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A PRIMAL-DUAL ALGORITHM FOR BOUNDED VARIABLES
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ABSTRACT

In spite of the considerable amount of research done in the field of linear programming, a method does not exist which can solve a linear programming problem, with some or all variables bounded from above, without resorting to the artificial variable techniques. This study proposes a new primal-dual algorithm for the bounded variable problem. The combination of the existing primal and dual simplex methods is used in such a way that the problem does not have to start either primal or dual feasible.

As a natural extension to the algorithm, the problem of sensitivity and parametric analysis for bounded variables is presented. Since the upper bounding techniques are extensively used in branch and bound algorithms of integer programming, the parametric analysis is extended to cover certain special cases of integer programming.

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CHAPTER I

INTRODUCTION

I. Background

In spite of the impressive progress in the area of linear programming, efforts have not provided an algorithm that can handle the general linear programming problem without resorting to artificial augmentations. To be specific, the general linear programming problem is defined as:

$$\text{maximize } x_0 = \underline{C}_1\ \underline{X}_1 + \underline{C}_2\ \underline{X}_2\ ,\ \underline{C}_1 \leqslant \underline{0}\ ,\ \underline{C}_2 > \underline{0},$$

subject to

$$\underline{A}_{11}\ \underline{X}_1 + \underline{A}_{12}\ \underline{X}_2 \leqslant \underline{b}_1\ ,\ \underline{b}_1 \geqslant \underline{0},$$
$$\underline{A}_{21}\ \underline{X}_1 + \underline{A}_{22}\ \underline{X}_2 \geqslant \underline{b}_2\ ,\ \underline{b}_2 \geqslant \underline{0},$$
$$\underline{A}_{31}\ \underline{X}_1 + \underline{A}_{32}\ \underline{X}_2 = \underline{b}_3,\ \underline{b}_3 \geqslant \underline{0},$$
$$\underline{X}_1 \geqslant \underline{0}\ ,\quad \underline{X}_2 \geqslant \underline{0}\ ,$$
$$\underline{X}_1^0 \leqslant \underline{U}_1,\ X_2^0 \leqslant \underline{U}_2\ ,\ \underline{U}_1 \geqslant \underline{0}\ ,\ \underline{U}_2 \geqslant \underline{0}$$

The elements of $\underline{X}_1^0$ and $\underline{X}_2^0$ are subsets of those of $\underline{X}_1$ and $\underline{X}_2$, respectively. The sizes of the constant vectors $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_3$ are given by m_1, m_2, and m_3, respectively. Also, the number of elements in $\underline{X}_1$ and $\underline{X}_2$ are equal to n_1 and n_2, respectively. Letting the number of elements in $\underline{X}_1^0$ and $\underline{X}_2^0$ be $\bar{n}_1$ and $\bar{n}_2$, then by definition, $\bar{n}_1 \leq n_1$ and $\bar{n}_2 \leq n_2$. For convenience, let $m = m_1 + m_2 + m_3$ and $n = n_1 + n_2$. The constant vectors $\underline{U}_1$ and $\underline{U}_2$ represent the upper bounds on $\underline{X}_1^0$ and $\underline{X}_2^0$, respectively.

The generality of the above problem is reflected by the fact that, in its present form, the problem does not readily possess a primal- or dual- feasible starting solution. Primal infeasibility is caused by the ($\geq$) and (=) constraints. Dual infeasibility, on the other hand, is created by the fact that the variables $\underline{X}_2$ have positive coefficients in the objective function, that is, $\underline{C}_2 > \underline{0}$. To complete the generality of the linear programming problem, the variables $\underline{X}_1^0$ and $\underline{X}_2^0$ are assumed to have finite upper bounds.

There are several methods which can be used to solve this problem. A straightforward method is to use Dantzig's regular simplex method [3]. This, however, requires extensive augmentations with artificial variables in order to provide a starting basic solution. Obviously, this method will involve extensive additional computations.

There are two main methods which can handle this problem more efficiently than the regular simplex method. These are:

1. Dantzig's primal simplex method for bounded variables (PSB), [5].

2. Wagner's dual simplex for bounded variables (DSB), [22].

The details of these two methods are given in Appendix A. Generally, the relative efficiency of the two methods stems from the fact that the upper bounding constraints are not considered explicitly in the calculations. Rather, the feasibility conditions of Dantzig's regular simplex method [3] and Lemke's regular dual simplex method [17] are modified so that the effect of upper bounding is considered only in an implicit sense. The main problems associated with the regular primal and dual methods are still present in both the PSB and DSB methods. Specifically, the PSB requires a starting primal feasible solution which can be secured for the above general problem only through artificial augmentation. Similarly, the DSB method necessitates the use of a starting dual-feasible solution, secured through a substitution of artificial upper bounds for the unbounded variables with positive coefficients in the objective function.

It may be remarked that a third method due to Zionts [24] has recently been developed which claims to treat the linear programming problem without artificial augmentations. The details of this method, called the criss-cross method, are also given in Appendix A. However, it can be shown by a counter-example that the criss-cross method may lead to erroneous results. In addition, since no provisions are made by the criss-cross method to handle upper bounded variables, the method is not general in the sense specified above.

II. Objective of the Research

The above discussion shows that the available methods have definite shortcomings in dealing with the general problem defined above. The objective of this research is to remedy these shortcomings. Specifically, a new algorithm, which combines all the features of the above three "general" methods is developed so that no artificial augmentations will be needed. In effect, the new algorithm will deal directly with the original problem so that no changes in the structure of the problem are needed.

Computational experience has shown that the <u>product</u> (revised) <u>form</u> of the simplex method [4] is more efficient. Consequently, this research will show how the proposed algorithm can be used with the product form calculations.

A natural extension of the new algorithm is to show how it can be used with sensitivity and parametric analysis. Since the upper bounding technique is employed extensively in the branch-and-bound algorithm of integer linear programming, the application of the parametric analysis is illustrated by considering special cases of <u>parametric integer programming</u>.

III. Organization of the Dissertation

Following the introduction in Chapter I, this dissertation is organized as follows: The new primal-dual algorithm for linear programs with bounded variables is presented in Chapter II. Chapter III introduces the parametric analysis of the bounded variable problem. The application of the parametric analysis to a class of integer programming problems is presented in Chapter IV.

CHAPTER II

A PRIMAL-DUAL ALGORITHM FOR LINEAR PROGRAMS WITH BOUNDED VARIABLES

Consider the maximization problem defined in Chapter I. Adding slack variables and using matrix notation, the problem can be represented as follows:

$$\text{maximize } \underline{x}_0 = (\underline{C}_1 \ , \ \underline{C}_2) \begin{pmatrix} \underline{X}_1 \\ \underline{X}_2 \end{pmatrix}$$

subject to

$$\begin{pmatrix} \underline{A}_{11} & \underline{A}_{12} & \underline{I} & \underline{0} & \underline{0} \\ -\underline{A}_{21} & -\underline{A}_{22} & \underline{0} & \underline{I} & \underline{0} \\ \underline{A}_{31} & \underline{A}_{32} & \underline{0} & \underline{0} & \underline{I} \end{pmatrix} \begin{pmatrix} \underline{X}_1 \\ \underline{X}_2 \\ \underline{X}_3 \\ \underline{X}_4 \\ \underline{X}_5 \end{pmatrix} = \begin{pmatrix} \underline{b}_1 \\ -\underline{b}_2 \\ \underline{b}_3 \end{pmatrix}$$

$$\underline{X}_i \geqslant \underline{0} \ , \quad i = 1, 2, \ldots , 5$$

$$\underline{X}_1 \leqslant \underline{U}_1 \ , \quad \underline{X}_2 \leqslant \underline{U}_2$$

Note that the equality constraints

$$\underline{A}_{31}\ \underline{X}_1 + \underline{A}_{32}\ \underline{X}_2 = \underline{b}_3$$

have been changed to

$$\underline{A}_{31}\ \underline{X}_1 + \underline{A}_{32}\ \underline{X}_2 + \underline{X}_5 = \underline{b}_3$$

by adding the new variables $\underline{X}_5$. These variables must vanish at any feasible solution. This can be guaranteed by adding zero upper bounds on all variables in $\underline{X}_5$, that is, the constraints

$$\underline{0} \leq \underline{X}_5 \leq \underline{0}$$

must be added to the problem. For ease of reference the variables $\underline{X}_5$ will be labelled "<u>excess</u> <u>variables</u>."

The problem can now be put in a more compact form.

maximize $x_0 = \underline{C}\ \underline{X}$

subject to

$$(\underline{A},\ \underline{I})\ \underline{X} = \underline{P}_0$$

$$\underline{0} \leq \underline{X} \leq \underline{U}$$

where $\underline{X}$ represents all the variables (including the slack and excess variables). For the variables without finite upper bounds, the corresponding element in $\underline{U}$ is assumed equal to infinity.

I. The Primal-Dual Algorithm

This algorithm uses a combination of both primal and dual simplex methods. Primal iterations are carried out while ignoring the primal infeasible constraints. If a variable can enter the solution as a result of a primal iteration, no attention is paid to the upper bound constraint. This is done in order to avoid the calculation and comparison of three different criteria as in the PSB method. If an entering variable has a finite upper bound but cannot be entered by the criterion of the regular primal simplex method, it is merely substituted at its upper bound. Similarly, dual iterations are carried out, when necessary, while ignoring dual infeasible constraints.

In the following discussion $z_j - c_j$ is used, as usual, to denote the optimality indicator of the variable x_j. Thus in a maximization problem, if $z_j - c_j$ is negative, x_j becomes a promising candidate to enter the solution. As in the regular simplex method, the variable having the most negative $z_j - c_j$ is selected to enter the basis. These details are given below.

Step 1: <u>Criterion for a variable to enter via primal</u>--

(a) If all $z_j - c_j$ are nonnegative, go to Step 2; otherwise go to (b).

(b) Select the most negative $z_j - c_j$ to enter the basis. If this variable can be entered (using the regular primal simplex feasibility criterion while neglecting primal infeasible and the upper bound constraints), complete the iteration and repeat Step 1; otherwise go to (c).

(c) Since the variable x_j cannot be entered via primal, check if x_j has a finite upper bound. If such a bound exists, substitute

$$x_j' = u_j - x_j$$

and go to Step 1; otherwise go to (d).

(d) Since the primal iteration cannot be carried out, neglect the most negative $z_j - c_j$ for this iteration only, and check if any other $z_j - c_j$ is nonpositive. If such $z_j - c_j$ exists return to (b); otherwise go to Step 4.

Step 2: Criterion for a variable to enter via dual--

(a) Check if the right hand side is nonnegative. If it is nonnegative, go to Step 3; otherwise go to (b).

(b) Carry out the dual simplex iteration on the most negative element of the right hand side. If this is done, return to Step 2; if this cannot be done the solution is infeasible.

Step 3: Check for upper bounds for the basic variables--

(a) Check if the basic variables are within their respective upper bounds. If this condition is satisfied the solution is optimum; otherwise substitute

$$x_j' = u_j - x_j$$

for all variables which exceed their upper bounds and return to Step 2.

Step 4: Procedure when primal iteration cannot be carried out--

(a) Check if the right hand side has at least one negative element. If there is at least one negative element go to (b); otherwise go to (d).

(b) Carry out a dual simplex iteration on the most negative element of the right hand side; while ignoring the dual infeasible columns. If this can be done return to Step 1; otherwise go to (c).

(c) Since the dual iteration cannot be carried out, ignore this most negative element for this iteration only, and return to (a).

(d) Check if any of the basic variables have a finite upper bound. If such bounds exist then substitute

$$x_j = u_j - x_j' ,$$

for all basic variables with finite upper bounds and go to Step 1. If all the basic variables have no finite upper bounds then either the solution is <u>infeasible</u> or <u>unbounded</u>.

The complete procedure is shown by a flowchart in Figure 1. Since the proposed algorithm uses both primal and dual simplex methods for bounded variables, it will be denoted by <u>PDSB.</u>

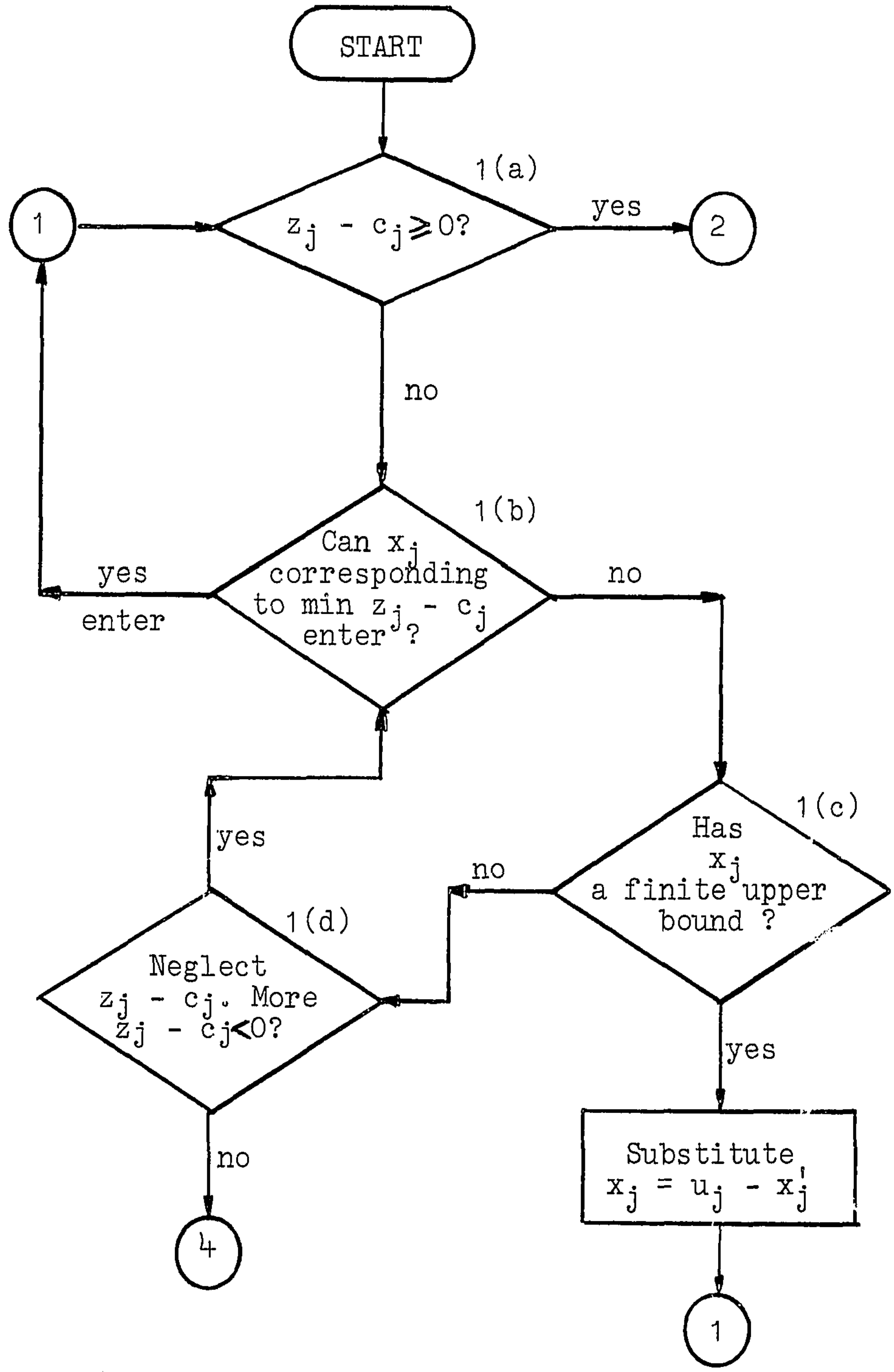

Figure 1.--Flowchart for the Primal-Dual Algorithm for Bounded Variables.

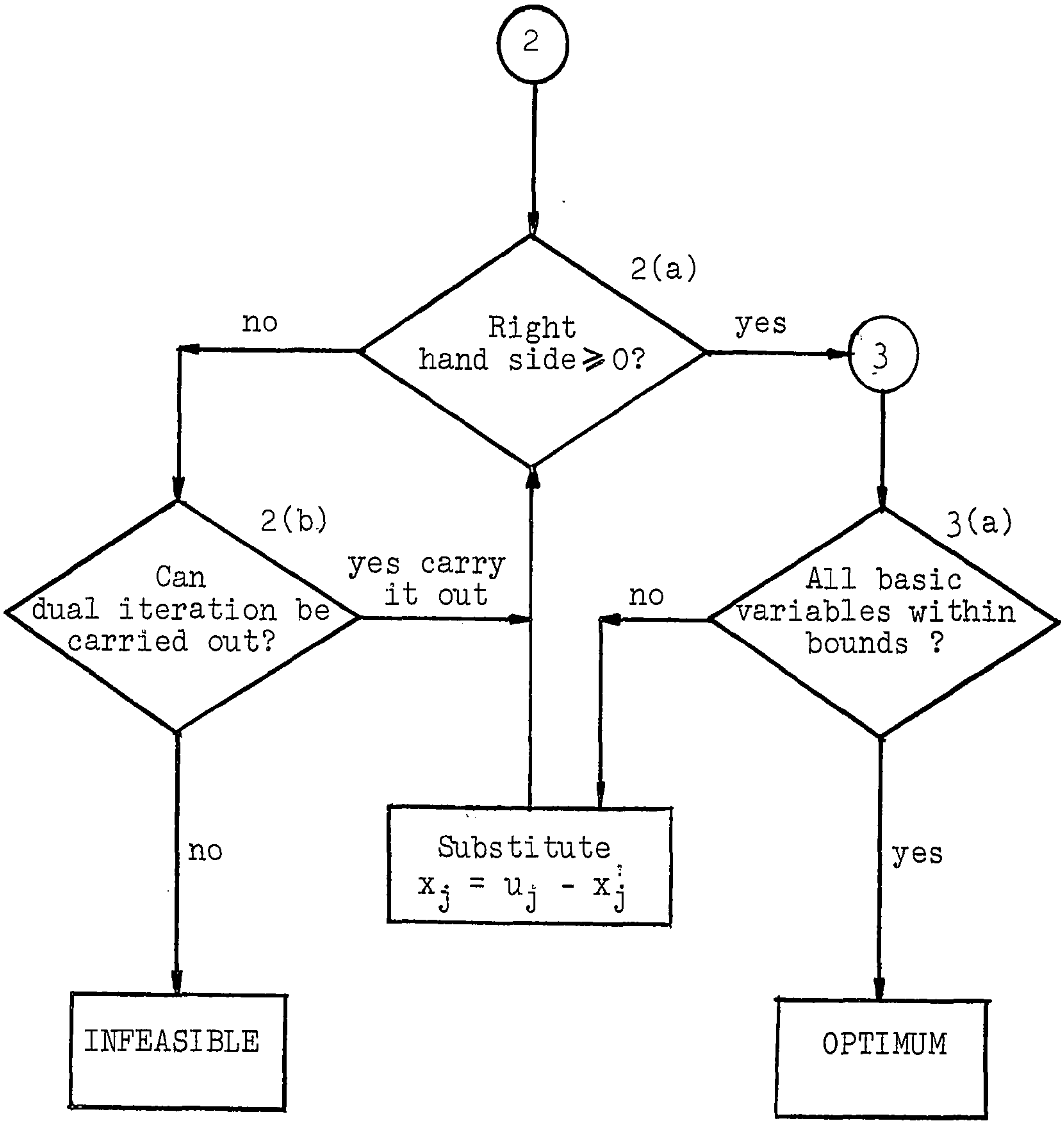

Figure 1.--Continued.

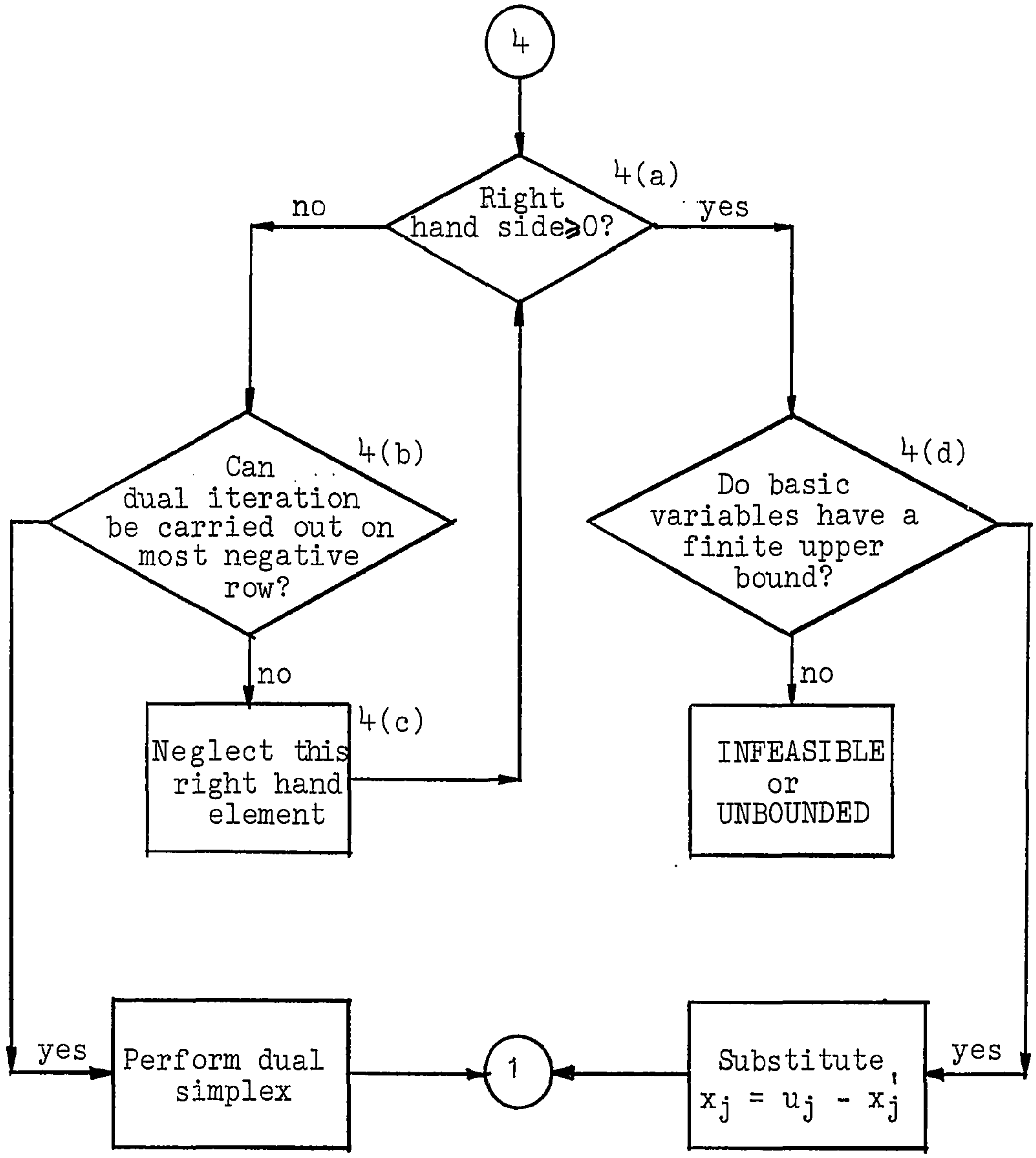

Figure 1.--<u>Continued</u>.

II. Theoretical Discussion

The above discussion shows that the new algorithm combines the principles of the regular primal simplex, the regular dual simplex, the PSB and the DSB method. Thus the proofs of feasibility, optimality and finiteness can be derived from these methods.

When the problem is at a stage such that neither primal nor dual iterations can be carried out and also all basic variables are unbounded [Step 4 (d)], then either the problem is infeasible or unbounded. This criterion is different from the criteria of the four methods mentioned above, but can be derived from Zionts' [24] criss cross method.

The following theorem explains the special characteristics of the "excess variables."

Theorem 1. At any iteration, once an excess variable becomes nonbasic it can never be made basic at a later iteration.

Proof--From the criteria of the regular primal simplex method a variable x_j is a candidate to enter if, and only if,

$$z_j - c_j \leq 0.$$

If x_r is such a candidate, then

$$z_r - c_r \leq 0.$$

Since x_r is currently nonbasic, it has a zero value and is at zero upper bound. Therefore, it can be substituted by

$$x_r' = u_r - x_r = -x_r$$

This substitution will keep the tableau unchanged except for the excess variable column which will have opposite signs. If $z_r' - c_r'$ denotes the new value of the coefficient corresponding to the excess variable in the objective row, then

$$z_r' - c_r' \geq 0.$$

Thus x_r' is no longer a candidate to enter the basis. If at any subsequent iteration the quantity $z_r' - c_r'$ becomes nonpositive, the same substitution will make it nonnegative. This completes the proof.

III. Numerical Examples

Two examples are presented in this section to illustrate various steps of the proposed primal-dual algorithm for bounded variable problems.

Example I

$$\text{maximize } x_0 = 2x_1 - x_2 + x_3$$

subject to

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 3 \\ -x_1 + 2x_2 - x_3 &\leq 10 \\ x_1 + x_3 &\geq 5 \end{aligned}$$

$$0 \leq x_1 \leq 4 \;,\; 0 \leq x_2 \leq 5 \;,\; 0 \leq x_3$$

Iteration 1

	x_1	x_2	x_3	
	-2	1	-1	0
x_4	1	1	-2	3
x_5	-1	2	-1	10
x_6	-1	0	-1	-5

Variable x_1 enters, excess variable x_4 leaves and its column is deleted [Step 1 (b)].

Iteration 2

	x_2	x_3	
	3	-5	6
x_1	1	-2	3
x_5	3	-3	13
x_6	1	-3	-2

Variable x_3 cannot enter via primal nor can x_6 leave via dual. Neither primal nor dual iteration can be carried out but basic variable x_1 has a finite upper bound. Hence x_1 is substituted at its upper bound [Step 4 (d)].

Iteration 3

	x_2	x_3	
	3	-5	6
x_1'	-1	2	1
x_5	3	-3	13
x_6	1	-3	-2

x_3 enters, x_1' leaves [Step 1 (b)].

Iteration 4

	x_2	x_1'	
	1/2	5/2	17/2
x_3	-1/2	1/2	1/2
x_5	3/2	3/2	29/2
x_6	-1/2	3/2	-1/2

Dual feasible, carry out dual iteration x_2 enters [Step 2 (b)].

Iteration 5

	x_6	x_1'	
	1	4	8
x_3	-1	-1	1
x_5	3	6	13
x_2	-2	-3	1

Optimum Solution $x_0 = 8$.

$x_1 = 4$, $x_2 = 1$, $x_3 = 1$.

Example II

$$\text{maximize } x_0 = 2x_1 + x_2 - x_3$$

subject to

$$x_1 - x_2 \leq 3$$
$$-x_1 + x_2 + 2x_3 = 4$$
$$0 \leq x_1,\ 0 \leq x_2 \leq 2,\ 0 \leq x_3 \leq 3$$

Iteration 1

	x_1	x_2	x_3	
	-2	-1	1	0
x_4	1	-1	0	3
x_5	-1	1	2	4

Variable x_1 enters, x_4 leaves [Step 1 (b)].

Iteration 2

	x_4	x_2	x_3	
	2	-3	1	6
x_1	1	-1	0	3
x_5	1	0	2	7

Variable x_2 cannot be entered via primal, but it has a finite upper bound. Substitute $x_2 = 2 - x_2'$ [Step 1 (c)].

Iteration 3

	x_4	x_2'	x_3	
	2	3	1	12
x_1	1	1	0	5
x_5	1	0	2	7

All $z_j - c_j$ are nonnegative, right hand side is also non-nonnegative but the variable x_5 has a finite upper bound equal to zero. Substitite

$$x_5 = -x_5'$$

[Step 3 (a)].

Iteration 4

	x_4	x_2'	x_3	
	2	3	1	12
x_1	1	1	0	5
x_5'	-1	0	-2	-7

Variable x_5' leaves the solution and x_3 enters the solution [Step 2 (b)]. Since x_5', the excess variable, becomes non-basic, x_5' can be deleted.

Iteration 5

	x_4	x_2'	
	3/2	3	17/2
x_1	1	1	5
x_3	1/2	0	7/2

Variable x_3 exceeds its upper bound. Replace x_3 by

$$x_3 = 3 - x_3'$$

[Step 3 (a)] and carry out dual simplex iteration [Step 2 (b)].

Iteration 6

	x_3'	x_2'	
	3	3	7
x_1	2	1	4
x_4	-2	0	1

Solution optimum. $x_0 = 7$.

$$x_1 = 4,\ x_2 = 2,\ x_3 = 3.$$

IV. Matrix Notation for Bounded Variables

Consider the general linear programming problem mentioned earlier (p. 1). At any iteration let $\underline{X}_B$ denote the vector of basic variables. Since the nonbasic variables may be at zero level or at their upper bounds, let $\underline{X}_z$ represent the nonbasic variables at zero level and let $\underline{X}_u$ represent the nonbasic variables at their upper bound. Partition the $\underline{C}$ vector by $\underline{C}_B$, $\underline{C}_z$ and $\underline{C}_u$ corresponding to $\underline{X}_B$, X_z and $\underline{X}_u$ respectively. Let $\underline{B}$ represent the basic matrix and let $\underline{D}_z$ and $\underline{D}_u$ represent the matrices corresponding to $\underline{X}_z$ and $\underline{X}_u$ respectively. The problem can then be represented by

$$\begin{pmatrix} 1 & -\underline{C}_B & -\underline{C}_z & -\underline{C}_u \\ \underline{0} & \underline{B} & \underline{D}_z & \underline{D}_u \end{pmatrix} \begin{pmatrix} x_0 \\ \underline{X}_B \\ \underline{X}_z \\ \underline{X}_u \end{pmatrix} = \begin{pmatrix} 0 \\ \underline{P}_0 \end{pmatrix}$$

Making the substitution

$$\underline{X}_u' = \underline{U}_u - \underline{X}_u$$

the tableau becomes

$$\begin{pmatrix} 1 & -\underline{C}_B & -\underline{C}_z & +\underline{C}_u \\ \underline{0} & \underline{B} & \underline{D}_z & -\underline{D}_u \end{pmatrix} \begin{pmatrix} x_0 \\ \underline{X}_B \\ \underline{X}_z \\ \underline{X}_u' \end{pmatrix} = \begin{pmatrix} \underline{C}_u\ \underline{U}_u \\ \underline{P}_0 - \underline{D}_u\ \underline{U}_u \end{pmatrix}$$

Thus, if at any iteration the basic variables $\underline{X}_B$ and the nonbasic variables $\underline{X}_u$ reaching their upper bounds are known, the entire tableau can be generated as follows.

1. The value of the objective function--

   This value of the objective function is

$$x_0 = \underline{C}_B\ \underline{B}^{-1}\ \underline{P}_0 + \underline{C}_u\ \underline{U}_u$$

2. The value of the right hand side--

   This value of the right hand side is given by

$$\underline{B}^{-1}\ (\underline{P}_0 - \underline{D}_u\ \underline{U}_u)$$

3. The value of the nonbasic coefficients $z_j - c_j$--

   (a) If the nonbasic variable is at zero level, then

$$z_j - c_j = \underline{C}_B\ \underline{B}^{-1}\ \underline{P}_j - c_j$$

where $\underline{P}_j$ denotes the column of coefficients of x_j from the constraints.

(b) If the nonbasic variable is at its upper bound then

$$z_j - c_j = c_j - \underline{C}_B \, \underline{B}^{-1} \, \underline{P}_j$$

4. The value of the coefficients of the nonbasic x_j at the current iteration for different constraints--

If $\underline{\alpha}^j$ denotes these coefficients, then

(a) If x_j is at zero level, then

$$\underline{\alpha}^j = \underline{B}^{-1} \, \underline{P}_j$$

(b) If x_j is at its upper bound, then

$$\underline{\alpha}^j = -\underline{B}^{-1} \, \underline{P}_j$$

Thus, it is clear that a knowledge of the basic variables and the variables that reach their upper bounds, is enough to generate the tableau. Hence, the proposed algorithm can be suitably modified to fit the product form.

CHAPTER III

PARAMETRIC ANALYSIS FOR LINEAR PROGRAMS WITH UPPER BOUNDS

Parametric programming is concerned with the study of variations in the optimum basic solution due to predetermined systematic changes in the different coefficients of a linear programming problem. The parametric linear programming problem can be defined generally as:

$$\text{maximize } x_0 = (\underline{C} + \Theta\underline{E})\underline{X}$$

subject to
$$(\underline{A},\underline{I})\underline{X} = \underline{P}_0 + \Theta\underline{R}_0$$
$$\underline{X} \geqslant \underline{0}$$

where $\underline{E}$ and $\underline{R}$ are given constant vectors and Θ is a real parameter.

The variation with Θ of the optimum solution to this problem has been studied by Gass and Saaty [9, 10] and Manne [18]. The study is based on the fact that (1) changes in the objective function can only affect the optimality of the problem (or the feasibility of its dual) and (2) changes in the right-hand side can only affect the feasibility of the problem (or the optimality of its dual). Thus, by checking these conditions systematically, the values of Θ at which the optimum basic solution changes can be determined. These values of Θ are usually referred to as characteristic values. The parametric analysis can also be extended to the variations in the elements of the matrix $\underline{A}$ as defined in the linear programming problem. The analysis in this case, however, is not as general as in the case of the objective coefficients and the right-hand side elements of the objective.

In this chapter, the parametric analysis is developed for the upper-bounded problem. Thus, in addition to the regular parametric problem, the analysis will include a parameterization of the upper bounds. This parametric variation can be represented by

$$\underline{0} \leq \underline{X} \leq \underline{U} + \Theta \underline{V}$$

where $\underline{V}$ is a known constant vector.

I. Parametric Analysis for Bounded Variables

In the regular linear programming problems, where the variables are not bounded from above, the analysis of the effect of changes in the $\underline{P}_0$ vector alone on the optimum solution can be carried out in an identical manner to the analysis, where the changes are made in the $\underline{C}$ vector alone. This is possible because of the relationship between the primal and the dual problems. However, for the bounded variable problems the effect of upper bounds makes such analysis different. Furthermore, the variations in $\underline{P}_0$ are accompanied by the variations in the upper bounds $\underline{U}$. The following four cases are discussed in this chapter.

1. Variations in $\underline{P}_0$ and $\underline{U}$ vectors.
2. Variations in $\underline{C}$ vector.
3. Variations in the coefficients of nonbasic vectors $\underline{P}_j$.
4. Simultaneous variations in $\underline{P}_0$, $\underline{U}$ and $\underline{C}$.

In all these cases it will be assumed that only one parameter θ is involved and that the parameter θ varies continuously in a specified interval.

1. Variations in $\underline{P}_0$ and $\underline{U}$ vectors

Consider the problem given by the following.

maximize $x_0 = \underline{C}\ \underline{X}$

subject to

$$(\underline{A}, \underline{I})\underline{X} = \underline{P}_0 + \theta \underline{R}_0$$

$$\underline{0} \leqslant \underline{X} \leqslant \underline{U} + \theta \underline{V}$$

where $\underline{R}_0$ and $\underline{V}$ are known vectors of constants and $\theta \geqslant 0$. With only minor modifications this analysis can be extended to the range of θ from $-\infty$ to 0.

To start with, assume that an optimum solution exists for $\theta = 0$. Let $\underline{X}_B$ be the optimum basis. Then,

$$\underline{0} \leqslant \underline{X}_B = \underline{B}^{-1}\ (\underline{P}_0 - \underline{D}_u\ \underline{U}_u) \leqslant \underline{U}_B$$

For $\theta > 0$, the solution will remain optimum as long as

$$\underline{0} \leqslant \underline{X} = \underline{B}^{-1}\ [\underline{P}_0 - D_u(\underline{U}_u + \theta \underline{V}_u) + \theta \underline{R}_0] \leqslant \underline{U}_B + \theta \underline{V}_B$$

Let

$$\underline{P} = \underline{P}_0 - \underline{D}_u \underline{U}_u \quad \text{and} \quad \underline{R} = \underline{R}_0 - \underline{D}_u \underline{V}_u \ .$$

Hence,

$$\underline{0} \leqslant \underline{X}_B = \underline{B}^{-1} (\underline{P} + \Theta\underline{R}) \leqslant \underline{U}_B + \Theta\underline{V}_B$$

Let $(\underline{B}^{-1} \underline{P})_i$ and $(\underline{B}^{-1} \underline{R})_i$ denote the i^{th} elements of $(\underline{B}^{-1} \underline{P})$ and $(\underline{B}^{-1} \underline{R})$ respectively. Then,

i) if $\emptyset_1'$ represents the value of Θ at which an upper will be exceeded, this will be given by

$$\emptyset_1' = \min_i \left\{ \frac{[\underline{U}_B - \underline{B}^{-1} P]_i}{[\underline{B}^{-1} \underline{R} - V_B]_i} \middle| \ [\underline{B}^{-1} \underline{R} - \underline{V}_B]_i > 0 \right\}$$

If all $[\underline{B}^{-1} \underline{R} - \underline{V}_B]_i \leqslant 0$, or if x_i corresponding to all any $[\underline{B}^{-1} \underline{R} - \underline{V}_B]_i \geqslant 0$ does not have a finite upper bound $\emptyset_1'$ will have an infinite value.

ii) If $(\underline{B}^{-1} \underline{R})_i < 0$ for at least one value of i, then let

$$\emptyset_1'' = \min_i \left\{ \frac{-(\underline{B}^{-1} \underline{P})_i}{(B^{-1} \underline{R})_i} \middle| \ (\underline{B}^{-1} \underline{R})_i < 0 \right\}$$

This condition gives the value of Θ at which the nonnegativity constraint(s) will be violated.

Let $\Theta_0 = 0$, and let Θ_1 denote the first characteristic value of Θ. If $\emptyset_1$ is defined such that

$$\emptyset_1 = \min (\emptyset_1' \ , \ \emptyset_1'')$$

then Θ_1 will be given by

$$\Theta_1 = \Theta_0 + \emptyset_1$$

In general, let Θ_i be the ith characteristic value. Then,

$$\Theta_{i+1} = \Theta_i + \phi_{i+1}$$

where

$$\phi_{i+1} = \min\,(\phi'_{i+1} \;,\; \phi''_{i+1})$$

The values of ϕ'_{i+1} and ϕ''_{i+1} can be represented by the following.

$$\phi'_{i+1} = \min_j \left\{ \frac{[\underline{U}^i_B - (\underline{B}^i)^{-1}\ \underline{P}^i]_j}{[(\underline{B}^i)^{-1}\ \underline{R}^i - \underline{V}^i_B]_j} \;\middle|\; [(\underline{B}^i)^{-1}\ R^i - \underline{V}^i_B]_j > 0 \right\}$$

and

$$\phi''_{i+1} = \min_j \left\{ \frac{-[(\underline{B}^i)^{-1}\ \underline{P}^i]_j}{[(\underline{B}^i)^{-1}\ \underline{R}^i]_j} \;\middle|\; [(\underline{B}^i)^{-1}\ R^i]_j < 0 \right\}$$

where $\underline{B}^i$, $\underline{P}^i$, $\underline{R}^i$, $\underline{U}^i_B$, and $\underline{V}^i_B$ represent the values of the respective matrices immediately after the ith critical value. If at any stage ϕ_{i+1} is infinite, as shown by the calculations, the analysis is stopped and the current basic solution at that point will continue to remain basic thereafter. The analysis may also be stopped if the solution space becomes infeasible at some value of Θ.

A numerical example is now presented which illustrates the method.

Example I

Solve the following parametric problem for $\Theta \geq 0$.

$$\text{maximize } x_0 = 3x_1 + 5x_2 + 2x_3$$

subject to

$$x_1 + x_2 + 2x_3 \leqslant 10 - \theta$$
$$2x_1 + 4x_2 + 3x_3 \leqslant 16 + 2\theta$$
$$0 \leqslant x_1 \leqslant 4 + \theta, \quad 0 \leqslant x_2 \leqslant 3 + \theta, \quad 0 \leqslant x_3 \leqslant 3 - (1/4)\theta$$

Solution: At $\theta = 0$, the optimum solution can be obtained using the new algorithm, and is given by

	x_1'	x_5	x_3	
	1/2	5/4	7/4	22
x_4	-1/2	-1/4	5/4	4
x_2	-1/2	1/4	3/4	2

$$\underline{B}^{-1} = \begin{pmatrix} 1 & -1/4 \\ 0 & 1/4 \end{pmatrix}$$

$$\underline{D}_u \underline{U}_u = \begin{pmatrix} 1 \\ 2 \end{pmatrix} (4) = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

$$\underline{P} = \underline{P}_0 - \underline{D}_u \underline{U}_u = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

$$\underline{B}^{-1} \underline{P} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\underline{R} = \underline{R}_0 - \underline{D}_u \underline{V}_u = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$\underline{B}^{-1} \underline{R} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

Thus

$$\phi_1' = \infty \quad , \quad \text{and} \quad \phi_1'' = 2.$$

Hence

$$\theta_1 = \theta_0 + \phi_1 = 2.$$

The alternative tableaus at $\theta = 2$ will be

	x_1'	x_5	x_3	
	1/2	5/4	7/4	28
x_4	-1/2	-1/4	5/4	0
x_2	-1/2	1/4	3/4	2

	x_4	x_5	x_3	
	1	1	3	28
x_1	2	-1/2	5/2	6
x_2	-1	1/2	-1/2	2

Note that in the alternate tableau x_1' becomes the basic variable but has been replaced by x_1.

$$(\underline{B}^1)^{-1} = \begin{pmatrix} 2 & -1/2 \\ -1 & 1/2 \end{pmatrix}$$

$$\underline{P}_0^1 = \begin{pmatrix} 8 \\ 20 \end{pmatrix} \qquad \underline{P}^1 = \begin{pmatrix} 8 \\ 20 \end{pmatrix} \qquad \underline{R}^1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$(\underline{B}^1)^{-1}\,\underline{P}^1 = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \qquad (\underline{B}^1)^{-1}\,\underline{R}^1 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$\underline{U}_B^1 = (6 \;,\; 5)$$

$$\underline{V}_B^1 = (1 \;,\; 1)$$

Thus,

$$\phi_2' = [\tfrac{5-2}{1}] = 3 \qquad \text{and} \qquad \phi_2'' = [\tfrac{-6}{-3}] = 2$$

Hence, $\qquad \phi_2 = \min\,(3 \;,\; 2) = 2$

and

$$\theta_2 = \theta_1 + \phi_2 = 4$$

At this characteristic value the alternative tableaus can be calculated and the analysis can be carried out beyond $\theta = 4$. Note that the analysis in any case will be stopped beyond $\theta = 10$, because the first constraint becomes infeasible.

2. <u>Variations in $\underline{C}$ vector</u>

Consider the following problem.

maximize $x_0 = (\underline{C} + \theta\underline{E})\ \underline{X}$

subject to

$$(\underline{A}\ ,\ \underline{I})\ \underline{X} = \underline{P}_0$$

$$\underline{0} \leqslant \underline{X} \leqslant \underline{U}\ , \qquad \theta \geqslant 0$$

where $\underline{E}$ is a known vector of constants. As before, the problem is solved at $\theta = 0$.

Let the optimum solution at $\theta = 0$ be given by

$$\underline{X}_B = \underline{B}^{-1}\ (\underline{P}_0 - \underline{D}_u\ \underline{U}_u) = \underline{B}^{-1}\ \underline{P}$$

Also,

$$z_j - c_j = \underline{C}_B\ \underline{B}^{-1}\ \underline{P}_j - c_j \geqslant 0 \qquad j = 1, 2, \ldots, m+n$$

This analysis differs from the analysis for linear programs without bounded variables in the following way. If the basis $\underline{X}_B$ contains a variable x_j' then the corresponding element of $\underline{C}_B$ is replaced by $-c_j$. Similarly, if the rth variable in the optimum tableau is x_r' , then

$$z_r' - c_r' = \underline{C}_B \, \underline{B}^{-1} \, (-\underline{P}_r) - (-c_r) \geqslant 0$$

If the optimum solution at $\Theta = 0$ remains optimum for $\Theta > 0$, then

$$(\underline{C}_B + \Theta \underline{E}_B) \, \underline{B}^{-1} \, \underline{P}_j - (c_j + \Theta e_j) \geqslant 0$$

Hence

$$\Theta(\underline{E}_B \, \underline{B}^{-1} \, \underline{P}_j - e_j) \geqslant - (\underline{C}_B \, \underline{B}^{-1} \, \underline{P}_j - c_j)$$

Thus, if

i) $(\underline{E}_B \, \underline{B}^{-1} \, \underline{P}_j - e_j) \geqslant 0$ for all j, the optimum solution will remain optimum for all values of Θ.

ii) $(\underline{E}_B \, \underline{B}^{-1} \, \underline{P}_j - e_j) < 0$ for at least one value j, there exists a characteristic value. If

$$\underline{\phi}_1 = \min_j \left\{ \frac{-(\underline{C}_B \, \underline{B}^{-1} \, \underline{P}_j - c_j)}{(\underline{E}_B \, \underline{B}^{-1} \, \underline{P}_j - e_j)} \;\middle|\; (\underline{E}_B \, \underline{B}^{-1} \, \underline{P}_j - e_j) < 0 \right\}$$

then

$$\Theta_1 = \Theta_0 + \phi_1 \quad \text{where } \Theta_0 = 0$$

Similar expressions can also be derived for the $i+1^{th}$ characteristic value.

3. <u>Variations in the coefficients of nonbasic vectors $\underline{P}_j$</u>

This analysis represents only a partial solution of an important problem in which the entire $\underline{A}$ matrix coefficients of a linear programming problem with upper bounds are varied with a parameter Θ. All the work done so far in the area of parametric programming is based on the fact that between two consecutive characteristic values of any problem, the $\underline{B}^{-1}$ matrix does not change, and a change in the optimum solution with θ can be calculated from this $\underline{B}^{-1}$ matrix. However, when the $\underline{A}$ matrix varies with θ, the $\underline{B}$ matrix will also vary with θ if it contains the elements of the $\underline{A}$ matrix, and no general formulas exist which directly give the value of $\underline{B}^{-1}$. Thus, by considering the changes in only the nonbasic vectors $\underline{P}_j$, the $\underline{B}^{-1}$ matrix will be preserved. Furthermore, such analysis will be carried out for only a limited range of θ. After calculating a characteristic value of θ, if it is found that a nonbasic $\underline{P}_j$ (which varies with θ) becomes basic, the problem thereafter lends itself poorly to a systematic analysis for the reason given above, and the analysis will be terminated.

The variations in nonbasic $\underline{P}_j$ can only affect the optimality of the problem if the variables of the problem are without upper bound constraints. However, for the problem with upper bounds, the feasibility of the problem may also be affected as shown below.

Let the variation in a nonbasic vector $\underline{P}_j$ be given by

$$\underline{P}_j + \theta \underline{F}_j$$

where

$$\underline{F}_j = (f_{1j} , f_{2j} , \cdots , f_{mj})^T$$

is a known vector of constants. Assume that the optimum solution for $\theta = 0$ has been calculated. The characteristic value of θ depends on two different criteria as follows.

a) Optimality: For $\theta > 0$, the solution will remain optimum as long as

$$\underline{C}_B\ B^{-1}\ (\underline{P}_j + \theta \underline{F}_j) - c_j > 0.$$

From this, the characteristic value based on optimality only, can be calculated. Let

$$\phi_1' = \min_j \left\{ \frac{-[\underline{C}_B\ \underline{B}^{-1}\ \underline{P}_j - c_j]}{\underline{C}_B\ \underline{B}^{-1}\ F_j} \;\middle|\; \underline{C}_B\ \underline{B}^{-1}\ \underline{F}_j < 0 \right\}$$

This value of ϕ_1' will be compared with a value obtained from the feasibility criterion.

b) Feasibility: At $\theta = 0$, the optimum solution is given by

$$\underline{0} \leqslant \underline{X}_B = \underline{B}^{-1}\ (\underline{P}_0 - \underline{D}_u\ \underline{U}_u) \leqslant \underline{U}_B$$

Since $\underline{D}_u$ consists of nonbasic vectors, for $\theta > 0$ the solution will remain optimum as long as

$$\underline{0} \leqslant \underline{X}_B = \underline{B}^{-1}\ (\underline{P}_0 - D_u\ U_u - \theta\ \underline{F}_u\ \underline{U}_u) \leqslant \underline{U}_B$$

From the expression above a value ϕ_1'' can be calculated so that

$$\theta_1 = \phi_1 = \min\ (\phi_1', \phi_1'')$$

The method will be illustrated by a numerical example.

Example II

Re-solve Example I (page 27) with the following changes. $\underline{P}_0$ and $\underline{U}$ vectors do not vary but the nonbasic vectors $\underline{P}_j$ vary as follows.

$$(1+\theta)x_1 + x_2 + (2-\theta)x_3 \leq 10$$

$$(2+\theta)x_1 + 4x_2 + [3-(1/2)\theta]x_3 \leq 16$$

Solution: At $\theta = 0$, the optimum solution idèntical to the previous example and is given by

	x_1'	x_5	x_3	
	1/2	5/4	7/4	22
x_4	-1/2	-1/4	5/4	4
x_2	-1/2	1/4	3/4	2

From the optimum tableau,

$$\underline{C}_B\,\underline{B}^{-1} = (0,5)\begin{pmatrix}1 & -1/4\\ 0 & 1/4\end{pmatrix} = (0,5/4)$$

$$\underline{C}_B\,\underline{B}^{-1}\,(\underline{F}_1',\ \underline{F}_5,\ \underline{F}_3) = (-5/4, 0, -5/8)$$

$$\phi_1' = \min\left\{\frac{1/2}{5/4},\ \frac{7/4}{5/8}\right\} = 2/5$$

$$-\underline{B}^{-1}\,\underline{F}_u\,\underline{U}_u = \begin{pmatrix}-3\\ -1\end{pmatrix}$$

Thus

$$\underline{X}_B = \begin{pmatrix}4\\ 2\end{pmatrix} + \begin{pmatrix}-3\theta\\ -1\theta\end{pmatrix}$$

This shows that the upper bounds of the current $\underline{X}_B$ will not be exceeded for any value of θ; on the other hand, $\underline{X}_B$ may become negative at ϕ_1'' where

$$\phi_1'' = \min\left\{\frac{4}{3},\ \frac{2}{1}\right\} = 4/3$$

Hence the characteristic value θ_1 is given by

$$\theta = \min (\phi_1' \ , \ \phi_1'') = 2/5$$

The alternate tableaus at $\theta = 2/5$ are given by

	x_1'	x_5	x_3	
	0	5/4	3/2	20
x_4	-4/5	-1/4	9/10	14/5
x_2	-3/5	1/4	7/10	8/5

	x_2'	x_5	x_3	
	0	5/4	3/2	20
x_4	4/3	-7/12	-1/30	14/3
x_1'	5/3	-5/12	-7/6	7/3

Since the nonbasic vector $\underline{P}_1$, which varies with θ has entered the basis, the analysis is stopped.

4. Simultaneous variations in $\underline{P}_0$, $\underline{U}$ and $\underline{C}$

It has been shown so far, that the variations in $\underline{P}_0$ and $\underline{U}$ affect only the feasibility of the problem, that is, a variable may either become negative or exceed its upper bound. Similarly, the variations in $\underline{C}$ affects only the optimality of the problem. Hence, if $\overline{\theta}_1$ represents the first characteristic value by varying only $\underline{P}_0$ and $\underline{U}$, and if $\overline{\overline{\theta}}_1$ represents the value by varying $\underline{C}$ only, the true characteristic value θ_1 is given by

$$\theta_1 = \min (\overline{\theta}_1 \ , \ \overline{\overline{\theta}}_1)$$

Similar considerations can be used to calculate the succeeding characteristic values.

Example III

Calculate the first characteristic value and the new solution for Example I (page 27) if the objective function is given by $x_0 = (3+\theta)x_1 + (5-\theta)x_2 + 2x_3$

Solution: The optimum tableau at $\theta = 0$ is identical to that of Example I and is not reproduced here. From the same example $\bar{\theta}_1$, the characteristic value when only $\underline{P}_0$ and $\underline{U}$ are changed is equal to 2. Also,

$$\underline{X}_B = (x_4 , x_2) \qquad \underline{E}_B = (0 , -1)$$

$$\underline{E}_B \, \underline{B}^{-1} (\underline{A} , \underline{I}) - \underline{E} = (3/2,0,-3/4,0,-1/4)$$

from which $\bar{\bar{\theta}}$ can be calculated and is equal to 7/3. Thus,

$$\theta_1 = \min(2,7/3) = 2$$

The alternate tableaus at $\theta = 2$ are given by

	x_1'	x_5	x_3	
	7/2	3/4	1/4	36
x_4	-1/2	-1/4	5/4	0
x_2	-1/2	1/4	3/4	2

	x_1'	x_4	x_3	
	2	3	4	36
x_5	2	-4	-5	0
x_2	-1	0	2	2

In the next chapter the parametric analysis for certain integer programming problems will be presented.

CHAPTER IV

APPLICATION OF PARAMETRIC PROGRAMMING FOR BOUNDED VARIABLES TO A CLASS OF PARAMETRIC INTEGER PROGRAMMING PROBLEMS

Integer programming is presently considered to be a fast moving field and many algorithms have been proposed to tackle this type of problem. There are four approaches capable of solving real problems to a varying degree of effectiveness. These approaches are:

1. Cutting plane methods.
2. Primal methods.
3. Branch and bound methods.
4. Partial enumeration methods.

Of these, only branch and bound methods will be discussed, since they involve upper bound techniques. (See [1, 2, 16].)

Branch and bound methods involve setting up a "tree" of linear programming problems. In every branch either an upper or a lower bound on some variable is imposed while ignoring all integrality constraints. The solution is obtained when a feasible integer solution is encountered on some branch of the tree, while the other branches have solutions of smaller magnitude (feasible or infeasible). Since consideration of any branch involves a bounded variable linear programming problem (without integer constraints), such a problem can be tackled by the new algorithm presented in this dissertation.

A natural extension of the development in the previous two chapters would be to conduct the parametric analysis for the integer problems. This discussion will be primarily restricted to the branch and bound methods, since the new algorithm can be directly applied in these methods.

Previous work in this area is almost nonexistent, probably due to the fact that as yet, the methods for solving integer programming problems themselves are not that powerful. Jensen [14] mentions this problem and states that the techniques for carrying out the sensitivity analysis of the integer problems probably will not exist until more efficient methods of solving integer models are discovered. He further states that in performing such analysis it may be necessary to rely on intuition and ingenuity rather than systematic procedure.

The difficulty involved in the analysis arises from the fact that the optimum integer solution may not coincide with an extreme point of the solution space. In order to use the regular linear programming methods, auxiliary integral valued upper and lower bound constraints must be used to force the optimum solution to be an extreme point of the new solution space. However, when the parametric analysis of such integer problems is considered, the parameter θ varies continuously; whereas the bounds on the auxillary constraints take only discrete integral values. Therefore, the new optimum integer point may not be an extreme point. The analysis presented here will only suggest some possibilities for a systematic analysis.

A. Linear variations in $\underline{P}_0$ vector

Let the problem be represented by

$$\text{maximize} \quad x_0 = \underline{C}\,\underline{X}$$

subject to

$$(\underline{A}\ ,\ \underline{I})\ \underline{X} = \underline{P}_0 + \theta \underline{R}_0$$

$$\underline{X} \geqslant \underline{0}\ , \quad \underline{X}\ \text{integer}, \quad \theta \geqslant 0.$$

Since this is a discrete programming problem, the parametric analysis for the regular linear programming problems does not apply here. This fact is best illustrated by a numerical example.

<u>Example I</u>

Solve the following parametric integer programming problem.

$$\text{maximize } x_0 = 7x_1 + 9x_2$$

subject to

$$-x_1 + 3x_2 \leqslant 6 + 5\theta$$

$$7x_1 + x_2 \leqslant 35 - 3\theta$$

$$x_1 \geqslant 0\ ,\ x_2 \geqslant 0; \quad x_1\ ,\ x_2\ \text{integers}; \quad \theta \geqslant 0$$

Solution: The problem is first solved for $\theta = 0$, using the branch and bound procedure. The optimum solution occurs in the branch

$x_1 \leq 4$, and $x_2 \leq 3$.

	x_1'	x_2'	
	7	9	55
x_3	1	-3	1
x_4	-7	-1	4

If the same optimum solution holds for $\theta \geq 0$, it will be given by

	x_1'	x_2'	
	7	9	55
x_3	1	-3	$1 + 5\theta$
x_4	-7	-1	$4 - 3\theta$

From the above tableau, the first characteristic value θ_1 = 4/3. The value of the objective function at this characteristic value will still be equal to 55.

The fallacy in the above solution is shown as follows. If this problem is re-solved for $\theta = 2/5$, it is found that the optimum integer solution is given by

$$x_0 = 64 \ , \ x_1 = 4 \ , \text{ and } x_2 = 4.$$

For $\theta = 4/3$, the optimum integer solution is

$$x_0 = 66 \ , \ x_1 = 3 \ , \text{ and } x_2 = 5.$$

It is clear from the above example that a new approach must be used. To explore any new method, a first step might be to analyze why the regular parametric linear programming method fails. This may shed some light on discovering other possible approaches. Two prime reasons can be stated. They are:

i) The auxillary upper and lower bound constraints hold only for a specific problem. Thus, if the value of $\underline{P}_0$ is changed these auxillary constraints may fail to enclose the feasible integer valued points of the new problem.

ii) When the values of the basic variables change at any value of Θ, the basis itself may not change, but the new integer optimum point may not be on the convex hull. With these facts in mind, the following analysis is presented.

The parametric integer programming problem may fall in one of the following two categories:

1. The parametric variation in $\underline{P}_0$ may be such that for all values of Θ under consideration, no new integer point is added to the solution space.

2. At some value of Θ under consideration, a new integer point may enter the solution space.

Since the analysis is preliminary, no attempt has been made to determine how any given parametric integer programming problem can be put in one of the two categories above. However, if a problem falls in the first category; that is, if the variation in $\underline{P}_0$ does not include any new integer point in the solution space, it is possible to determine the value of θ up to which the same optimum solution will hold, as shown below.

Consider a parametric integer programming problem in which the optimum solution for $\theta = 0$ is known and is represented by $\underline{X}^*$. If $\underline{X}'$ denotes a feasible integer point, then

$$\underline{C}\ \underline{X}^* \geqslant \underline{C}\ \underline{X}'$$

For $\theta > 0$ if $\underline{X}^*$ is still feasible, then it must also be optimum, since no new integer point entered the solution space. If for some value of $\theta > 0$ the optimum point $\underline{X}^*$ ceases to be optimum, then it must also be infeasible. Also, since the variation in θ is continuous, for some value of $\theta = \theta_1$, $\underline{X}^*$ must be on the boundary of the solution space. Initially for $\theta = 0$, $\underline{X}^*$ must be an extreme point of the solution space formed by the addition of the auxiliary constraints. If the auxiliary constraints are not affected while $\underline{P}_0$ changes, $\underline{X}^*$ will continue to be an extreme point as long as it is feasible and hence optimum. If $\underline{X}^*$ becomes infeasible, it will no longer be an extreme point. Therefore, the regular parametric linear programming analysis can be applied to calculate θ_1. However, once θ_1 is determined the new optimum solution must be calculated at $\theta_1 + \epsilon$. The new solution cannot be obtained from the previous tableau directly, and the problem has to be re-solved. The value θ_1 does not represent a characteristic value since the basic variables at $\theta = 0$ may still be basic for $\theta > \theta_1$. To distinguish this, the values of θ where the optimum integer solution shifts from one integer point to another may be called a "critical value."

Example II

Calculate the first critical value for Example I (page 39) if the first constraint is changed to

$$-x_1 + 3x_2 \leqslant 6 - \theta .$$

Solution: For this example the solution space does not add a new integer point as θ increases. The optimum solution for $0 \leqslant \theta \leqslant \theta_1$ will be given by

	x_1'	x_2''	
	7	9	55
x_3	1	-3	$1 - \theta$
x_4	-7	-1	$4 - 3\theta$

From the tableau

$$\theta_1 = \min\ (1/1\ ,\ 4/3) = 1.$$

For $\theta > 1$, the optimum integer point will shift from $x_1 = 4$, and $x_2 = 3$. However, the new optimum solution has to be calculated by branch and bound method by substituting $\theta = 1 + \epsilon$.

This analysis shows that if many critical values of θ exist, the problem may become quite lengthy. Probably, a more useful purpose of this procedure would be to give an idea of the range of θ for which a given integer point remains optimum.

If the integer programming problem is a pure integer problem with an all integer starting tableau , the following two procedures could be explored.

1. Let the i^{th} constraint of the problem be given by

$$a_{i1}\, x_1 + a_{i2}\, x_2 + \dots\, a_{i,m+n}\;\; x_{m+n} = b_i + \theta\, r_i$$

The optimum integer solution will change its value only when $b_i + \theta r_i$ is an integer. Therefore, θ must be a multiple of

$$\frac{1}{|r_i|}$$

Thus, the first value of θ which is likely to be a critical value is given by

$$\theta = \min_i \left\{ \frac{1}{|r_i|} \right\}$$

In Example I (page 39) the right hand side elements of the constraints are $6+5\theta$ and $35-3\theta$ respectively. Therefore the following value of θ will be checked in the given sequence.

$$\theta = 1/5\;,\; 1/3,\; 2/5,\; 3/5,\; 2/3,\; \dots$$

As it turns out, the first critical value is $\theta = 2/5$. Hence, the search for a new solution for $\theta = 1/5 + \epsilon$, and $\theta = 1/3 + \epsilon$ will indicate that the solution has not changed. Note that at the first critical value of θ the new right hand side is $8+5\theta$ and $33\ 4/5-3\theta$, which has a noninteger element. Thus further modifications are necessary before the procedure can continue.

2. When $\underline{P}_0$ varies with Θ, a new integer point will enter the solution space at a certain value of Θ only when one of the constraint passes through this point. Hence, if a procedure can be developed which indicates that a constraint is passing through a new feasible integer point, this point can be checked for optimality.

B. Linear variations in $\underline{C}$ vector

Similar difficulties exist when the changes in $\underline{C}$ vector are considered. If the convex polyhydron formed by the addition of auxiliary constraints has all integer extreme points only, this analysis would present no problem. Unfortunately, the auxiliary constraints derived by the existing methods, both branch and bound and cutting plane, do not give rise to this polyhydron. Some work has been done in this respect by Gomory [12] and Hu [13], though not for the purpose of parametric analysis.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

I. Summary

This dissertation has presented a primal-dual algorithm for solving the general linear programming problem with upper bounds. By combining the rules of the existing primal and dual methods artificial augmentations are eliminated. The product form of the algorithm is also developed. This should assist in improving its computational efficiency.

Parametric analysis for bounded variables has been developed for changes in the coefficients of the objective function, the right hand side elements, and the upper bounds.

The problem of parametric integer programming has been investigated as a natural extension of parametric programming for upper-bounded variables. The investigation in this case is limited only to changes in the right-hand side elements of the problem. The analysis of the other cases gives rise to many difficulties which are created by the nonconvexity of the integer solution space.

II. Recommendations for further Research

The comparison presented in Table 1 at the end of this chapter (page 50) presents an overall view of the size of the problem for different methods, and illustrates the possible advantages of the new algorithm in terms of the number of variables and the number of constraints. However, further judgement cannot be made unless a comparison on the basis of solving some test problems is made. Wolfe's [23] studies, which compare different linear programming methods on the basis of iterations, operations, and passes, give no conclusive results. Similar results are likely to occur when PSB, DSB, and PDSB are compared. However, nothing can be said about this, before such tests are carried out. Furthermore, these tests may present some useful information.

Usually, a primal-dual algorithm and parametric linear programming are considered to be equivalent. Kelly [15] states that the fact that the parametric programming algorithm and the primal-dual algorithm had different motivations for their developments makes this equivalence of some interest. Therefore, the parametric analysis presented here was a natural extension of the new primal-dual algorithm. The development of the parametric analysis for the A matrix hinges on being able to calculate an inverse of $\underline{B}$ from the old value of the inverse.

Some points not considered in the developments are as follows. In the discussion of the new algorithm, the problem of degeneracy and of variables which are unrestricted in sign was not considered. However, these problems can be handled in the same way as for the problems without upper bounds. In the parametric analysis the effect of variations by more than one parameter might produce some interesting results.

Dantzig [6] discusses the problem of artificial variables in the dual simplex method. This idea could be probably extended as follows. Given a regular linear programming problem, add the slack and the artificial variables as in the regular primal simplex method [3]. Solve the problem by the primal simplex method with the following two modifications:

i) Since the artificial variables can be considered as variables bounded from above and below by zero, they can be dropped out of the solution once they leave the basis.

ii) If the problem becomes dual feasible and the artificial variables are in the basis, carry out the dual iterations to remove them. Similarly when the primal iteration cannot be carried out try a dual iteration.

Complete details of this procedure can be developed on the similar lines to that of the new primal-dual algorithm.

The analysis presented for the integer programming problems is by no means complete. It is hoped that it may give rise to some new results. However, further work needs to be done before any significant claims can be made.

TABLE 1

COMPARISON OF DIFFERENT ALGORITHMS FOR BOUNDED VARIABLES

Algorithm	Number of Variables	Number of Constraints	Number of Artificial Variables	Number of Implicit Upper Bound Constraints	Size of the Tableau Tucker's Scheme [21]
Primal Algorithm PSB Dantzig [5]	$m+n+m_2+m_3$	m	m_2+m_3	$\bar{n}_1+\bar{n}_2$	$mX(n+m_2+m_3)$
Dual Algorithm DSB Wagner [22]	$m+n+m_3$	$m+m_3$	$n_2-\bar{n}_2$	$\bar{n}_1+\bar{n}_2$	$(m+m_3)x(n)$
Proposed Algorithm PDSB	$m+n$	m	0	$\bar{n}_1+\bar{n}_2+m_3$	mxn

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APPENDIX A

DESCRIPTION OF EXISTING ALGORITHMS FOR BOUNDED VARIABLES AND ZIONTS' CRISS-CROSS METHOD

I. Dantzig's Primal Simplex Algorithm for Bounded Variables (PSB) [5]

This algorithm is almost like the simplex method except for a few modifications. The upper bound constraints are treated implicitely and the tableau consists of m constraints. The criterion for entering a variable is identical to the simplex criterion, but attention is paid to the following three quantities

a) the upper bound of the entering variable

b) the level at which the new variable can enter by the feasibility criterion identical to the simplex method

c) the level at which the new variable can enter without violating the upper bounds of other basic variables. Wagner [22] states that this procedure is not simple and the criterion is more involved conceptually and computationally.

Besides these disadvantages, artificial variables have to be used if an initial feasible solution is not available. In fact, this is a basic disadvantage with the simplex method itself.

II. Wagner's Dual Simplex Algorithm for Bounded Variables (DSB) [22]

In this algorithm each equality constraint is converted into two inequality constraints so that the problem can be represented as follows:

$$\text{maximize } x_0 = \underline{C}_1 \underline{X}_1 + \underline{C}_2 \underline{X}_2 \qquad \underline{C}_1 \leqslant 0, \; \underline{C}_2 > 0$$

$$\text{subject to} \qquad \underline{A}_1 \underline{X}_1 + \underline{A}_2 \underline{X}_2 \leqslant \underline{P}_0 \, ,$$

$$0 \leqslant \underline{X}_1 \leqslant \underline{U}_1 \; , \quad 0 \leqslant \underline{X}_2 \leqslant \underline{U}_2$$

where the elements of $\underline{U}_1$ and $\underline{U}_2$ have infinite value (or a very large positive value) corresponding to the variables which do not have upper bounds.

Before any iteration is carried out the following substitution is made in the problem.

$$\underline{X}_2' = \underline{U}_2 - \underline{X}_2$$

The upper bound constraints are treated implicitly and the substitution shown above makes the problem dual feasible. The optimum solution can be obtained from the initial tableau by using a slightly modified dual simplex method.

Wagner [22] has stated a few minor drawbacks of the algorithm and states that these will be substantially outweighed by the relative simplicity of PSB over DSB. However, there are additional drawbacks described as follows:

a) Every equality constraint must be converted into two inequalities. This can make the problem, and hence the number of iterations, large if many equality constraints are present.

b) Substitution of any unbounded variable by a large upper bound value (say M) makes the problem dual feasible but the right hand side becomes a function of this large value M.

III. Zionts' Criss-Cross Method [24]

This method works on almost a similar set of rules compared to that of the proposed PDSB algorithm except for the following.

No provision is made for variables with upper bounds. Even if a problem contains no upper bounds the excess variables corresponding to the equality constraints are always treated as bounded variables in PDSB. No such special device is used in the criss-cross method. The criss-cross method can handle the unrestricted variables and a similar provision, although not incorporated at the present time, can be made in the PSDB method.

The criss-cross method also consists of switching back and forth from primal to dual iterations as necessity arises and the rules for terminating the procedure are also very similar to the PDSB method.

Consider the following problem:

$$\text{Max } x_0 = 2x_1 - x_2$$

subject to

$$-4x_1 + 5x_2 = 13$$
$$-3x_1 + 4x_2 = 11$$
$$x_1 - 2x_2 \leq 2$$
$$x_1 \geq 0 \ , \ x_2 \geq 0$$

The optimum solution to the problem is given by $x_1=3$ and $x_2=5$. Using Ziont's criss-cross method the solution is given by the following tableaus.

	x_1	x_2	
	-2	1	0
x_4	-4	5	13
x_5	-3	4	11
x_6	1	-2	2

Variables x_4 and x_5 are arbitrary, x_1 enters, x_6 leaves.

	x_6	x_2	
	2	-3	4
x_4	4	-3	21
x_5	3	-2	17
x_1	1	-2	2

No variable can enter from primal, no variable can leave via dual. Therefore, the solution is infeasible/unbounded.