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# DEHN FUNCTIONS OF SUBGROUPS IN RIGHT-ANGLED ARTIN GROUPS

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DEHN FUNCTIONS OF SUBGROUPS IN
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A DISSERTATION APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

BY

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DEDICATION

to

My wife Hayat, our sons Vikenty and Nikolai, and

my parents Eduard and Yekaterina Soroko

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# Abstract

The question of what is a possible range for the Dehn functions (a.k.a. isoperimetric spectrum) for certain classes of groups is a natural and interesting one. Due to works of many authors starting with Gromov, we know a lot about the isoperimetric spectrum for the class of all finitely presented groups. Much less is known for other natural classes of groups, such as subgroups of CAT(0) groups or of right-angled Artin groups. The isoperimetric spectrum for the subgroups of right-angled Artin groups, known so far, consists of polynomials up to degree 4 and exponential functions. We extend the knowledge of this spectrum to contain the set of all positive integers. We start by constructing a series of free-by-cyclic groups whose monodromy automorphisms grow as $n^k$, which admit a virtual embedding into suitable right-angled Artin groups. As a consequence we produce examples of right-angled Artin groups containing finitely presented subgroups whose Dehn functions grow as $n^{k+2}$.

# Chapter 1

# Introduction

## 1.1 Background

Right-angled Artin groups are a remarkable class of groups which can be thought of as an interpolation between free groups and free abelian groups. That is the reason why they are also known under the name of 'partially commutative groups'. They are the groups given by a finite number of generators $x_1, \ldots, x_n$ subject only to the commutator relations between a certain subset of pairs of generators. This structure can be conveniently represented by a graph $\Gamma$ whose vertices correspond to generators $x_1, \ldots, x_n$ and whose edges $\{x_i, x_j\}$ encode the commutation relations $x_i x_j = x_j x_i$. For example, a complete graph on $n$ vertices corresponds to the free abelian group $\mathbb{Z}^n$, and an empty graph (i.e. graph on $n$ vertices with no edges) corresponds to $F_n$, the free group of rank $n$. In other words, there is no other relations between generators of right-angled Artin group, except commutators of a certain subset of all pairs of generators. Figure 1.1 gives few typical examples of right-angled Artin groups. Note that the last graph (the pentagon) gives rise to the right-angled Artin group $A(\Gamma)$ which is not isomorphic to a non-trivial direct or free product of other known groups in any obvious way.

Despite their seemingly easy presentations, right-angled Artin groups (which we will abbreviate 'RAAGs' from now on) have revealed a very rich structure. An early result of Servatius, Droms and Servatius [29] showed that the RAAG with the length $n$ cycle as its defining graph contains as a subgroup the fun-

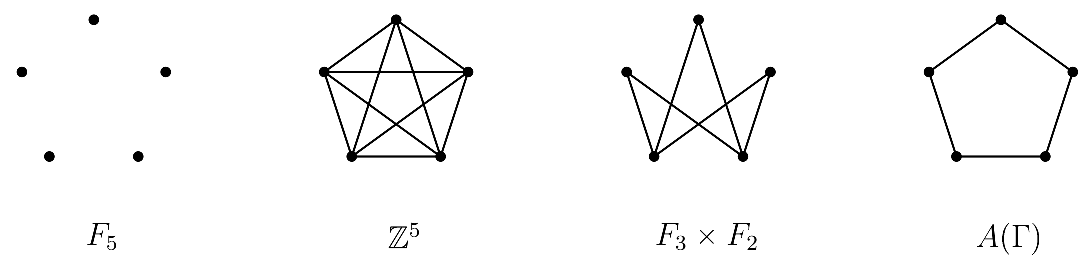


Figure 1.1: Examples of right-angled Artin groups.

damental group of an orientable surface of genus $1 + (n-4)2^{n-3}$. This came up quite unexpectedly and showed that there is a beautiful interplay between the combinatorics of commutation relations and the geometry of non-positively curved spaces.

In 1997, Bestvina and Brady [5] showed that certain RAAGs contain remarkable subgroups with properties not known before. The authors considered kernels of certain maps from RAAGs to $\mathbb{Z}$. They showed that among these kernels there are groups of types $FP_n$ but not $F_n$, groups that have infinite relation gap, and also groups that serve as potential counterexamples to the Eilenberg–Ganea conjecture.

In recent years interest in subgroups of RAAGs was reignited due to the spectacular breakthrough of Ian Agol, Dani Wise and others on the virtual fibering problem in 3–manifold topology. In [2] Agol showed that if $M^3$ is a compact, oriented, irreducible 3–manifold with $\chi(M) = 0$ and $\pi_1(M)$ is a subgroup of a RAAG, then $M$ virtually fibers. In their seminal paper [23] Haglund and Wise showed that fundamental groups of special cubical complexes are subgroups of RAAGs. Building on this machinery, Agol went on to solve the virtual fibering conjecture in [3]. The fundamental result in [3] is that non-positively curved cubical complexes with hyperbolic fundamental groups are virtually special, which implies that all these fundamental groups virtually embed into certain RAAGs.

## 1.2 Summary of Results

In this dissertation we consider two questions about subgroups of RAAGs.

The first question asks which free-by-cyclic groups virtually embed in RAAGs. In [21, 22] Hagen and Wise show that hyperbolic free-by-cyclic groups virtually embed in RAAGs. It is also known that $F_2 \rtimes \mathbb{Z}$ groups virtually embed in RAAGs. The hyperbolic free-by-cyclic examples all have exponentially growing monodromy automorphisms and the $F_2 \rtimes \mathbb{Z}$ groups have exponential or linear monodromy automorphisms. In [20], Gersten gives an explicit example of an $F_3 \rtimes \mathbb{Z}$ group which does not virtually embed in a RAAG. The group considered by Gersten is not a CAT(0) group, and this prompts the following open question. *Does every CAT(0) free-by-cyclic group virtually embed in a RAAG?* The family of the so-called Hydra groups considered in [19] provides a test case for this question where the monodromy automorphisms grow polynomially with arbitrary degree. While we haven't proved that Hydra groups virtually embed in RAAGs (this is a topic of an ongoing research of the author), we construct analogues of the Hydra groups (where the base $\mathbb{Z}^2$ subgroup is replaced by a more complicated RAAG) which are CAT(0) free-by-cyclic with polynomially growing monodromy automorphisms of arbitrary degree and which are virtually special. We prove

**Theorem A.** *For each positive* even *integer $m$ there exist virtually special free-by-cyclic groups $G_{m,m} \cong F_{2m} \rtimes_\phi \mathbb{Z}$ with the monodromy growth function $\mathrm{gr}_\phi(n) \sim n^m$ and $G_{m,m-1} \cong F_{2m-1} \rtimes_{\phi'} \mathbb{Z}$ with the monodromy growth function $\mathrm{gr}_{\phi'}(n) \sim n^{m-1}$.*

Since finite index subgroups of free-by-cyclic groups are again free-by-cyclic and since special groups embed into RAAGs we obtain the following corollary.

**Corollary A.** *For each positive integer $k$ there exist a right-angled Artin group containing a free-by-cyclic subgroup whose monodromy automorphism has growth function $\sim n^k$.*

The second question asks what kinds of functions arise as Dehn functions of finitely presented subgroups of RAAGs. Recall that Dehn functions capture the isoperimetric behavior of Cayley complexes of groups. A lot is known about Dehn functions of arbitrary finitely presented groups (see [9, 6, 28]). For example, a group is hyperbolic if and only if its Dehn function is linear. CAT(0) groups and, in particular, RAAGs, have Dehn functions which are either quadratic or linear. In [7] there are examples of CAT(0) groups which contain finitely presented subgroups whose Dehn functions are of the form $n^\alpha$ for a dense set of $\alpha \in [2, \infty)$. Restricting to the case where the ambient groups are RAAGs it gets harder to find examples of subgroups with a wide variety of Dehn functions. In [8] there are examples of finitely presented Bestvina–Brady kernels of RAAGs which have polynomial Dehn functions of degree 3 or 4. In [18] it is shown that the Dehn function of such kernels of RAAGs are at most quartic. In [12] Bridson provides an example of a RAAG containing a finitely presented group with exponential Dehn function. This is the extent of what is known about the isoperimetric behavior of subgroups of RAAGs. Our second result shows that there are finitely presented subgroups of RAAGs whose Dehn functions are polynomial of arbitrary degree.

**Theorem B.** *For each positive integer $k$ there exists a $3$–dimensional right-angled Artin group which contains a finitely presented subgroup with Dehn function $\simeq n^k$.*

An interesting open question remains: *Do there exist subgroups in RAAGs with other types of Dehn functions?*

## 1.3 Outline of the Thesis

In chapter 2 we introduce the growth functions of automorphisms and prove a folklore result (Proposition 2.4) that the growth of an automorphism of a free group is invariant under taking powers of the automorphism and under passing to a subgroup of finite index. In doing that, we rely on the Gilbert Levitt's Growth Theorem [26] (whose proof uses train-track machinery). We also provide an example due to Yves Cornulier which demonstrates that for arbitrary groups the invariance of the growth function under passing to a subgroup of finite index does not hold.

Chapter 3 is devoted to providing estimates for the Dehn function of the double of a free-by-cyclic group in terms of the growth of the monodromy automorphism. We use Bridson's lower bound from [13] (Proposition 3.3) and adapt the Bridson–Pittet's proof of the upper bound, given in [17] for the abelian-by-cyclic setting, to the case of free-by-free groups needed for our construction (Proposition 3.4). Using these estimates, we show later in chapter 7 that for the polynomially growing monodromy automorphisms involved in our construction, the upper and lower bounds on the Dehn function of the double actually coincide. If the monodromy automorphism has polynomial growth of order $n^k$, then the Dehn function of the double grows like $n^{k+2}$.

In chapter 4 we recollect all the relevant definitions related to the Morse theory on groups and special cubical complexes, which will be used in chapters 5 and 6.

In chapter 5 we introduce the free-by-cyclic groups $G_{m,k}$, which play the central role in our construction. We define these groups through LOG notation, which is a graphical tool to encode conjugation relations. We prove that the group $G_{m,k}$ is CAT(0) and free-by-cyclic, and exhibit explicit formulas for its monodromy automorphism (Proposition 5.5). We defer until chapter 8 the proof that this automorphism has growth $\sim n^k$.

The goal of chapter 6 is to exhibit a finite special cover for the presentation complex $K_{m,m}$ of the group $G_{m,m}$, for arbitrary even $m$. The construction is done in several stages. First, for arbitrary $m$, we engineer a certain right action of

$G_{m,m}$ on a set of cardinality $2^{2m+1}$, which may be thought of as the 0–skeleton of a $(2m+1)$–dimensional torus $\mathcal{T}_{2m+1}$. This action defines a finite cover $\widehat{K}_m \to K_{m,m}$, which cellularly embeds into the 2–skeleton of $\mathcal{T}_{2m+1}$ (Proposition 6.2). Since $\widehat{K}_m$ is a subcomplex of a product of graphs, it is free from three out of four hyperplane pathologies in the definition of a special cube complex (Proposition 6.4). To eliminate the fourth hyperplane pathology we observe that for even values of $m$, the complex $K_{m,m}$ is a $VH$-complex in the terminology of [23]. It follows that there exist another finite cover $\overline{K}_m \to \widehat{K}_m$, such that $\overline{K}_m$ is a special square complex (Proposition 6.6).

In chapter 7 we bring all the pieces together and prove Theorems A and B. For even values of $m$, groups $G_{m,m}$ are virtually special free-by-cyclic with the monodromy automorphism growing as $n^m$. To obtain growth functions of odd degree, we observe that the presentation 2–complex $K_{m,m-1}$ of the free-by-cyclic group $G_{m,m-1}$ is a combinatorial subcomplex of $K_{m,m-1}$ and it is obtained by deleting the 2–cells meeting the hyperplane corresponding to the last generator $a_{2m+1}$. Thus the pullback of $K_{m,m-1}$ in $\overline{K}_m$ is a finite special square complex covering $K_{m,m-1}$. This makes $G_{m,m-1}$ a virtually special free-by-cyclic group with the monodromy automorphism growing as $n^{m-1}$.

To prove theorem B, we look at the double of the special (free-by-cyclic) finite index subgroups $H$ of $G_{m,m}$ ($G_{m,m-1}$), and prove that the lower and the upper bounds for its Dehn function coincide, and are of the order $n^{m+2}$ (resp., $n^{m+1}$). This double naturally embeds into a RAAG, whose underlying graph is the join of the underlying graph for the RAAG containing $H$ and the empty graph on two vertices.

In chapter 8 we provide the computation of the growth function for the monodromy automorphism of $G_{m,k}$ and its abelianization. And finally, in chapter 9 we list open questions related to the study in this dissertation.

# Chapter 2

# Preliminaries on Growth

In what follows we will consider functions up to the following equivalence relations.

**Definition 2.1.** Two functions $f, g\colon [0,\infty) \to [0,\infty)$ are said to be $\sim$ equivalent if $f \preceq g$ and $g \preceq f$, where $f \preceq g$ means that there exist constants $A > 0$ and $B \geqslant 0$ such that $f(n) \leqslant Ag(n) + B$ for all $n \geqslant 0$.

**Definition 2.2.** Two functions $f, g\colon [0,\infty) \to [0,\infty)$ are said to be $\simeq$ equivalent if $f \leqq g$ and $g \leqq f$, where $f \leqq g$ means that there exist constants $A, B > 0$ and $C, D, E \geqslant 0$ such that $f(n) \leqslant Ag(Bn + C) + Dn + E$ for all $n \geqslant 0$.

We extend these equivalence relations to functions $\mathbb{N} \to [0,\infty)$ by assuming them to be constant on each interval $[n, n+1)$.

**Remark 2.3.** Notice that $f \preceq g$ implies $f \leqq g$ and $f \sim g$ implies $f \simeq g$. However, the relation $\sim$ is strictly finer than $\simeq$, as the latter identifies all single exponential functions, i.e. $k^n \simeq K^n$ for $k, K > 1$, whereas the relation $\sim$ does not. We will use the relations $\sim$, $\preceq$ when dealing with growth functions of automorphisms and the relations $\simeq$, $\leqq$ when discussing Dehn functions of groups (as is done traditionally). The term $Dn$ in the above definition of $\leqq$ is essential for proving the equivalence of Dehn functions under quasi-isometries.

Let $F$ be a free group of finite rank $k$ with a finite generating set $\mathcal{A}$. Let $d_{\mathcal{A}}(x, y)$ be the associated word metric on $F$. If $\psi\colon G \to G$ is an automorphism, we define

$$\mathrm{gr}_{\psi,\mathcal{A}}(n) := \max_{a\in\mathcal{A}} \|\psi^n(a)\|_{\mathcal{A}},$$

where $\|g\|_{\mathcal{A}}$ is equal to $d_{\mathcal{A}}(1,g)$ for $g \in F$.

The following properties of $\mathrm{gr}_{\psi,\mathcal{A}}$ will be used in the sequel.

**Proposition 2.4.** *Let $F$ be a free group of finite rank with a finite generating set $\mathcal{A}$, and let $\psi$ be an automorphism of $F$. Then*

*(i) for each finite generating set $\mathcal{B}$ of $F$, $\mathrm{gr}_{\psi,\mathcal{B}} \sim \mathrm{gr}_{\psi,\mathcal{A}}$;*

*(ii) for each $d \in \mathbb{N}$, $\mathrm{gr}_{\psi,\mathcal{A}} \sim \mathrm{gr}_{\psi^d,\mathcal{A}}$;*

*(iii) for each finite index subgroup $H \leqslant F$ invariant under $\psi$ with a finite generating set $\mathcal{B} \subset H$, we have $\mathrm{gr}_{\psi,\mathcal{A}} \sim \mathrm{gr}_{\psi|_H,\mathcal{B}}$.*

*Proof.* Let $\mathcal{A} = \{a_1,\dots,a_N\}$, $\mathcal{B} = \{b_1,\dots,b_M\}$. Since both sets $\mathcal{A}$ and $\mathcal{B}$ generate $F$, there exist words $w_i$ and $v_j$ such that $a_i = w_i(b_1,\dots,b_M)$ and $b_j = v_j(a_1,\dots,a_N)$, for all $1 \leqslant i \leqslant N$, $1 \leqslant j \leqslant M$. Let constants $K$ and $L$ denote the maximal lengths of $w_i$, $v_j$, respectively, i.e. $K = \max_{1\leqslant i\leqslant N} \|w_i\|_{\mathcal{B}}$, $L = \max_{1\leqslant j\leqslant M} \|v_j\|_{\mathcal{A}}$. Then, obviously, for all $i,j,n$, one has:

$$\|\psi^n(a_i)\|_{\mathcal{B}} \leqslant K \cdot \|\psi^n(a_i)\|_{\mathcal{A}} \qquad \text{and} \qquad \|\psi^n(b_j)\|_{\mathcal{A}} \leqslant L \cdot \|\psi^n(b_i)\|_{\mathcal{B}}.$$

Now fix arbitrary $1 \leqslant j \leqslant M$ and assume without loss of generality that $v_j(a_1,\dots,a_N) = a_{i_1}^{\varepsilon_1}\dots a_{i_L}^{\varepsilon_L}$, for values $1 \leqslant i_\ell \leqslant N$ and $\varepsilon_\ell = \pm 1$ or $0$. Then

$$\|\psi^n(b_j)\|_{\mathcal{B}} = \|\psi^n(a_{i_1}^{\varepsilon_1}\dots a_{i_L}^{\varepsilon_L})\|_{\mathcal{B}} \leqslant \|\psi^n(a_{i_1})\|_{\mathcal{B}} + \dots + \|\psi^n(a_{i_L})\|_{\mathcal{B}} \leqslant$$
$$K\|\psi^n(a_{i_1})\|_{\mathcal{A}} + \dots + K\|\psi^n(a_{i_L})\|_{\mathcal{A}} \leqslant KL \max_{1\leqslant i\leqslant N} \|\psi^n(a_i)\|_{\mathcal{A}} = KL\,\mathrm{gr}_{\psi,\mathcal{A}}(n).$$

Therefore, $\mathrm{gr}_{\psi,\mathcal{B}}(n) = \max_{1\leqslant j\leqslant M} \|\psi^n(b_j)\|_{\mathcal{B}} \leqslant KL\,\mathrm{gr}_{\psi,\mathcal{A}}(n)$, and, by symmetry, $\mathrm{gr}_{\psi,\mathcal{A}}(n) \leqslant LK\,\mathrm{gr}_{\psi,\mathcal{B}}(n)$. This proves (i).

Before proving parts (ii) and (iii), we state following remarkable result of Gilbert Levitt:

**Levitt's Growth Theorem** ([26, Cor. 6.3])**.** *Let $F$ be a free group of finite rank with a free generating set $\mathcal{A}$. Given $\alpha \in \mathrm{Aut}(F)$ and $g \in F$, there exist $\lambda \geqslant 1$, an integer $m \geqslant 0$ and constants $A$, $B > 0$ such that the word length $\|\alpha^n(g)\|_{\mathcal{A}}$ satisfies:*

$$\forall n \in \mathbb{N}, \qquad A\lambda^n n^m \leqslant \|\alpha^n(g)\|_{\mathcal{A}} \leqslant B\lambda^n n^m. \qquad \square$$

Consider a sequence of growth parameters $(\lambda_i, m_i)$ from the Levitt's Growth Theorem corresponding to the generators $a_1, \ldots, a_N$, so that for each $1 \leqslant i \leqslant N$ there exist constants $A_i, B_i > 0$ such that

$$A_i\lambda_i^n n^{m_i} \leqslant \|\psi^n(a_i)\|_{\mathcal{A}} \leqslant B_i\lambda_i^n n^{m_i} \qquad \text{for all } n \in \mathbb{N}.$$

Order these parameters lexicographically: $(\lambda_i, m_i) < (\lambda_j, m_j)$ if and only if $\lambda_i < \lambda_j$ or $\lambda_i = \lambda_j$ and $m_i < m_j$. Clearly, $(\lambda_i, m_i) < (\lambda_j, m_j)$ if and only if $\lambda_j^n n^{m_j}/\lambda_i^n n^{m_i} \to \infty$ as $n \to \infty$. Pick $1 \leqslant i_0 \leqslant N$ such that $(\lambda_{i_0}, m_{i_0})$ is maximal with respect to this order. Then for any constants $C_1, C_2 > 0$ and arbitrary $1 \leqslant i \leqslant N$ we have:

$$C_1\lambda_i^n n^{m_i} \lll C_2\lambda_{i_0}^n n^{m_{i_0}},$$

which means that the left-hand side is less than or equal to the right-hand side for all large enough $n \in \mathbb{N}$.

To prove (ii) in one direction, notice that for any $1 \leqslant i \leqslant N$,

$$\|(\psi^d)^n(a_i)\|_{\mathcal{A}} \leqslant B_i\lambda_i^{dn}(dn)^{m_i} = (B_i\lambda_i^d d^{m_i})\lambda_i^n n^{m_i} \lll \tfrac{B_i\lambda_i^d d^{m_i}}{A_{i_0}}\big(A_{i_0}\lambda_{i_0}^n n^{m_{i_0}}\big) \leqslant \tfrac{B_i\lambda_i^d d^{m_i}}{A_{i_0}}\|\psi^n(a_{i_0})\|_{\mathcal{A}}.$$

Hence, there exist a constant $C_{big} \geqslant 0$ such that

$$\mathrm{gr}_{\psi^d,\mathcal{A}}(n) = \max_{1\leqslant i\leqslant N} \|(\psi^d)^n(a_i)\|_{\mathcal{A}} \leqslant D\,\mathrm{gr}_{\psi,\mathcal{A}}(n) + C_{big},$$

where $D = \max_{1\leqslant i\leqslant N} B_i\lambda_i^d d^{m_i}/A_{i_0}$. Thus, $\mathrm{gr}_{\psi^d,\mathcal{A}} \preceq \mathrm{gr}_{\psi,\mathcal{A}}$.

In the opposite direction, for any $1 \leqslant i \leqslant N$ we have:

$$\|\psi^n(a_i)\|_{\mathcal{A}} \leqslant B_i\lambda_i^n n^{m_i} \leqslant B_i\lambda_i^{dn}(dn)^{m_i} \ll \tfrac{B_i}{A_{i_0}}\big(A_{i_0}\lambda_{i_0}^{dn}(dn)^{m_{i_0}}\big) \leqslant \tfrac{B_i}{A_{i_0}}\|\psi^{dn}(a_{i_0})\|_{\mathcal{A}}.$$

By taking maximum, we get for arbitrary $n \in \mathbb{N}$:

$$\mathrm{gr}_{\psi,\mathcal{A}}(n) = \max_{1\leqslant i\leqslant N} \|\psi^n(a_i)\|_{\mathcal{A}} \leqslant D\|\psi^{dn}(a_{i_0})\|_{\mathcal{A}} + C_{big} \leqslant D\,\mathrm{gr}_{\psi^d,\mathcal{A}}(n) + C_{big}$$

for $D = \max_i B_i/A_{i_0}$ and some $C_{big} \geqslant 0$. This proves that $\mathrm{gr}_{\psi,\mathcal{A}} \preceq \mathrm{gr}_{\psi^d,\mathcal{A}}$ and hence that $\mathrm{gr}_{\psi,\mathcal{A}} \sim \mathrm{gr}_{\psi^d,\mathcal{A}}$.

To prove (iii) in one direction, notice first that $H$, being of finite index in $F$, is quasi-convex in $F$ (see e.g. [16, III.3.5]). Hence for any $h \in H$, one has $\|h\|_{\mathcal{B}} \leqslant C\|h\|_{\mathcal{A}}$ for some $C > 0$. Writing each $b_j \in \mathcal{B}$ as a word $b_j = v_j(a_1, \ldots, a_N)$ and setting $L = \max_{1\leqslant j\leqslant M} \|v_j\|_{\mathcal{A}}$, we obtain for arbitrary $1 \leqslant j \leqslant M$:

$$\|\psi^n(b_j)\|_{\mathcal{B}} \leqslant C\|\psi^n(b_j)\|_{\mathcal{A}} \leqslant CL \max_{1\leqslant i\leqslant N} \|\psi^n(a_i)\|_{\mathcal{A}} = CL\,\mathrm{gr}_{\psi,\mathcal{A}}(n),$$

so that $\mathrm{gr}_{\psi|_H,\mathcal{B}}(n) = \max_{1\leqslant j\leqslant M} \|\psi^n(b_j)\|_{\mathcal{B}} \leqslant CL\,\mathrm{gr}_{\psi,\mathcal{A}}(n)$, i.e. $\mathrm{gr}_{\psi|_H,\mathcal{B}} \preceq \mathrm{gr}_{\psi,\mathcal{A}}$.

In the opposite direction, notice that there exist an integer $p > 0$ such that for every $a_i \in \mathcal{A}$, we have $a_i^p \in H$. As above, let $(\lambda_i, m_i)$, $A_i, B_i > 0$ be a sequence of growth parameters for the generators $a_1, \ldots, a_N$, and let $(\lambda_{i_0}, m_{i_0})$ be maximal. Consider a new generating set $\mathcal{B}'$ for $H$, $\mathcal{B}' = \mathcal{B} \cup \{a_{i_0}^p\}$. Then for arbitrary

$1 \leqslant i \leqslant N$ we have:

$$\|\psi^n(a_i)\|_{\mathcal{A}} \leqslant B_i \lambda_i^n n^{m_i} \ll \tfrac{B_i}{A_{i_0}} \big(A_{i_0} \lambda_{i_0}^n n^{m_{i_0}}\big) \leqslant \tfrac{B_i}{A_{i_0}} \|\psi^n(a_{i_0})\|_{\mathcal{A}} \leqslant \\ \tfrac{B_i}{A_{i_0}} \|\psi^n(a_{i_0}^p)\|_{\mathcal{A}} \leqslant \tfrac{B_i}{A_{i_0}} L \|\psi^n(a_{i_0}^p)\|_{\mathcal{B}'},$$

where $L$ has a similar meaning as above. Here the fourth inequality holds since, in general, for any automorphism $\alpha \in \mathrm{Aut}(F)$, any $g \in F$ and any $p > 0$, one has $\|\alpha(g)\|_{\mathcal{A}} \leqslant \|\alpha(g^p)\|_{\mathcal{A}}$. (Indeed, one can write $\alpha(g) = uvu^{-1}$ with $v$ cyclically reduced. Then $\|\alpha(g)\| = 2\|u\| + \|v\|$, whereas $\|\alpha(g^p)\| = \|uv^pu^{-1}\| = 2\|u\| + p\|v\|$.)

By taking maximum, we get for arbitrary $n \in \mathbb{N}$:

$$\mathrm{gr}_{\psi,\mathcal{A}}(n) = \max_{1 \leqslant i \leqslant N} \|\psi^n(a_i)\|_{\mathcal{A}} \leqslant DL\|\psi^n(a_{i_0}^p)\|_{\mathcal{B}'} + C_{big} \leqslant DL\,\mathrm{gr}_{\psi|_H,\mathcal{B}'}(n) + C_{big}$$

for $D = \max_i B_i/A_{i_0}$ and some $C_{big} \geqslant 0$. This means that $\mathrm{gr}_{\psi,\mathcal{A}} \preceq \mathrm{gr}_{\psi|_H,\mathcal{B}'}$. Since, by part (i), $\mathrm{gr}_{\psi|_H,\mathcal{B}'} \sim \mathrm{gr}_{\psi|_H,\mathcal{B}}$, this proves that $\mathrm{gr}_{\psi,\mathcal{A}} \preceq \mathrm{gr}_{\psi|_H,\mathcal{B}}$ and part (iii) is proved. $\square$

**Remark 2.5.** The proof of property (i) of Proposition 2.4 works for automorphisms of arbitrary finitely generated groups.

**Remark 2.6.** Property (iii) of Proposition 2.4 does not hold for arbitrary finitely generated groups. We are grateful to Yves Cornulier for providing the following example. Let $G = \langle a, b \mid aba^{-1} = b^{-1}, a^2 = 1\rangle$ be the infinite dihedral group. Then the inner automorphism $i_b\colon x \mapsto bxb^{-1}$ has linear growth, but its restriction to the index 2 subgroup $\langle b\rangle$ is trivial. To see that, observe that $aba^{-1} = b^{-1}$ implies $ab = b^{-1}a$ and hence $ba = ab^{-1}$. Thus $bab^{-1} = ab^{-2}$ and $i_b^n(a) = b^nab^{-n} = ab^{-2n}$. Looking at the Cayley graph of $G$ shows that the element $g = ab^{-2n}$ is at distance $2n + 1$ from 1, so that $ab^{-2n}$ is a word of minimal length representing element $g$, and $i_b$ indeed grows linearly on $G$.

In view of item (i) in Proposition 2.4, we will suppress the dependence on the generating set and adopt the notation

$$\mathrm{gr}_{\psi}(n) := \mathrm{gr}_{\psi,\mathcal{A}}(n),$$

for an arbitrary generating set $\mathcal{A} \subset F$.

For the abelianization $F_{ab} = F/[F,F] \cong \mathbb{Z}^k$ we consider the induced automorphism $\psi^{ab}\colon F_{ab} \to F_{ab}$ and denote $\{\bar{e}_i\}$ be the generating set of $F_{ab}$ corresponding to $\mathcal{A}$: $\bar{e}_i = a_i[F,F]$, $a_i \in \mathcal{A}$, $i = 1, \dots, k$. For any $v \in \mathbb{Z}^k$ let $|v|_1$ denote the $\ell_1$-norm on $\mathbb{Z}^k$ viewed as a subset of $\mathbb{C}^k$: if $v = \sum_{i=1}^k c_i \bar{e}_i$, then $|v|_1 = \sum_{i=1}^k |c_i|$. Define

$$\mathrm{gr}_{\psi^{ab}}(n) := \max_{i=1,\dots,k} |(\psi^{ab})^n(\bar{e}_i)|_1.$$

Then the following is true:

**Lemma 2.7.**

$$\mathrm{gr}_{\psi}(n) \geqslant \mathrm{gr}_{\psi^{ab}}(n).$$

*Proof.* Let $\varepsilon\colon F \to F_{ab}$ be the natural homomorphism. Then

$$(\psi^{ab})^n(\bar{e}_i) = \varepsilon(\psi^n(a_i))$$

and hence the length of the shortest word in generators $\{\bar{e}_i\}_{i=1}^k$ of the element $(\psi^{ab})^n(\bar{e}_i) \in \mathbb{Z}^k$ is no bigger than $\|\psi^n(a_i)\|_{\mathcal{A}}$. But the former is equal to $|(\psi^{ab})^n(\bar{e}_i)|_1$ hence $|(\psi^{ab})^n(\bar{e}_i)|_1 \leqslant \|\psi^n(a_i)\|_{\mathcal{A}}$ for all $i = 1, \dots, k$. By taking maximum, we get the required inequality. $\square$

By embedding $\mathbb{Z}^k$ into $\mathbb{C}^k$ we may consider $\mathbb{C}^k$ as a vector space with the basis $\{\bar{e}_i\}_{i=1}^k$. Now let $A$ be a linear operator on $\mathbb{C}^k$ given in basis $\{\bar{e}_i\}_{i=1}^k$ by the matrix $(a_{ij})_{i,j=1}^k$, and let $|v|_\infty$ denote the $\ell_\infty$-norm on $\mathbb{C}^k$: if $v = \sum_{i=1}^k c_i \bar{e}_i$, then

$|v|_\infty = \max_{i=1,\dots,k} |c_i|$. Consider two norms on $\mathrm{End}(\mathbb{C}^k)$, one is the operator norm with respect to $\ell_\infty$:

$$\|A\|_{op} = \sup_{v\neq 0} \frac{|Av|_\infty}{|v|_\infty} = \sup_{|v|_\infty=1} |Av|_\infty,$$

and another one is the supremum norm, which is the $\ell_\infty$-norm on the space $\mathbb{C}^{k^2}$:

$$\|A\|_{sup} = \max_{i,j=1,\dots,k} |a_{ij}|.$$

**Lemma 2.8.**

$$\max_{i=1,\dots,k} |A\bar{e}_i|_1 \geqslant \|A\|_{sup}.$$

*Proof.*

$$\max_i |A\bar{e}_i|_1 \geqslant \max_i |A\bar{e}_i|_\infty = \max_{i,j} |a_{ij}| = \|A\|_{sup}.$$ □

The following fact is well-known (see [24, Cor. 5.4.5]):

**Lemma 2.9.** *There exist constants $C_1$, $C_2 > 0$ such that*

$$C_1\|A\|_{op} \leqslant \|A\|_{sup} \leqslant C_2\|A\|_{op}.$$ □

**Corollary 2.10.** *The growth function*

$$\mathrm{gr}_A^{sup} \colon n \longmapsto \|A^n\|_{sup}$$

*is $\sim$ equivalent to the growth function*

$$\mathrm{gr}_A^{op} \colon n \longmapsto \|A^n\|_{op}.$$ □

The following results are proved in [14, Proof of Th. 2.1]:

**Lemma 2.11.** *The $\sim$ equivalence class of the function $\mathrm{gr}_A^{op}$ depends only on the conjugacy class of $A$ in $GL(k, \mathbb{C})$.* $\square$

In view of Corollary 2.10 and Lemma 2.11, we need only to consider the growth of the Jordan normal forms of matrices $A$.

**Lemma 2.12** ([14, Th. 2.1])**.** *Suppose that $J$ is a matrix in the Jordan normal form with all eigenvalues equal to 1. Then $\mathrm{gr}_J^{sup}(n) \sim n^{c-1}$, where c is the maximal size of Jordan blocks of $J$.* $\square$

Combining all of the above, we get:

**Corollary 2.13.** *Let $\psi$ be an automorphism of a free group $F$. If the abelianization $\psi^{ab}$ has all eigenvalues equal to 1, and $c$ is the size of the largest Jordan block in the Jordan normal form $J$ for $\psi^{ab}$, then*

$$\mathrm{gr}_\psi(n) \succeq n^{c-1} \quad \textit{and} \quad \mathrm{gr}_{\psi^{ab}}(n) \succeq n^{c-1}.$$

*Proof.* Indeed,

$$\begin{aligned}
\mathrm{gr}_\psi(n) &\geqslant \mathrm{gr}_{\psi^{ab}}(n) && \text{(by Lemma 2.7)} \\
&\geqslant \|(\psi^{ab})^n\|_{sup} && \text{(by Lemma 2.8)} \\
&\geqslant C_1\|(\psi^{ab})^n\|_{op} && \text{(for some } C_1 > 0\text{, by Lemma 2.9)} \\
&\sim \|J^n\|_{op} && \text{(by Lemma 2.11)} \\
&\sim n^{c-1} && \text{(by Lemma 2.12).}
\end{aligned}$$

$\square$

# Chapter 3

# Dehn Function of the Double

In this chapter we outline what is known about the upper and the lower bounds for the Dehn function of the double of a free-by-cyclic group, in terms of the growth of its monodromy automorphism.

## 3.1 Relevant Definitions

**Definition 3.1.** (The double) Let $G$ be a free-by-cyclic group $G = F \rtimes_\psi \mathbb{Z}$. The *double* of $G$ is the group $\Gamma(G) = G \ast_F G$.

If $G \cong \langle \mathcal{A}, t \mid tat^{-1} = \psi(a) \text{ for all } a \in \mathcal{A}\rangle$ then $\Gamma(G) \cong \langle \mathcal{A}, s, t \mid sas^{-1} = \psi(a),\ tat^{-1} = \psi(a) \text{ for all } a \in \mathcal{A}\rangle$. If one denotes $F(s,t)$ the free group on the generating set $\{s,t\}$ and $(\psi)\colon F(s,t) \to \mathrm{Aut}(F(\mathcal{A}))$ the homomorphism given on the generators by $s \mapsto \psi$, $t \mapsto \psi$, then $\Gamma(G) \cong F(\mathcal{A}) \rtimes_{(\psi)} F(s,t)$.

**Definition 3.2.** (Dehn function) Let a group $\Gamma$ be given by a finite presentation $P = \langle \mathcal{A} \mid R\rangle$. For each word $w$ lying in the normal closure of $R$ in the free group $F(\mathcal{A})$, define

$$\mathrm{Area}(w) := \min\Big\{N \mid w \underset{F(\mathcal{A})}{=} \prod_{i=1}^{N} x_i^{-1} r_i x_i \text{ with } x_i \in F(\mathcal{A}), r_i \in R^{\pm}\Big\}.$$

The *Dehn function* of $P$ is the function $\delta_P \colon \mathbb{N} \to \mathbb{N}$ defined by

$$\delta_P(n) := \max\{\mathrm{Area}(w) \mid w \underset{\Gamma}{=} 1, \|w\|_{\mathcal{A}} \leqslant n\}.$$

where $\|w\|_{\mathcal{A}}$ denotes the length of the word $w$ in generators $\mathcal{A}^{\pm}$.

Viewed up to $\simeq$ equivalence, the Dehn functions are independent of the choice of the presentation (see [11, 1.3.3]), so we denote $\delta_P(n)$ as $\delta_\Gamma(n)$.

## 3.2 Bounding the Dehn Function of the Double

The lower bound for the double of a free-by-cyclic group was established in [13, Lemma 1.5] (see Proposition 3.3 below). The argument for the upper bound (see Proposition 3.4 below) follows the outline of [17, Theorem 5.1]. In the latter paper the argument is given in the setting of abelian-by-cyclic groups; we adapt this reasoning to the free-by-free setting.

**Proposition 3.3** ([13, Lemma 1.5] and [11, Proposition 7.2.2])**.** *Let $\psi$ be an automorphism of $F$ and $\|.\|$ denote the word length with respect to a fixed generating set of $F$. Then for the Dehn function $\delta_\Gamma(n)$ of the double $\Gamma$ of $F \rtimes_\psi \mathbb{Z}$ one has*

$$n \cdot \max_{\substack{\|b\| \leqslant n \\ b \in F}} \|\psi^n(b)\| \leqq \delta_\Gamma(n). \qquad \square$$

**Proposition 3.4.** *Let $\psi$ be an automorphism of a free group $F$ and assume that* $\mathrm{gr}_\psi(n) \preceq n^d$ *and* $\mathrm{gr}_{\psi^{-1}}(n) \preceq n^d$. *Then for the Dehn function $\delta_\Gamma(n)$ of the double* $\Gamma = \Gamma(F \rtimes_\psi \mathbb{Z})$ *one has* $\delta_\Gamma(n) \leqq n^{d+2}$.

Later in chapter 7 we will use Proposition 3.3 to show that a particular double has Dehn function growing at least as $n^{d+2}$. Together with Proposition 3.4 this will imply that the Dehn function grows as $\simeq n^{d+2}$.

**Remark 3.5.** It can be proved using train-tracks that $\mathrm{gr}_{\psi^{-1}}(n) \sim n^d$ if and only if $\mathrm{gr}_\psi(n) \sim n^d$ (see e.g. [27, Th. 0.4]). However, for the reader who is unfamiliar with the train-track machinery we make the exposition independent of this result.

Instead, in what follows we will apply Proposition 3.4 to the automorphisms $\phi$ whose growth functions $\mathrm{gr}_\phi(n)$ and $\mathrm{gr}_{\phi^{-1}}(n)$ are computed in chapter 8 and are shown to be $\sim$ equivalent to each other.

### 3.2.1 Proof of the Upper Bound

In order to prove Proposition 3.4, we need some preliminary results on combings of groups. We start with some definitions from [10] and [17].

Let $\Gamma$ be a group with finite generating set $\mathcal{A}$ and $d_{\mathcal{A}}(x, y)$ be the associated word metric.

**Definition 3.6.** A *combing* (normal form) for $\Gamma$ is a set of words $\{\sigma_g \mid g \in \Gamma\}$ in the letters $\mathcal{A}^{\pm}$ such that $\sigma_g = g$ in $\Gamma$. We denote by $|\sigma_g|$ or $|\sigma_g|_{\mathcal{A}}$ the length of the word $\sigma_g$ in the free monoid on $\mathcal{A}^{\pm}$.

**Definition 3.7.** Let

$$\mathcal{R} = \{\rho\colon \mathbb{N} \to \mathbb{N} \mid \rho(0) = 0;\ \rho(n+1) \in \{\rho(n), \rho(n)+1\}\ \forall n;\ \rho \text{ unbounded}\}.$$

Given eventually constant paths $p_1, p_2\colon \mathbb{N} \to (\Gamma, d)$ we define

$$D(p_1, p_2) = \min_{\rho, \rho' \in \mathcal{R}} \big\{ \max_{t \in \mathbb{N}} \{d_{\mathcal{A}}(p_1(\rho(t)), p_2(\rho'(t))\} \big\}.$$

**Definition 3.8.** Given a combing $\sigma$ for $\Gamma$, the *asynchronous width* of $\sigma$ is the function $\Phi_\sigma\colon \mathbb{N} \to \mathbb{N}$ defined by

$$\Phi_\sigma(n) = \max\big\{ D(\sigma_g, \sigma_h) \mid d_{\mathcal{A}}(1, g), d_{\mathcal{A}}(1, h) \leqslant n;\ d_{\mathcal{A}}(g, h) = 1 \big\}.$$

**Definition 3.9.** A finitely generated group $\Gamma$ is said to be *asynchronously combable* if there exists a combing $\sigma$ for $\Gamma$ and a constant $K > 0$ such that $\Phi_\sigma(n) \leqslant K$

for all $n \in \mathbb{N}$.

**Definition 3.10.** The *length* of a combing $\sigma$ for $\Gamma$ is the function $L\colon \mathbb{N} \to \mathbb{N}$ given by:

$$L(n) = \max \{ |\sigma_g| \mid d_{\mathcal{A}}(1, g) \leqslant n \}.$$

The relation of combings to Dehn functions is manifested in the following result:

**Proposition 3.11** ([17, Lemma 4.1]). *Let $\Gamma$ be a group with a finite set of semi-group generators $\mathcal{A}^{\pm}$. If there exists a combing $\sigma$ for $\Gamma$ whose asynchronous width is bounded by a constant and whose length is bounded by the function $L(n)$, then the Dehn function $\delta_\Gamma(n)$ for any presentation of $\Gamma$ satisfies $\delta_\Gamma(n) \leqq nL(n)$.* □

In connection to the groups which are doubles, the following result from [10] is useful.

**Theorem 3.12** ([10, Theorem B]). *If $G$ is word-hyperbolic and $H$ is asynchronously combable then every split extension*

$$1 \longrightarrow G \longrightarrow G \rtimes H \longrightarrow H \longrightarrow 1$$

*of $G$ by $H$ is asynchronously combable.* □

**Remark 3.13.** From the proof of this result in [10] it follows that if groups $G$ and $H$ have combings $\sigma^G$ and $\sigma^H$ whose asynchronous width is bounded by some constants, then the combing for the split extension $G \rtimes H$ of $G$ by $H$, whose length is bounded by a constant, can be taken as the product (concatenation) $\sigma^H\sigma^G$ of combings $\sigma^H$ and $\sigma^G$, meaning that we traverse path $\sigma^H$ first, then path $\sigma^G$. Note that the product of combings in the opposite order, $\sigma^G\sigma^H$, may not have bounded asynchronous width, as the example of Baumslag–Solitar groups shows.

Now let again $\Gamma = F \rtimes_{(\psi)} F(s,t)$ be the double of $G = F \rtimes_{\psi} \mathbb{Z}$, where $F$ is a free group on the set of free generators $\mathcal{A}$.

Our goal is to obtain an upper bound on the length $L(n)$ of the combing $\sigma^{F(s,t)} \cdot \sigma^{F(\mathcal{A})}$ in terms of the growth of the automorphism $\psi$. (Here we treat a combing on a free group as a unique reduced word in a fixed system of generators which represents the given element of the group.) We prove the following proposition, adapting the reasoning for the abelian-by-cyclic groups from [17, Theorem 5.1] to the case of free-by-free groups.

**Proposition 3.14.** *Let $P(n)$ be an increasing function bounding the growth of both $\psi$ and $\psi^{-1}$, i.e. $d_{\mathcal{A}}(1, \psi^n(a)) \leqslant P(|n|)$ for all $a \in \mathcal{A}$, $n \in \mathbb{Z}$. Then the length $L(n)$ of the combing $\sigma^{F(s,t)} \cdot \sigma^{F(\mathcal{A})}$ of the group $\Gamma = F(\mathcal{A}) \rtimes_{(\psi)} F(s,t)$ satisfies*

$$L(n) \leqslant nP(n) + n.$$

*Proof.* Take arbitrary $\gamma \in \Gamma$ and write it as $\gamma = u \cdot g$, where $u \in F(s,t)$, $g \in F(\mathcal{A})$. Let $n_0 = d_{\mathcal{A} \cup \{s,t\}}(1, \gamma)$ be the length of the shortest word in generators $(\mathcal{A} \cup \{s,t\})^{\pm}$ representing element $\gamma$ in $\Gamma$. We would like to show that

$$|\sigma_\gamma| = |\sigma_u \cdot \sigma_g| = d_{\{s,t\}}(1, u) + d_{\mathcal{A}}(1, g) \leqslant n_0 P(n_0) + n_0.$$

Considering the natural homomorphism $\eta\colon F(\mathcal{A}) \rtimes_{(\psi)} F(s,t) \to F(s,t)$, one observes that $u = \eta(\gamma)$ and hence $d_{\{s,t\}}(1, u) \leqslant n_0$. Therefore it suffices to show that

$$d_{\mathcal{A}}(1, g) \leqslant n_0 P(n_0).$$

Denote $w_0$ the shortest word in generators $(\mathcal{A} \cup \{s,t\})^{\pm}$ such that $w_0 = \gamma$ in

$\Gamma$, so that $|w_0|_{\mathcal{A}\cup\{s,t\}} = n_0$. Then $w_0$ can be written as

$$w_0 = u_1 w_1 u_2 w_2 \cdot \dots \cdot u_r w_r,$$

where $u_i \in F(s,t)$, $w_i \in F(\mathcal{A})$ for all $i$. Then

$$n_0 = |w_0|_{\mathcal{A}\cup\{s,t\}} = \sum_{i=1}^{r} |u_i|_{\{s,t\}} + \sum_{i=1}^{r} |w_i|_{\mathcal{A}} \tag{*}$$

and $u = u_1 \dots u_r$.

Denote

$$v_i = \prod_{j=1}^{i} u_j \cdot w_i \cdot \Big(\prod_{j=1}^{i} u_j\Big)^{-1}, \qquad i = 1, \dots, r.$$

Then, as one easily checks,

$$v_1 v_2 \dots v_r = u_1 w_1 u_2 w_2 \dots u_r w_r \cdot \Big(\prod_{j=1}^{i} u_j\Big)^{-1}$$

so that

$$\gamma = w_0 = v_1 v_2 \dots v_r \cdot \Big(\prod_{j=1}^{i} u_j\Big).$$

Hence

$$g = u^{-1}\gamma = \Big(\prod_{j=1}^{r} u_j\Big)^{-1} \cdot v_1 v_2 \dots v_r \cdot \Big(\prod_{j=1}^{r} u_j\Big) =$$
$$\prod_{i=1}^{r} \Big[\Big(\prod_{j=1}^{r} u_j\Big)^{-1} \cdot \prod_{j=1}^{i} u_j \cdot w_i \cdot \Big(\prod_{j=1}^{i} u_j\Big)^{-1} \cdot \prod_{j=1}^{r} u_j\Big] =$$
$$\prod_{i=1}^{r} \big(u_r^{-1} u_{r-1}^{-1} \dots u_{i+1}^{-1}\big) \cdot w_i \cdot \big(u_{i+1} \dots u_r\big).$$

If we denote by $\varepsilon\colon F(s,t) \to \mathbb{Z}$ the homomorphism defined on the generators as: $s \mapsto 1$, $t \mapsto 1$, then for any $g \in F(\mathcal{A})$ and any $u \in F(s,t)$ we have $ugu^{-1} =$

$\psi^{\varepsilon(u)}(g)$. Therefore,

$$g = \prod_{i=1}^{r} \psi^{-\varepsilon(u_{i+1}\dots u_r)}(w_i) = \prod_{i=1}^{r} \psi^{-\sum_{j=i+1}^{r}\varepsilon(u_j)}(w_i).$$

On the other hand,

$$\Big|\sum_{j=i+1}^{r} \varepsilon(u_j)\Big| \leqslant \sum_{j=i+1}^{r} |\varepsilon(u_j)| \leqslant \sum_{j=1}^{r} |\varepsilon(u_j)| \leqslant \sum_{j=1}^{r} |u_j|_{\{s,t\}} = |u|_{\{s,t\}} \leqslant n_0 \qquad (**)$$

by the observation above.

Moreover, since for any $a \in \mathcal{A}$ we have $d_{\mathcal{A}}(1, \psi^n(a)) \leqslant P(|n|)$, then for any $w_i \in F(\mathcal{A})$ we get

$$d_{\mathcal{A}}\big(1, \psi^n(w_i)\big) \leqslant P(|n|) \cdot d_{\mathcal{A}}(1, w_i).$$

Finally, we get for the element $g$ the estimate:

$$d_{\mathcal{A}}\big(1, g\big) \leqslant \sum_{i=1}^{r} d_{\mathcal{A}}\big(1, \psi^{-\sum_{j=i+1}^{r}\varepsilon(u_j)}(w_i)\big) \leqslant \sum_{i=1}^{r} P\Big(\Big|\sum_{j=i+1}^{r} \varepsilon(u_j)\Big|\Big) \cdot d_{\mathcal{A}}(1, w_i)$$
$$\leqslant [\text{by } (**)] \leqslant \sum_{i=1}^{r} P(n_0) \cdot d_{\mathcal{A}}(1, w_i) = P(n_0) \cdot \sum_{i=1}^{r} d_{\mathcal{A}}(1, w_i) \leqslant [\text{by } (*)] \leqslant P(n_0)n_0.$$

This shows that $|\sigma_\gamma| \leqslant n_0 P(n_0) + n_0$ and finishes the proof of the Proposition. □

Now we are ready to prove the upper bound for the Dehn function of the double.

*Proof of Proposition 3.4.* As was noted above, $\Gamma = \Gamma(F \rtimes_\psi \mathbb{Z}) \cong F \rtimes_{(\psi)} F(s,t)$. As a free group, $F$ is asynchronously combable (with constant $K = 1$) and $F(s,t)$ is also word-hyperbolic. Therefore by Theorem 3.12, $\Gamma$ is asynchronously combable and hence, by Proposition 3.11, $\delta_\Gamma(n) \leqq nL(n)$. But due to Proposition 3.14, $L(n) \leqq n^{d+1}$, and therefore $\delta_\Gamma(n) \leqq n^{d+2}$. □

## 3.3 Comparing Lower and Upper Bounds

We will show later in chapter 7 by ad hoc methods that for the automorphisms involved in our construction, the lower and upper bounds on the Dehn functions given above actually coincide. It might be desirable, if possible, to find criteria when the upper and lower bounds from Propositions 3.3 and 3.4 are the same.

Recall that Propositions 3.3 and 3.4 give us the following double inequality for the Dehn function of $\Gamma = \Gamma(F \rtimes_{\psi} \mathbb{Z})$:

$$n \cdot \max_{\substack{\|b\| \leqslant n \\ b \in F}} \|\psi^n(b)\| \leqq \delta_{\Gamma}(n) \leqq \mathrm{gr}_{\psi}(n) \cdot n^2$$

The function $n \mapsto \max_{\|b\| \leqslant n} \|\psi^n(b)\|$ measures how fast the ball of radius $n$ in the word metric grows under the $n$-th power of an automorphism $\psi$. On the other hand, the function $\mathrm{gr}_{\psi}(n) = \max_{\|b\|=1} \|\psi^n(b)\|$ measures how fast the sphere of radius $n$ grows under the $n$-th iterate of $\psi$. Obviously, for all $\psi \in \mathrm{Aut}(F)$,

$$\max_{\|b\| \leqslant n} \|\psi^n(b)\| \preceq n \cdot \mathrm{gr}_{\psi}(n),$$

and for the above lower and upper bounds to meet we need this inequality to become an $\simeq$ equality. For some natural classes of automorphisms (e.g. if $\mathrm{gr}_{\psi^{ab}}(n) \simeq \mathrm{gr}_{\psi}(n)$) this actually happens, and the conclusions of Propositions 3.3 and 3.4 in these cases can be written in a more elegant form: $\delta_{\Gamma}(n) \simeq \mathrm{gr}_{\psi}(n) \cdot n^2$.

However in general the above inequality is strict. Gilbert Levitt [26, p. 1128] gives the following example of an automorphism from a paper of Bridson and Groves [15, p. 36], whose growth is one degree less than the growth of another representative of its outer automorphism class. This implies that the lower and upper bounds above are not equal:

**Example 3.15.** Let $\phi$ be an automorphism of a rank 2 free group $F_2 = F(x, y)$

acting on the generators $x, y$ as follows: $\phi(x) = x$, $\phi(y) = yx$. Let $i_y$ be the inner automorphism associated to $y$ and set $\psi = i_y \circ \phi$. Thus, $\psi(x) = yxy^{-1}$, $\psi(y) = y^2xy^{-1}$. Bridson and Groves notice (and this is easily proved using the formula (†) below) that $\mathrm{gr}_\phi(n) \simeq n$, but $\mathrm{gr}_\psi(n) \simeq n^2$.

We claim that the doubles of $G_\phi = F_2 \rtimes_\phi \mathbb{Z}$ and of $G_\psi = F_2 \rtimes_\psi \mathbb{Z}$ are isomorphic. Indeed, let $\Gamma(G_\phi) \cong F_2 \rtimes_{(\phi)} F(s,t)$ and $\Gamma(G_\psi) \cong F_2 \rtimes_{(i_y \circ \phi)} F(s_1, t_1)$. Consider a homomorphism $\mu\colon \Gamma(G_\phi) \to \Gamma(G_\psi)$ defined identically on $F_2$ and sending $s \mapsto y^{-1}s_1$, $t \mapsto y^{-1}t_1$. Relations in $G_\phi$ are satisfied:

$$\mu(sgs^{-1}\phi(g)^{-1}) = y^{-1}s_1gs_1^{-1}y\phi(g)^{-1} = y^{-1}(i_y \circ \phi(g))y\phi(g)^{-1} = 1$$

and similarly for $t$. Obviously, $\mu$ is surjective, and it is easy to see that $\mu$ is also injective. Indeed, if $\mu(g \cdot w(s,t)) = 1$ then $1 = g \cdot w(y^{-1}s_1, y^{-1}t_1) = gg' \cdot w(s_1, t_1)$ for some $g' \in F_2$, $w(s_1, t_1) \in F(s_1, t_1)$, and we conclude that $w(s_1, t_1) = 1$ and hence $w(s,t) = 1$ and $g = 1$.

Applying Propositions 3.3 and 3.4 to $\Gamma \cong \Gamma(G_\phi)$, we get:

$$n \cdot \max_{\|b\| \leqslant n} \|\phi^n(b)\| \leqq \delta_\Gamma(n) \leqq \mathrm{gr}_\phi(n) \cdot n^2.$$

By looking at the growth of a word $b = y^n$ we see that

$$\|\phi^n(y^n)\| = \|\phi^n(y)^n\| = \|(yx^n)^n\| = n(n+1)$$

Thus, $n \cdot \max_{\|b\| \leqslant n} \|\phi^n(b)\| \succeq n \cdot \|\phi^n(y^n)\| \simeq n^3$. On the other hand, $\mathrm{gr}_\phi(n) \cdot n^2 \simeq n \cdot n^2 = n^3$, and we conclude that $\delta_\Gamma(n) \simeq n^3$.

Now applying Propositions 3.3 and 3.4 to $\Gamma \cong \Gamma(G_\psi)$, we get:

$$n \cdot \max_{\|b\| \leqslant n} \|\psi^n(b)\| \leqq \delta_\Gamma(n) \precneqq \mathrm{gr}_\psi(n) \cdot n^2 \simeq n^4,$$

and, in particular, $\max_{\|b\| \leqslant n} \|\psi^n(b)\| \lneqq \delta_\Gamma(n) \cdot n^{-1} \simeq n^2 \simeq \mathrm{gr}_\psi(n) \precnsim n \cdot \mathrm{gr}_\psi(n)$.

This phenomenon that $\max_{\|b\| \leqslant n} \|\psi^n(b)\| \simeq \max_{\|b\|=1} \|\psi^n(b)\|$ can be explained by looking at the growth of $n$-th powers of generators of $F$. Bridson and Groves notice in [15] that for any $\phi \in \mathrm{Aut}(F)$, for any $w \in F$ and any inner automorphism $i_u$, the following formula holds:

$$(i_u \circ \phi)^n(w) = u^{-1}\phi(u^{-1}) \ldots \phi^{n-1}(u^{-1}) \cdot \phi^n(w) \cdot \phi^{n-1}(u) \ldots \phi(u)u. \qquad (\dagger)$$

In particular, for the automorphism $\psi$ defined above, we have $\psi^n(y) = U_n^{-1} V_n U_n$, where $V_n = \phi^n(y) = yx^n$ has length growing linearly, whereas

$$U_n = \phi^{n-1}(y) \ldots \phi(y)y = yx^{n-1} \cdot \ldots \cdot yx \cdot y$$

has length $n + \frac{n(n-1)}{2}$, quadratic in $n$. In particular, in the expansion for $\psi^n(y)^n$, the long conjugating elements $U_n^{-1}$, $U_n$ cancel each other, all except the first and the last ones, and the length of $\psi^n(y^n) = U_n^{-1} V_n^n U_n$ is quadratic, but not cubic.

# Chapter 4

# Cube Complexes

In their article [23] Haglund and Wise established that the fundamental groups of the so-called special cube complexes admit embeddings into right-angled Artin groups. This gives us a natural class of subgroups of right-angled Artin groups and suggests that we construct our examples within this class. We summarize the relevant definitions and results about special cube complexes in this chapter.

## 4.1 Piecewise Euclidean Cube Complexes

Note that there are different notions of a cube complex in the literature. The definition given in [16] seems to be too restrictive, as it prevents the torus with the standard CW structure having a single square 2–cell to be a cube complex. We adopt the approach from [8, 23].

Informally, a finite-dimensional *cube complex* is a CW complex which is obtained from a collection of standard cubes of dimension at most $m$ by identifying their faces via isometries. More formal definition involves the following ingredients.

**Definition 4.1.** ($n$–cube) Given a non-negative integer $n$, a *standard $n$–cube* is the product $[0,1]^n \subset \mathbb{R}^n$ viewed as a metric space with the usual Euclidean metric of $\mathbb{R}^n$. An *$n$–cube in* $\mathbb{R}^n$ is the image of the standard $n$–cube under an isometry $g\colon \mathbb{R}^n \to \mathbb{R}^n$. By embedding $\mathbb{R}^n \hookrightarrow \mathbb{R}^m$ ($n \leqslant m$) and composing with isometries $g\colon \mathbb{R}^m \to \mathbb{R}^m$, we get the notion of an *$n$–cube in* $\mathbb{R}^m$. We will call '$n$–cubes in

$\mathbb{R}^m$' just '$n$–cubes' or even 'cubes' for brevity.

**Definition 4.2.** (Face) Let $0 \leqslant k \leqslant n$ be an integer. A *$k$–face*, or just *face*, of a standard $n$–cube is a product $\prod_{i=1}^{n} J_i \subset [0,1]^n$, where $J_i = [0,1]$ for some $k$ of the factors and $J_i = \{0\}$ or $\{1\}$ for the remaining $n-k$ of the factors. A $k$–face of an $n$–cube $g([0,1]^n)$ in $\mathbb{R}^m$ is the $g$ image of an $k$–face of $[0,1]^n$.

**Definition 4.3.** (Cube complex) A *finite-dimensional piecewise Euclidean cube complex*, or simply a *cube complex*, $X$ is a CW complex with the following additional structure.

- [The cells] There exist a positive integer $m$ such that each cell of $X$ is an $n$–cube for some $n \leqslant m$.
- [Admissible maps] Let $D^n_\alpha$ be an $n$–cube in $\mathbb{R}^m$, and let $f_\alpha \colon D^n_\alpha \to X$ be the characteristic map of the $n$–cell $e^n_\alpha$ of $X$. If $g$ is an isometry of $\mathbb{R}^m$, then the composition

  $$f_\alpha \circ g \colon g^{-1}(D^n_\alpha) \to X$$

  is called an *admissible characteristic map* for the $n$–cell $e^n_\alpha$. (Note that $g^{-1}(D^n_\alpha)$ is an $n$–cube in $\mathbb{R}^m$.)
- [The gluing maps] For each $n$–cell $D^n_\alpha$ of $X$, the restriction of the characteristic map $f_\alpha$ to any $k$–face of $D^n_\alpha$ is an admissible characteristic map for a $k$–cell of $X$.

If all cubes of a cube complex are 2–cubes (i.e. $m = 2$ in the definition), such cube complex is called a *square complex*.

**Definition 4.4.** (Simple, flag, npc) A cube complex is *simple* if the link of every vertex of it is a simplicial complex. A simplicial complex is *flag* if any collection of $k+1$ pairwise adjacent vertices spans a $k$–simplex. A cube complex is *non-positively curved* if the link of each vertex is a flag simplicial complex.

## 4.2 Special Cube Complexes

**Definition 4.5.** (Hyperplane) A *midcube* of an $n$–cube $[0,1]^n$ is a subset obtained by restricting one of the coordinates to $\frac{1}{2}$. A *hyperplane* of a cube complex $X$ is a connected component of a new cube complex $Y$ which is formed as follows:

- the cubes of $Y$ are the midcubes of $X$;
- the restriction of a $(k+1)$–cell of $X$ to a midcube of $[0,1]^k$ defines the attaching map of a $k$–cell in $Y$.

An edge $a$ of $X$ is *dual* to some hyperplane $H$ if the midpoint of $a$ is a vertex of $H$.

**Definition 4.6.** (Parallelism, Walls) Two oriented edges $a$, $b$ of a cube complex $X$ are called *elementary parallel* if there is a square of $X$ containing $a$ and $b$ and such that the attaching map sends two opposite edges of $[0,1] \times [0,1]$ with the same orientation to $a$ and $b$ respectively. Define the *parallelism* on oriented edges of $X$ as the equivalence relation generated by elementary parallelism. An *(oriented) wall* of $X$ is a parallelism class of oriented edges. Note that every hyperplane $H$ in $X$ defines a pair of oriented walls consisting of edges dual to $H$.

Now we describe four pathologies for interaction of hyperplanes in a cube complex, which are forbidden for special cube complexes.

**Definition 4.7.** (Self-intersection) A hyperplane $H$ in $X$ *self-intersects*, if it contains more than one midcube from the same cube of $X$.

**Definition 4.8.** (One-sided) A hyperplane $H$ is *two-sided* if there exists a combinatorial map of CW complexes $H \times [0,1] \to X$ mapping $H \times \{\frac{1}{2}\}$ identically to $H$. (Recall that a cellular map $f\colon X \to Y$ of CW complexes is *combinatorial* if the restriction of $f$ to each open cell of $X$ is a homeomorphism onto its image.) A hyperplane $H$ in $X$ is called *one-sided* if it is not two-sided.

**Definition 4.9.** (Self-osculating) A hyperplane $H$ in $X$ is *self-osculating* if there are two edges $a$, $b$ dual to $H$ which do not belong to a common square of $X$ but share a common vertex. If in addition there is a consistent choice of orientation on the edges dual to $H$ which makes the common vertex for $a$, $b$ their origin or terminus, then the hyperplane $H$ is called *directly self-osculating*.

**Definition 4.10.** (Inter-osculating) Two distinct hyperplanes $H_1$, $H_2$ of $X$ are *inter-osculating* if they intersect and there are edges $a_1$ dual to $H_1$ and $a_2$ dual to $H_2$ which do not belong to the same square of $X$ but share a common vertex.

**Definition 4.11.** (Special cube complex) A non-positively curved cube complex is called *special* if its hyperplanes are all two-sided, with no self-intersections, self-osculations or inter-osculations.

**Definition 4.12.** (Virtually special group) A group $G$ is called *special* if there exists a special cube complex $X$ whose fundamental group is isomorphic to $G$. A group $G$ is *virtually special* if there exists a special cube complex $X$ and a finite index subgroup $H \leqslant G$ such that $H$ is isomorphic to the fundamental group of $X$.

**Definition 4.13.** (Right-angled Artin group) Let $\Delta$ be a finite simplicial graph. The *right-angled Artin group*, or RAAG, associated to $\Delta$, is a finitely presented group $A(\Delta)$ given by the presentation:

$$A(\Delta) = \langle a_i \in \text{Vertices}(\Delta) \mid [a_i, a_j] = 1 \text{ if } (a_i, a_j) \in \text{Edges}(\Delta) \rangle.$$

**Definition 4.14.** (Salvetti complex) Given a right-angled Artin group $A(\Delta)$, the *Salvetti complex* associated to $A(\Delta)$ is a non-positively curved cube complex $S_\Delta$ defined as follows. For each $a_i \in \text{Vertices}(\Delta)$ let $S^1_{a_i}$ be a circle endowed with a structure of a CW complex having a single 0–cell and a single

1–cell. Let $n = \mathrm{Card}(\mathrm{Vertices}(\Delta))$ and let $T = \prod_{j=1}^{n} S^1_{a_j}$ be an $n$–dimensional torus with the product CW structure. For every full subgraph $K \subset \Delta$ with $\mathrm{Vertices}(K) = \{a_{i_1}, \ldots, a_{i_k}\}$ define a $k$–dimensional torus $T_K$ as a Cartesian product of CW complexes: $T_K = \prod_{j=1}^{k} S^1_{a_{i_j}}$ and observe that $T_K$ can be identified as a combinatorial subcomplex of $T$. Then the *Salvetti complex associated with* $A(\Delta)$ is

$$S_\Delta = \bigcup \big\{ T_K \subset T \mid K \text{ a full subgraph of } \Delta \big\}.$$

Thus $S_\Delta$ has a single 0–cell and $n$ 1–cells. Each edge $(a_i, a_j) \in \mathrm{Edges}(\Delta)$ contributes a square 2–cell to $S_\Delta$ with the attaching map $a_i a_j a_i^{-1} a_j^{-1}$. And in general each full subgraph $K \subset \Delta$ contributes a $k$–dimensional cell to $S_\Delta$ where $k = \mathrm{Card}(\mathrm{Vertices}(K))$.

**Theorem 4.15** ([23],Th. 4.2)**.** *A cube complex is special if and only if it admits a local isometry into the Salvetti complex of some right-angled Artin group.*

Since local isometries of CAT(0) spaces are $\pi_1$-injective, one gets the following

**Corollary 4.16.** *The fundamental group of a special cube complex is isomorphic to a subgroup of a right-angled Artin group.*

## 4.3 Morse Functions on Cube Complexes

We will use the following definitions from [4, 5].

**Definition 4.17.** (Morse function) A map $f\colon X \to \mathbb{R}$ defined on a cube complex $X$ is a *Morse function* if

- for every cell $e$ of $X$, with the characteristic map $\chi_e\colon [0,1]^m \to e$, the composition $f\chi_e\colon [0,1]^m \to \mathbb{R}$ extends to an affine map $\mathbb{R}^m \to \mathbb{R}$ and $f\chi_e$ is constant only when $\dim e = 0$;

- the image of the 0–skeleton of $X$ is discrete in $\mathbb{R}$.

**Definition 4.18.** (Circle-valued Morse function) A *circle-valued Morse function* on a cube complex $X$ is a cellular map $f\colon X \to S^1$ with the property that $f$ lifts to a Morse function between universal covers.

**Definition 4.19.** (Ascending and descending links) Suppose $X$ is a cube complex, $f\colon X \to S^1$ is a circle-valued Morse function and $\tilde{f}\colon \tilde{X} \to \mathbb{R}$ is the corresponding Morse function. Let $v \in X^{(0)}$ and note that the link of $v$ in $X$ is naturally isomorphic to the link of any lift $\tilde{v}$ of $v$ in $\tilde{X}$. We say that a cell $\tilde{e} \subset \tilde{X}$ contributes to the *ascending* (respectively *descending*) link of $\tilde{v}$ if $\tilde{v} \in \tilde{e}$ and $\tilde{f}|_{\tilde{e}}$ achieves its minimum (respectively, maximum) value at $\tilde{v}$. The ascending (respectively, descending) link of $v$ is then defined to be the subset of the link $\mathrm{Lk}(v, X)$ naturally identified with the ascending (respectively, descending) link of $\tilde{v}$. Note that in the case when $X$ is a square complex, all ascending, descending and entire links are graphs.

For 2–dimensional complexes, we have the following characterization of free-by-cyclic groups, which was proven in [4] (see also [25, Th. 10.1]).

**Theorem 4.20** ([4], Proposition 2.5). *If $f\colon X \to S^1$ is a circle-valued Morse function on a 2–complex $X$ all of whose ascending and descending links are trees, then $X$ is aspherical and $\pi_1(X)$ is free-by-cyclic. This means that there is a short exact sequence*

$$1 \longrightarrow F_m \longrightarrow \pi_1(X) \longrightarrow \mathbb{Z} \longrightarrow 1,$$

*where the free group $F_m$ is isomorphic to $\pi_1(f^{-1}(\mathrm{pt}))$, pt being any point on $S^1$.*

# Chapter 5

# Groups $G_{m,k}$

In this chapter we define a sequence of groups $G_{m,k}$ and study their presentation complex. We show that it is a non-positively curved square complex and that the groups are free-by-cyclic.

## 5.1 LOG Definition

**Definition 5.1.** (LOG) A *labeled, oriented graph*, or LOG, consists of a finite, directed graph with labels on the vertices and edges satisfying the following: the vertices have distinct labels, and the edge labels are chosen from the set of vertex labels.

A LOG determines a finite presentation as follows. The set of generators is the set of vertex labels. The set of relations is in one-to-one correspondence with the set of edges; there is a relation of the form $a^{-1}ua = v$ for each oriented edge labeled $a$ from vertex $u$ to vertex $v$.

Let $m \in \mathbb{N}$, $m \geqslant 1$. For $k = 0, \ldots, m$, let $G_{m,k}$ be a group defined by the LOG presentation in the Figure 5.1:
i.e.

$$G_{m,k} = \langle\, a_1, \ldots, a_{m+k+1} \mid [a_i, a_{i+1}] = 1, \quad i = 1, \ldots, m, \\ a_{m+j+1}^{-1} a_j a_{m+j+1} = a_{m+j}, \quad j = 1, \ldots, k \,\rangle.$$

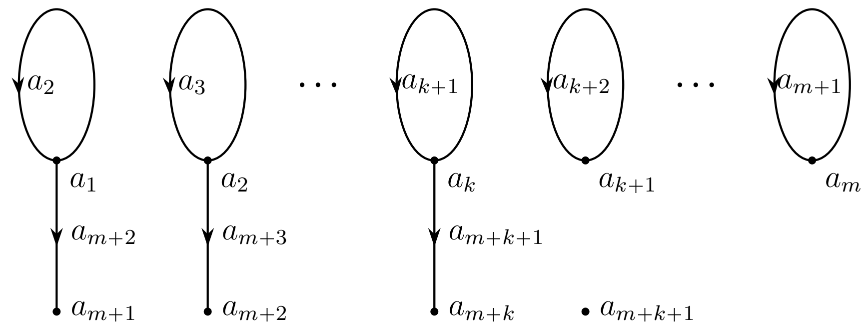


Figure 5.1: The LOG description of $G_{m,k}$.

Clearly $G_{m,k}$ is an HNN extension of $G_{m,k-1}$ with the stable letter $a_{m+k+1}$ so there is a natural tower of inclusions

$$G_{m,0} \subset G_{m,1} \subset G_{m,2} \subset \cdots \subset G_{m,m}.$$

## 5.2 CAT(0) Structure for $G_{m,k}$

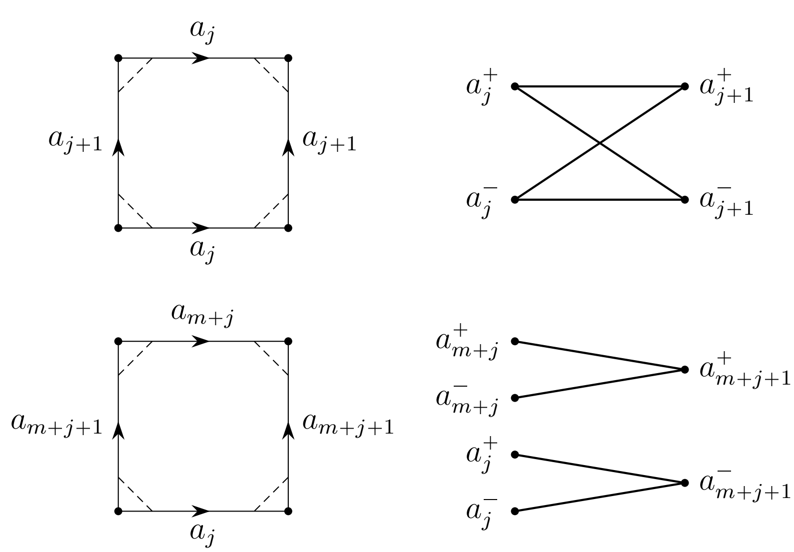


Figure 5.2: The contribution of the relations $a_{j+1}^{-1}a_ja_{j+1} = a_j$, $1 \leqslant j \leqslant m$, and $a_{m+j+1}^{-1}a_ja_{m+j+1} = a_{m+j}$, $1 \leqslant j \leqslant k$, to the link of the 0–cell of the presentation complex $K_{m,k}$.

One way of producing a CAT(0) structure on groups $G_{m,k}$ is to verify that the presentation 2–complex corresponding to their LOG presentation can be metrized so that it is a non-positively curved, piecewise euclidean (PE) complex.

Let $K_{m,k}$ denote the presentation 2–complex corresponding to the LOG presentation above of $G_{m,k}$. It has one 0–cell, $(m+k+1)$ 1–cells labeled by $a_1, \dots, a_{m+k+1}$, and $(m+k)$ 2–cells corresponding to the relations $a_{j+1}^{-1} a_j a_{j+1} = a_j$ for $1 \leqslant j \leqslant m$ and $a_{m+j+1}^{-1} a_j a_{m+j+1} = a_{m+j}$ for $1 \leqslant j \leqslant k$. By construction, $K_{m,k}$ is a subcomplex of $K_{m,k+1}$. We endow $K_{m,k}$ with a PE structure by using regular euclidean squares for the 2–cells, and using local isometric embedding attaching maps.

**Proposition 5.2.** *The presentation complex $K_{m,k}$ defined above is a non-positively curved PE complex.*

*Proof.* We need to check the Gromov link condition [16, Th. II.5.20]. For the square 2–cells, it reduces to a purely combinatorial requirement that the link of every 0–cell has no circuits of combinatorial length less than 4. Figure 5.2 shows the contributions of the relations of $G_{m,k}$ to the link $L$ of the unique 0–cell of $K_{m,k}$. We adopt the following notation: if a 1–cell $a$ originates at 0–cell $u$ and terminates at 0–cell $v$, then it contributes a vertex denoted $a^-$ to the link of $u$ and a vertex denoted $a^+$ to the link of $v$.

We see that the link $L$ can be obtained as a union of a sequence of graphs:

$$L_1 \subset L_2 \subset \cdots \subset L_{m+k+1} = L,$$

where $L_1$ is just a pair of disjoint vertices $a_1^+$, $a_1^-$, and $L_{i+1}$ is obtained from $L_i$ by adding a new pair of disjoint vertices $a_{i+1}^+$, $a_{i+1}^-$ and connecting each one of them to some pair $a_s^+$, $a_s^-$ with $s < i+1$. We observe that this procedure preserves the following property: "for every $l$, vertices $a_l^+$, $a_l^-$ are non-adjacent". Indeed, the shortest path between the "old" vertices $a_s^+$, $a_s^-$ has length two, and the shortest path between the newly added vertices $a_{i+1}^+$, $a_{i+1}^-$ is at least two. This shows that at each step we cannot create cycles of lengths two and three. Therefore the link $L$ has no cycles of length less than four. $\square$

**Corollary 5.3.** *Groups $G_{m,k}$ are CAT(0).*

*Proof.* Indeed, the universal cover $\widetilde{K}_{m,k}$ of non-positively curved square complex $K_{m,k}$ is a CAT(0) complex and $G_{m,k}$ acts on it by isometries, properly discontinuously and cocompactly. $\square$

## 5.3 Free-by-Cyclic Structure

Notice that all the relations of groups $G_{m,k}$ have the form: $a_i^{a_j} = a_l$. This implies that there exists a well-defined epimorphism $G_{m,k} \to \mathbb{Z}$, sending every $a_i$ to a fixed generator of $\mathbb{Z}$. This epimorphism can be realized geometrically by a circle-valued Morse function $f\colon K_{m,k} \to S^1$, which can be defined as follows. Consider a CW structure on $S^1$ consisting of one 0–cell and one 1–cell. Then $f$ takes the 0–cell of $K_{m,k}$ to the 0–cell of $S^1$, maps 1–cells of $K_{m,k}$ map homeomorphically onto the target 1–cell of $S^1$, and extends linearly over the 2–cells. Here by 'extends linearly' we mean that $f$ lifts to a map of the universal covers in the way depicted in the Figure 5.3. (Note that, by the non-positive curvature, characteristic maps of cells lift to embeddings in the universal cover.)

**Proposition 5.4.** *The (circle-valued) Morse function $f\colon K_{m,k} \to S^1$ induces a short exact sequence*

$$1 \longrightarrow F_{m+k} \longrightarrow G_{m,k} \longrightarrow \mathbb{Z} \longrightarrow 1,$$

*where $F_{m+k}$ is a free group of rank $m + k$.*

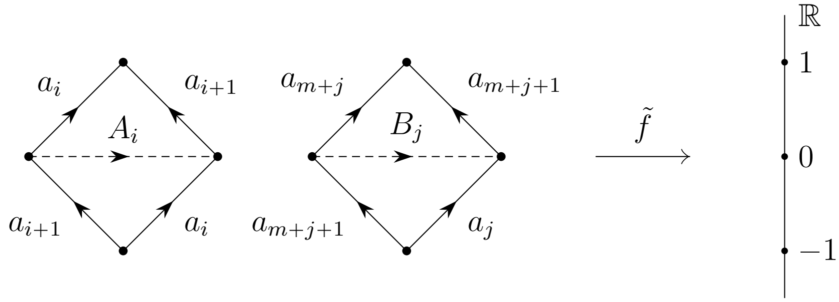


Figure 5.3: The Morse function on each 2–cell and the preimage set of 0.

*Proof.* By Theorem 4.20 it suffices to show that the ascending and the descending links of the 0–cell in $K_{m,k}$ are trees.

The ascending link of the 0–cell of $K_{m,k}$ is formed by those corners of 2–cells of $K_{m,k}$ which are formed by a pair of originating edges (labeled $a_\bullet^-$ in Figure 5.2). Similarly, the descending link of the 0–cell of $K_{m,k}$ is formed by those corners of 2–cells of $K_{m,k}$ which are formed by a pair of terminating edges (labeled $a_\bullet^+$, in Figure 5.2).

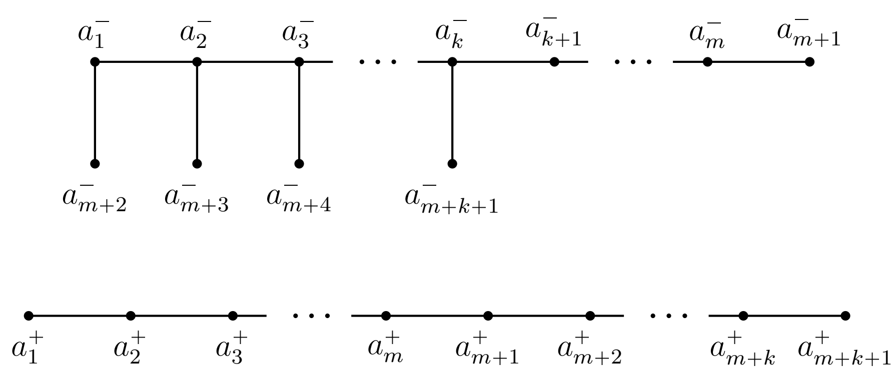


Figure 5.4: The ascending and the descending links for the Morse function $f\colon K_{m,k} \to \mathbb{R}/\mathbb{Z}$.

From Figure 5.4 we observe that the ascending and the descending links of the 0–cell of $K_{m,k}$ are indeed trees. By the definition of $f$, each 2–cell of $K_{m,k}$ contributes its diagonal loop to $f^{-1}$(0–cell). Furthermore, $f^{-1}$(0–cell) is a bouquet of these diagonal loops. Hence $f^{-1}$(0–cell) is a graph having a single 0–cell and $(m+k)$ 1–cells which are denoted in the Figure 5.3 by $A_i$, $B_j$. $\square$

The above Proposition implies that $G_{m,k} \cong F_{m+k} \rtimes_{\phi_{m,k}} \mathbb{Z}$ for some monodromy automorphism $\phi_{m,k}$. We shall determine explicitly the automorphism $\phi_{m,k}$ for a particular choice of basis for $F_{m+k}$. Let $A_i$, $B_j$ be the diagonals of the 2–cells of $K_{m,k}$, as shown in Figure 5.3. Note that they have the following expressions in the generators of $G_{m,k}$:

$$A_i = a_{i+1}^{-1} a_i, \quad 1 \leqslant i \leqslant m; \qquad B_j = a_{m+j+1}^{-1} a_j, \quad 1 \leqslant j \leqslant k.$$

**Proposition 5.5.** *For $0 \leqslant k \leqslant m$, $G_{m,k}$ has the following explicit free-by-cyclic structure:*

$$G_{m,k} \cong F_{m+k} \rtimes_{\phi_{m,k}} \mathbb{Z}$$

*where*

$$F_{m+k} = \langle A_1, \dots, A_m, B_1, \dots, B_k \rangle; \qquad \mathbb{Z} = \langle a_1 \rangle$$

*and the monodromy automorphism $\phi_{m,k}$ acts as follows (here overbar denotes the inverse):*

$$\begin{aligned}
\phi_{m,k}\colon \quad & A_1 \longmapsto A_1 \\
& A_2 \longmapsto A_1 \, (A_2) \, \overline{A}_1 \\
& A_3 \longmapsto A_1 A_2 \, (A_3) \, \overline{A}_2 \overline{A}_1 \\
& \dots \\
& A_m \longmapsto A_1 A_2 \dots A_{m-1} \, (A_m) \, \overline{A}_{m-1} \dots \overline{A}_1 \\
& \\
& B_1 \longmapsto A_1 A_2 \dots A_m \, (B_1) \\
& B_2 \longmapsto A_1 A_2 \dots A_m \, (B_1 B_2) \, \overline{A}_1 \\
& B_3 \longmapsto A_1 A_2 \dots A_m \, (B_1 B_2 B_3) \, \overline{A}_2 \overline{A}_1
\end{aligned}$$

$\ldots$

$$B_k \longmapsto A_1 A_2 \ldots A_m \left(B_1 B_2 \ldots B_k\right) \bar{A}_{k-1} \bar{A}_{k-2} \ldots \bar{A}_2 \bar{A}_1.$$

*Furthermore, $\phi_{m,k}$ is the restriction of $\phi_{m,m}$ to $F_{m+k}$.*

*Proof.* In the proof of Proposition 5.4 it was shown that $F_{m+k}$ is freely generated by all elements $A_i$, $B_j$.

As a generator of the $\mathbb{Z}$ factor we are free to choose any element that maps to a generator of $\pi_1(S^1)$; without loss of generality, we may take $\mathbb{Z} = \langle a_1 \rangle$.

To get the action of the monodromy automorphism $\phi$ on the generators $A_i$, $B_j$ of $F_{m+k}$ we need to compute the conjugations $a_1 A_i a_1^{-1}$ and $a_1 B_j a_1^{-1}$. That is, we need to find words in generators $A_i$, $B_j$ which are equal to $a_1 A_i a_1^{-1}$ and $a_1 B_j a_1^{-1}$ in $K_{m,k}$.

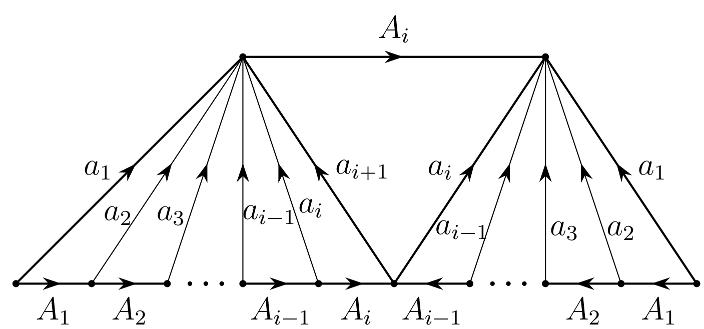


Figure 5.5: The action of the monodromy automorphism on $A_i$.

For $a_1 A_i a_1^{-1}$, we start with the triangle having $A_i$ on top and 1–cells $a_{i+1}$, $a_i$ forming two bottom sides. We would like to express $a_1 a_{i+1}^{-1}$ and $a_i a_1^{-1}$ as products of free generators $A_i$, $B_j$. Since the descending link of the 0–cell in $K_{m,k}$ is a tree, there exists a unique path in it connecting $a_1^+$ to $a_{i+1}^+$ and a unique path connecting $a_i^+$ to $a_1^+$. These paths correspond to paths $A_1 A_2 \ldots A_{i-1} A_i$ and

$\overline{A}_{i-1} \ldots \overline{A}_2\overline{A}_1$, respectively, see Figure 5.5. Thus,

$$a_1 A_i a_1^{-1} = A_1 A_2 \ldots A_{i-1} (A_i) \overline{A}_{i-1} \ldots \overline{A}_2 \overline{A}_1.$$

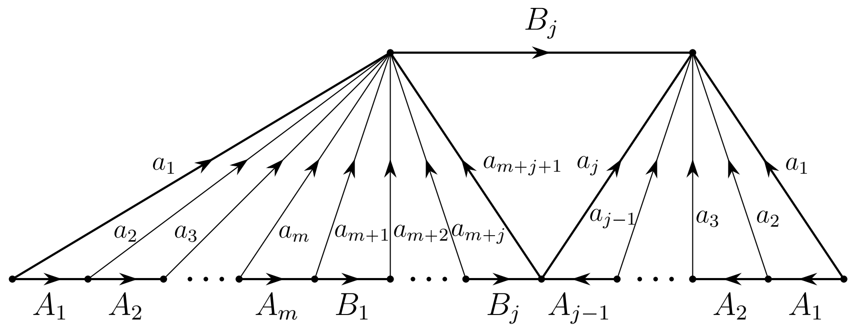


Figure 5.6: The action of the monodromy automorphism on $B_j$.

Similarly, for $a_1 B_j a_1^{-1}$, we start with the triangle having $B_j$ on top and $a_{m+j+1}$, $a_j$ forming two bottom sides. Again, there are unique paths in the descending link from $a_1^+$ to $a_{m+j+1}^+$ and from $a_j^+$ to $a_1^+$. They correspond to words $A_1 A_2 \ldots A_m B_1 \ldots B_j$ and $\overline{A}_{j-1} \ldots \overline{A}_2 \overline{A}_1$, respectively, see Figure 5.6. Hence,

$$a_1 B_j a_1^{-1} = A_1 A_2 \ldots A_m (B_1 \ldots B_j) \overline{A}_{j-1} \ldots \overline{A}_2 \overline{A}_1. \qquad \square$$

# Chapter 6

# Constructing a Special Cover for $G_{m,m}$

In this chapter we construct a certain permutation representation for a group $G_{m,m}$ and show that it defines a finite cover for its presentation 2–complex $K_{m,m}$, which can be embedded in an $(2m+1)$–dimensional torus. This allows us to construct a finite special cover for $K_{m,m}$, for all *even* values of $m$.

## 6.1 The Permutation Representation

We now define a right transitive action of $G_{m,m}$ on a certain set $H_{2m+1}$ of cardinality $2^{2m+1}$.

*Action set.* For any $n = 1, \dots, 2m+1$ denote $H_n$ to be the set of all tuples of length $n$ consisting of 0's and 1's, i.e. $H_n = \prod_{i=1}^{n}\{0,1\}$. There are natural inclusions

$$H_n \hookrightarrow H_{n+1}, \quad (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0),$$

and we identify $H_n$ with its image in $H_{n+1}$ under these inclusions. Also denote $H_n^*$ a subset of $H_{n+1}$ consisting of all tuples with the last coordinate 1:

$$H_n^* = \{(x_1, \dots, x_n, 1)\} \subset H_{n+1}.$$

With the above identifications, we have $H_{n+1} = H_n \sqcup H_n^*$ (disjoint union).

To define a *right action* of a group $G$ on a set $X$, it suffices to associate to each $g \in G$ a permutation $\pi(g)$ of $X$ such that

$$\pi(gh) = \pi(h)\pi(g) \quad \text{for all } g, h \in G.$$

Equivalently, a right action of $G$ on $X$ is a homomorphism of the opposite group $G^{\circ}$ to $\mathrm{Sym}(X)$, the group of all permutations of $X$, where $G^{\circ}$ equals $G$ as a set, with the new operation $\circ$ defined as

$$a \circ b := ba.$$

We adopt the latter approach and construct the homomorphism from $G^{\circ}_{m,m}$ to $\mathrm{Sym}(H_{2m+1})$.

Recall that we have a natural tower of inclusions

$$G_{m,0} \subset G_{m,1} \subset \cdots \subset G_{m,m}.$$

Since there are also inclusions

$$H_{m+1} \subset H_{m+2} \subset \cdots \subset H_{(m+1)+m} = H_{2m+1}$$

this allows us to define the homomorphism $\pi\colon G^{\circ}_{m,m} \to \mathrm{Sym}(H_{2m+1})$ inductively by repeatedly extending the homomorphisms $G^{\circ}_{m,k-1} \to \mathrm{Sym}(H_{m+k})$ to $G^{\circ}_{m,k} \to \mathrm{Sym}(H_{m+k+1})$ for $k = 1, \dots, m$ as follows.

*Base of induction.* Let the $m+1$ generators $a_1, \dots, a_{m+1}$ of $G_{m,0}$ act on $H_{m+1}$ as flips in the respective coordinates, i.e. for $i = 1, \dots, m+1$, set

$$\pi(a_i)|_{H_{m+1}} := \beta_i \tag{$A_0$}$$

where $\beta_i \colon H_{2m+1} \to H_{2m+1}$ given by

$$\beta_i(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots) = (x_1, \dots, x_{i-1}, 1 - x_i, x_{i+1}, \dots)$$

is the operator that changes the $i$-th coordinate from 0 to 1 and vice versa, fixing all others.

All the relations in $G_{m,0}$ (and $G^\circ_{m,0}$) are commutators $[a_i, a_{i+1}] = 1$, $i = 1, \dots, m$. Clearly, they are satisfied in $\mathrm{Sym}(H_{m+1})$ since operators $\beta_i$ pairwise commute. Thus we have a well-defined homomorphism $\pi \colon G^\circ_{m,0} \to \mathrm{Sym}(H_{m+1})$.

*Inductive step.* For a fixed $k \in \{1, \dots, m\}$, suppose that

$$\pi \colon G^\circ_{m,k-1} \to \mathrm{Sym}(H_{m+k})$$

is already defined. In particular, this implies that $H_{m+k}$ is invariant under $\pi(a_j)$ for all $j = 1, \dots, m+k$. Also suppose that the following property holds:

$$\text{for all } j = k, \dots, m, \quad \pi(a_j)|_{H_{m+k}} = \beta_j|_{H_{m+k}}. \tag{$P_k$}$$

The base of induction above guarantees that these suppositions are true for $k = 1$. Our goal is to extend the homomorphism $\pi|_{G^\circ_{m,k-1}}$ to $\pi \colon G^\circ_{m,k} \to \mathrm{Sym}(H_{m+k+1})$. Since $H_{m+k+1} = H_{m+k} \sqcup H^*_{m+k}$, it will suffice to define $\pi(a_j)|_{H^*_{m+k}}$ for $j = 1, \dots, m+k$, and $\pi(a_{m+k+1})|_{H_{m+k} \sqcup H^*_{m+k}}$.

To this end, we set

$$\pi(a_{m+k+1})|_{H_{m+k} \sqcup H^*_{m+k}} := \beta_{m+k+1} \tag{$A_1$}$$

and for all $1 \leqslant j \leqslant m+k$,

$$\pi(a_j)|_{H^*_{m+k}} := \beta_{m+k+1} \cdot \varphi_{k,m+k} \cdot \pi(a_j)|_{H_{m+k}} \cdot \varphi_{k,m+k} \cdot \beta_{m+k+1}, \qquad (A_2)$$

where $\cdot$ denotes the composition and $\varphi_{k,m+k} \colon H_{m+k+1} \to H_{m+k+1}$ is the involution that interchanges $k$-th and $(m+k)$-th coordinates leaving all other coordinates fixed:

$$\varphi_{k,m+k} \colon (x_1, \dots, x_k, \dots, x_{m+k}, x_{m+k+1}) \mapsto (x_1, \dots, x_{m+k}, \dots, x_k, x_{m+k+1}).$$

In other words, we transfer the action of $G_{m,k-1}$ from $H_{m+k}$ to $H^*_{m+k}$ while twisting it with $\varphi_{k,m+k}$. Notice that, with the above definitions, both sets $H_{m+k}$ and $H^*_{m+k}$ are invariant under $\pi(a_j)$ for $j = 1, \dots, m+k$. All the relations involving generators $a_1, \dots, a_{m+k}$ are satisfied on $H_{m+k+1} = H_{m+k} \sqcup H^*_{m+k}$ since they hold true on $H_{m+k}$ and the conjugation by $\beta_{m+k+1} \cdot \varphi_{k,m+k}$ is a homomorphism between permutation groups on $H_{m+k}$ and $H^*_{m+k}$.

The only relation in $G_{m,k}$ involving the last generator $a_{m+k+1}$ is

$$a^{-1}_{m+k+1} a_k a_{m+k+1} = a_{m+k},$$

which translates to

$$a_{m+k+1} \circ a_k \circ a^{-1}_{m+k+1} = a_{m+k}$$

in $G^\circ_{m,k}$.

Since $\beta_{m+k+1}$ sends $H_{m+k}$ to $H^*_{m+k}$, the left-hand side of this relation acts on $H_{m+k}$ as follows:

$$\pi(a_{m+k+1}) \cdot \pi(a_k) \cdot \pi(a^{-1}_{m+k+1})|_{H_{m+k}}$$

$$= \pi(a_{m+k+1}) \cdot \pi(a_k) \cdot \beta_{m+k+1}|_{H_{m+k}}$$

$$= \pi(a_{m+k+1}) \cdot \pi(a_k)|_{H^*_{m+k}} \cdot \beta_{m+k+1}|_{H_{m+k}}$$

$$= \pi(a_{m+k+1}) \cdot (\beta_{m+k+1} \cdot \varphi_{k,m+k} \cdot \pi(a_k)|_{H_{m+k}} \cdot \varphi_{k,m+k} \cdot \beta_{m+k+1}) \cdot \beta_{m+k+1}|_{H_{m+k}}$$

$$= \pi(a_{m+k+1}) \cdot \beta_{m+k+1} \cdot \varphi_{k,m+k} \cdot \pi(a_k)|_{H_{m+k}} \cdot \varphi_{k,m+k}|_{H_{m+k}}$$

$$= (\pi(a_{m+k+1}) \cdot \beta_{m+k+1}) \cdot \varphi_{k,m+k} \cdot \pi(a_k)|_{H_{m+k}} \cdot \varphi_{k,m+k}|_{H_{m+k}}$$

$$= \mathrm{id} \cdot \varphi_{k,m+k} \cdot \pi(a_k)|_{H_{m+k}} \cdot \varphi_{k,m+k}|_{H_{m+k}} = [\text{by } (P_k)]$$

$$= \varphi_{k,m+k} \cdot \beta_k|_{H_{m+k}} \cdot \varphi_{k,m+k}|_{H_{m+k}} = \beta_{m+k}|_{H_{m+k}} = \pi(a_{m+k})|_{H_{m+k}},$$

where the last equality holds due to the inductive definition of $\pi$. Thus, the both sides of the above relation act the same on $H_{m+k}$.

Analogously, on $H^*_{m+k}$, the left-hand side acts as:

$$\pi(a_{m+k+1}) \cdot \pi(a_k) \cdot \pi(a^{-1}_{m+k+1})|_{H^*_{m+k}} = \beta_{m+k+1} \cdot \pi(a_k)|_{H_{m+k}} \cdot \beta_{m+k+1}|_{H^*_{m+k}}$$
$$= [\text{by } (P_k)] = \beta_{m+k+1} \cdot \beta_k|_{H_{m+k}} \cdot \beta_{m+k+1}|_{H^*_{m+k}} = \beta_k|_{H^*_{m+k}},$$

and the right-hand side:

$$\pi(a_{m+k})|_{H^*_{m+k}} = \beta_{m+k+1} \cdot \varphi_{k,m+k} \cdot \pi(a_{m+k})|_{H_{m+k}} \cdot \varphi_{k,m+k} \cdot \beta_{m+k+1}|_{H^*_{m+k}}$$
$$= [\text{by the inductive definition}] = \beta_{m+k+1} \cdot \varphi_{k,m+k} \cdot \beta_{m+k} \cdot \varphi_{k,m+k} \cdot \beta_{m+k+1}|_{H^*_{m+k}}$$
$$= \beta_{m+k+1} \cdot \beta_k \cdot \beta_{m+k+1}|_{H^*_{m+k}} = \beta_k|_{H^*_{m+k}}.$$

Since the two sides act the same on $H_{m+k} \sqcup H^*_{m+k} = H_{m+k+1}$, the above relation is satisfied in $\mathrm{Sym}(H_{m+k+1})$, which proves that $\pi$ is well-defined on $G^\circ_{m,k}$.

It remains to be proved that the auxiliary condition $(P_k)$ is preserved under the inductive step, i.e. that $(P_k)$ implies $(P_{k+1})$.

Indeed, $(P_k)$ means that $\pi(a_j)|_{H_{m+k}} = \beta_j|_{H_{m+k}}$ for $j = k, \ldots, m$. Thus, for

$j > k$,

$$\begin{aligned}\pi(a_j)|_{H^*_{m+k}} &= \beta_{m+k+1}\cdot\varphi_{k,m+k}\cdot\pi(a_j)|_{H_{m+k}}\cdot\varphi_{k,m+k}\cdot\beta_{m+k+1}|_{H^*_{m+k}}\\ &= \beta_{m+k+1}\cdot\varphi_{k,m+k}\cdot\beta_j|_{H_{m+k}}\cdot\varphi_{k,m+k}\cdot\beta_{m+k+1}|_{H^*_{m+k}}\\ &= [\text{since } j\neq k,\ m+k] = \beta_j|_{H^*_{m+k}}.\end{aligned}$$

This proves that $\pi(a_j)|_{H_{m+k+1}} = \beta_j|_{H_{m+k+1}}$ for $j = k+1,\ldots,m$, i.e. that $(P_{k+1})$ holds.

This finishes the inductive construction of the homomorphism

$$\pi\colon G^{\circ}_{m,m} \to \mathrm{Sym}(H_{2m+1})$$

and the proof that it is well-defined. Thus one gets a right action of $G_{m,m}$ on $H_{2m+1}$ which will also be denoted $\pi$.

Figure 6.1 shows the permutation representation for $G_{2,2}$.

**Proposition 6.1.** *The right action $\pi$ of $G_{m,m}$ on $H_{2m+1}$, defined above, has the following properties:*

1. *$G_{m,k}$ acts transitively on $H_{m+k+1}$ for all $k = 0,\ldots,m$.*
2. *Each generator $a_i$, $i = 1,\ldots,2m+1$, acts as an involution on $H_{2m+1}$.*
3. *For any $v \in H_{2m+1}$, and any $a_i$, $i = 1,\ldots,2m+1$, $\pi(a_i)v$ differs from $v$ in exactly one coordinate. In particular, $\pi(a_i)$ has no fixpoints.*
4. *For any $i \neq j$, $\pi(a_ia_j)$ has no fixpoints.*

*Proof.* (1) An easy induction. The case $k = 1$ is obvious since $G_{m,0}$ acts on $H_{m+1}$ by coordinate flips. So one can start with any $(m+1)$–tuple of 0,1's and obtain any other $(m+1)$–tuple by changing one coordinate at a time. Suppose now that

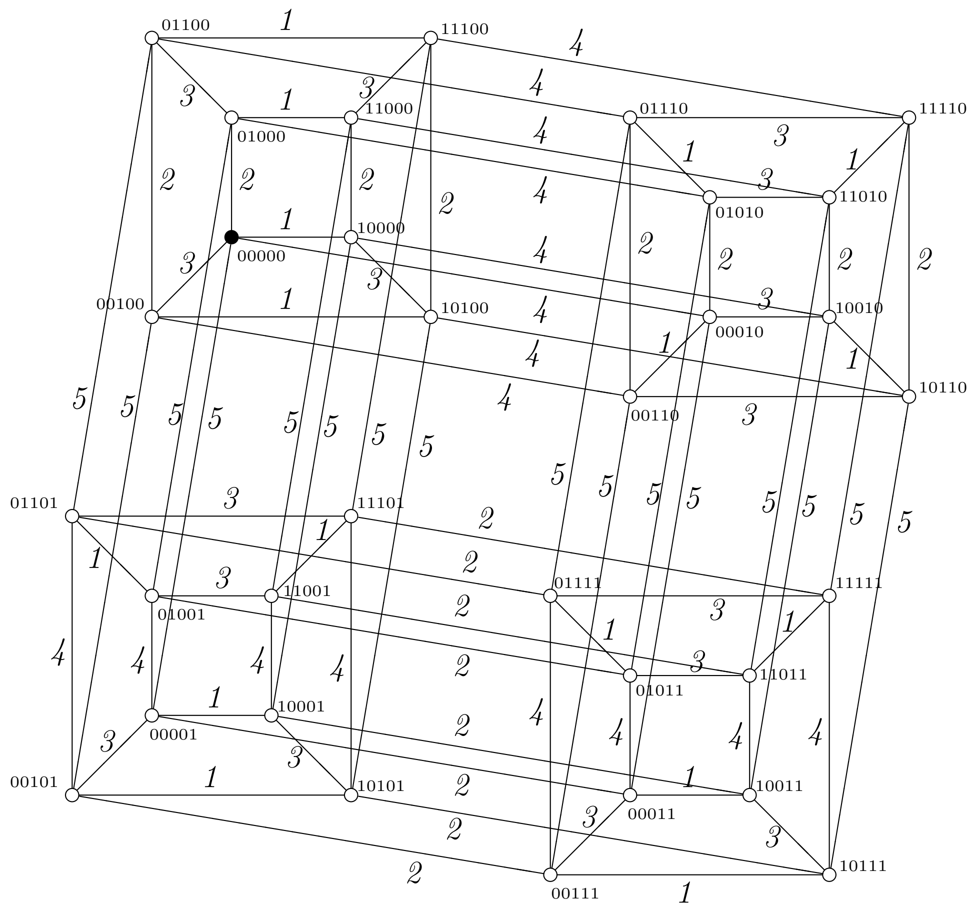


Figure 6.1: The case of $m = 2$, $k = 2$: the action of $G_{2,2} = \langle a_1, a_2, a_3, a_4, a_5 \mid [a_1, a_2] = 1, [a_2, a_3] = 1, a_4^{-1}a_1a_4 = a_3, a_5^{-1}a_2a_5 = a_4\rangle$ on $H_5$. Elements of $H_5$ are arranged at vertices of the hypercube graph marked with the corresponding tuples of 0,1's. (Thus, each edge of this graph corresponds to a pair of opposite edges in the 1–skeleton of the 5–dimensional torus $\mathcal{T}_5$ defined below.) The subset $H_3$ is represented by the upper left-hand corner subgraph, and $H_4$ by the upper half of the picture. If $\pi(a_i)$ interchanges vertices $u$ and $v$ we mark the edge $uv$ with the italicized digit $i$.

$G_{m,k-1}$ acts transitively on $H_{m+k}$. Then by $(A_2)$, $H^*_{m+k}$ comprises another orbit for $G_{m,k-1}$ and $a_{m+k+1}$ glues $H_{m+k}$ and $H^*_{m+k}$ into one orbit for $G_{m,k}$ by $(A_1)$ thus proving that $G_{m,k}$ is transitive on $H_{m+k+1} = H_{m+k} \sqcup H^*_{m+k}$.

(2),(3) Follow by induction from formulas $(A_0)$–$(A_2)$.

(4) Again, this is obvious for $a_i$, $a_j$ with $1 \leqslant i, j \leqslant m+1$ acting on $H_{m+1}$ since they act as different coordinate flips $\beta_i$, $\beta_j$. Suppose that the statement is proven for some $k \in \{1, \ldots, m+1\}$, for all $a_i$, $a_j$, $1 \leqslant i, j \leqslant m+k$ acting on $H_{m+k}$. Then $\pi(a_i a_j)$ has no fixpoints on $H^*_{m+k}$ either, since otherwise if $v \in H^*_{m+k}$ is such a fixpoint, then by $(A_2)$, $\varphi_{k,m+k} \cdot \beta_{m+k+1}(v)$ would be a fixpoint for $\pi(a_i a_j)$ in $H_{m+k}$. Finally, if, say, $i = m+k+1$ then $\pi(a_i)$ changes the last, $(m+k+1)$-st, coordinate on $H_{m+k+1}$, whereas for $j < i$, $\pi(a_j)$ preserves both subsets $H_{m+k}$ and $H^*_{m+k}$, so it doesn't change the $(m+k+1)$-st coordinate. Hence, the composition of $\pi(a_j)$ and $\pi(a_i)$ has no fixpoints. □

## 6.2 A $(2m+1)$–Torus Cover

Let $C_2 = \mathbb{R}/2\mathbb{Z}$ be a 1–dimensional CW complex with the following CW structure: its 0–cells are $0 + 2\mathbb{Z}$ and $1 + 2\mathbb{Z}$, which we denote by 0 and 1 respectively. The 1–cells are $[0,1] + 2\mathbb{Z}$ and $[1,2] + 2\mathbb{Z}$, which we denote by $e_0$ and $e_1$ respectively, see Figure 6.2.

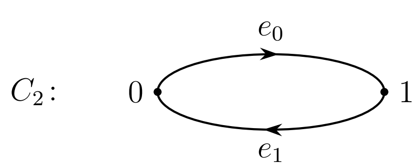


Figure 6.2: The CW complex $C_2$.

We denote by $\mathcal{T}_n$ the CW complex $\mathbb{R}^n/(2\mathbb{Z})^n \cong (\mathbb{R}/2\mathbb{Z})^n$ with the product CW structure. Notice that the natural action of $(2\mathbb{Z})^n$ on $\mathbb{R}^n$ preserves the standard unit cubulation of $\mathbb{R}^n$, hence induces the structure of a cubical complex on $\mathcal{T}_n$. Observe that $\mathcal{T}_n$ is homeomorphic to an $n$–dimensional torus.

In what follows, it will be convenient to parametrize points of $\mathcal{T}_n$ by $n$–tuples $(x_1, \ldots, x_n)$ of numbers from $[0, 2]$ viewed up to the identification $0 \sim 2$.

The 0–skeleton of $\mathcal{T}_n$ is naturally identified with the set $H_n$ of all $n$-tuples of $\{0, 1\}$ introduced before.

The 1–cells of $\mathcal{T}_n$ are formed by fixing an edge $e_0$ or $e_1$ in some factor of $\mathcal{T}_n = C_2 \times C_2 \times \cdots \times C_2$, say, in position $i$, and taking product with vertices 0 or 1 in all other positions. (So if two 0–cells of $\mathcal{T}_0$ differ in only one coordinate, then there is a unique directed edge in $\mathcal{T}_n^{(1)}$ from the first 0–cell to the second one and a unique directed edge from the second one to the first one.) Thus, a typical 1–cell in $\mathcal{T}_n$ can be identified with a product of the form

$$v_1 \times v_2 \times \cdots \times v_{i-1} \times e_\alpha \times v_{i+1} \times \cdots \times v_n,$$

where each $v_j \in \{0, 1\}$ and $\alpha = 0$ or 1.

Similarly, an arbitrary 2–cell of $\mathcal{T}_n$ is a product

$$v_1 \times \cdots \times v_{i-1} \times e_\alpha \times v_{i+1} \times \cdots \times v_{j-1} \times e_\beta \times v_{j+1} \times \cdots \times v_n$$

for some choice of $1 \leqslant i, j \leqslant n$ $(i \neq j)$, with each $v_k \in \{0, 1\}$, and $\alpha, \beta \in \{0, 1\}$.

Let $K_{m,m}$ be the presentation 2–complex for $G_{m,m}$. Recall that it consists of one 0–cell, $(2m+1)$ 1–cells corresponding to the generators $a_1, \ldots, a_{2m+1}$ of $G_{m,m}$ and $2m$ 2–cells corresponding to the relations of $G_{m,m}$.

Consider the right action $\pi \colon G^\circ_{m,m} \to \mathrm{Sym}(H_{2m+1})$ constructed in the previous section and denote

$$S = \{g \in G_{m,m} \mid \pi(g)(0, 0, \ldots, 0) = (0, 0, \ldots, 0)\}$$

the stabilizer of the point $(0, 0, \dots, 0)$ in $G_{m,m}$. Subgroup $S$ defines a finite covering $\widehat{K}_m \to K_{m,m}$ whose properties we now describe.

**Proposition 6.2.** *The covering space $\widehat{K}_m$ cellularly embeds into the 2–skeleton of $\mathcal{T}_{2m+1}$.*

*Proof.* The 0–cells of $\widehat{K}_m$ are in one-to-one correspondence with the right cosets $S\backslash G_{m,m}$. Since the action of $G_{m,m}$ is transitive on $H_{2m+1}$ by Proposition 6.1(1), $\widehat{K}_m^{(0)}$ consists of $|G_{m,m} : S| = 2^{2m+1}$ vertices which we can identify with $\mathcal{T}_{2m+1}^{(0)}$, the 0–skeleton of $\mathcal{T}_{2m+1}$, which was earlier identified with the set $H_{2m+1}$ of all $(2m + 1)$–tuples consisting of $\{0, 1\}$.

The 1–cells of $\widehat{K}_m$ are in one-to-one correspondence with pairs of right cosets $(Sg, Sga_i)$ where $a_i$, $i = 1, \dots, 2m + 1$ runs through all the generators of $G_{m,m}$. Proposition 6.1(3) guarantees that each such 1–cell is not a loop, and it actually belongs to the 1–skeleton of the $(2m + 1)$–torus $\mathcal{T}_{2m+1}$.

The 2–cells of $\widehat{K}_m$ are lifts of the 2–cells in $K_{m,m}$. Each such 2–cell is uniquely determined by the base vertex (a lift of the base vertex of $K_{m,m}$) and by the fixed cyclic order of the relator word of $G_{m,m}$, which defines the attaching map of a 2–cell in $K_{m,m}$. Indeed, since $\pi\colon G^{\circ}_{m,m} \to \mathrm{Sym}(H_{2m+1})$ is a homomorphism, every relator word $w$ of $G_{m,m}$ acts as the identical permutation. There are two types of relators in the presentation for $G_{m,m}$:

$$a_i a_{i+1} a_i^{-1} a_{i+1}^{-1} = 1 \quad \text{for } i = 1, \dots, m, \quad \text{and}$$
$$a_{m+j+1}^{-1} a_j a_{m+j+1} a_{m+j}^{-1} = 1 \quad \text{for } j = 1, \dots, m,$$

each of which has length 4. Thus they define length 4 loops in $\widehat{K}_m$, based at every vertex, each of such loops has to be filled with a 2–cell because these loops must be nullhomotopic when projected to $K_{m,m}$. Thus, each 2–cell of $\widehat{K}_m$ can be given by a word $a_i a_j a_l^{-1} a_j^{-1}$ for some values $i \neq j$, $j \neq l$, see Figure 6.3.

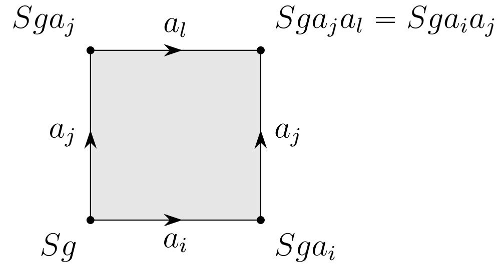


Figure 6.3: A typical 2–cell in $\widehat{K}_m$.

If we identify cosets $S\backslash G_{m,m}$ with $\mathcal{T}^{(0)}_{2m+1} \equiv H_{2m+1}$, Proposition 6.1(3) shows that for any $i = 1, \ldots, 2m+1$, every edge $(Sg, Sga_i)$ changes only one coordinate of the $(2m+1)$–tuple of $\{0, 1\}$ representing vertex $Sg$, therefore it maps to a suitable 1–cell of $\mathcal{T}_{2m+1}$.

Let's show that each 2–cell of the above form at a vertex $Sg \in \widehat{K}^{(0)}_m$ naturally embeds into $\mathcal{T}^{(2)}_{2m+1}$ under the embedding induced by the embeddings of $\widehat{K}^{(0)}_m$ and $\widehat{K}^{(1)}_m$ to $\mathcal{T}^{(1)}_{2m+1}$ introduced above. Denote $p, q, r, s$ the positions in $\{1, \ldots, 2m+1\}$ in which the endpoints of the following edges differ: $(Sg, Sga_i)$, $(Sga_i, Sga_ia_j)$, $(Sg, Sga_j)$, $(Sga_j, Sga_ja_l)$, respectively. By Proposition 6.1(4), $a_ia_j$ and $a_ja_l$ have no fixpoints on $H_{2m+1}$, hence $p \neq q$ and $r \neq s$. And since the square above is commutative, we conclude that 2–element sets $\{p, q\}$ and $\{r, s\}$ are equal. Again, by Proposition 6.1(2), $a_i$ and $a_j$ act as involutions, hence $p \neq r$, since otherwise $a_i^{-1}a_j = a_ia_j$ would have a fixpoint $Sga_i$. Therefore, $p = s$, $q = r$, and the 2–cell under consideration actually belongs to $\mathcal{T}^{(2)}_{2m+1}$ since each pair of its parallel edges changes coordinates of vertices in the same position, one in position $p = s$ and another in $q = r$. $\square$

## 6.3 Exploring Hyperplane Pathologies

We will need the following description of hyperplanes and walls in the $n$–torus $\mathcal{T}_n$.

**Lemma 6.3.** *The hyperplanes and oriented walls in $\mathcal{T}_n$ are in* 1–1 *correspondence with pairs $(i, e_\alpha)$, where $1 \leqslant i \leqslant n$ and $\alpha \in \{0,1\}$. More explicitly:*

*(1) the hyperplane corresponding to the pair $(i, e_\alpha)$ is a subset of $\mathcal{T}_n$ of one of the following two types:*

$$\left\{ (x_1, \ldots x_{i-1}, \tfrac{1}{2}, x_{i+1}, \ldots, x_n) \mid x_j \in [0,2] \right\}$$

*if $e_\alpha = e_0$, and*

$$\left\{ (x_1, \ldots x_{i-1}, \tfrac{3}{2}, x_{i+1}, \ldots, x_n) \mid x_j \in [0,2] \right\}$$

*if $e_\alpha = e_1$ (with the identification $0 \sim 2$).*

*(2) the oriented wall through a* 1*–cell*

$$v_1 \times \cdots \times v_{i-1} \times e_\alpha \times v_{i+1} \times \cdots \times v_n$$

*consists of all* 1*–cells*

$$u_1 \times \cdots \times u_{i-1} \times e_\alpha \times u_{i+1} \times \cdots \times u_n$$

*with $i$, $e_\alpha$ fixed, and $u_k$'s taking all possible values of $\{0,1\}$.*

*The oriented wall in* (2) *is dual to the corresponding hyperplane in* (1).

*Proof.* (1) Recall that the structure of a cube complex on $\mathcal{T}_n \cong \mathbb{R}^n/(2\mathbb{Z})^n$ is induced by the standard cubulation of $\mathbb{R}^n$. Hyperplanes in $\mathbb{R}^n$ are subsets of the form

$$H_{i,k} = \left\{ (x_1, \ldots, x_{i-1}, k + \tfrac{1}{2}, x_{i+1}, \ldots, x_n) \mid x_j \in \mathbb{R} \right\}, \quad 1 \leqslant i \leqslant n, \quad k \in \mathbb{Z}.$$

Modding out by the action of $(2\mathbb{Z})^n$ yields the result.

(2) Recall that the oriented wall containing a 1–cell $a$ of a cube complex $X$ is the class of all oriented 1–cells of $X$ which are connected to $a$ through a sequence of elementary parallelisms via the 2–cells of $X$. Notice that an arbitrary 1–cell of $\mathcal{T}_n$ is a product of the form: $v_1 \times \cdots \times v_{i-1} \times e_\alpha \times v_{i+1} \times \cdots \times v_n$, where each vertex $v_k \in \{0,1\}$ and $\alpha = 0$ or 1, and an arbitrary 2–cell of $\mathcal{T}_n$ is a product

$$v_1 \times \cdots \times v_{i-1} \times e_\alpha \times v_{i+1} \times \cdots \times v_{j-1} \times e_\beta \times v_{j+1} \times \cdots \times v_n$$

for some choice of $1 \leqslant i, j \leqslant n$ $(i \neq j)$ and $\alpha, \beta \in \{0,1\}$. Thus, the elementary parallelism via the above 2–cell establishes equivalence of the 1–cells

$$v_1 \times \cdots \times v_{i-1} \times e_\alpha \times v_{i+1} \times \cdots \times v_j \times \cdots \times v_n$$

and

$$v_1 \times \cdots \times v_{i-1} \times e_\alpha \times v_{i+1} \times \cdots \times (1 - v_j) \times \cdots \times v_n.$$

Since index $j$ varies independently of $i$, we conclude that any 1–cell $u_1 \times \cdots \times u_{i-1} \times e_\alpha \times u_{i+1} \times \cdots \times u_n$, $u_k \in \{0,1\}$, is contained in the parallelism class of $v_1 \times \cdots \times v_{i-1} \times e_\alpha \times v_{i+1} \times \cdots \times v_n$. $\square$

Now we show that the complex $\widehat{K}_m$ does not have three of the four pathologies in the definition of a special cube complex.

**Proposition 6.4.**

*(a) Hyperplanes of $\widehat{K}_m$ do not self-intersect.*

*(b) Hyperplanes of $\widehat{K}_m$ do not self-osculate.*

*(c) Hyperplanes of $\widehat{K}_m$ are two-sided.*

*Proof.* It is convenient to work with the oriented walls dual to hyperplanes. Since, by Proposition 6.2, $\widehat{K}_m$ is a square subcomplex of $\mathcal{T}_{2m+1}$, every wall of $\widehat{K}_m$ is a subset of some wall of $\mathcal{T}_{2m+1}$. By Lemma 6.3, the walls in $\mathcal{T}_{2m+1}$ consist of all 1–cells of the form $u_1 \times \cdots \times u_{i-1} \times e_\alpha \times u_{i+1} \times \cdots \times u_{2m+1}$ for a fixed $i$, $e_\alpha$, and arbitrary $u_k \in \{0, 1\}$.

If a hyperplane of $\widehat{K}_m$ were self-intersecting, the corresponding wall would contain edges with $e_\alpha$ in two different coordinate positions $i$ and $j$, which is impossible. This proves (a).

If a hyperplane of $\widehat{K}_m$ were self-osculating, the corresponding wall would contain a pair of edges $u_1 \times u_2 \times \cdots \times u_{i-1} \times e_\alpha \times u_{i+1} \times \cdots \times u_{2m+1}$ and $v_1 \times v_2 \times \cdots \times v_{i-1} \times e_\alpha \times v_{i+1} \times \cdots \times v_{2m+1}$ with common extremities: either their origins or their termini coincide (for direct self-osculation), or the origin of one edge coincides with the terminus of the other (for indirect self-osculation). In either case the tuples $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{2m+1})$ and $(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{2m+1})$ are equal, which means that the original 1–cells are equal and there is actually no self-osculation happening. This proves (b).

To prove (c) we observe that a hyperplane $H$ of $\widehat{K}_m$ lies in a unique hyperplane in $\mathcal{T}_{2m+1}$. In particular, by the above lemma, in the coordinate system on $\mathcal{T}_{2m+1}$, the hyperplane $H$ has the following description:

$$H = \left\{ (x_1, \ldots x_{i-1}, t, x_{i+1}, \ldots, x_{2m+1}) \right\}$$

for some $1 \leqslant i \leqslant 2m + 1$, $t = \frac{1}{2}$ or $\frac{3}{2}$, and some values from $[0, 2]$ for the rest of the variables. Since $\widehat{K}_m$ is a square complex, $H$ is the union of mid-cubes of some set of square 2–cells. Hence each point $z$ of $H$ belongs to a square 2–cell $C$

of $\widehat{K}_m$ of the form

$$C = v_1 \times \cdots \times v_{i-1} \times e_\alpha \times v_{i+1} \times \cdots \times v_{j-1} \times e_\beta \times v_{j+1} \times \cdots \times v_{2m+1}$$

for some $1 \leqslant j \leqslant 2m + 1$, where all $v_k$'s are 0 or 1. (The index $j$ may be less than or bigger than $i$.)

Suppose that $e_\alpha = e_0$ so that $t = \frac{1}{2}$. Then $z$ actually has coordinates:

$$z = (v_1, \ldots v_{i-1}, \tfrac{1}{2}, v_{i+1}, \ldots, v_{j-1}, s, v_{j+1}, \ldots, v_{2m+1})$$

where $s$ is some value from $[0, 2]$. We see that the set

$$z \times [0, 1] = \{ (v_1, \ldots v_{i-1}, t, v_{i+1}, \ldots, v_{j-1}, s, v_{j+1}, \ldots, v_{2m+1}) \mid t \in [0, 1] \}$$

also belongs to the same square $C$ above. We conclude that the product

$$H \times [0, 1] = \{ (x_1, \ldots x_{i-1}, t, x_{i+1}, \ldots, x_n) \mid t \in [0, 1] \}$$

is a union of 2–cells of $\widehat{K}_m$. This defines a combinatorial map $H \times [0, 1] \to \widehat{K}_m$ (actually, an embedding) such that $H \times \{\frac{1}{2}\}$ is identified with $H$ itself.

A similar reasoning applies if $e_\alpha = e_1$, $t = \frac{3}{2}$ (we parametrize $H \times [0, 1]$ by $t \in [1, 2]$).

This proves that every hyperplane of $\widehat{K}_m$ is two-sided. $\square$

Unfortunately, cube subcomplexes of a Cartesian product of three or more graphs can have inter-osculating hyperplanes, as the example in the Figure 6.4 shows. The 2–complex in Figure 6.4 is a subcomplex of the product of two segments of length one and a segment of length two.

However, Haglund and Wise have proved in [23, Th. 5.7] that in the case

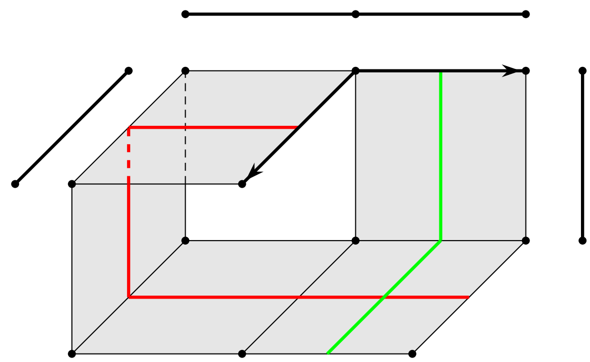

Figure 6.4: A subcomplex in a product of three graphs with inter-osculating hyperplanes.

when the square complex is a so-called VH-complex, the absence of the first three hyperplane pathologies guarantees the existence of a finite special cover.

**Definition 6.5.** (VH-complex) A simple square complex is called a *VH-complex* if its edges are divided into two disjoint classes: *vertical* and *horizontal*, such that the attaching map of each square is of the form $vhv'h'$ where $v, v'$ are vertical and $h, h'$ are horizontal edges.

**Proposition 6.6.** *For all* even *integers $m$, the complexes $K_{m,m}$ and $\widehat{K}_m$ are VH-complexes. Hence there exist a finite special cover $\overline{K}_m \to \widehat{K}_m$.*

*Proof.* From the LOG definition (see section 5.1) of groups $G_{m,m}$ we observe that, for the even integers $m$, the odd-indexed and the even-indexed generators form two classes $V$ and $H$ ('vertical' and 'horizontal') such that all relators of $G_{m,m}$ have the form: $v_1^h = v_2$ or $h_1^v = h_2$ for some $v, v_1, v_2 \in V$, $h, h_1, h_2 \in H$. This implies that the complex $K_{m,m}$ is a VH-complex.

The complex $\widehat{K}_m$, being a finite cover of $K_{m,m}$, inherits the structure of a VH-complex from $K_{m,m}$. Indeed, the preimages of the vertical and horizontal edges in $K_{m,m}$ under the covering map $p\colon \widehat{K}_m \to K_{m,m}$ form two disjoint classes $\widehat{V} = p^{-1}(V)$ and $\widehat{H} = p^{-1}(H)$, and all edges of $\widehat{K}_m$ are contained in $\widehat{V} \sqcup \widehat{H}$. The link of every vertex of $K_{m,m}$ is a bipartite graph corresponding to parts $V$

and $H$, and links of vertices are mapped isomorphically under covering maps. Thus all links of vertices in $\widehat{K}_m$ are bipartite with respect to parts $\widehat{V}$ and $\widehat{H}$. Therefore all 2–cells of $\widehat{K}_m$ have boundaries of the form $v_1 h_1 v_2 h_2$ with $v_1, v_2 \in \widehat{V}$, $h_1, h_2 \in \widehat{H}$. By Proposition 6.4, hyperplanes of $\widehat{K}_m$ have no self-intersections and no self-osculations. Hence, by Theorem 5.7 in [23], there exist a special cube complex $\overline{K}_m$ and a finite cover $\overline{K}_m \to \widehat{K}_m$. $\square$

# Chapter 7

# Proof of the Main Theorems

Now we are ready to prove our main results.

**Theorem A.** *For each positive* even *integer $m$ there exist virtually special free-by-cyclic groups $G_{m,m} \cong F_{2m} \rtimes_\phi \mathbb{Z}$ with the monodromy growth function $\mathrm{gr}_\phi(n) \sim n^m$ and $G_{m,m-1} \cong F_{2m-1} \rtimes_{\phi'} \mathbb{Z}$ with the monodromy growth function $\mathrm{gr}_{\phi'}(n) \sim n^{m-1}$.*

*Proof.* We have seen in Proposition 6.6 that for each even integer $m > 0$, there exist a special cover $\overline{K}_m \to K_{m,m}$ for the presentation complex $K_{m,m}$ of the group $G_{m,m} \cong F_{2m} \rtimes_\phi \mathbb{Z}$. In Propositions 8.1 and 8.12 of chapter 8 we show that the growth function for $\phi = \phi_{m,m}$ is $\sim n^m$. This proves the first part of Theorem A.

For the second part, recall that $G_{m,m-1} = F_{2m-1} \rtimes_{\phi'} \mathbb{Z}$, where $\phi' = \phi_{m,m-1}$ is the restriction of $\phi$ on the free subgroup on the first $2m-1$ generators. By construction, the presentation complex $K_{m,m-1}$ for $G_{m,m-1}$ is a subcomplex of $K_{m,m}$, and is actually obtained from $K_{m,m}$ by deleting the loop corresponding to the last generator $a_{2m+1}$ and also the single open 2–cell adjacent to $a_{2m+1}$ (i.e. which have $a_{2m+1}$ as one of their sides).

Let $\bar{p}\colon \overline{K}_m \to K_{m,m}$ be the special cover of $K_{m,m}$ from Proposition 6.6. Consider a square subcomplex $\overline{K}'_m \subset \overline{K}_m$ which is obtained by:

1. deleting all 1–cells of $\overline{K}_m$ which map under $\bar{p}$ onto the loop labeled $a_{2m+1}$ in $K_{m,m}$;

2. deleting all open 2–cells of $\overline{K}_m$ which have 1–cells from (1) as one of their sides;

3. taking a connected component of the resulting complex.

We claim that $\bar{p}\colon \overline{K}'_m \to K_{m,m-1}$ is a finite special cover of $K_{m,m-1}$.

Indeed, by construction, $\bar{p}(\overline{K}'_m)$ lies in $K_{m,m-1}$. The hyperplanes of $\overline{K}'_m$ are two-sided, do not self-intersect and do not self-osculate, since they are subcomplexes of the corresponding hyperplanes in the special complex $\overline{K}_m$.

To see that the complex $\overline{K}'_m$ has no inter-osculating hyperplanes, observe that in steps (1), (2) above we deleted only the hyperplanes which are dual to the 1–cells corresponding to the last generator $a_{2m+1}$. This doesn't change the absence of inter-osculation of the remaining hyperplanes of $\overline{K}_m$. Therefore, the hyperplanes in $\overline{K}'_m$ do not inter-osculate either, and $\overline{K}'_m$ is special.

Again, that the growth of $\phi'$ is $\sim n^{m-1}$ is shown in Propositions 8.1 and 8.12 in chapter 8, since $\phi' = \phi_{m,m-1}$. □

**Corollary A.** *For each positive integer $k$ there exist a right-angled Artin group containing a free-by-cyclic subgroup whose monodromy automorphism has growth function $\sim n^k$.*

*Proof.* In Theorem A we have proved that for any positive integer $k$ (where $k = m$ or $m-1$ for arbitrary even $m$) there exists a free-by-cyclic group $G = F \rtimes_\psi \mathbb{Z}$ with $\mathrm{gr}_\psi(n) \sim n^k$, such that some finite index subgroup $H \leqslant G$ is isomorphic to a fundamental group of a special cube complex. By Haglund and Wise's celebrated result (see Corollary 4.16), there exists a right-angled Artin group $A(\Delta)$ with $H$ isomorphic to a subgroup of $A(\Delta)$.

Let's prove that $H$ is free-by-cyclic itself. Indeed, we have a commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & F \arrow[r] & F \rtimes_\psi \mathbb{Z} \arrow[r, "\pi"] & \mathbb{Z} \arrow[r] & 1 \\
1 \arrow[r] & N \arrow[r] \arrow[u, hook] & H \arrow[r] \arrow[u, hook] & \ell\mathbb{Z} \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

Here $N = H \cap F$ and $\ell\mathbb{Z}$ is the image of $H$ under $\pi$. Since $H$ has finite index in $G$, $\ell \neq 0$. Hence the subgroup $N$ is invariant under $\psi^\ell$ and is a free group. Since $F \lhd G$, $FH$ is a subgroup of $G$, and

$$|F : N| = |F : H \cap F| = |FH : H| \leqslant |G : H| < \infty.$$

Therefore, $N$ is a finitely generated free group, and $H \cong N \rtimes_{\psi^\ell} \mathbb{Z}$, a free-by-cyclic group.

Parts (ii) and (iii) of Proposition 2.4 tell us now that

$$\mathrm{gr}_{\psi^\ell|_N}(n) \sim \mathrm{gr}_\psi(n) \sim n^k. \qquad \square$$

**Theorem B.** *For each positive integer $k$ there exists a $3$–dimensional right-angled Artin group $A$ which contains a finitely presented subgroup $\Gamma$ with Dehn function $\simeq n^k$.*

*Proof.* In Theorem A we proved that, for all even integers $m$, the free-by-cyclic group $G_{m,m} = F_{2m} \rtimes_\phi \mathbb{Z}$ (resp. $G_{m,m-1} = F_{2m-1} \rtimes_{\phi'} \mathbb{Z}$) has the following properties: $\mathrm{gr}_\phi(n) \sim n^m$ (resp. $\mathrm{gr}_{\phi'}(n) \sim n^{m-1}$), and it contains a finite index subgroup $H$ which embeds into a right-angled Artin group. In Corollary A we showed that $H$ is itself free-by-cyclic: $H \cong N \rtimes_{\phi^\ell} \mathbb{Z}$ (resp. $H \cong N \rtimes_{\phi'^\ell} \mathbb{Z}$) for some finite index subgroup $N \leqslant F_{2m}$ (resp. $N \leqslant F_{2m-1}$) with the monodromy automorphism being $\phi^\ell$ (resp. $(\phi')^\ell$) for some $\ell > 0$.

We claim that *the double of $H$, $\Gamma(H) = H \ast_F H$, has Dehn function $\delta_\Gamma(n) \simeq$ equivalent to $\mathrm{gr}_\phi(n) \cdot n^2$ (resp. $\mathrm{gr}_{\phi'}(n) \cdot n^2$), and itself embeds into a RAAG.*

We now prove this claim for the case of subgroup $H \leqslant G_{m,m}$, the monodromy automorphism $\phi^\ell$, and $k = m$, and notice that the case of $H \leqslant G_{m,m-1}$, the monodromy automorphism ${\phi'}^\ell$, and $k = m - 1$, is proved in a similar fashion.

The upper bound: $\delta_\Gamma(n) \leqq n^{k+2}$ is established as follows. By Propositions 8.1, and 8.12, $\mathrm{gr}_\phi(n) \sim n^k$ and $\mathrm{gr}_{\phi^{-1}}(n) \sim n^k$. Proposition 2.4(ii) implies that $\mathrm{gr}_{\phi^\ell}(n) \sim n^k$ and $\mathrm{gr}_{(\phi^\ell)^{-1}}(n) = \mathrm{gr}_{(\phi^{-1})^\ell}(n) \sim n^k$, and so the upper bound follows from Proposition 3.4.

The lower bound: $n \cdot \max_{\|b\| \leqslant n,\, b \in N} \|\phi^{\ell n}(b)\| \leqq \delta_\Gamma(n)$ was given in Proposition 3.3. If we show that $\max_{\|b\| \leqslant n,\, b \in N} \|\phi^{\ell n}(b)\| \geqq n^{k+1}$, it will follow, in view of the above, that $\delta_\Gamma(n) \simeq n^{k+2}$.

We prove in chapter 8 that in the group $G_{m,m} = F_{2m} \rtimes_\phi \mathbb{Z}$, containing $H$, the maximum in the definition of the growth functions $\mathrm{gr}_\phi(n)$ and $\mathrm{gr}_{\phi^{ab}}(n)$ is achieved at the generator $B_m$ (see Corollary 8.14): $\left\|\phi^n(B_m)\right\| \sim \left|(\phi^{ab}_{m,k})^n(\bar{B}_k)\right|_1 \sim n^m = n^k$. (Here the bar over an element of $F_{2m}$ denotes its image in the abelianization of $F_{2m}$.)

Since the subgroup $N$ is of finite index in $F_{2m}$, there exists an integer $p > 0$ such that $B_m^p \in N$. Then we have:

$$\max_{\substack{\|b\| \leqslant n \\ b \in N}} \left\|\phi^{\ell n}(b)\right\| \geqslant \left\|\phi^{\ell n}(B_m^{pn})\right\| \geqslant pn \cdot \left|(\phi^{ab})^{\ell n}(\bar{B}_m)\right|_1 \geq pn(\ell n)^m \sim n^{m+1}.$$

Since $k = m$, this proves that $\delta_\Gamma(n) \simeq n^{k+2}$.

To prove that $\Gamma$ embeds into a RAAG, consider a homomorphism $\mu\colon \Gamma \to H \times F(u,v)$, where $F(u,v)$ is a free group of rank 2 on free generators $u$, $v$ (a modification of the Bieri embedding), which is described as follows. Recall that $H \cong N \rtimes_{\phi^\ell} \mathbb{Z}$, where $N$ is a free group of finite rank. Denote for brevity $\psi = \phi^\ell$ and take three copies of $\mathbb{Z}$ with generators $s$, $t$ and $\tau$. Then $\Gamma = \big(N \rtimes_\psi \langle s \rangle\big) *_N \big(N \rtimes_\psi \langle t \rangle\big)$ and $H \times F(u,v) = \big(N \rtimes_\psi \langle \tau \rangle\big) \times F(u,v)$. Define $\mu$ on $\Gamma$ as follows:

$$\mu|_N = \mathrm{id}_N, \qquad s \mapsto \tau u^{-\ell}, \qquad t \mapsto \tau v^{-\ell}. \tag{‡}$$

We check at once that $\mu$ is a homomorphism, and it is easily proved using the normal forms of elements in free amalgamated products, that $\mu$ is injective.

Thus, if we denote the right-angled Artin group containing $H$ as $A(\Delta)$, for some graph $\Delta$, then $\Gamma \subset H \times F_2$ is a subgroup of $A(\Delta) \times F_2$, which is itself a RAAG (corresponding to the graph join of $\Delta$ and the empty graph on two vertices).

To prove that $A(\Delta) \times F_2$ has 3–dimensional Salvetti complex, notice that the graph $\Delta$ is the intersection graph of hyperplanes of $\overline{K}_m$. The latter being a $VH$-complex implies that $\Delta$ is bipartite, hence triangle-free. This makes the Salvetti complex for $A(\Delta)$ 2–dimensional, and the one for $A(\Delta) \times F_2$ 3–dimensional.

$\square$

**Corollary 7.1.** *The group $\Gamma$ from the above proof lies in the Bestvina–Brady kernel of the natural map $\varepsilon \colon A(\Delta) \times F_2 \to \mathbb{Z}$ that sends each generator to* 1.

*Proof.* Indeed, denote $\iota \colon H \hookrightarrow A(\Delta)$ the embedding of fundamental groups of special square complexes, existing due to Corollary 4.16. We have:

$$\Gamma = H *_N H \xhookrightarrow{\mu} H \times F_2(u, v) \xhookrightarrow{\iota \times \mathrm{id}} A(\Delta) \times F_2(u, v) \xrightarrow{\varepsilon} \mathbb{Z}$$

and we need to show that $\varepsilon \circ (\iota \times \mathrm{id}) \circ \mu(\Gamma) = 0$. Recall that $H = N \rtimes \mathbb{Z}$ is the fundamental group of a square complex $\overline{K}_m$ which covers $K_{m,m}$, see the proof of theorem A. (For simplicity, we consider the cover of $K_{m,m}$ only. The situation with $K_{m,m-1}$ is completely analogous.) The Morse function on $K_{m,m}$ lifts to that of $\overline{K}_m$, denote it $f$. Arguing exactly as in the proof of Proposition 5.4, we conclude that the preimage $f^{-1}(0)$ is the union of the diagonals of some square 2–cells of $\overline{K}_m$. Now recall the Haglund–Wise's construction of the RAAG $A(\Delta)$ into which the fundamental group of the special cube complex $\overline{K}_m$ embeds [23, Th. 4.2]: generators $\{\alpha_i\}$ of $A(\Delta)$ correspond to hyperplanes of $\overline{K}_m$, with any two of them commuting if and only if the respective hyperplanes intersect. Taking inverses of generators if necessary, we may assume that each generator of $A(\Delta)$ corresponds to an oriented wall consisting of 1–cells oriented positively with respect to the Morse function $f$. It follows that (up to taking inverse) each loop in $f^{-1}(0)$ embeds (via $\iota$) into $A(\Delta)$ as a product of words of the form $\alpha_i\alpha_j^{-1}$ (corresponding to diagonals of certain 2–cells). Thus if $N = \pi_1(f^{-1}(0))$ then $\varepsilon(\iota(N)) = 0$. Also, each 1–cell dual to a hyperplane of $\overline{K}_m$ maps via $f$ to $S^1$ as a degree 1 map. However, the generator $\tau$ of the $\mathbb{Z}$ factor of $H$ corresponds to the $\ell$-th power of automorphism $\phi$, hence we may assume that $\tau$ maps (via $\iota$) to some $\alpha_j^\ell$ in $A(\Delta)$. Hence, $\varepsilon(\iota(\tau)) = \ell$. Now looking at formulas ‡ for $\mu$, we see that $(\varepsilon \circ (\iota \times \mathrm{id}) \circ \mu)(s) = (\varepsilon \circ (\iota \times \mathrm{id}) \circ \mu)(t) = 0$, which finishes the proof.

□

**Remark 7.2.** Following the proof of Proposition 3.3 given in [13, Lemma 1.5] (see also [11, Proposition 7.2.2]), we can exhibit an explicit sequence of words $w_n = [(st^{-1})^n, t^{\ell n} B_k^{pn} t^{-\ell n}]$ (where $k = m$ or $m-1$, as above) which realize the lower bound for the Dehn function $\delta_\Gamma(n)$. To understand what van Kampen diagrams for these words look like, the reader is referred to the proof of [13, Lemma 1.5]. In the notation of [13], $\beta = B_k^{pn}$, $t_1 = t$, $t_2 = s$.

# Chapter 8

# Growth of $\phi$ and $\phi^{-1}$

In Proposition 5.5 we have shown that, for all $1 \leqslant k \leqslant m$, $G_{m,k} = F_{m+k} \rtimes_{\phi_{m,k}} \mathbb{Z}$, where $F_{m+k}$ is a free group with generators $A_1, \ldots, A_m, B_1, \ldots, B_k$. For convenience, in what follows we adopt the notation:

$$\phi = \phi_{m,m}$$

and use the fact that $\phi_{m,k}$ is the restriction of $\phi$ on the first $m+k$ generators. The goal of this chapter is to prove that $\phi_{m,k}$ and $(\phi_{m,k})^{-1}$ have growth $\sim n^k$. In particular, $\mathrm{gr}_\phi(n) \sim \mathrm{gr}_{\phi^{-1}}(n) \sim n^m$.

Throughout this chapter, $\|.\|$ will denote the word length in $F_{2m}$ with respect to the system of free generators $\{A_i, B_j\}$.

Recall (see Proposition 5.5) that the automorphism $\phi$ is given by the formulas (where the overbar denotes the inverse):

$$\begin{aligned}
\phi = \phi_{m,m}\colon \quad & A_1 \longmapsto A_1 \qquad (1)\\
& A_2 \longmapsto A_1\,(A_2)\,\overline{A}_1\\
& A_3 \longmapsto A_1 A_2\,(A_3)\,\overline{A}_2 \overline{A}_1\\
& \ldots\\
& A_m \longmapsto A_1 A_2 \ldots A_{m-1}\,(A_m)\,\overline{A}_{m-1} \ldots \overline{A}_1\\
& B_1 \longmapsto A_1 A_2 \ldots A_m\,(B_1)
\end{aligned}$$

$$B_2 \longmapsto A_1 A_2 \dots A_m \, (B_1 B_2) \, \overline{A}_1$$

$$B_3 \longmapsto A_1 A_2 \dots A_m \, (B_1 B_2 B_3) \, \overline{A}_2 \overline{A}_1$$

$$\dots$$

$$B_m \longmapsto A_1 A_2 \dots A_m \, (B_1 B_2 \dots B_m) \, \overline{A}_{m-1} \overline{A}_{m-2} \dots \overline{A}_2 \overline{A}_1.$$

## 8.1 Upper Bounds for the Growth of $\phi$, $\phi^{-1}$

**Proposition 8.1.** *For the automorphism $\phi_{m,k}$ we have:*

$$\mathrm{gr}_{\phi_{m,k}}(n) \preceq n^k \quad \textit{and} \quad \mathrm{gr}_{(\phi_{m,k})^{-1}}(n) \preceq n^k.$$

We will need few basic lemmas.

**Lemma 8.2.** *For $i = 1, \dots, m$,*

$$\phi^n(A_i) = A_1^n A_2^n \dots A_{i-1}^n \cdot A_i \cdot \overline{A}_{i-1}^n \dots \overline{A}_2^n \overline{A}_1^n.$$

*Proof.* We prove the statement by induction on $n$, observing that it is true for $n = 0, 1$:

$$\begin{aligned}
\phi^{n+1}(A_i) &= \phi(\phi^n(A_i)) = \phi(A_1^n A_2^n \dots A_{i-1}^n) \cdot \phi(A_i) \cdot \phi(\overline{A}_{i-1}^n \dots \overline{A}_2^n \overline{A}_1^n) \\
&= \phi(A_1^n)\phi(A_2^n)\phi(A_3^n) \dots \phi(A_{i-1}^n) \cdot \phi(A_i) \cdot \phi(\overline{A}_{i-1}^n) \dots \phi(\overline{A}_3^n)\phi(\overline{A}_2^n)\phi(\overline{A}_1^n) \\
&= (A_1^n)(A_1 A_2^n \overline{A}_1)(A_1 A_2 A_3^n \overline{A}_2 \overline{A}_1) \dots (A_1 \dots A_{i-2} A_{i-1}^n \overline{A}_{i-2} \dots \overline{A}_1) \\
&\times (A_1 \dots A_{i-1} A_i \overline{A}_{i-1} \dots \overline{A}_1) \times (A_1 \dots A_{i-2} \overline{A}_{i-1}^n \overline{A}_{i-2} \dots \overline{A}_1) \dots (A_1 A_2 \overline{A}_3^n \overline{A}_2 \overline{A}_1) \\
&\times (A_1 \overline{A}_2^n \overline{A}_1)(\overline{A}_1^n) = A_1^{n+1} A_2^{n+1} \dots A_{i-1}^{n+1} \cdot A_i \cdot \overline{A}_{i-1}^{n+1} \dots \overline{A}_2^{n+1} \overline{A}_1^{n+1}.
\end{aligned}$$

$\square$

**Corollary 8.3.** *For $i = 1, \dots, m$,*

$$\|\phi^n(A_i)\| = 2(i-1)n + 1. \qquad \square$$

**Lemma 8.4.** $\phi^n(B_1) = A_1^n A_2^n \dots A_m^n \cdot B_1$.

*Proof.* The statement is true for $n = 0, 1$. By induction,

$$\begin{aligned}
\phi^{n+1}(B_1) = \phi(\phi^n(B_1)) &= \phi(A_1^n A_2^n \dots A_m^n) \cdot \phi(B_1) \\
&= (A_1^n)(A_1 A_2^n \overline{A}_1)(A_1 A_2 A_3^n \overline{A}_2 \overline{A}_1) \dots (A_1 \dots A_{m-1} A_m^n \overline{A}_{m-1} \dots \overline{A}_1) \\
&\quad \times (A_1 A_2 \dots A_m \cdot B_1) = A_1^{n+1} A_2^{n+1} \dots A_m^{n+1} \cdot B_1. \qquad \square
\end{aligned}$$

**Corollary 8.5.** $\|\phi^n(B_1)\| = mn + 1$. $\square$

**Lemma 8.6.** $\phi^n(A_1 A_2 \dots A_m) = A_1^{n+1} A_2^{n+1} \dots A_{m-1}^{n+1} \cdot A_m \cdot \overline{A}_{m-1}^n \dots \overline{A}_2^n \overline{A}_1^n$.

*Proof.* We do induction on $n$, the case $n = 0$ being evident:

$$\begin{aligned}
&\phi^{n+1}(A_1 A_2 \dots A_m) = \phi(\phi^n(A_1 A_2 \dots A_m)) = \phi(A_1^{n+1})\phi(A_2^{n+1}) \dots \phi(A_{m-1}^{n+1}) \\
&\times \phi(A_m) \cdot \phi(\overline{A}_{m-1}^n) \dots \phi(\overline{A}_2^n)\phi(\overline{A}_1^n) = (A_1^{n+1})(A_1 A_2^{n+1} \overline{A}_1)(A_1 A_2 A_3^{n+1} \overline{A}_2 \overline{A}_1) \dots \\
&\quad \times (A_1 \dots A_{m-2} A_{m-1}^{n+1} \overline{A}_{m-2} \dots \overline{A}_1) \cdot (A_1 \dots A_{m-1} A_m \overline{A}_{m-1} \dots \overline{A}_1) \\
&\quad \times (A_1 \dots A_{m-2} \overline{A}_{m-1}^n \overline{A}_{m-2} \dots \overline{A}_1) \dots (A_1 A_2 \overline{A}_3^n \overline{A}_2 \overline{A}_1) \cdot (A_1 \overline{A}_2^n \overline{A}_1)(\overline{A}_1^n) \\
&\quad = A_1^{n+2} A_2^{n+2} \dots A_{m-1}^{n+2} \cdot A_m \cdot \overline{A}_{m-1}^{n+1} \dots \overline{A}_2^{n+1} \overline{A}_1^{n+1}. \qquad \square
\end{aligned}$$

**Lemma 8.7.** $\|\phi^n(B_2)\| = \frac{m}{2}n^2 + (\frac{m}{2} + 2)n + 1$.

*Proof.* One observes that

$$\phi(B_2) = \phi(B_1) \cdot B_2 \cdot \overline{A}_1.$$

This gives by induction in view of Lemma 8.4:

$$\begin{aligned}\phi^n(B_2) &= \phi^n(B_1)\phi^{n-1}(B_1)\ldots\phi(B_1)\cdot B_1\cdot \overline{A}_1^n\\ &= (A_1^nA_2^n\ldots A_m^nB_1)\cdot(A_1^{n-1}A_2^{n-1}\ldots A_m^{n-1}B_1)\ldots(A_1A_2\ldots A_mB_1)\cdot B_1\cdot\overline{A}_1^n.\end{aligned}$$

Hence,

$$\begin{aligned}\|\phi^n(B_2)\| &= [mn+1]+[m(n-1)+1]+\ldots+[m+1]+1+n\\ &= m\tfrac{n(n+1)}{2}+2n+1=\tfrac{m}{2}n^2+\left(\tfrac{m}{2}+2\right)n+1. \qquad \square\end{aligned}$$

**Claim 8.8.** *For $k=1,\ldots,m$,*

$$\|\phi^n(B_k)\|\le n^k.$$

*Proof.* From the formulas (1) we get for all $k\geqslant 2$,

$$\phi(B_{k+1})=\phi(B_k)\cdot(A_1\ldots A_{k-1})\cdot B_{k+1}\cdot\phi(\overline{A}_k\ldots\overline{A}_1).$$

Therefore,

$$\phi^n(B_{k+1})=\phi^n(B_k)\cdot\phi^{n-1}(A_1\ldots A_{k-1})\cdot\phi^{n-1}(B_{k+1})\cdot\phi^{n-1}(A_1\ldots A_k)^{-1}. \tag{2}$$

Lemma 8.6 gives:

$$\begin{aligned}\|\phi^{n-1}(A_1\ldots A_{k-1})\| &= 2(k-2)(n-1)+(k-1),\\ \|\phi^{n-1}(A_1\ldots A_k)^{-1}\| &= 2(k-1)(n-1)+k,\end{aligned}$$

so that the total length of $\phi^n(B_{k+1})$ is bounded above by

$$\|\phi^n(B_k)\| + \|\phi^{n-1}(B_{k+1})\| + [(4k-6)n - (2k-5)].$$

Now if we denote

$$f(k,n) := \|\phi^n(B_k)\|,$$

we will have

- $f(k,0) = 1$;
- $f(1,n) = mn + 1$, by Corollary 8.5;
- $f(2,n) = \frac{m}{2}n^2 + (\frac{m}{2} + 2) + 1$, by Lemma 8.7;

and for $k \geqslant 2$,

$$f(k+1,n) \leqslant f(k,n) + f(k+1,n-1) + [(4k-6)n - (2k-5)].$$

We have an inequality here (instead of an equality) because in the formula (2) there can be some cancellations. Let's define another function $g(k,n)$ as follows:

- $g(k,0) = 1$;
- $g(1,n) = f(1,n) = mn + 1$,
- $g(2,n) = f(2,n) = \frac{m}{2}n^2 + (\frac{m}{2} + 2) + 1$,
- $g(k+1,n) = g(k,n) + g(k+1,n-1) + [(4k-6)n - (2k-5)]$, for $k \geqslant 2$.

Obviously, $g$ is well-defined in a recurrent fashion. An easy induction shows that

$$f(k,n) \leqslant g(k,n) \quad \text{for all } k \geqslant 1,\ n \geqslant 0,$$

so that $g(k,n)$ gives an upper bound for the growth of $\|\phi^n(B_k)\|$.

To estimate the order of growth of $g(k,n)$, let's look at finite differences in $n$:

$$g(k+1,n) - g(k+1,n-1) = g(k,n) + [(4k-6)n - (2k-5)]$$

so if we assume by induction that $g(k,n)$ is a polynomial in $n$ of degree $k$, then we conclude that $g(k+1,n)$ is a polynomial of degree $k+1$ in $n$.

Since this assumption is true for $k=1$ and 2, this proves Claim 8.8. $\square$

Now we establish a similar bound for $\phi^{-1}$. One could use the train-track machinery (along the lines of [27, Th. 0.4]) to prove that the growth of $\phi^{-1}$ is the same as the growth of $\phi$, when it is polynomial, but we give here a simple direct proof.

One easily checks that the inverse automorphism $\phi^{-1}$ acts as follows:

$$\begin{aligned}
\phi^{-1}\colon \quad & A_1 \longmapsto A_1 \\
& A_2 \longmapsto \overline{A}_1\,(A_2)\,A_1 \\
& A_3 \longmapsto \overline{A}_1\overline{A}_2\,(A_3)\,A_2A_1 \\
& \dots \\
& A_m \longmapsto \overline{A}_1\overline{A}_2\dots\overline{A}_{m-1}\,(A_m)\,A_{m-1}\dots A_2A_1 \\
& \\
& B_1 \longmapsto \overline{A}_1\overline{A}_2\dots\overline{A}_{m-1}\overline{A}_m\cdot B_1 \\
& B_2 \longmapsto (\overline{B}_1B_2)A_1 \\
& B_3 \longmapsto \overline{A}_1\,(\overline{B}_2B_3)\,A_2A_1 \\
& \dots \\
& B_m \longmapsto \overline{A}_1\overline{A}_2\dots\overline{A}_{m-2}\,(\overline{B}_{m-1}B_m)\,A_{m-1}\dots A_2A_1.
\end{aligned} \tag{3}$$

**Lemma 8.9.** *For $i = 1, \dots, m$,*

$$\|\phi^{-n}(A_i)\| = \|\phi^n(A_i)\| = 2(i-1)n + 1.$$

*Proof.* Define an automorphism $\iota\colon H \longrightarrow H$ of the subgroup $H = \langle A_1, \dots, A_m \rangle$ given by $\iota\colon A_j \mapsto \overline{A}_j$, $j = 1, \dots, m$. One easily checks that

$$\phi^{-1}|_H = \iota \circ (\phi|_H) \circ \iota^{-1},$$

therefore $\|\phi^{-n}(A_i)\| = \|\phi^n(A_i)\|$ and the result follows from Corollary 8.3. $\square$

**Claim 8.10.** *For $k = 1, \dots, m$,*

$$\|\phi^{-n}(B_k)\| \preceq n^k.$$

*Proof.* Denote for any $i \geqslant 0$, $k \geqslant 2$:

$$\begin{aligned} T_{1,i} &:= B_1, \\ T_{k,i} &:= B_k \cdot A_{k-1}^i A_{k-2}^i \dots A_2^i A_1^i, \\ S_i &:= \overline{A}_1^i \overline{A}_2^i \dots \overline{A}_{m-1}^i \overline{A}_m^i. \end{aligned}$$

In this notation, the action of $\phi^{-1}$ can be written as follows:

$$\begin{aligned} \phi^{-1}(T_{k,i}) &= \overline{T_{k-1,1}} \cdot T_{k,i+2}, \\ \phi^{-1}(S_i \cdot T_{1,1}) &= S_{i+1} \cdot T_{1,1}. \end{aligned} \tag{4}$$

The first relation is obvious, and the second one follows by easy induction.

**Lemma 8.11.** $\|\phi^{-n}(B_1)\| = mn + 1.$

*Proof.* Indeed,

$$\begin{aligned}\phi^{-n}(B_1) = \phi^{-n}(S_0 \cdot T_{1,1}) = \phi^{-(n-1)}(S_1 \cdot T_{1,1}) = \phi^{-(n-2)}(S_2 \cdot T_{1,1}) = \dots \\ = \phi^{-2}(S_{n-2} \cdot T_{1,1}) = \phi^{-1}(S_{n-1} \cdot T_{1,1}) = S_n \cdot T_{1,1}. \qquad \square\end{aligned}$$

Define for any $k \geqslant 1$, $i \geqslant 0$, $n \geqslant 0$ a function $f(k, i, n)$ as follows:

- $f(1, i, n) = mn + 1$;
- $f(k, i, n) = \|\phi^{-n}(T_{k,i})\|$, for $k \geqslant 2$.

Then relations (4) imply

$$f(k, i, n + 1) \leqslant f(k - 1, 1, n) + f(k, i + 1, n),$$

where we have an inequality (but not an equality) because of possible cancellations in the reduced expression for $\phi^{-n}(T_{k,i})$.

To obtain an upper bound on $f(k, i, n)$ we introduce a function $g(k, i, n)$ defined recurrently as follows:

- $g(k, i, 0) = \|T_{k,i}\| = (k - 1)i + 1$, for all $k \geqslant 1$, $i \geqslant 0$;
- $g(1, i, n) = \|\phi^{-n}(B_1)\| = mn + 1$, for all $i \geqslant 0$, $n \geqslant 0$;
- $g(k, i, n + 1) = g(k - 1, 1, n) + g(k, i + 1, n)$, for all $k \geqslant 2$, $i \geqslant 0$, $n \geqslant 0$.

These formulas define $g(k, i, n)$ recurrently for all values of $k \geqslant 1$, $i \geqslant 0$, $n \geqslant 0$. Indeed, one proceeds by layers numbered by $n$, with the case $n = 0$ given by the first formula, and the case of arbitrary $n$ given by the third one, which is valid for $k \geqslant 2$. The remaining case $k = 1$ is given by the second formula.

Clearly, $f(k,i,n) \leqslant g(k,i,n)$ for all $k \geqslant 1$, $i \geqslant 0$, $n \geqslant 0$, so that the function $g$ can be used to establish the upper bound for $\|\phi^{-n}(B_k)\|$:

$$\|\phi^{-n}(B_k)\| = \|\phi^{-n}(T_{k,0})\| = f(k,0,n) \leqslant g(k,0,n).$$

To estimate the growth of $g(k,i,n)$, consider the finite difference $g(k,i,n+1) - g(k,i,n)$. Applying the recurrent relation several times, we get:

$$\begin{aligned} g(k,i,n+1) &= g(k-1,1,n) + g(k,i+1,n) \\ &= g(k-1,1,n) + g(k-1,1,n-1) + g(k,i+2,n-1) \\ &\dots \\ &= [g(k-1,1,n) + g(k-1,1,n-1) + \cdots + g(k-1,1,0)] + g(k,i+n+1,0). \end{aligned}$$

Similarly,

$$g(k,i,n) = [g(k-1,1,n-1) + g(k-1,1,n-2) + \cdots + g(k-1,1,0)] + g(k,i+n,0).$$

Since by definition

$$\begin{aligned} g(k,i+n+1,0) &= (k-1)(i+n+1)+1, \\ g(k,i+n,0) &= (k-1)(i+n)+1, \end{aligned}$$

we have for all $k \geqslant 2$, $i \geqslant 0$, $n \geqslant 0$:

$$g(k,i,n+1) - g(k,i,n) = g(k-1,1,n) + (k-1). \tag{5}$$

In particular,

$$g(k,1,n+1) - g(k,1,n) = g(k-1,1,n) + (k-1).$$

If we assume by induction on $k$ that $g(k-1,1,n)$ is a polynomial function in $n$ of degree $k-1$ (which is true for $k=2$ since $g(1,i,n) = mn+1$), then we conclude at once that $g(k,1,n)$ is a polynomial function in $n$ of degree $k$.

Now the formula (5) similarly implies that $g(k,i,n)$ is a polynomial in $n$ of degree $k$.

Therefore, $\|\phi^{-n}(B_k)\| \leqslant g(k,0,n) \sim n^k$, which finishes the proof of Claim 8.10.

□

*Proof of Proposition 8.1.* According to Corollary 8.3 and Lemma 8.9, $\|\phi^{\pm n}(A_i)\| \preceq n$ for $i = 1, \ldots, m$, and according to Claims 8.8 and 8.10, $\|\phi^{\pm n}(B_k)\| \preceq n^k$, for $k = 1, \ldots, m$. Therefore $\mathrm{gr}_{\phi_{m,k}}(n) \preceq n^k$ and $\mathrm{gr}_{(\phi_{m,k})^{-1}}(n) \preceq n^k$. □

## 8.2 Lower Bounds for the Growth of $\phi$, $\phi^{-1}$

**Proposition 8.12.** *For the automorphism $\phi_{m,k}$ and $\phi_{m,k}^{-1}$, we have:*

$$\mathrm{gr}_{\phi_{m,k}}(n) \succeq n^k \quad \textit{and} \quad \mathrm{gr}_{(\phi_{m,k})^{-1}}(n) \succeq n^k.$$

**Claim 8.13.** *The size of the largest Jordan block of the Jordan normal form for both $\phi_{m,k}^{ab}$, $(\phi_{m,k}^{-1})^{ab}$ is $k+1$.*

*Proof.* It is sufficient to prove the claim just for $\phi_{m,k}^{ab}$, as $(\phi_{m,k}^{-1})^{ab} = (\phi_{m,k}^{ab})^{-1}$.

By direct inspection of formulas (1), we see that $\phi_{m,k}^{ab}$ is represented by the following $(m+k) \times (m+k)$ matrix:

$$\phi^{ab}_{m,k} = \left[\begin{array}{c|c} I_m & D_{mk} \\ \hline O_{km} & C_{kk} \end{array}\right],$$

where $I_m$ is the identity $m \times m$ matrix, $O_{km}$ is the zero $k \times m$ matrix, and $D_{mk}$ and $C_{kk}$ are $m \times k$ and $k \times k$ matrices, respectively, given by the formulas:

$$D_{mk} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{bmatrix}, \qquad C_{kk} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

It is known that the number of Jordan blocks of a matrix $A \in GL(m+k, \mathbb{C})$ with all eigenvalues 1 is given by the number

$$\dim \ker(A - I_{m+k}) = m + k - \operatorname{rank}(A - I_{m+k}),$$

and the number of Jordan blocks of $A$ with all eigenvalues 1 and size at least 2 is given by

$$\dim \ker[(A - I_{m+k})^2] - \dim \ker(A - I_{m+k}) = \operatorname{rank}(A - I_{m+k}) - \operatorname{rank}[(A - I_{m+k})^2].$$

An easy computation shows that

$$\phi^{ab}_{m,k} - I_{m+k} = \left[\begin{array}{c|c} O_{mm} & D_{mk} \\ \hline O_{km} & C'_{kk} \end{array}\right], \quad \text{where} \quad C'_{kk} = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

and

$$(\phi^{ab}_{m,k} - I_{m+k})^2 = \left[\begin{array}{c|c} O_{mm} & D'_{mk} \\ \hline O_{km} & C''_{kk} \end{array}\right],$$

where

$$D'_{mk} = \begin{bmatrix} 0 & 1 & 2 & \dots & k-1 \\ 0 & 1 & 2 & \dots & k-1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & 2 & \dots & k-1 \end{bmatrix}, \qquad C''_{kk} = \begin{bmatrix} 0 & 0 & 1 & 2 & \dots & k-2 \\ 0 & 0 & 0 & 1 & \dots & k-3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Note that $\operatorname{rank} C'_{kk} = k-1$, $\operatorname{rank} C''_{kk} = k-2$, hence $\operatorname{rank}(\phi^{ab}_{m,k} - I_{m+k}) = k$ and $\operatorname{rank}[(\phi^{ab}_{m,k} - I_{m+k})^2] = k-1$. Therefore,

$$\text{the number of Jordan blocks for } \phi^{ab}_{m,k} = (m+k) - k = m,$$
$$\text{the number of Jordan blocks of size} \geqslant 2 \text{ for } \phi^{ab}_{m,k} = k - (k-1) = 1.$$

This means that there is only one block of size bigger than 1, let's denote this size $c$, and there are $m-1$ blocks of size 1. Hence, $m + k = c + (m-1)\cdot 1$ so that $c = k + 1$. $\square$

*Proof of Proposition 8.12.* The Proposition follows now from Corollary 2.13 and Claim 8.13. $\square$

## 8.3 Lower Bounds for the Growth of $\phi^{ab}$

In the proof of Theorem B in chapter 7 we needed a certificate for the growth of the abelianization of $\phi_{m,k}$, i.e. an element of the basis that realizes the maximum in the definition of the growth functions. Now we can provide it:

**Corollary 8.14.** *With the above notation, let $\bar{B}_k$ be the image of the generator $B_k$ in the abelianization of $F$. Then*

$$\left\|(\phi_{m,k})^n(B_k)\right\| \sim \left|(\phi^{ab}_{m,k})^n(\bar{B}_k)\right|_1 \sim \left|(\phi^{ab}_{m,k})^n(\bar{B}_k)\right|_\infty \sim n^k.$$

*Proof.* An elementary computation with the matrix from the proof of Claim 8.13 shows that $(\phi^{ab}_{m,k})^n$ is represented by the matrix with the following structure:

$$(\phi^{ab}_{m,k})^n = \left[\begin{array}{c|c} I_m & P_{mk} \\ \hline O_{km} & Q_{kk} \end{array}\right],$$

where $I_m$ is the identity $m \times m$ matrix, $O_{km}$ is the zero $k \times m$ matrix, and $P_{mk}$

and $Q_{kk}$ are $m \times k$ and $k \times k$ matrices, respectively, given by the formulas:

$$P_{mk} = \begin{bmatrix} c_{1n} & c_{2n} & \dots & c_{kn} \\ c_{1n} & c_{2n} & \dots & c_{kn} \\ \dots & \dots & \dots & \dots \\ c_{1n} & c_{2n} & \dots & c_{kn} \end{bmatrix}, \qquad Q_{kk} = \begin{bmatrix} 1 & c_{1n} & \dots & c_{k-2,n} & c_{k-1,n} \\ 0 & 1 & \dots & c_{k-3,n} & c_{k-2,n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & c_{1n} \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Here $P_{mk}$ has all rows equal to each other and $Q_{kk}$ is upper triangular with the same number on each diagonal sequence of entries parallel to the main diagonal. The numbers $c_{1n}, c_{2n}, \dots, c_{kn}$ satisfy the following identity, which follows from the matrix multiplication rule:

$$c_{\ell,n+1} = \sum_{i=0}^{\ell} c_{in},$$

with the convention that $c_{0n} = 1$. We now show by double induction on pairs $(i,n)$ that $c_{in} = \binom{n+i-1}{i}$. Indeed, this equality is true for pairs $(i,n) = (0,n)$ with arbitrary $n$, since $c_{0n} = 1$ by our convention, and for $(i,n) = (i,1)$ with arbitrary $i$, since $c_{i1} = 1$ in the matrix representation for $\phi^{ab}_{m,k}$ (see the proof of Claim 8.13). Now suppose that the equality $c_{in} = \binom{n+i-1}{i}$ is already proved for all pairs $(i,n)$ with $n$ fixed and $i$ arbitrary, and for all pairs $(i,n+1)$ with $0 \leqslant i \leqslant \ell - 1$. Then for the pair $(\ell, n+1)$ we get:

$$c_{\ell,n+1} = \sum_{i=0}^{\ell-1} c_{in} + c_{\ell,n} = c_{\ell-1,n+1} + c_{\ell,n} = \binom{n+\ell-1}{\ell-1} + \binom{n+\ell-1}{\ell} = \binom{n+\ell}{\ell},$$

as needed.

In particular, the coefficients of the vector $(\phi^{ab}_{m,k})^n(\bar{B}_k)$ are: $1, c_{1n}, \dots, c_{kn}$, with $c_{kn} = \binom{n+k-1}{k}$ being a polynomial $\sim n^k$.

Since, by Claim 8.8, $\|\phi^n_{m,k}(B_k)\| \preceq n^k$, we have:

$$n^k \succeq \left\|\phi^n_{m,k}(B_k)\right\| \geqslant \left|(\phi^{ab}_{m,k})^n(\bar{B}_k)\right|_1 \geqslant \left|(\phi^{ab}_{m,k})^n(\bar{B}_k)\right|_\infty \succeq n^k,$$

and we conclude that

$$\left\|\phi^n_{m,k}(B_k)\right\| \sim \left|(\phi^{ab}_{m,k})^n(\bar{B}_k)\right|_1 \sim \left|(\phi^{ab}_{m,k})^n(\bar{B}_k)\right|_\infty \sim n^k. \qquad \square$$

**Remark 8.15.** It can be proved in a similar manner that the same elements $B_k$ and $\bar{B}_k$ also serve as certificates for the growth of automorphisms $\phi^{-1}_{m,k}$ and $(\phi^{ab}_{m,k})^{-1}$, respectively. However the formulas involved are more complicated, and we don't need this result for our construction.

# Chapter 9

# Open Questions

We list here some open questions related to the topics of this work.

**Question 1.** Do there exist finitely presented subgroups of right-angled Artin groups whose Dehn functions are either super-exponential or sub-exponential but not polynomial?

**Question 2.** Does every CAT(0) free-by-cyclic group virtually embed into a right-angled Artin group?

The next question the author learned from Martin Bridson.

**Question 3.** Does any free-by-cyclic group $F_k \rtimes_\phi \mathbb{Z}$ with $\phi$ of maximal possible polynomial growth ($\sim n^{k-1}$) virtually embed into a right-angled Artin group?

A good test case for the last two questions are the Hydra groups $G_k$ studied in [19]:

**Question 4.** Do Hydra groups $G_k$ virtually embed into right-angled Artin groups?

The author has been able to prove that for $k \leqslant 4$ Hydra groups $G_k$ do embed into suitable RAAGs.

**Question 5.** For an arbitrary group $G$, an arbitrary automorphism $\alpha \in \mathrm{Aut}(G)$ and an $\alpha$–invariant subgroup $H \leqslant G$ of finite index, what is the range for possible gaps between the growth rate of an automorphism $\alpha$, $\mathrm{gr}_\alpha(n)$, and the growth of its restriction to $H$, $\alpha|_H$, $\mathrm{gr}_{\alpha|_H}(n)$?

See chapter 2 for the proof that these growth functions are the same if $G$ is free, and an example due to Yves Cornullier when the gap is one polynomial degree for a dihedral group $G$ (linear–constant). In an ongoing project, the author and Noel Brady have produced other examples of this sort, in particular, the ones realizing cubic–linear gap.

**Question 6.** In Corollary 7.1 we proved that the groups $\Gamma$ having polynomial Dehn functions of arbitrary degree lie in the Bestvina–Brady kernel (call it $BBK$) of the corresponding RAAG $A(\Delta) \times F_2$. Dison proved in [18] that Bestvina–Brady kernels have Dehn functions bounded by $n^4$. Also, the RAAG itself (being a CAT(0) group) has at most quadratic Dehn function. This tells us that in the series of inclusions:

$$\Gamma \leqslant BBK \leqslant \mathrm{RAAG}$$

we observe a “distortion of areas” phenomenon. On the other hand, it can be proven using technique from [1] that the subgroup distortion (i.e. the “distortion of lengths”) of $BBK$ in RAAG is at most quadratic. What is the subgroup distortion of $\Gamma$ in $BBK$?

**Question 7.** Our construction of groups $\Gamma$ with polynomial Dehn functions places them inside RAAGs with 3–dimensional Salvetti complex. Do there exist subgroups with polynomial Dehn functions of arbitrary degrees inside 2–dimensional RAAGs?

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