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FORMULATION OF STATIC AND DYNAMIC LAYERED BEAM SYSTEMS WITH AN INVERSE PROBLEM

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FORMULATION OF STATIC AND DYNAMIC LAYERED BEAM SYSTEMS WITH AN INVERSE PROBLEM

A Dissertation APPROVED FOR THE
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ABSTRACT

Mathematical models are developed to examine the interfaces of horizontally-layered beam systems. The application examined is that of concrete road overlays. We are particularly interested in how the normal displacement of the beams is affected by the shearing interface coefficient, K_S. We ignore any effect due to friction.

The static two-beam model gives results which were validated by laboratory data. We show that a unique solution exists and that this solution continuously depends on the parameter K_S. We also formulate the problem for multilayered systems. For the dynamic model, we show the existence of a unique soution to the weak form of the problem. We then consider a numerical example of an inverse problem in which we attempt to recover K_S using data generated from the forward problem.

Both the static and dynamic models developed in this paper allow for cavitation between the two beams. This in effect corrects the problem of one beam penetrating the other and allows us to try to predict where the two beams may acutally separate. Also, with the addition of the time-dependent model, we can add a moving mass or rolling load.

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0. Introduction

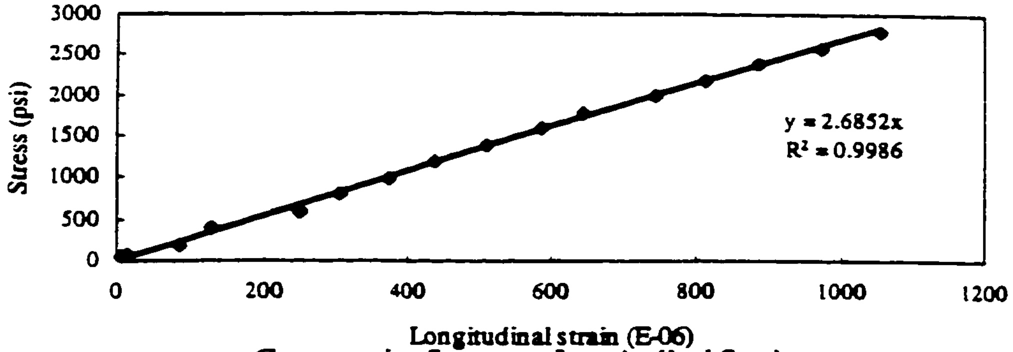
There are many practical applications that involve horizontally layered media. This paper develops a mathematical model which can be used to evaluate what happens at the interfaces of such layers. In particular, we will focus our discussion to that of concrete road overlays as we have data with which to compare our results.[5]

The stability and maintenance of concrete road surfaces is an important topic in our modern-day infrastructure. A current method of extending the life of a road is to pour a new concrete block on top of the original concrete surface. Thus, it makes sense to examine what happens at the interface of these two surfaces. In this paper, we will develop simple mathematical models based on a linear theory and the Timoshenko beam which are an extension of a previous model.[14]

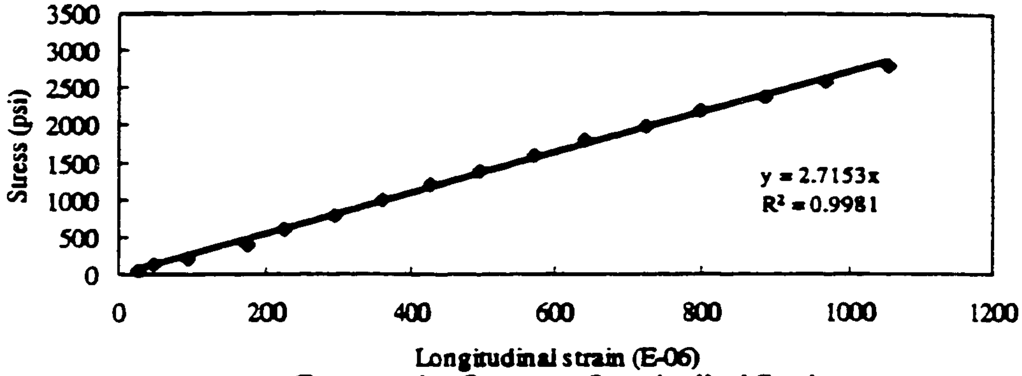
The Timoshenko beam models developed here are based on energy considerations assuming interfacial conditions and an elastic foundation. In addition, the models will incorporate the idea that the two surfaces may separate. This could be due to climate conditions or poor bonding. It is also important to note that while concrete is a very complicated nonlinear material, we are assuming linear elasticity conditions hold. This assumption is based on a laboratory experiment which showed a linear stress-strain relationship for the cases of interest.[5](see fig. 1)

We are particularly interested in the relation between the interfacial stiffness coefficient, K_S, and normal displacement. In chapter 1, we develop the static 2-beam model and give a validation of the results based on laboratory data. We show that a unique solution exists and this solution continuously depends on the parameter K_S. For $K_S = [7, 2]$, we get a relative error of .1059 and for the optimal K_S we get a relative error of .0253. In addition, we look at where the model predicts that separation of the beams may occur. We then give an extension to multilayered systems.

Chapter 2 discusses the time-dependent 2-beam model. The main theorem here is the existence of a unique solution to the weak form of the problem. We then formulate a simplified inverse problem where we generate data using the forward problem. In a numerical experiment, we generate a surface describing the residuals between the predicted deformations and a given data set where the predicted deformations are a function of the interfacial stiffness and we assume $K_S = [\,K_S(1), \quad K_S(2)\,]$. This surface represents the error between model predictions and experimental data. The time-dependent case is significant for developing models to predict fatigue and degeneration properties under transient loads.



Compressive Stress vs. Longitudinal Strain



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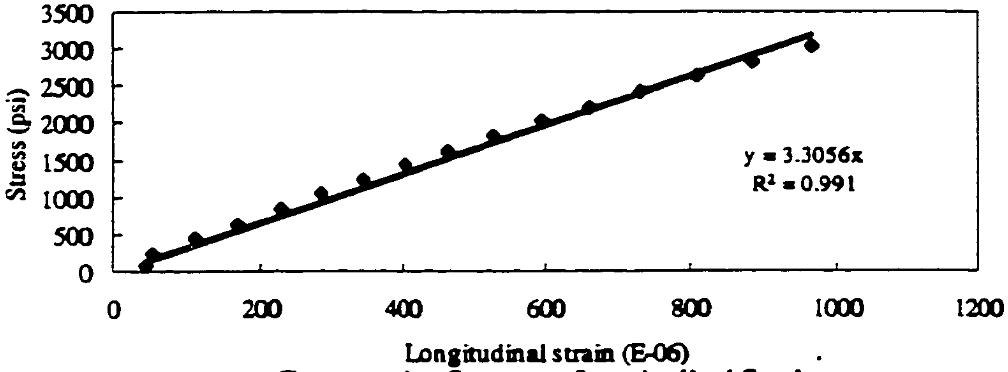

Compressive Stress vs. Longitudinal Strain

Figure 1

1. Static Case

1.1 Basic Model

The mathematical model we will use for analysis is the Timoshenko beam model.[13] A beam model is chosen in order to better approximate actual experimental data and results. In particular, the Timoshenko beam model is chosen because it is the simplest beam model which couples normal displacements with shearing displacements.

Suppose the beam is represented by a domain

$$\Omega = \{(x, y, z) \ : \ 0 \leq x \leq L, \ -k \leq y \leq k, \ -h \leq z \leq h\}.$$

where h is small enough so that we may assume the stress σ_{33} is incorporated in the body force. Then, under the small deformation gradient assumption[Hunter], we can write the stress-strain relationships as

$$\begin{aligned} \sigma_{11} &= \frac{E}{1-\mu^2}[\epsilon_{11} + \mu\epsilon_{22}] \\ \sigma_{22} &= \frac{E}{1-\mu^2}[\mu\epsilon_{11} + \epsilon_{22}] \\ \sigma_{33} &= 0 \\ \sigma_{12} &= G\epsilon_{12}, \ \sigma_{13} = G\epsilon_{13}, \ \sigma_{23} = G\epsilon_{23} \end{aligned} \tag{1.1}$$

where E is Young's modulus, μ is Poisson's ratio, and $G = \frac{E}{1+\mu}$ is the shear modulus. We denote the displacements in the $x-$, $y-$, and $z-$ directions by the functions $U = U(x, y, z)$, $V = V(x, y, z)$, and $W = W(x, y, z)$, respectively. The strain-displacement relations are

$$\begin{aligned} \epsilon_{11} = \frac{\partial U}{\partial x}, \ \epsilon_{12} = \frac{1}{2}(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}), \ \epsilon_{13} = \frac{1}{2}(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}) \\ \epsilon_{22} = \frac{\partial V}{\partial y}, \ \epsilon_{23} = \frac{1}{2}(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}), \ \epsilon_{33} = \frac{\partial W}{\partial z}. \end{aligned} \tag{1.2}$$

The displacement assumptions associated with a Timoshenko beam are

$$
\begin{aligned}
U(x,y,z) &= z\phi(x) \\
V(x,y,z) &= 0 \\
W(x,y,z) &= \omega(x).
\end{aligned}
\tag{1.3}
$$

We define the strain energy as

$$
\mathcal{V}(U,V,W) = \frac{1}{2}\int_0^L \int_{-k}^{k}\int_{-h}^{h} \{\sigma_{11}\epsilon_{11} + \sigma_{12}\epsilon_{12} + \sigma_{13}\epsilon_{13} + \sigma_{22}\epsilon_{22} + \sigma_{23}\epsilon_{23}\}\, dzdydx \tag{1.4}
$$

and the Lagrangian by

$$
\mathcal{L}(U,V,W) = \mathcal{V}(U,V,W) - \int_0^L f(x)W(x)\,dx \tag{1.4}
$$

where the last integral represents the work done by body forces and applied forces.

Substituting the stress-strain relations (1.1), the strain-displacement relations (1.2), and the displacement assumptions (1.3), the Lagrangian can be expressed as

$$
\mathcal{L}(U,V,W) = \frac{1}{2}\int_0^L \left\{ \frac{4h^3kE}{3(1-\mu^2)}\phi_x^2 + \frac{2hkE}{1+\mu}(\phi+\omega_x)^2 \right\} dx - \int_0^L f\omega\, dx \tag{1.6}
$$

1.2 Two-Beam System

Suppose now, that there are two bodies that we view as beam 1 and beam 2. Beam 1 occupies the set $\Omega_1 = [0,L] \times [-k,k] \times [0,2h_1]$ and beam 2 occupies the set $\Omega_2 = [0,L] \times [-k,k] \times [-2h_2, 0]$. We assign local coordinate systems 1 and 2 such that the relationships of the local coordinate systems to the global coordinate system are $x_1 = x,\ y_1 = y,$ and $z_1 = z - h_1$ for coordinate system 1 and $x_2 = x,\ y_2 = y,$ and $z_2 = z + h_2$ for coordinate system 2.(see fig. 2) The displacement functions for the ith beam are

$$U_i(x, y, z_i) = z_i \phi_i(x)$$

$$V_i(x, y, z_i) = 0$$

$$W_i(x, y, z_i) = \omega_i(x).$$

As in (1.6), the strain energy for the ith beam is given by

$$\mathcal{V}_i(\phi_i, \omega_i) = \frac{1}{2} \int_0^L \left\{ \frac{4h_i^3 k E_i}{3(1-\mu_i^2)} \phi_{ix}^2 + \frac{2h_i k E_i}{1+\mu_i} (\phi_i + \omega_{ix})^2 \right\} dx \tag{1.7}$$

The total potential energy of the system is the sum of the individual strain energies,

$$\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$$

and the Lagrangian for the system is

$$\mathcal{L}_0 = \mathcal{V} - \int_0^L f\omega_1 \, dx - \int_0^L f\omega_2 \, dx$$

where we have assumed here that the displacements of the two beams are independent.

Next, we impose interface conditions to relate the displacements of the two beams. If we assume that the displacements at the interface of the two beams are equal, we get

$$h_1\phi_1(x) + h_2\phi_2(x) = 0$$

and

$$\omega_1(x) - \omega_2(x) = 0.$$

For linear theory, we use the integrals

$$\mathcal{V}_x = \frac{1}{2} \int_0^L K_S(x)(h_1\phi_1(x) + h_2\phi_2(x))^2 \, dx$$

and

$$\mathcal{V}_z = \frac{1}{2}\int_0^L K_N(x)(w_1(x) - w_2(x))^2\, dx$$

where K_S and K_N are L^∞ functions introduced as elastic coefficients at the interface. The elastic interface between the beams is then modeled by

$$\mathcal{V}_I = \mathcal{V}_x + \mathcal{V}_z = \frac{1}{2}\int_0^L [K_S(x)(h_1\phi_1 + h_2\phi_2)^2 + K_N(x)(\omega_1 - \omega_2)^2]\, dx.$$

We also assume that the beams are sitting on an elastic foundation. This is modeled by

$$\mathcal{V}_F = \frac{1}{2}\int_0^L (K_{FS}(x)\phi_2^2 + K_{FN}(x)\omega_2^2)\, dx.$$

Consider two types of forces acting on the beam system: the body forces and the applied forces. The work due to gravity for the ith beam is given by

$$\mathcal{W}_i = \int_0^L \int_{-k}^{k} \int_{-h_i}^{h_i} \rho_i g \omega_i \, dz_i dy dx$$

and the work done by the body forces for the system is

$$\mathcal{W}_B = \mathcal{W}_1 + \mathcal{W}_2 = 4gk\int_0^L (\rho_1 h_1 \omega_1 + \rho_2 h_2 \omega_2)\, dx.$$

The work due to a static applied load, f_a, is

$$\mathcal{W}_a = \int_0^L f_a(x)\omega_1(x)\, dx.$$

Thus, the Lagrangian of the 2-beam system with an interface on an elastic foundation is given by

$$\mathcal{L}(\phi_1, \omega_1, \phi_2, \omega_2) = \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_I + \mathcal{V}_F - \mathcal{W}_B - \mathcal{W}_a. \tag{1.8}$$

To simplify notation, we make the following assignments:

$$\alpha_i = \frac{4h_i^3 k E_i}{3(1-\mu_i^2)},$$

$$\beta_i = \frac{2h_i k E_i}{1+\mu_i},$$

$$\mathcal{E} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\mathcal{A} = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \beta_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & \beta_2 \end{bmatrix},$$

$$\mathcal{A}_S = \begin{bmatrix} h_1^2 & 0 & h_1 h_2 & 0 \\ 0 & 0 & 0 & 0 \\ h_1 h_2 & 0 & h_2^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{A}_N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

$$\mathcal{A}_{FS} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\mathcal{A}_{FN} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In addition, the displacement vector is defined by

$$u = \begin{bmatrix} \phi_1 \\ \omega_1 \\ \phi_2 \\ \omega_2 \end{bmatrix}$$

and the forcing vectors are defined by

$$F_B = \begin{bmatrix} 0 \\ 4gk\rho_1 h_1 \\ 0 \\ 4gk\rho_2 h_2 \end{bmatrix}$$

and

$$F_a = \begin{bmatrix} 0 \\ f_a \\ 0 \\ 0 \end{bmatrix}.$$

Using the above notation, the Lagrangian (1.8) may be written as

$$\begin{aligned} \mathcal{L}(u) = \frac{1}{2}\int_0^L \{(u_x + \mathcal{E}u)^T \mathcal{A}(u_x + \mathcal{E}u) + u^T(K_S\mathcal{A}_S + K_N\mathcal{A}_N + K_{FS}\mathcal{A}_{FS} \\ +K_{FN}\mathcal{A}_{FN})u\}\, dx - \int_0^L (F_a + F_B)^T u\, dx. \end{aligned} \tag{1.9}$$

Remark: We want to develop a model which agrees with the experimental results. In this paper, we ignore nonlinearities associated with contact forces and friction.

1.3 Minimization Problem

Hamilton's Principle says that the displacement assumed by the beam system is the minimizer of the Lagrangian.[Meirovitch] The previous model[14] enforces the interface condition by penalizing the relative displacements at the interface. While this model seems to give good results, it is physically unrealistic in the sense that the two beams may actually penetrate each other. In this section, we avoid this penetration by restricting the relative normal displacements to allow for separation of the two beams. We will formulate this minimization problem and prove the existence of a unique solution.

We define the following Hilbert spaces with their usual norms and inner products:

$$H = L^2([0,L], \mathbf{R}^4),$$

$$V = H^1([0,L]; \mathbf{R}^4),$$

$$V_0 = \{V : \omega_1(0) = \omega_1(L) = 0\},$$

and the bilinear form on $V \times V$

$$a(u,v) = \frac{1}{2}\int_0^L \{(u_x + \mathcal{E}u)^T \mathcal{A}(v_x + \mathcal{E}v) \\ + u^T(K_S\mathcal{A}_S + K_N\mathcal{A}_N + K_{FN}\mathcal{A}_{FN} + K_{FS}\mathcal{A}_{FS})v\, dx.$$

Using this notation and denoting the inner product on H by $<,>$, the Lagrangian can be written as

$$\mathcal{L}(u) = a(u,u) - < F, u > \tag{1.10}$$

where $F = F_a + F_B$.

Lemma 1.1 Let K_S, K_N, K_{FN}, and $K_{FS} \geq 0$. The bilinear form $a(u,v)$ is an inner product on V_0.

Proof: We only need to show that $a(u,u) = 0$ implies that $u = 0$. This follows from [14] that there exists a positive constant κ_0 such that

$$a(u,u) \geq \kappa_0 \|u\|_V^2. \tag{1.11}$$

To enforce that the relative displacement is positive, we define the set

$$P = \{u \in V_0 : \omega_1(x) - \omega_2(x) \geq 0 \text{ for any } x \in [0,L]\}.$$

Lemma 1.2: P is a closed convex cone in V_0.

Proof: Let $u \in P$. Then $u^T = [\,\phi_1 \quad \omega_1 \quad \phi_2 \quad \omega_2\,]$ such that $\omega_1(x) - \omega_2(x) \geq 0$. Let $\alpha \geq 0$. Then $\alpha u^T = [\,\alpha\phi_1 \quad \alpha\omega_1 \quad \alpha\phi_2 \quad \alpha\omega_2\,]$ and $\alpha\omega_1 - \alpha\omega_2 = \alpha(\omega_1 - \omega_2) \geq 0$. Thus $\alpha u \in P$ and P is a cone.

Suppose v is another element of P and $v^T = [\,\xi_1 \quad \eta_1 \quad \xi_2 \quad \eta_2\,]$. Let $0 < \alpha < 1$. Then

$$\alpha u + (1-\alpha)v = \begin{bmatrix} \alpha\phi_1 + (1-\alpha)\xi_1 \\ \alpha\omega_1 + (1-\alpha)\eta_1 \\ \alpha\phi_2 + (1-\alpha)\xi_2 \\ \alpha\omega_2 + (1-\alpha)\eta_2 \end{bmatrix} \in P$$

since

$$(\alpha\omega_1 + (1-\alpha)\eta_1) - (\alpha\omega_2 + (1-\alpha)\eta_2) = \alpha(\omega_1 - \omega_2) + (1-\alpha)(\eta_1 - \eta_2) \geq 0.$$

Hence, P is convex.

Let $\{u^{(k)}\}$ be a sequence in P that converges to an element $u^{(0)}$ in V. We need to show that $u^{(0)}$ is in P. Since $u^{(k)} \in P$ for all k, we have

$$\omega_1^{(k)}(x) - \omega_2^{(k)}(x) \geq 0 \text{ for any } x \in [0, L].$$

Since V_0 imbeds in the space of continuous functions[4], we have pointwise convergence or

$$u^{(k)}(x) \longrightarrow u^{(0)}(x) \qquad \text{for each } x \in [0, L].$$

This implies that $\omega_1^{(k)}(x) \to \omega_1^{(0)}(x)$ and $\omega_2^{(k)}(x) \to \omega_2^{(0)}(x)$. Since

$$(\omega_1^{(k)} - \omega_2^{(k)})(x) \geq 0 \qquad \text{for all } k > 0,$$

the limit

$$\lim_{k\to\infty} (\omega_1^{(k)} - \omega_2^{(k)})(x) = (\omega_1^{(0)} - \omega_2^{(0)})(x) \geq 0.$$

Thus, $u^{(0)} \in P$ and P is closed proving the lemma.

We pose the minimization problem: Find $u_0 \in P$ such that

$$\mathcal{L}(u_0) = \text{minimum } \mathcal{L}(v) \qquad \text{for any } v \in P. \tag{MP}$$

Theorem 1.1: There exists a unique solution to the above mimization problem.

Proof: First, we show that $\mathcal{L}(v)$ is bounded below for any $v \in P$. Recall (1.11) $a(v,v) \geq \kappa_0 \|v\|_V^2$ for some $\kappa_0 > 0$ and the Lagrangian (1.10)

$$\mathcal{L}(v) = a(v,v) - < F, v > .$$

Also,

$$< F, v > \leq \|F\| \|v\|_H \leq M\|v\|_H \leq M\|v\|_V.$$

Thus,

$$\mathcal{L}(v) \geq \kappa_0 \|v\|_V^2 - M\|v\|_V.$$

The right hand side is a parabola in $\|v\|_V$ which has a minimum when $\|v\|_V = \frac{M}{2\kappa_0}$. Hence, $\{\mathcal{L}(v) : v \in P\}$ is bounded below and $d = \inf\{\mathcal{L}(v) : v \in P\}$ exists.

Also, note the parallelogram identity

$$\mathcal{L}\left(\frac{u_1 + u_2}{2}\right) + a\left(\frac{u_1 - u_2}{2}, \frac{u_1 - u_2}{2}\right) = \frac{1}{2}\mathcal{L}(u_1) + \frac{1}{2}\mathcal{L}(u_2) \tag{1.12}$$

holds for all u_1 and u_2 in V.

Let $\{u_n\} \subseteq P$ be a minimizing sequence. (i.e. $\mathcal{L}(u_n) \to d$.) We show that $\{u_n\}$ is Cauchy. From (1.12)

$$\mathcal{L}\left(\frac{u_n + u_m}{2}\right) + a\left(\frac{u_n - u_m}{2}, \frac{u_n - u_m}{2}\right) = \frac{1}{2}\mathcal{L}(u_n) + \frac{1}{2}\mathcal{L}(u_m)$$

or

$$a\left(\frac{u_n - u_m}{2}, \frac{u_n - u_m}{2}\right) = \frac{1}{2}\mathcal{L}(u_n) + \frac{1}{2}\mathcal{L}(u_m) - \mathcal{L}\left(\frac{u_n + u_m}{2}\right).$$

Since P is convex, $(u_n + u_m)/2 \in P$ and $\mathcal{L}(\frac{u_n+u_m}{2}) \geq d$. Thus,

$$a\left(\frac{u_n - u_m}{2}, \frac{u_n - u_m}{2}\right) \leq \frac{1}{2}\mathcal{L}(u_n) + \frac{1}{2}\mathcal{L}(u_m) - d.$$

Since the right hand side converges to zero, we have $a\left(\frac{u_n-u_m}{2}, \frac{u_n-u_m}{2}\right)$ converges to zero. Now,

$$0 \leq \kappa_0 \left\|\frac{u_n - u_m}{2}\right\|_V^2 \leq a\left(\frac{u_n - u_m}{2}, \frac{u_n - u_m}{2}\right).$$

Hence, $\|u_n - u_m\|_V$ converges to 0 and $\{u_n\}$ is Cauchy in V_0. $\{u_n\}$ Cauchy implies that $\{u_n\}$ converges to some $u \in V_0$. In fact $u \in P$ since P is closed. Also, $\mathcal{L}$ is continuous and $\mathcal{L}(u_n)$ converges to $\mathcal{L}(u)$. Since the limit is unique, $\mathcal{L}(u) = d$ and we have existence.

For uniqueness, suppose $u_0, v \in P$ are both solutions to the minimization problem. Then $\mathcal{L}(u_0) = \mathcal{L}(v) = d$. (1.12) implies

$$\mathcal{L}\left(\frac{u_0 + v}{2}\right) + a\left(\frac{u_0 - v}{2}, \frac{u_0 - v}{2}\right) = \frac{1}{2}\mathcal{L}(u_0) + \frac{1}{2}\mathcal{L}(v) = d.$$

$$0 \leq \kappa_0 \left\|\frac{u_0 - v}{2}\right\|^2 \leq a\left(\frac{u_0 - v}{2}, \frac{u_0 - v}{2}\right) = d - \mathcal{L}(\frac{u_0 + v}{2}) \leq d - d = 0.$$

Hence, $\|u_0 - v\| = 0$ or $v = u_0$ and u_0 is unique.

The associated variational problem can be stated as follows: Find $u_0 \in P$ such that

$$a(u_0, v - u_0) - < F, v - u_0 > \geq 0 \qquad \text{for any } v \in P. \qquad (VP)$$

It is easy to show that the minimization and variational problems are equivalent.[see Luenberger, p. 178]

Suppose now that the displacement u_0 is a function of K_S, the elastic coefficient at the interface. Since $K_S \in L^\infty$, we can approximate K_S in the L^2 sense by step functions. Thus, it suffices to look at K_S as a vector in $\mathbf{R}^n$ such that K_S lies in a bounded set.

Theorem 1.2: The mapping $K_S \to u_0(K_S)$ is continuous from $\mathbf{R}^n$ into P. (That is we assume K_S are bounded and piecewise constant.)

First, we need to prove some claims.

Claim 1: Fix $v \in P$. Then $K_S \to \mathcal{L}(v; K_S)$ is continuous. We assume K_S is piecewise constant.

Proof: The proof of claim 1 is straight-forward and is omitted.

Claim 2: $\{u_0(K_S) : K_S \text{ bounded in } \mathbf{R}^n\}$ is bounded in V_0. That is, we assume K_S are bounded and piecewise constant.

Proof: From Claim 1, for a fixed $v \in P$, $K_S \to \mathcal{L}(v; K_S)$ is continuous. Also, $\mathcal{L}$ is linear in K_S. Thus, assuming K_S are bounded, the map $K_S \to \mathcal{L}(v; K_S)$ is bounded. Recall that $u_0(K_S)$ satisfies the minimization problem

$$\mathcal{L}(u_0(K_S); K_S) = \min_{v \in P} \mathcal{L}(v; K_S) \leq \mathcal{L}(0; K_S) = 0$$

since $0 \in P$. This implies that

$$a(u_0(K_S), u_0(K_S); K_S) - < F, u_0(K_S) > \leq 0.$$

From the coercivity condition (1.11) we get

$$\kappa_0 \|u_0(K_S)\|_V^2 - M\|u_0(K_S)\|_V \leq a(u_0(K_S), u_0(K_S); K_S) - < F, u_0(K_S) > \leq 0.$$

In order for $\|u_0(K_S)\|(\kappa_0\|u_0(K_S)\| - M) \leq 0$, we need

$$\|u_0(K_S)\| \leq \frac{M}{\kappa_0}.$$

Hence, $\{u_0(K_S) : K_S \text{ bounded in } \mathbf{R}^n\}$ is bounded in V_0.

Claim 3: If $K_S \to K$ in $\mathbf{R}^n$ with K_S bounded then

$$u_0(K_S) \xrightarrow{w} u_0(K) \qquad \text{in } V_0.$$

Proof: From claim 2, $\{u_0(K_S) : K_S \text{ bounded in } \mathbf{R}^n\}$ is a bounded set in V_0. Since V_0 is a Hilbert space and reflexive, $\{u_0(K_S) : K_S \text{ bounded in } \mathbf{R}^n\}$ is relatively weakly compact. Thus, there is a subsequence K_{S_i} such that

$$u_0(K_{S_i}) \xrightarrow{w} u$$

for some u in V_0. In fact, $u \in P$ since P is closed, convex and hence weakly closed.

Let $v \in P$ be arbitrary. Then $\mathcal{L}(v; K_{S_i}) \geq \mathcal{L}(u_0(K_{S_i}); K_{S_i})$ or

$$a(v, v; K_{S_i}) - < F, v > \geq a(u_0(K_{S_i}), u_0(K_{S_i}); K_{S_i}) - < F, u_0(K_{S_i}) > .$$

Taking the limit inferior of both sides, we get from weak lower semicontinuity of $\mathcal{L}$

$$\begin{aligned} a(v, v; K) - < F, v > &\geq \liminf [a(u_0(K_{S_i}), u_0(K_{S_i}); K_{S_i}) - < F, u_0(K_{S_i}) >] \\ &\geq a(u, u; K) - < F, u > \end{aligned}$$

Thus, $\mathcal{L}(v; K) \geq \mathcal{L}(u; K)$ for any v in P and

$$\mathcal{L}(u; K) = \min_{v \in P} \mathcal{L}(v; K).$$

But we've shown the solution to this minimization problem is unique. Hence, $u = u_0(K)$ and $u_0(K_{S_i}) \xrightarrow{w} u_0(K)$. It is easy to show that in fact,

$$u_0(K_S) \xrightarrow{w} u_0(K).$$

Claim 4: If $K_S \to K$ in $\mathbf{R}^n$ where K_S is bounded and piecewise constant then

$$a(u_0(K_S), u_0(K); K_S) \longrightarrow a(u_0(K), u_0(K); K).$$

Proof: Consider

$$\begin{aligned}
&|a(u_0(K_S), u_0(K); K_S) - a(u_0(K), u_0(K); K)| \\
&\qquad\qquad \leq |a(u_0(K_S), u_0(K); K_S) - a(u_0(K_S), u_0(K); K)| \\
&\qquad\qquad\quad + |a(u_0(K_S), u_0(K); K) - a(u_0(K), u_0(K); K)|.
\end{aligned}$$

Since $a(u, v; K)$ is an inner product on V_0 and by claim 3 $u_0(K_S) \stackrel{w}{\to} u_0(K)$ in V_0, we have that

$$a(u_0(K_S), v; K) \longrightarrow a(u_0(K), v; K) \qquad \text{for any } v \in V_0.$$

Since $u_0(K) \in V_0$, $a(u_0(K_S), u_0(K); K) \to a(u_0(K), u_0(K); K)$. Hence,

$$|a(u_0(K_S), u_0(K); K) - a(u_0(K), u_0(K); K)| \longrightarrow 0.$$

Now,

$$\begin{aligned}
&|a(u_0(K_S), u_0(K); K_S) - a(u_0(K_S), u_0(K); K)| \\
&\qquad = \frac{1}{2}\left|\sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} u_0(K_S)^T (K_S^{(j)} - K^{(j)}) \mathcal{A}_S u_0(K)\, dx\right| \\
&\qquad \leq \frac{1}{2} \sum_{j=1}^{n} |K_S^{(j)} - K^{(j)}| \int_{x_{j-1}}^{x_j} |u_0(K_S)^T \mathcal{A}_S u_0(K)|\, dx \\
&\qquad \leq \frac{1}{2} \|K_S - K\|_\infty \int_0^L |u_0(K_S)^T \mathcal{A}_S u_0(K)|\, dx.
\end{aligned}$$

Taking the limit as K_S goes to K, we get

$$\int_0^L |u_0(K_S)^T \mathcal{A}_S u_0(K)|\, dx \longrightarrow \int_0^L |u_0(K)^T \mathcal{A}_S u_0(K)|\, dx$$

and $\|K_S - K\|_\infty \to 0$. Hence,

$$|a(u_0(K_S), u_0(K); K_S) - a(u_0(K_S), u_0(K); K)| \longrightarrow 0.$$

and claim 4 is proved.

Proof of Theorem 1.2: Suppose $K_S \to K$ in $\mathbf{R}^n$. We need to show that $u_0(K_S) \to u_0(K)$ strongly in V_0. Consider

$$a(u_0(K_S), u_0(K); K_S) = a(u_0(K_S), u_0(K) - u_0(K_S); K_S) + a(u_0(K_S); u_0(K_S); K_S)$$

and recall the variational condition

$$a(u_0, v - u_0) - < F, v - u_0 > \geq 0 \qquad \text{for any } v \in P.$$

By claim 3, $u_0(K_S) \xrightarrow{w} u_0(K)$ in V_0. Since P is closed and convex, P is weakly closed and $u_0(K) \in P$. Thus,

$$a(u_0(K_S), u_0(K) - u_0(K_S); K_S) - < F, u_0(K) - u_0(K_S) > \geq 0$$

and we have

$$\begin{aligned} a(u_0(K_S), u_0(K_S); K_S) &\leq [a(u_0(K_S), u_0(K) - u_0(K_S); K_S) \\ &\qquad - < F, u_0(K) - u_0(K_S) >] + a(u_0(K_S), u_0(K_S), K_S) \\ &= a(u_0(K_S); u_0(K); K_S) - < F, u_0(K) - u_0(K_S) > . \end{aligned}$$

Taking limits, we get

$$\begin{aligned} a(u_0(K), u_0(K); K) &\leq \underline{\lim}\, a(u_0(K_S), u_0(K_S); K_S) \\ &\leq \overline{\lim}\, a(u_0(K_S), u_0(K_S); K_S) \\ &\leq \lim_{K_S \to K} (a(u_0(K_S), u_0(K); K_S) - < F, u_0(K) - u_0(K_S) >) \\ &= a(u_0(K), u_0(K); K) - 0 \end{aligned}$$

by claim 4. Therefore, $a(u_0(K_S), u_0(K_S); K_S) \to a(u_0(K), u_0(K); K)$. But

$$a(u_0(K), u_0(K); K) = \|u_0(K)\|_{V_0}^2.$$

Thus, we have

$$\|u_0(K_S)\|_{V_0} \longrightarrow \|u_0(K)\|_{V_0}.$$

This implies since V_0 is a Hilbert Space that $\|u_0(K_S) - u_0(K)\|_{V_0} \to 0$ or that $u_0(K_S) \to u_0(K)$ in V_0. Thus, the mapping $K_S \to u_0(K_S)$ is continuous.

1.4 Numerical Model and Validation

We want to test the above model against experimental data obtained in the laboratory. In particular, we want to compare normal displacements of beam 1 with the displacement observed on a concrete sample in the laboratory. For this purpose we ignore any foundation terms and assume the ends of the beam are fixed.

We need to write the discrete version of the Lagrangian (1.9) and the minimization problem (MP). We approximate u_0 by piecewise linear elements as follows: We partition $[0, L]$ into N subintervals of width L/N. Let

$\underline{c}^T = [\, c_1 \quad c_2 \quad \ldots \quad c_{4(N+1)} \,]$ and

$$B(x) = \begin{bmatrix} \underline{b}(x) & 0 & 0 & 0 \\ 0 & \underline{b}(x) & 0 & 0 \\ 0 & 0 & \underline{b}(x) & 0 \\ 0 & 0 & 0 & \underline{b}(x) \end{bmatrix}$$

where $B(x)$ is a $4 \times 4(N+1)$ matrix with $\underline{b}(x) = [\, b_1(x) \quad b_2(x) \quad \ldots \quad b_{N+1}(x) \,]$ and $b_i(x)$ is the piecewise linear element which is 1 at $x_i = \frac{L}{N} i$ and 0 at all other partition points. Then

$$u^{(N)}(x) = B(x)\underline{c}$$

is an approximation of u_0. Define the matrices

$$G_A = \int_0^L (B_x + \mathcal{E}B)^T \mathcal{A}(B_x + \mathcal{E}B)\, dx,$$

$$G_S = \int_0^L K_S(x)B(x)^T \mathcal{A}_S B(x)\, dx,$$

and

$$G_N = \int_0^L K_N B(x)^T \mathcal{A}_N B(x)\, dx.$$

Also, define the vectors

$$\mathcal{F}_a = \int_0^L F_a B(x)\, dx$$

and

$$\mathcal{F}_B = \int_0^L F_B B(x)\, dx.$$

Our goal is to examine what is happening at the interface between the two beams. Thus, we assume that the shear interface coefficient, K_S, is a function of x, the location along the length of the beam. For this analysis we assume that $K_S(x)$ is piecewise constant with p pieces where p is a factor of N. Let $G = G_A + G_S + G_N$ and $\mathcal{F} = \mathcal{F}_a + \mathcal{F}_B$. Then the Lagrangian becomes

$$\mathcal{L}(c) = \frac{1}{2}\underline{c}^T G\underline{c} - \mathcal{F}\underline{c}.$$

From Hamilton's principle, the displacement of the beam system occurs at the minimum of the Lagrangian. For the discrete case, the minimization problem (MP) becomes: Find $\underline{c}_0 \in \mathbf{R}^{4(N+1)}$ such that $B(x)\underline{c}_0 \in P$ and

$$\mathcal{L}(\underline{c}_0) = \min\{\mathcal{L}(\underline{c}) :\ B(x)\underline{c} \in P\} \qquad (MPN)$$

where we are now minimizing the Lagrangian over a finite dimensional subset of P. The following theorem shows that in fact the solutions to the discrete problem converge to the solution of (MP).

Theorem 1.3 Let $u_0^N = B(x)\underline{c}_0$ be the unique solution to (MPN). Then as $N \to \infty$,

$$u_0^N \longrightarrow u_0 \qquad \text{in } V_0$$

where $u_0 \in P$ is the solution to the original minimization problem(MP).

Proof: Define the set

$$P^N = \left\{ B(x)\underline{c} : \sum_{i=1}^{N+1} b_i(x) c_{N+1+i} - \sum_{i=1}^{N+1} b_i(x) c_{3(N+1)+i} \geq 0 \right\}.$$

Then $P^N \subset P$ for all N. Since u_0^N is the solution to (MPN) and $0 \in P^N$ for all N

$$\mathcal{L}(u_0^N) = \min\{\mathcal{L}(v) : \; v \in P^N\} \leq 0.$$

Also, from (1.11), we have

$$\kappa_0 \|u_0^N\|_V^2 - M\|u_0^N\|_V \leq \mathcal{L}(u_0^N) = a(u_0^N, u_0^N) - < F, u_0^N > \leq 0.$$

For this to hold, we need $\|u_0^N\|_V \leq M/\kappa_0$ for all N which says that $\{u_0^N\}$ are bounded in V_0. But this implies that $\{u_0^N\}$ is weak compact. Thus, there exists a subsequence such that $u_0^N \xrightarrow{w} u \in P$.

Let $v \in P$ and let v^N be its piecewise linear approximation. Then $v^N \to v$ and $\mathcal{L}(v^N) \to \mathcal{L}(v)$ as $N \to \infty$. Since $v^N \in P^N$, we have that $\mathcal{L}(v^N) \geq \mathcal{L}(u_0^N)$. From weak lower semicontinuity of $\mathcal{L}$, we get

$$\mathcal{L}(v) = \underline{\lim}\; \mathcal{L}(v^N) \geq \underline{\lim}\; \mathcal{L}(u_0^N) \geq \mathcal{L}(u)$$

and u solves (MP). But the solution to (MP) is unique, hence $u = u_0$ and the theorem is proved.

If we were to ignore the cone condition, the minimum of $\mathcal{L}(\underline{c})$ is achieved when $D\mathcal{L}(\underline{c}) = 0$ which happens when $G\underline{c} = \mathcal{F}^T$. Suppose $u^N = B\underline{c}$ is not in P. Then we change the coefficients c_i to get a new point $B\hat{\underline{c}}$ which is in P where $\hat{\underline{c}}$ is determined by comparing the discrete coefficients corresponding to $\omega_1(x)$ and $\omega_2(x)$ and adjusting them so that we are still in the cone. In other words, if $\omega_1(x) = \sum_{i=1}^{N+1} c_{N+i+1} b_i$ and $\omega_2(x) = \sum_{i=1}^{N+1} c_{3(N+1)+i} b_i$, and $c_{N+1+i} - c_{3(N+1)+i} < 0$, we let $\hat{c}_{3(N+1)+i} = c_{N+1+i}$. Thus $\hat{\underline{c}}$ is just modified from $\underline{c}$ so that we remain in the cone. We then use Newton's method to find the minimum, $\underline{c}_0$.

For the numerical model, we use data from the laboratory experiment for the 3 inch overlay. The length of the beam is

$$L = 37$$

and the loads were applied along the length of the beam at

$$x = L/3 \qquad \text{and} \qquad x = 2L/3.$$

The observed displacements were

$$w_d = \begin{bmatrix} 0 \\ -.0014 \\ -.00245 \\ -.0017 \\ 0 \end{bmatrix}$$

at locations

$$x_d = [0, \quad 9.25, \quad 18.5, \quad 27.75, \quad 37].$$

We want to see how well our model fits with the experimental data. For the graph in fig. 3, we assume $N = 10$ subintervals of length $L/10$ and we choose the shear interface coefficient to consist of $p = 2$ pieces

$$K_S = \begin{cases} 7 & \text{if } 0 \leq x \leq L/2 \\ 2 & \text{if } L/2 < x \leq L. \end{cases}$$

Consider the fit-to-data functional

$$J(K_S) = \sum_{i=1}^{N_d} (\omega_1(x_d(i)) - w_d(i))^2$$

where N_d are the number of data points. In order to write this in matrix form, let

$$\mathcal{C}\underline{c} = \begin{bmatrix} \omega_1(x_d(1)) \\ \vdots \\ \omega_1(x_d(N_d)) \end{bmatrix}$$

where $\mathcal{C} : \mathbf{R}^{4(N+1)} \to \mathbf{R}^{N_d}$ is defined by

$$\mathcal{C} = \begin{bmatrix} 0 & \begin{matrix} \underline{b}(x_d(1)) \\ \vdots \\ \underline{b}(x_d(N_d)) \end{matrix} & 0 & 0 \end{bmatrix}.$$

Then the fit-to-data functional can be written as

$$J(K_s) = ||\mathcal{C}\underline{c}(K_S) - w_d||^2.$$

The relative error of our model is calculated by

$$RE = \frac{\sqrt{J(K_S)}}{||w_d||}$$

and for the above data

$$RE = .1059.$$

These are as good as the original model with two pieces. Consider the relative displacements of the two beams.(see figs. 4 and 5) In fig. 4 for the previous model, beam 1 actually penetrates beam 2 which is not physically possible. In fig. 5, for the present case where the two beams can separate, there is no such difficulty.

We can improve on these results as we let K_S consist of more than two pieces. For the graph in figure 6, we assume $N = 10$ and

$$K_S = [\, 13.3 \quad 9.5 \quad .85 \quad 0 \quad 3.2 \quad 0.66 \quad 0 \quad 0 \quad 0.3 \quad 6.9 \,]$$

which was the optimal vector for the previous case.[14] In this case, we get a relative error of

$$RE = .0253.$$

If we look at the relative displacements of the two beams(see fig. 7), the model predicts that the beams separate between $x = 11$ and $x = 26$. Note we are not as concerned about the endpoints as they are assumed to be fixed.

1.5 Existence of Lagrange Multipliers

Recall the minimization problem:

$$\text{Find } u_0 \in P \text{ such that } \mathcal{L}(u_0) = \min_{v \in P} \mathcal{L}(v).$$

Let $\Gamma = [\, 0 \quad 1 \quad 0 \quad -1 \,]$. Then the above minimization problem can be rewritten as

$$\text{Find } u_0 \in P \text{ such that } \mathcal{L}(u_0) = \min\{\mathcal{L}(v) : -\Gamma v \leq 0, \; v \in V_0\}.$$

In other words, we want to find the minimum of the Lagrangian (1.10) subject to the inequality constraint $\omega_2(x) - \omega_1(x) \leq 0$. Note that $\mathcal{L} : V_0 \to \mathbf{R}$ is Gateaux differentiable and $-\Gamma : V_0 \to \hat{V}_0$ is also Gateaux differentiable where

$$\hat{V}_0 = \{\omega \in H^1([0, L]; \mathbf{R}) : \omega(0) = \omega(\mathrm{L}) = 0\}.$$

Clearly the Gateaux differentials of $\mathcal{L}$ and $-\Gamma$ are linear in their increments. Define

$$\hat{P} = \{\omega \in \hat{V}_0 : \omega(x) \geq 0 \ a.e.\}$$

Lemma 1.3: $\hat{P}$ is a positive cone with nonempty interior.

Proof: If $\omega_1 \geq \omega_2$ then $\omega_1 - \omega_2 \in \hat{P}$ and $\hat{P}$ is a positive cone.

Since $H^1([0,L];\mathbf{R})$ imbeds in $C([0,L];\mathbf{R})$, it suffices to show that

$$P_0 = \{f \in C(0,L) : f(x) \geq 0\}$$

has nonempty interior. Let $g \in P_0$ such that $g(x) > 0$ on $[0,L]$ and let $\delta = \min_{x\in[0,L]} |g(x)| > 0$. We show that the ball centered at g of radius $\frac{\delta}{2}$, denoted by $B(g; \frac{\delta}{2})$, is contained in P_0.

Let $f \in B(g; \frac{\delta}{2})$. Then $\|f - g\| = \max_{x\in[0,L]} |f(x) - g(x)| < \frac{\delta}{2}$. But this implies that $|f(x) - g(x)| < \frac{\delta}{2}$ for any $x \in [0,L]$. Thus,

$$f(x) \in (g(x) - \frac{\delta}{2}, g(x) + \frac{\delta}{2}) \qquad \text{for any } x \in [0,L]$$

and $f(x) \geq 0$ for any $x \in [0,L]$. But this says that $f \in P_0$. Hence, the interior of P_0 is nonempty and the lemma is proved.

Let u_0 be the unique solution to the minimization problem.

Lemma 1.4: u_0 is a regular point of the inequality $-\Gamma v \leq 0$.

Proof: We need to show that

(*i*) $-\Gamma u_0 \leq 0$ and

(*ii*) There exists $h \in V_0$ such that $-\Gamma u_0 + \delta[-\Gamma(u_0; h)] < 0$.

By definition, if $u_0 \in P$, then $-\Gamma u_0 \leq 0$. For (ii)

$$\begin{aligned}
\delta[-\Gamma(u_0;h)] &= \lim_{\alpha\to 0}\frac{1}{\alpha}[-\Gamma(u_0+\alpha h)-(-\Gamma(u_0))] \\
&= \lim_{\alpha\to 0}\frac{1}{\alpha}\left[\begin{bmatrix}0 & -1 & 0 & 1\end{bmatrix}\begin{bmatrix}\phi_1+\alpha\xi_1\\ \omega_1+\alpha\eta_1\\ \phi_2+\alpha\xi_2\\ \omega_2+\alpha\eta_2\end{bmatrix}-\begin{bmatrix}0 & -1 & 0 & 1\end{bmatrix}\begin{bmatrix}\phi_1\\ \omega_1\\ \phi_2\\ \omega_2\end{bmatrix}\right] \\
&= \lim_{\alpha\to 0}\frac{1}{\alpha}[-(\omega_1+\alpha\eta_1)+(\omega_2+\alpha\eta_2)+\omega_1-\omega_2] \\
&= \lim_{\alpha\to 0}\frac{\alpha(\eta_2-\eta_1)}{\alpha} \\
&= \eta_2-\eta_1 = -\Gamma h.
\end{aligned}$$

Choose $h \in V_0$ such that $-\Gamma u_0 + (-\Gamma h) < 0$ or $\eta_2 - \eta_1 > \omega_2 - \omega_1$ and (ii) is proved. Thus, u_0 is a regular point of the inequality $-\Gamma v \leq 0$.

Theorem 1.4: There is a Lagrange multiplier, $\lambda \in (\hat{V}_0)^*$, where $\lambda \geq 0$ such that $L(v;\lambda) = \mathcal{L}(u) + \lambda(-\Gamma u)$ is stationary at u_0 i.e.

$$\delta L(u_0;h) = 0 \qquad \text{for all } h \in V_0.$$

Also, $\lambda(-\Gamma u_0) = 0$.

Proof: This follows from Lemma 1.3, 1.4 and the Kuhn-Tucker Theorem.

1.6 M-Beam System

Many applications may involve more than two layers. For example, we could apply the techniques used in this paper to extend the current seismic models to include shearing. Thus, it makes sense to look at a model for such multi-layered media.

Suppose that we have M Timoshenko beams lying on top of one another. If we let

$$\mathcal{D} = \{(x,y) : 0 \leq x \leq L,\ -k \leq y \leq k\}$$

and let beam 1 occupy the set $\mathcal{D} \times [0, 2h_1]$, beam 2 occupy the set $\mathcal{D} \times [2h_1, 2h_1 + 2h_2]$, etc. then the ith beam occupies the set $\mathcal{D} \times [\sum_{j=1}^{i-1} 2h_j, \sum_{j=1}^{i} 2h_j]$.(see fig. 9) We define the local coordinates for beam 1 as

$$x_1 = x, \; y_1 = y, \; \text{and } z_1 = z - h_1$$

and for the ith beam, we have

$$x_i = x, \; y_i = y, \; \text{and } z_i = z - [\sum_{j=1}^{i-1} 2h_j + h_i] \qquad \text{when } 2 \le i \le M.$$

For each beam, we assume the following displacements:

$$U_i(x, y, z_i) = z_i \phi_i(x)$$
$$V_i(x, y, z_i) = 0$$
$$W_i(x, y, z_i) = \omega_i(x).$$

As in the two-beam case, we can write the strain energy for the ith beam as

$$\mathcal{V}_i = \frac{1}{2} \int_0^L \int_{-k}^{k} \int_{-h_i}^{h_i} \{\sigma_{11}\epsilon_{11} + \sigma_{12}\epsilon_{12} + \sigma_{13}\epsilon_{13} + \sigma_{22}\epsilon_{22} + \sigma_{23}\epsilon_{23}\} \, dz_i dy dx$$

and simplifying with α_i and β_i defined as in the 2 beam system we get

$$\mathcal{V}_i = \frac{1}{2} \int_0^L \{\alpha_i \phi_{ix}^2 + \beta_i(\phi_i + \omega_{ix})^2\} dx.$$

The strain energy for the M-beam system is then given by

$$\mathcal{V} = \sum_{i=1}^{M} \mathcal{V}_i(U_i, V_i, W_i).$$

We need to relate the displacements between adjacent beams. For M beams we have 2(M-1) interface conditions and coefficients. We model the elastic properties of the interfaces by

$$\mathcal{V}_I = \frac{1}{2}\int_0^L \left\{ \sum_{i=1}^{M-1} K_{S_i}(x)(h_i\phi_i(x) + h_{i+1}\phi_{i+1}(x))^2 \right.$$
$$\left. + \sum_{i=1}^{M-1} K_{N_i}(x)(\omega_{i+1}(x) - \omega_i(x))^2 \right\} dx.$$

As before, we assume that the beams are sitting on an elastic foundation. This is modeled by

$$\mathcal{V}_F = \frac{1}{2}\int_0^L (K_{FS}(x)\phi_1^2 + K_{FN}(x)\omega_1^2)\, dx.$$

We still assume that there are both body forces and applied forces acting on the beam system. The work due to body forces is given by

$$\mathcal{W}_B = \int_0^L 4gk \sum_{i=1}^{M} \rho_i h_i \omega_i(x)\, dx$$

and the work due to applied static forces is given by

$$\mathcal{W}_a = \int_0^L f_a \omega_M(x)\, dx.$$

Thus, the Lagrangian for the M-beam system with interface and foundation terms is

$$\mathcal{L} = \mathcal{V} + \mathcal{V}_I + \mathcal{V}_F - \mathcal{W}_B - \mathcal{W}_a.$$

For simplification, define the matrices

$$\mathcal{A}_M = \begin{bmatrix} \alpha_1 & 0 & 0 & \dots & 0 \\ 0 & \beta_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \alpha_M & 0 \\ 0 & & \dots & 0 & \beta_M \end{bmatrix},$$

$$E_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$\mathcal{E}_M = \begin{bmatrix} E_0 & 0 & \dots & 0 \\ 0 & E_0 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & E_0 \end{bmatrix},$$

$$\mathcal{A}_{FN} = \begin{bmatrix} 0 & 0 & \dots & & 0 \\ 0 & 1 & & & \vdots \\ & & 0 & & \\ \vdots & & & \ddots & \\ 0 & \dots & & & 0 \end{bmatrix},$$

and

$$\mathcal{A}_{FS} = \begin{bmatrix} 1 & 0 & \dots & & 0 \\ 0 & 0 & & & \vdots \\ & & 0 & & \\ \vdots & & & \ddots & \\ 0 & \dots & & & 0 \end{bmatrix}.$$

For the interface conditions, let $\mathcal{A}_{S_i}$ be the $2M \times 2M$ matrix such that for $1 \leq i \leq M-1$

$$\mathcal{A}_{S_i}(2i-1, 2i-1) = h_i^2$$

$$\mathcal{A}_{S_i}(2i-1, 2i+1) = h_i h_{i+1}$$

$$\mathcal{A}_{S_i}(2i+1, 2i-1) = h_i h_{i+1}$$

$$\mathcal{A}_{S_i}(2i+1, 2i+1) = h_{i+1}^2$$

and let

$$G_S = \sum_{i=1}^{M-1} K_{S_i}(x)\mathcal{A}_{S_i}.$$

Similarly, let $\mathcal{A}_{N_i}$ be the $2M \times 2M$ matrix such that for $1 \leq i \leq M-1$

$$\mathcal{A}_{N_i}(2i, 2i) = 1$$

$$\mathcal{A}_{N_i}(2i, 2i+2) = -1$$

$$\mathcal{A}_{N_i}(2i+2, 2i) = -1$$

$$\mathcal{A}_{N_i}(2i+2, 2i+2) = 1$$

and

$$G_N = \sum_{i=1}^{M-1} K_{N_i}(x)\mathcal{A}_{N_i}.$$

We define the displacement vector by

$$u = \begin{bmatrix} \phi_1 \\ \omega_1 \\ \phi_2 \\ \omega_2 \\ \vdots \\ \phi_M \\ \omega_M \end{bmatrix}.$$

The vectors associated with the body forces and applied forces for M beams are defined by

$$F_B = \begin{bmatrix} 0 \\ 4gk\rho_1 h_1 \\ 0 \\ 4gk\rho_2 h_2 \\ \vdots \\ 0 \\ 4gk\rho_M h_M \end{bmatrix}$$

and

$$F_a = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f_a \end{bmatrix}.$$

Using this notation, the Lagrangian for the M beams on a foundation with elastic interfaces and a static applied force becomes

$$\begin{aligned} \mathcal{L}_M(u) = \frac{1}{2}\int_0^L \Big\{ &(u_x + \mathcal{E}_M u)^T \mathcal{A}_M (u_x + \mathcal{E}_M u) + u^T (G_S + G_N \\ &+ K_{FN}\mathcal{A}_{FN} + K_{FS}\mathcal{A}_{FS}) u \Big\}\, dx - \int_0^L (F_B + F_a) u \, dx. \end{aligned}$$

We formulate the minimization problem for the M-beam system. Define the following Hilbert spaces with their usual norms and inner products

$$H = L^2([0, L], \mathbf{R}^{2\mathrm{M}}),$$

$$V = H^1([0,L];\mathbf{R}^{2M}),$$

$$V_0 = \{V : \omega_i(0) = \omega_i(L) = 0 \qquad 1 \le i \le M\},$$

and the bilinear form on $V \times V$ by

$$a(u,v) = \frac{1}{2}\int_0^L \{(u_x + \mathcal{E}_M u)^T \mathcal{A}_{\mathcal{M}}(v_x + \mathcal{E}_M v) \\ + u^T(G_S + G_N + K_{FN}\mathcal{A}_{FN} + K_{FS}\mathcal{A}_{FS})v\,dx.$$

Define the sets for $1 \le i \le M$

$$P_i = \{u \in V_0 : \omega_{i+1}(x) - \omega_i(x) \ge 0\}$$

where each set P_i is a closed convex cone in V_0. Then

$$\hat{P} = \bigcap_{i=1}^{M} P_i$$

is also a closed convex cone in V_0.

The minimization problem for the M-beam system is :

$$\text{Find } u_0 \in \hat{P} \text{ such that } \mathcal{L}_M(u_0) = \min\{\mathcal{L}_M(v) : v \in \hat{P}\}.$$

The following theorem follows from the two beam case.

Theorem 1.5: There exists a unique solution to the above minimization problem.

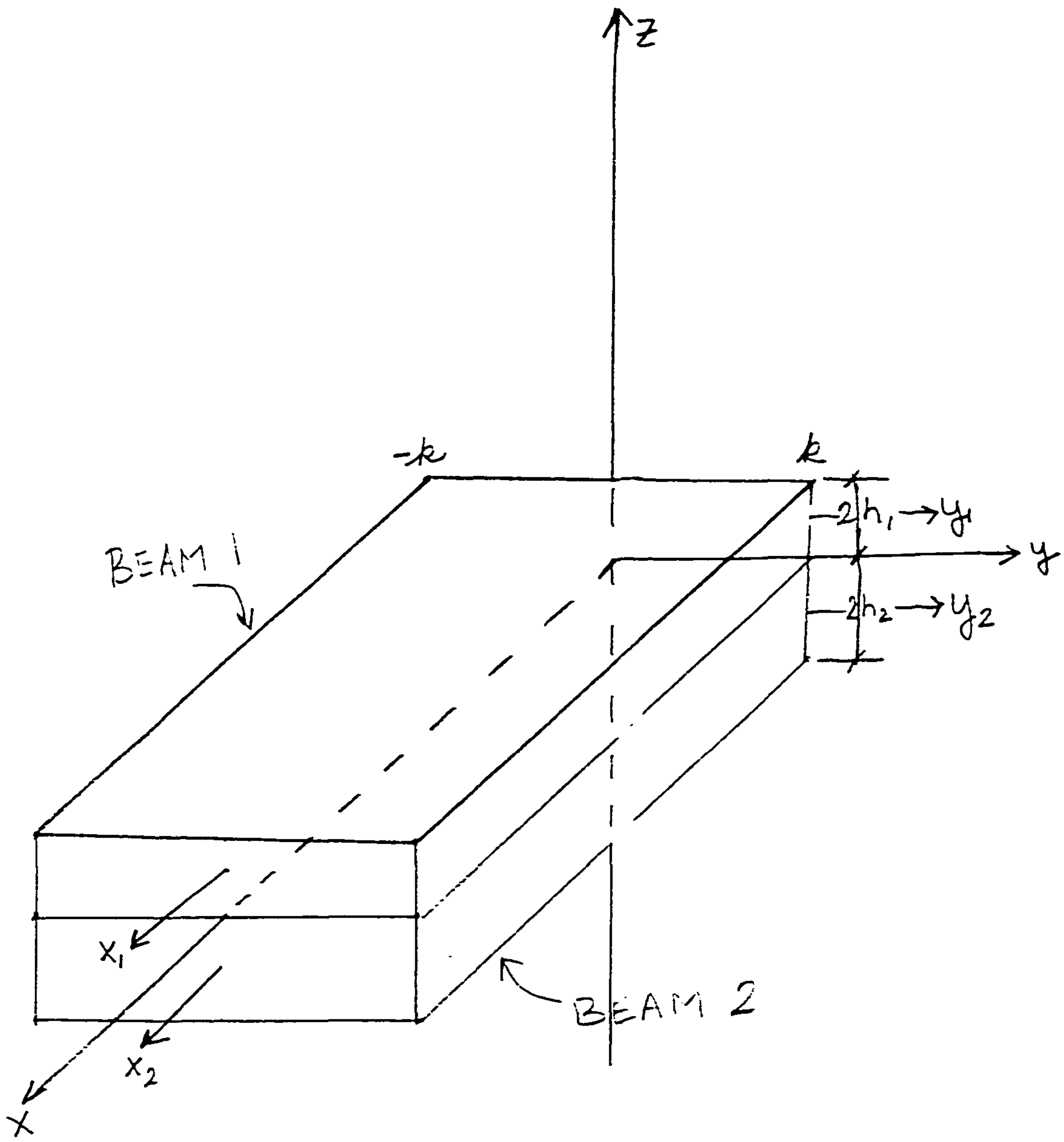

Figure 2: Orientation of the Two-Beam System

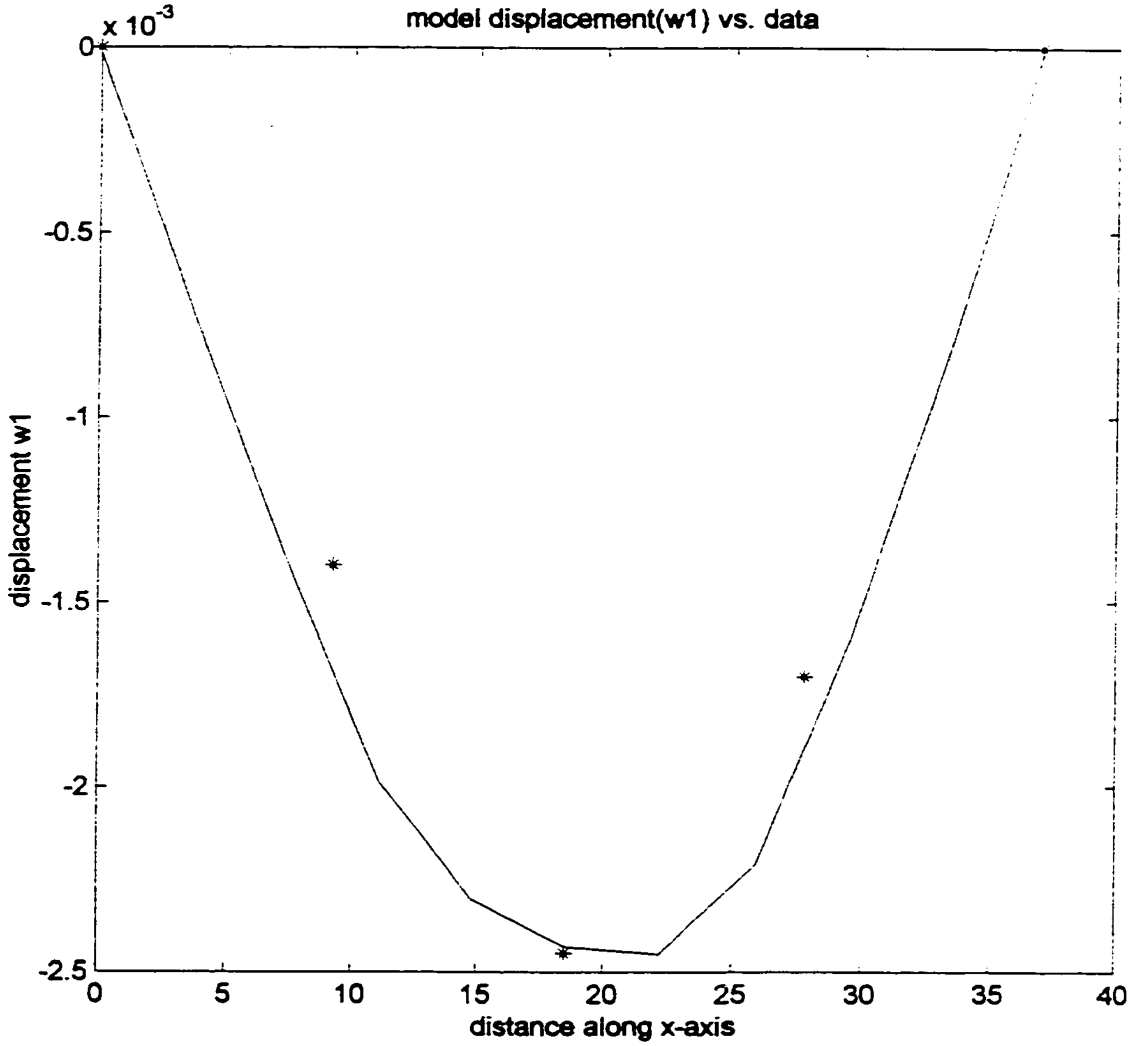

Figure 3: Model Displacement vs. Given Data for the case $K_S = [7 \quad 2]$

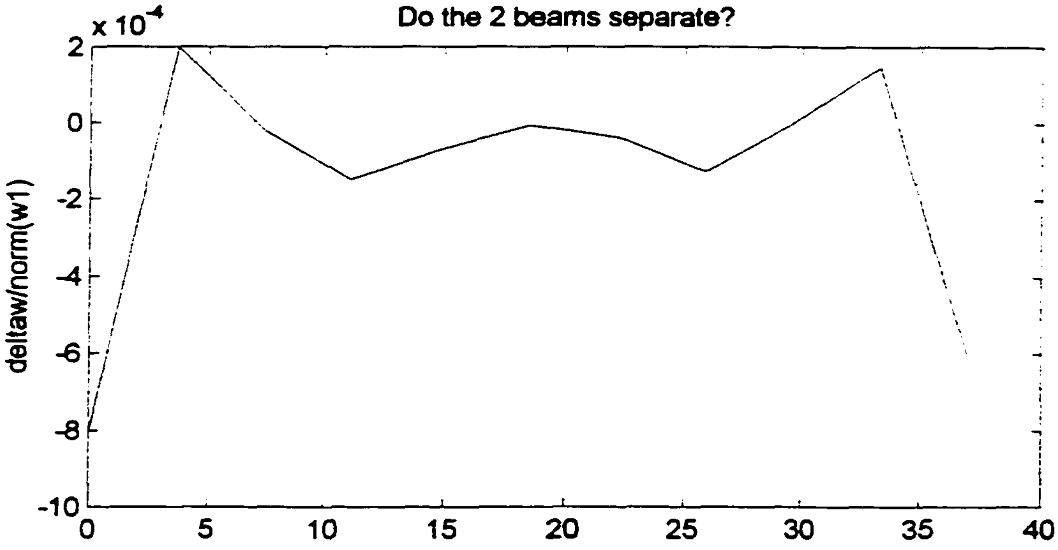

Figure 4: Relative displacement without cone condition for $K_S = [7 \quad 2]$

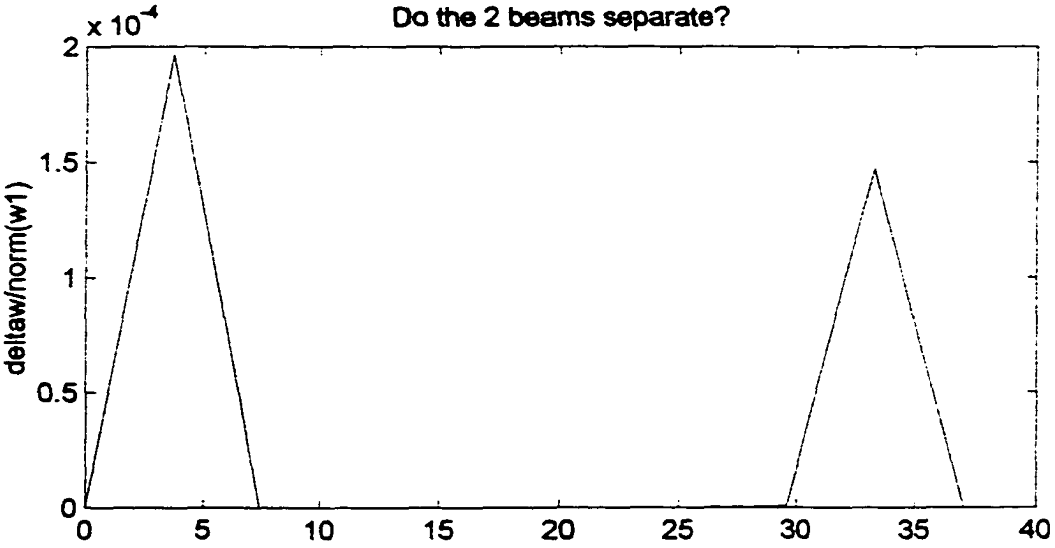

Figure 5: Relative displacement with cone condition for $K_S = [7 \quad 2]$

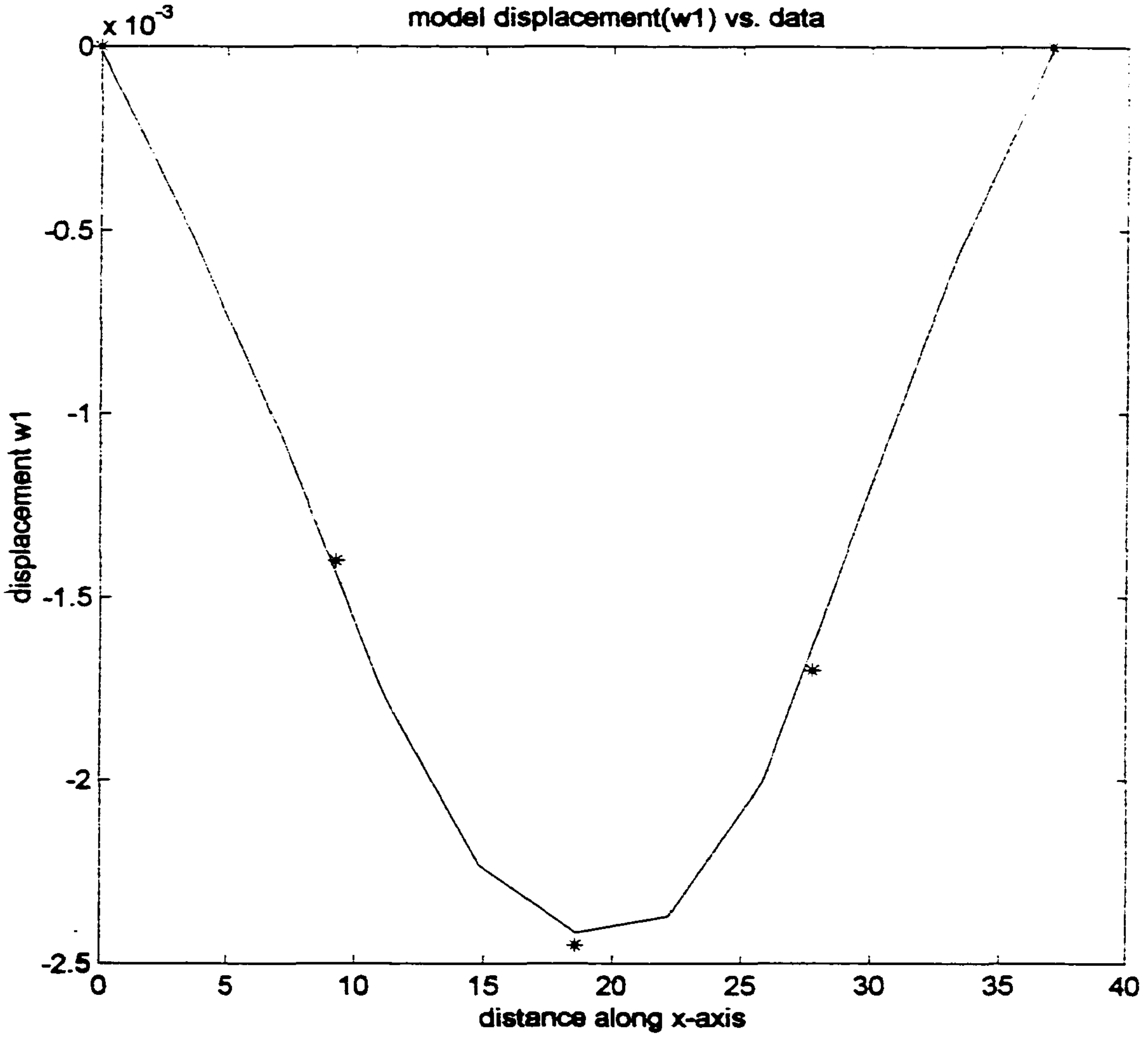

Figure 6: Model Displacement vs. Given Data for the optimal K_S

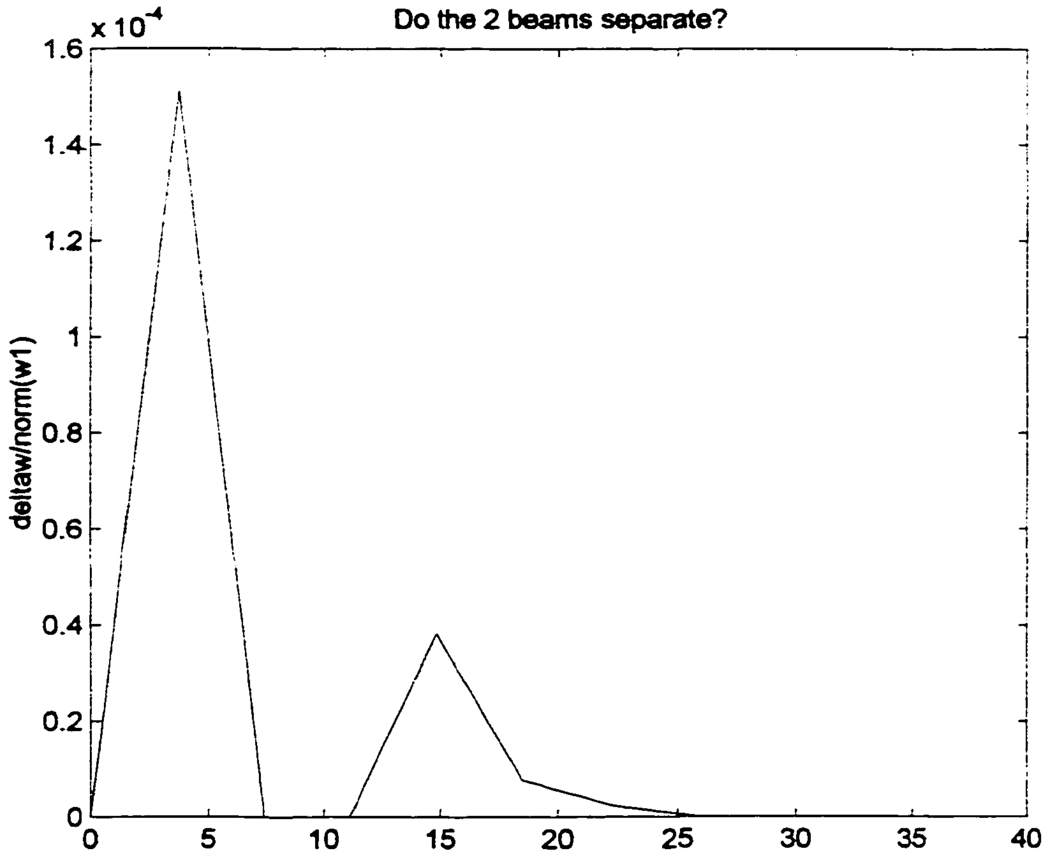

Figure 7: Relative displacement with cone condition for the optimal K_S

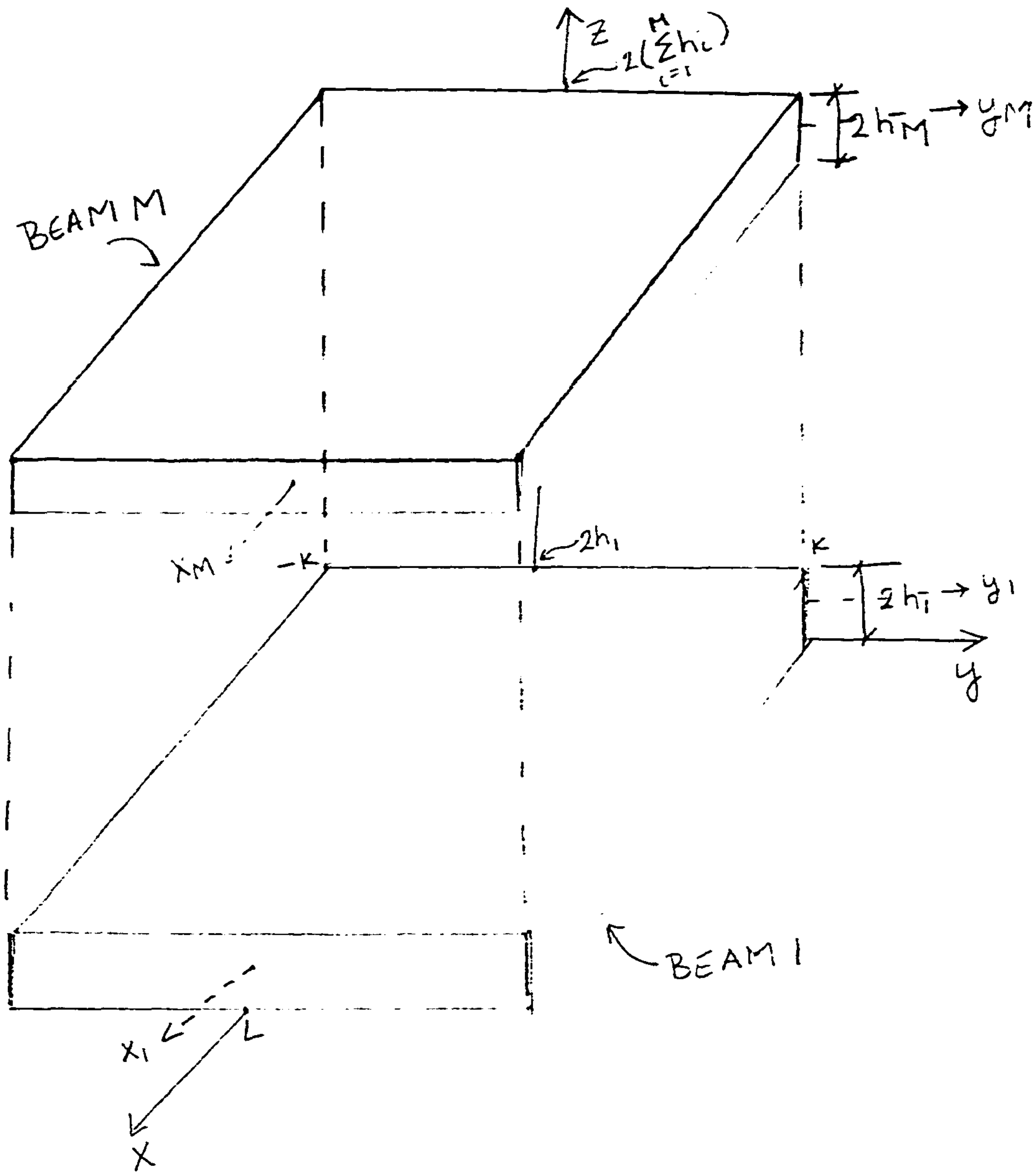

Figure 8: M-Beam Orientation

2. Dynamic Model

2.1 Two-Beam System

In this section we develop models of time-dependent 2-beam systems. The models will include structural damping and elastic interface conditions. As in the static case, we model the road and its overlay as two Timoshenko beams with the displacement for the ith beam in the x-, y- and z-directions defined by

$$\begin{aligned} U_i(x,y,z_i,t) &= z_i\phi_i(x,t) \\ V_i(x,y,z_i,t) &= 0 \\ W_i(x,y,z_i,t) &= \omega_i(x,t). \end{aligned} \tag{2.1}$$

For the dynamic model, we introduce a kinetic energy term. The kinetic energy for the ith beam is

$$\mathcal{T}_i = \frac{1}{2}\int_0^L \int_{-k}^{k} \int_{-h_i}^{h_i} \rho_i \left(U_{it}^2 + V_{it}^2 + W_{it}^2\right)\, dz_i\, dy\, dx$$

where ρ_i is the density of the ith beam. Substituting for U, V, and W and performing the integrations, we obtain

$$\mathcal{T}_i = \frac{1}{2}\int_0^L \rho_i \left(\frac{4h_i^3 k}{3}\phi_{it}^2 + 4h_i k\omega_{it}^2\right)\, dx \tag{2.2}$$

as an expression of the kinetic energy for the ith beam. The total kinetic energy of the system is then given by $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$. In order to simplify the notation, let

$$a_i = \frac{4h_i^3 k\rho_i}{3} \text{ and } b_i = 4h_i k\rho_i$$

and define

$$\mathcal{M} = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & b_2 \end{bmatrix}.$$

Then the total kinetic energy can be written as

$$\mathcal{T} = \frac{1}{2}\int_0^L u_t^T \mathcal{M} u_t \, dx \tag{2.3}$$

where $u = u(x,t)$ is the displacement vector as defined in Chapter 1.

The potential energy, $\mathcal{V}$, of the 2 beam system is obtained by

$$\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_I + \mathcal{V}_F + \mathcal{G} - \mathcal{W}_B - \mathcal{W}_a \tag{2.4}$$

where $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_F$, and $\mathcal{W}_B$ are defined as for the static case.

$\mathcal{W}_a = \int_0^L f_a(x,t)\omega_1(x,t)\, dx$ is the work due to applied forces where the applied forces are now assumed to be functions of time as well as location. For the interfacial conditions

$$\mathcal{V}_I = \frac{1}{2}\int_0^L K_S(x)(h_1\phi_1 + h_2\phi_2)^2 \, dx$$

is the shear component at the interface and

$$\mathcal{G} = \frac{1}{2}\int_0^L K_N(x)[(\omega_1 - \omega_2)_-]^2 \, dx$$

is a modeling term that reflects the elastic properties at the interface and ensures we stay in the cone P as defined for the static case. If we set $\Gamma = [\,0 \quad 1 \quad 0 \quad -1\,]$, then

$$\mathcal{G} = \mathcal{G}(u) = \frac{1}{2}\int_0^L K_N[(\Gamma u)_-]^2 \, dx. \tag{2.5}$$

From (1.9) we can express the potential energy by

$$\begin{aligned}\mathcal{V} = \frac{1}{2}\int_0^L \Big\{ &(u_x + \mathcal{E}u)^T \mathcal{A}(u_x + \mathcal{E}u) + u^T (K_S(x)\mathcal{A}_S + K_{FS}\mathcal{A}_{FS} \\ &+ K_{FN}\mathcal{A}_{FN})u \Big\} \, dx - \int_0^L (F_B + F_a)^T u \, dx + \mathcal{G}.\end{aligned} \tag{2.6}$$

The Lagrangian is defined by $\mathcal{L} = \int_0^{t_f} (\mathcal{T} - \mathcal{V})\, dt$ or

$$\mathcal{L} = \int_0^{t_f} [\mathcal{T} - (\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_I + \mathcal{V}_F + \mathcal{G} - \mathcal{W}_B - \mathcal{W}_a)] \, dt. \tag{2.7}$$

Substituting (2.3), (2.6), and (2.5), we can write the Lagrangian as

$$\mathcal{L}(u) = \int_0^{t_f} \int_0^L \Big[\frac{1}{2} \Big\{ u_t^T \mathcal{M} u_t - (u_x + \mathcal{E}u)^T \mathcal{A}(u_x + \mathcal{E}u) - u^T (K_S(x)\mathcal{A}_S + K_{FS}\mathcal{A}_{FS} + K_{FN}\mathcal{A}_{FN})u \Big\} - \frac{1}{2} K_N [(\Gamma u)_-]^2 + (F_B + F_a)^T u \Big] \, dx \, dt.$$

Let H, V, and V_0 denote the Hilbert spaces defined in Chapter 1. We define a bilinear form on $V \times V$ by

$$a(u,v) = \frac{1}{2} \int_0^L \{ (u_x + \mathcal{E}u)^T A (v_x + \mathcal{E}v) + u^T (K_S\mathcal{A}_S + K_{FS}\mathcal{A}_{FS} + K_{FN}\mathcal{A}_{FN})v \} \, dx. \tag{2.8}$$

As in the static case, if K_S, K_{FS}, $K_{FN} > 0$, we can find $\kappa_0 > 0$ such that

$$a(u,u) \geq \kappa_0 \|u\|_V^2. \tag{2.9}$$

According to Hamilton's Principle, the displacement occurs at a critical point of the Lagrangian. Thus, calculating the variation of the Lagrangian, we obtain

$$\begin{aligned}
\delta\mathcal{L} &= \int_0^{t_f} (\delta\mathcal{T} - \delta\mathcal{V}) \, dt \\
&= \int_0^{t_f} \int_0^L \Big[-(a_1 \phi_{1tt} \delta\phi_1 + b_1 \omega_{1tt} \delta\omega_1 + a_2 \phi_{2tt} \delta\phi_2 + b_2 \omega_{2tt} \delta\omega_2) \\
&+ (\alpha_1 \phi_{1xx} \delta\phi_1 + \beta_1 (\phi_1 + \omega_{1x})_x \delta\omega_1 + \alpha_2 \phi_{2xx} \delta\phi_2 + \beta_2 (\phi_2 + \omega_{2x})_x \delta\omega_2) \\
&- \Big(\beta_1 (\phi_1 + \omega_{1x}) \delta\phi_1 + \beta_2 (\phi_2 + \omega_{2x}) \delta\phi_2 + K_S (h_1\phi_1 + h_2\phi_2) h_1 \delta\phi_1 \\
&\quad + K_S (h_1\phi_1 + h_2\phi_2) h_2 \delta\phi_2 + K_{FS} \phi_2 \delta\phi_2 + K_{FN} \omega_2 \delta\omega_2 \\
&\quad + \frac{1}{2} K_N [(\Gamma u)_-] \left(\frac{\Gamma u}{|\Gamma u|} - 1 \right) \delta\omega_1 - \frac{1}{2} K_N [(\Gamma u)_-] \left(\frac{\Gamma u}{|\Gamma u|} - 1 \right) \delta\omega_2 \Big) \\
&+ (4gk\rho_1 h_1 \delta\omega_1 + 4gk\rho_2 h_2 \delta\omega_2 + f_a(x,t) \delta\omega_1) \Big] \, dx \, dt.
\end{aligned}$$

The minimum occurs when $\delta\mathcal{L} = 0$ or when each of the coefficients of the variations $\delta\phi_1$, $\delta\omega_1$, $\delta\phi_2$, $\delta\omega_2$ are equal to zero. This gives the following system of differential equations:

$$a_1\phi_{1tt} - \alpha_1\phi_{1xx} + \beta_1(\phi_1 + \omega_{1x}) + K_S h_1^2\phi_1 + K_S h_1 h_2 \phi_2 = 0$$

$$b_1\omega_{1tt} - \beta_1(\phi_1 + \omega_{1x})_x + \frac{1}{2}K_N[(\Gamma u)_-]\left(\frac{\Gamma u}{|\Gamma u|} - 1\right) = 4gk\rho_1 h_1 + f_a(x,t)$$

$$a_2\phi_{2tt} - \alpha_2\phi_{2xx} + \beta_2(\phi_2 + \omega_{2x}) + K_S h_2^2\phi_2 + K_S h_1 h_2 \phi_1 + K_{FS}\phi_2 = 0$$

$$b_2\omega_{2tt} - \beta_2(\phi_2 + \omega_{2x})_x + K_{FN}\omega_2 - \frac{1}{2}K_N[(\Gamma u)_-]\left(\frac{\Gamma u}{|\Gamma u|} - 1\right) = 4gk\rho_2 h_2$$

Lemma 2.1:

$$\frac{1}{2}[(\Gamma u)_-]\left(\frac{\Gamma u}{|\Gamma u|} - 1\right) = -(\Gamma u)_-$$

Proof: This follows from

$$\frac{\Gamma u}{|\Gamma u|} - 1 = \begin{cases} -2 & \text{if } \Gamma u < 0 \\ 0 & \text{if } \Gamma u > 0. \end{cases}$$

Using matrix notation and Lemma 2.1, we can rewrite the initial value problem as

$$\mathcal{M}u_{tt} - A(u_x + \mathcal{E}u)_x + \mathcal{E}^T A\mathcal{E}(u + \mathcal{E}^T u_x) + (K_S\mathcal{A}_S + K_{FS}\mathcal{A}_{FS} + K_{FN}\mathcal{A}_{FN})u$$
$$- K_N\Gamma^T(\Gamma u)_- = F \qquad (2.10)$$
$$u(x,0) = 0$$
$$u_t(x,0) = 0$$

where $F = F_a + F_B$.

Remark 1: These equations may also be obtained using conservation of energy. If we express the total energy at time t as the sum of the kinetic and the potential energy, then conservation of energy states that the total energy is a constant function of time.

Remark 2: To incorporate the strong damping properties of the concrete structures of interest we include a structural damping term to obtain the weak form of the initial value problem: Find $u \in L^2(0,t_f;V_0)$ such that for any $\varphi \in V_0$

$$\int_0^L \varphi^T \mathcal{M} u_{tt}\,dx + a(\varphi,u_t) + a(\varphi,u) - \int_0^L K_N \varphi^T \Gamma^T[(\Gamma u)_-]\,dx = <\varphi, F> \quad (2.11)$$

for almost all $t \in [0,t_f]$ with initial conditions

$$u(x,0)=0 \qquad \text{and} \qquad u_t(x,0)=0$$

where we have added a term, $a(\varphi,u_t)$, for structural damping of the concrete beams.

Theorem 2.1: There exists a unique solution to the weak formulation of the initial value problem.

Proof: We use the Galerkin Technique. Suppose $u = u^N = B(x)c(t)$ where

$$B = \begin{bmatrix} \underline{\psi} & 0 & 0 & 0 \\ \underline{0} & \underline{\psi} & 0 & 0 \\ 0 & \underline{0} & \underline{\psi} & 0 \\ 0 & 0 & \underline{0} & \underline{\psi} \end{bmatrix},$$

$\underline{\psi} = [\psi_1 \quad \psi_2 \quad \ldots \quad \psi_N]$, and ψ_j are independent functions in $H^1(0,L;\mathbf{R})$. In (2.11), set $\varphi = B$ and $u = u^N$ to get

$$\left[\int_0^L B^T \mathcal{M} B\,dx\right] c_{tt} + \left[\int_0^L B^T(K_S\mathcal{A}_S + K_{FS}\mathcal{A}_{FS} + K_{FN}\mathcal{A}_{FN})B\,dx\right] c_t$$
$$+ \left[\int_0^L (B_x + \mathcal{E}B)^T A(B_x+\mathcal{E}B)\,dx\right] c_t + \left[\int_0^L (B_x+\mathcal{E}B)^T A(B_x+\mathcal{E}B)\,dx\right] c$$
$$+ \left[\int_0^L (B^T K_S\mathcal{A}_S + K_{FS}\mathcal{A}_{FS} + K_{FN}\mathcal{A}_{FN})B\,dx\right] c - \left[\int_0^L K_N B^T\Gamma^T[(\Gamma Bc)_-]\,dx\right]$$
$$= \int_0^L B^T F\,dx. \quad (2.12)$$

Let

$$G_0 = \int_0^L B^T \mathcal{M} B\, dx,$$

$$G_A = \int_0^L (B_x + \mathcal{E}B)^T A (B_x + \mathcal{E}B)\, dx,$$

$$G_S = \int_0^L B^T K_S \mathcal{A}_S B\, dx,$$

$$G_F = \int_0^L B^T (K_{FS}\mathcal{A}_{FS} + K_{FN}\mathcal{A}_{FN}) B\, dx,$$

$$H(c) = \int_0^L K_N B^T \Gamma^T [(\Gamma B c)_-]\, dx,$$

$$\underline{F} = \int_0^L B^T (F_a + F_B)\, dx,$$

and

$$G = G_A + G_S + G_F.$$

Then (2.12) can be rewritten as

$$G_0 c_{tt} + G c_t + G c - H(c) - \underline{F} = 0$$

and as a first-order system as

$$\begin{bmatrix} I & 0 \\ 0 & G_0 \end{bmatrix} \begin{bmatrix} c \\ c_t \end{bmatrix}_t + \begin{bmatrix} 0 & -I \\ G & G \end{bmatrix} \begin{bmatrix} c \\ c_t \end{bmatrix} + \begin{bmatrix} 0 \\ -H(c) \end{bmatrix} - \begin{bmatrix} 0 \\ \underline{F} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $(\xi)_- = \frac{1}{2}(|\xi| - \xi)$, it is clear that the negative part of a Lipschitz continuous function is Lipschitz. Also note that $H(c)$ is Lipschitz since it is a composition of Lipschitz functions. Thus from the theory of differential equations, there exists a unique solution to the initial value problem

$$G_0 c_{tt} + G c_t + G c - H(c) - \underline{F} = 0 \qquad c(0) = 0, \qquad c_t(0) = 0.$$

Using a Gronwall argument, we develop a priori estimates for the system. From equation (2.10), multiply by u_t^T and integrate over $[0, L]$ to get

$$\int_0^L \Big\{ u_t^T \mathcal{M} u_{tt} - u_t^T A(u_x + \mathcal{E}u)_x + u_t^T \mathcal{E}^T A\mathcal{E}(u + \mathcal{E}^T u_x) + u_t^T (K_S \mathcal{A}_S + K_{FS} \mathcal{A}_{FS} + K_{FN} \mathcal{A}_{FN}) u - K_N u_t^T \Gamma^T [(\Gamma u)_-] - u_t^T (F_B + F_a) \Big\}\, dx = 0.$$

With $u = u^N$, rewrite this as

$$\frac{d}{dt}\left\{ \frac{1}{2}\int_0^L \left(u_t^T \mathcal{M} u_t + K_N[(\Gamma u)_-]^2\right)\, dx + a(u,u) \right\} = \int_0^L (F_B + F_a)^T u_t \, dx$$

and

$$\frac{d}{dt}\left\{ \int_0^L \left(u_t^T \mathcal{M} u_t + K_N[(\Gamma u)_-]^2\right) dx + 2a(u,u) \right\} = 2\int_0^L (F_B + F_a)^T u_t \, dx$$
$$\leq \int_0^L (F_B + F_a)^T \mathcal{M}^{-1} (F_B + F_a)\, dx + \int_0^L u_t^T \mathcal{M} u_t \, dx.$$

Set

$$E(u) = \frac{1}{2}\int_0^L \left(u_t^T \mathcal{M} u_t + K_N[(\Gamma u)_-]^2\right)\, dx + a(u,u)$$

which is positive provided K_S, K_N, K_{FS}, and K_{FN} are positive. Then we get

$$\frac{d}{dt}E(u) \leq \frac{1}{2}\int_0^L (F_B + F_a)^T \mathcal{M}^{-1} (F_B + F_a)\, dx + \frac{1}{2}\int_0^L u_t^T \mathcal{M} u_t \, dx$$
$$\leq \frac{1}{2}\int_0^L (F_B + F_a)^T \mathcal{M}^{-1} (F_B + F_a)\, dx + E(u).$$

We can rewrite this as

$$\frac{d}{dt}(e^{-t}E(u)) \leq \frac{e^{-t}}{2}\int_0^L (F_B + F_a)^T \mathcal{M}^{-1} (F_B + F_a)\, dx.$$

Integrating from 0 to t, we get

$$e^{-t}E(u)(t) \leq E(u)(0) + \int_0^t \frac{e^{-s}}{2}\int_0^L (F_B + F_a)^T \mathcal{M}^{-1} (F_B + F_a)\, dx\, ds$$

or

$$E(u)(t) \leq e^t E(u)(0) + e^t \int_0^t \frac{e^{-s}}{2} \int_0^L (F_B + F_a)^T \mathcal{M}^{-1}(F_B + F_a)\,dx\,ds.$$

We notice that the right hand side of this inequality is independent of N. Thus, for a fixed t and every N

$$E(u^N) \leq \kappa$$

where κ is independent of N. From the definition of $E(u)$ and the above inequality it is easy to see that

$$u^N \text{ is bounded in } L^2(0, t_f; V_0)$$

and

$$u_t^N \text{ and } u_x^N \text{ are bounded in } L^2(0, t_f; H).$$

Thus, there are subsequences such that

$$u_t^N \to u_t \text{ weakly in } L^2(0, t_f; H) \tag{2.13}$$

and

$$u_x^N \to u_x \text{ weakly in } L^2(0, t_f; H). \tag{2.14}$$

Define the set

$$\mathcal{Y} = \{v \in L^2(0, t_f; V_0), v_t \in L^2(0, t_f; H)\}.$$

Since $V_0 \subset H \subset H$ and V_0 imbeds compactly in H, we know that the map

$$\mathcal{Y} \to L^2(0, t_f; H)$$

is compact.[Temam] Since $u^N \in L^2(0,t_f;V_0)$ and $u_t^N \in L^2(0,t_f;H)$, then $u^N \in \mathcal{Y}$ and there exists a subsequence that converges strongly in $L^2(0,t_f;H)$, i.e. $u^N \to u$. In fact, since $u_x^N \xrightarrow{w} u_x$ in $L^2(0,t_f;H)$,

$$u^N \longrightarrow u \text{ weakly in } L^2(0,t_f;V_0).$$

Let $\{b_i\}_{i=1}^{\infty}$ be a basis in V_0 and let $\alpha \in C_0^\infty(0,t_f)$. Then from (2.11) we have

$$\int_0^{t_f}\left\{\int_0^L \left\{b_i^T \mathcal{M} u_{tt}^N - K_N b_i^T \Gamma^T[(\Gamma u^N)_-]\right\} dx \right. \\ \left. + a(b_i,u_t^N) + a(b_i,u^N) - < b_i, F > \right\}\alpha\, dt = 0 \tag{2.15}$$

Suppose $u^N \xrightarrow{w} u$ in $L^2(0,t_f;V_0)$. From (2.13) and (2.14), we know that $u_x^N \xrightarrow{w} u_x$ in $L^2(0,t_f;H)$ and $u_t^N \xrightarrow{w} u_t$ in $L^2(0,t_f;H)$. Thus, as $N \to \infty$, $a(b_i,u_t^N) \to a(b_i,u_t)$ and $a(b_i,u^N) \to a(b_i,u)$. Also,

$$\int_0^{t_f}\left\{\int_0^L b_i^T \mathcal{M} u_{tt}^N\, dx\right\}\alpha\, dt = -\int_0^{t_f}\left\{\int_0^L b_i^T \mathcal{M} u_t^N\, dx\right\}\alpha_t\, dt.$$

We need to show that $[(\Gamma u^N)_-] \to [(\Gamma u)_-]$. Recall that the negative part of a function can be written as $(\xi)_- = \frac{1}{2}(|\xi| - \xi)$ and

$$\int_0^{t_f}\int_0^L (|\Gamma u^N| - |\Gamma u|)^2\, dx\, dt \le \int_0^{t_f}\int_0^L (\Gamma u^N - \Gamma u)^2\, dx\, dt \to 0.$$

Taking the limit of (2.15) as $N \to \infty$ we get

$$\int_0^{t_f}\left\{\int_0^L \left\{b_i^T \mathcal{M} u_{tt} - K_N b_i^T \Gamma^T[(\Gamma u)_-]\right\} dx \right. \\ \left. + a(b_i,u_t) + a(b_i,u) - < b_i, F > \right\}\alpha\, dt = 0$$

for any $\alpha \in C_0^\infty(0,t_f)$.

Since $\{b_i\}_{i=1}^{\infty}$ spans V_0 and $\alpha \in C_0^\infty(0,t_f)$, we have for any $\varphi \in V_0$

$$\int_0^L \left\{\varphi^T \mathcal{M} u_{tt} - K_N \varphi^T \Gamma^T[(\Gamma u)_-]\right\} dx + a(\varphi,u_t) + a(\varphi,u) - < \varphi, F > = 0.$$

Thus, u is a solution of the weak version of the initial value problem and we have existence.

For uniqueness, suppose u and $\tilde{u}$ are weak solutions of the problem (2.11). Then for any $\varphi \in V_0$, u satisfies

$$\int_0^L \{\varphi^T \mathcal{M} u_{tt} - K_N \varphi^T \Gamma^T [(\Gamma u)_-]\} \, dx + a(\varphi, u_t) + a(\varphi, u) - <\varphi, F> = 0$$

and $\tilde{u}$ satisfies

$$\int_0^L \{\varphi^T \mathcal{M} \tilde{u}_{tt} - K_N \varphi^T \Gamma^T [(\Gamma \tilde{u})_-]\} \, dx + a(\varphi, \tilde{u}_t) + a(\varphi, \tilde{u}) - <v, F> = 0.$$

Subtracting we get

$$\int_0^L \Big\{\varphi^T \mathcal{M} (u - \tilde{u})_{tt} - K_N \varphi^T \Gamma^T ([(\Gamma u)_-] - [(\Gamma \tilde{u})_-])\Big\} \, dx \\ + a(\varphi, (u - \tilde{u})_t) + a(\varphi, u - \tilde{u}) = 0$$

for any $\varphi \in V_0$. Let $w = u - \tilde{u}$ and choose $\varphi = w_t \in L^2(0, t_f; V_0)$ to get

$$\int_0^L \{w_t^T \mathcal{M} w_{tt} - K_N w_t^T \Gamma^T ([(\Gamma u)_-] - [(\Gamma \tilde{u})_-])\} \, dx + a(w_t, w_t) + a(w_t, w) = 0.$$

We can rewrite this as

$$\frac{d}{dt}\frac{1}{2}\left\{\int_0^L w_t^T \mathcal{M} w_t \, dx + a(w, w)\right\} + a(w_t, w_t) \\ = \int_0^L K_N w_t^T \Gamma^T ([(\Gamma u)_-] - [(\Gamma \tilde{u})_-]) \, dx$$

or

$$\begin{aligned} \frac{d}{dt}\left\{\int_0^L w_t^T \mathcal{M} w_t \, dx + a(w, w)\right\} &+ 2a(w_t, w_t) \\ &= 2\int_0^L K_N (\Gamma w_t)^T ([(\Gamma u)_-] - [(\Gamma \tilde{u})_-]) \, dx \\ &\leq 2\|K_N\| \|\Gamma\| \|w_t\|_H \|((\Gamma u)_-) - ((\Gamma \tilde{u})_-)\| \\ &\leq \delta \|w_t\|_H^2 + \frac{1}{\delta} \|K_N\|^2 \|\Gamma\|^2 \|((\Gamma u)_-) - ((\Gamma \tilde{u})_-)\|^2 \end{aligned}$$

which holds for any $\delta > 0$. Also, from (2.9) there is $\kappa_0 > 0$ such that

$$2a(w_t, w_t) \geq 2\kappa_0 \|w_t\|_V^2.$$

Thus, we get

$$\begin{aligned} \frac{d}{dt}\left\{\int_0^L w_t^T \mathcal{M} w_t \, dx + a(w,w)\right\} &+ (2\kappa_0 - \delta)\|w_t\|_V^2 \\ &\leq \frac{1}{\delta}\|K_N\|^2 \|\Gamma\|^2 \|((\Gamma u)_-) - ((\Gamma \tilde{u})_-)\|^2 \end{aligned} \tag{2.16}$$

where we choose δ such that $2\kappa_0 - \delta > 0$.

For the right-hand side, recall that the negative part of a function can be written as $(\xi)_- = \frac{1}{2}(|\xi| - \xi)$ and consider

$$\begin{aligned} \|((\Gamma u)_-) - ((\Gamma \tilde{u})_-)\| &= \left\{\int_0^L ([(\Gamma u)_-] - [(\Gamma \tilde{u})_-])^2 \, dx\right\}^{\frac{1}{2}} \\ &= \frac{1}{2}\left\{\int_0^L [(|\Gamma u| - |\Gamma \tilde{u}|) - (\Gamma u - \Gamma \tilde{u})]^2 \, dx\right\}^{\frac{1}{2}} \\ &\leq \frac{1}{2}\left\{\||\Gamma u| - |\Gamma \tilde{u}|\| + \|\Gamma u - \Gamma \tilde{u}\|\right\}. \end{aligned}$$

Also,

$$0 \leq (|\Gamma u| - |\Gamma \tilde{u}|)^2 \leq |\Gamma u - \Gamma \tilde{u}|^2.$$

Hence

$$\begin{aligned} \||\Gamma u| - |\Gamma \tilde{u}|\| &= \left\{\int_0^L (|\Gamma u| - |\Gamma \tilde{u}|)^2 \, dx\right\}^{\frac{1}{2}} \\ &\leq \left\{\int_0^L (\Gamma u - \Gamma \tilde{u})^2 \, dx\right\}^{\frac{1}{2}} = \|\Gamma u - \Gamma \tilde{u}\| \end{aligned}$$

and we have that

$$\|((\Gamma u)_-) - ((\Gamma \tilde{u})_-)\|^2 \leq \|\Gamma u - \Gamma \tilde{u}\|^2 = \|\Gamma w\|^2. \tag{2.18}$$

It follows from (2.16) and (2.17) that

$$\frac{d}{dt}\left\{\int_0^L w_t^T \mathcal{M} w_t\,dx + a(w,w)\right\} + (2\kappa_0 - \delta)\|w_t\|_V^2 \le \frac{1}{\delta}\|K_N\|^2\|\Gamma\|^2\|\Gamma w\|^2.$$

Since $2\kappa_0 - \delta > 0$, we get

$$\frac{d}{dt}\left\{\int_0^L w_t^T \mathcal{M} w_t\,dx + a(w,w)\right\} \le \frac{1}{\delta}\|K_N\|^2\|\Gamma\|^4\|w\|_H^2.$$

Recall that $a(u,u) \ge \kappa_0\|u\|_V^2 \ge \kappa_0\|u\|_H^2$ and let $\mathcal{K}(t) = \int_0^L w_t^T \mathcal{M} w_t\,dx + a(w,w)$. Thus, we have

$$\frac{d}{dt}\mathcal{K} \le \hat{\kappa} a(w,w) \le \hat{\kappa}\left(a(w,w) + \int_0^L w_t^T \mathcal{M} w_t\,dx\right) = \hat{\kappa}\mathcal{K}$$

where $\hat{\kappa} > 0$ is a constant. The solution to $\frac{d}{dt}\mathcal{K} \le \hat{\kappa}\mathcal{K}$ is

$$0 \le \mathcal{K} \le \mathcal{K}(0)e^{\hat{\kappa}t} = 0.$$

Hence, $\mathcal{K} = 0$ which implies that $a(w,w) = 0$. But this implies that $w = 0$ or $u = \tilde{u}$ and we have uniqueness.

2.2 Estimation Problem

To formulate the inverse problem, it is convenient to make additional simplifying assumptions. In particular, let

$$\begin{aligned}\mathcal{L}(u) = \int_0^{t_f}\int_0^L \Big[\frac{1}{2}\{u_t^T \mathcal{M} u_t - (u_x + \mathcal{E}u)^T \mathcal{A}(u_x + \mathcal{E}u) - K_S(x)u^T \mathcal{A}_S u \\ - K_N u^T \mathcal{A}_N u\} - (F_B + F_a)u\Big]\,dx\,dt\end{aligned} \tag{2.18}$$

where we have ignored any foundation terms and replaced the cone condition by a term that forces the two beams together with $\mathcal{A}_N$ defined as in Chapter 1.

The weak formulation of the minimization problem can be written as: Find $u \in L^2(0, t_f, V_0)$ such that

$$\int_0^L \{\varphi^T \mathcal{M} u_{tt} + (\varphi_x + \mathcal{E}\varphi)^T \mathcal{A}(u_x + \mathcal{E}u) + \varphi^T(K_S\mathcal{A}_S + K_N\mathcal{A}_N)u \\ - \varphi^T(F_B + F_a)\}\, dx = 0 \tag{2.19}$$

for any $\varphi \in V_0$. Note that we are ignoring damping here.

Let $u(x,t) = B(x)c(t)$ where

$$B = \begin{bmatrix} \underline{\ell} & 0 & 0 & 0 \\ 0 & \underline{\ell} & 0 & 0 \\ 0 & 0 & \underline{\ell} & 0 \\ 0 & 0 & 0 & \underline{\ell} \end{bmatrix}$$

with $\underline{\ell} = [\,\ell_1 \quad \ell_2 \quad \ldots \quad \ell_{N+1}\,]$ and ℓ_j are hat functions. In (2.19), set $\varphi = B$ to get

$$\left[\int_0^L B^T \mathcal{M} B\, dx\right] c_{tt} + \left[\int_0^L (B_x + \mathcal{E}B)^T \mathcal{A}(B_x + \mathcal{E}B)\, dx\right] c \\ + \left[\int_0^L B^T(K_S\mathcal{A}_S + K_N\mathcal{A}_N)B\, dx\right] c - \left[\int_0^L B^T(F_B + F_a)\, dx\right] = 0.$$

Let

$$G_0 = \int_0^L B^T \mathcal{M} B\, dx,$$

$$G_A = \int_0^L (B_x + \mathcal{E}B)^T A(B_x + \mathcal{E}B)\, dx,$$

$$G_S = \int_0^L B^T K_S \mathcal{A}_S B\, dx,$$

$$G_N = \int_0^L B^T K_N \mathcal{A}_N B\, dx,$$

and

$$\underline{F}(t) = \int_0^L B^T(F_B + F_a)\, dx.$$

We need to solve

$$G_0 \frac{d^2 c}{dt^2} + Gc = \underline{F}(t) \tag{2.20}$$

where

$$G = G(K_S) = G_A + G_S + G_N.$$

We approximate

$$\frac{d^2 c}{dt^2} = \frac{1}{(\Delta t)^2}(c_{i-1} - 2c_i + c_{i+1})$$

and

$$c = \frac{c_{i+1} + 2c_i + c_{i-1}}{4}$$

where $c_i = c(t_i)$ the value of c at time t_i. Substituting these into (2.20) and letting $\underline{F}_i = \underline{F}(t_i)$ we get

$$G_0 \left[\frac{1}{(\Delta t)^2}(c_{i-1} - 2c_i + c_{i+1})\right] + G\left[\frac{1}{4}(c_{i+1} + 2c_i + c_{i-1})\right] = \underline{F}_i$$

or

$$\left(G_0 + \frac{(\Delta t)^2}{4}G\right) c_{i-1} + 2\left(-G_0 + \frac{(\Delta t)^2}{4}G\right) c_i + \left(G_0 + \frac{(\Delta t)^2}{4}G\right) c_{i+1} = (\Delta t)^2 \underline{F}_i$$

which holds for $1 \le i \le N_t - 1$. Define

$$A_1 = \left(G_0 + \frac{(\Delta t)^2}{4}G\right) \text{ and } A_2 = 2\left(-G_0 + \frac{(\Delta t)^2}{4}G\right).$$

We get the following system of equations:

$$\begin{aligned} c_0 &= c_0 \\ c_1 - c_0 &= (\Delta t) c_t(0) \\ A_1 c_0 + A_2 c_1 + A_1 c_2 &= (\Delta t)^2 \underline{F}_1 \\ &\cdots \\ A_1 c_{N_t} + A_2 c_{N_t - 1} + A_1 c_{N_t - 2} &= (\Delta t)^2 \underline{F}_{N_t - 1}. \end{aligned} \tag{2.21}$$

Define

$$\bar{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{N_t} \end{bmatrix},$$

$$\bar{G} = \begin{bmatrix} I & & & & & \\ -I & I & & & & \\ A_1 & A_2 & A_1 & & & \\ & A_1 & A_2 & A_1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & A_1 & A_2 & A_1 \end{bmatrix},$$

and

$$\bar{F} = \begin{bmatrix} c_0 \\ (\Delta t)c_t(0) \\ (\Delta t)^2 \underline{F}_1 \\ \vdots \\ (\Delta t)^2 \underline{F}_{N_t - 1} \end{bmatrix}.$$

Using this notation, the system of equations (2.21) can be written

$$\bar{G}(K_S)\bar{c}(K_S) = \bar{F}. \tag{2.22}$$

We define the fit-to-data functional by

$$J(K_S) = \int_0^{t_f} \sum_{i=1}^{N_d} \left(\Pi_{\omega_1} B^{(i)} c(K_S, t) - \omega_1^{(i)} \right)^2 dt$$

where $x_d^{(i)}$ are the observation points; N_d is the number of observation points; Π_{ω_1} is the projection of u onto ω_1; $B^{(i)} = B(x_d^{(i)})$, and $\omega_1^{(i)} = \omega_1(x_d^{(i)})$ is the observed data. Define $C : \mathbf{R}^{4n} \longrightarrow \mathbf{R}^{N_d}$ by

$$C = \begin{bmatrix} \Pi_{\omega_1} B^{(1)} \\ \Pi_{\omega_1} B^{(2)} \\ \vdots \\ \Pi_{\omega_1} B^{(N_d)} \end{bmatrix}$$

and the observation vector

$$z(t) = \begin{bmatrix} \omega_1^{(1)}(t) \\ \omega_1^{(2)}(t) \\ \vdots \\ \omega_1^{(N_d)}(t) \end{bmatrix}.$$

Thus the fit-to-data functional may be written as

$$J(K_S) = \int_0^{t_f} [\mathcal{C}c(t) - z(t)]^T [\mathcal{C}c(t) - z(t)]\, dt. \tag{2.23}$$

We use the trapezoidal method to write the discrete version of the fit-to-data functional.

$$\begin{aligned} J(K_S) = \frac{\Delta t}{2} \{ &[\mathcal{C}c(0) - z(0)]^T [\mathcal{C}c(0) - z(0)] + 2[\mathcal{C}c(t_1) - z(t_1)]^T [\mathcal{C}c(t_1) - z(t_1)] \\ &+ \ldots + 2[\mathcal{C}c(t_{N_t-1}) - z(t_{N_t-1})]^T [\mathcal{C}c(t_{N_t-1}) - z(t_{N_t-1})] \\ &+ [\mathcal{C}c(t_{N_t}) - z(t_{N_t})]^T [\mathcal{C}c(t_{N_t}) - z(t_{N_t})] \} \end{aligned}$$

or rewriting we get

$$J(K_S) = \frac{\Delta t}{2} [\hat{M}\bar{c}(K_S) - \hat{z}]^T M [\hat{M}\bar{c}(K_S) - \hat{z}]$$

where

$$\hat{M} = \begin{bmatrix} \mathcal{C} & & & \\ & \mathcal{C} & & \\ & & \ddots & \\ & & & \mathcal{C} \end{bmatrix},$$

$$M = \begin{bmatrix} I & & & & \\ & 2I & & & \\ & & \ddots & & \\ & & & 2I & \\ & & & & I \end{bmatrix},$$

and

$$\hat{z} = \begin{bmatrix} z(0) \\ z(t_1) \\ \vdots \\ z(N_t) \end{bmatrix}.$$

The fit-to data functional was plotted in Matlab assuming K_S to be piecewise constant with two pieces using a time increment $\Delta t = .2$ and 20 time steps. It was assumed that the K_S values fell between .01 and 6.76. In addition, we assume that the applied force depends on both time and the location along the length of the beam. In particular, we assume that $f_a = f\sin(\pi t)$ occuring at $x = L/3$ and $x = 2L/3$. Since there is no actual data for this case, data was generated using the forward model to see if we could retrieve a given $K_S = [\, K_S(1) \quad K_S(2)\,]$.

For this purpose, the values were chosen to be $K_S(1) = 1$ and $K_S(2) = 5$. Fig. 9 shows the contour plot of $J(K_S(1), K_S(2))$. Notice the minimum falls within a trough. We can look at the individual matrix entries to see that the minimum is 1.985×10^{-7} which occurs at

$$K_S(1) = .76 \text{ and } K_S(2) = 5.26$$

which are close to the desired values.

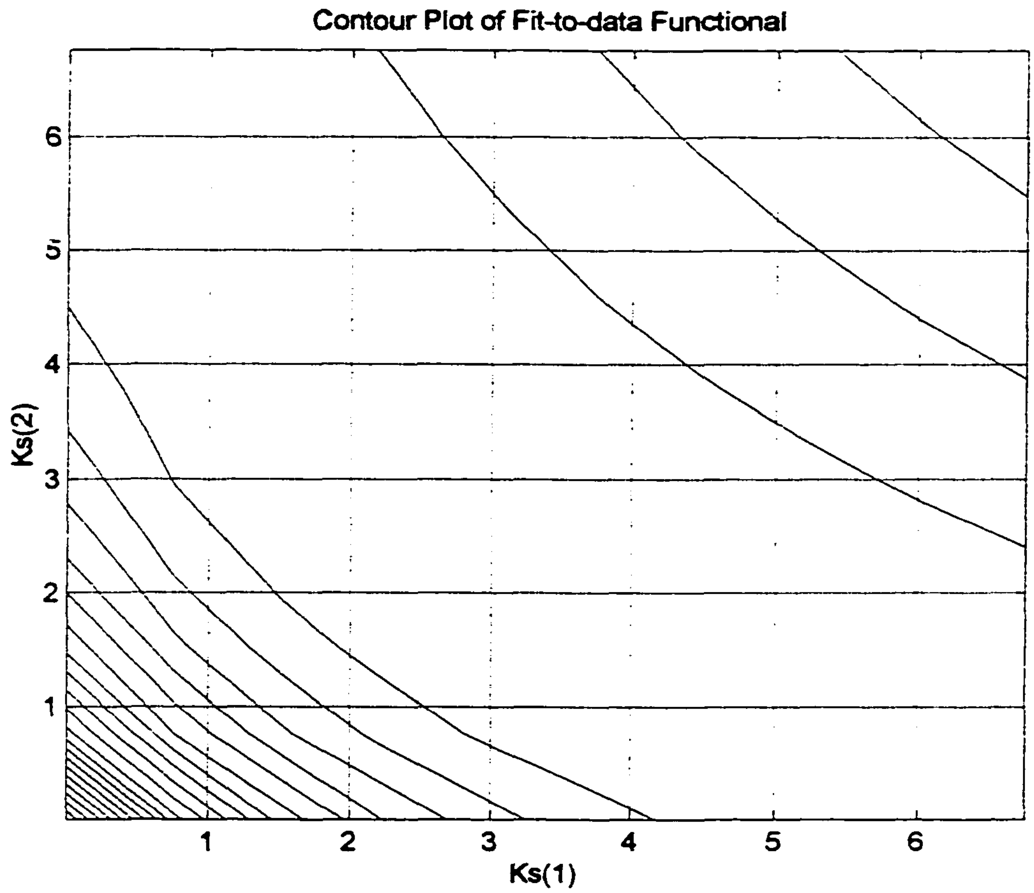

Figure 9: Contour Plot of Fit-to-Data Functional
Note: The minimum occurs at Ks=[.76 5.26]

Conclusion

In this paper, we have developed mathematical models to examine the interfaces of layered beam systems. The particular application examined was that of concrete road overlays. We were particularly interested in how normal displacement of the beams was affected by the shearing interface coefficient, K_S. We ignored any effect due to friction.

For the static two-beam case, we observed that for $K_S = [7 \quad 2]$, we got reasonably good correspondence to laboratory data with a relative error of .1059. We are able to improve on these results as we refine the vector K_S and for the optimal case we got a relative error of .0253. For the time-dependent case, we incorporated a structural damping term in the model. We then considered the inverse problem to verify that we could recover a given K_S using data generated from the forward problem.

Both the static and dynamic models developed in this paper allowed for separation of the two beams. This in effect corrected the problem of one beam penetrating the other and allows us to try to predict where the two beams may acutally separate. Also, with the addition of the time-dependent model, we can now add a moving mass or rolling load which more closely approximate real world applications.

References

[1] *Annual Book of ASTM Standards*. Vol. 04.02, American Standard for Testing Materials, 1916 Race Street, Philadelphia, PA 19103-1187.

[2] J. B. Conway, *A Course in Functional Analysis, Second Edition*, Springer-Verlag, New York, 1990.

[3] G. Duvaut and J. L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin 1976.

[4] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer- Verlag, Berlin, 1977.

[5] Z. Hossain, *Behavior of Thin Boded Overlay on Rigid Pavements: An Experimental Approach*, Masters Thesis, University of Oklahoma, 1998.

[6] S.C. Hunter, *Mechanics of Continuous Media*, Ellis Horwood, New York, 1976.

[7] David G. Luenberger, *Optimization by Vector Space Methods*, John Wiley & Sons, New York, 1969.

[8] L. Meirovitch, *Analytical Methods in Vibrations*, The MacMillan Company, New York, 1967.

[9] R. D. Mindlin, "Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic Elastic Plates," *J. Appl. Mech.*, March(1951), pp. 31-38.

[10] H. L. Royden, *Real Analysis, 3rd ed.*, Macmillan Publishing Company, New York, 1988.

[11] M. Schultz, *Spline Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1973.

[12] R. Temam, *Navier Stokes Equations*, North Holland, Amsterdam, 1984.

[13] S. Timoshenko and W. Woinowsky-Krieger, *Theory of Plates and Shells*, McGraw-Hill, New York, 1974.

[14] L. White and M. Zaman, "Coupling Elastic Models Through Interfacial conditions with Application to Concrete Pavement Overlay", *J. Appl. Math. and Comp.* to appear.

[15] L. White, "Control of Dynamic Models of Beams and Plates with Small Damping: Location of Actuators," *J. Appl. Anal.*, **28**(1988), 297-327.

[16] L. White, "Estimation of Flexural Rigidity in a Dynamic Plate Model," *J. Math. Analysis and Applications*, 15(1989), 275-303.

[17] L. White and J. T. Oden, "Dynamics and Control of Viscoelastic Solids with Contact and Friction Effects," *J. of Nonlinear Analysis and Applications*, 4(1989), 459-474.

[18] L. White, "Estimation of Higher Order Damping Terms in Linear Plate Models," *J. Appl. Math. and Comp.*, **33**(1989), 89-122.

[19] L. White, "Output Least Squares Estimation of Power-laws in Damping Models: Beams," Control of Boundaries and Stabilization, Lecture Notes in Control and Information Sciences, J. Simon ed., Springer-Verlag, 1989.

[20] L. White, "Estimation of Elastic and Damping Parameters in Dynamic Nonlinear Plate Models," *J. Appl. Math. and Comp.*, **54**, No. 1(1993), 25-63.

[21] L. White and D. L. Russell, "Formulation and Validation of Dynamical Models for Narrow Plate Motion," *J. Appl. Math. and Comp.*, **58**, No. 2 and 3(1993), 103-141.

Appendix

This appendix contains the Matlab programs and functions used to generate the graphs in this paper. Ovrlay2 is the numerical model for the static two-beam problem. Ovrlyte0 is the estimation problem for the dynamic case. zdata is the forward problem for the dynamic case which generates the data used in Ovrlyte0.

```
%ovrlay2: timoshenko beam with two layers.
%The purpose of this program is to minimize the Lagrangian
%for 2 Timoshenko beams.  In order to minimize when the 2
%beams can separate, put cone=1.
%
clear
cone=1;
pen=1.0e+03;
Kn=1.0e+03;
force=1400;
f1=1;
f2=1;
%
%beam and material parameters
grav=32;
h1=9/2;
h2=3/2;
k=6;
L=37;
E1=3.63;
E2=2.9;
mu1=0.14;
mu2=0.12;
rho1=141/(12^3);
rho2=150/(12^3);
%Displacement data
Nd=5;
wd(1)=0;
wd(2)=-0.0014;
wd(3)=-0.00245;
wd(4)=-.0017;
wd(5)=0;
wd=wd;
xd(1)=0;
xd(2)=9.25;
xd(3)=18.5;
xd(4)=27.75;
xd(5)=37;
%
N=10;
M=2;
%The Ks vector is of dimension M. It penalizes the interface
%condition on each of the N subintervals in the beam.
%
Ks(1)=7;
Ks(2)=2;
%
n=N+1;
m=4*n;
hs=L/N;
%
% set basic matrices for piecewise linear basis functions
%
g0=eye(n);
g1=zeros(n);
g2=eye(n);
g0(1,1)=2;
```

```
g0(n,n)=2;
g1(1,1)=-0.5;
g1(n,n)=0.5;
for i=2:n-1
   g0(i,i)=4;
   g2(i,i)=2;
end
for i=1:n-1
   g0(i,i+1)=1;
   g0(i+1,i)=1;
   g1(i,i+1)=-0.5;
   g1(i+1,i)=0.5;
   g2(i,i+1)=-1;
   g2(i+1,i)=-1;
end
g0=hs/6*g0;
g2=1/hs*g2;
%
%Strain Energy
%
alpha1=(4*k*h1^3*E1)/(3*(1-mu1^2));
alpha2=(4*k*h2^3*E2)/(3*(1-mu2^2));
beta1=2*k*h1*E1/(1+mu1);
beta2=2*k*h2*E2/(1+mu2);
%
GA=[alpha1*g2+beta1*g0 beta1*g1' zeros(n)            zeros(n)
    beta1*g1           beta1*g2  zeros(n)            zeros(n)
    zeros(n)           zeros(n)  alpha2*g2+beta2*g0 beta2*g1'
    zeros(n)           zeros(n)  beta2*g1           beta2*g2];
%
%penalizes displacement at x=0 and x=L to zero
%
for i=1:4
for j=1:m
   bL(i,j)=0;
   b0(i,j)=0;
end
end
%
e0=[0 0 0 0
    0 1 0 0
    0 0 0 0
    0 0 0 1];
%
bL(1,n)=1;
bL(2,2*n)=1;
bL(3,3*n)=1;
bL(4,4*n)=1;
b0(1,1)=1;
b0(2,n+1)=1;
b0(3,2*n+1)=1;
b0(4,3*n+1)=1;
BL=bL'*e0*bL;
B0=b0'*e0*b0;
%
% The is the stiffness matrix and the boundary condition.
%
```

```
Bdry=B0+BL;
G=GA+pen*Bdry;
%
% Interface conditions
%
BS=zeros(4);
BN=zeros(4);
BS(1,1)=h1^2;
BS(1,3)=h1*h2;
BS(3,1)=h2*h1;
BS(3,3)=h2^2;
BN(2,2)=1;
BN(2,4)=-1;
BN(4,2)=-1;
BN(4,4)=1;
%
%matrix that describes the penalization over the interface
%between the beams
%
Gs=ela(Ks,n,N,M);
GI=kron(Kn*BN,g0)+kron(BS,Gs);
%
G=G+GI;
%
%Work Terms
%
fb=[0 4*k*grav*rho1*h1 0 4*k*grav*rho2*h2];
%
for i=1:4
for j=1:m
   fbody0(i,j)=0;
end
end
%
for j=1:n
   fbody0(2,n+j)=1;
   fbody0(4,3*n+j)=1;
end
%
fbody0(2,n+1)=0.5;
fbody0(2,2*n)=0.5;
fbody0(4,3*n+1)=0.5;
fbody0(4,4*n)=0.5;
fbody0=hs*fbody0;
fbody=fb*fbody0;
%
for i=1:4
for j=1:m
   fapld0(i,j)=0;
end
end
for j=1:n
   fapld0(2,j+n)=f1*b(L/3,j-1,N+1,L)+f2*b(2*L/3,j-1,N+1,L);
end
%
fa=[0 2*k*force 0 0];
fapld=fa*fapld0;
```

```
f=fbody+fapld;
f=1.0e-06*f;
c=-G\f';
%
sign=0;
if cone==1
%
%Test to see if solution is in the cone.
%
for i=1:n
    if c(n+i)-c(3*n+i)<0
   c(3*n+i)=c(n+i);
   sign=sign+1;
    end
end
sign
end
%
for i=1:n
   cw(i)=c(i+n);
   x(i)=(i-1)*hs;
end
%
%Compute w1(x) and compare with data.
%
for j=1:Nd
     w1(j)=sovrly(cw,xd(j),n,L);
end
plot(xd,w1,xd,wd,'*')
title('w1,wd, in cone before minimizing')
pause
%
%Use Newton's Method to minimize Lagrangian.
if sign>0
   for j=1:20
%
      gradL=G'*c-f';
      grad2L=G';
      lagr(j)=.5*c'*G*c-f*c;
%
      c=c-inv(grad2L)*gradL;
c=-c;
if cone==1
%
%Test to see if solution is in the cone.
%
for i=1:n
    if c(n+i)-c(3*n+i)<0
   c(3*n+i)=c(n+i);
    end
end
end
    end
end
%
for i=1:n
   cw(i)=c(i+n);
```

```
    cw2(i)=c(3*n+i);
    x(i)=(i-1)*hs;
end
%Compute w1(x) and w2(x) and compare with data.
%
for j=1:n
      w1(j)=sovrly(cw,x(j),n,L);
      w2(j)=sovrly(cw2,x(j),n,L);
      deltaw(j)=w1(j)-w2(j);
end
plot(x,w1,xd,wd,'*')
ylabel('displacement w1')
xlabel('distance along x-axis')
title('model displacement(w1) vs. data')
pause
%plot(x,w1,'+',x,w2)
dwn=deltaw/norm(w1);
plot(x,dwn)
ylabel('deltaw/norm(w1)')
title('Do the 2 beams separate?')
%
%
%Find the relative error
cwd=0;
reler=0;
for i=1:Nd
  wev(i)=sovrly(cw,xd(i),n,L);
  reler=reler+(wev(i)-wd(i))^2;
  cwd=cwd+wd(i)^2;
end
reler=sqrt(reler/cwd)
```

```
%ovrlyte0.m: timoshenko beam with two layers.
%The purpose of this program is to plot the discrete
%version of the fit-to-data functional (jfnl) as a
%function of Ks.
%
clear
format long e
cone=0;
pen=1.0e+03;
Kn=1.0e+03;
force=1400;
f1=1;
f2=1;
%
%beam and material parameters
grav=32;
h1=9/2;
h2=3/2;
k=6;
L=37;
E1=3.63;
E2=2.9;
mu1=0.14;
mu2=0.12;
rho1=141/(12^3);
rho2=150/(12^3);
%
N=4;
M=2;
%The Ks vector is of dimension M. It penalizes the
%interface condition on each of the N subintervals in
%the beam interface.
%
Ks(1)=1;
Ks(2)=5;
%
n=N+1;
m=4*n;
hs=L/N;
Nt=20;
mfactor=10^(-6);
%
z1=zdata(N,M,Ks,Nt,mfactor);
for j=1:Nt+1
   zhat((j-1)*n+1:j*n)=z1(:,j);
end
%
%set basic matrices for piecewise linear basis functions
%
g0=eye(n);
g1=zeros(n);
g2=eye(n);
g0(1,1)=2;
g0(n,n)=2;
g1(1,1)=-0.5;
g1(n,n)=0.5;
for i=2:n-1
```

```
    g0(i,i)=4;
    g2(i,i)=2;
 end
 for i=1:n-1
    g0(i,i+1)=1;
    g0(i+1,i)=1;
    g1(i,i+1)=-0.5;
    g1(i+1,i)=0.5;
    g2(i,i+1)=-1;
    g2(i+1,i)=-1;
 end
 g0=hs/6*g0;
 g2=1/hs*g2;
 %
 for ix=1:10
    for iy=1:10
       Ks(1)=.01+0.75*(ix-1);
       Ks(2)=.01+0.75*(iy-1);
       K1v(ix)=Ks(1);
       K2v(iy)=Ks(2);
 %
 a1=4*h1^3*k*rho1/3;
 a2=4*h2^3*k*rho2/3;
 b1=4*h1*k*rho1;
 b2=4*h2*k*rho2;
 %
 Gmass=[ a1*g0    zeros(n) zeros(n) zeros(n)
         zeros(n)  b1*g0   zeros(n) zeros(n)
         zeros(n) zeros(n)  a2*g0   zeros(n)
         zeros(n) zeros(n) zeros(n)  b2*g0];
 Gmass=mfactor*Gmass;
 %
 %
 alpha1=(4*k*h1^3*E1)/(3*(1-mu1^2));
 alpha2=(4*k*h2^3*E2)/(3*(1-mu2^2));
 beta1=2*k*h1*E1/(1+mu1);
 beta2=2*k*h2*E2/(1+mu2);
 %
 GA=[alpha1*g2+beta1*g0 beta1*g1' zeros(n)             zeros(n)
     beta1*g1            beta1*g2  zeros(n)            zeros(n)
     zeros(n)            zeros(n)  alpha2*g2+beta2*g0 beta2*g1'
     zeros(n)            zeros(n)  beta2*g1            beta2*g2];
 %
 %
 for i=1:4
 for j=1:m
    bL(i,j)=0;
    b0(i,j)=0;
 end
 end
 %
 e0=[0 0 0 0
     0 1 0 0
     0 0 0 0
     0 0 0 1];
 %
 bL(1,n)=1;
```

```
bL(2,2*n)=1;
bL(3,3*n)=1;
bL(4,4*n)=1;
b0(1,1)=1;
b0(2,n+1)=1;
b0(3,2*n+1)=1;
b0(4,3*n+1)=1;
BL=bL'*e0*bL;
B0=b0'*e0*b0;
%
%The is the stiffness matrix and the boundary condition.
%
Bdry=B0+BL;
G=GA+pen*Bdry;
%
%Interface conditions
%
BS=zeros(4);
BN=zeros(4);
BS(1,1)=h1^2;
BS(1,3)=h1*h2;
BS(3,1)=h2*h1;
BS(3,3)=h2^2;
BN(2,2)=1;
BN(2,4)=-1;
BN(4,2)=-1;
BN(4,4)=1;
%
%matrix that describes the penalization over the interface
%between the beams
%
Gs=ela(Ks,n,N,M);
GI=kron(Kn*BN,g0)+kron(BS,Gs);
%
G=G+GI;
%
fb=[0 4*k*grav*rho1*h1 0 4*k*grav*rho2*h2];
%
for i=1:4
for j=1:m
   fbody0(i,j)=0;
end
end
%
for j=1:n
   fbody0(2,n+j)=1;
   fbody0(4,3*n+j)=1;
end
%
fbody0(2,n+1)=0.5;
fbody0(2,2*n)=0.5;
fbody0(4,3*n+1)=0.5;
fbody0(4,4*n)=0.5;
fbody0=hs*fbody0;
fbody=fb*fbody0;
%
for i=1:4
```

```
 for j=1:m
    fapld0(i,j)=0;
 end
 end
 for j=1:n
    fapld0(2,j+n)=f1*b(L/3,j-1,N+1,L)+f2*b(2*L/3,j-1,N+1,L);
 end
 %
 fa=[0 2*k*force 0 0];
 fapld=fa*fapld0;
 f=fbody+fapld;
 f=1.0e-06*f;
 %
 %We need to solve Gmass*c"(t) + G*c(t) = f.
 %Note that f is adjusted below to depend on time.
 %
 dt=.2;
 GM=(1/(dt)^2)*Gmass+(1/4)*G;
 alphaM1=-.5*G+(2/(dt)^2)*Gmass;
 alphaM2=-(.25*G+(1/(dt)^2)*Gmass);
 ci=zeros(4*n,1);
 ciml=zeros(4*n,1);
 %
 omega=pi;
 for it=0:1
    t(it+1)=it*dt;
    c=ci;
    chat(:,it+1)=c;
    for i=1:n
       cw(i)=c(i+n);
       x(i)=(i-1)*hs;
    end
    for j=1:n
       w1(j)=sovrly(cw,x(j),n,L);
    end
    w1hat(it*n+1:(it+1)*n)=w1;
 end
 %
 for it=1:Nt-1
    t(it+1)=(it+1)*dt;
 %
    fa=fa*sin(omega*t(it));
    fapld=fa*fapld0;
    f=fbody+fapld;
    f=1.0e-06*f;
 %
    cip1=GM\(f'+alphaM1*ci+alphaM2*ciml);
    ciml=ci;
    ci=cip1;
    c=-cip1;
    chat(:,it+2)=c;
    for i=1:n
       cw(i)=c(i+n);
       x(i)=(i-1)*hs;
    end
 %
 %Compute w1.
```

```
%
   for j=1:n
      w1(j)=sovrly(cw,x(j),n,L);
   end
   w1hat((it+1)*n+1:(it+2)*n)=w1;
end
%
jfnl(ix,iy)=jovrlyt(dt,w1hat,zhat,n,Nt);
end %iy loop
end %ix loop
mesh(jfnl)
pause
contour(K1v,K2v,jfnl,20)
xlabel('K_1 ')
ylabel('K_2 ')
title('Contour Plot of Fit-to-data Functional')
```

```
%evaluates the i-th piecewise linear basis function at the
%point x along a beam of length L
%
function y=b(x,i,m,L)
y=0;
h0=L/(m-1);
xi=i*h0;
xip1=(i+1)*h0;
xim1=(i-1)*h0;
if x>=xim1
   y=(x-xim1)/h0;
end
if x>=xi
   y= -(x-xip1)/h0;
end
if x>=xip1
   y=0;
end
```

```
function Gs=ela(K,n,N,M)
% ela provides the interface stiffness in ovrlay2.m
%
r=N/M;
Gs=zeros(n);
for i=1:M
   for j=(i-1)*r+1:i*r
        Gsi=elai(j,n,N,M);
        Gs=Gs+K(i)*Gsi;
   end
end
```

```
function Gsi=elai(i,n,N,M)
%elai provides the component interface stiffness in ovrlay2.m
%
Gsi=zeros(n);
Gsi(i,i)=2;
Gsi(i,i+1)=1;
Gsi(i+1,i)=1;
Gsi(i+1,i+1)=2;
```

```
%evaluates the displacement for the overlay problem
function w=sovrly(c,x,n,L)
w=0;
for i=1:n
    w=w+c(i)*b(x,i-1,n,L);
end
```

```
%overlay: timoshenko beam with two layers.
%This function generates data for use in time dependent
%case.  The data is put in a matrix (N+1)x Nt.  N is the
%number of subintervals and M is the dimension of Ks.
%
function z=zdata(N,M,Ks,Nt,mfactor)
cone=0;
pen=1.0e+03;
Kn=1.0e+03;
force=2000;
f1=1;
f2=1;
%
%beam and material parameters
grav=32;
h1=9/2;
h2=3/2;
k=6;
L=37;
E1=3.63;
E2=2.9;
mu1=0.14;
mu2=0.12;
rho1=141/(12^3);
rho2=150/(12^3);
%
%
n=N+1;
m=4*n;
hs=L/N;
%
% set basic matrices for piecewise linear basis functions
%
g0=eye(n);
g1=zeros(n);
g2=eye(n);
g0(1,1)=2;
g0(n,n)=2;
g1(1,1)=-0.5;
g1(n,n)=0.5;
for i=2:n-1
   g0(i,i)=4;
   g2(i,i)=2;
end
for i=1:n-1
   g0(i,i+1)=1;
   g0(i+1,i)=1;
   g1(i,i+1)=-0.5;
   g1(i+1,i)=0.5;
   g2(i,i+1)=-1;
   g2(i+1,i)=-1;
end
g0=hs/6*g0;
g2=1/hs*g2;
%
%
a1=4*h1^3*k*rho1/3;
```

```
a2=4*h2^3*k*rho2/3;
b1=4*h1*k*rho1;
b2=4*h2*k*rho2;
%
Gmass=[ a1*g0   zeros(n) zeros(n) zeros(n)
       zeros(n)  b1*g0   zeros(n) zeros(n)
       zeros(n) zeros(n)  a2*g0   zeros(n)
       zeros(n) zeros(n) zeros(n)  b2*g0];
Gmass=mfactor*Gmass;
%
%
alpha1=(4*k*h1^3*E1)/(3*(1-mu1^2));
alpha2=(4*k*h2^3*E2)/(3*(1-mu2^2));
beta1=2*k*h1*E1/(1+mu1);
beta2=2*k*h2*E2/(1+mu2);
%
GA=[alpha1*g2+beta1*g0 beta1*g1' zeros(n)           zeros(n)
    beta1*g1           beta1*g2  zeros(n)           zeros(n)
    zeros(n)           zeros(n)  alpha2*g2+beta2*g0 beta2*g1'
    zeros(n)           zeros(n)  beta2*g1           beta2*g2];
%
%
for i=1:4
for j=1:m
   bL(i,j)=0;
   b0(i,j)=0;
end
end
%
e0=[0 0 0 0
    0 1 0 0
    0 0 0 0
    0 0 0 1];
%
bL(1,n)=1;
bL(2,2*n)=1;
bL(3,3*n)=1;
bL(4,4*n)=1;
b0(1,1)=1;
b0(2,n+1)=1;
b0(3,2*n+1)=1;
b0(4,3*n+1)=1;
BL=bL'*e0*bL;
B0=b0'*e0*b0;
%
% The is the stiffness matrix and the boundary condition.
%
Bdry=B0+BL;
G=GA+pen*Bdry;
%
% Interface conditions
%
BS=zeros(4);
BN=zeros(4);
BS(1,1)=h1^2;
BS(1,3)=h1*h2;
BS(3,1)=h2*h1;
```

```
BS(3,3)=h2^2;
BN(2,2)=1;
BN(2,4)=-1;
BN(4,2)=-1;
BN(4,4)=1;
%
%matrix that describes the penalization over the interface
between
% the beams
%
Gs=ela(Ks,n,N,M);
GI=kron(Kn*BN,g0)+kron(BS,Gs);
%
G=G+GI;
%
fb=[0 4*k*grav*rho1*h1 0 4*k*grav*rho2*h2];
%
for i=1:4
for j=1:m
   fbody0(i,j)=0;
end
end
%
for j=1:n
   fbody0(2,n+j)=1;
   fbody0(4,3*n+j)=1;
end
%
fbody0(2,n+1)=0.5;
fbody0(2,2*n)=0.5;
fbody0(4,3*n+1)=0.5;
fbody0(4,4*n)=0.5;
fbody0=hs*fbody0;
fbody=fb*fbody0;
%
for i=1:4
for j=1:m
   fapld0(i,j)=0;
end
end
for j=1:n
   fapld0(2,j+n)=f1*b(L/3,j-1,N+1,L)+f2*b(2*L/3,j-1,N+1,L);
end
%
fa=[0 2*k*force 0 0];
fapld=fa*fapld0;
f=fbody+fapld;
f=1.0e-06*f;
%
%We need to solve Gmass*c"(t) + G*c(t) = f.
%
dt=.2;
GM=(1/(dt)^2)*Gmass+(1/4)*G;
alphaM1=-.5*G+(2/(dt)^2)*Gmass;
alphaM2=-(.25*G+(1/(dt)^2)*Gmass);
ci=zeros(4*n,1);
cim1=zeros(4*n,1);
```

```
omega=pi;
for i=1:n
   x(i)=(i-1)*hs;
end
for it=0:1
   t(it+1)=it*dt;
   c=ci;
   for i=1:n
      cw(i)=c(i+n);
   end
%
%Compute w1(x).  This will be the data used in the
%inverse problem.
%
for j=1:n
    z(j,it+1)=sovrly(cw,x(j),n,L);
end
end
for it=1:Nt-1
   t(it+1)=(it+1)*dt;
%change applied force to depend on time
   fa=fa*sin(omega*t(it));
   fapld=fa*fapld0;
   f=fbody+fapld;
   f=1.0e-06*f;
%
%
   cip1=GM\(f'+alphaM1*ci+alphaM2*cim1);
   cim1=ci;
   ci=cip1;
   c=-cip1;
   for i=1:n
      cw(i)=c(i+n);
   end
%
%Compute w1(x).  This will be the data used in the
%inverse problem.
%
for j=1:n
    z(j,it+2)=sovrly(cw,x(j),n,L);
end
end
```

```
%fit-to-data functional for the overlay problem time dependent.
function valj=jovrlyt(dt,wl,z,n,Nt)
M=eye((Nt+1)*n);
for i=n+1:Nt*n
   M(i,i)=2;
end
valj=(dt/2)*(wl-z)*M*(wl-z)';
```