand $(\epsilon_1, \epsilon_2)$ and the investment vector $(K_1, K_2, K_f)$, the optimal production quantities for the Stage II problem are given as* *follows:*

$$
\begin{aligned}
\Omega_1: &\quad Q_1 = \epsilon_1/2\alpha, Q_2 = \epsilon_2/2\alpha;\\
\Omega_2: &\quad Q_1 = \frac{\epsilon_1 - c_1}{2\alpha}, Q_2 = \frac{\epsilon_2 + \gamma c_1}{2\alpha};\\
\Omega_3: &\quad Q_1 = \frac{\epsilon_1 + \gamma c_2}{2\alpha}, Q_2 = \frac{\epsilon_2 - c_2}{2\alpha};\\
\Omega_4: &\quad Q_1 = \frac{\epsilon_1 - c_1 + \gamma c_2}{2\alpha}, Q_2 = \frac{\epsilon_2 - c_2 + \gamma c_1}{2\alpha};\\
\Omega_5: &\quad Q_1 = \frac{\epsilon_1 + \gamma\epsilon_2 - (1-\gamma^2)c_1}{2\alpha} - \gamma K_2, Q_2 = K_2;\\
\Omega_6: &\quad Q_1 = K_1, Q_2 = \frac{\epsilon_2 + \gamma\epsilon_1 - (1-\gamma^2)c_2}{2\alpha} - \gamma K_1;\\
\Omega_7: &\quad Q_1 = K_1 + K_f, Q_2 = \frac{\epsilon_2 + \gamma\epsilon_1}{2\alpha} - \gamma(K_1 + K_f);\\
\Omega_8: &\quad Q_1 = \frac{\epsilon_1 + \gamma\epsilon_2}{2\alpha} - \gamma(K_2 + K_f), Q_2 = K_2 + K_f;\\
\Omega_9: &\quad Q_1 = \frac{K_1 + K_2 + K_f}{2} + \frac{\epsilon_1 - \epsilon_2 - (1+\gamma)(c_1 - c_2)}{4\alpha},\\
&\quad Q_2 = \frac{K_1 + K_2 + K_f}{2} - \frac{\epsilon_1 - \epsilon_2 - (1+\gamma)(c_1 - c_2)}{4\alpha};\\
\Omega_{10}: &\quad Q_1 = K_1 + K_f, Q_2 = K_2;\\
\Omega_{11}: &\quad Q_1 = K_1, Q_2 = K_2 + K_f.
\end{aligned}
$$

Please refer to the appendix for all the proofs in this chapter.

By plugging in the above quantities into equation (3.5) we obtain closed-form solutions for optimal revenue of the Stage II problem $R^\gamma(\epsilon_1, \epsilon_2)$. Please refer to the appendix for the closed form solutions for $R^\gamma$.

### 3.3.2 Optimal Capacity Investment Decisions for $\gamma$-demand model

We now discuss the optimality conditions for the Stage I problem. The lemma below implies the existence of a unique optimal capacity solution $(K_1, K_2, K_f)$.

**Lemma 3.1** *The Stage I objective function $\Pi^\gamma$ is strictly jointly concave in $(K_1, K_2, K_f)$*

**Theorem 3.2** *The Stage I optimal capacity investment vector $(K_1, K_2, K_f)$ is optimal if and only if the Lagrangian multipliers given by $\lambda_i, i = 1, 2, f$ exist and satisfy the following condition:*

$$\frac{\partial R^{\gamma}(K_1,K_2,K_f)}{\partial K_i} = g_i - \lambda_i \ for \ i \ \in \{1,2,f\} \tag{3.7}$$

$$and \qquad K_i\lambda_i = 0 \tag{3.8}$$

*Hence, conditioning on* $(\mathcal{E}_1, \mathcal{E}_2)$ *we obtain,*

$$\begin{Bmatrix} g_1-\lambda_1 \\ g_2-\lambda_2 \\ g_f-\lambda_f \end{Bmatrix} = Pr(\Omega_2)\mathbf{E}\begin{Bmatrix} c_1 \\ 0 \\ 0 \end{Bmatrix} + Pr(\Omega_3)\mathbf{E}\begin{Bmatrix} 0 \\ c_2 \\ 0 \end{Bmatrix} + Pr(\Omega_4)\mathbf{E}\begin{Bmatrix} c_1 \\ c_2 \\ 0 \end{Bmatrix}$$

$$+ Pr(\Omega_5)\mathbf{E}\begin{Bmatrix} c_1 \\ \mathcal{E}_2 - 2\alpha K_2 + c_1\gamma \\ 0 \end{Bmatrix} + Pr(\Omega_6)\mathbf{E}\begin{Bmatrix} \mathcal{E}_1 - 2\alpha K_1 + c_2\gamma \\ c_2 \\ 0 \end{Bmatrix}$$

$$+ Pr(\Omega_7)\mathbf{E}\begin{Bmatrix} c_1 \\ \mathcal{E}_2 - 2\alpha(K_2+K_f) + \gamma c_1 \\ \mathcal{E}_2 - 2\alpha(K_2+K_f) + \gamma c_1 - c_2 \end{Bmatrix}$$

$$+ Pr(\Omega_8)\mathbf{E}\begin{Bmatrix} \mathcal{E}_1 - 2\alpha(K_1+K_f) + \gamma c_2 \\ c_2 \\ \mathcal{E}_1 - 2\alpha(K_1+K_f) + \gamma c_2 - c_1 \end{Bmatrix}$$

$$+ Pr(\Omega_9)\mathbf{E}\begin{Bmatrix} \frac{\mathcal{E}_1+\mathcal{E}_2-2\alpha(K_1+K_2+K_f)}{2(1-\gamma)} + \frac{c_1-c_2}{2} \\ \frac{\mathcal{E}_1+\mathcal{E}_2-2\alpha(K_1+K_2+K_f)}{2(1-\gamma)} - \frac{c_1-c_2}{2} \\ \frac{\mathcal{E}_1+\mathcal{E}_2-2\alpha(K_1+K_2+K_f)}{2(1-\gamma)} - \frac{c_1+c_2}{2} \end{Bmatrix}$$

$$+ Pr(\Omega_{10})\mathbf{E}\begin{Bmatrix} \frac{1}{1-\gamma^2}[\mathcal{E}_1 + \gamma\mathcal{E}_2 - 2\alpha(K_1+K_f) - 2\alpha\gamma K_2] \\ \frac{1}{1-\gamma^2}[\mathcal{E}_1\gamma - 2\alpha\gamma(K_1+K_f) + \mathcal{E}_2 - 2\alpha K_2] \\ \frac{1}{1-\gamma^2}[\mathcal{E}_1 + \gamma\mathcal{E}_2 - 2\alpha(K_1+K_f) - 2\alpha\gamma K_2] - c_1 \end{Bmatrix}$$

$$+ Pr(\Omega_{11})\mathbf{E}\begin{Bmatrix} \frac{1}{1-\gamma^2}[\mathcal{E}_1 - 2\alpha K_1 - 2\alpha\gamma(K_2+K_f) + \gamma\mathcal{E}_2] \\ \frac{1}{1-\gamma^2}[\mathcal{E}_1\gamma - 2\alpha\gamma K_1 - 2\alpha(K_2+K_f) + \mathcal{E}_2] \\ \frac{1}{1-\gamma^2}[\mathcal{E}_1\gamma - 2\alpha\gamma K_1 - 2\alpha(K_2+K_f) + \mathcal{E}_2] - c_2 \end{Bmatrix} \quad \blacksquare$$

When the contingent capacity is too costly, the firm may be better off with just investing in the dedicated resources. As shown in the following lemma, the decision to invest in contingent capacity is of a threshold type, and it depends on both the initial investment cost $g_f$ and the additional production costs $c_i$ for $i$=1, 2 using the contingent capacity.

**Lemma 3.2** *The firm invests in the flexible contingent capacity if and only if $g_f < g'_f$, where $g'_f$ is given as*

$$
\begin{aligned}
g'_f = g_f - \lambda_f = Pr(\Omega'_7)\mathbf{E}\left\{ \ \mathcal{E}_2 - 2\alpha K_2 + \gamma c_1 - c_2 \ \right\} + Pr(\Omega'_8)\mathbf{E}\left\{ \ \mathcal{E}_1 - 2\alpha K_1 + \gamma c_2 - c_1 \ \right\} \\
+ Pr(\Omega'_9)\mathbf{E}\left\{ \ \tfrac{\mathcal{E}_1+\mathcal{E}_2-2\alpha(K_1+K_2)}{2(1-\gamma)} - \tfrac{c_1+c_2}{2} \ \right\} \\
+ Pr(\Omega'_{10})\mathbf{E}\left\{ \ \tfrac{1}{1-\gamma^2}[\mathcal{E}_1 + \gamma\mathcal{E}_2 - 2\alpha K_1 - 2\alpha\gamma K_2] - c_1 \ \right\} \\
+ Pr(\Omega'_{11})\mathbf{E}\left\{ \ \tfrac{1}{1-\gamma^2}[\mathcal{E}_1\gamma - 2\alpha\gamma K_1 - 2\alpha K_2 + \mathcal{E}_2] - c_2 \ \right\}
\end{aligned}
$$

■

### 3.4 Solution for the $b$-demand Model

In this section we analyze the Stage II and Stage I decisions of the firm utilizing the $b$-demand model. We solve the two stage problem by first solving the Stage II formulation subject to capacity constraints.

The solution methodology employed for the $b$-demand model very similar to the methodology described in Section 3.3 for the $\gamma$-demand model. The Stage II decision variables $Q_1$ and $Q_2$ are functions of demand realizations ($a_1$, $a_2$), invested capacities ($K_1$,$K_2$,$K_f$), additional production costs $c_1, c_2$ as well as product substitutability $b$.

The Stage II objective function could have four possible forms depending on the four feasible regions as shown in Figure 3.2-Right: (1) In Region I, no capacity flexibility is required, and thus no additional production cost is incurred. $Q_1 \leq K_1$ and $Q_2 \leq K_2$ and $Q_1 \leq K_1 + K_2 + K_f$ (2) In region II, demand realization of product 1 is higher than that of product 2 and contingent capacity is used to produce product 1, and thus an additional production cost $c_1$ is incurred. $Q_1 > K_1$ and $Q_1 \leq K_1 + K_f$ while $Q_2 \leq K_2$. (3) Symmetrically, in region III, contingent capacity is used to produce product 2, and thus an additional production cost $c_2$ is incurred. We have $Q_2 > K_2$ and $Q_2 \leq K_2 + K_f$ while $Q_1 \leq K_1$. (4) In region IV, contingent capacity is used to produce both products as demand for each product exceeds its own dedicated capacity. $Q_1 \geq K_1$, $Q_2 \geq K_2$ and $Q_1 + Q_2 \leq K_1 + K_2 + K_f$. The partition of the feasible region enables us to solve the individual optimization problems and obtain closed form solutions to the production quantities in many regions using KKT conditions. At boundaries we have either $Q_1 = K_1$ or $Q_2 = K_2$ where the objective function is non-differentiable. The Stage II problem in those regions can be solved without using KKT optimality conditions.

The demand space can be partitioned into eleven different regions (Figure 3.2-Left) where each region belongs to one of the scenarios described above. The partition of the feasible region enables us to solve the individual optimization problems and obtain closed form solutions to the production quantities as well as avoid non-differentiability at the boundaries.

### 3.4.1 Stage II Solution

We apply the KKT conditions (cf. see Bazaraa et al. 1993) to solve the individual Stage II problems under the $b$-demand model. At the boundaries (regions $\Omega_5$,$\Omega_6$,$\Omega_{10}$,$\Omega_{11}$) the Stage II problem can be solved without using KKT optimality conditions. The constraints can be binding /non-binding in eleven different combinations and hence the state space of $\mathbf{A}$ is partitioned into eleven regions, each corresponding to one of these combinations. Imposing the conditions $A_1 - bA_2 \geq 0$, $A_2 - bA_1 \geq 0$ to enforce positive demand realizations (see, e.g., Singh and Vives 1984), the demand regions are shown in Figure 3.2-left.

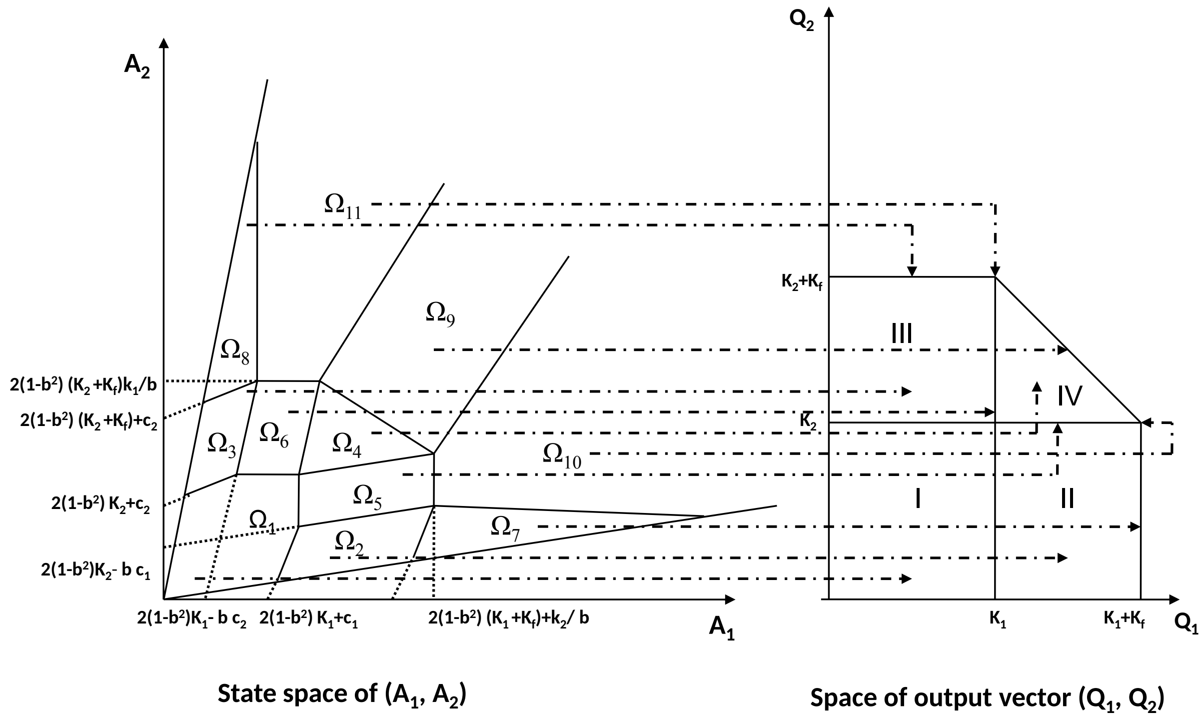


Figure 3.2: Mapping of the state space of $\mathbf{A}$ into the output space

The demand regions are defined as follows:

$$\Omega_1^b: \quad \{0 \le a_1 - ba_2 \le 2(1-b^2)K_1 + c_1, 0 \le a_2 - ba_1 \le 2(1-b^2)K_2 + c_2,$$
$$a_1 \le 2(K_1 + bK_2) + c_1, a_2 \le 2(bK_1 + K_2) + c_2\};$$

$$\Omega_2^b: \quad \{2(1-b^2)K_1 + c_1 \le a_1 - ba_2 \le 2(1-b^2)(K_1 + K_f) + c_1,$$
$$0 \le a_2 - ba_1 \le 2(1-b^2)K_2 - c_1\};$$

$$\Omega_3^b: \quad \{0 \le a_1 - ba_2 \le 2(1-b^2)K_1 - bc_2,$$
$$2(1-b^2)K_2 + c_2 \le a_2 - ba_1 \le 2(1-b^2)(K_2 + K_f) + c_2\};$$

$$\Omega_4^b: \quad \{a_1 - ba_2 \ge 2(1-b^2)K_1 + c_1 - bc_2, a_2 - ba_1 \ge 2(1-b^2)K_2 + c_2 - bc_1,$$
$$a_1 + a_2 \le 2(1+b)(K_1 + K_2 + K_f) + (c_1 + c_2)\};$$

$$\Omega_5^b: \quad \{2(1-b^2)K_2 - bc_1 \le a_2 - ba_1 \le 2(1-b^2)K_2 - c_2 - bc_1,$$
$$2(K_1 + bK_2) + c_1 \le a_1 \le 2((K_1 + bK_2 + K_f) + c_1\};$$

$$\Omega_6^b: \quad \{2(1-b^2)K_1 - bc_2 \le a_1 - ba_2 \le 2(1-b^2)K_1 - c_1 - bc_2$$
$$2(K_2 + bK_1) + c_2 \le a_2 \le 2(bK_1 + K_2 + K_f) + c_2\};$$

$$\Omega_7^b: \quad \{a_2 - ba_1 \geq 2(1-b^2)(K_2+K_f)+c_2, a_1 \geq ba_2, a_1 \leq 2b(K_2+K_f)+2K_1\};$$

$$\Omega_8^b: \quad \{a_1 - ba_2 \geq 2(1-b^2)(K_1+K_f)+c_1, a_2 \geq ba_1, a_2 \leq 2b(K_1+K_f)+2K_2\};$$

$$\Omega_9^b: \quad \{a_1 + a_2 \geq 2(K_1+K_2+K_f)(1+b)+c_1+c_2,$$

$$2(1-b)(K_2-K_1+K_f)-(c_1-c_2) \leq a_1-a_2 \leq 2(1-b)(K_1-K_2+K_f)+c_1-c_2\};$$

$$\Omega_{10}^b: \quad \{a_2 \geq ba_1, a_2 \geq 2b(K_1+K_f)+2K_2,$$

$$a_1 \geq 2(K_1+bK_2+K_f)+c_1, a_1-a_2 \geq 2(1-b)(K_1-K_2+K_f)+c_1-c_2\};$$

$$\Omega_{11}^b: \quad \{a_1 \geq ba_2, a_1 \geq 2b(K_2+K_f)+2K_1,$$

$$a_2 \geq 2(K_2+bK_1+K_f)+c_2, a_1-a_2 \leq 2(1-b)(K_2-K_1+K_f)+c_2-c_1\}$$

For the $b$-demand model, the following theorem gives out the optimal production quantities in Stage II that maximize $R^b$.

**Theorem 3.3** *Given the realizations of the price intercepts $(a_1, a_2)$ and the capacities*

*$(K_1, K_2, K_f)$, the optimal production quantities of the Stage II problem are as follows:*

$$
\begin{aligned}
\Omega_1: &\quad Q_1 = \frac{a_1 - ba_2}{2(1-b^2)}, Q_2 = \frac{a_2 - ba_1}{2(1-b^2)};\\
\Omega_2: &\quad Q_1 = \frac{a_1 - ba_2 - c_1}{2(1-b^2)}, Q_2 = \frac{a_2 - ba_1 + bc_1}{2(1-b^2)};\\
\Omega_3: &\quad Q_1 = \frac{a_1 - ba_2 + bc_2}{2(1-b^2)}, Q_2 = \frac{a_2 - ba_1 - c_2}{2(1-b^2)};\\
\Omega_4: &\quad Q_1 = \frac{a_1 - ba_2 + bc_2 - c_1}{2(1-b^2)}, Q_2 = \frac{a_2 - ba_1 - c_2 + bc_1}{2(1-b^2)};\\
\Omega_5: &\quad Q_1 = \frac{a_1 - c_1}{2} - bK_2, Q_2 = K_2;\\
\Omega_6: &\quad Q_1 = K_1, Q_2 = \frac{a_2 - c_2}{2} - bK_1;\\
\Omega_7: &\quad Q_1 = K_1 + K_f, Q_2 = \frac{a_2}{2} - b(K_1 + K_f);\\
\Omega_8: &\quad Q_1 = \frac{a_1}{2} - b(K_2 + K_f), Q_2 = K_2 + K_f;\\
\Omega_9: &\quad Q_1 = \frac{K_1 + K_2 + K_f}{2} + \frac{a_1 - a_2 - (c_1 - c_2)}{4(1-b^2)},\\
&\quad Q_2 = \frac{K_1 + K_2 + K_f}{2} - \frac{a_1 - a_2 - (c_1 - c_2)}{4(1-b^2)};\\
\Omega_{10}: &\quad Q_1 = K_1 + K_f, Q_2 = K_2;\\
\Omega_{11}: &\quad Q_1 = K_1, Q_2 = K_2 + K_f.
\end{aligned}
$$

The closed-form solution for the Stage II problem $R^b(a_1, a_2)$ can be obtained by inserting the $Q_1$ and $Q_2$ values in equation (3.6). Please refer to Appendix A for the Stage II optimal revenues under the $b$-demand model.

### 3.4.2 Optimal Capacity Investment Decisions for $b$-demand model

Using the above results, we now analyze the firm's optimal investment decisions under the $b$-demand model.

**Lemma 3.3** *The Stage I objective function $\Pi^b$ is jointly concave in $(K_1, K_2, K_f)$*

Hence, the following theorem establishes the necessary and sufficient conditions for the optimal capacity investment vector $(K_1, K_2, K_f)$.

**Theorem 3.4** *The Stage I optimal capacity investment vector $(K_1, K_2, K_f)$ is optimal if and only if the Lagrangian multipliers given by $\lambda_i$ exist and satisfy the following condition:*

$$\frac{\partial R^b(K_1, K_2, K_f)}{\partial K_i} = g_i - \lambda_i \ for \ i \ \in \{1, 2, f\} \tag{3.9}$$

$$and \qquad K_i \lambda_i = 0. \tag{3.10}$$

*Hence, conditioning on $(A_1, A_2)$, we obtain,*

$$\begin{Bmatrix} g_1 - \lambda_1 \\ g_2 - \lambda_2 \\ g_f - \lambda_f \end{Bmatrix} = Pr(\Omega_2)\mathbf{E}\begin{Bmatrix} c_1 \\ 0 \\ 0 \end{Bmatrix} + Pr(\Omega_3)\mathbf{E}\begin{Bmatrix} 0 \\ c_2 \\ 0 \end{Bmatrix} + Pr(\Omega_4)\mathbf{E}\begin{Bmatrix} c_1 \\ c_2 \\ 0 \end{Bmatrix}$$
$$+ Pr(\Omega_5^b)\mathbf{E}\begin{Bmatrix} c_1 \\ A_2 - bA_1 - 2(1-b^2)K_2 + c_1 b \\ 0 \end{Bmatrix}$$
$$+ Pr(\Omega_6^b)\mathbf{E}\begin{Bmatrix} A_1 - bA_2 - 2(1-b^2)K_1 + c_2 b \\ c_2 \\ 0 \end{Bmatrix}$$

$$+Pr(\Omega_7^b)\mathbf{E}\left\{\begin{array}{c} c_1 \\ A_2-bA_1-2(1-b^2)(K_2+K_f)+bc_1 \\ A_2-bA_1-2(1-b^2)(K_2+K_f)+bc_1-c_2 \end{array}\right\}$$

$$+Pr(\Omega_8^b)\mathbf{E}\left\{\begin{array}{c} A_1-bA_2-2(1-b^2)(K_1+K_f)+bc_2 \\ c_2 \\ A_1-bA_2-2(1-b^2)(K_1+K_f)+bc_2-c_1 \end{array}\right\}$$

$$+Pr(\Omega_9^b)\mathbf{E}\left\{\begin{array}{c} \frac{A_1+A_2-2(1+b)(K_1+K_2+K_f)}{2}+\frac{c_1-c_2}{2} \\ \frac{A_1+A_2-2(1+b)(K_1+K_2+K_f)}{2}-\frac{c_1-c_2}{2} \\ \frac{A_1+A_2-2(1+b)(K_1+K_2+K_f)}{2}-\frac{c_1+c_2}{2} \end{array}\right\}$$

$$+Pr(\Omega_{10}^b)\mathbf{E}\left\{\begin{array}{c} A_1-2(K_1+K_f)-2bK_2 \\ A_2-2b(K_1+K_f)-2K_2 \\ A_1-2(K_1+K_f)-2bK_2-c_1 \end{array}\right\}$$

$$+Pr(\Omega_{11}^b)\mathbf{E}\left\{\begin{array}{c} A_1-2b(K_2+K_f)-2K_1 \\ A_2-2bK_1-2(K_2+K_f) \\ A_2-2bK_1-2(K_2+K_f)-c_2 \end{array}\right\} \quad \blacksquare$$

### 3.5 Optimal Contingent and Dedicated Capacities under clearance

In this section, we analytically characterize the impacts of product substitution factors ($\gamma$ and $b$) on the optimal contingent flexible capacity ($K_f$) and the optimal dedicated capacities ($K_1$ and $K_2$). We note here that the firm faces only demand uncertainties and hence we drop the supply disruption RVs making $\theta_1 K_1 = K_1$ and $\theta_2 K_2 = K_2$. Subsequently we use this notation to obtain insights on the optimal contingent flexible and dedicated capacities.

The first order conditions for the optimal capacity portfolio derived in Theorems 2 and 4 are implicit functions. We need a closed form expression to show the impacts of $\gamma$ and $b$ on $K_f$, $K_1$ and $K_2$ clearly. Hence, we make the following two assumptions: (1) Probability distributions of demand intercepts are independent, i.e., $\Psi(\mathcal{E}_1, \mathcal{E}_2) = \Psi_1(\mathcal{E}_1)\Psi_2(\mathcal{E}_2)$; (2) The firm always produces to both the dedicated and flexible capacity levels irrespective of any demand intercept realization. The second assumption is known in literature as *clearance* (please refer to Van Mieghem and Dada 1999 for details on *clearance* and *holdback* strategies). Note that although committed contingent capacity must be completely utilized, the firm still has the flexibility to allocate the resource between the two products.

Obviously, clearance is sub-optimal behavior since the firm is forced to produce at the capacity level for any demand intercept realization. However, this is not unrealistic as Goyal and Netessine (2007) note "...firms often find it difficult to produce below capacity in view of large fixed costs...car makers have been forced to slash prices to keep lines running as models fall out of favor with the public, rather than reduce production". Chod and Rudi (2005) numerically investigate the impact of the clearance assumption and conclude that they generally yield close-to-optimal solutions.

In our model, in regions $\Omega_9$, $\Omega_{10}$ and $\Omega_{11}$ the capacities are completely utilized to produce either one or both products under both the $\gamma$ and $b$-demand models. Hence the clearance assumption is satisfied naturally in these regions. Each region has its own expressions for optimal contingent and dedicated capacities. However, as shown in Figures 3.1 and 3.2, the capacity allocation solutions in $\Omega_{10}$ and $\Omega_{11}$ are special cases of that in $\Omega_9$ at the boundaries. Therefore, we use the solution for $\Omega_9$ to conduct analysis under the clearance assumption.

### 3.5.1 Optimal Contingent Flexible Capacity

In this sub-section, we compare the impacts of $\gamma$ and $b$ on optimal contingent capacity $K_f$ under the clearance assumption, given the dedicated resource investment $K_1$ and $K_2$.

In many industries firms already have a basic level of capacity built up as a long-term investment in plant and resource acquisitions which incur significant capital. For short term decisions such as single period selling seasons, the only decision variable may be the level of *additional* resource to acquire in addition to already existing resources. Moreno and Terwiesch (2015) report several industries that already have a sunk capital cost in existing plants and invest in either adding capacity to already existing plants or invest in new resources altogether. Further, related to cross-production, Li et al. (2014) cite the example of Intel's wafer fabrication units where capacity in the form of new production lines can be purchased in addition to converting existing production lines to produce multiple products.

We hence study this setting and assume that dedicated capacity levels $K_1$ and $K_2$ are exogenous and the only decision variable in Stage I is the level of contingent flexible capacity $K_f$. The other advantage in studying this setting is that with the closed form expressions we can explicitly compare the impact of product substitutability $\gamma$ and $b$ (which forms the basis of the respective demand models) on the contingent flexible capacity levels.

**Theorem 3.5** *Given $K_1$, $K_2$, and $g_f$ ($g_f < g'_f$), the optimal contingent capacity*

$$K_f = \frac{\epsilon_1 + \epsilon_2}{2\alpha} - (K_1 + K_2) - \frac{(1-\gamma)(c_1 + c_2 + 2g_f)}{2\alpha} \quad \textit{in the } \gamma\textit{-demand model};$$

$$K_f = \frac{\mu_1 + \mu_2}{2(1+b)} - (K_1 + K_2) - \frac{c_1 + c_2 + 2g_f}{2(1+b)} \quad \textit{in the } b\textit{-demand model.}$$

The closed-form expressions for $K_f$ under both demand models have three terms. The first term could be interpreted as the impact of market size because a higher expected mean $\mu_i$ for product $i$ also implies a higher expected demand for the product. The second term involves the available dedicated capacities of the firm. The third is the impact of the various production costs on the flexible capacity.

Some interesting results can be derived from Theorem 3.5.

**Corollary 3.1** *In the $\gamma$-demand model the optimal contingent capacity $K_f$ increases as $\gamma$ increases, i.e., $\frac{\partial K_f}{\partial \gamma} > 0$.*

Corollary 3.1 reveals that under the $\gamma$-demand model, as products become closer substitutes optimal $K_f$ increases. Notice that as $\gamma$ increases, the first term representing the total market size remains unaffected. This is *inconsistent* with the intuition that more differentiated (less substitutable) products reach a larger customer base (see, e.g., Talluri and van Ryzin 2005).

**Lemma 3.4** *In the b-demand model if the optimal contingent capacity $K_f > 0$, then $K_f$ decreases as $b$ increases, i.e., $\frac{\partial K_f}{\partial b} < 0$.*

Lemma 3.4 implies that if the firm invests in flexible capacity under the $b$-demand model, then the optimal flexible capacity level decreases if the two products become closer substitutes. Intuitively, as products become closer substitutes consumers tend to be more sensitive to price changes. Hence, a small price increase of one product, say product 1, would allow the firm to shift a large portion of product 1's demand to its substitute- product 2. Responsive pricing reduces the investment in contingent capacity.

Thus the $b$-demand model reveals that contingent capacity is less desired, and responsive product pricing may be more beneficial to the firm when products become more substitutable. This is *consistent* with the intuition that consumers are less price-sensitive when purchasing a unique item and more differentiated (less substitutable) products reach a larger customer base (see, e.g., Talluri and van Ryzin 2005).

If we denote the optimal solution under $\gamma$ demand model as $K_f^\gamma$ and that under $b$-demand model as $K_f^b$, the following corollary helps compare the level of optimal $K_f$ under both the demand models.

**Corollary 3.2** *Given the same model parameters and existing dedicated capacities, the optimal contingent capacity $K_f^\gamma$ is higher than the corresponding capacity $K_f^b$ for $(\gamma = b) \in (0, 1)$.*

When $\gamma = b = 0$ (independent products) it is easy to see that $K_f^\gamma = K_f^b$. For

$(\gamma = b) \in (0, 1)$ we have, from corollary 3.1 and lemma 3.4, that $K_f^\gamma > K_f^b$. Thus, corollary 3.2 reveals that the $\gamma$-demand model suggests a higher investment in contingent capacity than the $b$-demand model under same model parameters. Hence, our results presented in this section indicate that the $\gamma$-demand model, which lacks the support of a utility function, may not be an accurate representation of product substitution relationships, and maybe misleading if used in practice.

### 3.5.2 Optimal Dedicated Capacity

In this sub-section, we focus on the impact of product substitutability ($\gamma$ and $b$) on the optimal dedicated capacity under the clearance assumption. Without loss of generality we assume that the firm tries to decide the level of investment in dedicated capacity $K_2$, given that $K_1$ and $K_f$ are fixed. By doing so, we are able to study the difference between the investment structures between the $b$ and $\gamma$-demand models. This enables to evaluate which functional form represents reality more closely: more differentiated products reach a larger customer base and consumers are less price sensitive when purchasing highly differentiated item. Consequently, as product substitutability increases, the overall market size decreases (Talluri and van Ryzin 2005). Under a high degree of product substitutability, we expect the need for overall capacity levels to decrease due to lower market size.

**Theorem 3.6** *Given $K_1$, $K_f$, the optimal dedicated capacity*

$$K_2 = \frac{\epsilon_1 + \epsilon_2}{2\alpha} - (K_1 + K_f) - \frac{(1-\gamma)(c_1 - c_2 + 2g_2)}{2\alpha} \quad \textit{in the } \gamma\textit{-demand model;}$$
$$K_2 = \frac{\mu_1 + \mu_2}{2(1+b)} - (K_1 + K_f) - \frac{(c_1 - c_2 + 2g_2)}{2(1+b)} \quad \textit{in the } b\textit{-demand model.}$$

Some interesting results can be derived from Theorem 3.6.

**Corollary 3.3** *In the $\gamma$-demand model $K_2$ increases as $\gamma$ increases, i.e., $\frac{\partial K_2}{\partial \gamma} > 0$.*

Corollary 3.3 reveals that the investment trend in optimal dedicated capacity considering the $\gamma$-demand model is very similar to the result for contingent flexible capacity using this model. As $\gamma$ increases the investment in dedicated capacity increases. Hence, the $\gamma$-demand model suggests an increased investment in $K_2$ as products become closer substitutes.

**Lemma 3.5** *In the b-demand model if the optimal dedicated capacity $K_2 > 0$, then $K_2$ decreases as $b$ increases, i.e., $\frac{\partial K_2}{\partial b} < 0$.*

Similar to the previous sub-section, the $b$-demand model exhibits a consistent trend in the optimal dedicated capacity investment i.e., the need for dedicated capacity decreases as products become closer substitutes. This is again consistent with the intuition that the overall market demand decreases as products are highly substitutes.

If we denote the optimal solution under $\gamma$ demand model as $K_2^\gamma$ and that under $b$-demand model as $K_2^b$, the following corollary helps compare the level of optimal $K_2$ under both the demand models.

**Corollary 3.4** *Given the same model parameters and existing capacities $K_1$ and $K_f$, the optimal dedicated capacity $K_2^\gamma$ is higher than the corresponding capacity $K_2^b$ for $(\gamma = b) \in (0, 1)$.*

Corollary 3.4 reveals that the $\gamma$-demand model suggests a higher investment in dedicated capacity than the $b$-demand model under same model parameters. This result is similar to corollary 3.2 under optimal contingent capacity.

The $\gamma$-demand model and the $b$-demand model suggest completely different capacity investment strategies. Literature using the $\gamma$ measure of product substitutability (see, e.g., Choi 1991, Birge et al. 1998, Chod and Rudi 2005 and Biller et al. 2006) conclude that as products become more substitutable, flexible capacity is highly preferred and that optimal prices and profits increase. However, we show that when using the $b$-demand model, the investment in contingent flexible capacity decreases as product substitutability increases. Hence, the demand model using $\gamma$ as the product substitutability factor may not be appropriate to study the optimal capacity investment decision problem when products are substitutes.

### 3.6 Summary

In this chapter we study the pricing and capacity investment decisions for a firm that faces stochastic price-dependent demand for two substitutable products. We examined two related but different functional form of linear demand models, namely the $\gamma$-demand model and the $b$-demand model. We show that the selection of correct demand models plays a crucial role in deciding the optimal dedicated and flexible capacity for substitutable products. Specifically, we demonstrate that under the $b$-demand function the firm always invests less in contingent flexible capacity as products become substitutable. In this case responsive pricing can be used as an effective strategy hence reducing the need to invest in costly flexibility. However, under the $\gamma$-demand model, the firm prefers to invest more in contingent flexible capacity as products become more substitutes. The $b$-demand model has the foundational support of a consumer utility function which reflects the fact that as products become more substitutable (i.e., as $b$ increases), customers become more sensitive to price changes and the overall market size decreases. When using $\gamma$ instead of $b$ as product substitutability factor in the demand functions, an increase in $\gamma$ does not impact own price effect or thew total market potential. Therefore, in subsequent chapters of this dissertation we will use the $b$-demand model which correctly captures the impact of product substitutability on market potentials as well as price and cross-price effects, to investigate the impact of different types of efficiency loss.

# CHAPTER 4

# CAPACITY, FLEXIBILITY AND PRICING DECISIONS UNDER CROSS-PRODUCTION: THE SHRINKING CAPACITY MODEL

## 4.1 Introduction

In this chapter we study the capacity, flexibility and pricing decisions of a price-setting firm that uses cross-production to produce two substitutable products. In addition to facing demand uncertainties at the capacity investment stage (Stage I), the firm may also face additional uncertainties in production stage (Stage II) such as capacity uncertainties which may lower the yield or 0-1 type disruptions that may shut down an entire resource altogether.

Recall from Chapter 1 that under cross-production there is no separate flexible resource. The firm produces two products each with it's own dedicated resource and if capacity is insufficient for a product, it can use the other resource to produce the product with a certain degree of efficiency loss. This efficiency loss may be caused by machine specialization, changeover time, or shortage of cross-trained workers. In the capacity investment stage, under demand uncertainty, the firm decides on the optimal dedicated capacities and the optimal degree of flexibility. The higher the degree of flexibility investment in Stage I the lower the efficiency loss in Stage II. However, flexibility investment is costly and thus a moderate degree of flexibility that hedges against the three uncertainties to an acceptable extent might be desirable.

The rest of the chapter is organized as follows. A two-stage optimization model with capacity, flexibility, and pricing decisions is formulated in Section 4.2 and its solutions and analytical properties are derived in Section 4.3. Section 4.4 presents the numerical analysis to obtain managerial insights into the impacts of uncertainties, disruption risks and resource investment costs on the optimal capacity and flexibility investment, as well as pricing decisions. Finally, Section 4.5 concludes the chapter.

## 4.2 Model

We first introduce the linear price-demand relationship that is the basis of this dissertation. Demand for a product is modeled as a downward sloping function of its own price with the effect of its substitute product. This demand model is derived by maximizing the utility function given by Singh and Vives (1984) as follows:

$$\max_{Q_1,Q_2} U(Q_1, Q_2) = a_1 Q_1 + a_2 Q_2 - (\vartheta_1 Q_1^2 + 2bQ_1Q_2 + \vartheta_2 Q_2^2)/2$$

where $Q_i$ is the amount of product i, for $i = 1, 2$. The parameter $b \in [0, 1)$ is the measure of product substitutability, $a_i$ are the observed/realized market potentials, for $i = 1, 2$, with $\vartheta_2 a_1 \geq ba_2$ and $\vartheta_1 a_2 \geq ba_1$ (to enforce positive demand realizations). $\vartheta_i$ for $i = 1, 2$, are the demand sensitivity parameters for each product and we assume $\vartheta_1\vartheta_2 - b^2 >$ to ensure strict concavity of the utility function.

By maximizing the utility function, we obtain the following linear demand function:

$$Q_1(\mathbf{p}, \boldsymbol{a}) = \frac{\vartheta_2 a_1 - ba_2}{\vartheta_1\vartheta_2 - b^2} - \frac{\vartheta_2}{\vartheta_1\vartheta_2 - b^2} p_1 + \frac{b}{\vartheta_1\vartheta_2 - b^2} p_2 \tag{4.1}$$

$$Q_2(\mathbf{p}, \boldsymbol{a}) = \frac{\vartheta_1 a_2 - ba_1}{\vartheta_1\vartheta_2 - b^2} - \frac{\vartheta_1}{\vartheta_1\vartheta_2 - b^2} p_2 + \frac{b}{\vartheta_1\vartheta_2 - b^2} p_1 \tag{4.2}$$

where $p_i$ is the price for product $i$, where $i = 1, 2$. To simplify expressions in subsequent sections we assume parameters $\vartheta_1$=$\vartheta_2$=1 as they can be easily incorporated into the model without changing any of the insights derived. The inverse of the demand function is as follows,

$$p_1 = a_1 - Q_1 - bQ_2; \qquad p_2 = a_2 - Q_2 - bQ_1. \tag{4.3}$$

This demand function captures the fact that as products become more substitutable (i.e., as $b$ increases), customers become more sensitive to price changes and the overall market size decreases (Talluri and van Ryzin 2005).

We consider a two-stage optimization problem for a firm producing two products, indexed by $i = 1, 2$. In Stage I, while the demands for the two products are uncertain, the firm determines the optimal capacity investment levels for each product denoted by $K_1$ and $K_2$. In addition the firm also decides on the level of flexibility captured as the degree of efficiency loss as described below.

**Cross-production efficiency loss:** We denote $\beta_{ij} < 1$, for $i \neq j$, as the partial flexibility factor when using resource $j$ to produce product $i$. That is one unit of resource $j$ (which normally can produce one unit of product $j$) can only be used to produce $\beta_{ij}$ units of product $i$. For simplicity, in the following, we denote $\beta_{12}$ as $\beta_1$ and $\beta_{21}$ as $\beta_2$. The firm can now choose the level of (partial) flexibility by deciding on optimal $\beta_1$ and $\beta_2$ in Stage I. A higher degree of flexibility investment in Stage I leads to lower efficiency loss during cross-production in Stage II.

**Demand uncertainties:** The product demand uncertainty in Stage I is modeled by random demand intercepts $\boldsymbol{A} = (A_1,A_2)$ with positive support and the mean of their marginal distribution is denoted by $\mu_i$ for $i = 1, 2$. Their joint distribution is denoted by $\Psi(A_1, A_2)$. Essentially, demand uncertainty is modeled by assuming that the demand curve intercepts $A_1$,$A_2$ have a continuous probability distribution characterized by the joint density function described above. Their variance is denoted by $\sigma_2^2$, $\sigma_2^2$, for products 1 and 2, respectively. Using $\rho \in [-1, 1]$ as the correlation coefficient, the covariance of the joint distribution is $\sigma_{12} = \rho\sigma_1\sigma_2$. We denote the total demand uncertainty of the two produces by $\sigma_T^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$. The firm thus decides $K_1$,$K_2$, $\beta_1$ and $\beta_2$ in Stage I based on the distribution of $A_1$ and $A_2$.

In Stage II the firm observes the *realizations* of product demands denoted by $a_1$ and $a_2$ corresponding to RVs $A_1$ and $A_2$, respectively, and available capacity levels $K_1$ and $K_2$ contingent on the level of resource uncertainties. Given a level of product substitutability $b$ and other Stage II parameters ($K_1$,$K_2$,$\beta_1$,$\beta_2$,$a_1$,$a_2$) the firm then determines the optimal production quantities $Q_1$,$Q_2$ of the two products while maximizing its profit.

Under responsive pricing, the price is set only after the demand curve intercepts are realized. Since we assume a monopolistic price-setting firm the prices and sold quantities are duals and we hence use the firm's decision variables in Stage II to be the production quantities $Q_1$,$Q_2$ which may be sold at market clearing prices.

Let $\widetilde{\Pi}$ denote the expected profit in Stage I and $\widetilde{R}$ denote the revenue in Stage II. The two-stage decision problem for the firm can be formulated as follows:

$$\textit{Stage I}: \qquad \widetilde{\Pi} = \max_{\bar{K_1},\bar{K_2},\beta_1,\beta_2} \boldsymbol{E}[\widetilde{R}(\bar{K_1},\bar{K_2},\beta_1,\beta_2)] - \sum_{i=1,2} g_i\bar{K_i} - g_3(\beta_1\bar{K_1}+\beta_2\bar{K_2}) \tag{4.4}$$

$$\textit{subject to}: \ \bar{K_1},\bar{K_2} \geq 0$$

$$0 \leq \beta_1,\beta_2 \leq 1$$

where $\bar{K_1}$ and $\bar{K_2}$ are the capacity invested, and $g_1$ and $g_2$ are the unit capacity costs. In the third term, $g_3$ is the additional cost for a resource with the flexibility degree (shrinking capacity factor) equal to $\beta_1$ and $\beta_2$.

At the beginning of Stage II, the market potentials, $a_1$ and $a_2$, and the available capacities, $K_1 = \theta_1\bar{K_1}$ and $K_2 = \theta_2\bar{K_2}$, are realized. In the numerical analysis in Section 4.4 the resource uncertainty factor $\Theta_i$ follows a Normal distribution in the capacity uncertainty case and is modeled as a 0-1 binary variable through a Bernoulli distribution in the supply disruption case. In Stage II, the firm makes production decisions to maximize its profit. By Equation (4.3), the prices are obtained from the optimal production quantities as they are linearly related.

$$\text{Stage II}: \qquad \widetilde{R} = \max_{Q_1, Q_2} [\mathbf{pQ}] \tag{4.5}$$

$$\text{subject to}: \; Q_1 \leq K_1 + \beta_1(K_2 - Q_2) \tag{4.6}$$

$$Q_2 \leq K_2 + \beta_2(K_1 - Q_1) \tag{4.7}$$

$$Q_1, Q_2 \geq 0$$

## 4.3 Analytical Results

### 4.3.1 Solution Methodology

We study the capacity, flexibility and pricing decision problems under a two-stage stochastic program framework where the *demand intercepts* $A_i$, $i$=1, 2 are *random variables*. We solve the problem backwards by first solving the Stage II revenue maximization problem utilizing a state-space decomposition approach of the feasible region to obtain optimal production quantities (and prices). Once we obtain the optimal Stage II solutions, we use those as inputs to obtain the optimal capacity and flexibility levels in Stage I as well as the optimal expected profit. This solution methodology is similar to Fine and Freund (1990), Chod and Rudi (2005), Bish and Wang (2004), Goyal and Netessine (2007) and Lus and Muriel (2009).

The problem formulation described in equation 4.4 consists of two inter-linked stages. In Stage I, before the demand intercepts are realized (i.e., under demand uncertainties), the firm decides the capacity levels ($K_1K_2$) and degrees of flexibility ($\beta_1$,$\beta_2$). In this stage only the distributions of these random variables are assumed to be known i.e., the mean of the demand intercepts denoted by $\mu_i$, $i$=1, 2 and the standard deviations denoted by $\sigma_i$, $i$=1, 2. The expectation $\boldsymbol{E}$ is with respect to random demand intercepts $A_i$, $i$=1, 2. In Stage II, after observing the corresponding realizations of the demand intercepts $a_i$, $i$=1, 2, the production quantities $Q_1, Q_2$ are determined. The prices $p_1, p_2$ are linearly related to the production quantities as given by equation (4.3) and hence can be easily obtained. The Stage II decision variables are a function of the observed demand realizations as well as invested capacities in Stage I. While we analytically investigate the impact of demand uncertainties, we numerically investigate the impact of capacity uncertainties and supply disruptions.

We solve this two-stage problem backwards by first considering the Stage II problem and determining the optimal solutions and their properties. Since the realizations of the demand intercepts $a_i$, $i$=$1, 2$, for the two products could be above or below their invested capacities $K_i$, $i$=$1, 2$, we must consider all possible cases together with the capacity constraints. We note here that under only demand uncertainties the firm does not face any resource supply uncertainties and the invested capacities in Stage I $(\bar{K_1}, \bar{K_2})$ are the same as the available capacities $K_1, K_2$ in Stage II. To simplify the exposition we hence use $K_1$ and $K_2$ as the capacity decision variables in Stage I under only demand uncertainties.

Given a resource investment vector $(K_1, K_2, \beta_1, \beta_2)$, demand realizations $a_1, a_2$ and degree of product substitutability $b$, we partition the demand intercept space into disjoint sets also called demand regions. Each case (demand region) corresponds to an optimization problem in Stage II and can be solved in closed form and the production quantities can be obtained.

In the Shrinking Capacity model, there are 5 possible cases as described below:

**Case (1):** The available capacities for both products are individually sufficient to meet the respective product demands and the firm does not need to use all the invested capacities nor invest in any cross-production. $Q_1 < K_1 + \beta_1(K_2 - Q_2)$ and $Q_2 < K_2 + \beta_2(K_1 - Q_1)$ and the unconstrained solution is feasible.

**Case (2):** Demand realization of product 1 ($a_1$) is high and available dedicated capacity for product 1 is insufficient i.e., $Q_1 \geq K_1$. Demand realization of product 2 ($a_2$) is low and does not utilize all of its dedicated capacity and hence $Q_2 < K_2 + \beta_2(K_1 - Q_1)$. In this case firm needs to cross-produce product 1 in using resource 2 and hence constraint 4.6 is binding $Q_1 = K_1 + \beta_1(K_2 - Q_2)$.

**Case (3):** This case is symmetric to the previous case where demand realization of product 2 ($a_2$) is high while demand observed for product 1 ($a_1$) is lower than available dedicated capacity and the firm chooses to cross-produce product 2 using resource 1. We have $Q_2 = K_2 + \beta_2(K_1 - Q_1)$ and $Q_1 < K_1 + \beta_1(K_2 - Q_2)$.

**Case (4):** The demand realization for product 1 ($a_1$) is much higher than the demand realization for product 2 ($a_2$) so the firm uses all of its capacities to make product 1 which means $Q_1 = K_1 + \beta_1 K_2$ and $Q_2 = 0$. The firm hence uses cross-production in this scenario and sells only product 1.

**Case (5):** This case is symmetric to the previous case as demand for product 2 ($a_2$) is much higher than the demand realization for product 1 ($a_1$) and the optimal production quantities are $Q_2 = K_2 + \beta_2 K_1$ and $Q_1 = 0$.

In the next section we obtain the optimal Stage II solutions for each of the cases considered above.

### 4.3.2 Stage II Optimal Production Quantities

Before solving the Stage II problem, we first algebraically define the demand regions corresponding to each case described in the previous section:

$$
\begin{aligned}
\Omega_1: \quad & a_1(1-b\beta_1) + a_2(\beta_1 - b) \leq 2(1-b^2)(K_1 + \beta_1 K_2), \\
& a_2(1-b\beta_2) + a_1(\beta_2 - b) \leq 2(1-b^2)(K_2 + \beta_2 K_1); \\
\Omega_2: \quad & 2(1-b)(K_1 - K_2) \leq a_1 - a_2 \leq 2(1-b)(K_1 + \beta_1 K_2), \\
& a_1 - ba_2 + \beta_1(a_2 - ba_1) \geq 2(1-b^2)(K_1 + \beta_1 K_2); \\
\Omega_3: \quad & 2(1-b)(K_2 - K_1) \leq a_2 - a_1 \leq 2(1-b)(K_2 + \beta_2 K_1), \\
& a_2 - ba_1 + \beta_2(a_1 - ba_2) \geq 2(1-b^2)(K_2 + \beta_2 K_1); \\
\Omega_4: \quad & a_1 - a_2 \geq 2(1-b)(K_1 + \beta_1 K_2); \\
\Omega_5: \quad & a_2 - a_1 \geq 2(1-b)(K_2 + \beta_2 K_1);
\end{aligned}
$$

The demand regions for the independent products case (i.e., when $b$=0) are shown in Figure 4.1-left and the corresponding optimal production quantities on the right. In region $\Omega_1$ the demand curve intercepts for both the products are low and hence the unconstrained solution is achieved. In regions $\Omega_2$ and $\Omega_3$ the firm sells positive quantities of both products. However, since demand is higher for one product in a region the corresponding optimal production quantity is also higher. In regions $\Omega_4$ and $\Omega_5$ the demand for one product is significantly higher than the other and hence the firm utilizes all of the available capacities to make this product.

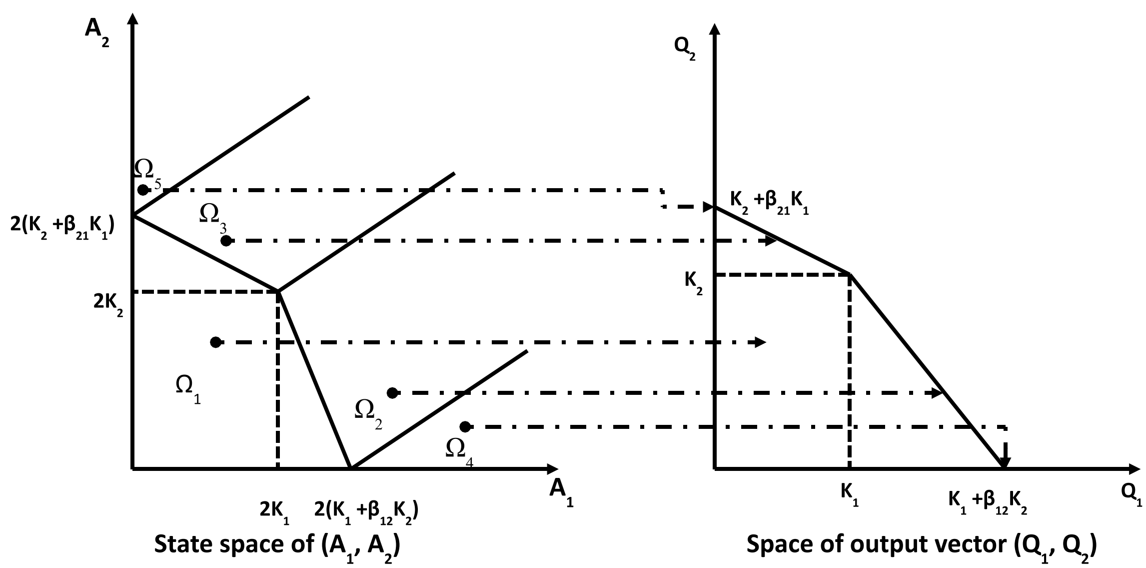


Figure 4.1: Mapping of the state space of $\mathbf{A}$ into the output space for $b = 0$ case

**Theorem 4.1** *Given the realizations of the market potentials* $(a_1, a_2)$ *and the capacities* $(K_1, K_2)$*, the optimal production quantities for the Stage II problem are given as follows:*

$$\Omega_1: \quad Q_1 = \frac{a_1 - ba_2}{2(1-b^2)}, Q_2 = \frac{a_2 - ba_1}{2(1-b^2)};$$

$$\Omega_2: \quad Q_1 = \frac{(a_1 - a_2)\beta_1 + 2(1-b)(K_1 + \beta_1 K_2)}{2(1-b)(1+\beta_1)}, Q_2 = \frac{a_2 - a_1 + 2(1-b)(K_1 + \beta_1 K_2)}{2(1-b)(1+\beta_1)};$$

$$\Omega_3: \quad Q_1 = \frac{a_1 - a_2 + 2(1-b)(K_2 + \beta_2 K_1)}{2(1-b)(1+\beta_2)}, Q_2 = \frac{(a_2 - a_1)\beta_2 + 2(1-b)(K_2 + \beta_2 K_1)}{2(1-b)(1+\beta_2)};$$

$$\Omega_4: \quad Q_1 = K_1 + \beta_1 K_2, Q_2 = 0;$$

$$\Omega_5: \quad Q_1 = 0, Q_2 = K_2 + \beta_2 K_1;$$

Please refer to Appendix B for all the proofs in this chapter.

In each region, by solving the Stage II problem, the optimal revenue $R_i(a_1, a_2)$ equals,

$$\Omega_1: \quad R_1 = \frac{a_1^2 + a_2^2 - 2ba_1a_2}{4(1-b^2)};$$

$$\Omega_2: \quad R_2 = \frac{a_2^2 + a_1^2\beta_1^2 - 4a_1(b\beta_1 - 1)(K_1 + \beta_1 K_2) + 4(b^2-1)(K_1 + \beta_1 K_2)^2}{4(1+\beta_1^2) - 8b\beta_1} - \frac{2a_2[a_1\beta_1 + 2(b - \beta_1)(K_1 + \beta_1 K_2)]}{4(1+\beta_1^2) - 8b\beta_1};$$

$$\Omega_3: \quad R_3 = \frac{a_1^2 + a_2^2\beta_2^2 - 4a_2(b\beta_2 - 1)(K_2 + \beta_2 K_1) + 4(b^2-1)(K_2 + \beta_2 K_1)^2}{4(1+\beta_2^2) - 8b\beta_2} - \frac{2a_1[a_2\beta_2 + 2(b - \beta_2)(K_2 + \beta_2 K_1)]}{4(1+\beta_2^2) - 8b\beta_2};$$

$$\Omega_4: \quad R_4 = a_1(K_1 + \beta_1 K_2) - (K_1 + \beta_1 K_2)^2;$$

$$\Omega_5: \quad R_5 = a_2(K_2 + \beta_2 K_1) - (K_2 + \beta_2 K_1)^2;$$

### 4.3.3 Stage I Optimal Capacity and Flexibility Investment

Using the above results, we now analyze the firm's optimal capacity investment portfolio $(K_1, K_2, \beta_1, \beta_2)$ in Stage I. To simplify the Stage I problem, we assume $\beta_1 = \beta_2 = \beta$ indicating that any increase in flexibility for one resource also increases the flexibility for both resources (and hence incurs costs). While this assumption impacts the results on degree of flexibility of the individual resource, it does not impact our primary objectives of understanding the impact of increasing uncertainties, demand correlation $\rho$ as well as increasing $b$ as our numerical studies show.

**Lemma 4.1** *The Stage I objective function $\widetilde{\Pi}$ is strictly jointly concave in $(K_1, K_2, \beta)$*

Hence, the following theorem provides the necessary and sufficient conditions for the optimal capacity investment portfolio $(K_1, K_2, \beta)$.

**Theorem 4.2** *The Stage I optimal capacity investment vector $(K_1, K_2, \beta)$ is optimal if and only if the Lagrangian multipliers given by $\lambda_i$ exist and satisfy the following conditions:*

$$\frac{\partial \widetilde{R}(K_1, K_2, \beta)}{\partial K_i} = g_i - \lambda_i \ for \ i \in \{1, 2\},$$

$$\frac{\partial \widetilde{R}(K_1, K_2, \beta)}{\partial \beta} = g_3 - \lambda_3,$$

$$K_i \lambda_i = 0 \ for \ i \in \{1, 2\}, \ \beta \lambda_3 = 0.$$

*Hence, conditioning on $(A_1, A_2)$, we obtain,*

$$\left\{ \begin{array}{c} g_1 - \lambda_1 \\ g_2 - \lambda_2 \\ g_3 - \lambda_3 \end{array} \right\} = Pr(\Omega_2)\mathbf{E} \left\{ \begin{array}{c} \frac{2(A_1(1-b\beta)+A_2(\beta-b)-2(1-b^2)(K_1+\beta K_2))}{(1-b)(1+\beta)^2} \\ \frac{2\beta(A_1(1-b\beta)+A_2(\beta-b)-2(1-b^2)(K_1+\beta K_2))}{(1-b)(1+\beta)^2} \\ \frac{(A_1-A_2+2(b-1)(K_1-K_2))(A_1(1-b\beta)+A_2(\beta-b)+2(b^2-1)(K_1+\beta K_2))}{(b-1)^2(1+\beta)^3} \end{array} \right\}$$

$$+ Pr(\Omega_3)\mathbf{E} \left\{ \begin{array}{c} \frac{2\beta(A_2(1-b\beta)+A_1(\beta-b)-2(1-b^2)(K_2+\beta K_1)}{(1-b)(1+\beta)^2} \\ \frac{2(A_2(1-b\beta)+A_1(\beta-b)-2(1-b^2)(K_2+\beta K_1))}{(1-b)(1+\beta)^2} \\ \frac{(A_2-A_1+2(b-1)(K_2-K_1))(A_2(1-b\beta)+A_1(\beta-b)+2(b^2-1)(K_2+\beta K_1))}{(b-1)^2(1+\beta)^3} \end{array} \right\}$$

$$+ Pr(\Omega_4)\mathbf{E} \left\{ \begin{array}{c} [A_1 - 2(K_1 + \beta K_2)] \\ \beta[A_1 - 2(K_1 + \beta K_2)] \\ K_2[A_1 - 2(K_1 + \beta K_2)] \end{array} \right\}$$

$$+ Pr(\Omega_5)\mathbf{E} \left\{ \begin{array}{c} \beta[A_2 - 2(K_2 + \beta K_1)] \\ A_2 - 2(K_2 + \beta K_1) \\ K_1[A_2 - 2(K_2 + \beta K_1)] \end{array} \right\} \quad \blacksquare$$

### 4.3.4 Properties of Optimal Solutions

**Theorem 4.3** *The firm's optimal expected profit decreases as product substitutability $b$ increases, i.e.,* $\frac{\partial \widetilde{\Pi}}{\partial b} < 0$.

Theorem 4.3 states that as products become increasingly substitutable, the optimal expected profit decreases. This is not so surprising since we know that as $b$ increases, customers become more sensitive to price changes and the overall market size decreases (Talluri and van Ryzin 2005).

Next we study the impact of $b$ on the optimal capacities $K_1$ and $K_2$.

**Theorem 4.4** *Given $\beta$ and $K_j$, the optimal capacity $K_i$ decreases with $b$, i.e.,* $\frac{\partial K_i}{\partial b} < 0$, $i, j = \{1, 2\}$ *and* $i \neq j$.

Investment in capacity is costly as compared to utilizing responsive pricing to shift demand. When products are substitutes the demand from one product can be efficiently shifted to the other using pricing and hence an additional investment in capacities is discouraged. Hence demand management through responsive pricing is an effective strategy compared to investment in capacity. This trend exists even $\beta$ and $K_j$ are endogenous as verified by our numerical analysis in Section 4.4.

### 4.3.5 Optimal solutions under clearance assumption

We now analyze the impact of the optimal capacity and flexibility investments using the clearance assumption. We assume that the firm always produces to available capacity $K_i$, $i = 1, 2$, irrespective of any demand intercept realization and that the production quantities $Q_i$, $i = 1, 2$ are always positive (please refer to Van Mieghem and Dada 1999 for details on *clearance* and *hold-back* strategies). We derive closed form expressions of capacities $K_i$ to better understand the impacts of other parameters.

Obviously, clearance is sub-optimal behavior since the firm is forced to produce at the capacity level for any demand intercept realization. However, this is not unrealistic as Goyal and Netessine (2007) note "...firms often find it difficult to produce below capacity in view of large fixed costs...Car makers have been forced to slash prices to keep lines running as models fall out of favor with the public, rather than reduce production". Chod and Rudi (2005) numerically investigate the impact of the clearance assumption and conclude that they generally yield close-to-optimal solutions.

In our demand model, the clearance assumption for capacities $K_1$ ($K_2$) is satisfied naturally in regions $\Omega_2$ ($\Omega_3$) and $\Omega_4(\Omega_5)$. When $\boldsymbol{A} \in \Omega_2$ dedicated capacity $K_1$ is used completely to produce product 1. Assuming a symmetric capacity investment ($K_1$=$K_2$=$K$), symmetric investment cost ($g_1$=$g_2$=$g$) and exogenous $\beta$, we have the following theorem that characterizes the relationship between optimal clearance capacity $K$ and other parameters in the model.

**Theorem 4.5** *Given $\beta$, $g$ and $b$, the optimal clearance capacity*

$$\hat{K}^* = \frac{\mu_1 + \mu_2 - 2g}{4(1+b)} + \frac{(\mu_1 - \mu_2)(1-\beta)}{4(1-b)(1+\beta)}$$

The first term in Theorem 4.5 reveals that optimal capacity increases in total market size ($\mu_1 + \mu_2$) and decreases in its investment cost. The second term reveals that the optimal capacity increases with the difference between the market size ($\mu_1 - \mu_2$) which is in turn impacted by the level of product differentiation $b$ and flexibility $\beta$. As $b$ increases the market size difference is amplified which increases investment in capacity. However this increase is tempered by the level of inherent flexibility $\beta$: A firm with very high efficiency loss (low $\beta$) invests in a higher amount of dedicated capacity whereas a firm with low efficiency loss (high $\beta$) invests in much lower dedicated capacity. Term 1 in conjunction with term 2 reveals that as product substitutability increases the overall optimal capacity investment decreases because when products are less differentiated the two markets overlap more and hence the total expected market size is lower (i.e., term 1 dominates).

Due to the assumptions made in clearance, viz., the firm produces up to capacity levels and production quantities are always positive, the optimal capacities are independent of variability or correlation of the demand intercepts. The total capacities translate directly into expected prices. In Section 5, we endogenize $\beta$ and numerically study the effect of different parameters when the firm simultaneously invests in $K_1$,$K_2$ and $\beta$ under increasing demand uncertainty and correlation (without clearance assumption).

Next we study the impact of increasing $b$ on the reconfiguration loss or shrinking capacity factor $\beta$.

**Theorem 4.6** *Given $K_1, K_2$, the optimal shrinking capacity factor $\beta^*$ decreases as products become more substitutable i.e.,* $\frac{\partial \beta^*}{\partial b} < 0$

Theorem 4.6 reveals that as the products become less differentiated (more substitutable) the firm prefers to invest in lower degree of resource flexibility. As the level of product substitutability $b$ increases, the price potentials (intercepts) $A_i$ for the two products tend to be highly correlated, reducing the benefit associated with shifting production. In addition, demand for the substitutes can be easily managed through pricing, which reduces the need for flexibility to balance supply and demand. Hence, the optimal flexibility levels decrease with product substitutability.

Our extensive numerical analysis in Section 4.4 reveals that these trends remain the same when product substitutability $b$ increases across *all* demand realizations.

## 4.4 Numerical Analysis

In this section we report the key results of extensive numerical experiments conducted to understand the impact of the nature and severity of demand and capacity uncertainties. In 4.4.1 we study the impact of high demand uncertainty by changing the coefficient of variation (CV) of the Normal distribution of the demand. Next in 4.4.2 we investigate the impact of capacity uncertainties by changing the CV of the Normal distribution of the yields. In 4.4.3 we study the impact of supply disruptions (also called 0-1 disruptions) by changing the disruption probability of the Bernoulli distribution of the yields. Under supply disruptions, a resource is either completely available or completely unavailable in the production stage. We use the scenario-based stochastic programming approach to study the impacts of product substitutability, demand correlation, demand uncertainty, capacity uncertainty, and supply disruptions on the expected profit and the capacity and the flexibility investment decisions. In 4.4.4 we investigate the impact of responsive pricing by studying the case of a firm that does not have any pricing power. The firm faces capacity and demand uncertainties and price is exogenous in Stage I. Finally, in section 4.4.5 we evaluate the sensitivity of the investment decisions in sections 4.4.1-4.4.3 to unit capacity investment cost as well as relative cost of flexibility.

**Experimental Design:** In the first four sub-sections, we generate demand scenarios from a multivariate Normal distribution with means $\frac{\vartheta_2 E[A_1]-bE[A_2]}{\vartheta_1\vartheta_2-b^2}$ and $\frac{\vartheta_1 E[A_2]-bE[A_1]}{\vartheta_1\vartheta_2-b^2}$ as recommended in Lus and Muriel (2009). Notice that the distribution means are calculated by Equation (4.2) with symmetric slope intercepts ($\vartheta_1 = \vartheta_2 = 1$), symmetric market potential ($E[A_1] = E[A_2] = 2500$), and nonnegative product substitutability $b \geq 0$. We set the product demand correlation $\rho$ at -0.5, 0 and +0.5 for each of the cases and compare the impact of increasing $b$ on the optimal total capacity ($K_t^* = K_1^* + K_2^*$), the partial flexibility degree ($\beta^*$), and the optimal expected profit. The cost of a unit of dedicated capacity for each product ($g_1, g_2$) is 1 while the additional cost rate for the flexibility degree ($g_3$) is 0.2. Please note that the above parameter setting satisfies the conditions in the demand model: $\vartheta_2 A_1 - bA_2 > 0$, $\vartheta_1 A_2 - bA_1 > 0$, $\vartheta_1\vartheta_2 - b^2 > 0$ and $\vartheta_1, \vartheta_2 > b$. In addition, the normalized ratio $\frac{b^2}{\vartheta_1\vartheta_2}$

represents the relative degree of substitutability between the two products: it is 0 when $b = 0$ (independent) and 1 when $A_1 \approx A_2$ and $b = \vartheta_1 = \vartheta_2$ (perfect substitutes) theoretically.

In the first case, we study the impact of the demand uncertainty by increasing the CV of demand from 0.1 to 0.4. To be comparable with Cases 2 and 3, where the capacity yields are uncertain, we fix the capacity yield rates $\theta_i = 0.5, i = 1, 2$ in Case 1.

In the second case, the capacity uncertainty is modeled with independent and Normally distributed yield rates $\Theta_i, i = 1, 2$, with both means equal to 0.5. We study the impact of the capacity uncertainty by increasing the CV from 0.1 to 0.4. The demand follows Normal distribution with CV equal to 0.2 with other parameters as stated above, and we assume that the capacity uncertainties are independent of the demand uncertainties.

In the third case, supply disruption is modeled with identically and independently distributed Bernoulli random variables $\Theta_i \in \{0, 1\}, i = 1, 2$. A resource either has full yield (realization $\theta_i$=1) or is completely disrupted ($\theta_i$=0) in the production stage. There are four scenarios corresponding to the two resources: (0,0), (0,1), (1,0) and (1,1). The probability of each scenario reflects the disruption risk. We study the impact of disruption by increasing the CV. A higher CV indicates a higher risk of disruption. The demands for the two products are normally distributed with CV=0.2, and we assume that the the supply disruptions are independent of demand uncertainties.

Under the no responsive pricing scenario, the firm first determines the optimal prices to be set in Stage I assuming that the demand and the yield rates are deterministic at the expected value. We analyze this scenario using the capacity uncertainty case with $\Theta_i, i = 1, 2$ as 0.5 and varying the CV from 0.1 to 0.4. Once the prices are set we subsequently calculate capacities, reconfiguration levels and expected profits by the methodology described in the experimental design.

To analyze the sensitivity of results to the investment cost, we increase $g$=$g_1$=$g_2$ from 1 to 9 and for each value of $g$ we study the effect of increasing the cost of flexibility $g_3$ . This is done for each of the three types of uncertainties noted earlier. With this analysis we are able to comprehensively compare how the firm's optimal portfolio changes as investment cost increases.

### 4.4.1 Impacts of Demand Uncertainties

We study the impact of demand uncertainties modeled by increasing the CV of normally distributed product demands under different correlation scenarios. We also consider the demand correlation, as it is a key factor influencing capacity investment decisions in operational and financial hedging models (Chod et al. 2010). We fix the correlation of the demand intercepts and study the effect of product substitutability for each demand correlation level. We report the representative cases where $\rho$ was fixed at -0.5, 0 and +0.5. In addition, to compare these results with capacity uncertainties, we restrict the capacities available in the production stage to $0.5K_i$, $i = 1, 2$. While this restriction affects the capacities numerically it ensures that the investment trends, which is the major focus of this research, are unaltered. In Case 1, we assume the firm does not face any yield uncertainties or risk of supply disruptions, i.e., $\theta_1^m = \theta_2^m = 1$.

The results for the demand uncertainty case are shown in Table 4.1, and they are summarized as follows: (1) Flexibility is preferred only under very high demand CV for highly differentiated products under negative demand correlation $\rho$. (2) As the demand CV increases, the total dedicated capacity investment $K_t^*$ and the optimal profits increase. (3) With a fixed positive product substitutability $b$, total capacity $K_t^*$ and the optimal profits *increase* with $\rho$. (4) As the product substitutability $b$ increases, the total capacities and the expected profits decrease.

Table 4.1: Impacts of Demand Uncertainties

| | | CV=0.1 | | | CV=0.2 | | | CV=0.4 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\rho$ | $b$ | $K_t^*$ | $\beta^*$ | Profit | $K_t^*$ | $\beta^*$ | Profit | $K_t^*$ | $\beta^*$ | Profit |
| -0.5 | 0 | 4960 | 0.001 | 57737 | 5378 | 0.001 | 58579 | 5748 | 0.49 | 63183 |
| | 0.2 | 4092 | 0.001 | 48044 | 4382 | 0.001 | 48445 | 4834 | 0.41 | 51190 |
| | 0.4 | 3482 | 0.001 | 41142 | 3684 | 0.001 | 41379 | 4198 | 0.23 | 43016 |
| | 0.6 | 3036 | 0.001 | 36039 | 3170 | 0.001 | 36109 | 3624 | 0.001 | 37178 |
| | 0.8 | 2688 | 0.001 | 31967 | 2784 | 0.001 | 32072 | 3100 | 0.001 | 32799 |
| 0 | 0 | 4964 | 0.001 | 57743 | 5392 | 0.001 | 58615 | 6324 | 0.15 | 63095 |
| | 0.2 | 4108 | 0.001 | 48068 | 4416 | 0.001 | 48517 | 5276 | 0.001 | 51585 |
| | 0.4 | 3504 | 0.001 | 41164 | 3742 | 0.001 | 41531 | 4406 | 0.001 | 43666 |
| | 0.6 | 3058 | 0.001 | 36061 | 3234 | 0.001 | 36211 | 3772 | 0.001 | 37881 |
| | 0.8 | 2708 | 0.001 | 31985 | 2854 | 0.001 | 32195 | 3282 | 0.001 | 33465 |
| 0.5 | 0 | 4960 | 0.001 | 57737 | 5378 | 0.001 | 58578 | 6470 | 0.001 | 62889 |
| | 0.2 | 4116 | 0.001 | 48088 | 4430 | 0.001 | 48557 | 5294 | 0.001 | 51877 |
| | 0.4 | 3518 | 0.001 | 41183 | 3778 | 0.001 | 41651 | 4476 | 0.001 | 44163 |
| | 0.6 | 3076 | 0.001 | 36073 | 3284 | 0.001 | 36308 | 3868 | 0.001 | 38458 |
| | 0.8 | 2728 | 0.001 | 32003 | 2910 | 0.001 | 32311 | 3406 | 0.001 | 34058 |

A major difference in the results between our cross-production model and literature is the value of flexibility. Flexibility is usually assumed free and thus is often recommended in the literature (e.g., Chod and Rudi 2005, Goyal and Netessine 2007, Goyal and Netessine 2010) even when uncertainty in demand is low. However, when flexibility is not free, with just a moderate cost ($g_3 = 0.2$), investment in flexibility is not recommended in most of the cases as shown in Table 4.1. Flexibility is recommended only when the demand uncertainty is very high *and* the demand correlation is negative or zero. In those settings, the manufacturer frequently needs cross production to increase the quantity of higher demand product. As we would see in the next sub-sections, this investment trend dramatically changes when the type of supply uncertainty changes.

In the literature (e.g., Lus and Muriel 2009), the total dedicated capacity usually decreases (while flexible capacity increases) with the demand uncertainty. In contrast, Table 4.1 shows that the total dedicated capacity investment $K_t^*$ and the optimal expected profit always increase as the demand uncertainty increases. To cope with demand variability the firm simply invests in higher amount of dedicated capacities rather than the higher level of flexibility. It is much more economical to increase the investment levels of dedicated capacities and invest in *partially* reconfiguring them as opposed to investing in full flexibility. In our numerical results, we find that if the demand for a product is very low then the firm employs a hold-back strategy, i.e., it sells only a restricted quantity of the product at a higher price than selling all of it at much lower market-clearing prices, mitigating the over-investment risk.

For a fixed positive product substitutability ($b > 0$), the optimal expected profit increases as the demand correlation $\rho$ increases. This trend under partially flexible resources is consistent with literature that considers completely flexible resources (e.g., Lus and Muriel 2009, Goyal and Netessine 2007). Lus and Muriel (2009) showed that as $\rho$ increases the optimal profit increases (or decreases) under high (or low) product substitutability $b$. In their model, flexibility was highly valuable under low product substitutability but also incurred a higher investment cost. With high values of $b$, the price intercepts are generally highly correlated since customers are more sensitive to product price changes which makes flexibility less valuable. In addition, Oi (1961) showed that a firm using responsive pricing makes more money in the high demand states than it loses in the low demand states. Hence, as $\rho$ increases the probabilities of both demands being high increases due to which the firm benefits from higher expected profit. In our model, even when products are highly differentiated and under negative demand correlation, the investment in flexibility is generally zero (CV=0.1

and 0.2 cases). This is because if $\beta^*$ were to increase to produce say product 1 (with higher demand) on resource 2, it also implicitly reconfigures resource 1 (since $\beta_1^* = \beta_2^* = \beta^*$), a very costly effort given the cost structure.

Our results on limited flexibility is also substantiated by Tang and Tomlin (2009) who argue that "The higher the degree of flexibility required the more costly the investment and, therefore, the more likely it is that a precise ROI analysis will be required to justify the investment. The fact that a relatively low degree of flexibility is often sufficient may enable managers to justify flexibility investments more readily, even if precise estimates of costs, impacts, and likelihoods are not available."

### 4.4.2 Impacts of Capacity Uncertainties

Next we investigate the optimal investment decisions when the firm faces capacity uncertainties. The results are shown in Table 4.2.

Our study reveals that: (1) A moderate degree of flexibility hedges the firm against *high* capacity uncertainty (CV=0.4) only under negative demand correlation and low product substitutability. Flexibility is generally unnecessary when capacity uncertainty is low. (2) The total capacity investment $K_t^*$ and the profit *increase* with $\rho$. This trend is also observed in Table 4.1 under only demand uncertainty. (3) For a fixed $b$, the total capacity investment $K_t^*$ increases as capacity uncertainty increases. The profit, however, decreases with the capacity uncertainty, which is different from Table 4.1.

When the degree of capacity uncertainties is low or moderate, no degree of flexibility is required because there is always a certain amount of realized capacity still available for each product. An increased investment in cheaper dedicated capacities in planning stage ensures that the firm has enough realized capacity in the selling stage. As product substitutability $b$ increases, the firm can also adjust the prices to match the demands for the two products with the available capacities. Investing in

Table 4.2: Impacts of Capacity Uncertainties

| | | CV=0.1 | | | CV=0.2 | | | CV=0.4 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\rho$ | $b$ | $K_t^*$ | $\beta^*$ | Profit | $K_t^*$ | $\beta^*$ | Profit | $K_t^*$ | $\beta^*$ | Profit |
| -0.5 | 0 | 5414 | 0.001 | 58478 | 5528 | 0.04 | 58177 | 5662 | 0.49 | 57071 |
| | 0.2 | 4416 | 0.001 | 48368 | 4520 | 0.001 | 48137 | 4830 | 0.13 | 47188 |
| | 0.4 | 3714 | 0.001 | 41318 | 3798 | 0.001 | 41136 | 4086 | 0.001 | 40395 |
| | 0.6 | 3194 | 0.001 | 36060 | 3266 | 0.001 | 35912 | 3508 | 0.001 | 35324 |
| | 0.8 | 2804 | 0.001 | 32030 | 2864 | 0.001 | 31907 | 3074 | 0.001 | 31432 |
| 0 | 0 | 5428 | 0.001 | 58514 | 5564 | 0.001 | 58211 | 5826 | 0.33 | 57021 |
| | 0.2 | 4448 | 0.001 | 48440 | 4550 | 0.001 | 48210 | 4896 | 0.001 | 47253 |
| | 0.4 | 3768 | 0.001 | 41471 | 3848 | 0.001 | 41291 | 4128 | 0.001 | 40554 |
| | 0.6 | 3256 | 0.001 | 36164 | 3322 | 0.001 | 36021 | 3556 | 0.001 | 35441 |
| | 0.8 | 2872 | 0.001 | 32157 | 2926 | 0.001 | 32041 | 3126 | 0.001 | 31577 |
| 0.5 | 0 | 5414 | 0.001 | 58477 | 5552 | 0.001 | 58175 | 5934 | 0.17 | 56922 |
| | 0.2 | 4460 | 0.001 | 48481 | 4560 | 0.001 | 48251 | 4904 | 0.001 | 47295 |
| | 0.4 | 3802 | 0.001 | 41592 | 3880 | 0.001 | 41414 | 4154 | 0.001 | 40679 |
| | 0.6 | 3302 | 0.001 | 36262 | 3364 | 0.001 | 36122 | 3590 | 0.001 | 35547 |
| | 0.8 | 2924 | 0.001 | 32275 | 2974 | 0.001 | 32162 | 3170 | 0.001 | 31705 |

additional capacities is relatively cheaper than improving the flexibility for the whole resource capacity.

When facing high capacity uncertainties, the firm invests in moderate flexibility in both resources. As risk increases, the firm simultaneously increases *both* $\beta^*$ and $K_i^*$ in the planning stage to compensate for the capacity loss during production. The firm can transfer capacity from the product with low demand to the product with higher demand through cross-production. Further, with low $b$, responsive pricing is not a viable strategy and hence demand switching is ineffective.

Despite the increase in the overall capacity investment, the firm's profit decreases as the capacity uncertainty increases. This trend is different from Table 4.1 and the literature where the total capacity and the profit increase under only demand uncertainty. Under capacity uncertainties, only a proportion of the invested capacities is available in the production stage. As the capacity uncertainty increases, the capacity available and hence the production quantities decrease, and thus the firm's revenue decreases.

### 4.4.3 Impacts of Supply Disruptions

Now we investigate the impact of supply disruptions on the capacity and flexibility decisions by modeling $\Theta_1$ and $\Theta_2$ with Bernoulli random variables. To reflect the increasing level of uncertainty in supply disruptions, we increase the probabilities of scenarios with disruption. Table 4.3 shows the optimal capacities, the flexibility levels, and the profits for a firm facing capacity disruptions.

From Table 4.3 we find that: (1) Under moderate or high supply disruptions, the firm invests in a high degree of flexibility making cross-production an excellent risk hedging strategy for all levels product demand correlations. Under a low degree of supply disruption the firm invests in flexibility only when the two products are not substitutable to each other, i.e., $b = 0$ and the demand correlations are low. (2) For a fixed $b$, total capacity investment $K_t^*$ increases as supply disruption risk increases. However, the optimal expected profit decreases. (3) When the two products are not substitutable to each other, the firm's profit decreases as the demand correlation increases. However, with higher levels of product substitutability, the profit increases with the demand correlation.

The nature of uncertainty plays a key role in determining if flexibility is desirable or not. The risk is more severe under 0-1 disruptions than under capacity uncertainties, and the firm responds by investing in higher level of flexibility. As the level of disruption increases, the optimal expected profit decreases while the total capacity

Table 4.3: Impacts of Supply Disruptions

| | | CV=0.1 | | | CV=0.2 | | | CV=0.4 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\rho$ | $b$ | $K_t^*$ | $\beta^*$ | Profit | $K_t^*$ | $\beta^*$ | Profit | $K_t^*$ | $\beta^*$ | Profit |
| -0.5 | 0 | 2644 | 0.8 | 60845 | 2748 | 0.99 | 59932 | 4022 | 0.99 | 57575 |
| | 0.2 | 2344 | 0.001 | 50268 | 2368 | 0.99 | 49492 | 3350 | 0.99 | 47536 |
| | 0.4 | 1982 | 0.001 | 42971 | 2028 | 0.99 | 42031 | 2872 | 0.99 | 40534 |
| | 0.6 | 1706 | 0.001 | 37509 | 1778 | 0.001 | 36814 | 2512 | 0.99 | 35302 |
| | 0.8 | 1488 | 0.001 | 33318 | 1578 | 0.001 | 32780 | 2086 | 0.001 | 31155 |
| 0 | 0 | 2754 | 0.66 | 60789 | 2854 | 0.99 | 59903 | 4034 | 0.99 | 57578 |
| | 0.2 | 2362 | 0.001 | 50356 | 2372 | 0.99 | 49527 | 3356 | 0.99 | 47555 |
| | 0.4 | 2008 | 0.001 | 43149 | 2038 | 0.99 | 42349 | 2878 | 0.99 | 40660 |
| | 0.6 | 1739 | 0.001 | 37649 | 1808 | 0.001 | 36956 | 2516 | 0.99 | 35406 |
| | 0.8 | 1534 | 0.001 | 33486 | 1620 | 0.001 | 32960 | 2098 | 0.001 | 31311 |
| 0.5 | 0 | 2865 | 0.001 | 60705 | 2924 | 0.99 | 59790 | 4048 | 0.99 | 57391 |
| | 0.2 | 2366 | 0.001 | 50400 | 2430 | 0.99 | 49524 | 3368 | 0.99 | 47562 |
| | 0.4 | 2025 | 0.001 | 43286 | 2068 | 0.001 | 42339 | 2890 | 0.99 | 40745 |
| | 0.6 | 1764 | 0.001 | 37771 | 1828 | 0.001 | 37078 | 2526 | 0.99 | 35494 |
| | 0.8 | 1566 | 0.001 | 33635 | 1646 | 0.001 | 33114 | 2110 | 0.001 | 31446 |

investment increases. This is similar to the capacity uncertainty case except that the firm does not invest in any degree of flexibility in Table 4.2, whereas it invests in complete flexibility under the risk of disruptions. This can be explained as follows: As the disruption levels increase, the probability of both resources being up (1,1) decreases. In addition, due to the disruptions of the resources being independent Bernoulli random variables, the probability of only one resource being down, (0,1) or (1,0), increases faster than that of both resources being down, (0,0). Thus the firm invests in a high degree of flexibility in *both* resources in the planning stage.

In the production stage, when one resource is disrupted completely, then the firm is left with no other choices but to produce the two products from the other resource. Without resource flexibility, more capacity cannot provide more help. As $\rho$ increases, the probability of higher demand realizations of both products increase. If resource flexibility levels are very low, then the firm would end up selling mostly only one product although market potentials are high for both products. If, however, the other resource has a sufficient capacity *and* with a high flexibility level, the firm can produce a substantial amount of both products. Hence in the cross-production model, flexibility is beneficial even with positively correlated demands.

Our results on the value of flexibility under 0-1 disruptions are very different from Tomlin and Wang (2005). In their research, the firm's preference for flexibility decreased as resource investments became less reliable and the firm leaned more toward utilizing the dedicated capacities. Also, as demands became more negatively correlated flexibility preference increased in general. In our cross-production model, as the failure probabilities of the resources increased, flexibility was highly preferred due to reasons stated above. Furthermore, accounting for the degree of product substitutability, we find that flexibility is not beneficial when the value of $b$ is very high under any level of disruption even when product demands are highly negatively correlated.

To summarize, under supply disruptions, flexibility is a potent tool in spite of its higher investment cost. This is in contrast to results considering just demand uncertainties (see for e.g., Lus and Muriel 2009, Goyal and Netessine 2010). Further, the optimal profit decreases as the capacity uncertainty (Table 4.2) or the disruption risk increases (Table 4.3) as compared to the demand variability case (Table 4.1). The *nature* of capacity risk faced by a firm plays a key role in evaluating the effectiveness of the hedging strategies. Cross production is not a viable strategy under low demand or capacity uncertainties and when the product substitutability is very high. When the firm faces low demand or capacity uncertainties, it is better off adjusting the prices than reconfiguring its resources.

### 4.4.4 Impact of Pricing

In this section we examine the firm's optimal capacity and flexibility decisions without responsive pricing. Specifically, the firm decides on capacity levels ($K_1$, $K_2$) and reconfiguration factor $\beta$ in Stage I where product prices ($p_1$, $p_2$) are exogenous. Table 4.4 summarizes optimal total capacities, flexibility levels and expected profit for a firm facing increasing capacity uncertainties.

Table 4.4: Impact of Capacity Uncertainties under No Responsive Pricing

| | | CV=0.1 | | | CV=0.2 | | | CV=0.4 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\rho$ | $b$ | $K_t^*$ | $\beta^*$ | Profit | $K_t^*$ | $\beta^*$ | Profit | $K_t^*$ | $\beta^*$ | Profit |
| -0.5 | 0 | 10922 | 0.19 | 51061 | 11811 | 0.94 | 48363 | 16733 | 0.99 | 42354 |
| | 0.2 | 9018 | 0.19 | 42566 | 9751 | 0.94 | 40339 | 13814 | 0.99 | 35377 |
| | 0.4 | 7804 | 0.19 | 36409 | 8439 | 0.94 | 34482 | 11956 | 0.99 | 30188 |
| | 0.6 | 6785 | 0.19 | 31883 | 7337 | 0.94 | 30207 | 10395 | 0.99 | 26474 |
| | 0.8 | 6058 | 0.19 | 28328 | 6551 | 0.94 | 26832 | 9281 | 0.99 | 23498 |
| 0 | 0 | 10922 | 0.19 | 51061 | 11811 | 0.94 | 48363 | 16733 | 0.99 | 42354 |
| | 0.2 | 9018 | 0.19 | 42520 | 9751 | 0.94 | 40293 | 13814 | 0.99 | 35331 |
| | 0.4 | 7804 | 0.19 | 36421 | 8439 | 0.94 | 34494 | 11956 | 0.99 | 30200 |
| | 0.6 | 6785 | 0.19 | 31843 | 7337 | 0.94 | 30168 | 10395 | 0.99 | 26434 |
| | 0.8 | 6058 | 0.19 | 28325 | 6551 | 0.94 | 26829 | 9281 | 0.99 | 23496 |
| 0.5 | 0 | 10922 | 0.19 | 51061 | 11811 | 0.94 | 48363 | 16733 | 0.99 | 42354 |
| | 0.2 | 9018 | 0.19 | 42492 | 9751 | 0.94 | 40265 | 13814 | 0.99 | 35304 |
| | 0.4 | 7804 | 0.19 | 36436 | 8439 | 0.94 | 34509 | 11956 | 0.99 | 30215 |
| | 0.6 | 6785 | 0.19 | 31819 | 7337 | 0.94 | 30144 | 10395 | 0.99 | 26410 |
| | 0.8 | 6058 | 0.19 | 28324 | 6551 | 0.94 | 26828 | 9281 | 0.99 | 23494 |

When the degree of capacity uncertainties is low (CV=0.1) the average ex ante capacity investment is about 52% higher and average profit is 14% lower than corresponding values in Table 4.2. When capacity yield is highly uncertain (CV=0.4) the average capacity investment in Table 4.4 is about 66% higher and average profit is 34% lower than corresponding (average) values in Table 4.2. The firm without responsive pricing invests in a very high degree of flexibility for all levels of capacity uncertainties under any demand correlation $\rho$. In addition, $\beta$ does not change with $b$ and remains constant for all levels of product substitutability. Recall that as $b$ increases the price premiums of the two products tend to be closer to each other and hence it was inexpensive to shift demand through pricing making flexibility less desirable under responsive pricing. However, faced with exogenous prices, the firm invests in higher levels of reconfiguration under all levels of $b$ (and $\rho$) which undermines its overall expected profit. These results clearly indicate that responsive pricing is a very potent tool to mitigate demand uncertainties, capacity uncertainties as well as supply disruptions.

### 4.4.5 Sensitivity to Flexibility Cost

We now investigate the sensitivity of the investment decisions in Sections 4.4.1-4.4.3 to unit capacity cost $g$ and the relative cost of cross-production i.e, $g_3/g$. In the interest of space, we report the key results of this analysis in Table 4.5 only for the 0-1 disruption case with CV=0.4 while summarizing the results for other cases below.

For the firm facing *low* demand uncertainties we find that for a fixed $g$ as $g_3/g$ increases total capacity $K_t^*$ remains unchanged while the profit decreases. However, when demand uncertainty is *high*, $K_t^*$ increases as cost of flexibility increases. This is not surprising given that the firm does not invest in any degree of flexibility when demand uncertainty is low (Table 4.1). The firm invests in flexibility only under high demand uncertainty and when the cost of flexibility investment increases, it mitigates demand uncertainties by increasing the capacity investment. We also note that when dedicated capacity cost $g$ increases the firm's optimal price increases as the firm uses responsive pricing to maintain its margins as opposed to increasing its flexibility investment.

The investment decision of the firm facing capacity uncertainties is very similar to the case of demand uncertainties described above. For a fixed $g$, as $g_3/g$ increases, only under high capacity uncertainty does $K_t^*$ increase and expected profit decrease. This is because the firm does not invest in any degree of flexibility under low capacity uncertainties (Table 4.2) and relies on responsive pricing to mitigate capacity risk.

While the capacity investment trends of the firm facing low degree of supply disruptions are similar to the demand and capacity uncertainty cases, it is completely reversed under high levels of 0-1 disruptions. Table 4.5 presents a snapshot of the results under negatively correlated demands ($\rho = -0.5$) for different values of $g$ with a disruption CV of 0.4. For a given $g$, the firm's total capacity $K_t^*$ *decreases* as relative cost of flexibility increases. Recall that when the probability of disruption increases, the probability of one of the resources being down (0,1) and (1,0) increases faster than that of both resources (0,0). If the resource producing the product with a higher demand survives, a higher capacity of this resource may benefit the firm. However, a lower capacity level can be managed through responsive pricing mitigating the cost of under-investment. If the resource producing the product with a lower demand survives then without a high level of flexibility a higher capacity of this resource alone cannot produce the more popular product. Consequently, the cost of over-investment in the latter case (added with increased flexibility cost) dominates the under-investment cost inducing the firm to invest less in overall capacity levels.

## 4.5 Summary

This chapter examines the interplay between the cost of investing in flexibility, the efficiency loss due to cross-production as well as the responsive pricing for substitutable

Table 4.5: Impact of cost under $\rho = -0.5$ and 0-1 disruption CV=0.4

| | | $g_3/g = 0.1$ | | | $g_3/g = 0.3$ | | | $g_3/g = 0.5$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $g$ | $b$ | $K_t^*$ | $\beta^*$ | Profit | $K_t^*$ | $\beta^*$ | Profit | $K_t^*$ | $\beta^*$ | Profit |
| 1 | 0 | 4108 | 0.99 | 57977 | 3938 | 0.99 | 57180 | 3772 | 0.99 | 56417 |
| | 0.2 | 3420 | 0.99 | 47869 | 3280 | 0.99 | 47205 | 3140 | 0.99 | 46570 |
| | 0.4 | 2930 | 0.99 | 40820 | 2810 | 0.99 | 40251 | 2692 | 0.001 | 38766 |
| | 0.6 | 2564 | 0.99 | 35555 | 2458 | 0.99 | 35057 | 2354 | 0.001 | 34468 |
| 3 | 0 | 2670 | 0.99 | 50829 | 2568 | 0.99 | 49276 | 2510 | 0.001 | 47057 |
| | 0.2 | 2222 | 0.99 | 41919 | 2136 | 0.99 | 40626 | 2114 | 0.001 | 39931 |
| | 0.4 | 1906 | 0.99 | 35719 | 1848 | 0.001 | 34796 | 1848 | 0.001 | 34793 |
| | 0.6 | 1666 | 0.99 | 31093 | 1648 | 0.001 | 30825 | 1648 | 0.001 | 30823 |
| 5 | 0 | 2390 | 0.99 | 45312 | 2308 | 0.001 | 42254 | 2308 | 0.001 | 42252 |
| | 0.2 | 1990 | 0.99 | 37326 | 1944 | 0.001 | 35887 | 1942 | 0.001 | 35885 |
| | 0.4 | 1706 | 0.99 | 31781 | 1688 | 0.001 | 31269 | 1688 | 0.001 | 31268 |
| | 0.6 | 1494 | 0.001 | 27699 | 1494 | 0.001 | 27697 | 1494 | 0.001 | 27684 |
| 7 | 0 | 2220 | 0.99 | 40252 | 2146 | 0.001 | 37806 | 2146 | 0.001 | 37803 |
| | 0.2 | 1848 | 0.99 | 33113 | 1810 | 0.001 | 32137 | 1810 | 0.001 | 32135 |
| | 0.4 | 1584 | 0.99 | 28169 | 1572 | 0.001 | 28010 | 1572 | 0.001 | 28008 |
| | 0.6 | 1392 | 0.001 | 24816 | 1392 | 0.001 | 24814 | 1392 | 0.001 | 24813 |

products. We model a firm producing two products with two partially flexible resources and facing three types of uncertainties separately: demand uncertainty, yield uncertainty, and supply disruptions. The firm can choose the level of resource flexibility in the investment stage. The higher flexibility level, the less efficiency loss will be incurred when switching a resource for one product to produce another product. We investigate how the type *and* severity of the uncertainties affect capacity investment and resource reconfiguration decisions. The impacts of demand correlation and product substitutability on the firm's investment decisions are also examined.

A firm facing demand or yield uncertainties does not benefit much by investing in resource flexibility. Investing in additional dedicated capacities is relatively cheaper than improving the flexibility for the whole resource capacity. There is a certain amount of capacity available for each resource, and the firm can mitigate the impact of demand and capacity uncertainties through responsive pricing. So flexibility is not very valuable.

When the firm faces supply disruptions, however, flexibility becomes extremely valuable. This is because, in the 0-1 disruption case, when one resource is completely unavailable, a higher flexibility level of the other resource ensures that demands for both products are met in conjunction with pricing. This is true irrespective of the nature of demand correlation. Hence one product does not completely cannibalize the other product and the firm can sell both products while mitigating the impact of disruption through pricing. Even though investment in flexibility is costly, it is an effective hedging strategy under severe capacity disruptions.

# CHAPTER 5

# CAPACITY, FLEXIBILITY AND PRICING DECISIONS UNDER CROSS-PRODUCTION: THE ADDITIONAL COST MODEL

## 5.1 Introduction

In Chapter 4 we studied *one* form of efficiency loss in cross-production through explicit modeling of the partial flexibility factor. Partial flexibility implies that fewer units of a product will be produced if a resource originally specialized for one type of product is switched to produce another type (i.e., many-to-one mapping). Even if the dedicated resource can produce all units of another product (i.e., one-to-one mapping) there may be an increase in unit production cost during cross-production. This additional cost may be caused by machine reconfiguration, additional manufacturing processes, changeover cost, overtime working, or extra training for workers. Li et al. (2014) cite examples of wafer manufacturing and testing units at Intel where retro-fitting is usually done at a much higher expense to produce the newer or more popular product.

In this chapter we model the trade-off between investing in costly flexibility in the planning stage and incurring additional production cost in the selling stage. Specifically, if the degree of flexibility invested in Stage I is higher then the additional cost of cross-production incurred in Stage II is lower. The firm faces uncertain demand in Stage I and may also face capacity uncertainties or 0-1 type resource disruptions in Stage II.

The rest of the chapter is organized as follows. A two-stage optimization model with additional production cost is formulated in Section 5.2 and its solutions and analytical properties are derived in Section 5.3. Section 5.4 presents the numerical analysis to obtain managerial insights into the impacts of uncertainties and resource investment costs on the optimal capacity and flexibility investment decisions. Finally, Section 5.5 concludes the chapter.

## 5.2 Model

We model a price-setting firm manufacturing two products, $i$ and $j$. Demands are observed before production and the inverse demand curve for the products are assumed to be linear and of the form $p_i = A_i - Q_i - bQ_j$ and $p_j = A_j - Q_j - bQ_i$. $(A_i, A_j)$ are the demand curve intercepts that indicate the customer's willingness-to-pay and $(Q_i, Q_j)$ are production quantities for products $i$ and $j$ sold by the firm that sets prices $(p_i, p_j)$. The parameter $b \in [0, 1]$ is called the product substitutability parameter and indicates that the demand for a product (say $i$) increases with increase in the price of product $j$. In Stage I $(A_i, A_j)$ are random variables from a bi-variate continuous distribution $F(.,.)$ with density function $\Psi(.,.)$. The mean of the marginal distribution is denoted by $\mu_i$ and $\mu_j$ and the variance by $\sigma_i^2$, $\sigma_j^2$, for products $i$ and $j$, respectively. Using $\rho \in [-1, 1]$ as the correlation coefficient, the covariance of the joint distribution is $\sigma_{12} = \rho\sigma_1\sigma_2$. We denote the total demand uncertainty of the two produces by $\sigma_T^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$. In Stage II the firm observes the demand realizations $(a_i, a_j)$ after the uncertainty in market conditions is resolved but before production begins.

We denote $c_{ij}$ as the additional unit production cost incurred if resource $j$ is used to produce product $i$ and $\mathbf{C} = (c_1, c_2)'$. In some parts of the research to simplify the exposition we use a symmetric cost $c$=$c_1$=$c_2$ without losing any insights. Let $\widetilde{\Pi}$ denote the expected profit in Stage I and $\widetilde{R}$ denote the revenue in Stage II. We also assume $c >g_3$ i.e., the cross-production cost in Stage II is higher than the flexibility investment cost in Stage I. Otherwise the firm will not invest in flexibility as it may never be used. The two-stage decision problem for the firm can be formulated as follows:

$$\text{Stage } I: \qquad \widetilde{\Pi} = \mathbf{max}\ \boldsymbol{E}[\widetilde{R}(\bar{K_1}, \bar{K_2}, f)] - \sum_{i=1,2} g_i \bar{K_i} - g_3(\bar{K_1} + \bar{K_2})f \tag{5.1}$$

$$\text{subject to}: \ \bar{K_1}, \bar{K_2} \geq 0$$

$$0 \leq f \leq 1$$

$$\text{Stage } II: \qquad \widetilde{R} = \mathbf{max}\ [\mathbf{pQ}] - ((\mathbf{Q} - \mathbf{K})^{+})'\mathbf{c}(1\text{-}f) \tag{5.2}$$

$$\text{subject to}: \ Q_1 + Q_2 \leq K_1 + K_2$$

$$Q_1, Q_2 \geq 0$$

where $\bar{K_1}$ and $\bar{K_2}$ are the capacity invested, and $g_1$ and $g_2$ are the unit capacity costs. Note that if the firm does not invest in any flexibility (i.e., $f$=0) then it incurs an additional cost $c$ to cross-produce in Stage II. As flexibility investment increases the effect of additional cost decreases and under full flexibility($f$=1) there is no additional cost.

A key difference between this model and the Shrinking Capacity model in Chapter 4 is that here if the firm did not invest in any flexibility in Stage I ($f$=0), it can still use cross-production in Stage II although at a much higher cost. However, if $\beta$=0 in that formulation, the Shrinking Capacity model reduces to a pure dedicated system and hence cannot use any cross-production in Stage II.

In Stage I the firm selects its capacities $\bar{K_1}$ and $\bar{K_2}$, and the degree of flexibility $f$ under demand uncertainties. In Stage II the firm selects the production quantities $(Q_1, Q_2)$ constrained by the capacity and flexibility decisions in Stage I and the additional production cost in Stage II. This model captures the investment trade-off between cost savings in the capacity planning stage to higher cross-production cost in the production stage. In Stage I, the firm decides on the optimal flexibility level and maximizes its expected revenue. In this stage, the firm incurs a cost to invest in flexibility $g_3(\bar{K_1} + \bar{K_2})f$ which is incremental in the capacity levels. Specifically, $g_3$

is incurred in addition to unit cost of capacity $g_1$ or $g_2$ and to avoid trivialities we also assume that the cost of a fully flexible resource ($f$=1) is greater than the unit investment cost of a dedicated resource but is lesser than the cost of two units of the dedicated resource.

At the beginning of Stage II, the market potentials, $A_1$ and $A_2$, and the available capacities, $K_1 = \theta_1 \bar{K}_1$ and $K_2 = \theta_2 \bar{K}_2$, are realized. In the numerical analysis in Section 5.4 the resource uncertainty factor $\Theta_i$ follows a Normal distribution in the capacity uncertainty case and is modeled as a Bernoulli Random Variable in the supply disruption case. In Stage II, the firm makes production decisions to maximize its profit.

## 5.3 Analytical Results

In this section we analyze the Stage II and Stage I decisions of the firm with additional production cost. We first start by describing the solution methodology.

### 5.3.1 Solution Methodology

We solve the two stage problem by first solving the Stage II formulation subject to capacity constraints. The solution methodology employed for the Additional Cost model is very similar to the methodology described in Section 4.3 for the Shrinking Capacity model. The Stage II decision variables $Q_1$ and $Q_2$ are functions of demand realizations ($a_1$, $a_2$), available capacities ($K_1$,$K_2$), degree of flexibility ($f$) as well as product substitutability $b$. We note here that under only demand uncertainties the firm does not face any resource supply uncertainties and the invested capacities in Stage I $(\bar{K}_1, \bar{K}_2)$ are the same as the available capacities $K_1, K_2$ in Stage II. For clarity we hence use $K_1$ and $K_2$ as the capacity decision variables in Stage I under only demand uncertainties.

The Stage II objective function could have three possible forms under different scenarios: (1) If $Q_1 \leq K_1$ and $Q_2 \leq K_2$, i.e., no cross-production is required, no additional cost is incurred; (2) If $Q_1 > K_1$ and $Q_2 < K_2$, an additional production cost $c_1$ is incurred for each unit of product 1 that is produced using resource 2; (3) If $Q_2 > K_2$ and $Q_1 < K_1$, an additional cost $c_2$ is incurred. The demand space can be partitioned into 10 different regions where each region belongs to one of the scenarios. The partition of the feasible region enables us to solve the individual optimization problems and obtain closed form solutions to the production quantities in many regions using KKT conditions. At some boundaries when the objective function is non-differentiable, we have either $Q_1 = K_1$ or $Q_2 = K_2$. The Stage II problem can then be solved at those boundaries without using KKT optimality conditions.

The three scenarios (10 cases) corresponding to the three different forms of the Stage II objective function are described below:

**Scenario 1:** We have $Q_1 \leq K_1$ and $Q_2 \leq K_2$. This leads to *four* possible cases:

- **Case (1)**: The demand realizations of both the products are so low that the firm does not use up even the available dedicated capacities. $Q_1 < K_1$ and $Q_2 < K_2$. Notice that the objective function is devoid of the cross production terms and the unconstrained solution is optimal.

- **Case (2)**: The demand for product 1 is high enough to use its dedicated capacity completely while demand for product 2 is lower than its available capacity. $Q_1 = K_1$ and $Q_2 < K_2$. No cross-production is required and hence the objective function does not contain any of the $(Q_i - K_i)^+$ terms, $i$=1, 2.

- **Case (3)**: This case is symmetric to Case 2: The demand for product 2 is high enough to use its dedicated capacity completely while demand for product 1 is lower than its available capacity. $Q_1 < K_1$ and $Q_2 = K_2$. No cross-production is required and hence the objective function does not contain any of the $(Q_i - K_i)^+$

terms, $i$=1, 2.

- **Case (4)**: When demand realizations for both products are very high then cross production does not benefit either product. The firm simply produces up capacity levels for both products and $Q_1 = K_1$ and $Q_2 = K_2$. Since no cross-production is required we still do not incur any additional cost and the objective function does not contain any of the $(Q_i - K_i)^+$ terms, $i$=1, 2.

**Scenario 2:** We have $Q_1 > K_1$ as the demand for product is is relatively high compared to demand for product 2, $Q_2 < K_2$. This leads to *three* possible cases as the firm incurs additional production cost $c_1$:

- **Case (1)**: Cross-production cost $c_1$ is incurred as $Q_1 > K_1$. However demand for product 2 is very low and total capacity is not used by the firm i.e., $Q_1 + Q_2 < K_1 + K_2$. The Stage II objective function includes the cross-cost term for resource 1 and the revenue is denoted by $\widetilde{R} = p_1Q_1 + p_2Q_2 - (Q_1 - K_1)c_1(1 - f)$.

- **Case (2)**: Demand for product 1 is such that the firm uses all its available capacities $Q_1 + Q_2 = K_1 + K_2$ while still cross-producing product 1. Demand for product 2 is low enough to not use its entire dedicated capacity and hence the Stage II objective function only includes the cross-cost term for resource 1 similar to Case 1 in this scenario.

- **Case (3)**: The demand realization of product 1 is so high relative to product 2 that the firm decides to allocate capacity $K_2$ to produce only product 1. Cross-production cost $c_1$ is incurred as $Q_1 > K_1$ but in this case $Q_2 = 0$ while $Q_1 + Q_2 = K_1 + K_2$.

**Scenario 3:** This scenario is symmetric to Scenario 2 where demand realization of product 2 is higher than that of product 1. We have $Q_1 < K_1$ and $Q_2 > K_2$. This leads to *three* possible cases as the firm incurs additional production cost $c_2$:

- **Case (1)**: Cross-production cost $c_2$ is incurred as $Q_2 > K_2$. However demand for product 1 is very low and total capacity is not used by the firm i.e., $Q_1 + Q_2 < K_1 + K_2$. The Stage II objective function includes the cross-cost term for resource 2 and the revenue is denoted by $\widetilde{R} = p_1Q_1 + p_2Q_2 - (Q_2 - K_2)c_2(1-f)$.

- **Case (2)**: Demand for product 2 is such that the firm uses all its available capacities $Q_1 + Q_2 = K_1 + K_2$ while still cross-producing product 2. Demand for product 1 is low enough to not use its entire dedicated capacity and hence the Stage II objective function only includes the cross-cost term for resource 2 similar to Case 1 in this scenario.

- **Case (3)**: The demand realization of product 2 is so high relative to product 1 that the firm decides to allocate capacity $K_1$ to produce only product 2. Cross-production cost $c_2$ is incurred as $Q_2 > K_2$ but in this case $Q_1 = 0$ while $Q_1 + Q_2 = K_1 + K_2$.

For each of the cases described above under the three scenarios we next mathematically define the feasible regions and proceed to solve the Stage II and Stage I problems.

### 5.3.2 Stage II Optimal Production Quantities

Since the objective function of the Stage II problem is jointly concave in $Q_1$ and $Q_2$ and the constraint is linear, the first order KKT conditions are necessary and sufficient for optimality. At boundaries (regions $\Omega_2$,$\Omega_3$,$\Omega_8$) where the objective function is non-differentiable, we have either $Q_1 = K_1$ or $Q_2 = K_2$ or $Q_1 = K_1$ and $Q_2 = K_2$ . The Stage II problem can then be solved in those regions without using KKT optimality conditions.

The constraint can be binding /non-binding in ten different combinations and hence the state space of **A** is partitioned into ten regions, each corresponding to one of these combinations. Imposing the conditions $A_1 - bA_2 \geq 0$, $A_2 - bA_1 \geq 0$ to enforce positive demand realizations (see, e.g., Singh and Vives 1984), the demand regions are shown in Figure 5.1-Left.

The corresponding optimal production quantities are shown in Figure 5.1-Right.

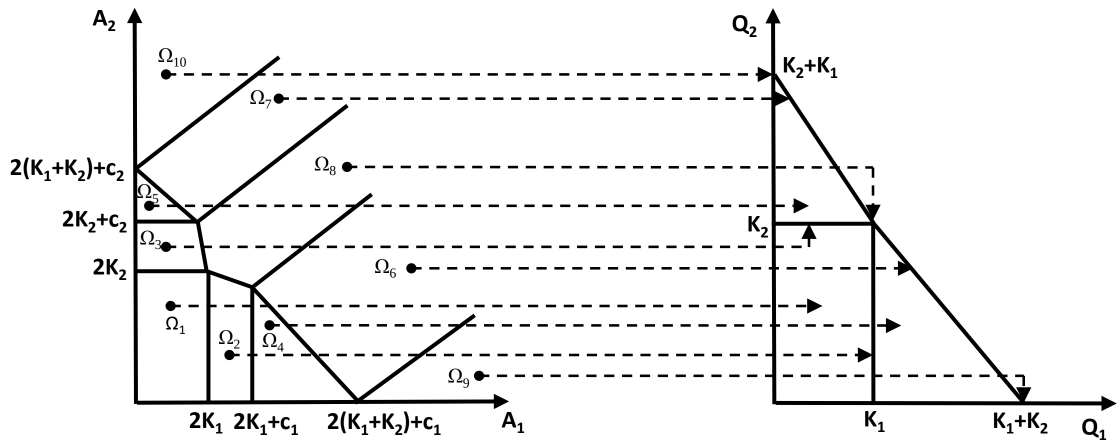


Figure 5.1: Mapping of the state space of $\mathbf{A}$ into the output space for $b = 0$ case

The demand regions are defined as follows:

$$\Omega_1: \quad 0 \le a_1 - ba_2 \le 2(1-b^2)K_1, 0 \le a_2 - ba_1 \le 2(1-b^2)K_2;$$

$$\Omega_2: \quad 2(1-b^2)K_1 \le a_1 - ba_2 \le 2(1-b^2)K_1 + c_1(1-f), a_2 \le 2(K_2 + bK_1);$$

$$\Omega_3: \quad 2(1-b^2)K_2 \le a_2 - ba_1 \le 2(1-b^2)K_2 + c_2(1-f), a_1 \le 2(K_1 + bK_2);$$

$$\Omega_4: \quad a_1 - ba_2 > 2(1-b^2)K_1 + c_1(1-f), a_1 + a_2 \le 2(1+b)(K_1+K_2) + c_1(1-f);$$

$$\Omega_5: \quad a_2 - ba_2 > 2(1-b^2)K_2 + c_2(1-f), a_1 + a_2 \le 2(1+b)(K_1+K_2) + c_2(1-f);$$

$$\Omega_6: \quad a_1 + a_2 > 2(1+b)(K_1+K_2) + c_1(1-f),$$
$$2(1-b)(K_1-K_2) + c_1(1-f) < a_1 - a_2 \le 2(1-b)(K_1+K_2) + c_1(1-f);$$

$$\Omega_7: \quad a_1 + a_2 > 2(1+b)(K_1+K_2) + c_2(1-f),$$
$$2(1-b)(K_2-K_1) + c_2(1-f) < a_2 - a_1 \le 2(1-b)(K_1+K_2) + c_2(1-f);$$

$$\Omega_8: \quad a_1 \ge 2(bK_2 + K_1), a_2 \ge 2(bK_1 + K_2),$$
$$2(1-b)(K_1-K_2) - c_2(1-f) \le a_1 - a_2 \le 2(1-b)(K_1+K_2) + c_1(1-f);$$

$$\Omega_9: \quad a_1 - a_2 > 2(1-b)(K_1+K_2) + c_1(1-f);$$

$$\Omega_{10}: \quad a_2 - a_1 > 2(1-b)(K_1+K_2) + c_2(1-f);$$

The following theorem characterizes the optimal solution of the Stage II problem.

**Theorem 5.1** *Given the realizations of the market potentials $(a_1, a_2)$ and the investment vector $(K_1, K_2, f)$, the optimal production quantities for the Stage II problem* *are given as follows:*

$$
\begin{aligned}
\Omega_1: &\quad Q_1 = \frac{a_1 - ba_2}{2(1-b^2)}, Q_2 = \frac{a_2 - ba_1}{2(1-b^2)};\\
\Omega_2: &\quad Q_1 = K_1, Q_2 = \frac{a_2}{2} - bK_1;\\
\Omega_3: &\quad Q_1 = \frac{a_1}{2} - bK_2, Q_2 = K_2;\\
\Omega_4: &\quad Q_1 = \frac{a_1 - ba_2 - c_1(1-f)}{2(1-b^2)}, Q_2 = \frac{a_2 - ba_1 + bc_1(1-f)}{2(1-b^2)};\\
\Omega_5: &\quad Q_1 = \frac{a_1 - ba_2 + bc_2(1-f)}{2(1-b^2)}, Q_2 = \frac{a_2 - ba_1 - c_2(1-f)}{2(1-b^2)};\\
\Omega_6: &\quad Q_1 = \frac{K_1+K_2}{2} + \frac{a_1 - a_2 - c_1(1-f)}{4(1-b)}, Q_2 = \frac{K_1+K_2}{2} - \frac{a_1 - a_2 - c_1(1-f)}{4(1-b)};\\
\Omega_7: &\quad Q_1 = \frac{K_1+K_2}{2} - \frac{a_2 - a_1 - c_2(1-f)}{4(1-b)}, Q_2 = \frac{K_1+K_2}{2} + \frac{a_2 - a_1 - c_2(1-f)}{4(1-b)};\\
\Omega_8: &\quad Q_1 = K_1, Q_2 = K_2;\\
\Omega_9: &\quad Q_1 = K_1 + K_2, Q_2 = 0;\\
\Omega_{10}: &\quad Q_1 = 0, Q_2 = K_1 + K_2;
\end{aligned}
$$

Please refer to Appendix C for all the proofs in this chapter.

In each region, by solving the Stage II problem, the optimal revenue $R_i(a_1, a_2)$

equals,

$$\Omega_1: \qquad R_1 = \frac{a_1^2 + a_2^2 - 2ba_1a_2}{4(1-b^2)};$$

$$\Omega_2: \qquad R_2 = \frac{a_2^2}{4} + (a_1 - ba_2)K_1 - (1-b^2)K_1^2;$$

$$\Omega_3: \qquad R_3 = \frac{a_1^2}{4} + (a_2 - ba_1)K_2 - (1-b^2)K_2^2;$$

$$\Omega_4: \qquad R_4 = \frac{a_1^2 + a_2^2 - 2a_1(a_2b + c_1(1-f)) + c_1(1-f)(c_1(1-f) + 4K_1 + 2b(a_2 - 2bK_1))}{4(1-b^2)};$$

$$\Omega_5: \qquad R_5 = \frac{a_1^2 + a_2^2 - 2a_2(a_1b + c_2(1-f)) + c_2(1-f)(c_2(1-f) + 4K_2 + 2b(a_1 - 2bK_2))}{4(1-b^2)};$$

$$\begin{aligned}\Omega_6: \qquad R_6 = {} & \frac{a_1^2}{8(1-b)} + \frac{a_2^2}{8(1-b)} + \frac{a_2(c_1(1-f) + 2(1-b)(K_1+K_2))}{4(1-b)} \\ & + \frac{c_1^2(1-f)^2 + 4(1-b)c_1(1-f)(K_1-K_2) - 4(1-b^2)(K_1+K_2)^2}{8(1-b)} \\ & - \frac{a_1(a_2 + c_1(1-f) - 2(1-b)(K_1+K_2))}{4(1-b)};\end{aligned}$$

$$\begin{aligned}\Omega_7: \qquad R_7 = {} & \frac{a_1^2}{8(1-b)} + \frac{a_2^2}{8(1-b)} - \frac{a_2(c_2(1-f) - 2(1-b)(K_1+K_2))}{4(1-b)} \\ & + \frac{c_2^2(1-f)^2 - 4(1-b)c_2(1-f)(K_1-K_2) - 4(1-b^2)(K_1+K_2)^2}{8(1-b)} \\ & - \frac{a_1(a_2 - c_2(1-f) - 2(1-b)(K_1+K_2))}{4(1-b)};\end{aligned}$$

$$\Omega_8: \qquad R_8 = a_1K_1 + a_2K_2 - (K_1^2 + K_2^2 + 2bK_1K_2);$$

$$\Omega_9: \qquad R_9 = (a_1 - (K_1+K_2))(K_1+K_2) - K_2(c_1(1-f));$$

$$\Omega_{10}: \qquad R_{10} = (a_2 - (K_1+K_2))(K_1+K_2) - K_1(c_2(1-f));$$

### 5.3.3 Stage I Optimal Capacity and Flexibility Investment

Using the above results, we now analyze the firm's optimal capacity investment portfolio $(K_1, K_2, f)$ in Stage I. To simplify the exposition we assume that the firm invests in capacity $K{=}K_1{=}K_2$. Our analytical insights are unaffected by this assumption as shown by our numerical studies.

**Lemma 5.1** *The Stage I objective function $\widetilde{\Pi}$ is strictly jointly concave in $(K_1, K_2, f)$*

Hence, the following theorem provides the necessary and sufficient conditions for the optimal capacity investment portfolio $(K_1, K_2, f)$.

**Theorem 5.2** *The Stage I optimal capacity investment vector $(K_1, K_2, f)$ is optimal if and only if the Lagrangian multipliers given by $\nu_i$ and $\tau$ exist and satisfy the following conditions:*

$$\frac{\partial\widetilde{\Pi}(K_1, K_2, f)}{\partial K_i} = g_i - \nu_i \ for\ i\ \in \{1, 2\}, \tag{5.3}$$

$$\frac{\partial\widetilde{\Pi}(K_1, K_2, f)}{\partial f} = g_3 - \tau, \tag{5.4}$$

$$K_i\nu_i = 0, \quad for\ i\ \in \{1, 2\}, \tag{5.5}$$

$$(1 - f)\tau = 0. \tag{5.6}$$

*Hence, conditioning on $(A_1, A_2)$, we obtain,*

$$\begin{aligned}
\left\{\begin{array}{c} g_1 + g_3 f - \nu_1 \\ g_2 + g_3 f - \nu_2 \\ \frac{g_3(K_1+K_2)}{2f} + \tau \end{array}\right\} =& Pr(\Omega_2)\mathbf{E}\left\{\begin{array}{c} A_1 - bA_2 - 2(1-b^2)K_1 \\ 0 \\ 0 \end{array}\right\} \\
&+ Pr(\Omega_3)\mathbf{E}\left\{\begin{array}{c} 0 \\ A_2 - bA_1 - 2(1-b^2)K_2 \\ 0 \end{array}\right\} \\
&+ Pr(\Omega_4)\mathbf{E}\left\{\begin{array}{c} c_1(1-f) \\ 0 \\ \frac{c_1[A_1 - bA_2 - c_1(1-f) - 2K_1(1-b^2)]}{4(1-b^2)f} \end{array}\right\} \\
&+ Pr(\Omega_5)\mathbf{E}\left\{\begin{array}{c} 0 \\ c_2(1-f) \\ \frac{c_2[A_2 - bA_1 - c_2(1-f) - 2K_2(1-b^2)]}{4(1-b^2)f} \end{array}\right\}
\end{aligned}$$

$$+ Pr(\Omega_6)\mathbf{E}\left\{\begin{array}{c} \frac{[A_1+A_2+c_1(1-f)-2(1+b)(K_1+K_2)]}{2} \\ \frac{[A_1+A_2-c_1(1-f)-2(1+b)(K_1+K_2)]}{2} \\ \frac{c_1[A_1-A_2-c_1(1-f)-4(1-b)(K_1+K_2)]}{2} \end{array}\right\}$$

$$+ Pr(\Omega_7)\mathbf{E}\left\{\begin{array}{c} \frac{[A_1+A_2-c_2(1-f)-2(1+b)(K_1+K_2)]}{2} \\ \frac{[A_1+A_2+c_2(1-f)-2(1+b)(K_1+K_2)]}{2} \\ \frac{c_2[A_2-A_1-c_2(1-f)-4(1-b)(K_1+K_2)]}{2} \end{array}\right\}$$

$$+ Pr(\Omega_8)\mathbf{E}\left\{\begin{array}{c} A_1 - 2(K_1 + bK_2) \\ A_2 - 2(K_2 + bK_1) \\ 0 \end{array}\right\}$$

$$+ Pr(\Omega_9)\mathbf{E}\left\{\begin{array}{c} A_1 - 2(K_1 + K_2) \\ A_1 - 2(K_1 + K_2) - c_1(1-f) \\ \frac{c_1 K_2}{2f} \end{array}\right\}$$

$$+ Pr(\Omega_{10})\mathbf{E}\left\{\begin{array}{c} A_2 - 2(K_1 + K_2) - c_2(1-f) \\ A_2 - 2(K_1 + K_2) \\ \frac{c_2 K_1}{2f} \end{array}\right\} \quad \blacksquare$$

### 5.3.4 Optimal solutions under clearance assumption

We now analyze the impact of the optimal capacity and flexibility investments using the clearance assumption. We assume that the firm always produces to available capacity $K_i$, $i = 1, 2$, irrespective of any demand intercept realization and that the production quantities $Q_i$, $i = 1, 2$ are always positive (please refer to Van Mieghem and Dada 1999 for details on *clearance* and *hold-back* strategies). We derive closed form expressions of optimal clearance capacities $K_i$, flexibility level $f$ as well as expected profit to better understand the impacts of other parameters. Despite being generally sub-optimal, an approximation of clearance can be solved in closed form and analyzed for *any* demand distribution. We supplement the analytical solutions with extensive numerical analysis to corroborate the insights gained from clearance.

In our additional cost model, the clearance assumption for capacities $K_1$, $K_2$ is satisfied naturally in regions $\Omega_6$, $\Omega_7$ and $\Omega_8$. However, since we are interested in the optimal flexibility levels and region $\Omega_8$ is a pure dedicated solution (no flexibility component or cross production cost $c$), we restrict our analysis to $\Omega_6$ (results for region $\Omega_7$ can be easily derived as they are symmetric).

**Theorem 5.3** *Given $g_3$, $c$=$c_1$=$c_2$, $g$=$g_1$=$g_2$ and $b$, the closed form solutions for optimal clearance capacities $\hat{K_i}^*$ $i \in 1, 2$, optimal flexibility level $f^*$ and total capacity investment $\hat{K_1}^*$+$\hat{K_2}^*$=$\hat{K_t}^*$ is given by*

$$\begin{aligned}
\hat{K_1}^* &= \frac{\mu_1 - b\mu_2}{2(1-b^2)} - \frac{g_3}{2c(1+b)}[\mu_1 + \mu_2 + c - 2(g+g_3)] - \frac{g}{2(1+b)} \\
\hat{K_2}^* &= \frac{\mu_2 - b\mu_1}{2(1-b^2)} + \frac{g_3}{2c(1+b)}[\mu_1 + \mu_2 - (c + 2(g+g_3))] - \frac{g}{2(1+b)} \\
\hat{f}^* &= 1 \\
\hat{K_t}^* &= \frac{\mu_1 + \mu_2 - 2(g+g_3)}{2(1+b)}
\end{aligned}$$

Some interesting insights can be derived from the above results.

**Corollary 5.1** *The total capacity investment $\hat{K_t}^*$* increases *with overall market size $\mu_1 + \mu_2$.*

Corollary 5.1 reveals that as total market size increases the firm gains by investing in and hence selling a higher quantity of both products. The firm takes advantage of the high demand state by pricing higher and hence gains through an overall higher profit.

**Corollary 5.2** $\hat{K_t}^*$ decreases *with product substitutability $b$*

Corollary 5.2 shows that as products become less differentiated (high $b$) demand can be managed through pricing and hence responsive pricing is more effective than investing in higher capacity. Recall that consumers are less price-sensitive when purchasing a unique item and more differentiated (less substitutable) products reach a larger customer base (see, e.g., Talluri and van Ryzin 2005). However, as $b$ increases the market sizes for the products overlap and hence results in lower effective total market potentials. With overall lower market potentials the firm does not gain by investing in higher capacities as they may go unused. Hence $\hat{K_t}^*$ *decreases* with product substitutability $b$.

We also note that $\hat{K_t}^*$ *decreases* with unit cost of capacity ($g$) *and* flexibility investment ($g_3$). We also note that $\hat{K_t}^*$ is *independent* of cross-production cost $c$ while $\hat{K_1}^*$ and $\hat{K_2}^*$ are dependent on $c$.

When the firm's optimal portfolio includes flexibility investment ($f > 0$), clearance suggests that it *always* invest in full flexibility. We note here that the actual solution under clearance for flexibility is $\hat{f}^*$=$1 + \frac{g_2-g_1}{c}$ in region $\Omega_6$, $\hat{f}^*$=$1 + \frac{g_1-g_2}{c}$ in region $\Omega_7$ or $\hat{f}^*$=0 in region $\Omega_8$. Clearance is generally sub-optimal as the firm is forced to utilize all it's available capacities even by offering products at market clearing prices. A lower total capacity level helps the firm to produce and sell what it may just need thus deterring the firm from investing in cheaper dedicated capacities. Hence the firm invests in a high degree of flexibility in the planning stage.

The above analysis also suggests that the clearance solution maybe closer to the no-clearance optimal solution under negatively correlated demands. To verify this and the other results we conduct a numerical analysis in Section 5.4.5 to compare the solutions under clearance and no-clearance cases. While we observe that clearance is closer to the optimal solution when demands are negatively correlated, investment in full flexibility is recommended only when the *relative cost* of flexibility is cheap.

We next proceed to analyze the impact of parameters on on the optimal profit under clearance. By plugging in the above solutions from Theorem 5.3 into the Stage I profit function, we obtain the following characterization of the optimal profit under clearance ($\widetilde{\Pi^c}$):

**Theorem 5.4** *Given $g_3$, $c$=$c_1$=$c_2$, $g$=$g_1$=$g_2$ and $b$, total demand variability $\sigma_T^2$ =*

$\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2$ *and* $b$*, the optimal profit under the clearance assumption is given by*

$$\begin{aligned}\widetilde{\Pi^c} = \quad & \frac{(\mu_1+\mu_2)^2}{8(1+b)} - \frac{(\mu_1+\mu_2)g_3(c-2g)}{2c(1+b)} + \frac{(\mu_1-\mu_2)(1+4g)}{8(1-b)} \\ & + \frac{\sigma_T^2}{8(1-b)} + \frac{g_3^3-g^2}{2(1+b)} + \frac{2gg_3^2+2g^2g_3}{c(1+b)}\end{aligned}$$

Our results on the impact of total market size $(\mu_1+\mu_2)$ on optimal profit is similar to what was observed in Chapter 4: as total market size increases the firm's profit also increases. However, the presence of additional cross-production cost $c$ plays a role in moderating this benefit: under a low value of $c$ (say, $c$=$g$), the component related to market size is $\frac{(\mu_1+\mu_2)^2}{8(1+b)}+\frac{(\mu_1+\mu_2)g_3}{2(1+b)}$. If, however, $c$ is large (say, $c$=$3g$) then the market size component becomes $\frac{(\mu_1+\mu_2)^2}{8(1+b)}-\frac{(\mu_1+\mu_2)g_3}{6(1+b)}$ indicating that the firm experiences a lower profit. We also observe here that a higher level of product substitutability $b$ lowers the effect of the market size. This is consistent with literature (for e.g., Goyal and Netessine (2007)) because as products become closer substitutes the two markets overlap more and the total market size becomes smaller.

**Corollary 5.3** $\widetilde{\Pi^c}$ increases *with the difference in the market size* $\mu_1-\mu_2$.

Corollary 5.3 shows that profit also depends on the *difference* between the market sizes $\mu_1-\mu_2$ and the nature of product substitutability. When the total market size $\mu_1+\mu_2$ is held fixed, the firm prefers to maximize the difference in two market sizes. Hence profit increases with difference in market sizes. The firm can now use responsive pricing to charge higher prices for the product with the larger market to compensate for the lower price for the product in the smaller market. A higher degree of product substitutability $b$ simply adds to this effect because an increase in demand for one product decreases the demand for the other.

**Corollary 5.4** $\widetilde{\Pi}^c$ increases *with total demand variability* $\sigma_T^2$.

Corollary 5.4 shows that the optimal profit under clearance increases with total demand variability $\sigma_T^2$. As degree of demand uncertainty (variance) increases so does the realization of high demand states. Oi (1961) showed that a firm using responsive pricing makes more money in the high demand states than it loses in the low demand states. Under clearance the firm always invests in full flexibility and hence it can easily allocate capacity from one market to another enabling it to take advantage of the higher demand states. A higher degree of substitutability $b$ between the products amplifies this effect benefitting the firm with a higher profit.

Our comprehensive numerical analysis in the next section reveals that a majority of the insights derived from the solutions assuming clearance still hold in the general case.

## 5.4 Numerical Analysis

We now investigate numerically how changes in the degree of the three types of uncertainties viz., demand uncertainties (Section 5.4.1), capacity uncertainties (Section 5.4.2) and 0-1 disruptions (Section 5.4.3) affect the capacity and flexibility investment decisions of the firm using cross-production. We also analyze the impact of parameters such as demand correlation and the degree of product substitutability. Finally, we also investigate the sensitivity of the investment decisions to increase in the additional cost of cross-production (Section 5.4.4).

**Experimental Design:** We generate demand scenarios as described in Chapter 4 (Section 4.4) and hence avoid repetition. The cost of a unit of dedicated capacity for each product $(g_1, g_2)$ is 1 while the additional cost rate for the flexibility degree $(g_3)$ is 0.2. The additional cost of cross-production $c$ was set at 2 for the experiments on increasing uncertainties. To study the impact of the additional cost $c$, we increase $c$

from 3 to 7 (relatively very high) and report results on the high demand uncertainty (Coefficient of Variation CV=0.4), high capacity uncertainties (Normally distributed uncertainties with CV=0.4) as well as high supply disruption scenarios (Bernoulli RVs corresponding to CV=0.4).

### 5.4.1 Impacts of Demand Uncertainties

We study the impact of demand uncertainties modeled by increasing the CV of normally distributed product demands under different correlation scenarios. We also consider the demand correlation, as it is a key factor influencing capacity investment decisions in operational and financial hedging models (Chod et al. 2010). To compare these results with capacity uncertainties, we restrict the capacities available in the production stage to $0.5K_i$, $i = 1, 2$. While this restriction affects the capacities numerically it ensures that the investment trends, which is the major focus of this research, are unaltered. In this scenario, we assume the firm does not face any yield uncertainties or disruptions, i.e., $\theta_1^m = \theta_2^m = 1$.

The results for the demand uncertainty case shown in Table 5.1 reveal that: (1) Flexibility is *never* preferred under any demand CV and under any demand correlation $\rho$ when relative cost of dedicated capacities is cheaper. (2) As the demand CV increases, the total dedicated capacity investment $K_t^*$ and the optimal profits increase. (3) With a fixed positive product substitutability $b$, total capacity $K_t^*$ and the optimal profits *increase* with $\rho$. (4) As the product substitutability $b$ increases, the total capacities and the expected profits decrease.

The Additional Cost model shows that despite a moderate cost of flexibility ($g_3 = 0.2$), flexibility is not recommended in any of the cases as shown in Table 5.1. The firm prefers to incur the cost of cross-production rather than invest in flexibility. These results also generally support the results from the Shrinking Capacity model discussed in Chapter 4 (Table 4.1). In general, to cope with demand variability the

Table 5.1: Impacts of Demand Uncertainties

| | | CV=0.1 | | | CV=0.2 | | | CV=0.4 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\rho$ | $b$ | $K_t^*$ | $f^*$ | Profit | $K_t^*$ | $f^*$ | Profit | $K_t^*$ | $f^*$ | Profit |
| -0.5 | 0 | 4852 | 0.001 | 57752 | 5038 | 0.001 | 58875 | 5518 | 0.001 | 63693 |
| | 0.2 | 4042 | 0.001 | 48024 | 4187 | 0.001 | 48585 | 4598 | 0.001 | 51595 |
| | 0.4 | 3461 | 0.001 | 41099 | 3588 | 0.001 | 41424 | 3941 | 0.001 | 43322 |
| | 0.6 | 3029 | 0.001 | 35984 | 3133 | 0.001 | 36087 | 3448 | 0.001 | 37323 |
| | 0.8 | 2684 | 0.001 | 31915 | 2776 | 0.001 | 31994 | 3058 | 0.001 | 32791 |
| 0 | 0 | 4891 | 0.001 | 57716 | 5153 | 0.001 | 58714 | 5904 | 0.001 | 63541 |
| | 0.2 | 4074 | 0.001 | 48023 | 4285 | 0.001 | 48574 | 4920 | 0.001 | 51879 |
| | 0.4 | 3489 | 0.001 | 41109 | 3677 | 0.001 | 41506 | 4217 | 0.001 | 43816 |
| | 0.6 | 3053 | 0.001 | 36003 | 3212 | 0.001 | 36203 | 3677 | 0.001 | 37919 |
| | 0.8 | 2705 | 0.001 | 31932 | 2848 | 0.001 | 32141 | 3254 | 0.001 | 33423 |
| 0.5 | 0 | 4923 | 0.001 | 57672 | 5249 | 0.001 | 58639 | 6125 | 0.001 | 63100 |
| | 0.2 | 4099 | 0.001 | 48021 | 4365 | 0.001 | 48530 | 5100 | 0.001 | 51978 |
| | 0.4 | 3510 | 0.001 | 41118 | 3748 | 0.001 | 41606 | 4368 | 0.001 | 44187 |
| | 0.6 | 3072 | 0.001 | 36013 | 3271 | 0.001 | 36259 | 3819 | 0.001 | 38427 |
| | 0.8 | 2724 | 0.001 | 31949 | 2904 | 0.001 | 32268 | 3391 | 0.001 | 33997 |

firm simply invests in higher amount of relatively cheaper dedicated capacities rather than a higher level of flexibility. In addition responsive pricing is a more cost effective strategy as demand for substitutable products can be easily managed through pricing.

Table 5.1 also shows that the optimal expected profit always increases as the demand uncertainty increases. Recall from Theorem 5.4 that under clearance the firm's profit increased with total demand variability $\sigma_T^2$. Even without clearance the firm benefits from increased demand variability because of its ability to charge higher prices in the larger market. High realizations of the demand leads to a higher marginal revenue while marginal revenue is constant for low realizations of demand.

This asymmetry causes the expected profit to increase as the total demand variability increases. If the demand for a product is very low then the firm employs a hold-back strategy, i.e., it sells only a restricted quantity of the product at a higher price than selling all of it at much lower market-clearing prices, mitigating the over-investment risk.

### 5.4.2 Impacts of Capacity Uncertainties

Next we investigate the optimal investment decisions when the firm faces capacity uncertainties by modeling $\Delta_1$ and $\Delta_2$ with independent Normal random variables. To reflect increasing degree of capacity uncertainties, we increase the CV of these distributions. The results are shown in Table 5.2.

Our study reveals that: (1) Flexibility is not preferred under any capacity CV and under any demand correlation $\rho$. (2) The total capacity investment $K_t^*$ and the profit *increase* with $\rho$. This trend was also also observed in Table 5.1 under only demand uncertainty. (3) For a fixed $b$, the total capacity investment $K_t^*$ increases as capacity uncertainty increases. The profit, however, decreases with the capacity uncertainty, which is different from Table 5.1 but similar to the results under Shrinking Capacity model in Table 4.2.

Flexibility is not required under capacity uncertainties because there is always a certain amount of realized capacity still available for each product. The firm increases it's investment in cheaper dedicated capacity hence protecting itself against uncertainties in the selling stage. As product substitutability $b$ increases, the firm can also adjust the prices to match the demands for the two products with the available capacities. Investing in additional capacities is relatively cheaper than improving the flexibility for the whole resource capacity. Compared to the results of Shrinking Capacity model in Table 4.2 even when facing high capacity uncertainties, flexibility is not recommended. Unlike the Shrinking Capacity model, the firm can cross-produce

Table 5.2: Impacts of Capacity Uncertainties

| | | CV=0.1 | | | CV=0.2 | | | CV=0.4 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\rho$ | $b$ | $K_t^*$ | $f^*$ | Profit | $K_t^*$ | $f^*$ | Profit | $K_t^*$ | $f^*$ | Profit |
| -0.5 | 0 | 5071 | 0.001 | 58628 | 5199 | 0.001 | 58353 | 5822 | 0.001 | 56419 |
| | 0.2 | 4225 | 0.001 | 48470 | 4331 | 0.001 | 48241 | 4852 | 0.001 | 46729 |
| | 0.4 | 3623 | 0.001 | 41354 | 3714 | 0.001 | 41156 | 4167 | 0.001 | 39894 |
| | 0.6 | 3167 | 0.001 | 36078 | 3251 | 0.001 | 35905 | 3654 | 0.001 | 34810 |
| | 0.8 | 2803 | 0.001 | 31894 | 2877 | 0.001 | 31741 | 3240 | 0.001 | 30772 |
| 0 | 0 | 5213 | 0.001 | 58540 | 5303 | 0.001 | 58271 | 5910 | 0.001 | 56369 |
| | 0.2 | 4340 | 0.001 | 48505 | 4419 | 0.001 | 48281 | 4931 | 0.001 | 46788 |
| | 0.4 | 3715 | 0.001 | 41442 | 3788 | 0.001 | 41250 | 4237 | 0.001 | 40000 |
| | 0.6 | 3246 | 0.001 | 36193 | 3316 | 0.001 | 36026 | 3715 | 0.001 | 34942 |
| | 0.8 | 2872 | 0.001 | 32023 | 2934 | 0.001 | 31878 | 3291 | 0.001 | 30921 |
| 0.5 | 0 | 5299 | 0.001 | 58396 | 5416 | 0.001 | 58125 | 5979 | 0.001 | 56247 |
| | 0.2 | 4413 | 0.001 | 48503 | 4513 | 0.001 | 48279 | 4995 | 0.001 | 46795 |
| | 0.4 | 3780 | 0.001 | 41511 | 3869 | 0.001 | 41318 | 4292 | 0.001 | 40074 |
| | 0.6 | 3306 | 0.001 | 36297 | 3383 | 0.001 | 36131 | 3763 | 0.001 | 35050 |
| | 0.8 | 2929 | 0.001 | 32144 | 2984 | 0.001 | 32001 | 3333 | 0.001 | 31050 |

in the Additional Cost model without investing in any degree of costly flexibility. Hence even under conditions where flexibility is typically preferred i.e., negatively correlated demands ($\rho$ <0) and high degree of product differentiation (low $b$) the firm never invests in any degree of flexibility.

Despite the increase in the overall capacity investment, the firm's profit decreases as the capacity uncertainty increases. This trend is different from Table 5.1 and the literature where the total capacity and the profit increase under only demand uncertainty. Under capacity uncertainties, only a proportion of the invested capacities is available in the production stage. As the capacity uncertainty increases, the capacity available and hence the production quantities decrease, and thus the firm's revenue decreases.

### 5.4.3 Impacts of Supply Disruptions

Table 5.3 shows the optimal capacities, the flexibility levels, and the profits for a firm facing increasing capacity disruptions.

Table 5.3: Impacts of Supply Disruptions

| | | CV=0.1 | | | CV=0.2 | | | CV=0.4 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\rho$ | $b$ | $K_t^*$ | $f^*$ | Profit | $K_t^*$ | $f^*$ | Profit | $K_t^*$ | $f^*$ | Profit |
| -0.5 | 0 | 2707 | 0.001 | 61124 | 2832 | 0.001 | 60093 | 3176 | 0.001 | 56415 |
| | 0.2 | 2274 | 0.001 | 50574 | 2380 | 0.001 | 49805 | 2882 | 0.001 | 47230 |
| | 0.4 | 1957 | 0.001 | 43163 | 2052 | 0.001 | 42530 | 2670 | 0.001 | 40602 |
| | 0.6 | 1696 | 0.001 | 37662 | 1798 | 0.001 | 37109 | 2464 | 0.001 | 35561 |
| | 0.8 | 1481 | 0.001 | 33294 | 1592 | 0.001 | 32798 | 2248 | 0.001 | 31475 |
| 0 | 0 | 2810 | 0.001 | 61127 | 2860 | 0.001 | 60109 | 3192 | 0.001 | 56449 |
| | 0.2 | 2340 | 0.001 | 50685 | 2410 | 0.001 | 49922 | 2896 | 0.001 | 47343 |
| | 0.4 | 2001 | 0.001 | 43315 | 2096 | 0.001 | 42686 | 2687 | 0.001 | 40741 |
| | 0.6 | 1744 | 0.001 | 37830 | 1849 | 0.001 | 37285 | 2480 | 0.001 | 35711 |
| | 0.8 | 1539 | 0.001 | 33468 | 1634 | 0.001 | 32984 | 2261 | 0.001 | 31634 |
| 0.5 | 0 | 2838 | 0.001 | 61048 | 2889 | 0.001 | 60043 | 3193 | 0.001 | 56411 |
| | 0.2 | 2368 | 0.001 | 50736 | 2462 | 0.001 | 49980 | 2907 | 0.001 | 47403 |
| | 0.4 | 2029 | 0.001 | 43427 | 2141 | 0.001 | 42804 | 2703 | 0.001 | 40846 |
| | 0.6 | 1773 | 0.001 | 37971 | 1881 | 0.001 | 37431 | 2498 | 0.001 | 35836 |
| | 0.8 | 1570 | 0.001 | 33623 | 1666 | 0.001 | 33146 | 2277 | 0.001 | 31770 |

From Table 5.3 we find that: (1) Even under *high* supply disruptions, the firm does not invest in costly flexibility. (2) For a fixed $b$, total capacity investment $K_t^*$ increases as supply disruption risk increases. However, the optimal expected profit decreases. (3) When the two products are not substitutable to each other, the firm's profit decreases as the demand correlation increases. However, with higher levels of product substitutability, the profit increases with the demand correlation.

In the Shrinking Capacity model in Chapter 4 (Table 4.3) we saw that the firm invested in a high degree of flexibility when the degree of disruption was very high. However the Additional Cost model suggests that when the relative cost of flexibility is expensive then flexibility is not required for any level of supply disruption. When one resource is completely unavailable then the firm can still incur the cross-production cost in Stage II and produce the other product. The firm would rather rely on a higher investment in (cheaper) dedicated capacities in Stage I and cross-produce in Stage II rather than investing in flexibility.

Our results from the Additional Cost model supports the results from Shrinking Capacity model on the value of flexibility under low and moderate 0-1 disruptions. Under very high disruptions the *type* of efficiency loss as well as relative cost parameters play a key role in determining the value of flexibility. Specifically, when the loss is in terms of number of units (partial flexibility factor $\beta$ in the Shrinking Capacity model) the firm then invests in a high degree of flexibility to allow cross-production. When the efficiency loss is in terms of the cost of cross-production ($c$ in Additional Cost model) the firm is willing to forego investment in costly flexibility in the planning stage and instead incur additional cost in the selling stage.

### 5.4.4 Impact of Additional Cost of Cross-Production

To better understand the trade-off between flexibility investment in Stage I and cross-production cost in Stage II we study the impact of increasing $c$ under all three types of uncertainties. Results for demand uncertainties are shown in Table 5.4 while results for capacity uncertainties are shown in Table 5.5. The results for the 0-1 uncertainty case are omitted as they are very similar to the capacity uncertainty scenario.

Table 5.4: Impact of Cross Production Cost $c$ under CV=0.4 Demand Uncertainties

| | | $c$=3 | | | $c$=5 | | | $c$=7 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\rho$ | $b$ | $K_t^*$ | $f^*$ | Profit | $K_t^*$ | $f^*$ | Profit | $K_t^*$ | $f^*$ | Profit |
| -0.5 | 0 | 5578 | 0.001 | 63553 | 5780 | 0.001 | 63323 | 5868 | 0.001 | 63152 |
| | 0.2 | 4649 | 0.001 | 51484 | 4783 | 0.001 | 51310 | 4863 | 0.001 | 51183 |
| | 0.4 | 3985 | 0.001 | 43236 | 4065 | 0.001 | 43107 | 4141 | 0.001 | 43020 |
| | 0.6 | 3483 | 0.001 | 37262 | 3526 | 0.001 | 37179 | 3567 | 0.001 | 37136 |
| | 0.8 | 3073 | 0.001 | 32762 | 3087 | 0.001 | 32741 | 3092 | 0.001 | 32738 |
| 0 | 0 | 5962 | 0.001 | 63458 | 6015 | 0.001 | 63316 | 6095 | 0.001 | 63201 |
| | 0.2 | 4955 | 0.001 | 51814 | 5000 | 0.001 | 51706 | 5071 | 0.001 | 51627 |
| | 0.4 | 4235 | 0.001 | 43766 | 4273 | 0.001 | 43688 | 4326 | 0.001 | 43640 |
| | 0.6 | 3694 | 0.001 | 37883 | 3724 | 0.001 | 37839 | 3742 | 0.001 | 37817 |
| | 0.8 | 3265 | 0.001 | 33410 | 3273 | 0.001 | 33401 | 3274 | 0.001 | 33401 |
| 0.5 | 0 | 6144 | 0.001 | 63052 | 6184 | 0.001 | 62969 | 6237 | 0.001 | 62906 |
| | 0.2 | 5118 | 0.001 | 51941 | 5152 | 0.001 | 51882 | 5194 | 0.001 | 51842 |
| | 0.4 | 4384 | 0.001 | 44159 | 4416 | 0.001 | 44120 | 4428 | 0.001 | 44099 |
| | 0.6 | 3834 | 0.001 | 38410 | 3849 | 0.001 | 38392 | 3856 | 0.001 | 38384 |
| | 0.8 | 3394 | 0.001 | 33993 | 3397 | 0.001 | 33991 | 3397 | 0.001 | 33991 |

Under high demand uncertainty (CV=0.4) Table 5.4 reveals that as cost of cross-production increases the optimal total capacity $K_t^*$ increases under all levels of $\rho$. This increase, however, depends on the level of product substitutability $b$: When products are highly differentiated (low $b$) then increase in capacity is higher as $c$ increases. When products are highly substitutable (very high $b$) the increase in optimal total capacity is very minimal and for very high values of $c$ it is constant. The firm still does not invest in flexibility despite a very high cost of cross-production. This is because the relative cost of dedicated capacity is cheaper. In addition, the firm invests in higher dedicated capacities only when products are highly differentiated. If the products are very close substitutes then the overall market potential is lower and any investment in extra capacity is unnecessary. Further, demand can be easily managed through responsive pricing. This investment trend holds for all levels of demand correlation $\rho$.

Table 5.5 shows the results for increasing $c$ under high capacity uncertainties (CV=0.4). It is easy to see that despite an increase in additional cost of cross-production and the increased uncertainty in capacity availability, the investment trend is very similar to the demand uncertainty case in Table 5.4. Flexibility is never preferred under any degree of product substitutability $b$ or demand correlation $\rho$. The optimal total capacity $K_t^*$ increases as $c$ increases and the firm once again invests more in cheaper dedicated capacities under low $b$.

Table 5.5: Impact of Cross Production Cost $c$ under CV=0.4 Capacity Uncertainties

| | | $c$=3 | | | $c$=5 | | | $c$=7 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\rho$ | $b$ | $K_t^*$ | $f^*$ | Profit | $K_t^*$ | $f^*$ | Profit | $K_t^*$ | $f^*$ | Profit |
| -0.5 | 0 | 5852 | 0.001 | 56286 | 5929 | 0.001 | 56039 | 6043 | 0.001 | 55826 |
| | 0.2 | 4882 | 0.001 | 46616 | 4943 | 0.001 | 46416 | 5001 | 0.001 | 46247 |
| | 0.4 | 4193 | 0.001 | 39802 | 4235 | 0.001 | 39646 | 4264 | 0.001 | 39521 |
| | 0.6 | 3677 | 0.001 | 34740 | 3689 | 0.001 | 34632 | 3704 | 0.001 | 34560 |
| | 0.8 | 3245 | 0.001 | 30735 | 3248 | 0.001 | 30696 | 3248 | 0.001 | 30684 |
| 0 | 0 | 5940 | 0.001 | 56241 | 6031 | 0.001 | 56007 | 6117 | 0.001 | 55806 |
| | 0.2 | 4959 | 0.001 | 46681 | 5033 | 0.001 | 46491 | 5074 | 0.001 | 46331 |
| | 0.4 | 4261 | 0.001 | 39913 | 4296 | 0.001 | 39766 | 4327 | 0.001 | 39648 |
| | 0.6 | 3732 | 0.001 | 34876 | 3742 | 0.001 | 34774 | 3761 | 0.001 | 34707 |
| | 0.8 | 3295 | 0.001 | 30886 | 3295 | 0.001 | 30850 | 3295 | 0.001 | 30839 |
| 0.5 | 0 | 6008 | 0.001 | 56124 | 6112 | 0.001 | 55902 | 6172 | 0.001 | 55710 |
| | 0.2 | 5020 | 0.001 | 46692 | 5093 | 0.001 | 46513 | 5127 | 0.001 | 46361 |
| | 0.4 | 4312 | 0.001 | 39990 | 4349 | 0.001 | 39851 | 4377 | 0.001 | 39740 |
| | 0.6 | 3773 | 0.001 | 34988 | 3790 | 0.001 | 34892 | 3802 | 0.001 | 34831 |
| | 0.8 | 3335 | 0.001 | 31018 | 3335 | 0.001 | 30986 | 3335 | 0.001 | 30975 |

Next we study the impact of the clearance assumption.

### 5.4.5 Impact of Clearance

We now study the impact of clearance on the optimal capacity and flexibility investment decisions when the firm faces demand uncertainties. We report the results from moderate demand uncertainty case (CV=0.2) as the results for the other cases are similar. We study two settings of the cost parameters: (1) Table 5.6 shows results from setting $g_1$=$g_2$=1, $g_3$=0.2 and $c$=2. This corresponds to the settings in Sections 5.4.1, 5.4.2 and 5.4.3. (2) Table 5.7 shows results from setting $g_1$=$g_2$=2, $g_3$=0.1 and $c$ =3.

In addition to optimal total capacity $K_t^*$, flexibility level $f^*$ and profit, we also report the difference between the no-clearance and with-clearance cases in terms of Profit $\Delta$ and capacities $K_t^*$ $\Delta$. A positive $\Delta$ indicates that values under the no-clearance case are higher than the values under with-clearance case.

We find that under clearance full flexibility is never recommended for any values of $b$ or $\rho$ in Table 5.6 when the relative cost of flexibility is high. Under these parameters flexibility is expensive (for e.g., $g_3/g$=0.2 and $g_3/c$=0.1) and hence even under clearance the firm does not invest in any degree of flexibility. The results from Theorem 5.3, however, suggested $f$=1 indicating full flexibility. This is because in our analytical model clearance could be evaluated in three different regions in Figure 5.1: Region 6, Region 7 as well as Region 8. In Region 8, the firm simply produces up to its dedicated capacities and hence cross-production or flexibility was not required i.e, $f$=0 in that solution.

We also find that clearance solutions are generally more closer to optimal solutions under no-clearance when demands are negatively correlated: The capacities and profits under $\rho$ <0 are generally closer to the solutions under no-clearance compared to $\rho$ >0. Overall the capacity difference is about 6.6% lower on average while the profit difference is only about 0.77% lower compared to the no-clearance solution. Due to the assumptions made in clearance, viz., the firm produces up to capacity

Table 5.6: Impact of Clearance under Demand Uncertainties: $g$=1, $g_3$=0.2, $c$=2

| | | Without Clearance | | | With Clearance | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\rho$ | $b$ | $K_t^*$ | $f^*$ | Profit | $K_t^*$ | $f^*$ | Profit | Profit $\Delta$ | $K_t^*$ $\Delta$ |
| -0.5 | 0 | 5038 | 0.001 | 58875 | 4796 | 0.001 | 58641 | 0.40% | 4.8% |
| | 0.2 | 4187 | 0.001 | 48585 | 3994 | 0.001 | 48393 | 0.39% | 4.6% |
| | 0.4 | 3588 | 0.001 | 41424 | 3423 | 0.001 | 41261 | 0.39% | 4.6% |
| | 0.6 | 3133 | 0.001 | 36087 | 2994 | 0.001 | 35949 | 0.38% | 4.4% |
| | 0.8 | 2776 | 0.001 | 31994 | 2662 | 0.001 | 31880 | 0.36% | 4.1% |
| 0 | 0 | 5153 | 0.001 | 58714 | 4796 | 0.001 | 58242 | 0.80% | 6.9% |
| | 0.2 | 4285 | 0.001 | 48574 | 3994 | 0.001 | 48188 | 0.80% | 6.8% |
| | 0.4 | 3677 | 0.001 | 41506 | 3423 | 0.001 | 41175 | 0.80% | 6.9% |
| | 0.6 | 3212 | 0.001 | 36203 | 2994 | 0.001 | 35919 | 0.79% | 6.8% |
| | 0.8 | 2848 | 0.001 | 32141 | 2662 | 0.001 | 31902 | 0.74% | 6.5% |
| 0.5 | 0 | 5249 | 0.001 | 58639 | 4796 | 0.001 | 57948 | 1.18% | 8.6% |
| | 0.2 | 4365 | 0.001 | 48530 | 3994 | 0.001 | 47968 | 1.16% | 8.5% |
| | 0.4 | 3748 | 0.001 | 41606 | 3423 | 0.001 | 41121 | 1.16% | 8.7% |
| | 0.6 | 3271 | 0.001 | 36259 | 2994 | 0.001 | 35840 | 1.16% | 8.5% |
| | 0.8 | 2904 | 0.001 | 32268 | 2662 | 0.001 | 31910 | 1.11% | 8.3% |

levels and production quantities are always positive, the optimal capacity expressions are independent of variability or correlation of the demand intercepts.

When cost of flexibility is low relative to the cost of dedicated capacities or unit cost of cross-production (for e.g., $g_3/g$=0.05 and $g_3/c$=0.03) Table 5.7 shows that the firm is inclined to invest in *full* flexibility under negatively correlated demands and moderate degrees of product substitutability $b$. This can be explained as follows: Under clearance, having a higher capacity level may force the firm to lower prices to sell higher quantities of the products. With full flexibility, the firm will invest less in total dedicated capacity levels and hence has a better chance of utilizing all of it's installed capacities. In this situation, by investing in full flexibility the firm avoids any additional cost of cross-production. We also note that when demands are positively correlated the firm does not invest in any flexibility under clearance. Instead the firm responds to positive demand correlation by further lowering it's dedicated capacity investment as seen from the higher levels of $K_t^*$ $\Delta$.

Table 5.7: Impact of Clearance under Demand Uncertainties: $g$=2, $g_3$=0.1, $c$ =3

| | | Without Clearance | | | With Clearance | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\rho$ | $b$ | $K_t^*$ | $f^*$ | Profit | $K_t^*$ | $f^*$ | Profit | Profit $\Delta$ | $K_t^*$ $\Delta$ |
| -0.5 | 0 | 2478 | **1.0** | 58915 | 2395 | **1.0** | 58749 | **0.28%** | **3.4%** |
| | 0.2 | 2106 | 0.001 | 48591 | 1994 | **1.0** | 48474 | **0.24%** | **5.3%** |
| | 0.4 | 1804 | 0.001 | 41438 | 1709 | **1.0** | 41321 | **0.28%** | **5.2%** |
| | 0.6 | 1573 | 0.001 | 36112 | 1495 | **1.0** | 35984 | **0.35%** | **4.9%** |
| | 0.8 | 1391 | 0.001 | 32031 | 1332 | 0.001 | 31914 | **0.37%** | **4.3%** |
| 0 | 0 | 2592 | 0.001 | 58741 | 2395 | **1.0** | 58306 | 0.74% | 7.6% |
| | 0.2 | 2156 | 0.001 | 48603 | 1994 | **1.0** | 48233 | 0.76% | 7.5% |
| | 0.4 | 1850 | 0.001 | 41537 | 1710 | **1.0** | 41205 | 0.80% | 7.6% |
| | 0.6 | 1612 | 0.001 | 36239 | 1498 | 0.001 | 35940 | 0.82% | 7.1% |
| | 0.8 | 1427 | 0.001 | 32182 | 1332 | 0.001 | 31939 | 0.76% | 6.6% |
| 0.5 | 0 | 2640 | 0.001 | 58689 | 2400 | 0.001 | 57959 | 1.24% | 9.1% |
| | 0.2 | 2194 | 0.001 | 48577 | 1997 | 0.001 | 47986 | 1.22% | 9.0% |
| | 0.4 | 1880 | 0.001 | 41650 | 1713 | 0.001 | 41145 | 1.21% | 8.9% |
| | 0.6 | 1639 | 0.001 | 36303 | 1497 | 0.001 | 35874 | 1.18% | 8.7% |
| | 0.8 | 1454 | 0.001 | 32311 | 1332 | 0.001 | 31950 | 1.12% | 8.4% |

## 5.5 Summary

This chapter examines the interplay between the cost of investing in flexibility, the efficiency loss due to cross-production incurred through additional production cost as well as the responsive pricing for substitutable products. We model a firm producing two products with two partially flexible resources and facing three types of uncertainties separately: demand uncertainty, yield uncertainty, and supply disruptions. The firm can choose the level of resource flexibility and the level of dedicated resources in the investment stage. A higher flexibility level in the planning stage would incur a lower additional production cost in the selling stage. We investigate how the type *and* severity of the uncertainties affect capacity investment and resource reconfiguration decisions. The impacts of demand correlation, additional cost and product substitutability on the firm's investment decisions are also examined.

The results from the Additional Cost model are generally consistent with results from the Shrinking Capacity model: When the relative cost of flexibility is high, a firm facing demand or yield uncertainties does not benefit much by investing in resource flexibility. Investing in higher amount of dedicated capacities is relatively cheaper than investing in flexibility because the firm can mitigate the impact of cross-production cost through pricing. Even under very high capacity uncertainties the firm does not invest in flexibility under negatively correlated demands. This result is different from the Shrinking Capacity model because here the firm can still cross-produce without any investment in flexibility. It would simply incur an additional unit cost of cross-production in the selling stage. As there is a certain amount of capacity available for each resource, the firm can mitigate the impact of demand and capacity uncertainties through responsive pricing. So flexibility is not very valuable in these circumstances.

Similarly, under supply disruptions the firm prefers to invest in cheaper dedicated capacities and use cross-production without flexibility investment. If one resource were completely unavailable, a higher amount of dedicated capacity of the other resource combined with the ability to cross-produce deters the firm from investing in flexibility. This is true irrespective of the nature of demand correlation or the degree of product substitutability. Hence one product does not completely cannibalize the other product and the firm can sell both products while mitigating the impact of disruption through pricing.

# CHAPTER 6

# CONCLUSION

This research investigates the resource investment and pricing decisions for a profit-maximizing firm producing two substitutable products facing three types of uncertainties separately: demand uncertainties, capacity uncertainties and supply disruptions. In the planning stage (Stage I) the firm must decide on the optimal capacity and flexibility investment levels. Only the distribution of these uncertainties is typically known i.e., the firm makes these investment decisions under demand, capacity and supply uncertainties. In the production stage (Stage II) once the market potentials as well as realized capacity levels are revealed, the firm must then determine optimal production quantities and prices.

In the *Contingent Flexible Capacity* model the firm can invest in a separate contingent flexible capacity in addition to product dedicated capacities in the planning stage. In the *Shrinking Capacity* model the firm uses cross-production to switch a resource designed for one product to produce another product which incurs an efficiency loss in the production stage. In the *Additional Cost* model the firm incurs efficiency loss in cross-production through an increase in unit production cost in the selling stage. In the latter two models, if the firm invests in a higher degree of flexibility in the planning stage then the efficiency loss is lower in the production stage. This captures the resource investment trade-off that is typically faced by firms in practice.

The main findings from our research can be summarized as follows:

- From the Contingent Flexible Capacity model we find that utilizing the correct functional form of the demand-price relationship that captures the impact of product substitutability produces more realistic results that is in line with practice. We hence use this correct functional form in subsequent models to analyze the impact of cross-production.

- Utilizing the Shrinking Capacity model we show that:

  - Cross-production is not an optimal hedging strategy when facing low or moderate demand uncertainties. The firm prefers to invest in higher dedicated capacities as opposed to any level of investment in flexibility. However, if degree of demand uncertainties is very high, product demands are negatively correlated and level of product substitutability is low, then *partial* flexibility is necessary.

  - Only when facing high degree of capacity uncertainties the firm invests in any flexibility. In fact, cross-production is never preferred under low or moderate capacity uncertainties. Investment in *partial* flexibility is sufficient to mitigate high capacity uncertainties. As demand correlation increases flexibility is unnecessary as the firm can increase it's investment in cheaper dedicated capacities and avoid cross-production

  - When facing a high degree of supply disruptions, the firm's optimal investment strategy includes a *high* degree of investment in flexibility. This result is independent of the nature of demand correlation as well as product substitutability

- The Additional Cost model reveals that:

  - When the relative cost of flexibility is high, flexibility is generally unnecessary when facing any of the three types of uncertainties. This is because the firm can cross-produce even without any flexibility investment. It simply prefers to incur the additional cost of cross-production rather than investing in costly flexibility that may never be used.

  – Flexibility is also not recommended under any demand correlation or level of product substitutability. Investing in higher amounts of cheaper dedicated capacities and utilizing responsive pricing is a more economical way for the firm to mitigate these risks. However, if unit cost of dedicated capacities were higher then the firm may invest in flexibility under negatively correlated demands under clearance.

- As degree of demand uncertainties increases the firm's optimal expected profit increases. This was also observed in the reviewed literature. However, our cross-production models also show that as the degree of capacity uncertainties and supply disruptions increase the firm's optimal expected profit decreases.

Our results emphasize that product differentiation and responsive pricing are excellent risk hedging strategies even under capacity uncertainties and supply disruptions. Flexibility, on the other hand, is a potent tool depending on the *type* of efficiency loss incurred by the firm as well as the *type and degree* of uncertainty. While literature (for e.g., Tomlin and Wang 2005) has shown flexibility to be less beneficial as resource investments become less reliable, our research shows that this is not always true. A *partial* degree of flexibility may be desired depending on the type of loss. Further, our findings on demand uncertainties fully support the empirical evidence of Moreno and Terwiesch (2015) and extend the analysis to capacity uncertainties and supply disruptions. It also explains why completely flexible resources are still rare in industrial practice, although it has been highly advocated in academia.

This dissertation can be extended in several ways. We model a risk-neutral firm that maximizes it's expected profit. In practice there may be several types of decision makers: risk-neutral, risk-taking as well as risk-averse. *Conditional Value-at-Risk* (CVaR), also known as the expected shortfall, can be used to model the different types of these decision makers and their inclination towards risk. In our models the risk-neutral firm maximizes its expected profit in Stage I. To model a loss-averse investment, the firm may try to maximize the $CVaR_{\eta}$, the mean of the left $\eta$-tail of the profit function. The percentile $\eta \in (0,1)$ is the parameter that reflects the firm's inclination for downside risk. At $\eta = 1$ the firm is risk-neutral and for $\eta<1$ the firm maximizes the mean of the profit falling below a specified percentile level $\eta$. CVaR is a coherent risk measure (Rockafellar and Uryasev 2002) that also has some nice analytical properties that is amenable to closed form solutions as illustrated in Tomlin and Wang (2005).

In this research we have assumed that uncertainties or disruptions are independent of each other. Similar to demand correlation, disruptions or uncertainties may be correlated with each other. Hence it may be very beneficial for the firm to understand how the optimal portfolio changes with changes in these correlations. In Stage II when realization of capacity uncertainties or supply disruptions are revealed the nature of disruption correlation could have a key impact on the flexibility levels. Finally, we have also assumed that the additional cost of cross-production is deterministic. This may not always be true as lead times are much longer and currency valuations, labor costs, changing market conditions and entry/ exit of a competitor could impact the cost of cross-production. Hence modeling the cost as a random variable in the planning stage would be a worthwhile extension to this research.

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# APPENDIX A

## Proofs for Chapter 3: The Contingent Flexible Capacity Model

In this technical appendix we provide detailed solutions to the Stage II and Stage I problems.

**Proof of Theorem 3.1**

In Stage II given the invested capacities $K_1$,$K_2$,$K_f$ in Stage I, additional production costs $c_1$, $c_2$ and demand intercept realizations $\epsilon_1$ and $\epsilon_2$, the firm decides the production quantities. Recall from Section 3.3 that there are four different scenarios corresponding to eleven different regions depending on the demand realizations and available capacity levels. Since we partition the feasible region and solve the optimization problem within each region, this avoids any non-differentiability due to terms $(Q_i - K_i)^+$, $i$=$1, 2$. At these boundaries we have either $Q_1 = K_1$ or $Q_2 = K_2$ and the Stage II problem can be solved at those boundaries without using KKT conditions and we can directly obtain the demand region inequalities. The Stage II objective function consists of the revenue term $p_1Q_1 + p_2Q_2$ and additional terms depending on the demand realizations. The revenue term by itself can be shown to be strictly jointly concave in the production quantities (see, for example Appendix B) for the proof of $b$-demand model). The cross-production terms here are linear functions of $Q_1$ or $Q_2$ and the constraints are linear. In each region where the function is differentiable we can use KKT conditions which are both necessary and sufficient to obtain the production quantities in these regions. As stated above, the function is non-differentiable at the boundaries but since we have either $Q_1 = K_1$ or $Q_2 = K_2$ the Stage II problem can be solved at those boundaries without invoking KKT conditions.

Optimal production quantities in Stage II

Combinations of the Lagrange multipliers and slack variables gives rise to eleven different optimization problems each corresponding to the eleven feasible regions (denoted by $\Omega_1, .., \Omega_{11}$). In each region the Stage II problem can be solved in closed form.

We now solve eleven different cases corresponding to the four scenarios depending on whether the capacity constraints are binding or non-binding.

**Region** $\Omega_1$

The corresponding Lagrange multipliers $\lambda_1 = \lambda_2 = \lambda_3 = 0$ by complementary slackness and hence solving for quantities we obtain

$$\Omega_1: \quad Q_1 = \epsilon_1/2\alpha, \quad Q_2 = \epsilon_2/2\alpha$$

**Regions** $\Omega_{2,3}$

Here, $Q_1 \geq K_1$, $Q_2 \leq K_2$ and $Q_1 + Q_2 < K_1 + K_2 + K_f$ as the contingent capacity is not used fully. All three lagrangian multipliers ($\mu_1$, $\mu_2$ and $\mu_3$ ) are 0 which leads us to $Q_1 + Q_2 = \frac{\epsilon_1+\epsilon_2-(c_1+c_2)((1-\gamma}{2\alpha}$, which gives us the demand region $\epsilon_2 \leq 2\gamma(K_2 - \gamma c_1)$. For region 2, we have $Q_2 < K_2$, hence $c_2 = 0$ solving for quantities after setting up the equations yields,

$$\Omega_2: \quad Q_1 = \frac{\epsilon_1 - c_1}{2\alpha}, \quad Q_2 = \frac{\epsilon_2 + \gamma c_2}{2\alpha}$$

$$\Omega_3: \quad Q_2 = \frac{\epsilon_2 - c_2}{2\alpha}, \quad Q_1 = \frac{\epsilon_1 + \gamma c_1}{2\alpha}$$

**Region** $\Omega_4$

Only a portion of the contingent capacity is utilized and it incurs a cost to produce a positive quantity of both the products. Hence, we have

$$\Omega_4: \quad Q_1 = \frac{\epsilon_1 - c_1 + \gamma c_2}{2\alpha}, \quad Q_2 = \frac{\epsilon_2 - c_2 + \gamma c_1}{2\alpha};$$

From the binding constraints we obtain, after substituting the above optimal values, $\epsilon_1 \geq 2\alpha K_1 + c_1 - \gamma c_2, \epsilon_2 \geq 2\alpha K_2 + c_2 - \gamma c_1$ and $\epsilon_1 + \epsilon_2 \leq 2\alpha(K_1 + K_2 + K_f) + (1 - \gamma)(c_1 + c_2)$

**Regions** $\Omega_{5,6}$

We solve in region $\Omega_5$ first. Product 2 is produced only using the dedicated capacity i.e., $Q_2 = K_2$ but product 1 is produced using a portion of the contingent flexible capacity. $Q_1 < K_1 + K_f$ and $Q_1 \geq K_1$. Also the other two constraints are not binding since the contingent capacity is not utilized fully. This is similar to region $\Omega_2$ except that product 2 is produced fully utilizing the dedicated capacity $K_2$. Since $Q_2$=$K_2$ we only need to obtain solution of $Q_1$ which can be easily obtained from its first order conditions as the function is differentiable in $Q_1$ in this region. The solution yields $Q_1 = \frac{\epsilon_1 + \gamma\epsilon_2 - (1-\gamma^2)c_1}{2b} - \gamma K_2$ in region $\Omega_5$. We also obtain $2\alpha(K_1 + \gamma K_2) + (1-\gamma^2)c_1 \leq \epsilon_1 + \gamma\epsilon_2$ and $\epsilon_1 + \gamma\epsilon_2 \leq 2\alpha(K_1 + \gamma K_2 + K_f) + (1-\gamma^2)c_1$ from $Q_1 \leq K_1 + K_f$ and $Q_1 \geq K_1$.

Similarly we obtain,

$$\Omega_6: \quad Q_1 = K_1, \quad Q_2 = \frac{\epsilon_2 + \gamma\epsilon_1 - (1-\gamma^2)c_2}{2b} - \gamma K_1;$$

**Regions** $\Omega_{7,8}$

When the contingent capacity is utilized to produce product 1 completely (in addition to its dedicated capacity) while product 2 is produced utilizing the available dedicated capacity (region $\Omega_8$), $Q_1 = K_1 + K_f$, $Q_2 < K_2$ and $Q_1 + Q_2 < K_1 + K_2 + K_f$. Hence, $\lambda_1 \neq 0$, $\lambda_2, \lambda_3$ and $c_2 = 0$, yield

$$\Omega_8: \quad Q_1 = K_1 + K_f, \quad Q_2 = \frac{\epsilon_2 + \gamma\epsilon_1}{2\alpha} - \gamma(K_1 + K_f)$$

where the defining equations for the two regions are obtained by substituting the values of the optimal solutions and lagrangian multipliers to yield,

$$\Omega_8: \quad \{(\epsilon_1, \epsilon_2) : \epsilon_1 \geq 2\alpha(K_1 + K_f) + c_1, \epsilon_2 \geq 0, \gamma\epsilon_1 + \epsilon_2 \leq 2\alpha\gamma(K_1 + K_f) + 2\alpha K_2\}$$

and symmetrically

$$\Omega_7: \quad \{(\epsilon_1, \epsilon_2) : \epsilon_1 \geq 2\alpha(K_2 + K_f) + c_2, \epsilon_1 \geq 0, \gamma\epsilon_2 + \epsilon_1 \leq 2\alpha\gamma(K_2 + K_f) + 2\alpha K_1\}$$

**Region** $\Omega_9$

When both products are produced using the contingent flexible capacity in addition to their dedicated capacities, the third capacity constraint ($Q_1 + Q_2 = K_1 + K_2 + K_f$) is binding which gives rise to $\mu_3 = \frac{\xi_1+\xi_2-2\alpha(K_1+K_2+K_f)-(c_1+c_2)(1-\gamma)}{2(1-\gamma)} > 0$.

$$\begin{aligned} \Omega_9: \quad Q_1 &= \frac{K_1 + K_2 + K_f}{2} + \frac{\epsilon_1 - \epsilon_2 - (1+\gamma)(c_1 - c_2)}{4\alpha}, \\ Q_2 &= \frac{K_1 + K_2 + K_f}{2} - \frac{\epsilon_1 - \epsilon_2 - (1+\gamma)(c_1 - c_2)}{4\alpha}; \end{aligned}$$

after substituting the values for Lagrange multipliers and the optimal production quantities, we obtain the defining regions.

**Regions** $\Omega_{10,11}$

Here product $3 - i$ is not produced utilizing any of the contingent capacity. Hence only product $i$'s constraint is binding (where $i = 1$ for $\epsilon \in \Omega_{10}$ and $i = 2$ for $\epsilon \in \Omega_{11}$). We obtain,

$$\begin{aligned} \Omega_{10}: &\quad Q_1 = K_1 + K_f, \quad Q_2 = K_2; \\ \Omega_{11}: &\quad Q_1 = K_1, \quad Q_2 = K_2 + K_f \end{aligned}$$

It is easy to see that both quantities as well as prices are non-negative. The demand regions can be obtained from substituting these values into first order condition for $Q_1$ and we have . The other defining inequalities can be obtained directly from borders with region $\Omega_5$ as $\epsilon_1 + \gamma\epsilon_2 \geq 2\alpha(K_1 + \gamma K_2 + K_f) + (1 - \gamma^2)c_1$, region as $\Omega_7$ $\gamma\epsilon_1 + \epsilon_2 \geq 2\alpha\gamma(K_1 + K_f) + 2\alpha K_2$ and from region $\Omega_9$, $\epsilon_1 - \epsilon_2 \geq 2\alpha(K_1 - K_2 + K_f)$ .

We now briefly describe the Lagrangian for the four main scenarios in the $b$-demand model. The solution methodology is very similar to the $\gamma$-demand model.

The Lagrangian of the Stage II objective function $R^L$ has four possible forms depending on the four regions as it was shown in Figure 3.2-Right.

In Region I, contingent flexible capacity is not required , and thus no additional production cost is incurred. We have

$$\begin{aligned} R^{L_I} &= \epsilon_1 Q_1 + \epsilon_2 Q_2 - Q_1^2 - Q_2^2 - 2bQ_1Q_2 - \lambda_1[Q_1 - K_1 - K_f] - \lambda_2[Q_2 - K_2 - K_f] \\ &\quad -\lambda_3[Q_1 + Q_2 - K_1 - K_2 - K_f] + v_1 Q_1 + v_2 Q_2. \end{aligned}$$

$\lambda_1, \lambda_2,\ \lambda_3,\ v_1$ and $v_2$ are the corresponding Lagrange multipliers and slack variables for each of the capacity and non-negativity constraints.

In region II, demand realization of product 1 is higher than that of product 2 and contingent capacity is used to produce product 1, and thus an additional production cost $c_1$ is incurred. We have

$$\begin{aligned} R^{L_{II}} &= \epsilon_1 Q_1 + \epsilon_2 Q_2 - Q_1^2 - Q_2^2 - 2bQ_1Q_2 - (Q_1 - K_1)c_1 - \lambda_1[Q_1 - K_1 - K_f] \\ &\quad -\lambda_2[Q_2 - K_2 - K_f] - \lambda_3[Q_1 + Q_2 - K_1 - K_2 - K_f] + v_1 Q_1 + v_2 Q_2. \end{aligned}$$

In region III, contingent capacity is used to produce product 2, and thus an additional production cost $c_2$ is incurred. We have

$$\begin{aligned} R^{L_{III}} &= \epsilon_1 Q_1 + \epsilon_2 Q_2 - Q_1^2 - Q_2^2 - 2bQ_1Q_2 - (Q_2 - K_2)c_2 - \lambda_1[Q_1 - K_1 - K_f] \\ &\quad -\lambda_2[Q_2 - K_2 - K_f] - \lambda_3[Q_1 + Q_2 - K_1 - K_2 - K_f] + v_1 Q_1 + v_2 Q_2. \end{aligned}$$

In region IV, contingent capacity is used to produce both products as demand for each product exceeds its own dedicated capacity. We have

$$\begin{aligned} R^{L_{IV}} &= \epsilon_1 Q_1 + \epsilon_2 Q_2 - Q_1^2 - Q_2^2 - 2bQ_1Q_2 - (Q_1 - K_1)c_1 - (Q_2 - K_2)c_2 \\ &\quad -\lambda_1[Q_1 - K_1 - K_f] - \lambda_2[Q_2 - K_2 - K_f] \\ &\quad -\lambda_3[Q_1 + Q_2 - K_1 - K_2 - K_f] + v_1 Q_1 + v_2 Q_2. \end{aligned}$$

In summary, in each region, the optimal revenue of the Stage II problem $R^\gamma(\epsilon_1, \epsilon_2)$, are as follows:

$$\Omega_1: \qquad R_1 = \frac{\epsilon_1^2+\epsilon_2^2+2\gamma\epsilon_1\epsilon_2}{4\alpha(1-\gamma^2)};$$

$$\Omega_2: \qquad R_2 = \frac{(\epsilon_1-c_1+\gamma\epsilon_2+\gamma c_1)(\epsilon_1+c_1)+(\gamma\epsilon_1+\epsilon_2)(\epsilon_2+\gamma c_1)}{4\alpha(1-\gamma^2)} - \frac{c_1(\epsilon_1-c_1-2\alpha K_1)}{2\alpha};$$

$$\Omega_3: \qquad R_3 = \frac{(\epsilon_2-c_2+\gamma\epsilon_1+\gamma c_2)(\epsilon_2+c_2)+(\gamma\epsilon_2+\epsilon_1)(\epsilon_1+\gamma c_2)}{4\alpha(1-\gamma^2)} - \frac{c_2(\epsilon_2-c_2-2\alpha K_2)}{2\alpha};$$

$$\Omega_4: \qquad R_4 = \frac{(\epsilon_1+c_1-\gamma c_2)(\epsilon_1+\gamma\epsilon_2-c_1(1-\gamma^2))+(\epsilon_2+c_2-\gamma c_1)(\gamma\epsilon_1+\epsilon_2-c_2(1-\gamma^2))}{4\alpha(1-\gamma^2)}$$
$$+\frac{c_1(\epsilon_1-c_1+\gamma c_2-2\alpha K_1)}{2\alpha} - \frac{c_2(\epsilon_2-c_2+\gamma c_1-2\alpha K_2)}{2\alpha};$$

$$\Omega_5: \qquad R_5 = \frac{(\epsilon_1+\gamma\epsilon_2-(1-\gamma^2)c_1)(1-\gamma^2)(c_1-\gamma\epsilon_2)}{2\alpha}$$
$$+\frac{\gamma(\epsilon_1+\gamma\epsilon_2-(1-\gamma^2)c_1-2\alpha K_2\gamma+2\alpha K_2)(\epsilon_2-\alpha K_2)}{2\alpha}$$
$$-\frac{c_1(\epsilon_1+\gamma\epsilon_2-(1-\gamma^2)c_1-2\alpha\gamma K_2-2\alpha K_1)}{2\alpha};$$

$$\Omega_6: \qquad R_6 = \frac{(\epsilon_2+\gamma\epsilon_1-(1-\gamma^2)c_2)(1-\gamma^2)(c_2-\gamma\epsilon_1)}{2\alpha}$$
$$+\frac{\gamma(\epsilon_2+\gamma\epsilon_1-(1-\gamma^2)c_2-2\alpha K_1\gamma+2\alpha K_1)(\epsilon_1-\alpha K_1)}{2\alpha}$$
$$-\frac{c_2(\epsilon_2+\gamma\epsilon_1-(1-\gamma^2)c_2-2\alpha\gamma K_1-2\alpha K_2)}{2\alpha};$$

$$\Omega_7: \quad R_7 = \frac{1}{1-\gamma^2}[\frac{(\epsilon_1+\gamma\epsilon_2)(\epsilon_1-\gamma\epsilon_2+2\alpha\gamma(K_2+K_f))}{4\alpha}$$
$$+(\gamma\frac{\epsilon_1+\gamma\epsilon_2}{2\alpha}-(K_2+K_f)(1+\gamma^2))(\epsilon_2-\alpha(K_2+K_f))]$$
$$-c_1(\frac{\epsilon_1+\gamma\epsilon_2}{2\alpha}-\gamma(K_2+K_f)-K_1)-c_2K_f;$$
$$\Omega_8: \quad R_8 = \frac{1}{1-\gamma^2}[\frac{(\epsilon_2+\gamma\epsilon_1)(\epsilon_2-\gamma\epsilon_1+2\alpha\gamma(K_1+K_f))}{4\alpha}$$
$$+(\gamma\frac{\epsilon_2+\gamma\epsilon_1}{2\alpha}-(K_1+K_f)(1+\gamma^2))(\epsilon_1-\alpha(K_1+K_f))]$$
$$-c_2(\frac{\epsilon_2+\gamma\epsilon_1}{2\alpha}-\gamma(K_1+K_f)-K_2)-c_1K_f);$$
$$\Omega_9: \quad R_9 = \frac{K_1+K_2+K_f}{4(1-\gamma)}-\frac{c_1(2\alpha(K_1+K_2+K_f)+\epsilon_1-\epsilon_2-(1+\gamma)(c_1-c_2)-4\alpha K_1)}{4\alpha}$$
$$-\frac{c_2(2\alpha(K_1+K_2+K_f)-\epsilon_1-\epsilon_2-(1+\gamma)(c_1-c_2)-4\alpha K_2)}{4\alpha};$$
$$\Omega_{10}: \quad R_{10} = \frac{\epsilon_1(K_1+K_f+\gamma K_2)-\alpha(K_1+K_f)^2-2\alpha\gamma K_2(K_1+K_f)+\epsilon_2(\gamma(K_1+K_f)+K_2)}{1-\gamma^2}$$
$$\frac{-\alpha K_2^2}{1-\gamma^2}-c_1K_f$$
$$\Omega_{11}: \quad R_{11} = \frac{\epsilon_2(\gamma K_1+K_2+K_f)-\alpha(K_2+K_f)^2-2\alpha\gamma K_1(K_2+K_f)+\epsilon_1(K_1+\gamma(K_2+K_f)}{1-\gamma^2}$$
$$\frac{-\alpha K_1^2}{1-\gamma^2}-c_2K_f \quad \blacksquare$$

**Proof of Lemma 3.1**

In this section we prove that the stage I objective function $\Pi^\gamma$ is strictly jointly concave in the investment vector $\mathbf{K}$=$(K_1, K_2, K_f)$ for any continuous distribution of $\xi_i, i = 1, 2$ having positive support. Let $\Psi$ denote the joint pdf of the random variables $\xi_1$ and $\xi_2$. We define the following elements:

$$e_5 \equiv 2\alpha\int\int_{\Omega_5}\Psi(\epsilon_1,\epsilon_2)d\epsilon_1 d\epsilon_2; \quad e_6 \equiv 2\alpha\int\int_{\Omega_6}\Psi(\epsilon_1,\epsilon_2)d\epsilon_1 d\epsilon_2;$$
$$e_7 \equiv 2\alpha\int\int_{\Omega_7}\Psi(\epsilon_1,\epsilon_2)d\epsilon_1 d\epsilon_2; \quad e_8 \equiv 2\alpha\int\int_{\Omega_8}\Psi(\epsilon_1,\epsilon_2)d\epsilon_1 d\epsilon_2;$$
$$e_9 \equiv \frac{\alpha}{1-\gamma}\int\int_{\Omega_9}\Psi(\epsilon_1,\epsilon_2)d\epsilon_1 d\epsilon_2; \quad e_{10} \equiv \frac{2\alpha}{1-\gamma^2}\int\int_{\Omega_{10}}\Psi(\epsilon_1,\epsilon_2)d\epsilon_1 d\epsilon_2;$$
$$e_{11} \equiv \frac{2\alpha}{1-\gamma^2}\int\int_{\Omega_{11}}\Psi(\epsilon_1,\epsilon_2)d\epsilon_1 d\epsilon_2.$$

We further denote

$$A \equiv e_8 + e_9 + e_{10}; \quad B \equiv e_7 + e_9 + e_{11}.$$

We now write **-H**, negative of the Hessian matrix of $\Pi^\gamma(\mathbf{K})$ corresponding to $\mathbf{K}$, as follows:

$$\textbf{-H} = \begin{pmatrix} A + e_6 + e_{11} & e_9 + \gamma e_{10} + e_{11} & A + \gamma e_{11} \\ e_9 + \gamma e_{10} + e_{11} & B + e_5 + e_{10} & B + \gamma e_{10} \\ A + \gamma e_{11} & B + \gamma e_{10} & A + B - e_9 \end{pmatrix}$$

Next, we apply the super diagonalization theorem to check the positive definiteness of **-H**. Observing that all elements on the diagonal are positive, **-H** reduces to the following by elementary row operations:

$$\textbf{-H} = \begin{pmatrix} A + e_6 + e_{11} & e_9 + \gamma e_{10} + e_{11} & A + \gamma e_{11} \\ 0 & B + e_5 + e_{10} - \frac{(e_9+\gamma e_{10}+e_{11})^2}{A+e_6+e_{11}} & B + \gamma e_{10} - \frac{(A+\gamma e_{11})(e_9+\gamma e_{10}+e_{11})}{A+e_6+e_{11}} \\ 0 & B + \gamma e_{10} - \frac{(A+\gamma e_{11})(e_9+\gamma e_{10}+e_{11})}{A+e_6+e_{11}} & A + B - e_9 - \frac{(A+\gamma e_{11})^2}{A+e_6+e_{11}} \end{pmatrix}$$

We let

$$H_{sub} \equiv \begin{pmatrix} h_{22} & h_{23} \\ h_{32} & h_{33} \end{pmatrix}$$

represent the lower right sub-matrix.

Now, since $A + e_6 + e_{11} > 0$, -H is positive definite if and only if $H_{sub}$ is positive definite [using Bazaara, Sherali and Shetty (1993)]. Rewriting the matrix entries and canceling negative quantities we observe that for any values of $\gamma \in (-1, 1)$, the terms $h_{22}, h_{23}$ and $h_{22}h_{33} - h_{23}^2$ are all strictly positive. Hence, we conclude that $H_{sub}$ is positive definite, and therefore, **-H** is positive definite for any continuous distribution of $\xi_i, i = 1, 2$, having positive support. Thus, H is negative definite and $\Pi^\gamma$ is strictly jointly concave in investment vector $\mathbf{K}$. ■

**Proof of Theorem 3.2**

From Lemma 3.1, it is easy to see that optimal investment vector is unique and the first order KKT conditions given in Theorem 3.2 are necessary and sufficient for optimality. This completes the proof.

**Proof of Lemma 3.2**

The proof is straightforward and can be obtained by setting $K_f$=0 in the third optimality condition in Theorem 3.2. $\Omega'_i$, $i = 7, 8, \ldots, 11$, are the demand regions for the new solution (with $K_f$=0).

**Proof of Theorem 3.3**

The proof is similar to the one under the $\gamma$-demand model and thus omitted.

**Optimal stage II revenue under $b$-demand model**

In each region, by solving the Stage II problem of the $b$-demand model, the optimal revenue $R^b(\epsilon_1, \epsilon_2)$ equals,

$$\Omega_1: \qquad R_1 = \frac{a_1^2 + a_2^2 - 2ba_1a_2}{4(1-b^2)};$$

$$\Omega_2: \qquad R_2 = \frac{(a_1(1+b) - c_1))(a_1 - ba_2 + c_1)(1-b) + a_2(1-b^2)(a_2 - ba_1 + bc_1)}{4(1-b^2)^2}$$
$$- \frac{c_1(a_1 - ba_2 - c_1 - 2(1-b^2)K_1)}{2(1-b^2)};$$

$$\Omega_3: \qquad R_3 = \frac{(a_2(1+b) - c_2))(a_2 - ba_1 + c_2)(1-b) + a_1(1-b^2)(a_1 - ba_2 + bc_2)}{4(1-b^2)^2}$$
$$- \frac{c_2(a_2 - ba_1 - c_2 - 2(1-b^2)K_2)}{2(1-b^2)};$$

$$\Omega_4: \qquad R_4 = \frac{(a_1 - ba_2 + c_1 - bc_2)(a_1 - ba_1) - c_1(1-b^2)) + (a_2 - ba_1 + c_2 - bc_1)(a_2 - ba_2)}{4(1-b^2)^2}$$
$$- \frac{c_2(1-b^2)}{4(1-b^2)^2} + \frac{c_1(a_1 - ba_2 - c_1 + bc_2 - 2(1-b^2)K_1)}{2(1-b^2)}$$
$$- \frac{c_2(a_2 - ba_1 - c_2 + bc_1 - 2(1-b^2)K_2)}{2(1-b^2)};$$

$$\Omega_5: \qquad R_5 = \frac{(a_1 - ba_1 - (1-b^2)c_1)(1-b^2)(c_1 - ba_2 - ba_1)}{2(1-b^2)} +$$
$$\frac{(b(a_1 - ba_1 - (1-b^2)c_1 - 2(1-b^2)K_2b + 2(1-b^2)K_2)(a_2 - ba_1 - (1-b^2)K_2)}{2(1-b^2)}$$
$$- \frac{c_1(a_1 - ba_2 + ba_2 - ba_1 - (1-b^2)c_1 - 2(1-b^2)bK_2 - 2(1-b^2)K_1)}{2(1-b^2)};$$

$$\Omega_6: \qquad R_6 = \frac{(a_2 - ba_2 - (1-b^2)c_2)(1-b^2)(c_2 - ba_1 - ba_2)}{2(1-b^2)} +$$
$$\frac{(b(a_2 - ba_2 - (1-b^2)c_2 - 2(1-b^2)K_1b + 2(1-b^2)K_1)(a_1 - ba_2 - (1-b^2)K_1)}{2(1-b^2)}$$
$$- \frac{c_2(a_2 - ba_1 + ba_1 - ba_2 - (1-b^2)c_2 - 2(1-b^2)bK_1 - 2(1-b^2)K_2)}{2(1-b^2)};$$

$$\Omega_7: \qquad R_7 = \frac{1}{1-b^2}[\frac{(a_1-ba_1)(a_1-2ba_2-ba_1+2(1-b^2)b(K_2+K_f))}{4(1-b^2)}+$$
$$(b\frac{a_1-ba_1}{2(1-b^2)}-(K_2+K_f)(1+b^2))(a_2-ba_1-(1-b^2)(K_2+K_f))]$$
$$-c_1(\frac{a_1-ba_1}{2(1-b^2)}-b(K_2+K_f)-K_1)-c_2K_f;$$

$$\Omega_8: \qquad R_8 = \frac{1}{1-b^2}[\frac{(a_2-ba_2)(a_2-2ba_1-ba_2+2(1-b^2)b(K_1+K_f))}{4(1-b^2)}+$$
$$(b\frac{a_2-2ba_1-ba_2}{2(1-b^2)}-(K_1+K_f)(1+b^2))(a_1-ba_2-(1-b^2)(K_1+K_f))]$$
$$-c_2(\frac{a_2-ba_2}{2(1-b^2)}-b(K_1+K_f)-K_2)-c_1K_f);$$

$$\Omega_9: \qquad R_9 = \frac{(K_1+K_2+K_f)}{4(1-b)}-\frac{c_1(2(1-b)(K_2+K_f-K_1)+a_1-a_2-(c_1-c_2))}{4(1-b)}$$
$$-\frac{c_2(2(1-b)(K_1-K_2+K_f)-(a_1+a_2+(c_1-c_2)))}{4(1-b)};$$

$$\Omega_{10}: \qquad R_{10} = a_1(K_1+K_f)+a_2K_2-((K_1+K_f)(K_1+K_f+2bK_2)+K_2^2)-c_1K_f$$

$$\Omega_{11}: \qquad R_{11} = a_2(K_2+K_f)+a_1K_1-((K_2+K_f)(K_2+K_f+2bK_1)+K_2^2)-c_2K_f.$$

**Proof of Lemma 3.3**

The stage I problem under the $b$-demand model is a linear transformation of the $\gamma$-demand model. Hence the proof is similar to the proof of Lemma 3.1 and is omitted.

**Proof of Theorem 3.4**

The proof is similar to one under the $\gamma$-demand model and thus omitted.

**Proof of Theorem 3.5**

When $K_f > 0$, the lagrangian multiplier $\psi_g$ in both demand models (see theorems 3.2 and 3.4) is 0. Note that under the clearance assumption we use the solution for $\Omega_9$ to conduct our analysis which renders other regions as 0 probability regions. Hence, treating $K_f$ as a function of $K_1$ and $K_2$, i.e. for a given dedicated capacity investment, we rearrange the terms in the third vector to obtain the optimal contingent capacity expressions given in theorem 3.5.

**Proof of Lemma 3.4**

Since $K_f > 0$, it follows from proof of theorem 3.5 that $\dfrac{\mu_1 + \mu_2}{2(1+b)} > (K_1 + K_2) + \dfrac{c_1 + c_2 + 2g_f}{2(1+b)}$ in the $b$-demand model. Rearranging the terms we have $\frac{\mu_1+\mu_2-(c_1+c_2+2g_f)}{2(1+b)} > (K_1 + K_2)$. As long as $K_f > 0$, we then have $\frac{\partial K_f}{\partial b} < 0$ as stated in the lemma.

**Proof of Theorem 3.6**

In this case the firm invests in dedicated capacity $K_2$, i.e., $K_2 > 0$, if $g_2 < g_2'$ where $g_2'$ is defined similar to $g_f'$ in lemma 3.2. The proof is similar to the proof of theorem 3.5.

**Proof of Lemma 3.5**

The proof is similar to the proof of lemma 3.4 and thus omitted.

# APPENDIX B

## Proofs for Chapter 4: The Shrinking Capacity Model

### Proof of Theorem 4.1

The Hessian matrix for the Stage II objective function in terms of the quantities $Q_1$ and $Q_2$ ($\mathbf{H}^{SC}$) can be written as:

$$\mathbf{H}^{SC} = \begin{bmatrix} -2 & -2b \\ -2b & -2 \end{bmatrix}$$

Recall that the consumer utility model also required $\vartheta_1 \vartheta_2 - b^2 > 0$ to ensure its strict concavity and under our assumptions $\vartheta_1 = \vartheta_2 = 1$ the determinant is $4(1-b^2) > 0$. $\mathbf{H}^{SC}$ is negative definite and hence the Stage II objective function is strictly jointly concave, the constraints are linear in $Q_1$ and $Q_2$ and therefore the production quantities (and prices) can be uniquely determined.

The Lagrangian of the Stage II problem can be written as:

$$\begin{aligned} R^L &= a_1 Q_1 + a_2 Q_2 - Q_1^2 - Q_2^2 - 2bQ_1Q_2 - (Q_1 - K_1 - \beta_1(K_2 - Q_2))u_1 \\ &\quad -(Q_2 - K_2 - \beta_2(K_1 - Q_1))u_2 + v_1 Q_1 + v_2 Q_2. \end{aligned}$$

Let $u_1$ and $u_2$ be the Lagrangian multipliers corresponding to constraints $Q_1 \leq K_1 + \beta(K_2 - Q_2)$ and $Q_2 \leq K_2 + \beta(K_1 - Q_1)$ respectively. $v_1$, $v_2$ are the slack variables for the non-negativity constraints on the production quantities.

Different combinations of the Lagrange multipliers and slack variables gives rise to 5 different optimization problems each corresponding to the 5 cases described in the Section 4.3 under solution methodology. The feasible regions are denoted by $\Omega_1, .., \Omega_5$. In each region the Stage II problem can be solved in closed form using KKT conditions to obtain the corresponding optimal production quantities.

$$
\begin{aligned}
a_1 - 2Q_1 - 2bQ_2 - u_1 - \beta_2 u_2 + v_1 &= 0, \\
a_2 - 2bQ_1 - 2Q_2 - \beta_1 u_1 - u_2 + v_2 &= 0, \\
u_1(Q_1 - (K_1 + \beta_1(K_2 - Q_2)) &= 0, \\
u_2(Q_2 - (K_2 + \beta_2(K_1 - Q_1)) &= 0, \\
u_1, u_2 &\geq 0, \\
v_1, v_2 &\geq 0
\end{aligned}
$$

**Case 1: Region $\Omega_1$**

In this region capacities are not binding for any of the products i.e., $Q_i < K_i$ for $i = 1, 2$. The unconstrained solution lies in the interior of the feasible region and we have $u_1$=$u_2$=$v_1$=$v_2$=0. The conditions above reduce to solving the following two equations:

$$
\begin{aligned}
a_1 - 2Q_1 - 2bQ_2 &= 0, \\
a_2 - 2bQ_1 - 2Q_2 &= 0
\end{aligned}
$$

The optimal production quantities when demand realizations ($a_1$,$a_2$)$\in \Omega_1$ are given by:

$$
Q_1 = \frac{a_1 - ba_2}{2(1 - b^2)}, \quad Q_2 = \frac{a_2 - ba_1}{2(1 - b^2)};
$$

Recall that for the strict concavity of the consumer utility function it was required that $a_1 - ba_2 > 0$, $a_1 - ba_2 < 0$ and $1 - b^2 > 0$. Hence, quantities are positive. Prices are given by:

$$
p_1 = \frac{a_1}{2}, \quad p_2 = \frac{a_2}{2};
$$

**Case 2: Region $\Omega_2$**

Next we consider Case 2 where capacity is binding only for resource 1. The demand realization for product 1 is high compared to product 2 ($Q_2 < K_2$) and the firm uses cross-production to produce product 1 using resource 2. This corresponds to Region $\Omega_2$ where we have $Q_1 = K_1 + \beta_1(K_2 - Q_2)$ and hence $u_1 > 0$ and $u_2 = 0$. Solving for quantities we obtain,

$$Q_1 = \frac{(a_1 - a_2)\beta_1 + 2(1-b)(K_1 + \beta_1 K_2)}{2(1-b)(1+\beta_1)}, \quad Q_2 = \frac{a_2 - a_1 + 2(1-b)(K_1 + \beta_1 K_2)}{2(1-b)(1+\beta_1)}$$

If this solution also satisfies $Q_1 \geq K_1$ and $Q_2 \geq 0$ then it is optimal for the Stage II problem. That is, if $a_1 - a_2 \geq 2(1-b)(K_1 - K_2)$ and $a_1 - a_2 \leq 2(1-b)(K_1 + \beta_1 K_2)$ then the solutions are optimal. From the Lagrange multiplier we also obtain the defining equation for the Region 2 as $a_1 - ba_2 + \beta_1(a_2 - ba_1) \geq 2(2-b^2)(K_1 + \beta_1 K_2)$. The three inequalities together define the demand region where this solution is optimal.

In general, the expressions below represent production quantities, $Q_i$, and optimal prices, $p_i$, obtained from the KKT conditions.

$$Q_1 = \frac{(a_1 - ba_2) - u_1(1 - b\beta_1) + u_2(b - \beta_1)}{2(1-b^2)},$$
$$Q_2 = \frac{(a_2 - ba_1) - u_2(1 - b\beta_2) + u_1(b - \beta_2)}{2(1-b^2)},$$
$$p_1 = \frac{a_1 + u_1 + \beta u_2}{2} \geq 0, \quad p_2 = \frac{a_2 + u_2 + \beta u_1}{2} \geq 0$$

It is easy to see that optimal prices are always non-negative because $u_i \geq 0$ for $i \in 1, 2$. Quantities are also non-negative once we substitute the corresponding values of Lagrange multipliers in each region and under the assumption that $a_i - ba_j > 0$ for $i, j = 1, 2$ and $i \neq j$. In the absence of any cross-production (i.e., when the constraint set is only given by $Q_1 \leq K_1$ and $Q_2 \leq K_2$), we can show that the amount of product $i$ produced will be at least $min((a_i - ba_j)/(2(1-b^2)), K_i) \geq 0$ for $i, j = 1, 2$ and $i \neq j$ and the solution is always feasible.

**Case 3: Region $\Omega_3$**

The proof for region $\Omega_3$ under Case 3 is symmetric to Case 2 (region $\Omega_2$) and hence omitted.

**Case 4: Region $\Omega_4$**

Next we consider Case 4 where when the demand realization for product 1 is much higher compared to product 2 and the firm uses cross-production to produce *only* product 1 using both resources. We have $Q_1 = K_1 + \beta_1 K_2$ and $Q_2 = 0$. Therefore, $u_1 > 0$ and $v_2 > 0$ and we have $a_1 - a_2 \geq 2(1-b)(K_1 + \beta_1 K_2)$. Hence if the demand realizations are within this region, the solutions are optimal for the Stage II problem.

**Case 5: Region $\Omega_5$**

The proof for region $\Omega_5$ under Case 5 is symmetric to Case 4 (region $\Omega_4$) and hence omitted.

**Proof of Lemma 4.1**

In this section we prove that the Stage I objective function $\Pi$ is strictly jointly concave in the investment vector $(K, \beta)$ for any continuous distribution of $A_i, i = 1, 2$ having positive support. We assume a symmetric capacity investment where $K_1 = K_2 = K$ as the proof is shorter (the asymmetric case can be proved along similar lines). The profit function is continuous and the first derivatives are bounded in each region. We hence write **-H**, negative of the Hessian matrix of $\Pi(K, \beta)$ as:

$$\begin{pmatrix} \begin{array}{l} \iint\limits_{\Omega_2} \frac{\partial^2 \Pi_2}{\partial K^2} \Psi(A_1, A_2) + \iint\limits_{\Omega_3} \frac{\partial^2 \Pi_3}{\partial K^2} \Psi(A_1, A_2) \\ + \iint\limits_{\Omega_4} \frac{\partial^2 \Pi_4}{\partial K^2} \Psi(A_1, A_2) + \iint\limits_{\Omega_5} \frac{\partial^2 \Pi_5}{\partial K^2} \Psi(A_1, A_2) \end{array} & \begin{array}{l} \iint\limits_{\Omega_2} \frac{\partial^2 \Pi_2}{\partial K \partial \beta} \Psi(A_1, A_2) + \iint\limits_{\Omega_3} \frac{\partial^2 \Pi_3}{\partial K \partial \beta} \Psi(A_1, A_2) \\ + \iint\limits_{\Omega_4} \frac{\partial^2 \Pi_4}{\partial K \partial \beta} \Psi(A_1, A_2) + \iint\limits_{\Omega_5} \frac{\partial^2 \Pi_5}{\partial K \partial \beta} \Psi(A_1, A_2) \end{array} \\ \\ \begin{array}{l} \iint\limits_{\Omega_2} \frac{\partial^2 \Pi_2}{\partial K \partial \beta} \Psi(A_1, A_2) + \iint\limits_{\Omega_3} \frac{\partial^2 \Pi_3}{\partial K \partial \beta} \Psi(A_1, A_2) \\ + \iint\limits_{\Omega_4} \frac{\partial^2 \Pi_4}{\partial K \partial \beta} \Psi(A_1, A_2) + \iint\limits_{\Omega_5} \frac{\partial^2 \Pi_5}{\partial K \partial \beta} \Psi(A_1, A_2) \end{array} & \begin{array}{l} \iint\limits_{\Omega_2} \frac{\partial^2 \Pi_2}{\partial \beta^2} \Psi(A_1, A_2) + \iint\limits_{\Omega_3} \frac{\partial^2 \Pi_3}{\partial \beta^2} \Psi(A_1, A_2) \\ + \iint\limits_{\Omega_4} \frac{\partial^2 \Pi_4}{\partial \beta^2} \Psi(A_1, A_2) + \iint\limits_{\Omega_5} \frac{\partial^2 \Pi_5}{\partial \beta^2} \Psi(A_1, A_2) \end{array} \end{pmatrix}$$

The derivatives in each region are obtained as follows:

- $\Omega_1$: $\frac{\partial \Pi_1}{\partial K} = 0$; $\qquad \frac{\partial \Pi_1}{\partial \beta} = 0$.

- $\Omega_2$: $\frac{\partial \Pi_2}{\partial K} = \frac{2(A_1(1-b\beta)+A_2(\beta-b)-2(1-b^2)(1+\beta)K)}{(1-b)(1+\beta)}$

  $\frac{\partial^2 \Pi_2}{\partial K^2} = -4(1+b) < 0$.

  $\frac{\partial^2 \Pi_2}{\partial K \partial \beta} = \frac{2(1+b)(A_1-A_2)}{(-1+b)(1+\beta)^2} < 0$; $\frac{\partial^2 \Pi_2}{\partial \beta^2} = \frac{(A_1-A_2)(A_1(-3+b(-1+2\beta))+A_2(1+3b-2\beta)+4(1-b^2)K(1+\beta))}{(-1+b)^2(1+\beta)^4}$;

- $\Omega_3$: $\frac{\partial \Pi_3}{\partial K} = \frac{2(A_2(1-b\beta)+A_1(\beta-b)-2(1-b^2)(1+\beta)K)}{(1-b)(1+\beta)} < 0$; $\frac{\partial^2 \Pi_3}{\partial K^2} = -4(1+b) < 0$.

  $\frac{\partial^2 \Pi_3}{\partial K \partial \beta} = \frac{2(1+b)(A_2-A_1)}{(-1+b)(1+\beta)^2} - 2g < 0$; $\frac{\partial^2 \Pi_3}{\partial \beta^2} = \frac{(A_2-A_1)(A_2(-3+b(-1+2\beta))+A_1(1+3b-2\beta)+4(1-b^2)K(1+\beta))}{(-1+b)^2(1+\beta)^4}$;

- $\Omega_4$: $\frac{\partial \Pi_4}{\partial K} = A_1(1+\beta) - 2K(1+\beta)^2$ $\frac{\partial^2 \Pi_4}{\partial K^2} = -2(1+\beta)^2 < 0$.

  $\frac{\partial^2 \Pi_4}{\partial K \partial \beta} = A_1 - 4K(1+\beta) - 2g < 0$; $\qquad \frac{\partial^2 \Pi_4}{\partial \beta^2} = -2K^2 < 0$;

- $\Omega_5$: $\frac{\partial \Pi_5}{\partial K} = A_2(1+\beta) - 2K(1+\beta)^2$ $\frac{\partial^2 \Pi_5}{\partial K^2} = -2(1+\beta)^2 < 0$.

  $\frac{\partial^2 \Pi_5}{\partial K \partial \beta} = A_2 - 4K(1+\beta) < 0$; $\qquad \frac{\partial^2 \Pi_5}{\partial \beta^2} = -2K^2 < 0$;

Except $\frac{\partial^2 \Pi_2}{\partial \beta^2}$ and $\frac{\partial^2 \Pi_3}{\partial \beta^2}$ all other terms in the Hessian are strictly negative. Consider $\frac{\partial^2 \Pi_2}{\partial \beta^2}$.

When $K_1 = K_2 = K$ the inequalities defining $\Omega_2$ can be rewritten as follows: $A_1 - A_2 \geq 0$, $A_1 - bA_2 + \beta(A_2 - bA_1) \geq 2(1-b^2)K(1+\beta)$ and $A_1 - A_2 \leq 2K(1-b)(1+\beta)$. We rearrange the second term as $0 \geq -2(A_1 - bA_2) + 4(1-b^2)K(1+\beta) - 2\beta(A_2 - bA_1)$. In $\frac{\partial^2 \Pi_2}{\partial \beta^2}$ we have $A_1 - A_2 \geq 0$ and $(1-b)^2(1+\beta)^4 \geq 0$ from the definitions of the region. Hence the sign of the entire term depends upon the sign of $A_1(-3+b(-1+2\beta)) + A_2(1+3b-2\beta) + 4(1-b^2)K(1+\beta)$ which can be simplified as $-3(A_1 - bA_2) + (A_2 - bA_1)(1-2\beta) + 4K(1-b^2)(1+\beta)$. This can be re-written as $-2(A_1 - bA_2) + 4K(1-b^2)(1+\beta) - 2\beta(A_2 - bA_1) + (A_2 - A_1)(1+b)$. Comparing this term with the regional inequality $0 \geq -2(A_1 - bA_2) + 4K(1-b^2)(1+\beta) - 2\beta(A_2 - bA_1)$ and noting that in this region $A_2 - A_1 \leq 0$ we conclude that this term is negative. Proof for $\Omega_3$ is along similar lines. Since the terms in the Hessian as well as the determinant are negative, we conclude that the Stage I objective function is jointly concave in $(K, \beta)$.

**Proof of Theorem 4.2**

Using Lemma 4.1, we differentiate the Stage I objective function $\Pi$ w.r.t. capacities $K_1, K_2$ to obtain the first order conditions. ■

**Proof of Theorem 4.3**

It is easy to see that the Stage II optimal production quantities decrease as $b$ increases. Hence the Stage II profit decreases overall. Since the Stage I objective function is concave in the capacities and flexibility, this trend exists in expectation. Hence overall optimal expected profit decreases.

**Proof of Theorem 4.4**

We assume $\beta_1$=$\beta_2$=$\beta$. The optimal revenue $R(A_1, A_2)$ in regions 2 and 3 are symmetric and, regions 4 and 5 are symmetric. We prove for $K_1$ as proof for $K_2$ is along similar lines.

We evaluate $\frac{\partial K}{\partial b}$ which in turn is obtained by implicit differentiation as: $\frac{\partial}{\partial b}(\frac{\partial \Pi}{\partial K})$ $= \frac{\partial^2 \Pi}{\partial K^2}|_{K=K^*(b)}\ \frac{\partial K^*(b)}{\partial b} + \frac{\partial^2 \Pi(b)}{\partial K \partial b}|_{K=K^*(b)}$. Then, $\frac{\partial K^*(b)}{\partial b}$ = - $\frac{\frac{\partial^2 \Pi(b)}{\partial K \partial b}|_{K=K^*(b)}}{\frac{\partial^2 \Pi}{\partial K^2}|_{K=K^*(b)}}$. Since $\Pi(b)$ is strictly concave, $\frac{\partial^2 \Pi}{\partial K_1^2}|_{K=K^*(b)} < 0$. For $K^*(b) > 0$ , $\frac{\partial K}{\partial b}$ is of the same sign as $\frac{\partial^2 \Pi(b)}{\partial K \partial b}|_{K=K^*(b)}$. We apply Leibniz rule for differentiating under the integral sign and after tedious algebra obtain,

$$\frac{\partial K_1}{\partial b} = -4K_2 \int_{\Omega_4} \frac{A_1 - 2(K_1 + \beta K_2)}{1+\beta} h(A_1, A_1 - 2(K_1 + \beta K_2) dA_1 dA_2 +$$
$$\int\int_{\Omega_2} \frac{(A_2 - 2K_2)(1-\beta) - 2(A_1 - 2K_1)}{(1+\beta)^3} h(A_1, A_2) dA_1 dA_2$$

Clearly each term is $< 0$ (by definitions of the respective demand regions) and hence $\frac{\partial K_1}{\partial b} < 0$ in all regions. Similar proof follows for $K_2$.

**Proof of Theorem 4.5**

The clearance assumption is satisfied in Region 2. Using the first order conditions for the firm for $K_1$ in Region 2 we directly derive this result.

**Proof of Theorem 4.6**

Using the expression in Region 2 for first order conditions of the optimal reconfiguration level, and under the clearance assumption we use the logarithm of beta to obtain the following expression: $log(A_1 - A_2) + log(K_1 - K_2) + log(1 + \beta) + 10 log\beta + 2log(3g_3^2) = 2log(1 + b)$. It is easy to see that the optimal reconfiguration factor is linearly related to product substitutability $b$ and hence as $b$ increases optimal $\beta$ increases. $\blacksquare$

## APPENDIX C

### Proofs for Chapter 5: The Additional Cost Model

#### Proof of Theorem 5.1

In Stage II given the invested capacities $K_1$, $K_2$ and degree of flexibility $f$ in Stage I and demand intercept realizations $a_1$ and $a_2$, the firm decides the production quantities. Recall from Section 5.3 that there are three different scenarios and sub-cases corresponding to each scenario. Since we partition the feasible region and solve the optimization problem within that region, this avoids any non-differentiability due to terms $(Q_i - K_i)^+$, $i$=1, 2. Depending on the demand region we are either left with no cross-production terms (corresponding to the four cases in Scenario 1) or we have $(Q_1 - K_1)c_1(1 - f)$ (corresponding to the three cases under Scenario 2) or we have $(Q_2 - K_2)c_2(1 - f)$ (corresponding to the three cases under Scenario 3), which are linear terms and hence are differentiable. At the boundaries we do not use KKT conditions as the solutions are either $Q_1 = K_1$ or $Q_2 = K_2$ and hence the demand regions can be directly obtained from boundaries of other demand regions.

The Stage II objective function consists of the revenue term $p_1Q_1 + p_2Q_2$ and additional terms depending on the demand regions. The revenue term was shown to be strictly jointly concave in the proof for Shrinking Capacity model in Appendix B. The additional cross-production terms here are linear functions of $Q_1$ or $Q_2$. Thus the Stage II function is strictly jointly concave with linear constraints and in each region where the objective function is differentiable, we can use KKT conditions which are both necessary and sufficient to obtain the production quantities. As stated above, the function is non differentiable at the boundaries but since we have either $Q_1 = K_1$

or $Q_2 = K_2$ the Stage II problem can be solved at those boundaries without using KKT conditions.

Let $u$, $v_1$ and $v_2$ be the Lagrangian multipliers corresponding to constraints $Q_1 + Q_2 \leq K_1 + K_2$ and $Q_1 \geq 0$ and $Q_2 \geq 0$ respectively. Combinations of the Lagrange multipliers and slack variables gives rise to 10 different optimization problems each corresponding to the 10 feasible regions (denoted by $\Omega_1, .., \Omega_{10}$). In each region the Stage II problem can be solved in closed form.

**Scenario 1:** We have $Q_1 \leq K_1$ and $Q_2 \leq K_2$ leading to four different optimization problems. The Stage II objective function does not include any of the additional cost terms. We discuss the solutions of the *four* possible cases:

The Lagrangian of the Stage II problem for Scenario 1 (for each of the demand regions $\Omega_1, \Omega_2, \Omega_3, \Omega_8$) can be written as:

$$\begin{aligned} R^L &= a_1 Q_1 + a_2 Q_2 - Q_1^2 - Q_2^2 - 2bQ_1Q_2 \\ &\qquad\qquad -[Q_1 + Q_2 - (K_1 + K_2)]u + v_1 Q_1 + v_2 Q_2. \end{aligned}$$

**Case 1** corresponds to $Q_1 < K_1$ and $Q_2 < K_2$. In this region ($\Omega_1$) capacities are not binding for any of the products i.e., $Q_i < K_i$ for $i = 1, 2$. The unconstrained solution lies in the interior of the feasible region and we have $u = v_1 = v_2 = 0$. The conditions above reduce to solving the following two equations:

$$a_1 - 2Q_1 - 2bQ_2 = 0,$$

$$a_2 - 2bQ_1 - 2Q_2 = 0$$

The optimal production quantities when demand realizations $(a_1, a_2) \in \Omega_1$ are given by:

$$Q_1 = \frac{a_1 - ba_2}{2(1 - b^2)}, \quad Q_2 = \frac{a_2 - ba_1}{2(1 - b^2)};$$

Recall that for the strict concavity of the consumer utility function it was required that $A_1 - bA_2 > 0$, $A_1 - bA_2 < 0$ and $1 - b^2 > 0$. Hence, quantities are positive. Prices are given by:

$$p_1 = \frac{a_1}{2}, \quad p_2 = \frac{a_2}{2};$$

**Case 2** corresponds to $Q_1 = K_1$ and $Q_2 < K_2$. In this region $(\Omega_2)$ capacity is binding only for resource 1 but no cross production is used. $u$=$v_1$=$v_2$=0 and $Q_1 = K_1$. From the first order condition of quantity $Q_2$, $a_2 - 2bK_1 - 2Q_2 = 0$, we obtain $Q_2$.

$$Q_1 = K_1, Q_2 = \frac{a_2}{2} - bK_1;$$

This solution is optimal if and only if $Q_2 \leq K_2$ and we obtain the inequality $a_2 \leq 2(K_2 + bK_1)$.

**Case 3** is symmetric to Case 2 and hence the production quantities in this region $(\Omega_3)$ are given by:

$$Q_1 = \frac{a_2}{2} - bK_2, Q_2 = K_2;$$

**Case 4** corresponds to very high realizations of demands for both products (region $\Omega_8$) . In this case $Q_1 = K_1$ and $Q_2 = K_2$. Cross-production is not optimal and hence is never used.

**Scenario 2:** In this scenario we have $Q_1$>$K_1$ and $Q_2$ <$K_2$. This leads to *three* possible cases as the firm incurs additional production cost $c_1$. The Lagrangian of the Stage II problem for Scenario 2 for each of the three cases corresponding to demand regions $\Omega_4$,$\Omega_6$ and $\Omega_9$ can be written as:

$$\begin{aligned} R^L &= a_1Q_1 + a_2Q_2 - Q_1^2 - Q_2^2 - 2bQ_1Q_2 - (Q_1 - K_1)c_1(1-f) \\ &\quad -[Q_1 + Q_2 - (K_1 + K_2)]u + v_1Q_1 + v_2Q_2. \end{aligned}$$

Combinations of the Lagrangian multipliers and slack variables enables us to obtain closed form solutions for the production quantities in each of the three regions as shown below:

**Case 1**

In Region $\Omega_4$ we have demand for product 1 to be high and hence we use cross-production to produce product 1 using resource 2. We have $Q_1 > K_1$ implying $c_1(1-f) > 0$ and $Q_2 < K_2$ and $c_2(1-f) = 0$. When $Q_1 > 0$, $Q_2 > 0$ and constraint is not binding i.e, $Q_1 + Q_2 < K_1 + K_2$ the corresponding Lagrange multipliers $u = 0$, $v_1 = 0$ and $v_2 = 0$ by complementary slackness and hence solving for quantities using the above equations we obtain

$$Q_1 = \frac{a_1 - ba_2 - c_1(1-f)}{2(1-b^2)}, \quad Q_2 = \frac{a_2 - ba_1 + bc_1(1-f)}{2(1-b^2)};$$

If these solutions also satisfy the constraints $a_1 - ba_2 > 2(1-b^2)K_1 + c_1(1-f)$ obtained from $Q_1 \geq K_1$, and $a_1 + a_2 \leq 2(1+b)(K_1+K_2) + c_1(1-f)$ obtained from $Q_2 \leq K_2$ and $Q_2 \geq 0$, then they are optimal. Quantities and prices are non-negative (similar to proof of Theorem 4.1) once we substitute the corresponding values of Lagrange multipliers in each region and under the assumption that $a_i - ba_j > 0$ for $i, j = 1, 2$ and $i \neq j$.

**Case 2**

In Region $\Omega_6$ the constraint $Q_1 + Q_2 = K_1 + K_2$ is binding and $Q_1 > K_1$ implying $c_1(1-f) > 0$ and $Q_2 < K_2$ and $c_2(1-f) = 0$. Hence the corresponding Lagrange multipliers $u > 0$, $v_1 = 0$ and $v_2 = 0$ by complementary slackness. Solving for quantities using the three equations we obtain the production quantities as:

$$Q_1 = \frac{K_1 + K_2}{2} + \frac{a_1 - a_2 - c_1(1-f)}{4(1-b)},$$
$$Q_2 = \frac{K_1 + K_2}{2} - \frac{a_1 - a_2 - c_1(1-f)}{4(1-b)} \quad \blacksquare$$

From $Q_2 \leq K_2$, $Q_2 \geq 0$ and $Q_1 \geq K_1$ we obtain the defining equations for this region.

**Case 3**

In Region $\Omega_9$ the demand realization of product 1 is so high the firm uses all of its resources to produce only product 1. $Q_1 = K_1 + K_2$ and $Q_2 = 0$ and the firm incurs the additional cost of cross-production. These solutions are optimal as long as the difference between the demands are defined by $a_1 - a_2 > 2(1-b)(K_1+K_2)+c_1(1-f)$.

**Scenario 3:** In this scenario we have $Q_1<K_1$ and $Q_2 >K_2$. This leads to *three* possible cases as the firm incurs additional production cost $c_2$. The Lagrangian of the Stage II problem for Scenario 3 for each of the three cases corresponding to demand regions $\Omega_5$,$\Omega_7$ and $\Omega_{10}$ can be written as:

$$\begin{aligned} R^L &= a_1Q_1 + a_2Q_2 - Q_1^2 - Q_2^2 - 2bQ_1Q_2 - (Q_2 - K_2)c_2(1-f) \\ &\qquad -[Q_1 + Q_2 - (K_1 + K_2)]u + v_1Q_1 + v_2Q_2. \end{aligned}$$

The three cases in Scenario 3 are symmetric to the cases described in Scenario 2 for product 1. For each combination of the Lagrange multiplier and slack variables we can solve three different problems to obtain the production quantities in regions $\Omega_5$,$\Omega_7$ and $\Omega_{10}$. The proofs are along very similar lines as developed for Scenario 2 and hence omitted.

**Proof of Theorem 5.1**

To prove the concavity of the objective function, we let $h$=$f^2$. If there exists a unique solution for $h$, then there exists a unique solution for $f$. Assuming $K_1$=$K_2$=$K$ and $c_1$=$c_2$=$c$ the derivatives in each region are obtained as follows:

- $\Omega_1$: $\frac{\partial^2 \Pi_1}{\partial K^2} = 0$; $\qquad \frac{\partial^2 \Pi_1}{\partial h^2} = 0$.

- $\Omega_2$: $\frac{\partial \Pi_2}{\partial K}= A_1 - bA_2 - 2(1-b^2)K$

  $\frac{\partial^2 \Pi_2}{\partial K^2}=-2(1-b^2) < 0.$

  $\frac{\partial^2 \Pi_2}{\partial K \partial h}= 0$; $\frac{\partial^2 \Pi_2}{\partial h^2}=0$;

- $\Omega_3$: $\frac{\partial \Pi_3}{\partial K}= A_2 - bA_1 - 2(1-b^2)K$; $\frac{\partial^2 \Pi_3}{\partial K^2}=-2(1-b^2) < 0.$

  $\frac{\partial^2 \Pi_3}{\partial K \partial h}= 0$; $\frac{\partial^2 \Pi_3}{\partial h^2}=0$;

- $\Omega_4$: $\frac{\partial^2 \Pi_4}{\partial K^2}=0.$

  $\frac{\partial^2 \Pi_4}{\partial K \partial h}= \frac{-c}{2(\sqrt{h})} < 0$; $\frac{\partial^2 \Pi_4}{\partial h^2}=\frac{-c}{h^{\frac{3}{2}}8(1-b^2)}(A_1 - bA_2 - c - 2K(1-b^2)$;

- $\Omega_5$: $\frac{\partial^2 \Pi_5}{\partial K^2}=0.$

  $\frac{\partial^2 \Pi_5}{\partial K \partial h}= \frac{-c}{2(\sqrt{h})} < 0$; $\frac{\partial^2 \Pi_5}{\partial h^2}=\frac{-c}{h^{\frac{3}{2}}8(1-b^2)}(A_2 - bA_1 - c - 2K(1-b^2)$;

- $\Omega_6$: $\frac{\partial \Pi_6}{\partial K}= A_1 + A_2 + c(1-\sqrt{h}) - 4(1+b)K$ $\frac{\partial^2 \Pi_6}{\partial K^2}=-4(1+b) < 0.$

  $\frac{\partial^2 \Pi_6}{\partial K \partial h}= 0$; $\frac{\partial^2 \Pi_6}{\partial h^2}=\frac{-c}{h^{\frac{3}{2}}16(1-b)}(A_1 - A_2 - c)$;

- $\Omega_7$: $\frac{\partial \Pi_7}{\partial K}= A_1 + A_2 + c(1-\sqrt{h}) - 4(1+b)K$ $\frac{\partial^2 \Pi_7}{\partial K^2}=-4(1+b) < 0.$

  $\frac{\partial^2 \Pi_7}{\partial K \partial h}= 0$; $\frac{\partial^2 \Pi_7}{\partial h^2}=\frac{-c}{h^{\frac{3}{2}}16(1-b)}(A_2 - A_1 - c)$;

- $\Omega_8$: $\frac{\partial^2 \Pi_8}{\partial K^2}=-2 < 0.$

  $\frac{\partial^2 \Pi_8}{\partial K \partial h}= 0$; $\frac{\partial^2 \Pi_8}{\partial h^2}=0$;

- $\Omega_9$: $\frac{\partial \Pi_9}{\partial K}= 2A_1 - 8K$ $\frac{\partial^2 \Pi_9}{\partial K^2}=-8 < 0.$

  $\frac{\partial^2 \Pi_9}{\partial K \partial h}= \frac{c}{2\sqrt{h}}$; $\frac{\partial^2 \Pi_9}{\partial h^2}=\frac{-cK}{4h^{\frac{3}{2}}}$;

- $\Omega_{10}$: $\frac{\partial \Pi_{10}}{\partial K}= 2A_1 - 8K$ $\frac{\partial^2 \Pi_{10}}{\partial K^2}=-8 < 0.$

  $\frac{\partial^2 \Pi_{10}}{\partial K \partial h}= \frac{c}{2\sqrt{h}}$; $\frac{\partial^2 \Pi_{10}}{\partial h^2}=\frac{-cK}{4h^{\frac{3}{2}}}$;

Let $\Psi$ denote the joint pdf of the random variables $A_1$ and $A_2$. We define the following elements:

$$
\begin{aligned}
B &\equiv \int\int_{\Omega_4} \Psi(A_1, A_2) dA_1 dA_2 + \int\int_{\Omega_5} \Psi(A_1, A_2) dA_1 dA_2; \\
L &\equiv \int\int_{\Omega_4} A_1 - bA_2 - c(2K)(1-b^2)\Psi(A_1, A_2) dA_1 dA_2; \\
Z &\equiv \int\int_{\Omega_5} A_2 - bA_1 - c(2K)(1-b^2)\Psi(A_1, A_2) dA_1 dA_2; \\
X &\equiv \int\int_{\Omega_9} \Psi(A_1, A_2) dA_1 dA_2 + \int\int_{\Omega_{10}} \Psi(A_1, A_2) dA_1 dA_2; \\
Y &\equiv \int\int_{\Omega_2} \Psi(A_1, A_2) dA_1 dA_2 + \int\int_{\Omega_3} \Psi(A_1, A_2) dA_1 dA_2; \\
M &\equiv \int\int_{\Omega_6} \Psi(A_1, A_2) dA_1 dA_2 + \int\int_{\Omega_7} \Psi(A_1, A_2) dA_1 dA_2; \\
T &\equiv \int\int_{\Omega_8} \Psi(A_1, A_2) dA_1 dA_2; \\
D &\equiv \int\int_{\Omega_6} (A_1 - A_2 - c)\Psi(A_1, A_2) dA_1 dA_2; \\
E &\equiv \int\int_{\Omega_7} (A_2 - A_1 - c)\Psi(A_1, A_2) dA_1 dA_2;
\end{aligned}
$$

We hence write $\mathbf{H}$, the Hessian matrix of $\Pi(K, h)$ as:

$$
\begin{pmatrix}
-2(1-b^2)Y - 4(1+b)M - 2T - 8X & \frac{-c}{2\sqrt{h}}B + \frac{c}{2\sqrt{h}}X - \frac{g_3}{\sqrt{h}} \\
\frac{-c}{2\sqrt{h}}B + \frac{c}{2\sqrt{h}}X - \frac{g_3}{\sqrt{h}} & \frac{-c(L+Z)}{8(1-b^2)h^{1.5}} - \frac{c(D+E)}{16(1-b)h^{1.5}} - \frac{cKX}{4h^{1.5}} + \frac{g_3 K}{2h^{1.5}}
\end{pmatrix}
$$

Observe that all elements on the diagonal of **-H**, the negative of the Hessian, are positive due to the definitions of the regions. We now calculate the determinant of the Hessian **-H** as follows: Let

$$N \equiv \frac{1}{h^{1.5}}; S \equiv \frac{1}{\sqrt{h}};$$

Hence, the determinant can be simplified as:

$$\begin{aligned}\text{Det}(\textbf{-H}): \quad &= \left[\frac{cN}{16}\left(\frac{D+E}{1-b} + 4KX + \frac{2(L+Z)}{1-b^2} - \frac{g_3 N}{2}\right)\right]\left[2(Y(1-b^2) + 2(1+b)M + T + 4X\right] \\ &\quad - [Bc + 2g_3 - cX]^2 \frac{S^2}{4} > 0.\end{aligned}$$

We hence conclude that **-H** is positive definite for any continuous distribution of $A_1$ and $A_2$ and hence **H** is negative definite and the objective function is strictly jointly concave in $(K,\ f)$ and the first-order conditions are necessary and sufficient for optimality. ■

# APPENDIX D

## Notations

Table D.1: Common Notation

| | |
|---|---|
| RV | Random Variable |
| CV | Coefficient of Variation |
| $A_1, A_2$ | RVs denoting demand intercepts of products in **Stage I** |
| $a_1$,$a_2$ | Realizations of RVs $A_1$, $A_2$ in **Stage II** |
| $\Theta_1$,$\Theta_2$ | RVs denoting degree of uncertainties of resources in **Stage I** |
| $\theta_1$,$\theta_2$ | Realizations of RVs $\Theta_1$,$\Theta_2$ in **Stage II** |
| $\mu_1, \mu_2$ | Mean of RVs $A_1, A_2$ |
| $\sigma_1$, $\sigma_2$ | Standard Deviation of RVs $A_1$,$A_2$ |
| $\rho \in [-1, 1]$ | Correlation of demand intercepts of RVs $A_1$,$A_2$ |
| $b \in [0, 1)$ | Product substitutability factor in the $b$-demand model |
| $\vartheta_1, \vartheta_2$ | Price sensitivity parameters of products in $b$-demand model |
| $\bar{K}_1$,$\bar{K}_2$ | Invested resource capacities in **Stage I** |
| $K_1$,$K_2$ | Available resource capacities in **Stage II** |
| $Q_1$,$Q_2$ | Decision variables denoting selling quantities in **Stage II** |
| $p_1$,$p_2$ | Decision variables denoting unit selling prices in **Stage II** |
| $g_1$,$g_2$ | Unit capacity cost of resources |

Table D.2: Additional Notation for Contingent Flexible Capacity Model

| | |
|---|---|
| $\mathcal{E}_1$ and $\mathcal{E}_2$ | RVs of product demand intercepts in **Stage I** ($\gamma$-demand model) |
| $\epsilon_1, \epsilon_2$ | Realization of RVs $\mathcal{E}_1$ and $\mathcal{E}_2$ in **Stage II** ($\gamma$-demand model) |
| $\gamma \in [0, 1)$ | Product substitutability factor ($\gamma$-demand model) |
| $\alpha$ | Price sensitivity parameter ($\gamma$-demand model) |
| $g_f$ | Unit cost of flexibility |
| $c_1, c_2$ | Additional cost of using flexible resource in **Stage II** |

Table D.3: Additional Notation for Shrinking Capacity Model

| | |
|---|---|
| $\beta_1, \beta_2 \in (0, 1)$ | Decision variables denoting partial flexibility in **Stage I** |
| $g_3$ | Unit cost of flexibility |

Table D.4: Additional Notation for Additional Cost Model

| | |
|---|---|
| $f$ | Decision variable denoting degree of flexibility in **Stage I** |
| $c_1,\ c_2$ | Additional unit cost of cross-production in **Stage II** |
| $g_3$ | Unit cost of flexibility |

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We study the resource investment and pricing decisions for a profit-maximizing firm producing two substitutable products with partially flexible resources facing three types of uncertainties separately: demand uncertainties, capacity uncertainties and supply disruptions. The resources are partially flexible indicating efficiency losses when a resource designed for one type of product is used to produce another type (i.e., cross-production): The Shrinking Capacity model explicitly captures the fact that fewer units of products will be produced under cross-production. If the degree of flexibility is zero, the firm cannot cross-produce. The Additional Cost model captures the unit increase in production cost due to cross-production. Cross-production is possible even when degree of flexibility is zero but incurs a higher production cost.

We find that product substitutability, type and severity of uncertainties as well as type of efficiency loss play a key role in deciding the optimal investment strategies. When facing low or moderate demand and capacity uncertainties, flexibility is not required under both models. However, if demand or capacity uncertainties is high, a moderate degree of flexibility maybe beneficial under shrinking capacity if the products are highly differentiated and demands are negatively correlated. Shrinking capacity also suggests a moderate degree of flexibility that decreases with product substitutability under high capacity uncertainties. As the degree of supply disruptions increases flexibility is extremely valuable under any demand intercept correlation even for highly substitutable products. The additional cost model suggests investing in full flexibility only if the unit cost of dedicated capacities were higher, demands are negatively correlated and firm is forced to clear capacities. While literature has shown flexibility to be less beneficial as resource investments become less reliable, our research shows that this is not always true. It also explains why completely flexible resources are still rare in industrial practice, although it has been highly advocated in academia.

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