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SPECTRAL THEORY FOR SINGULAR LINEAR SYSTEMS

OF HAMILTONIAN DIFFERENTIAL EQUATIONS

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SPECTRAL THEORY FOR SINGULAR LINEAR SYSTEMS
OF HAMILTONIAN DIFFERENTIAL EQUATIONS

1. Introduction. In 1910 Weyl [11] inaugurated the modern theory of singular self-adjoint differential operators, considering, in particular, second order differential operators with real coefficients. Since then his results have been generalized by various authors to the case of arbitrary even order self-adjoint differential equations with real coefficients. Also, in 1954 Coddington [2] treated the case of an arbitrary n-th order self-adjoint differential equation with complex coefficients. He obtained the Parseval equality and spectral expansion associated with the singular case directly, through the consideration of such a problem as a limiting case of corresponding self-adjoint two-point boundary value problems on compact subintervals of the reals. Using the same general concept, Coddington and Levinson [3] derived a Green's function, which in turn enabled them to obtain the spectral matrix for a singular problem involving a linear homogeneous n-th order differential operator, and then proceeded to derive the Parseval equality and spectral expansions.

Recently Brauer [1] treated similar problems which involve a definitely self-adjoint vector differential operator, of the sort that has been treated by Reid [6] and others. He obtained the Green's matrix and spectral matrix through the use of the spectral theorem and the theory of direct integrals.

In the spirit of variational methods and principles, Reid [7,8] developed the general theory of self-adjoint two-point boundary value problems associated with a formally self-adjoint Hamiltonian vector differential operator defined on a given compact subinterval of the reals. The main purpose of the present paper is to employ the general method of Coddington and Levinson [3; Chapter 10, Sections 4,5] to derive the Green's matrix, spectral matrix, the Parseval equality, and spectral expansion for a singular linear Hamiltonian vector differential system which includes as special instances the Euler vector differential equation and a system that is equivalent to a self-adjoint 2n-th order scalar differential equation.

In Section 4 the existence and the properties of a Green's matrix are treated. In Section 5 there is obtained the spectral matrix, and finally, the Parseval equality and the expansion theorems are derived in Section 6.

Matrix notation is used throughout; in particular, matrices of one column are called vectors, and for a vector $y = (y_i)$, $(i = 1,\cdots,n)$, the norm $|y|$ is given by $\left(\sum_{i=1}^{n} |y_i|^2\right)^{1/2}$. We denote by E_n and 0_n the $n \times n$ identity and zero matrices, respectively, and when there is no ambiguity as to dimensions, we use merely the symbols E and 0. The conjugate transpose of a matrix M is designated by M^*. If A is an Hermitian matrix, then the symbol $A > 0$ ($A \geq 0$) signifies that A is positive (non-negative) definite; in general, $A > B$ ($A \geq B$) signifies that A and B are Hermitian matrices of the same dimension and such that $A - B$ is positive (non-negative) definite.

A vector function is called locally absolutely continuous on an open interval I of the reals when it is absolutely continuous on each compact sub-interval of I, and a matrix function is called continuous, differentiable, integrable, of bounded variation, locally absolutely continuous, etc., when each element of the matrix possesses the specified property. If $M(t) = [M_{ij}(t)]$ denotes an $n \times m$ matrix function, then we define $\int M(t)dt \equiv [\int M_{ij}(t)dt]$ and $M'(t) = [M'_{ij}(t)]$. For a given interval J of the real line, the symbols C^m_n, $_n$ and ^p_n are used to denote the class of n-dimensional vector functions y(t) on J which are respectively m times continuously differentiable, (Lebesgue) integrable, and (Lebesgue) measurable with $|y(t)|^p$ integrable on J. If M(t,s) is a matrix function of two variables t and s, then, for brevity, we use the symbols $M^{[1,0]}(t,s)$ and $M^{[0,1]}(t,s)$ to designate the partial derivative of M with respect to t and s, respectively. If $M \equiv [M_{jk}]$, $N = [N_{jk}]$, $(j = 1,\cdots,n;$ $k = 1,\cdots,m)$, are $n \times m$ matrices, for typographical simplicity the symbol (M;N) will denote the $2n \times m$ matrix $[S_{ik}]$ with $S_{ik} = M_{ik}$, $S_{n+i,k} = N_{ik}$, $(i = 1,\cdots,n;\ k = 1,\cdots,m)$. In particular, if $u = (u_i)$ and $v = (v_i)$ are n-dimensional vectors then (u;v) denotes the 2n-dimensional vector $y = (y_i)$ such that $y_i = u_i$ and $y_{n+i} = v_i$, $(i = 1,\cdots,n)$.

2. Hamiltonian vector differential system. We are concerned with a Hamiltonian vector differential system of the type

$$\text{(2.1)}\qquad \begin{aligned} L_1[u,v](t) &\equiv -v'(t) + C(t)u(t) - A^*(t)v(t) = 0, \\ L_2[u,v](t) &\equiv u'(t) - A(t)u(t) - B(t)v(t) = 0, \end{aligned} \qquad t \in I,$$

where $I = (c,d)$ is a given open interval of reals such that (2.1) is singular both at c and d, while u(t) and v(t) are n-dimensional complex-valued vector functions, and the coefficients A(t), B(t), and C(t) are given $n \times n$ complex-valued matrix functions.

By definition a solution of (2.1) is a pair of n-dimensional vector functions u(t) and v(t) belonging to $C_n^1(I)$, and satisfying (2.1) for $t \in I$. Throughout this paper we assume the following hypothesis.

(H_1) _On_ I _the matrix functions_ A(t), B(t), _and_ C(t) _are continuous_, A(t) _and_ B(t) _are continuously differentiable_, _and_ B(t), C(t) _are Hermitian_. _Moreover_, _the system_ (2.1) _is identically normal in the sense that if_ (u(t);v(t)) = (0;v(t)) _is a solution of_ (2.1) _on any non-degenerate sub-interval of_ I, _then_ $v(t) \equiv 0$ _on that interval_.

The system (2.1) may be expressed as a single 2n-dimensional vector differential equation

$$(2.2)\qquad \mathcal{L}[y](t) \equiv \mathcal{J}y'(t) + \mathcal{A}(t)y(t) = 0, \qquad t \in I,$$

where

$$y(t) = (u(t);v(t)), \qquad \mathcal{A}(t) = \begin{bmatrix} C(t) & -A^*(t) \\ -A(t) & -B(t) \end{bmatrix}, \qquad \mathcal{J} = \begin{bmatrix} 0 & -E_n \\ E_n & 0 \end{bmatrix}.$$

As $\mathcal{J} = -\mathcal{J}^*$ and $\mathcal{A}(t) = \mathcal{A}^*(t)$, the differential operator $\mathcal{L}$ is identical with its formal Lagrange adjoint

$$\mathcal{L}^*[y](t) = -\mathcal{J}^*y'(t) + \mathcal{A}^*(t)y(t) = 0, \qquad t \in I.$$

If y(t) and z(t) are of class $C_{2n}^1(I)$, then the Lagrange identity in integral form is given by

$$(2.3)\qquad \int_J (\mathcal{L}[y],z)dt - \int_J (y,\mathcal{L}[z])dt = z^*(t)\mathcal{J}y(t)\Big|_{t=a}^{t=b},$$

where J = [a,b] is an arbitrary compact subinterval of I. In particular, if we set $y(t) = (u_1(t);v_1(t))$, $z(t) = (u_2(t);v_2(t))$, and y, z are such that $L_2[u_i,v_i](t) = 0$, $(i = 1,2)$, on I, then relation (2.3) yields the equation

$$(2.4) \qquad \int_J (L_1[u_1,v_1],u_2)dt - \int_J (u_1,L_1[u_2,v_2])dt = (v_2^*u_1 - u_2^*v_1)\Big|_{t=a}^{t=b} .$$

Following Reid [7,8], if $I = (c,d)$ is a given open interval then we denote by $\mathfrak{D}[I]$ the set of n-dimensional vector functions $u(t)$ which are locally absolutely continuous on I and such that $L_2[u,v](t) = 0$ on I for some $v(t) \in \mathfrak{L}_n^2(I)$. Similarly, we denote by $\mathfrak{D}_0[I]$ the set of all $u(t) \in \mathfrak{D}[I]$ such that $u(t)$ has a compact support on I. If $u_i \in \mathfrak{D}_0[I]$, $(i = 1,2)$, with corresponding v_i which are of class $C_n^1(I)$, then from (2.4) we have

$$\int_I (L_1[u_1,v_1],u_2)dt = \int_I (u_1,L_1[u_2,v_2])dt.$$

A $2n \times 2n$ matrix function $Y(t)$ is called a <u>fundamental matrix</u> for (2.2) if the $2n$ column vector functions of $Y(t)$ are linearly independent solutions of (2.2) on I.

3. <u>Preliminary results</u>. Let $\delta = [a,b]$ be an arbitrary, but fixed, compact subinterval of $I = (c,d)$, and let $K(t)$ be a non-identically vanishing $n \times n$ matrix function defined on I which is continuous and Hermitian.

We now consider the two-point self-adjoint boundary value problem

$$(\mathcal{B}_\delta) \quad \begin{array}{ll} \text{(a)} & L_1[u,v|\lambda](\lambda) \equiv L_1[u,v](t) - \lambda K(t)u(t) = 0,\ L_2[u,v](t) = 0,\ t \in \delta, \\ \text{(b)} & \mathcal{M}_\delta[u,v] \equiv M_1u(a) + M_2v(a) + M_3u(b) + M_4v(b) = 0, \end{array}$$

involving the characteristic parameter λ, where each coefficient matrix M_i, $(i = 1,2,3,4)$, is of dimension $2n \times n$ and the $2n \times 4n$ matrix

$\mathcal{M}_\delta = [M_1\ M_2\ M_3\ M_4]$ has rank 2n. Corresponding to (2.2), system $(\mathcal{B}_\delta)$ may be expressed in terms of the 2n-dimensional vector function $y(t) = (u(t);v(t))$ as

$$(3.1)_\delta \quad \begin{array}{ll} \text{(a)} & \mathcal{L}[y|\lambda](t) \equiv \mathcal{L}[y](t) - \lambda\mathcal{K}(t)y(t) = 0, \qquad t \in \delta, \\ \text{(b)} & \mathcal{M}_\delta[y] \equiv \mathcal{M}_\delta[u,v] = 0, \end{array}$$

where $\mathcal{K}(t)$ is the $2n \times 2n$ matrix function defined by $\mathcal{K}(t) = \operatorname{diag}\{K(t),0\}$.

Since we are dealing with self-adjoint boundary problems, the boundary condition $\mathcal{M}_\delta[y] = 0$ must satisfy the self-adjointness condition; that is, if y and z belong to $C^1_{2n}(\delta)$ then

$$\int_\delta (\mathcal{L}[y],z)dt = \int_\delta (y, \mathcal{L}[z])dt$$

whenever $\mathcal{M}_\delta[y] = 0$ and $\mathcal{M}_\delta[z] = 0$. It is to be noted that in view of (2.3) the above relation is satisfied if and only if $z^*(b)\mathcal{J}y(b) - z^*(a)\mathcal{J}y(a) = 0$ whenever $\mathcal{M}_\delta[y] = 0$ and $\mathcal{M}_\delta[z] = 0$.

Throughout our discussion in this paper we assume also the following hypothesis.

(H_2) <u>The</u> $n \times n$ <u>Hermitian matrix function</u> $K(t)$ <u>is non-negative definite on</u> I, <u>and not all</u> λ <u>are proper values for</u> $(\mathcal{B}_\delta)$.

Following Reid [8; Sec. VII.10], for $u_i(t) \in \mathcal{L}^2_n(J)$, $(i = 1,2)$, where J is a compact subinterval of I, we define an Hermitian functional $\underset{\sim}{K}_J$ on $\mathcal{L}^2_n(J) \times \mathcal{L}^2_n(J)$ by

$$\underset{\sim}{K}_J[u_1,u_2] = \int_J (K(t)u_1(t),u_2(t))dt;$$

$$\underset{\sim}{K}_J[u_1] = \underset{\sim}{K}_J[u_1,u_1].$$

In view of hypothesis (H_2) we have the Cauchy-Schwarz inequality $|\underset{\sim}{K}_J[u_1,u_2]|^2 \le \underset{\sim}{K}_J[u_1]\underset{\sim}{K}_J[u_2]$. The non-negative square root of $\underset{\sim}{K}_J[u]$ is called the K-norm of $u(t)$ over J, and, for brevity, is denoted by $\|u\|_J$, or merely by $\|u\|$ when there is no ambiguity.

For later reference we make the following observations as to $(\mathcal{B}_\delta\text{-a})$:

(i) if the matrix function $B(t)$ is non-singular on δ, then the system $(\mathcal{B}_\delta\text{-a})$ is a canonical representation of the n-dimensional Euler vector differential equation

$$(3.1') \quad (R(t)u'(t)+Q(t)u(t))' - (Q^*(t)u'(t)+P(t)u(t)) = \lambda K(t)u(t), \ t \in \delta,$$

in terms of the canonical variables $u(t)$ and $v(t) = R(t)u'(t) + Q(t)u(t)$, where $R = B^{-1}$, $Q = -B^{-1}A$, $P = C + A^*B^{-1}A$, and K is non-singular;

(ii) the system $(\mathcal{B}_\delta\text{-a})$ also contains as a special instance a system which is equivalent to a self-adjoint 2n-th order scalar differential equation

$$(3.1'') \quad \begin{aligned}(-1)^n[r_n(t)u^{[n]}(t)]^{[n]} + (-1)^{n-1}[r_{n-1}(t)u^{[n-1]}(t)]^{[n-1]} + \cdots +\\ r_o(t)u(t) = \lambda k(t)u(t), \quad t \in \delta,\end{aligned}$$

with real coefficients, $r_m(t) \neq 0$ and $k(t) \neq 0$ for $t \in \delta$. In this case the corresponding $n \times n$ matrix functions $A(t)$, $B(t)$, $C(t)$, and $K(t)$ are given by $A_{i,i+1}(t) \equiv 1$, $(i = 1,\cdots,n-1)$, $A_{ij}(t) \equiv 0$ otherwise; $B(t) = \operatorname{diag}\{0_{n-1},[r_n(t)]^{-1}\}$; $C(t) = \operatorname{diag}\{r_o(t),r_1(t),\cdots,r_{n-1}(t)\}$; $K(t) = \operatorname{diag}\{k(t),0_{n-1}\}$.

Thus, the self-adjoint system $(\mathcal{B}_\delta)$ includes as special instances the corresponding two-point self-adjoint boundary problems of these two important cases.

Concerning the basic properties of the proper values, proper vector functions, and the Green's matrix associated with the self-adjoint boundary value problem $(\mathcal{B}_\delta)$, we state two theorems, whose proofs may be found in Reid [8; Theorems IV.5.1, VII.10.1].

THEOREM 3.1. *For a self-adjoint system* $(\mathcal{B}_\delta)$ *satisfying hypotheses* (H_1) *and* (H_2) *all proper values are real, and the set of the proper values is at most denumerably infinite with no finite limit point. If* $(u_i(t);v_i(t))$, $(i = 1,2)$, *are two proper vector functions of* $(\mathcal{B}_\delta)$ *corresponding to distinct proper values* λ_i, $(i = 1,2)$, *then they are* K_δ*-orthogonal in the sense that* $K_\delta[u_1,u_2] = 0$. *Moreover, if* λ *is a proper value of index* $m(\lambda)$, *then the linear vector space of solutions of* $(\mathcal{B}_\delta)$ *for this value* λ *has a basis* $(u_i(t);v_i(t))$ *which is* K_δ*-orthonormal in the sense that* $K_\delta[u_i,u_j] = \delta_{ij}$, $(i,j = 1,\cdots,m(\lambda))$.

It is to be noted that the last conclusion of the above theorem is implied by hypothesis (H_2), since if $y(t) = (u(t);v(t))$ is a proper solution of $(\mathcal{B}_\delta)$ for a proper value λ we have $0 \le K_\delta[u]$, and the equality sign would hold if and only if $K(t)u(t) = 0$ for t a.e. on δ, in which case all values of λ would be proper value of $(\mathcal{B}_\delta)$. In particular, if $y(t) = (u(t);v(t))$ is a proper solution of $(\mathcal{B}_\delta)$ for a proper value λ then $u(t) \not\equiv 0$ on $\delta = [a,b]$, so that the boundary value problem $(\mathcal{B}_\delta)$ is normal.

THEOREM 3.2. *If* $(\mathcal{B}_\delta)$ *satisfies hypotheses* (H_1) *and* (H_2), $\delta = [a,b]$, *and* λ *is not a proper value of* $(\mathcal{B}_\delta)$, *then there exists a unique* $2n \times 2n$ *matrix function* $\mathfrak{G}(t,s,\lambda)$, *called the Green's matrix, such that for arbitrary* $f_o(t) \in \mathcal{L}^2_{2n}(\delta)$ *the* 2n-*dimensional vector function* $y(t)$ *defined by*

$$y(t) = \int_\delta \mathfrak{G}(t,s,\lambda)f_o(s)ds$$

is the unique solution of the boundary problem

$$\mathcal{L}[y|\lambda](t) = f_o(t), \quad \mathcal{M}_\delta[y] = 0.$$

Moreover, the Green's matrix $\mathfrak{G}(t,s,\lambda)$ *possesses the following properties.*

(i) *On each of the triangular regions*

(3.2) $\Delta_1 = \{(t,s) \in \delta \times \delta \mid a \le s < t \le b$ and $\Delta_2 = \{(t,s) \in \delta \times \delta \mid a \le t < s \le b\}$,

(a) *the matrix* $\mathcal{G}(t,s,\lambda)$ *is continuous in* (t,s);

(b) *the partial derivative* $\mathcal{G}^{[1,0]}(t,s,\lambda)$ *exists, and is continuous in* (t,s); *moreover,* $\mathcal{G}(t,s,\lambda)$ *satisfies the differential equation* $\mathcal{L}[\mathcal{G}(\cdot,s,\lambda)|\lambda](t) = 0$;

(c) *as a function of the complex number* λ *the matrix* $\mathcal{G}(t,s,\lambda)$ *is regular at each point which is not a proper value; in particular,* $\mathcal{G}(t,s,\lambda)$ *is holomorphic in the half-planes* $\operatorname{Im}\lambda > 0$ *and* $\operatorname{Im}\lambda < 0$;

(d) *if* λ *is not a proper value, then*

(3.3)
$$\mathcal{G}(t,s,\lambda) = [\mathcal{G}(s,t,\bar{\lambda})]^*;$$

(e) *if* $(s_o,s_o) \in \delta \times \delta$, *then* $\mathcal{G}(t,s,\lambda)$ *has the finite limits*

(3.4) $\mathcal{G}_j(s_o,s_o,\lambda) \equiv \lim \mathcal{G}(t,s,\lambda)$, as $(t,s) \to (s_o,s_o)$, $(t,s) \in \Delta_j$, $(j = 1,2)$;

moreover, these two limits satisfy the relation

(3.5)
$$\mathcal{G}_1(s_o,s_o,\lambda) - \mathcal{G}_2(s_o,s_o,\lambda) = \mathcal{J}^*;$$

(ii) *on each of the triangular regions* $\{(t,s) \in \delta \times \delta \mid a < s < t \le b\}$ *and* $\{(t,s) \in \delta \times \delta \mid a \le t < s < b\}$, *as a function of* t, *the matrix* $\mathcal{G}(t,s,\lambda)$ *satisfies the boundary condition; that is,* $\mathcal{M}_\delta[\mathcal{G}(\cdot,s,\lambda)] = 0$.

It is to be noted that relations (i-b) and (i-d) imply that the matrix $\mathcal{G}^{[0,1]}(t,s,\lambda)$ exists, and is continuous in (t,s) on each of the triangular regions Δ_1 and Δ_2.

In the sequel a partitioned matrix form of $\mathcal{G}(t,s,\lambda)$ will be used frequently, and to this end we set

(3.6)
$$\mathcal{G}(t,s,\lambda) = \begin{bmatrix} G_{11}(t,s,\lambda) & G_{12}(t,s,\lambda) \\ G_{21}(t,s,\lambda) & G_{22}(t,s,\lambda) \end{bmatrix} = [G_{ij}(t,s,\lambda)], \quad (i,j = 1,2),$$

where each G_{ij} is an $n \times n$ matrix function. In particular, conclusion (i-d) of the above theorem is equivalent to the relations

$$G_{ij}(t,s,\lambda) = [G_{ji}(s,t,\bar{\lambda})]^*, \quad (i,j = 1,2).$$

If $f_o(t)$ in Theorem 3.2 is a particular vector function of the form $f_o(t) = (f(t);0)$, where $f(t) \in \mathcal{L}_n^2(\delta)$, then the following results are immediate.

COROLLARY. <u>If</u> Im $\lambda \neq 0$, <u>then for an arbitrary</u> $f(t) \in \mathcal{L}_n^2(\delta)$ <u>the</u> 2n-<u>dimensional vector function</u> $(u(t);v(t)) = (u^{(1)}(t);u^{(2)}(t))$ <u>defined by</u>

$$u^{(i)}(t) = \int_\delta G_{i1}(t,s,\lambda)f(s)ds, \quad (i = 1,2),$$

<u>is the solution of the differential system</u>

(a) $L_1[u,v|\lambda](t) = f(t), \quad L_2[u,v](t) = 0, \quad t \in \delta,$

(b) $\mathcal{M}_\delta[u,v] = 0.$

In connection with Theorem 3.2 the reader is referred to Reid [8; Theorem VII.8.2] for additional properties of the matrix functions $G_{i1}(t,s,\lambda)$, $(i = 1,2)$. In the following, <u>whenever it is desired to emphasize the dependence on</u> δ <u>of</u> $\mathfrak{G}(t,s,\lambda)$, <u>we write</u> $\mathfrak{G}(t,s,\lambda;\delta)$.

We shall now establish the following result which is basic to our discussion in this section.

LEMMA 3.3. <u>If</u> $g(t) \in \mathcal{L}_n^2(\delta)$, <u>then for</u> Im $\lambda \neq 0$ <u>the solution</u> $(u(t);v(t))$ <u>of the differential system</u>

(3.7) (a) $L_1[u,v|\lambda](t) = K(t)g(t), \quad L_2[u,v](t) = 0, \quad t \in \delta,$

(b) $\mathcal{M}_\delta[u,v] = 0,$

<u>satisfies the inequality</u>

(3.8) $$\|u\|_\delta \leq |\text{Im}\ \lambda|^{-1}\|g\|_\delta .$$

Suppose that $(u(t);v(t))$ is the solution of (3.7). Since the problem $(\mathcal{B}_\delta)$ is self-adjoint, it follows that

$$(3.9) \qquad \int_\delta (L_1[u,v],u)dt - \int_\delta (u,L_1[u,v])dt = 0,$$

and using the first relation of (3.7-a) in (3.9) we obtain the result $\operatorname{Im} \lambda\, \underset{\sim}{K}_\delta[u] + \operatorname{Im} \underset{\sim}{K}_\delta[g,u] = 0$. Now applying the Cauchy-Schwarz inequality to $\underset{\sim}{K}_\delta[g,u]$, and using the fact that $\underset{\sim}{K}_\delta[u] \geq 0$, we obtain (3.8).

If $Y(t)$ is a $2n \times 2n$ fundamental matrix for (2.2), then for $f_o(t) = (f(t);0)$ with $f(t) \in \mathcal{L}_n^2(I)$ a solution of $\mathcal{L}[y](t) = f_o(t)$, $t \in I$, can be obtained by the method of variation of parameters. That is, there exists a $2n \times 2n$ matrix function $\mathcal{H}(t,s)$ such that the 2n-dimensional vector function $y(t)$ defined by $y(t) = \int_\delta \mathcal{H}(t,s)f_o(s)ds$ is a solution of $\mathcal{L}[y](t) = f_o(t)$ on each compact subinterval δ of I; in fact, this result holds for $\mathcal{H}(t,s)$ defined as

$$(3.10) \qquad \mathcal{H}(t,s) = \frac{1}{2} Y(t)Y^{-1}(s)\mathcal{J}^* \operatorname{sgn}(t-s).$$

We observe that this matrix $\mathcal{H}(t,s)$ possesses the following properties.

(i) On each of the triangular regions $\Delta_1 = \{(t,s) \in I \times I \mid s < t\}$ and $\Delta_2 = \{(t,s) \in I \times I \mid t < s\}$ we have that:

(a) the matrix function $\mathcal{H}(t,s)$ is continuous in (t,s); moreover, $\mathcal{H}^{[1,0]}(t,s)$ and $\mathcal{H}^{[0,1]}(t,s)$ exist, and are continuous in (t,s);

(b) as a function of t the matrix function $\mathcal{H}(t,s) = [H_{ij}(t,s)]$, $(i,j = 1,2)$, satisfies the differential equation $\mathcal{L}[\mathcal{H}(\cdot,s)](t) = 0$; that is,

$$(3.11) \qquad L_i[H_{1j}(\cdot,s),H_{2j}(\cdot,s)](t) = 0, \qquad (i,j = 1,2);$$

(c) if $(s_o,s_o) \in I \times I$, then the limits $\mathcal{H}_j(s_o,s_o)$, $(j = 1,2)$, in the sense of (3.4), have finite values, and they satisfy the relation

$$(3.12) \qquad \mathcal{H}_1(s_o,s_o) - \mathcal{H}_2(s_o,s_o) = \mathcal{G}^*.$$

(ii) _If_ $\delta = [a,b] \subset I$ _and_ $(s_o,s_o) \in \delta \times \delta$, _then the matrix function_ $\mathcal{F}(t,s,\lambda) = \mathcal{G}(t,s,\lambda) - \mathcal{H}(t,s)$ _has finite limit values_ $\mathcal{F}_j(s_o,s_o,\lambda)$, $(j = 1,2)$, _in the sense of_ (3.4), _and they satisfy the relation_

$$\mathcal{F}_1(s_o,s_o,\lambda) - \mathcal{F}_2(s_o,s_o,\lambda) = 0.$$

(iii) _The partial derivative_ $\mathcal{F}^{[1,0]}(t,s,\lambda)$ _exists and is continuous on_ $\{(t,s) \in \delta \times \delta \mid t \neq s\} \times \Lambda$, _where_ Λ _is a subset of the complex plane which does not intersect the real-axis._

In view of (ii), if $\mathcal{F}(s_o,s_o,\lambda)$ is defined to be equal to $\mathcal{F}_1(s_o,s_o,\lambda)$ then $\mathcal{F}(t,s,\lambda)$ is defined and continuous on $\delta \times \delta \times \Lambda$.

4. _Green's matrix._ We shall introduce an additional hypothesis which will be assumed in this section and hereafter.

(H_2') _If_ X _denotes the set of real-valued scalar functions_ $\mu(t)$ _of class_ $C^{2n}(I)$, _then there exists a linear homogeneous operator_ $M[\mu]$ _on_ X _into the class of_ $n \times n$ _matrix functions on_ I _such that if_ $u(t)$, $v(t)$ _are_ n-_dimensional vector functions of class_ $C_n^1(I_o)$ _on a subinterval_ I_o _of_ I _which satisfy_ $L_2[u,v](t) = 0$ _on_ I_o _then:_

(i) _there exist_ n-_dimensional vector functions_ $q(t) = q(t|u,v;\mu) \in C_n(I_o)$, $z(t) = z(t|u,v;\mu) \in C_n^1(I_o)$ _such that_ $y(t) = M[\mu](t)u(t)$, $z(t)$ _are solutions of the differential equations_

$$L_1[y,z](t) = K(t)q(t), \quad L_2[y,z](t) = 0, \qquad t \in I_o;$$

(ii) $|(K(t)[u(t)-y(t)],u(t)-y(t))| \leq |1-\mu(t)|^2 |(K(t)u(t),u(t))|$, $t \in I_o$;

(iii) _if_ J_1 _is a non-degenerate subinterval of_ I _such that_ $\mu(t) \equiv 1$ _on_ J_1 _then_ $M[\mu](t) = E_n$ _and_ $z(t|u,v;\mu) \equiv v(t)$ _on_ J_1; _if_ J_2 _is a_ non-degenerate subinterval of I such that $\mu(t) \equiv 0$ on J_2, then $M[\mu](t) \equiv 0$, $z(t|u,v;\mu) \equiv 0$ and $q(t|u,v;\mu) \equiv 0$ on J_2.

The above hypothesis looks unduly restrictive, but it will be demonstrated that the two important instances of $(\mathfrak{B}_\delta)$ given in Section 3 do indeed satisfy the above hypothesis. First we establish the following result.

LEMMA 4.1. *Suppose that* $[a,b]$ *is a compact subinterval of* I, *and* S *is a set of* k *points of* $[a,b]$. *If on* $[a,b] - S$ *the* n-*dimensional vector functions* $u(t)$, $v(t)$ *are bounded, of class* C_n^1, *and satisfy* $L_2[u,v](t) = 0$, *then there exist sequences of functions* $y_m(t)$, $z_m(t)$ *belonging to* $C_n^1[a,b]$ *such that*

$$L_2[y_m,z_m](t) = 0,\ t \in [a,b];\ y_m(t) = 0 \text{ for } t \in S_o = S \cup \{a\} \cup \{b\};$$

$$\int_a^b (K(t)[y_m(t) - u(t)],\ y_m(t) - u(t))dt \to 0 \quad \text{as} \quad m \to \infty.$$

Let J_m be a set of at most $k+2$ non-overlapping open intervals covering $S_o = S \cup \{a\} \cup \{b\}$, and such that the sum of the lengths of these intervals is less than $1/m$. Let $\mu_m(t)$ be a real-valued scalar function of class $C^{2n}(I)$ such that $\mu_m(t) = 1$ on $[a,b] - J_m$, $\mu_m(t) = 0$ in a neighborhood of each point of S_o, while $0 \le \mu_m(t) \le 1$ for $t \in I$. Then in view of hypothesis $(H_2'\text{-i})$ there exist n-dimensional vector functions $z_m(t) = z(t|u,v;\mu_m) \in C_n^1$ on $[a,b]$, and $n \times n$ matrix functions $M[\mu_m](t)$ such that $L_2[y_m,z_m](t) = 0$, $(m = 1,2,\cdots)$, on $[a,b]$ where $y_m(t) = M[\mu_m](t)u(t)$. Since $\mu_m(t) = 0$ in the neighborhood of each point of S_o, it follows from $(H_2'\text{-iii})$ that $y_m(t) = 0$ for $t \in S_o$.

In view of $(H_2'\text{-ii})$ and the fact that $\mu_m(t) = 1$ on $[a,b] - J_m$, it follows that

$$\int_a^b (K(t)[y_m(t)-u(t)],y_m(t)-u(t))dt$$
$$\leq \int_{J_m} (1-\mu_m(t))^2(K(t)u(t),u(t))dt \leq \int_{J_m} (K(t)u(t),u(t))dt.$$

Since the functions K and u are bounded on $[a,b]$, the right-hand member of the above relation tends to zero as $m \to \infty$, thus establishing the conclusion of the lemma.

We shall now demonstrate that the two important cases stated in Section 3 satisfy the above hypothesis.

Example 1. Suppose that the matrix functions $B(t)$ and $K(t)$ are non-singular on I; that is, $(\mathcal{B}_\delta$-a) represents (3.1'). For $u(t)$, $v(t)$ n-dimensional vector functions of class $C_n^1(\delta)$ such that $L_2[u,v](t) = 0$ on δ, let $\mu(t)$ be a scalar function of X which is of class $C^{2n}(I)$ and such that $0 \leq \mu(t) \leq 1$ on I. Define an operator M on μ to be $M(t) \equiv M[\mu](t) \equiv \mu(t)E_n$. Then $(Mu)' - A(Mu) = B(B^{-1}M'u + Mv)$, which is equivalent to $L_2[y,z](t) = 0$, where $y(t) = M(t)u(t)$ and $z(t) = B^{-1}(t)M'(t)u(t) + M(t)v(t) \in C_n^1(\delta)$. Since $K(t)$ is non-singular, $(H_2'$-i) holds for these vector functions $y(t)$, $z(t)$, and $q(t) = K(t)^{-1}L_1[y,z](t)$. Conclusion $(H_2'$-iii) is quite obvious. Employing the relations $M(t)u(t) = \mu(t)u(t)$ and $0 \leq \mu(t) \leq 1$, property $(H_2'$-ii) follows readily.

Example 2. Suppose $(\mathcal{B}_\delta$-a) is equivalent to the 2n-th order differential equation (3.1"), with the corresponding coefficient matrix functions $A(t)$, $B(t)$, $C(t)$, and $K(t)$ given in the statement following relation (3.1"). Let $u(t) = (u_i(t))$, $v(t) = (v_i(t))$, $(i = 1,2,\cdots,n)$, be as in (H_2'), and $\mu(t)$ be as in Example 1. Now we define the operator M to be the $n \times n$ matrix function $M[\mu](t)$ whose i-th row is given by

$$\mu^{[i-1]} \quad {}_{i-1}C_1\mu^{[i-2]} \quad \cdots \quad \mu \quad 0 \cdots 0, \quad (i = 1,2,\cdots,n),$$

where ${}_rC_s$ is the binomial coefficient $r!/[s!(r-s)!]$.

With $y(t) = M(t)u(t)$, the first $n-1$ components of $y'(t) - A(t)y(t)$ are all zero, and for arbitrary vector function $z(t) = (z_i(t))$ the first $n-1$ components of $B(t)z(t)$ are all zero, so that the condition $y'(t) - A(t)y(t) = B(t)z(t)$ determines $z_n(t)$ uniquely as $[r_n(t)]^{-1}\hat{y}_n(t)$, where $\hat{y}_n(t)$ is the n-th component of $y'(t) - A(t)y(t)$. Also in terms of the components the vector equation $L_1[y,z](t) = K(t)q(t)$ reduces to

$$
\begin{aligned}
-z_1' \qquad\qquad + r_o y_1 &= k(t)q_1(t), \\
-z_2' - z_1 + r_1 y_2 &= 0, \\
&\vdots \\
-z_{n-1}' - z_{n-2} + r_{n-2}\, y_{n-1} &= 0, \\
-z_n' \quad - z_{n-1} + r_{n-1}\, y_n \quad &= 0,
\end{aligned}
\tag{4.1}
$$

and it follows that the last $n-1$ of these equations determine $z_{n-1}(t)$, $z_{n-2}(t), \cdots, z_1(t)$ uniquely. Finally, since $k(t) \neq 0$ for $t \in I$, the first equation of (4.1) specifies $q_1(t)$ uniquely. Condition $(H_2'\text{-ii})$ follows readily from the relations $(y-u)^*K(t)(y-u) = (\mu-1)^2\bar{u}_1 k(t)u_1 = (\mu-1)^2 u^*Ku$, and conclusion $(H_2'\text{-iii})$ is obvious from the definition of $\mu(t)$.

Let $Y(t) = (U(t);V(t))$ be a fundamental matrix for $\mathcal{L}[y](t) = 0$ on I, where U and V are $n \times 2n$ matrix functions. Then from relation (3.10) we have that

$$\mathcal{L}[\mathcal{H}(\cdot,s)](t) = \mathcal{L}[Y](t)\Phi(t,s) = 0, \qquad t \neq s,$$

where $\Phi(t,s)$ is the $2n \times 2n$ matrix function given by

$$\Phi(t,s) = \frac{1}{2}\, Y^{-1}(s)\mathcal{G}^*\operatorname{sgn}(t-s).$$

Equivalently, we may write

$$L_1[U,V](t)\Phi(t,s) = 0, \qquad t \in I.$$
$$L_2[U,V](t)\Phi(t,s) = 0,$$

If $\mu(t) \in X$, then hypothesis (H_2') implies that there exists an $n \times n$ matrix function $M(t) \equiv M[\mu](t)$, and $n \times 2n$ matrix functions $V_o(t) \in C^1$, $Q(t) \in C$ on I which satisfy the system of $n \times 2n$ matrix differential equations

$$(4.1') \qquad L_1[MU,V_o](t) = K(t)Q(t), \quad L_2[MU,V_o](t) = 0 \quad \text{on} \quad I.$$

Moreover, in view of $(H_2'\text{-iii})$ we have that if $\mu(t) \equiv 1$ on a non-degenerate subinterval J_1 of I then $K(t)Q(t) \equiv 0$ on J_1, and if $\mu(t) \equiv 0$ on a non-degenerate subinterval J_2 of I then $K(t)Q(t) \equiv 0$ on J_2. From (4.1') we also obtain the relations

$$L_1[MU,V_o](t)\Phi(t,s) = K(t)Q(t)\Phi(t,s),$$
$$L_2[MU,V_o](t)\Phi(t,s) = 0, \quad t \neq s.$$

If we set $\Phi(t,s) = [\Phi_1(t,s) \quad \Phi_2(t,s)]$, where each $\Phi_j(t,s)$, $(j = 1,2)$, is a $2n \times n$ matrix function, then the above system is equivalent to the system of $n \times n$ matrix differential equations

$$(4.2) \qquad \begin{array}{ll} \text{(a)} & L_1[N_{1j}(\cdot,s),N_{2j}(\cdot,s)](t) = K(t)Q_j(t,s), \\ \text{(b)} & L_2[N_{1j}(\cdot,s),N_{2j}(\cdot,s)](t) = 0, \quad (j = 1,2), \end{array}$$

where N_{ij} and Q_j are $n \times n$ matrix functions defined by

$$(4.3) \qquad \begin{array}{l} N_{1j}(t,s) = M(t)U(t)\Phi_j(t,s), \\ N_{2j}(t,s) = V_o(t)\Phi_j(t,s), \\ Q_j(t,s) = Q(t)\Phi_j(t,s), \qquad (j = 1,2). \end{array}$$

It is to be remarked that if $\mu(t) \equiv 1$ on an open interval J_o of I then the $n \times n$ matrix functions N_{ij} are such that

$$N_{ij}(t,s) \equiv H_{ij}(t,s) \text{ for } (t,s) \in J_o \times I, \quad t \neq s,$$

where $H_{ij}(t,s)$, $(i,j = 1,2)$, are the $n \times n$ partition matrix functions of $\mathcal{H}(t,s)$ in (3.10). Moreover, if $s_o \in J_o$ then in view of the limit relations (3.10) satisfied by $\mathcal{H}(t,s)$ we have that the functions $N_{ij}(t,s)$ have limits $N_{ij;1}(s_o,s_o)$ and $N_{ij;2}(s_o,s_o)$ in the sense of (3.4), and

$$N_{1j;1}(s_o,s_o) - N_{1j;2}(s_o,s_o) = \delta_{2j}E_n,$$

$$N_{2j;1}(s_o,s_o) - N_{2j;2}(s_o,s_o) = -\delta_{1j}E_n.$$

In particular, $N_{ii;1}(s_o,s_o) = N_{ii;2}(s_o,s_o)$, $(i = 1,2)$, and if $N_{ii}(s,s)$ is defined as equal to $N_{ii;1}(s,s)$ then $N_{ii}(t,s)$ is continuous in (t,s) on $J_o \times J_o$. Also, if $\mu(t) \equiv 0$ on a non-degenerate subinterval J_2 of I then $N_{ij}(t,s) \equiv 0$ for $t \in J_2$, $(i,j = 1,2)$.

As to the differentiability properties, it is now clear from the expression of $\Phi(t,s)$ and hypothesis (H_2') that each $N_{ij}(t,s)$, $(i,j = 1,2)$, has continuous partial derivatives with respect to t and s on the region $D(I) = \{(t,s) \in I \times I \mid t \neq s\}$, while each $Q_j(t,s)$, $(j = 1,2)$, has a continuous partial derivative with respect to s on $D(I)$.

We are ready to prove now a result that is basic for our consideration. First, we introduce some notations. Let Λ be a set which possesses the following property.

(4.4) Λ <u>is a non-empty compact set in the complex λ-plane which is symmetric with respect to the real-axis, and does not intersect the real-axis</u>.

If d is an arbitrary subinterval of I, then we define $D(d)$ and Ω by

(4.5)
(a) $D(d) = \{(t,s) \quad d \times d \mid t \neq s\}$,
(b) $\Omega = D(d) \times \Lambda$.

It is to be remarked that Ω is symmetric in the sense that if $(t,s,\lambda) \in \Omega$, then $(s,t,\bar{\lambda}) \in \Omega$. When the dependence of Ω on d and Λ is desired, we write $\Omega(d,\Lambda)$.

It is emphasized here that the notations Λ, $D(d)$, and $\Omega(d,\Lambda)$ described above will be used subsequently in our discussion in the remaining of this section as well as in the next section. In particular, the symbol Λ is reserved for an arbitrary set with the property listed in (4.4).

THEOREM 4.1. *Suppose that* $(\mathcal{B}_\delta)$ *satisfies hypotheses* (H_1), (H_2), (H_2'), *and let* $d_1 = [a_1,b_1]$ *be an arbitrary given non-degenerate compact subinterval of* I *and* $\varepsilon = \varepsilon(d_1) > 0$ *a value such that the closed interval* $d_2 = [a_1-\varepsilon,\ b_1+\varepsilon]$ *is contained in* I. *Let* δ *be an arbitrary compact subinterval of* I *such that* $d_2 \subset \delta$, *and for a given non-real complex number* λ *denote by* $\mathfrak{G}(t,s,\lambda;\delta) = [\mathfrak{G}_{\sigma\tau}(t,s,\lambda;\delta)]$, $(\sigma,\tau = 1,\cdots,2n)$, *the Green's matrix for the self-adjoint problem* $(\mathcal{B}_\delta)$. *Moreover, let* Λ *be a compact subset of the complex plane possessing the property* (4.4). *Then there exists a bound* $\kappa = \kappa(d_1,\Lambda) > 0$ *such that for all* $(t,s,\lambda) \in \Omega(d_1,\Lambda)$ *we have* $|\mathfrak{G}_{\sigma\tau}(t,s,\lambda;\delta)| \leq \kappa$, $(\sigma,\tau = 1,\cdots,2n)$, *independent of* δ.

Choose an open interval d_o such that $d_1 \subset d_o \subset d_2 \subset \delta$, and the closure of d_o is properly contained in the interior of d_2. Let $\mu(t)$ be a real-valued function of class $C^{2n}(I)$ such that $\mu(t) \equiv 1$ for $t \in d_o$, $\mu(t) \equiv 0$ for t outside d_2. Since the matrix function $\mathcal{H}(t,s) = [H_{ij}(t)]$, $(i,j = 1,2)$, defined in (3.10) satisfies the equation $\mathcal{L}[\mathcal{H}(\cdot,s)](t) = 0$ on D(I), there exist $n \times n$ matrix functions $M[\mu](t)$, $N_{2j}(t,s)$, and $Q_j(t,s)$, $(j = 1,2)$, which satisfy on D(I) the system of $n \times n$ matrix differential equations

$$(4.7)\quad \begin{aligned}&(a)\quad L_2[N_{1j}(\cdot,s),N_{2j}(\cdot,s)](t) = 0,\\ &(b)\quad L_1[N_{1j}(\cdot,s),N_{2j}(\cdot,s)](t) = K(t)Q_j(t,s),\ (j = 1,2),\end{aligned}$$

where N_{ij} and Q_j are given in (4.3). In particular, it is to be noted that $K(t)Q_j(t,s) \equiv 0$ for $t \in d_o$.

Define the $2n \times 2n$ matrix function $\mathcal{N}(t,s)$ as

$$(4.8)\qquad \mathcal{N}(t,s) = [N_{ij}(t,s)], \qquad (i,j = 1,2).$$

From the definition of $\mu(t)$ and condition (H_2'-iii) the matrix functions $N_{ij}(t,s)$ and $Q_j(t,s)$, $(j = 1,2)$, vanish for t outside d_2. Also, we observe that the matrix function $\mathcal{N}(t,s)$ possesses the following properties:

(4.9)

(i) $\mathcal{N}(t,s)$ is defined and continuous in (t,s) on $D(I) = \{(t,s) \in I \times I \mid t \neq s\}$;

(ii) $\mathcal{N}(t,s) = \mathcal{H}(t,s)$ for $(t,s) \in d_o \times I$, $t \neq s$; $\mathcal{N}(t,s) = 0$ for $(t,s) \in (I-d_2) \times I$;

(iii) $\mathcal{N}(t,s)$ has continuous partial derivatives with respect to t and s on $D(I)$.

Let $\mathcal{W}(t,s,\lambda;\delta)$ be the $2n \times 2n$ matrix function defined by

$$(4.10)\quad \mathcal{W}(t,s,\lambda;\delta) \equiv \mathcal{G}(t,s,\lambda;\delta) - \mathcal{N}(t,s) \text{ for } (t,s,\lambda) \in \delta \times d_1 \times \Lambda.$$

It is to be emphasized that in the consideration of the matrix function $\mathcal{W}(t,s,\lambda;\delta)$ defined by (4.10) the interval δ is always required to satisfy the condition specified in the statement of Theorem 4.1. In particular, the matrix function $\mathcal{N}(t,s)$ is defined by $\mathcal{H}(t,s)$ of (3.10) and the transformation matrix $M[\mu]$ which is identically zero on $I - d_2$, so that $\mathcal{N}(t,s) = 0$ for $t \in I - d_2$, and is independent of the compact interval δ which contains d_2. Also, in view of (3.5) and (3.12) the matrix function $\mathcal{W}$ has limits $\mathcal{W}_1$ and $\mathcal{W}_2$ in the sense of (3.4) and $\mathcal{W}_1(s,s,\lambda;\delta) = \mathcal{W}_2(s,s,\lambda;\delta)$. Consequently, if we define $\mathcal{W}(s,s,\lambda;\delta)$ to be equal to $\mathcal{W}_1(s,s,\lambda;\delta)$ then $\mathcal{W}(t,s,\lambda;\delta)$ is a continuous function of (t,s,λ) on $\delta \times d_1 \times \Lambda$. Throughout the following discussion it will be understood that $\mathcal{W}(t,s,\lambda;\delta)$ has been so defined.

For each $s \in d_1$ the matrix $\mathcal{W}(t,s,\lambda;\delta)$ has a continuous partial derivative relative to t on δ if $t \neq s$; furthermore, it satisfies the boundary condition $\mathcal{M}_\delta[\mathcal{W}(\cdot,s,\lambda;\delta)] = 0$ and the matrix differential equation

$$(4.11) \qquad \mathfrak{L}[\mathcal{W}(\cdot,s,\lambda;\delta)|\lambda](t) = -\mathfrak{L}[\mathcal{N}(\cdot,s)|\lambda](t) \text{ if } t \neq s,$$

and hence from Theorem 3.2 we have the relation

$$(4.12) \qquad \mathcal{W}(t,s,\lambda;\delta) = -\int_\delta \mathcal{G}(t,r,\lambda;\delta)\mathfrak{L}[\mathcal{N}(\cdot,s)|\lambda](r)dr.$$

It is to be remarked that the integral in (4.12) remains unchanged when δ is replaced by d_2, since the matrix functions $N_{ij}(t,s)$ vanish for t outside $d_2 \subset \delta$.

Upon setting $\mathcal{W}(t,s,\lambda;\delta) = [W_{ij}(t,s,\lambda;\delta)]$, $(i,j = 1,2)$, where each W_{ij} is an $n \times n$ partition matrix, in terms of the notations given by (3.6) and (4.8) the relation (4.12) is equivalent to the following four equations

$$(4.13) \qquad \begin{aligned} W_{ij}(t,s,\lambda;\delta) &\equiv G_{ij}(t,s,\lambda;\delta) - N_{ij}(t,s) \\ &= -\int_{d_2} G_{i1}(t,r,\lambda;\delta)L_1[N_{1j}(\cdot,s),N_{2j}(\cdot,s)|\lambda](r)dr, \end{aligned}$$

which in view of (4.7-b) can be written as

$$(4.14) \qquad W_{ij}(t,s,\lambda;\delta) = -\int_{d_2} G_{i1}(t,r,\lambda;\delta)K(r)R_j(r,s,\lambda)dr, \ (i,j = 1,2),$$

for $(t,s,\lambda) \in \delta \times d_1 \times \Lambda$. It may be verified readily that in (4.14) the $n \times n$ matrix functions

(4.15) $$R_j(t,s,\lambda) \equiv Q_j(t,s) - \lambda N_{1j}(t,s), \quad (j = 1,2),$$

have the following properties:

(i) $K(t)R_1(t,s,\lambda)$ _is continuous in_ (t,s,λ) _on_ $\delta \times d_1 \times \Lambda$, _while_ $K(t)R_2(t,s,\lambda)$ _is continuous on_ $\{(t,s) \in \delta \times d_1 \mid t \neq s\} \times \Lambda$; _moreover, if_ $(s_o,s_o) \in d_1 \times d_1$, _then with_ $\mathcal{R}(t,s,\lambda,j) = K(t)R_j(t,s,\lambda)$ _the limits_ $\mathcal{R}_i(s_o,s_o,\lambda;j)$, $(i = 1,2)$, _exist in the sense of_ (3.4), _are finite, and satisfy the relation_

(4.16)$_j$ $$\mathcal{R}_1(s_o,s_o,\lambda;j) - \mathcal{R}_2(s_o,s_o,\lambda;j) = -\lambda\delta_{2j}K(s_o), \quad (j = 1,2);$$

(ii) _on the set_ $\{(t,s) \in \delta \times d_1 \mid t \neq s\} \times \Lambda$, _the matrix functions_ $R_j(t,s,\lambda)$ _have continuous partial derivatives_ $R_j^{[0,1]}(t,s,\lambda)$ _with respect to_ s; _moreover,_ $R_j^{[0,1]}(t,s,\lambda)$, $(j = 1,2)$, _are continuous and bounded functions of_ (t,s,λ) _on_ $\{(t,s) \in \delta \times d_1 \mid t \neq s\} \times \Lambda$;

(iii) $R_j(t,s,\lambda) \equiv 0$, $(j = 1,2)$, _for_ $(t,s,\lambda) \in (I-d_2) \times I \times \Lambda$.

Relation (4.16)$_j$ and property (ii) follow readily from (4.15) and from the fact that if $s \in d_1$ then for t sufficiently close to s we have that t also belongs to d_o. Indeed, if $(t,s) \in d_o \times d_1$ and $t \neq s$, then $N_{ij}(t,s) = H_{ij}(t,s)$, $(i,j = 1,2)$, and hence $L_1[N_{1j}, N_{2j}](t) = 0$. Consequently, from (4.2-a) and (4.15) we obtain the relations

(4.17) $$-\lambda K(t)H_{1j}(t,s) = K(t)R_j(t,s,\lambda), \text{ for } (t,s) \in d_o \times d_1,\ t \neq s, \text{ and } j = 1,2.$$

It is then clear that (4.16)$_j$ follows from the above relation and (3.10). In view of the continuity property of H_{1j}, condition (i) also follows from (3.10) and (4.17). Property (iii) follows readily from the remark following relation (4.8).

It is easy to see that the two pairs $(U;V) = (W_{1j};W_{2j})$, $(j = 1,2)$, in (4.14) are respectively the solutions of the matrix differential systems

$$\begin{aligned} &L_1[U,V|\lambda](t) = -K(t)R_j(t,s,\lambda), \quad (j = 1,2),\\ &L_2[U,V](t) = 0, \qquad\qquad\qquad\qquad t \in \delta,\\ &\mathcal{M}_\delta[U,V] = 0. \end{aligned}$$

Consequently, we can apply Lemma 3.3 to each corresponding column vector of W_{1j} and R_j. To this end, let ξ be an arbitrary n-dimensional constant vector belonging to the unit ball $B_1 = \{\xi \,\big|\, |\xi| \le 1\}$. Then it follows from Lemma 3.3 that

$$\|W_{1j}(\cdot,s,\lambda;\delta)\xi\|_\delta \le |\mathrm{Im}\ \lambda|^{-1}\|R_j(\cdot,s,\lambda)\xi\|_{d_2}. \tag{4.18}$$

Applying the triangular inequality to $G_{1j}\xi = W_{1j}\xi + N_{1j}\xi$ we obtain from (4.18) the relation

$$\|G_{1j}(\cdot,s,\lambda;\delta)\xi\|_\delta \le \|N_{1j}(\cdot,s)\xi\|_{d_2} + |\mathrm{Im}\ \lambda|^{-1}\|R_j(\cdot,s,\lambda)\xi\|_{d_2}. \tag{4.19}$$

It is clear that for $j = 1,2$ the first and the second terms of the right-hand side in the above relation (4.19) are respectively continuous in (s,ξ) on $d_1 \times B_1$, and in (s,λ,ξ) on $d_1 \times \Lambda \times B_1$. Consequently, for $(s,\lambda) \in d_1 \times \Lambda$ there exists a constant $k(d_1, \Lambda)$ such that

$$\|G_{1j}(\cdot,s,\lambda;\delta)\xi\|_\delta \le k(d_1,\Lambda), \quad (j = 1,2), \tag{4.20}$$

for $(s,\lambda) \in d_1 \times \Lambda$ uniformly for $\xi \in B_1$, and independent of δ. In particular, upon choosing ξ equal to the unit vectors $\xi = (\delta_{ik})$, $(i,k = 1,\cdots,n)$, we have that as a function of t the $\underset{\sim}{K}_\delta$-norm of each column vector of $G_{1j}(t,s,\lambda;\delta)$ is bounded uniformly for $(s,\lambda) \in d_1 \times \Lambda$, independent of δ.

We recall that the $W_{ij} \equiv G_{ij} - N_{ij}$ and R_j, $(i,j = 1,2)$, appearing in relation (4.13) are $n \times n$ matrix functions. Let $G_{ij}^{\alpha\beta}$ and $N_{ij}^{\alpha\beta}$,

$(\alpha,\beta = 1,\cdots,n)$, denote the scalar elements in the α-th row and β-th column of G_{ij} and N_{ij}, respectively; similarly, let G^{β}_{ij} and R^{β}_{j}, $(\beta = 1,2,\cdots,n)$, denote the β-th column vectors of G_{ij} and R_j, respectively. As $G_{i1}(t,r,\lambda;\delta) = [G_{1i}(r,t,\bar{\lambda};\delta)]^*$, relation (4.14) is equivalent to

$$(4.21)\qquad G^{\alpha\beta}_{ij}(t,s,\lambda;\delta) = N^{\alpha\beta}_{ij}(t,s) - \underset{\sim}{K}_{d_2}[R^{\beta}_{j}(\cdot,s,\lambda),\ G^{\alpha}_{1i}(\cdot,t,\bar{\lambda};\delta)],$$

which, in view of the Cauchy-Schwarz inequality, implies that

$$|G^{\alpha\beta}_{ij}(t,s,\lambda;\delta)| \le |N^{\alpha\beta}_{ij}(t,s)| + \|G^{\alpha}_{1i}(\cdot,t,\bar{\lambda};\delta)\|_{\delta}\|R^{\beta}_{j}(\cdot,s,\lambda)\|_{d_2},$$

$(i,j = 1,2;\ \alpha,\beta = 1,2,\cdots,n)$. We shall estimate the right-hand members in the above relation. Each $N^{\alpha\beta}_{ij}(t,s)$ is clearly bounded for $(t,s) \in D(d_1)$, and for $(t,\lambda) \in d_1 \times \Lambda$ the $\underset{\sim}{K}$-norm of G^{α}_{1i} over δ is bounded by (4.20). Moreover, for $(s,\lambda) \in d_1 \times \Lambda$ the $\underset{\sim}{K}$-norm of R^{β}_{j} over d_2 is also bounded in view of the properties listed above for the matrix function R_j. Therefore, there exists a constant $\kappa = \kappa(d_1,\Lambda)$ such that for $(t,s,\lambda) \in \Omega(d_1,\Lambda)$ we have

$$|G^{\alpha\beta}_{ij}(t,s,\lambda;\delta)| \le \kappa,\quad (i,j = 1,2;\ \alpha,\beta = 1,\cdots,n),$$

independent of δ. This completes the proof of the theorem.

THEOREM 4.2. <u>Under the hypothesis of Theorem</u> 4.1 <u>the family of</u> $2n \times 2n$ <u>matrix functions</u> $\mathcal{W}(t,s,\lambda;\delta) \equiv \mathfrak{G}(t,s,\lambda;\delta) - \mathcal{N}(t,s)$ <u>is equicontinuous on each compact domain</u> $d_1 \times d_1 \times \Lambda$.

As noted in the paragraph following (4.10), the matrix function $\mathcal{W}(t,s,\lambda;\delta)$ is continuous in (t,s,λ) on $d_1 \times d_1 \times \Lambda$. From the integral expression (4.14) it follows that for $(t,s,\lambda) \in \delta \times d_1 \times \Lambda$, and $t \neq s$, the partial derivative with respect to s of $W_{ij}(t,s,\lambda;\delta)$ is given by

$$W_{ij}^{[0,1]}(t,s,\lambda;\delta) = G_{i1}(t,s,\lambda;\delta)K(s)[R_j(s^+,s,\lambda) - R_j(s^-,s,\lambda)] - \int_{d_2} G_{i1}(t,r,\lambda;\delta)K(r)R_j^{[0,1]}(r,s,\lambda)dr.$$

Using $(4.16)_j$, the above relation may be written as

$$W_{ij}^{[0,1]}(t,s,\lambda;\delta) = -\lambda\delta_{2j}G_{i1}(t,s,\lambda;\delta)K(s) - \int_{\delta} G_{i1}(t,r,\lambda;\delta)K(r)R_j^{[0,1]}(r,s,\lambda)dr. \tag{4.22}$$

We shall proceed to show that each element of $W_{ij}^{[0,1]}(t,s,\lambda;\delta)$ in (4.22) is bounded on $\Omega = \Omega(d_1, \Lambda) = \{(t,s) \in d_1 \times d_1 \mid t \neq s\} \times \Lambda$, independent of δ. In view of Theorem 4.1 the first term of the right-hand member in (4.22) is bounded on Ω. Next, in view of property (ii) for (4.15), it is easy to see that the matrix functions $R_j^{[0,1]}(r,s,\lambda)$ appearing in the integrand in (4.22) are such that $\|R_j^{[0,1]}(\cdot,s,\lambda)\xi\|_{d_2}$, $(j = 1,2)$, are continuous functions of (s,λ,ξ) on $d_1 \times \Lambda \times B_1$. Consequently, to relation (4.22) we can apply arguments similar to these used to show the boundedness of W_{ij}, and conclude that the matrix functions $W_{ij}^{[0,1]}(t,s,\lambda;\delta)$ are bounded on Ω, independent of δ.

From relation (4.22) it also follows that the matrix $\mathcal{W}^{[0,1]}(t,s,\lambda;\delta)$ has finite limits $\mathcal{W}_1^{[0,1]}(s_o,s_o,\lambda;\delta)$ and $\mathcal{W}_2^{[0,1]}(s_o,s_o,\lambda;\delta)$ in the sense of (3.4), which are bounded for $(s,\lambda) \in d_1 \times \Lambda$.

From the symmetry properties of $\mathfrak{J}(t,s,\lambda;\delta)$ and Ω it also follows that the matrix function $\mathcal{W}^{[1,0]}(t,s,\lambda;\delta)$ possesses similar properties. Consequently, we have that $\mathcal{W}(t,s,\lambda;\delta)$ is Lipschitzian in (t,s) on $d_1 \times d_1$, uniformly for $\lambda \in \Lambda$. Now for fixed $(t,s) \in d_1 \times d_1$ the matrix function $\mathcal{W}(t,s,\lambda;\delta)$ is a holomorphic function of λ on Λ, and in view of Theorem 4.1 this matrix function is bounded on $d_1 \times d_1 \times \Lambda$, independent of δ. With the aid of the Cauchy integral formula for the derivative of a holomorphic function it then follows that the partial derivative matrix function $\mathcal{W}_\lambda(t,s,\lambda;\delta)$ is also bounded on $d_1 \times d_1 \times \Lambda$, independent of δ. Combining these results, it then follows that the family of matrix functions $\mathcal{W}(t,s,\lambda;\delta)$ satisfies on $d_1 \times d_1 \times \Lambda$ a Lipschitz condition in (t,s,λ) with Lipschitz constant independent of δ, so that the family $\mathcal{W}(t,s,\lambda;\delta)$ is equicontinuous on $d_1 \times d_1 \times \Lambda$. This completes the proof of the theorem.

Now let d_{1k}, $(k = 1,2,\cdots)$, be a non-decreasing sequence of compact subintervals converging to I, and Λ_k a corresponding non-decreasing sequence of compact sets which with the property listed in (4.4) converges to C_o, the complex plane with the real-axis deleted. Corresponding to $d_1 = d_{1k}$ there exist intervals $d_o = d_{ok}$, $d_2 = d_{2k}$ satisfying the conditions specified in the proof of Theorem 4.1, and a corresponding matrix function $\mathcal{H}(t,s) = \mathcal{H}_k(t,s)$ satisfying the conditions (4.9); in particular, $\mathcal{H}_k(t,s)$ is independent of the choice of intervals $\delta \supset d_{2k}$. For each positive integer k we have that for $\delta \supset d_{2k}$ the set of matrix functions $\mathcal{W}(t,s,\lambda;\delta) = \mathcal{W}_k(t,s,\lambda;\delta) = \mathfrak{G}(t,s,\lambda;\delta) - \mathcal{H}_k(t,s)$ is equicontinuous on $d_{1k} \times d_{1k} \times \Lambda_k$; moreover, $\mathcal{H}_k(t,s) = \mathcal{H}(t,s)$ on $D(d_{1k}) = \{(t,s) \in d_{1k} \times d_{1k} \mid t \neq s\}$.

Now suppose that $\{\delta_h\}$ is a non-decreasing sequence of compact intervals converging to I as $h \to \infty$. In view of the Ascoli theorem there is a subsequence $\{\delta_{m1}\}$ of $\{\delta_h\}$ with $d_{21} \subset \delta_{m1}$, $(m = 1,2,\cdots)$, and such that $\mathcal{W}_1(t,s,\lambda;\delta_{m1}) = \mathfrak{G}(t,s,\lambda;\delta_{m1}) - \mathcal{H}_1(t,s)$ converges uniformly on $d_{11} \times d_{11} \times \Lambda_1$. Moreover, since $\mathcal{H}_1(t,s) = \mathcal{H}(t,s)$ on $D(d_{11})$, we have that the sequence $\{\mathfrak{G}(t,s,\lambda;\delta_{m1})\}$ converges uniformly on $\Omega(d_{11},\Lambda_1)$. Similarly, there is a subsequence $\{\delta_{m2}\}$ of $\{\delta_{m1}\}$ with $d_{22} \subset \delta_{m2}$, $(m = 1,2,\cdots)$, and $\mathcal{W}_2(t,s,\lambda;\delta_{m2}) = \mathfrak{G}(t,s,\lambda;\delta_{m2}) - \mathcal{H}_2(t,s)$ converges uniformly on

$d_{12} \times d_{12} \times \Lambda_2$; also, as $\mathcal{H}_2(t,s) = \mathcal{H}(t,s)$ on $D(d_{12})$ the sequence $\mathcal{G}(t,s,\lambda;\delta_{m2})$ converges uniformly on $\Omega(d_{12},\Lambda_2)$. Applying a diagonalization process, one obtains a subsequence $\{\delta_m\}$ of $\{\delta_h\}$ such that for $k = 1,2,\cdots$ the sequence $\mathcal{W}_k(t,s,\lambda;\delta_m) = \mathcal{G}(t,s,\lambda;\delta_m) - \mathcal{H}_k(t,s)$ converges uniformly on $d_{1k} \times d_{1k} \times \Lambda_k$, and correspondingly the sequence $\mathcal{G}(t,s,\lambda;\delta_m)$ converges uniformly on $\Omega(d_{1k},\Lambda_k)$.

Consequently, we have proved the following theorem.

THEOREM 4.3. *Suppose that the hypotheses of Theorem* 4.1 *are satisfied and* $\{\delta_h\}$ *is a non-decreasing sequence of compact subintervals converging to* I. *Then there exists a subsequence* $\{\delta_m\}$ *of* $\{\delta_h\}$ *such that the corresponding sequence of Green's matrices* $\{\mathcal{G}(t,s,\lambda;\delta_m)\}$ *converges on* $\Omega(I,C_o) = \{(t,s,\lambda) \mid t \in I,\ s \in I,\ t \neq s;\ \lambda \in C_o\}$ *to a limit matrix* $\mathcal{G}(t,s,\lambda)$. *Moreover, if* d *and* Λ *are respectively an arbitrary compact subinterval of* I, *and a compact set of the complex plane with the properties listed in* (4.4), *then the convergence of* $\{\mathcal{G}(t,s,\lambda;\delta_m)\}$ *is uniform on* $\Omega(d,\Lambda)$.

In view of the uniform convergence of $\{\mathcal{G}(t,s,\lambda;\delta_m)\}$ on an arbitrary set $\Omega(d,\Lambda)$, we have that on $\Omega(I,C_o)$ the limit matrix function $\mathcal{G}(t,s,\lambda)$ is continuous in (t,s,λ), and holomorphic in λ for fixed (t,s), moreover, the symmetry property of $\mathcal{G}(t,s,\lambda;\delta_m)$ for all m implies that

$$\mathcal{G}(t,s,\lambda) = [\mathcal{G}(s,t,\bar{\lambda})]^* \text{ on } \Omega(I,C_o). \tag{4.23}$$

Let $\mathcal{L}^2_n(I;K)$ be the Hilbert space consisting of all n-dimensional vector functions u(t) which are defined and measurable on I, and such that

$$\underset{\sim}{K}[u] \equiv \int_I (K(t)u(t),u(t))dt < \infty.$$

If $u_i(t)$, $(i = 1,2)$, are vector functions of $\mathcal{L}^2_n(I;K)$, then we define an inner product correspondingly as

$$K[u_1,u_2] = \int_I (K(t)u_1(t),u_2(t))dt.$$

The K-norm of u over I is denoted merely by $\|u\|$.

THEOREM 4.4. Suppose that hypotheses (H_1), (H_2), and (H_2') are satisfied, and for $\lambda \in C_o$ let $\mathcal{G}(t,s,\lambda) = [G_{ij}(t,s,\lambda)]$, $(i,j = 1,2)$, be the limit of a convergent sequence $\{\mathcal{G}(t,s,\lambda;\delta_m)\}$ of Green's matrix functions as in Theorem 4.3. Then $\mathcal{G}(t,s,\lambda)$ possesses the following properties.

(i) $\mathcal{G}^{[0,1]}(t,s,\lambda)$ and $\mathcal{G}^{[0,1]}(t,s,\lambda)$ are continuous functions of (t,s,λ) on $\Omega(I,C_o) = \{(t,s,\lambda) \in I \times I \times C_o \mid t \neq s\}$;

(ii) the sequences $\mathcal{G}^{[1,0]}(t,s,\lambda;\delta_m)$ and $\mathcal{G}^{[0,1]}(t,s,\lambda;\delta_m)$ converge uniformly on every bounded subregion $\Omega(d_1,\Lambda)$ of $\Omega(I,C_o)$ to $\mathcal{G}^{[1,0]}(t,s,\lambda)$ and $\mathcal{G}^{[0,1]}(t,s,\lambda)$, respectively;

(iii) as a function of t, the matrix $\mathcal{G}(t,s,\lambda)$ satisfies the differential equation $\mathcal{L}[\mathcal{G}(\cdot,s,\lambda)|\lambda](t) = 0$ if $t \neq s$;

(iv) for each fixed $(t,\lambda) \in I \times C_o$ each row vector of the $n \times n$ matrix functions $G_{i1}(t,s,\lambda)$, $(i = 1,2)$, as a function of s, belongs to $\mathcal{L}_n^2(I;K)$;

(v) if $(s_o,s_o) \in I \times I$, then the limits $\mathcal{G}_j(s_o,s_o,\lambda)$, $(j = 1,2)$, in the sense of (3.4) exist, and are finite; moreover;

$$\mathcal{G}_1(s_o,s_o,\lambda) - \mathcal{G}_2(s_o,s_o,\lambda) = \mathcal{J}^*;$$

(vi) if $f_o(t) = (f(t);0)$, $f \in \mathcal{L}_n^2(I;K)$, then the 2n-dimensional vector function y(t) defined by

$$y(t) = \int_I \mathcal{G}(t,s,\lambda)\mathcal{K}(s)f_o(s)ds$$

satisfies the differential equation $\mathcal{L}[y|\lambda](t) = \mathcal{K}(t)f_o(t)$ on I; moreover, if $y(t) = (u_1(t);u_2(t))$, then

(4.24) $$\|u_1\| \leq |\mathrm{Im}\ \lambda|^{-1}\|f\|.$$

Let the compact subintervals d_1, d_2, and δ be as described in the beginning of the proof of Theorem 4.1, and let Λ be a set with property (4.4). We consider the relation (4.14) for $(t,s,\lambda) \in d_1 \times d_1 \times \Lambda$. Upon taking $\delta_m = \delta$ and letting $\delta_m \to I$, in view of the uniform convergence of $\mathfrak{J}(t,s,\lambda;\delta_m)$ on $\Omega(d_1,\Lambda)$ to a limit matrix $\mathfrak{J}(t,s,\lambda)$ which is continuous and bounded, it follows that

(4.25) $$W_{ij}(t,s,\lambda) = -\int_{d_2} G_{i1}(t,r,\lambda)K(r)R_j(r,s,\lambda)dr,$$

where the $n \times n$ matrix functions

(4.26) $$W_{ij}(t,s,\lambda) = G_{ij}(t,s,\lambda) - H_{ij}(t,s),\ (i,j = 1,2),$$

are continuous in (t,s,λ) on $d_1 \times d_1 \times \Lambda$. It is easy to see that the integral in (4.25) possesses a continuous partial derivative with respect to s for $(t,s,\lambda) \in \Omega = \Omega(d_1,\Lambda)$, and hence $W_{ij}(t,s,\lambda)$ has the same property. Indeed, on $\Omega(d_1,\Lambda)$ we have that

(4.27) $$\begin{aligned} W_{ij}^{[0,1]}(t,s,\lambda) = &- \lambda\delta_{2j}G_{i1}(t,s,\lambda)K(s) \\ &- \int_{d_2} G_{i1}(t,r,\lambda)K(r)R_j^{[0,1]}(r,s,\lambda)dr. \end{aligned}$$

In view of the continuity on $\Omega(d_1,\Lambda)$ of $G_{i1}(t,s,\lambda)$ and property (ii) of the partial derivative matrix functions $R_j^{[0,1]}(r,s,\lambda)$, it follows from (4.27) that $W_{ij}^{[0,1]}(t,s,\lambda)$ is continuous in (t,s,λ) on $\Omega(d_1,\Lambda)$. From property (i-a) of the matrix function $\mathcal{H}(t,s)$ as stated following relation (3.10), the matrices $H_{ij}^{[0,1]}(t,s)$ are also continuous in (t,s) on $D(d_1)$. Consequently, from relation (4.26) and (4.27) it follows that each matrix function $G_{ij}^{[0,1]}(t,s,\lambda)$ exists and is a continuous function of (t,s,λ) on $\Omega(d_1,\Lambda)$. Also, we have the relation

(4.28) $G_{ij}^{[0,1]}(t,s,\lambda) = H_{ij}^{[0,1]}(t,s) + W_{ij}^{[0,1]}(t,s,\lambda)$, on $\Omega(d_1,\Lambda)$.

Since $\Omega(d_1,\Lambda)$ is arbitrary, conclusion (i) then follows from the symmetry properties of $\mathfrak{H}(t,s,\lambda)$ and that of Ω.

To prove conclusion (ii), we return to relation (4.22) with $\delta_m = \delta$ and $(t,s,\lambda) \in \Omega(d_1,\Lambda)$; that is,

$$(4.29) \quad W_{ij}^{[0,1]}(t,s,\lambda;\delta_m) = -\lambda\delta_{2j}G_{i1}(t,s,\lambda;\delta_m)K(s) - \int_{d_2} G_{i1}(t,r,\lambda;\delta_m)K(r)R_j^{[0,1]}(r,s,\lambda)dr,$$

where

$$W_{ij}^{[0,1]}(t,s,\lambda;\delta_m) = G_{ij}^{[0,1]}(t,s,\lambda;\delta_m) - H_{ij}^{[0,1]}(t,s) \text{ on } \Omega(d_1,\Lambda).$$

Let (t,s,λ) be an arbitrary, but fixed, point of $\Omega(d_1,\Lambda)$ such that, with $d_2 = [a_2,b_2]$ we have $a_2 < t < s < b_2$. We write the integral expression in (4.29) as the sum of the integrals over the intervals $J_1 = [a_2,t)$, $J_2 = (t,s)$, $J_3 = (s,b_2]$. Then in view of the continuity property of $K(t)$, the property (ii) following relation $(4.16)_j$, and the uniform convergence of $G_{ij}(t,s,\lambda;\delta_m)$ on $\Omega(d_1,\Lambda)$ to $G_{ij}(t,s,\lambda)$, it follows that each integral $\int_{J_k} G_{i1}(t,r,\lambda;\delta_m)K(r)R_j^{[0,1]}(r,s,\lambda)dr$ converges uniformly to $\int_{J_k} G_{i1}(t,r,\lambda)K(r)R_j^{[0,1]}(r,s,\lambda)dr$, $(k = 1,2,3)$. Consequently, each right-hand member in (4.29) converges uniformly on $\Omega(d_1,\Lambda)$ to the corresponding right-hand member in (4.27). Therefore, the left-hand member in (4.29) converges under the same condition to the left-hand member in (4.27), which from the symmetry properties of $\mathfrak{H}(t,s,\lambda;\delta_m)$, $\mathfrak{H}(t,s,\lambda)$, and Ω proves conclusion (ii) for all $(t,s,\lambda) \in \Omega(d_1,\Lambda)$ with $a_2 < t < s < b_2$.

The case in which $a_2 = t$ or $s = b_2$, or both $a_2 = t$ and $s = b_2$, or $s < t$ can be treated similarly.

In view of conclusion (i-b) of Theorem 3.2, for $\delta_m = \delta$ the matrix function $\mathfrak{G}(t,s,\lambda;\delta_m)$ satisfies the differential equation

$$\mathfrak{L}[\mathfrak{G}(\cdot,s,\lambda;\delta_m)|\lambda](t) = 0 \text{ on } \Omega = \Omega(d_1,\Lambda), \quad (m = 1,2,\cdots).$$

Upon letting $\delta_m \to I$ in the above relation, with the aid of conclusion (ii) we then obtain $\mathfrak{L}[\mathfrak{G}(t,s,\lambda)|\lambda](t) = 0$ on Ω. Since Ω is arbitrary, conclusion (iii) follows.

To prove conclusion (iv), we take $\delta_m = \delta$, and let $\delta_m \to I$ in relation (4.20). It is then clear that as a function of t the $\underset{\sim}{K}$-norm of each column vector of $G_{1j}(t,s,\lambda)$ over I is bounded uniformly for $(s,\lambda) \in d_1 \times \Lambda$. Since d_1 and Λ are arbitrary, the conclusion follows from the symmetry properties of $\mathfrak{G}(t,s,\lambda)$ and Ω.

Conclusion (v) follows directly from relations (3.12) and (4.26).

Finally, let $f_o(t) = (f(t);0)$ with $f \in \mathcal{L}_n^2(I;K)$. Then the 2n-dimensional vector function y(t) as defined in the theorem is meaningful, for the integral is equivalent to the condition that the n-dimensional vector functions $u_i(t)$ defined by $y(t) = (u_1(t);u_2(t))$ satisfy the equations

$$(4.30)_i \qquad u_i(t) = \int_I G_{i1}(t,s,\lambda)K(s)f(s)ds, \quad (i = 1,2),$$

which, in view of conclusion (v), are well-defined. Moreover, it follows readily with the aid of conclusion (v) that we have

$$y'(t) = [\mathfrak{G}(t,t^-,\lambda) - \mathfrak{G}(t,t^+,\lambda)]\mathcal{K}(t)f_o(t) + \int_I \mathfrak{G}^{[1,0]}(t,s,\lambda)\mathcal{K}(s)f_o(s)ds$$
$$= \mathcal{G}^*\mathcal{K}(t)f_o(t) + \int_I \mathfrak{G}^{[1,0]}(t,s,\lambda)\mathcal{K}(s)f_o(s)ds,$$

and with the aid of conclusion (iii) it follows that y(t) satisfies the differential equation. Finally, to establish (4.24) we define a vector function $g(t)$ by

$$g(t) = \begin{cases} f(t), & \text{for } t \in \delta, \\ 0, & \text{for } t \in I - \delta. \end{cases}$$

Application of (3.8) to $(4.30)_1$ then yields the relation

$$\|u_1\|_\delta \leq |\text{Im } \lambda|^{-1} \|f\|_\delta.$$

Since $f \in \mathcal{L}_n^2(I;K)$ we have $\|f\|_\delta \leq \|f\|$, and consequently $\|u_1\|_\delta \leq |\text{Im } \lambda|^{-1}\|f\|$, so that also $u_1 \in \mathcal{L}_n^2(I;K)$ and $\|u_1\| \leq |\text{Im } \lambda|^{-1}\|f\|$. This completes the proof of the theorem.

Let $\mathfrak{D}(\mathcal{L})$ denote the set of all 2n-dimensional vector functions $y(t) = (u(t);v(t))$ which are locally absolutely continuous on I, and such that $u(t) \in \mathcal{L}_n^2(I;K)$. In particular, it follows from relation (4.24) that the 2n-dimensional vector function $y(t)$ given in (vi) of Theorem 4.4 belongs to $\mathfrak{D}(\mathcal{L})$.

In order to prove the uniqueness of the limit matrix $\mathfrak{G}(t,s,\lambda)$ we introduce the following hypothesis.

(H_3) <u>There exists a non-real complex number</u> λ_o <u>such that</u> $\mathcal{L}[y](t) \pm \lambda_o \mathcal{K}(t)y(t) = 0$, $t \in I$, <u>has no non-trivial solution of class</u> $\mathfrak{D}(\mathcal{L})$.

THEOREM 4.5. <u>Under hypotheses</u> (H_i), $(i = 1,2,3)$, <u>and</u> (H_2') <u>for any complex number</u> λ <u>belonging to</u> C_o <u>the equation</u> $\mathcal{L}[y|\lambda](t) = 0$, $t \in I$, <u>has no non-trivial solution of class</u> $\mathfrak{D}(\mathcal{L})$; <u>that is,</u>

(4.31) $$\mathcal{L}[y|\lambda](t) = 0, \quad t \in I, \quad y \in \mathfrak{D}(\mathcal{L}),$$

<u>is an incompatible system.</u>

Suppose that hypothesis (H_3) holds for a complex number λ_o with $\text{Im } \lambda_o > 0$, and let λ be any complex number such that

(4.32) $$|\lambda - \lambda_o| < \text{Im}\,\lambda_o;$$

in particular, this implies that $\text{Im}\,\lambda > 0$. Now suppose that there exists a 2n-dimensional vector function $z(t) = (z_1(t);z_2(t)) \in \mathfrak{D}(\mathfrak{L})$ which is such that $y = z(t)$ is a non-trivial solution of (4.31) for a complex λ satisfying (4.32). We shall show that $z(t) \equiv 0$ on I. Set

(4.33) $$w(t) = z(t) - (\lambda - \lambda_o)y_o(t),$$

where $y_o(t) = \int_I \mathfrak{G}(t,s,\lambda_o)\mathcal{K}(s)z(s)ds$. As $z(t) \in \mathfrak{D}(\mathfrak{L})$, in view of conclusion (vi) of Theorem 4.4 we have that $y_o(t) \in \mathfrak{D}(\mathfrak{L})$, and from (4.33) we conclude that $w(t) \in \mathfrak{D}(\mathfrak{L})$. Since $y_o(t)$ is a solution of $\mathfrak{L}[y|\lambda_o](t) = \mathcal{K}(t)z(t)$ on I, it follows from the hypothesis on $z(t)$ that $\mathfrak{L}[w|\lambda_o](t) = 0$. As hypothes..s (H_3) implies that $w(t) \equiv 0$ on I, the relation (4.33) is reduced to

$$z(t) = (\lambda - \lambda_o)\int_I \mathfrak{G}(t,s,\lambda_o)\mathcal{K}(s)z(s)ds,$$

which is equivalent to the two equations

$$z_i(t) = (\lambda - \lambda_o)\int_I G_{i1}(t,s,\lambda_o)K(s)z_1(s)ds, \quad (i = 1,2).$$

From conclusion (vi) of Theorem 4.4 for $z(t) = (z_1(t);z_2(t))$, and relation (4.32), it follows that

$$\|z_1\| \leq |\lambda - \lambda_o| \; |\text{Im}\,\lambda_o|^{-1} \|z_1\| < \|z_1\|.$$

But this is impossible, since the normality condition for the system and the fact that $z(t)$ is a non-trivial solution of (4.31) imply that $z_1(t) \not\equiv 0$ on I, and consequently, that $\|z_1\| > 0$. Therefore, we conclude that $z(t) \equiv 0$ on I, thus completing the proof of the result that (4.31) is incompatible whenever λ satisfies (4.32). If λ is such that $\text{Im}\,\lambda < 0$ and $|\lambda - (-\lambda_o)| = |\lambda + \lambda_o| < |\text{Im}\,\lambda_o|$, then by a similar argument it follows that the system (4.31) is incompatible for this value of λ. That is, we have established that if hypothesis (H_3) holds for a non-real complex λ_o then hypothesis (H_3) also holds for any complex number, necessarily non-real, satisfying $|\lambda - \lambda_o| < |\mathrm{Im}\ \lambda_o|$ or $|\lambda - (-\lambda_o)| = |\lambda + \lambda_o| < |\mathrm{Im}\ \lambda_o|$.

If μ is any complex number with $\mathrm{Im}\ \mu \neq 0$, then after a finite number of iterations of the above argument, μ will eventually fall in a domain $|\mu-\lambda| < |\mathrm{Im}\ \lambda|$ or $|\mu + \lambda| < |\mathrm{Im}\ \lambda|$ for some λ for which hypothesis (H_3) holds, which in turn implies that (H_3) also holds for μ.

THEOREM 4.6. *If hypotheses* (H_i), $(i = 1,2,3)$, *and* (H_2') *are satisfied, then there is a unique* $2n \times 2n$ *matrix function* $\mathfrak{G}(t,s,\lambda)$ *satisfying the properties of Theorem* 4.4. *In particular, if* δ *denotes a general non-degenerate compact subinterval of* I, *and* $(\mathcal{B}_\delta)$ *is an associated self-adjoint boundary value problem* $(3.1)_\delta$ *involving a fixed, but arbitrary, set of two-point boundary conditions at the end-points of* δ *then for* λ *non-real the family of Green's matrix functions* $\mathfrak{G}(t,s,\lambda;\delta)$ *converges to* $\mathfrak{G}(t,s,\lambda)$ *as* δ *tends monotonically to* I, *and the convergence is uniform in* (t,s,λ) *on each region* $\Omega(d,\Lambda)$ *of the form* $\{(t,s,\lambda) \in d \times d \times \Lambda \mid t \neq s\}$, *where* d *is an arbitrary compact subinterval of* I *and* Λ *is an arbitrary set possessing the property listed in* (4.4).

Suppose there are two matrix functions $\mathfrak{G}^{(j)}(t,s,\lambda)$, $(j = 1,2)$, each of which satisfies the properties of Theorem 4.4. Then in view of conclusion (v) of Theorem 4.4 the $2n \times 2n$ matrix function $\Psi(t,s,\lambda) \equiv \mathfrak{G}^{(1)}(t,s,\lambda) - \mathfrak{G}^{(2)}(t,s,\lambda)$ is a continuous function of (t,s) in $I \times I$ upon defining this matrix function in a suitable fashion along the line $t = s$, and from conclusion (iii) of the theorem we have that

$[\Psi(\cdot,s,\lambda)|\lambda](t) = 0$ for t (c,s) and t (s,d) with $I = (c,d)$, so that as a function of t the matrix $\Psi(t,s,\lambda)$ is locally absolutely continuous on I. Moreover, from conclusions (i) and (iv) of Theorem 4.4, as a function of t each column vector of $\Psi_{1j}(t,s,\lambda)$, $(j = 1,2)$, where $\Psi(t,s,\lambda) = [\Psi_{ij}(t,s,\lambda)]$, $(i,j = 1,2)$, belongs to $\mathcal{L}^2_n(I;K)$. Consequently, as a function of t each column vector of $\Psi(t,s,\lambda)$ belongs to $\mathfrak{D}(\mathcal{L})$. But the incompatible system (4.31) implies that $\Psi(t,s,\lambda) \equiv 0$ for $t \in I$ and $(s,\lambda) \in I \times C_o$. Therefore, $\mathfrak{G}^{(1)}(t,s,\lambda) = \mathfrak{G}^{(2)}(t,s,\lambda)$ on $\Omega(I,C_o)$.

Corresponding to a non-decreasing sequence $\{\delta_h\}$ of compact subintervals converging to I as $h \to \infty$, let $\mathfrak{G}^{(1)}(t,s,\lambda)$ be a Green's matrix which is determined as in Theorem 4.3 and which possesses the properties of Theorem 4.4. If it is not true that the Green's matrix function $\mathfrak{G}(t,s,\lambda;\delta)$ tends to $\mathfrak{G}^{(1)}(t,s,\lambda)$ on $\Omega(I,C_o) = \{(t,s) \in I \times I | t \neq s\} \times C_o$ as $\delta \to I$, then there exists a point (t_o,s_o,λ_o) of $\Omega(I,C_o)$ such that $\mathfrak{G}(t_o,s_o,\lambda_o;\delta)$ does not tend to $\mathfrak{G}^{(1)}(t_o,s_o,\lambda_o)$ as $\delta \to I$. By a suitable argument it follows that there exist a $2n \times 2n$ constant matrix L with elements in the extended real number system such that $L \neq \mathfrak{G}^{(1)}(t_o,s_o,\lambda_o)$ and a sequence $\{\delta'_h\}$ of compact intervals tending monotonically to I as $h \to \infty$, and such that $\mathfrak{G}(t_o,s_o,\lambda_o;\delta'_h)$ tends to L as $h \to \infty$. By the preceding, there exists a subsequence $\{\delta_m\}$ of $\{\delta'_h\}$ such that $\mathfrak{G}(t,s,\lambda;\delta_m)$ tends to a limit matrix function $\mathfrak{G}^{(2)}(t,s,\lambda)$ on $\Omega(I,C_o)$ as $m \to \infty$ which possesses the properties of Theorem 4.4. As $\mathfrak{G}^{(1)}(t_o,s_o,\lambda_o) \neq \mathfrak{G}^{(2)}(t_o,s_o,\lambda_o)$, we thus have a contradiction to the uniqueness property of the limit matrix, hence completing the proof of the theorem.

Under the conditions of Theorem 4.6 the uniquely determined $2n \times 2n$ matrix function $\mathfrak{G}(t,s,\lambda)$ is called the <u>Green's matrix</u> for the incompatible system (4.31).

5. The existence of a spectral matrix. In this section we are concerned again with the self-adjoint problem $(\mathcal{B}_\delta)$, or equivalently $(3.1)_\delta$. Hypotheses (H_i), $(i = 1,2,3)$, and (H_2') are all assumed in this section.

First we employ the following theorem, which permits us to replace the self-adjoint boundary condition $\mathcal{M}_\delta[u,v] = 0$ in $(\mathcal{B}_\delta)$ by an alternate form. The reader is referred to Reid [8; Sec. VII.8] for a proof of this result.

THEOREM 5.1. *If hypotheses (H_i), $(i = 1,2,3)$, and (H_2') hold, and if $\delta = [a,b]$ is a compact subinterval of I, then a necessary and sufficient condition for a differential system $(\mathcal{B}_\delta)$ to be self-adjoint is that there exist a $2n \times 2n$ Hermitian matrix $Q = Q[\mathcal{B}_\delta]$ and a linear subspace $\mathcal{S} = \mathcal{S}[\mathcal{B}_\delta]$ of the 2n-dimensional Euclidean space C_{2n} over the complex field such that the 2n-dimensional vectors $\hat{u} = (u(a);u(b))$, $\hat{v} = (v(a);v(b))$ satisfy the boundary conditions $(\mathcal{B}_\delta\text{-b})$ if and only if*

$$\hat{u} \in \mathcal{S}, \qquad Q\hat{u} + D\hat{v} \in \mathcal{S}^{\perp},$$

where D is the $2n \times 2n$ matrix defined by $D = \operatorname{diag}\{-E_n, E_n\}$, and $\mathcal{S}^{\perp} = \mathcal{S}^{\perp}[\mathcal{B}_\delta]$ denotes the orthogonal complement of $\mathcal{S} = \mathcal{S}[\mathcal{B}_\delta]$ in C_{2n}.

Since the system $(\mathcal{B}_\delta)$ treated in Section 3 is self-adjoint, in view of the above theorem we may present this system as

$$(5.1)_\delta \qquad \begin{aligned} &\text{(a)} \quad L_1[u,v|\lambda](t) = 0, \quad L_2[u,v](t) = 0, \qquad t \in \delta, \\ &\text{(b)} \quad \hat{u} \in \mathcal{S}, \qquad Q\hat{u} + D\hat{v} \in \mathcal{S}^{\perp}. \end{aligned}$$

We denote by

$$(5.2) \qquad \{\lambda_j(\delta); y_j(t;\delta) = (u_j(t;\delta); v_j(t;\delta))\}, \qquad (j = 1,2,\cdots),$$

the sequence of the proper values and corresponding proper vector functions for $(5.1)_\delta$, as described in Theorem 3.1.

In this section and hereafter the following additional hypothesis is assumed throughout.

(H_4) The $n \times n$ matrix function $B(t)$ is non-negative definite Hermitian on I.

A consequence of the above hypothesis is the following result, which appears in Reid [8; Theorem VII.11.1].

THEOREM 5.2. If hypothesis (H_i), ($i = 1,2,3,4$), and (H_2') hold for a given compact subinterval δ of I then the proper values $\lambda_j(\delta)$, ($j = 1,2,\cdots$), for $(5.1)_\delta$ may be ordered as a sequence $\lambda_1(\delta) \leq \lambda_2(\delta) \leq \cdots$ with the corresponding proper solutions $(u(t);v(t)) \equiv (u_j(t;\delta);v_j(t;\delta))$ such that: (a) $K_\delta[u_i(\cdot;\delta),u_j(\cdot;\delta)] = \delta_{ij}$, ($i,j = 1,2,\cdots$); (b) the set of proper values $\lambda_j(\delta)$ is bounded below, and $\lambda_j(\delta) \to \infty$ as $j \to \infty$.

We denote by $\mathfrak{D}[\mathcal{B}_\delta]$ the space of all n-dimensional vector functions $\eta(t) \in \mathfrak{D}[\delta]$ such that $\hat{\eta} = (\eta(a);\eta(b)) \in \mathfrak{S}[\mathcal{B}_\delta]$, where $\mathfrak{D}[\delta]$ is defined in the paragraph following the relation (2.4) in Section 2. The space $\mathfrak{D}[\mathcal{B}_\delta]$ is clearly a linear space; moreover, it is an inner product space with inner product $K_\delta[\eta,\xi]$ and norm

$$\|\eta\|_\delta = (K_\delta[\eta,\eta])^{1/2}.$$

We denote by $\mathfrak{D}(\delta;K)$ the completion of this space, so that $\mathfrak{D}(\delta;K)$ is a Hilbert space.

Concerning the Parseval equality, we state the following theorem. A proof appears in Theorem VII.11.3 of Reid [8].

THEOREM 5.3. If $\eta(t)$, $\xi(t)$ are vector functions in $\mathfrak{D}[\mathcal{B}_\delta]$ then

$$K_\delta[\eta] = \sum_{j=1}^{\infty} |C_{\delta j}[\eta]|^2, \tag{5.3}$$

$$K_\delta[\eta,\xi] = \sum_{j=1}^{\infty} \overline{C_{\delta j}[\xi]}C_{\delta j}[\eta], \tag{5.4}$$

where $C_{\delta j}[\eta]$ *denotes the* j-th *Fourier coefficient of* η *defined by*

$$C_{\delta j}[\eta] = \int_\delta (K(t)\eta(t),\ u_j(t;\delta))dt, \qquad (j = 1,2,\cdots).$$

THEOREM 5.4. *If* $\eta(t)$, $\xi(t)$ *are vector functions in* $\mathfrak{D}(\delta;K)$, *then the relations* (5.3) *and* (5.4) *hold.*

If $\eta \in \mathfrak{D}(\delta;K)$, then for $\varepsilon > 0$ there exists an $\eta_o \in \mathfrak{D}[\mathcal{B}_\delta]$ such that

$$(5.5) \qquad \underset{\sim}{K}_\delta[\eta - \eta_o] < \varepsilon/4.$$

Let $r_o^{(m)}(t) = \sum_{j=1}^{m} C_{\delta j}[\eta_o]u_j(t)$. Then in view of the inequality $\underset{\sim}{K}_\delta[\eta+\xi] \leq 2\{\underset{\sim}{K}_\delta[\eta] + \underset{\sim}{K}_\delta[\xi]\}$, we have

$$(5.6) \qquad \underset{\sim}{K}_\delta[\eta - r_o^{(m)}] \leq 2\{\underset{\sim}{K}_\delta[\eta - \eta_o] + \underset{\sim}{K}_\delta[\eta_o - r_o^{(m)}]\}.$$

From Theorem 5.3 we have that for a sufficiently large m the term $\underset{\sim}{K}_\delta[\eta_o - r_o^{(m)}]$ is less than $\varepsilon/4$, so that for such an m it follows from relation (5.6) that

$$(5.7) \qquad \underset{\sim}{K}_\delta[\eta - r_o^{(m)}] < \varepsilon.$$

Set $w^{(m)}(t) = \eta(t) - \sum_{j=1}^{m} C_{\delta j}[\eta]u_j(t;\delta)$, $(m = 1,2,\cdots)$. Since the Bessel inequality for the $\underset{\sim}{K}_\delta$-orthonormal sequence $\{u_j(\delta)\}$ implies $\underset{\sim}{K}_\delta[w^{(m)}] \leq \underset{\sim}{K}_\delta[\eta - r_o^{(m)}]$, and $\underset{\sim}{K}_\delta[w^{(m)}] = \underset{\sim}{K}_\delta[\eta] - \sum_{j=1}^{m}|C_{\delta j}[\eta]|^2$, it then follows that $\underset{\sim}{K}_\delta[w^{(m)}] \to 0$ as $m \to \infty$, which proves (5.3) for $\eta \in \mathfrak{D}(\delta;K)$.

In the usual fashion the sesquilinear representation (5.4) follows from the quadratic relation (5.3).

In the proof of Theorem 5.4, we have actually established the following result.

COROLLARY. *If* δ *is a compact subinterval of* I *and* $\eta(t)$ *is an* n-*dimensional vector function belonging to* $\mathfrak{D}(\delta;K)$, *then*

$$(5.7') \qquad \eta(t) = \sum_{j=1}^{\infty} C_{\delta j}[\eta]u_j(t;\delta),$$

where the series converges in the norm of $\mathfrak{H}(\delta;K)$.

In what follows, for typographical simplification we write the proper vector functions for $(\mathcal{B}_\delta)$ as

(5.8) $\quad (u_j^{(1)}(t;\delta);u_j^{(2)}(t;\delta)) = (u_j(t;\delta);v_j(t;\delta)), \quad (j = 1,2,\cdots).$

For λ not a proper value of the problem $(5.1)_\delta$ let $\mathfrak{G}(t,s,\lambda;\delta) = [G_{ij}(t,s,\lambda;\delta)]$, $(i,j = 1,2)$, be the corresponding Green's matrix. Then clearly $(u;v) = (u_j^{(1)}(t;\delta);u_j^{(2)}(t;\delta))$, $(j = 1,2,\cdots)$, are solutions of the differential system

$$L_1[u,v|\bar{\lambda}](T) = (\lambda_j - \bar{\lambda})K(t)u_j(t;\delta), \quad L_2[u,v](t) = 0, \quad t \in \delta,$$
$$\hat{u} \in \mathcal{S}, \qquad Q\hat{u} + D\hat{v} \in \mathcal{S}^\perp.$$

Consequently, for $\beta = 1,2$ we have the relation

$$(\lambda_j - \bar{\lambda})^{-1}\, u_j^{(\beta)}(t;\delta) = \int_\delta G_{\beta 1}(t,s,\bar{\lambda};\delta)K(s)u_j^{(1)}(s;\delta)ds, \quad t \in \delta,$$

and hence for $\beta = 1,2$ and $j = 1,2,\cdots$ we have

(5.9) $\quad (\lambda_j - \lambda)^{-1}[u_j^{(\beta)}(t;\delta)]^* = \int_\delta [u_j^{(1)}(s;\delta)]^*K(s)G_{1\beta}(s,t,\lambda;\delta)ds.$

We now establish a result which is basic in what follows.

THEOREM 5.5. *Suppose that hypotheses* (H_i), $(i = 1,2,3,4)$, *and* (H_2') *hold, and* $\delta = [a,b]$ *is a compact subinterval of* I, *while* λ *is a complex number which is not a proper value for* $(5.1)_\delta$. *If* $\mathfrak{G}(t,s,\lambda;\delta) = [G_{ij}(t,s,\lambda;\delta)]$, $(i,j = 1,2)$, *is the corresponding* $2n \times 2n$ *Green's matrix for* $(5.1)_\delta$, *then as a function of* t *each column vector of* $G_{1j}(t,s,\lambda;\delta)$, $(j = 1,2)$, *belongs to* $\mathfrak{H}(\delta;K)$.

In view of Theorem VII.8.2 in Reid [8], as a function of t each column vector of $G_{11}(t,s,\lambda;\delta)$ belongs to $\mathfrak{D}[\mathcal{B}_\delta]$, and hence belongs to $\mathfrak{H}(\delta;K)$. Therefore, it suffices to show the similar result for

$G_{12}(t,s,\lambda;\delta)$. Let ξ be any n-dimensional constant vector, and s a point of $\delta = [a,b]$. Then from conclusion (i-b) of Theorem 3.2 we have that as a function of t the n-dimensional vector functions $G_{12}(t,s,\lambda;\delta)\xi$ and $G_{22}(t,s,\lambda;\delta)\xi$ are of class C_n^1 on $[a,b] - \{s\}$, and

$$L_2[G_{12}(\cdot,s,\lambda;\delta)\xi,\ G_{22}(\cdot,s,\lambda;\delta)\xi](t) = 0, \text{ for } t \in [a,b] - \{s\}.$$

Consequently, in view of Lemma 4.1, for each fixed $s \in \delta$ we have that there exist sequences $\{y_m(t,s,\lambda)\}$ and $\{z_m(t,s,\lambda)\}$ which, as a function of t, belong to $C_n^1[a,b]$ and are such that

$$\begin{aligned} &\text{(a)}\quad L_2[y_m(\cdot,s,\lambda),z_m(\cdot,s,\lambda)](t) = 0, \quad \text{for} \quad t \in [a,b];\\ \text{(5.10)}\quad &\text{(b)}\quad y_m(t,s,\lambda) = 0 \text{ for } t \in S_o = \{s\}\cup\{a\}\cup\{b\}, \ (m = 1,2,\cdots);\\ &\text{(c)}\quad \underset{\sim}{K}_\delta[y_m(\cdot,s,\lambda) - G_{12}(\cdot,s,\lambda;\delta)\xi] \to 0 \text{ as } m \to \infty. \end{aligned}$$

Relations (5.10-a,b) imply that as a function of t each $y_m(t,s,\lambda)$ belongs to $\mathfrak{D}[\mathfrak{B}_\delta]$, and in turn relation (5.10-c) implies that as a function of t the vector $G_{12}(t,s,\lambda;\delta)\xi$ belongs to $\mathfrak{D}(\delta;K)$. Since ξ is arbitrary, it follows, in particular, that as a function of t each column vector of $G_{12}(t,s,\lambda;\delta)$ belongs to $\mathfrak{D}(\delta;K)$.

THEOREM 5.6. <u>Under the hypothesis of Theorem</u> 5.5, <u>if the sequences of proper values and proper vector functions for</u> $(5.1)_\delta$ <u>are denoted by</u> (5.2) <u>together with</u> (5.8), <u>then for</u> $(t,s) \in \delta \times \delta$ <u>and</u> $\alpha,\beta = 1,2$ <u>we have that</u>

$$\int_\delta [G_{1\alpha}(r,t,\lambda;\delta)]^* K(r) G_{1\beta}(r,s,\lambda;\delta)dr$$
$$= \sum_{j=1}^\infty |\lambda_j(\delta) - \lambda|^{-2}\, u_j^{(\alpha)}(t;\delta)[u_j^{(\beta)}(s;\delta)]^*.$$

Let ξ be an arbitrary constant n-dimensional vector. Then, in view of Theorem 5.5, as a function of t the vector function $G_{1\alpha}(t,s,\lambda)\xi$ belongs to $\mathfrak{D}(\delta;K)$. Relation (5.11) then follows readily from Theorem 5.4

together with relation (5.9).

The four matrix equations in (5.11) can be expressed as a $2n \times 2n$ matrix relation

$$\int_\delta \mathfrak{G}^*(r,t,\lambda;\delta)\mathcal{K}(r)\mathfrak{G}(r,s,\lambda;\delta)dr = \sum_{j=1}^{\infty} |\lambda_j(\delta) - \lambda|^{-2} y_j(t;\delta)y_j^*(s;\delta). \tag{5.12}$$

If λ_1 and λ_2 are complex numbers which are not proper values of $(5.1)_\delta$, then (see, for example, Prob. IV.3:1 in Reid [8]) we have the well-known identity

$$\mathfrak{G}(t,s,\lambda_2;\delta) - \mathfrak{G}(t,s,\lambda_1;\delta) = (\lambda_2 - \lambda_1)\int_\delta \mathfrak{G}(t,r,\lambda_1;\delta)\mathcal{K}(r)\mathfrak{G}(r,s,\lambda_2;\delta)dr.$$

Upon setting $\lambda_2 = \lambda$ and $\lambda_1 = \bar{\lambda}$, where λ is a non-real complex number, and using the symmetry property of $\mathfrak{G}(t,s,\lambda;\delta)$, we obtain the relation

$$Q(t,s,\lambda;\delta) = 2i \operatorname{Im} \lambda \int_\delta \mathfrak{G}^*(r,t,\lambda;\delta)\mathcal{K}(r)\mathfrak{G}(r,s,\lambda;\delta)dr, \tag{5.13}$$

where

$$Q(t,s,\lambda;\delta) \equiv \mathfrak{G}(t,s,\lambda;\delta) - \mathfrak{G}(t,s,\bar{\lambda};\delta).$$

In view of property (i-e) of the Green's matrix it follows that $Q(t,s,\lambda;\delta)$ possesses limits $Q_j(s_0,s_0,\lambda;\delta)$, $(j = 1,2)$, in the sense of (3.4), and if $Q(s_0,s_0,\lambda;\delta)$ is defined as $Q_1(s_0,s_0,\lambda;\delta)$ then $Q(t,s,\lambda;\delta)$ is continuous in (t,s,λ) on $\delta \times \delta \times C_0$. Throughout the following discussion it will be assumed that $Q(t,s,\lambda;\delta)$ has been so defined. If we set

$$\mathcal{R}(t,s,\lambda;\delta) = \frac{1}{2i} Q(t,s,\lambda;\delta), \tag{5.14}$$

then conclusion (i-d) of Theorem 3.2 implies $\mathcal{R}^*(t,s,\lambda;\delta) = \mathcal{R}(s,t,\lambda;\delta)$, and it follows from relations (5.12) - (5.14) that

$$\mathcal{R}(t,s,\lambda;\delta) = \operatorname{Im} \lambda \sum_{j=1}^{\infty} |\lambda_j(\delta) - \lambda|^{-2} y_j(t;\delta)y_j^*(s;\delta). \tag{5.15}$$

Let $Y(t,\lambda)$ be a $2n \times 2n$ fundamental matrix for $[y|\lambda](t) = 0$ on I such that $Y(t_o,\lambda) = \mathcal{J}$ for some $t_o \in (a,b)$, where $\delta = [a,b]$. Each proper vector function $y_j(t;\delta)$ may be expressed as $y_j(t;\delta) = Y(t,\lambda_j)p_j(\delta)$, $(j = 1,2,\cdots)$, for some 2n-dimensional constant vector $p_j(\delta)$ which depends on δ as well as on $Y(t,\lambda)$, and thus using this relation and $\mathcal{J}^*\mathcal{J} = E_{2n}$ it follows from (5.15) that

$$(5.16) \qquad \mathcal{J}^*\mathcal{R}(t_o,t_o,\lambda;\delta)\mathcal{J} = \operatorname{Im}\lambda \sum_{j=1}^{\infty}|\lambda_j(\delta) - \lambda|^{-2}p_j(\delta)p_j^*(\delta).$$

We shall now express the right-hand member of (5.16) in terms of a Stieltjes integral. To this end, we introduce a $2n \times 2n$ matrix function $\rho(\mu;\delta)$ defined on $-\infty < \mu < \infty$ as

$$(5.16') \qquad \rho(\mu;\delta) = \begin{cases} -\sum_{\mu < \lambda_j(\delta) \le 0} p_j(\delta)p_j^*(\delta), & \text{for } \mu < 0, \\ \sum_{0 < \lambda_j(\delta) \le \mu} p_j(\delta)p_j^*(\delta), & \text{for } \mu \ge 0. \end{cases}$$

From this definiti n it is clear that $\rho(\mu;\delta)$ has the following properties:

(i) <u>if</u> $\Delta = (\mu,\nu]$, <u>and</u> $\mu < \nu$, <u>then</u>

$$\Delta\rho(\delta) \equiv \rho(\nu;\delta) - \rho(\mu;\delta) = \sum_{\mu < \lambda_j(\delta) \le \nu} p_j(\delta)p_j^*(\delta);$$

(ii) <u>if</u> λ <u>is a proper value</u>, <u>then</u>

$$\rho(\lambda;\delta) = \rho(\lambda^+;\delta) = \rho(\lambda^-;\delta) + \sum_{\lambda_j(\delta)=\lambda} p_j(\delta)p_j^*(\delta),$$

<u>while if</u> λ <u>is not a proper value then</u> ρ <u>is continuous at</u> λ. <u>Moreover</u>, $\rho(\mu;\delta)$ <u>is a step function which is discontinuous at each proper value</u>;

(iii) $\rho(0;\delta) = 0$;

(iv) $\rho(\mu;\delta)$ <u>is Hermitian</u>;

(v) $\Delta\rho(\delta)$ <u>is non-negative definite Hermitian if</u> $\Delta = (\mu,\nu]$ <u>and</u> $\mu < \nu$;

(vi) <u>the total variation of</u> $\rho(\mu;\delta) = [\rho_{jk}(\mu;\delta)]$, $(j,k = 1,2,\cdots,2n)$, <u>is finite on every bounded</u> Δ-<u>interval.</u>

Property (vi) follows from the fact that each diagonal element $\rho_{jj}(\mu;\delta)$ is a non-decreasing step function, and hence it is of bounded variation on each bounded Δ-interval, while for $j \neq k$ the Cauchy-Schwarz inequality implies

$$|\Delta\rho_{jk}(\delta)|^2 \leq \Delta\rho_{jj}(\delta)\Delta\rho_{kk}(\delta).$$

The $2n \times 2n$ matrix function $\rho(\mu;\delta)$ is called a <u>spectral matrix</u> for the self-adjoint problem $(5.1)_\delta$. In terms of this spectral matrix, relation (5.16) now may be written as

$$T(t_o,\lambda;\delta) = \operatorname{Im}\lambda \int_{-\infty}^{\infty} |\mu - \lambda|^{-2} d\rho(\mu;\delta), \tag{5.18}$$

where the $2n \times 2n$ matrix $T(t_o,\lambda;\delta)$ is given by

$$T(t_o,\lambda;\delta) \equiv \mathcal{G}^* \mathcal{R}(t_o,t_o,\lambda;\delta)\mathcal{G}. \tag{5.19}$$

Let d be an arbitrary compact subinterval of I containing the point t_o, and Λ an arbitrary compact subset of C_o with the properties in (4.4). Let $\{\mathcal{G}(t,s,\lambda;\delta_m)\}$ be a sequence of Green's matrix functions which converges uniformly on $\Omega(d,\Lambda)$ to a limit matrix $\mathcal{G}(t,s,\lambda)$ as determined in Theorem 4.3. Then the corresponding matrices $\mathcal{Q}(t,s,\lambda;\delta_m)$, $(m = 1,2,\cdots)$, and $\mathcal{Q}(t,s,\lambda) = \mathcal{G}(t,s,\lambda) - \mathcal{G}(t,s,\bar{\lambda})$ are continuous functions of (t,s,λ) on $d \times d \times \Lambda$. Consequently, the sequence $\mathcal{Q}(t_o,t_o,\lambda;\delta_m)$ converges to $\mathcal{Q}(t_o,t_o,\lambda)$ as $m \to \infty$, and hence the matrix sequence $T(t_o,\lambda;\delta_m)$ defined in (5.19) converges to

$$T(t_o,\lambda) \equiv \mathcal{G}^* \mathcal{R}(t_o,t_o,\lambda)\mathcal{G},$$

where $\mathcal{R}(t_o,t_o,\lambda) = \frac{1}{2i}\ \mathcal{Q}(t_o,t_o,\lambda)$.

THEOREM 5.7. Suppose that hypotheses (H_i), $(i = 1,2,3,4)$, and (H_2') hold, and $\{\mathcal{G}(t,s,\lambda;\delta_m)\}$ is a convergent sequence of Green's matrices as established in Theorem 4.3. Let $\rho(\mu;\delta_m) = [\rho_{jk}(\mu;\delta_m)]$, $(j,k = 1,\cdots,2n)$, be a spectral matrix for the problem $(\mathcal{B}_{\delta_m})$ possessing the properties (i) - (vi) of (5.17). Then there exists a $2n \times 2n$ matrix function $\rho(\mu) = [\rho_{jk}(\mu)]$ defined on $(-\infty,\infty)$, and such that:

(a) $\rho(0) = 0$;

(b) $\rho(\mu)$ is Hermitian;

(c) $\Delta\rho = \rho(\nu) - \rho(\mu)$ is non-negative definite Hermitian if $\Delta = (\mu,\nu]$ and $\mu < \nu$, and a subsequence of $\{\rho(\mu;\delta_m)\}$ which converges to $\rho(\mu)$ on $(-\infty,\infty)$.

If μ_1, μ_2 are points of continuity of $\rho(\mu)$, then

(i) $\frac{1}{\pi}\lim_{\varepsilon\to 0^+}\displaystyle\int_{\mu_1}^{\mu_2} T(t_o,\sigma+i\varepsilon)d\sigma = \rho(\mu_2) - \rho(\mu_1)$.

Relation (5.18) for $\lambda = i$ and $\delta = \delta_m$ implies that

$$T(t_o,i,\delta_m) = \int_{-\infty}^{\infty} (\mu^2+1)^{-1}d\rho(\mu;\delta_m).$$

Consequently, if σ is any positive real number then we have the matrix relation

$$T(t_o,i,\delta_m) \geq \int_{-\sigma}^{\sigma} (\mu^2+1)^{-1}d\rho(\mu;\delta_m).$$

Since $T(t_o,i,\delta_m)$ converges to $T(t_o,i)$, there exists a constant Hermitian matrix A such that

$$\int_{-\sigma}^{\sigma} (\mu^2+1)^{-1}d\rho(\mu;\delta_m) \leq A, \quad (m = 1,2,\cdots;\ 0 < \sigma < \infty).$$

In particular,

$$(5.20) \qquad \int_{-\sigma}^{\sigma} d\rho(\mu;\delta_m) \leq (\sigma^2+1)A, \quad (m = 1,2,\cdots;\ 0 < \sigma < \infty),$$

which shows that the sequence $\{\rho(\mu;\delta_m)\}$ of spectral matrices is uniformly of bounded variation on every compact subinterval of $(-\infty,\infty)$. From (5.20) it follows that

$$\rho(\sigma;\delta_m) - \rho(-\sigma;\delta_m) \leq (\sigma^2+1)A \quad \text{for} \quad 0 < \sigma < \infty,$$

and, in view of the condition $\rho(-\sigma;\delta_m) \leq 0$ for such σ the above relation implies that

$$(5.21) \qquad |\rho(\mu;\delta_m)| \leq (\mu^2+1)A, \quad (m = 1,2,\cdots;\ \mu \in (-\infty,\infty)),$$

where

$$|\rho(\mu;\delta_m)| = \begin{cases} \rho(\mu;\delta_m) & \text{for} \quad \mu \geq 0, \\ -\rho(\mu;\delta_m) & \text{for} \quad \mu < 0. \end{cases}$$

Applying Helly's selection theorem, (see, for example, Coddington and Levinson [3; p. 233]), to (5.21) one obtains the existence of a convergent subsequence of $\{\rho(\mu;\delta_m)\}$ and limit matrix function $\rho(\mu)$ possessing properties (a), (b), and (c).

To prove conclusion (i), we consider the case that $\mu_1 \neq \mu_2$, for if $\mu_1 = \mu_2$ then the conclusions hold trivially. We assume that $\mu_1 < \mu_2$ and let $\delta = \delta_m$ in (5.18), where the sequence $\{\delta_m\}$ is such that the corresponding sequence $\{\rho(\mu;\delta_m)\}$ converges to $\rho(\mu)$ for $\mu \in (-\infty,\infty)$. Since the set $\{\rho(\mu;\delta_m)\}$ is uniformly of bounded variation on every finite subinterval of $(-\infty,\infty)$, from the generalized Osgood theorem, (see, for example, Hildebrandt [4; p. 98]), it follows that

$$T(t_o,\lambda) = \operatorname{Im} \lambda \int_{-\infty}^{\infty} |\mu - \lambda|^{-2} d\rho(\mu).$$

If σ and ε are real variables with $\varepsilon > 0$, then upon setting $\lambda = \sigma + i\varepsilon$ in the above equation and integrating over $[\mu_1, \mu_2]$ with respect to σ, we obtain

$$H(\mu_1,\mu_2,\varepsilon) = \int_{\mu_1}^{\mu_2} \int_{-\infty}^{\infty} \varepsilon[(\mu-\sigma)^2 + \varepsilon^2]^{-1} d\rho(\mu) d\sigma,$$

where

$$(5.21') \qquad H(\mu_1,\mu_2,\varepsilon) = \int_{\mu_1}^{\mu_2} T(t_o,\sigma + i\varepsilon) d\sigma.$$

For $m > 0$ let

$$T_m(t_o,\sigma + i\varepsilon) \equiv \int_{-m}^{m} \varepsilon[(\mu-\sigma)^2 + \varepsilon^2]^{-1} d\rho(\mu).$$

Then $\{T_m\}$ is a monotone non-decreasing Hermitian matrix sequence with the limit matrix $T(t_o,\sigma + i\varepsilon)$. Hence, from the monotone convergence theorem it follows that

$$(5.22) \qquad H(\mu_1,\mu_2,\varepsilon) = \lim_{m \to \infty} \int_{\mu_1}^{\mu_2} T_m(t_o,\sigma + i\varepsilon) d\sigma.$$

On the other hand,

$$(5.23) \qquad \int_{\mu_1}^{\mu_2} T_m(t_o,\sigma + i\varepsilon) d\sigma \equiv \int_{\mu_1}^{\mu_2} \int_{-m}^{m} \varepsilon[(\mu-\sigma)^2 + \varepsilon^2]^{-1} d\rho(\mu) d\sigma$$

$$= \int_{-m}^{m} \int_{\mu_1}^{\mu_2} \varepsilon[(\mu-\sigma)^2 + \varepsilon^2]^{-1} d\sigma d\rho(\mu)$$

$$= \int_{-m}^{m} [\tan^{-1}(\mu_2-\mu)/\varepsilon - \tan^{-1}(\mu_1-\mu)/\varepsilon] d\rho(\mu).$$

Consequently, from (5.22) and (5.23) we have the relation

$$(5.24) \qquad H(\mu_1,\mu_2,\varepsilon) = \int_{-\infty}^{\infty} [\tan^{-1}(\mu_2-\mu)/\varepsilon - \tan^{-1}(\mu_1-\mu)/\varepsilon] d\rho(\mu).$$

Let e be a positive number such that $e < (\mu_2 - \mu_1)/3$, and express the integral in (5.24) as the sum of the integrals over $J_1 = (-\infty,\mu_1-e)$, $J_2 = [\mu_1-e,\mu_1+e)$, $J_3 = [\mu_1+e,\mu_2-e)$, $J_4 = [\mu_2-e,\mu_2+e)$, and $J_5 = [\mu_2+e,\infty)$. If for brevity we set $f(\mu,\varepsilon) = \tan^{-1}(\mu_2-\mu)/\varepsilon - \tan^{-1}(\mu_1-\mu)/\varepsilon$, then:

(a) it follows readily from (5.23) that there exists an $\varepsilon_o > 0$ such that for $0 < \varepsilon < \varepsilon_o$ the scalar function $f(\mu,\varepsilon)$ converges monotonically on $J_1 \cup J_5$ to zero as $\varepsilon \to 0$, and hence in view of the monotone convergence theorem we have that $\int_{J_1 \cup J_5} f(\mu,\varepsilon)d\rho(\mu) \to 0$ as $\varepsilon \to 0$;

(b) on $J_2 \cup J_4$, the inequality $|f(\mu,\varepsilon)| \leq \pi$ implies the matrix inequality

$$-\pi\kappa(e) \leq \int_{J_2 \cup J_4} f(\mu,\varepsilon)d\rho(\mu) \leq \pi\kappa(e),$$

where $\kappa(e) = [\rho(\mu_2+e) - \rho(\mu_2-e)] + [\rho(\mu_1+e) - \rho(\mu_1-e)]$;

(c) on the interval J_3 the function $f(\mu,\varepsilon) \to \pi$ as $\varepsilon \to 0$, and hence $\int_{J_3} f(\mu,\varepsilon)d\rho(\mu) \to \pi[\rho(\mu_2 - e) - \rho(\mu_1 + e)]$.

Consequently, in view of the continuity property of $\rho(\mu)$ at μ_1, μ_2 and the arbitrariness of e, it follows from the above (a), (b), and (c) that

$$\lim_{\varepsilon\to 0} H(\mu_1,\mu_2,\varepsilon) = \pi[\rho(\mu_2) - \rho(\mu_1)],$$

which, in view of (5.21'), is equivalent to relation (i).

For a $2n \times 2n$ matrix ρ as established in Theorem 5.7 we denote by $\mathfrak{H}$ the Hilbert space $\mathcal{L}^2_{2n}(\rho)$ consisting of all 2n-dimensional vector functions $w(\lambda)$ defined on $(-\infty,\infty)$ which are ρ-<u>measurable</u>, and are such that

$$\underset{\sim}{P}[w] \equiv \int_{-\infty}^{\infty} (d\rho(\lambda)w(\lambda),w(\lambda))d\lambda \equiv \int_{-\infty}^{\infty} w^*(\lambda)d\rho(\lambda)w(\lambda) < \infty.$$

If $w(\lambda)$ and $z(\lambda)$ are elements of $\mathcal{H}$, then the inner product of w and z, denoted by $\underset{\sim}{P}[w,z]$, is defined in the natural way by

$$\underset{\sim}{P}[w,z] = \int_{-\infty}^{\infty} z^*(\lambda)d\rho(\lambda)w(\lambda).$$

It is clear that we have the Cauchy-Schwarz inequality

$$|\underset{\sim}{P}[w,z]|^2 \leq \underset{\sim}{P}[w]\ \underset{\sim}{P}[z].$$

If J is a subinterval of $(-\infty,\infty)$, then, corresponding to the notation used for the Hermitian functional $\underset{\sim}{K}$, we denote by $\underset{\sim}{P}_J[w]$ the associated functional $\int_J w^*(\lambda)d\rho(\lambda)w(\lambda)$

6. The Parseval equality and expansion theorem. In this section we assume all the previous hypotheses (H_i), $(i = 1,2,3,4)$, together with (H_2'). Moreover, as in Section 5, if $\delta = [a,b]$ is a non-degenerate compact subinterval of I we denote by

$$\{\lambda_j(\delta);y_j(t;\delta) \equiv (u_j(t;\delta);v_j(t;\delta))\}, \quad (j = 1,2,\cdots),$$

a set of the proper values and the corresponding proper vector functions for $(5.1)_\delta$ satisfying the conditions of Theorem 5.2. Moreover, let $Y(t,\lambda) \equiv (U(t,\lambda);\ V(t,\lambda))$ be a fundamental matrix for $\mathcal{L}[y|\lambda](t) = 0$ on I such that $Y(t_o,\lambda) = \mathcal{I}$ for some $t_o \in (a,b)$, where $U(t,\lambda)$ and $V(t,\lambda)$ are $n \times 2n$ matrix functions. Then corresponding to Theorem 5.3 we have the following result.

THEOREM 6.1. Let $\eta(t)$ be an n-dimensional vector function defined on I such that $\eta(t) \in \mathcal{D}[\mathcal{B}_\delta]$ for $t \in \delta$ and $\eta(t) \equiv 0$ for t outside δ. Then there exists an element $w(\lambda|\eta) \in \mathcal{H}$ which depends on $\eta(t)$ such that

$$(6.1) \qquad \underset{\sim}{K}[\eta] = \int_{-\infty}^{\infty} w^*(\lambda|\eta)d\rho(\lambda)w(\lambda|\eta) \equiv \underset{\sim}{P}[w(\lambda|\eta)],$$

where $\rho(\lambda)$ is a $2n \times 2n$ matrix function determined in Theorem 5.7, and

$w(\lambda|\eta)$ *is given by*

(6.2) $$w(\lambda|\eta) = \int_I (K(t)\eta(t), U(t,\lambda))dt.$$

It is to be noted that the integral in (6.2) exists in the ordinary sense, since $\eta(t)$ vanishes outside δ.

Let $\eta(t)$ be an n-dimensional vector function as described in the theorem. Applying relation (5.3) to $\eta(t)$, we obtain the equation

$$\underset{\sim}{K}_\delta[\eta] = \sum_{j=1}^{\infty} \int_I \overline{(K(t)\eta(t), u_j(t;\delta))}dt \int_I (K(t)\eta(t), u_j(t;\delta))dt.$$

From the relation $y_j(t;\delta) = Y(t,\lambda_j)p_j(\delta)$, $(j = 1,2,\cdots)$, it follows that $u_j(t;\delta) = U(t,\lambda_j)p_j(\delta)$. Then, employing the spectral matrix defined by (5.16'), the above equation can be expressed as

$$\begin{aligned}\underset{\sim}{K}_\delta[\eta] &= \sum_{j=1}^{\infty}\left\{\int_I (K(t)U(t,\lambda_j), \eta(t))dt\right\}p_j(\delta)p_j^*(\delta)\left\{\int_I (K(t)\eta(t), U(t,\lambda_j))dt\right\} \\ &= \int_{-\infty}^{\infty} w^*(\lambda|\eta)d\rho(\lambda;\delta)w(\lambda|\eta).\end{aligned}$$

It is to be noted that the left-hand member $\underset{\sim}{K}_\delta[\eta]$ is independent of each compact interval containing δ. Now let $\{\delta_m\}$ be a sequence of compact intervals converging monotonically to I, and such that the corresponding sequence $\{\rho(\lambda;\delta_m)\}$ converges to a limit matrix $\rho(\lambda)$ as determined in Theorem 5.7. Then, in particular, the above relation holds for such δ_m; that is,

(6.3) $$\underset{\sim}{K}_{\delta_m}[\eta] = \int_{-\infty}^{\infty} w^*(\lambda|\eta)d\rho(\mu;\delta_m)w(\lambda|\eta), \quad (m = 1,2,\cdots).$$

Moreover, in view of relation (5.20), the set $\{\rho(\lambda;\delta_m)\}$ is uniformly of bounded variation on every compact λ-interval of $(-\infty,\infty)$. Also, for fixed η it follows readily that $w(\lambda|\eta) = \int_I (K(t)\eta(t), U(t,\lambda))dt$ is continuous in λ on $(-\infty,\infty)$. Relation (6.1) then follows upon letting $\delta_m \to I$ in (6.3), and applying the generalized Osgood theorem (see, for example, Hildebrandt) [4, p. 98]). Since $\underset{\sim}{K}[\eta] = \underset{\sim}{K}_{\delta_m}[\eta] = \underset{\sim}{K}_\delta[\eta]$ whenever $\delta_m \supset \delta$, relation (6.1) implies that $w(\lambda|\eta) \in \mathcal{H}$.

Denote by $\mathfrak{D}(I;K)$ the space consisting of all n-dimensional vector functions $\eta(t)$ which are defined and measurable on I, and such that:

(i) $\underset{\sim}{K}[\eta] \equiv \int_I (K(t)\eta(t),\eta(t))dt < \infty$;

(ii) if d is any compact subinterval of I, then the restriction of $\eta(t)$ on d belong to $\mathfrak{D}(d;K)$.

We shall now extend the result of Theorem 6.1 to the space $\mathfrak{D}(I;K)$.

THEOREM 6.2. *If* $\eta(t) \in \mathfrak{D}(I;K)$, *then*

$$\underset{\sim}{K}[\eta] = \underset{\sim}{P}[w(\lambda|\eta)], \tag{6.4}$$

where $w(\lambda|\eta) \in \mathcal{H}$ *is determined by* $\eta(t)$, *and given by*

$$w(\lambda|\eta) = \int_I (K(t)\eta(t),U(t,\lambda))dt. \tag{6.5}$$

The integral in (6.5) *converges in the norm of* $\mathcal{H}$; *that is,*

$$\underset{\sim}{P}[w(\lambda|\eta) - \int_{d(m)} (K(t)\eta(t),U(t,\lambda))dt] \to 0 \tag{6.6}$$

as $m \to \infty$, *where* $\{d(m)\}$ *is a sequence of intervals converging monotonically to* I *as* $m \to \infty$. *Correspondingly, if* $\eta(t)$ *and* $\xi(t)$ *belong to* $\mathfrak{D}(I;K)$ *then*

$$\underset{\sim}{K}[\eta,\xi] = \underset{\sim}{P}[w(\lambda|\eta),w(\lambda|\xi)]. \tag{6.7}$$

The proof given here is a standard one. Following Coddington and Levinson [3; pp. 236-7], we first prove relation (6.4) for $\eta(t) \in \mathfrak{D}_0(I;K)$, where $\mathfrak{D}_0(I;K)$ consists of those $\eta(t) \in \mathfrak{D}(I;K)$ for which $\eta(t) \equiv 0$ outside some compact subinterval of I. If $\eta(t) \in \mathfrak{D}_0(I;K)$, and δ is a compact subinterval of I such that $\eta(t) \equiv 0$ outside δ, then clearly $\eta(t) \in \mathfrak{D}(\delta;K)$. Consequently, there exists a sequence of elements $\eta_m(t) \in \mathfrak{D}[\mathcal{B}_\delta]$ such that

$\underset{\sim}{K}_\delta[\eta - \eta_m] \to 0$ as $m \to \infty$. Next, application of (6.1) to $\eta_m - \eta_n$ yields the conclusion that

$$\text{(6.8)} \qquad \begin{aligned} \underset{\sim}{K}_\delta[\eta_m - \eta_n] &= \underset{\sim}{P}[w(\lambda|\eta_m - \eta_n)] \\ &= \underset{\sim}{P}[w(\lambda|\eta_m) - w(\lambda|\eta_n)]. \end{aligned}$$

Since $\{\eta_m\}$ is a Cauchy sequence in $\mathfrak{D}(I;K)$, it follows from (6.8) that the sequence $\{w(\lambda|\eta_m)\}$ is also Cauchy in $\mathcal{H}$. Therefore, there exists a unique 2n-dimensional vector function $w(\lambda) \in \mathcal{H}$ such that

$$\text{(6.9)} \qquad \underset{\sim}{P}[w(\lambda|\eta_m) - w(\lambda)] \to 0 \text{ as } m \to \infty.$$

On the other hand, if we denote by $w_i(\lambda|\eta)$ and $U_i(t,\lambda)$, $(i = 1,2,\cdots,2n)$, the i-th element of $w(\lambda|\eta)$ and i-th column vector of $U(t,\lambda)$, respectively, then from (6.5) we have for $i = 1,2,\cdots,2n$ the equation

$$w_i(\lambda|\eta) - w_i(\lambda|\eta_m) = \int_\delta (K(t)[\eta(t) - \eta_m(t)], U_i(t,\lambda))dt.$$

Consequently, for each fixed $\lambda \in (-\infty,\infty)$ it follows that

$$\text{(6.10)} \qquad |w_i(\lambda|\eta) - w_i(\lambda|\eta_m)|^2 \le \underset{\sim}{K}_\delta[\eta-\eta_m]\underset{\sim}{K}_\delta[U_i(\cdot,\lambda)] \to 0$$

as $m \to \infty$; that is $w(\lambda|\eta_m)$ converges pointwise to $w(\lambda|\eta)$ as $m \to \infty$. From a well-known result of integration theory (see, for example, Rudin [10; Theorem 3.12, p. 67]) the relations (6.9) and (6.10) then imply that $w(\lambda) = w(\lambda|\eta)$ a.e. in $\mathcal{H}$. Since the convergence in the mean implies the convergence of the norm, it follows that

$$\underset{\sim}{K}_\delta[\eta] = \lim_{m\to\infty}\underset{\sim}{K}_\delta[\eta_m] = \lim_{m\to\infty}\underset{\sim}{P}[w(\lambda|\eta_m)] = \underset{\sim}{P}[w(\lambda|\eta)],$$

which proves relation (6.4) for all $\eta(t) \in \mathfrak{D}_o(I;K)$.

Let $\{d(m)\}$ be a sequence of intervals as described in the theorem. If $\eta(t) \in \mathfrak{D}(I;K)$, then we define $\eta^{(m)}(t) \equiv \eta(t)\chi_{d(m)}$, where $\chi_{d(m)}$ is the characteristic function of d(m), and

$$(6.11) \qquad w(\lambda|\eta^{(m)}) \equiv \int_I (K(t)\eta^{(m)}(t), U(t,\lambda))dt$$

$$= \int_{d(m)} (K(t)\eta(t), U(t,\lambda))dt.$$

For $n > m$ it then follows that

$$w(\lambda|\eta^{(n)}) - w(\lambda|\eta^{(m)}) = \int_{d(n,m)} (K(t)\eta(t), U(t,\lambda))dt,$$

where for brevity we set $d(n,m) = d(n) - d(m)$. Consequently, $\underset{\sim}{P}[w(\lambda|\eta^{(n)}) - w(\lambda|\eta^{(m)})] = \underset{\sim}{K}_{d(n,m)}[\eta]$, and since the right-hand member tends to 0 as $m, n \to \infty$, the corresponding sequence $\{w(\lambda|\eta^{(m)})\}$ tends in the norm of $\mathcal{H}$ to an element $w(\lambda)$ of $\mathcal{H}$ as $m \to \infty$. Therefore, $\lim_{m \to \infty} \underset{\sim}{P}[w(\lambda|\eta^{(m)})]$ exists and is finite. Moreover, in view of (6.11) we have the equation

$$\underset{\sim}{K}_{d(m)}[\eta] = \underset{\sim}{P}[w(\lambda|\eta^{(m)})], \quad (m = 1,2,\cdots),$$

and (6.4) results upon letting $m \to \infty$. From the relation

$$\underset{\sim}{P}[w(\lambda|\eta) - w(\lambda|\eta^{(m)})] = \int_{-\infty}^{\infty} w^*(\lambda|\eta^{(-m)})d\rho(\lambda)w(\lambda|\eta^{(-m)})$$

$$= \underset{\sim}{K}_{-d(m)}[\eta],$$

where $-d(m) = I - d(m)$, and $\eta^{(-m)}(t) = \eta(t)\chi_{-d(m)}$, we obtain (6.6) upon letting $m \to \infty$. Finally, relation (6.7) follows from (6.4).

Corresponding to the Corollary to Theorem 5.4 we have the following result.

THEOREM 6.3. _If_ $\eta(t) \in \mathfrak{L}(I;K)$ _then_

$$(6.12) \qquad \eta(t) = \int_{-\infty}^{\infty} U(t,\lambda)d\rho(\lambda)w(\lambda|\eta),$$

where the integral converges in the norm of $\mathfrak{L}(I;K)$; _that is, if we_ denote by [m] the closed interval [-m,m], then

$$\underset{\sim}{K}[\eta - \int_{[m]} U(t,\lambda)d\rho(\lambda)w(\lambda|\eta)] \to 0 \text{ as } m \to \infty.$$

For $\eta(t) \in \mathfrak{D}(I;K)$ we define

$$\eta_m(t) \equiv \int_{[m]} U(t,\lambda)d\rho(\lambda)w(\lambda|\eta). \tag{6.13}$$

If η_1 and η_2 belong to $\mathfrak{D}(I;K)$, then from (6.7) we have

$$\underset{\sim}{K}[\eta_1,\eta_2] = \underset{\sim}{P}[w(\lambda|\eta_1),w(\lambda|\eta_2)]. \tag{6.14}$$

Now let $y(t) \in \mathfrak{D}(I;K)$ be such that $y(t) \equiv 0$ outside a compact subinterval δ of I. Using (6.13), it then follows that

$$\begin{aligned}\underset{\sim}{K}_\delta[\eta_m,y] &= \int_\delta (K(t)\eta_m(t),y(t))dt \\ &= \int_\delta y^*(t)K(t)\{\int_{[m]} U(t,\lambda)d\rho(\lambda)w(\lambda|\eta)\}dt \\ &= \underset{\sim}{P}_{[m]}[w(\lambda|\eta),w(\lambda|y)].\end{aligned}$$

On the other hand, for $\eta(t) \in \mathfrak{D}(I;K)$ the application of (6.14) implies that

$$\underset{\sim}{K}_\delta[\eta,y] = \underset{\sim}{P}[w(\lambda|\eta),w(\lambda|y)],$$

and hence

$$\underset{\sim}{K}_\delta[\eta - \eta_m,y] = \underset{\sim}{P}_{-[m]}[w(\lambda|\eta),w(\lambda|y)],$$

where $-[m] = (-\infty,\infty) - [-m,m]$. Upon applying the Cauchy-Schwarz inequality we obtain the relation

$$\begin{aligned}|\underset{\sim}{K}_\delta[\eta - \eta_m,y]|^2 &\le \underset{\sim}{P}_{-[m]}[w(\lambda|y)]Q \\ &\le \underset{\sim}{P}[w(\lambda|y)]Q \\ &= \underset{\sim}{K}_\delta[y]Q,\end{aligned} \tag{6.15}$$

where $Q = \underset{\sim}{P}_{-[m]}[w(\lambda|\eta)]$. In particular, let y(t) be defined as

$$y(t) = \begin{cases} \eta(t) - \eta_m(t), & \text{if } t \in \delta, \\ 0, & \text{if } t \notin \delta. \end{cases}$$

Since $\underset{\sim}{K}_\delta[\eta-\eta_m] \geq 0$, it follows from (6.15) that

$$\underset{\sim}{K}_\delta[\eta - \eta_m] \leq \underset{\sim}{P}_{-[m]}[w(\lambda|\eta)],$$

and as the right-hand member is independent of δ we have the relation

$$\underset{\sim}{K}[\eta - \eta_m] \leq \underset{\sim}{P}_{-[m]}[w(\lambda|\eta)].$$

The right-hand member tends to zero as $m \to \infty$, thus completing the proof.

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