A THEORY FOR DESIGN

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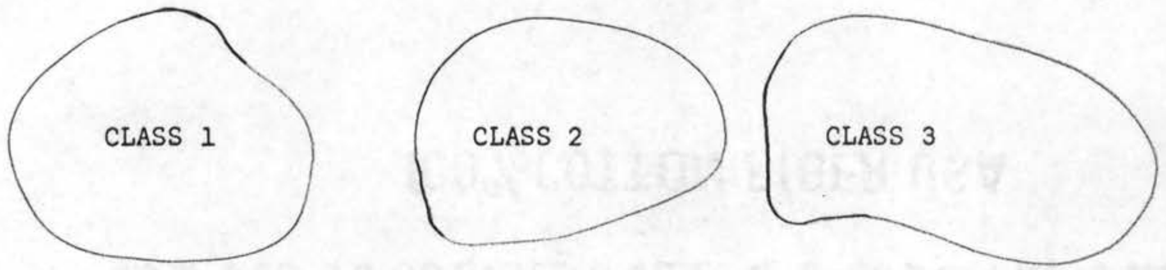
PREFACE

This thesis is the result of the author's search for a method to look at engineering design from an analytical point of view. That the engineer's primary job is design, seems to be commonly accepted. Also, it seems to be commonly agreed that there is no accepted general theory by which to study this process.

The author was working with Professors Charles F. Cameron and Daniel D. Lingelbach when a practical need for a general design procedure became necessary. The requirements of the problem indicated a need for general characterizations of quantities such as "physical system", "specification", and "solution". These quantities are a part of each design problem. In addition, these characterizations had to be quickly related to a particular mathematical model (e.g., several functions of n real variables). Yet, it was felt that the characterizations should not be associated with any particular mathematical model (e.g. linear equations, Boolean algebra models, statistical models, etc.). That is, a theory for design, relatively independent of the "state of the art", was desired.

The reason for the engineer not having a separate theory is probably imbedded in his constant association with ordered sets. The ordering of a set is an important phase of the scientific process illustrated below:

Assume the existence of the three distinct non-degenerate sets of elements as indicated.



Assume that the three classes are logically related in some manner. The scientific process is concerned with the problem of finding a relation between these classes based on some "observable" criteria.

An interesting assumption always seems to take place at the first stage of the above process. A common indexing set is chosen for the three distinct classes (i.e., the elements of each class are associated with some abstract set of "numbers"). This seems to be dictated by the "observable" criteria. This is where the order theory (relative value) seems to take control. If each of the above classes can be associated with a single set A which can be ordered, then a method of solving the problem is available (i.e., if the "observable" criteria is converted to some ordering scheme, a method for deductive prediction is available).

The point to be illustrated by the above is that the classes do not seem to prefer which indexing sets are chosen. They do seem to have a preference when an indexing set and "observable" criteria are selected.

This thesis leaves the indexing set arbitrary. Also, the indexing set is made disjoint from the indexed set. The separation of the abstract common indexing set from the classes of interest allows a

general foundation for looking at all design problems.

The author is indebted to the many who helped during his search for a formal way to look at the design problem of engineering. Many of the concepts of a theoretical system used in this thesis should be particularly vivid to John C. Paul who helped the author immeasurably. The encouragement and guidance of Professors Charles F. Cameron, Daniel D. Lingelbach, and William A. Blackwell to seek a theory for design helped make this thesis a reality. Also, the criticisms of Professor Jeanne L. Agnew greatly helped the author in the area of technical consistency and conceptual clarity.

To the author's wife, Rose, and sons, Charles, Bryan, and James, this thesis is dedicated.

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CHAPTER I

INTRODUCTION

The design problems of the engineer are increasing in complexity at a rapid rate. The common term associated with this trend is that of the "system". Most of the engineering problems are now looked at from a systems point of view. (1). This being the case, there have been attempts to generalize, in order to have a single abstract representation for all systems. (2). If this could be done, then a classification process, distinguishing the typical cases would be possible.

Some of the more notable attempts to obtain general approaches to systems were presented in the last decade. These were the generalized energy equations and the linear graph theory characterizations of systems. (3, 4). A strong trend at present is to look at systems from a statistical point of view.

Exactly when the generalized system approach got started, the author cannot say. However, it seems to be sometime in the late 1940's. The reason for this trend is fairly clear. Until this "modern trend" in engineering was initiated, the engineer learned to design by studying particular systems (e.g., a motor). When the number of particular systems to be designed became large, it was no longer feasible to use this method. Hence, the search for a way to look at many systems was launched.

This thesis is a result of looking at the original problem--that of finding a more efficient way to design systems than the "case" approach. Once the author became interested in this problem, it was clear that many people were working in this area. (5). A particular case is noted due to its "different" approach. A book entitled Systems Philosophy was published in 1962. (5). This book, which used terminology of modern mathematics, defined a system symbolically. In addition, the design problem was defined symbolically. The actual definitions are of no importance here, but the fact that a symbolic (mathematical) definition was given is thought to be noteworthy.

The fact that there are few general definitions of systems might seem questionable in view of what has previously been said. However, the only ones of interest and considered important are those which are symbolic definitions involving the tools of the engineer. That is, a system which is unrelated to the tools of the engineer is unlikely to yield positive results.

This problem is approached from a rather fundamental point of view. Generally speaking, a system is thought of as a "real" or "abstract" set of interrelated objects. In either case there is the problem of defining a system.

It appears to be commonly accepted in engineering that an "abstract" system is a "mathematical model" which represents a "real" system. Also, design is usually associated with the "real" system. Hence, in this thesis the concern is towards the "real" system rather than the "abstract" or "mathematical model". However, this is done from the point of view that every "real" system will be represented by some "mathematical model". Thus a "real" system formulation of the design problem must be directly related to available theories of mathematics. (These are the tools referred to earlier.)

To accomplish the above with an established theory of mathematics would be desirable. However, the only general theory of mathematics is that of set theory. (6, 7). This approach to the general design problem has been attempted. (8). Unfortunately, the generality of set theory seems to be both its power and its weakness. The set notation is used extensively in this thesis. However, just the ability to formally abstract a set of elements is not considered powerful enough to gain a useful design theory.

From set theory the particular areas of mathematics are developed and can be classified as the "number theories" and the "operational theories". "Number theories" refer to the construction and classification of the types of "counting" theories (e.g., integers (I), reals (R), complex (C), etc.). These "numbers" can all be thought of as indexing sets. (For example, to use the properties of a number theory, a set of objects must be indexed by these numbers.) In this thesis the terminology of the indexing set is used to imply that some number theory is available.

The "operational theories" are the investigations and classifications of operations. (For example, $f(x, y) = z$ denotes some type of abstract operation involving the elements x and y.) The major concern is to insure that the operations have meaning. (9). Also the operations in a particular number system are made to conform to the properties of that number system. (For example, $a \cdot b = c \rightarrow$ if a, b are reals, c is real and given by the natural order of reals.) This does not mean that other operations cannot be defined on the reals, just that they must conform to the properties of the reals. The ability to make the many different operations, which can be defined on a set of elements, conform to the ones already defined is a primary concern in mathematics. Without this logical consistency, the formulas (logically consistent operations) used in the applied sciences would be much less powerful. It is only because these formulas are definite that the approach used in this thesis can be applied. Also, it is the modern approaches to discussing mathematical relations which have discarded the formula as used in this thesis. However, from a practical (applied) point of view the formula offers a conceptual advantage over the more abstract approach mentioned above.

The formula, as seen and used by the engineer is a "relation" among disjoint sets. (For example, E = IR denotes a relation among the disjoint classes E, I, and R.) However, to state any particular relation in a precise manner, the disjoint property is abandoned for the "well defined" operations of some common indexing set. This process would be acceptable if a single indexing set could be used for all "relations" of interest. However, there is an increased application of finite indexing sets and transform methods (i.e., change of indexing set such as the Laplace transform commonly used in engineering). Thus, a practical system for the engineer should not involve a particular indexing set (e.g., reals, complex, etc.).

This thesis develops a general system by reversing the process indicated earlier. That is, instead of looking at a "real" system by some particular indexing set, the freedom to "project" these disjoint sets onto any indexing set is reserved. This approach allows the properties of all "real" physical systems to be formulated "independent" of the indexing set which will be used in a particular mathematical model.

The system, as defined in this thesis, can be thought of as a theory. These theories are based on a set of axioms called "unordered relations". These "unordered relations" are the result of having definite formulas on elements from disjoint sets. (6). These formulas are assumed because of the well defined mathematical operations discussed earlier.

The "unordered relation" used in this thesis is obtained by removing the ordered set restraints of the ordered relation theory used in mathematics. (6). In discussing functions of more than two variables, this formalism starts to become burdensome. This is illustrated as follows:

Consider f $(x_1, x_2, x_3) = x_4$ a real valued function of three real variables. The fact that it is a function implies that for each x_1, x_2, x_3, each a real number, there is a unique x_4, a real number. Hence, the above symbolizes a method for taking three real numbers and looking at how these are related to a fourth. However, the above is defined in terms of the dyadic relation concept of ordered pairs. The fact that there were three real numbers used instead of one was immaterial in the above idea of a function. If, in the above function, x_1 in terms of x_2, x_3, and x_4 was of interest, a new discussion involving a binary relation g $(x_2, x_3, x_4) = x_1$ would be needed. Conditions for obtaining g in terms of the properties of f are discussed in most mathematical analysis courses. (10).

This illustrates a problem associated with trying to use the standard methods already developed. Conceptually, it is clear that several functions of n variables is "just" an extension of the ordered pair concept. However, this is seldom studied on a formal basis other than the concept of $R_1 \times R_2 \times R_3 \ldots R_n$ representing the n-dimensional product space of ordered n-tuples $(r_1, r_2, \ldots r_n)$. It is clear that to define quantities formally as done for the binary relations (e.g., relations, mappings, converse relations, etc.) would be a tremendous job. This becomes quite evident when it is considered that the number of distinct orderings is n!. Hence, most of the n variable work is done in terms of the ordered pair concepts.

Formally, the ordered pair is not necessary when classifying relations. (6). This can be replaced by a "definite" formula. (The Appendix gives a more formal discussion of the ordered pair.) However, ordered pairs can be well defined in terms of a set of elements. (6). Thus, to relate a particular set of elements the ordered pair has been adopted in works dealing with the foundations of mathematics.

To indicate the connection between the ordered and unordered relations, Chapter II develops the fundamental ideas of the unordered relation starting with the ordered n-tuple. The disjoint sets on which the unordered relations are defined are called parameters. The unordered relation definitions parallel the ordered relation ideas where possible. The definitions are slanted towards systems (Chapter III) and the design theory of Chapter IV. The idea of a parameter model is introduced. These models have the basic properties of the systems which are used in engineering design. The results of Chapter II are used in Chapter III to develop the general system as viewed by this thesis.

The system has a simple interpretation in terms of the terminology constructed in Chapter II. Basically, each system is a theory defined on n parameters (logic classes) in v "parts". The "parts" refer to the number of unordered relations which are involved in postulating a system. Each system can be viewed as a collection of unordered relations which has the potential of containing logical information. A piece of logical information can be viewed in an order if "projected" into the ordering structure of its indexing set. In fact, the "projection" of a system into an indexing set returns the system concept of this thesis to the ordinary functional concepts of ordered relations. This seems to be a practical approach in engineering. This gives an engineering design theory which is relatively independent of the "state of the art". (This does not mean that design is independent of the "state of the art". On the contrary, any particular design solution is in terms of the "state of the art".) This design theory is able to look at all of these "state of art" models.

Chapter IV makes use of the two preceding chapters in developing a theory for design. The contents of this chapter are based on the "tools" developed in Chapters II and III. There is one fundamental axiom which is used in this chapter. This axiom is called the real system axiom. The axiom assumes the existence of a particular type of system. Involved in this axiom is a uniqueness between systems relative to the "logical information" of a system. This approach allows the design theory to be applicable to the "best" available models.

The final chapter indicates some of the areas of application of a design theory as developed in this thesis. In developing a basis for the design theory, many areas of application looked promising. The scope of the foundation work did not permit the author to give detailed examples. Three major applications which look promising are discussed in the final chapter. Throughout the thesis discussions concerning possible interpretations of the material are given. Also several examples are given which directly indicate areas of application.

CHAPTER II

PARAMETERS AND UNORDERED RELATIONS

The discussion of Chapter I indicated that the basic ingredients of a system, as viewed by this thesis, are the parameter and the relation. This chapter is devoted to the definition of these and related concepts. Motivation for many of the definitions and facts presented in this chapter stems from ideas involved in engineering design. For this reason, some of the material in this chapter may seem unmotivated. When possible, reference is made to the area where the concept is to be used.

In conjunction with this, and later chapters, an Appendix has been included covering the basic mathematical concepts repeatedly utilized. The discussion of binary relations presented in the Appendix is somewhat more extensive than the coverage of set theory concepts. The reason for this is that the general concept of a relation, as given in this chapter, is basic to that which follows. The discussion of the binary relation in the Appendix is slanted towards the approach taken in this thesis with unordered relations.

Throughout the remainder of this thesis, the parameter is used to denote an indexed set with at least two elements called scalars. In engineering a quantity with these properties goes by the same name.

D2-1: A set P is called a parameter if and only if

(1) There is a set $A \ni A \cap P = \phi$ and $\forall\, a \,\varepsilon\, A$

$\exists$ a unique $P_a \varepsilon\, P$ and conversely,

(2) A has at least two distinct elements.

The set A which is equivalent to P (1-1 mapping exists) is called the indexing set of P. An element of P is called a scalar and denoted by p_a where (a) denotes the element of A which corresponds to (indexes) the element $p_a \varepsilon\, P$. For convenience, and unless confusion might arise, the indexing subscript will be dropped and $p \varepsilon P$ will be used to denote a scalar of P. The value of a scalar p is the element a of A which indexes p.

Defining a parameter as in D2-1 might seem unusual and unnecessary. Yet, when dealing with abstract quantities such as voltage (E), current (I), and gain (K), it is important not to confuse the scalars of (E) with those of (I), etc. This is so, even though the same indexing set, usually the real numbers, is used for each parameter. In other words, if E and I were said to be the same as their indexing sets, then E = I in a logical sense. But not only is $E \neq I$ in a logical sense, they can be (and are) considered disjoint (no elements equal) in a set sense, by letting them be parameters (i.e., $P_1 \cap P_2 = \phi$ can follow from D2-1 even though the indexing sets for P_1 and P_2 are the same).

It is noted that since P and A are equivalent and A has at least two elements, a parameter has at least two scalars. Hence, every parameter has a proper subset. In connection with the set notation in this thesis, the definitions of DA-3 and DA-5 of the Appendix are used.

D2-2: $\overline{P}$ denotes a proper subset of a parameter P.

It is also noted that if a parameter P exists, then a parameter A exists which is disjoint from P. This follows from (1) of D2-1.

Throughout the remainder of this thesis, parameters with different subscripts (e.g., P_1, P_2, ...) will be considered disjoint unless otherwise stated. This does not mean their indexing sets are disjoint. On the contrary, their indexing sets will usually be the same. Also, capital letters will be used for sets and lower case for elements.

In order to relate a set of parameters, the scalar n-tuple is defined.

D2-3: Given the parameters P_1, P_2, ... P_n, $n > 0$, in the order P_{1i}, P_{2i}, ... P_{ni}, the set $\pi_{ni}\ (p) = (p_{1i}, p_{2i}, \dots p_{ni})$, where $p_{1i} \varepsilon P_{1i} \dots p_{ni} \varepsilon P_{ni}$ is called a scalar n-tuple of order (i). ($P_{1i} \dots P_{ni}$ are not necessarily disjoint.)

In D2-3 the subscripts refer to the parameters to which the scalars belong and have nothing to do with their indexing sets. Two n-tuples are considered the same $\pi_{ni}\ p = \pi_{nj}\ p'$ if and only if $p_{1i} = p'_{1j}$, $p_{2i} = p'_{2j} \dots p_{ni} = p'_{nj}$. That is, scalars in identical "slots" must be identical even if they come from different parameters. Hence, if the parameters P_1, ... P_n are disjoint (each disjoint with the rest), there are n! distinct n-tuples for the n scalars p_1, p_2 ... p_n. Each of the distinct n-tuples corresponds to an "order" (i). The n-tuple is called an ordered pair in the case $n = 2$. (6). It is noted that another approach is available which uses ordered sets to define the n-tuple. (6).

The n-tuple gives a precise way of denoting a "relation" between a set of parameters. This is most conveniently done by using the product set for the order (i).

D2-4: The totality of scalar n-tuples, denoted by $\Pi_{ni}\, P$ representing the order P_{1i}, P_{2i} ... P_{ni}, is called the scalar product set (Cartesian product) for the order (i). (P's not necessarily disjoint.) Sometimes $\Pi_{ni}\, P$ is denoted by $P_1 \times P_2 \ldots \times P_n$.

The product set is used to define the graph of a relation. As discussed in the Appendix, there is, conceptually at least, a difference between a relation and the graph of a relation. In this work it is more convenient to consider the relation as the "formula" ϕ which generates a graph. In the works dealing with foundations of mathematics, it is the graph which is called the relation. (6).

The approach taken here is slanted towards the application of mathematical formulas rather than precise methods for constructing useful formulas. However, the connection between the formula and its graph is necessary in order to have a "precise" meaning of a formula.

A formula is a logical concept used to separate objects. The logical properties all formulas must have is that of being "definite". (6). That is, let x be the set for which a formula ϕ can be applied. Then for each x either ϕ (x) is true or ϕ (x) is false, but not both. The set $\bar{x}$ for which ϕ (x) is true is called the graph of the formula ϕ. From the above it is clear why a formula cannot be separated from its graph. Also it shows the conceptual difference between the two.

D2-5: A set $R_i\,(P_n)$, $n > 0$ is said to be a n parameter ordered graph of the relation ϕ on the set $\{P_n\}$ if

$$R_i^{\circ}(P_n) = \{x/x \in \Pi_{ni} P \text{ and } \phi(x) \text{ is true}\}.$$

The formula ϕ is not allowed to involve the set $\bar{x}$. (6).

From D2-5 the graph of every relation is a subset of a product set. For consistency with set theory, the one parameter graphs ($n = 1$), which are possible by D2-5, denote particular subsets of $\{P_1\}$ which are specified by ϕ. (That is, from set theory the only relation p_1 can have with P_1 is "$p_1 \in P_1$"; and a collection of elements, each belonging to the same set, is called a subset DA-3.) Note that if $\phi(x) \leftrightarrow x \in \emptyset$ is used for the relation, then its graph is the empty set.

The theory of dyadic relations ($n = 2$) is that commonly found in the foundations work of mathematics. Polyadic relations are less common for reasons indicated in Chapter I. Also, they are theoretically obtainable from dyadic relations. (6). As mentioned earlier, the common approach to the relation is to let each distinct subset of a product set be a relation. The formula ϕ is used to denote a relation in this work.

One advantage of this approach is that the "useful" formulas are independent of the order of the graphs they produce. That is, the formulas, at least in a practical sense, usually produce more than one graph. This is illustrated in the following example:

Consider the parameters P_1 and P_2 which are indexed by the real numbers (i.e., $x \in R$ $p_{1x} \in P_1$, $p_{1x'} = p_{1x} \leftrightarrow x = x'$) and $P_1 \cap P_2 = \emptyset$.

Consider the formula: $\phi[(P_{1x}, P_{2y})] \leftrightarrow x_{P1} = {y_{P2}}^2$ and the graph of ϕ relative to $P_1 \times P_2$ and $P_2 \times P_1$.

$$R_1\ (P_n) = \{x/x\ \varepsilon\ P_1 \times P_2 \text{ and } \phi\ (x)\}$$

$$R_2\ (P_n) = \{x/x\ \varepsilon\ P_2 \times P_1 \text{ and } \phi\ (x)\}$$

From D2-3 and the fact that $P_1 \bigcap P_2 = \emptyset$ the above graphs are completely disjoint, yet represent the same relation ϕ, between the parameters P_1 and P_2.

In the theory of dyadic relations, R_1 and R_2 would be converse to each other and equal only under special conditions. One obvious necessary condition is that $P_1 \bigcap P_2 \neq \emptyset$, excluding the empty relation. It is noted that the scalar values of P_{1x} and P_{2y} are equal for the scalars P_{11} and P_{21}. This illustrates the distinction which has been made between the scalars and their scalar values. The formulas relating the scalars are usually given in terms of the indexing set (scalar values). Yet, there is no need to consider a parameter to have a unique indexing set. Indeed, many transformations used in the applied sciences are performed on indexing sets, not the parameters themselves (e.g., the Laplace transformation changes the indexing set of linear voltages from the real numbers to the complex numbers. However, voltage itself can be thought of as a parameter independent of the indexing set).

This is a very convenient way of looking at quantities involved in engineering design. For example, it might be more convenient to think of voltages as parameters indexed with a set having only two elements. These voltages might then be related by the rules of Boolean algebra rather than ordinary algebra.

To remove formally the order from an n-tuple the following classification is convenient:

D2-6: A relation ϕ on the set of parameters $\{P_n\}$ $n > 1$ is said to be consistent if and only if

(1) $\forall$ order of Π_{ni} $P \exists$ a graph R_i (P_n) of ϕ.

(2) $\forall$ $i \neq j$, R_i $(P_n) = R_{j'}$ (P_n) where j' represents an order permutation performed on the n-tuples of R_j (P_n) to obtain the n-tuples of $R_{j'}(P_n)$, and order (i) = order (j').

(P's not necessarily disjoint.)

D2-6 in effect says a relation is consistent if the graphs are equal when brought into corresponding order. This says that if the parameters are disjoint, then a consistent relation ϕ is not dependent on its graph's order. This definition conforms with the logical restriction for a consistent formula.

D2-7: A relation ϕ is said to be independent of order if

$$\forall\, x \in \bigcup_{i=1}^{n!} \Pi_{ni}\, P \ni \phi\,(x)$$

then ϕ (y) if $y = x'$. Here the prime denotes an interchange of order to give the n-tuple x the same order as the n-tuple y.

T2-1: A consistent relation ϕ_c on the parameters $\{P_n\}$ is independent of order if the P's are disjoint.

Proof: From D2-3, D2-6, and D2-7, the above follows.

The above theorem indicates that only the unordered sets $\{p_1, \ldots p_n\}$ need be considered in ϕ_c when the parameters $\{P_n\}$ are disjoint. This, of course, was the purpose of indexing a collection of disjoint sets. The indexed set can be any of the well-known counting sets (e.g., R, C, I, etc.). The formulas can (as is common) be defined in terms of these indexing sets using well-defined operations.

Since the operations used in most formulas are required to be consistent and these operations are classified in algebraic studies of mathematics, it seems appropriate to call consistent relations, algebraic. Also, it is required that the same set be used when discussing algebraic operations. Hence, for consistent relations on parameters indexed by the same set, the following classification seems appropriate.

D2-8: Let ϕ be a consistent relation on a set of parameters $\{P_n\}$. Let $\{A_n\}$ be the indexing sets corresponding to $\{P_n\}$. Then if for some suitable ordering $A_1, \supsetneq A_2, \ldots \supsetneq A_n,$ ϕ is said to be algebraic.

The importance of algebraic relations cannot be overemphasized. It is the common indexing sets which allow a relation to have a definite "value" meaning (i.e., unless a common base [indexing set] can be found, such that each parameter of concern can be indexed by this set, a definite "value" relation seems impossible). The property of D2-8 is given to the relations considered in this thesis.

As mentioned earlier, the main purpose for working with parameters is to remove the ordered set problems associated with classical relation theory. From T2-1 it is natural to consider the unordered graph set R (P_n) which can be associated with any ϕ_c on $\{P_n\}$. (From now on, every relation will be consistent unless specified, and ϕ will be used in place of ϕ_c.)

D2-9: For the relation ϕ, the set $R(P_n)$ is called the unordered graph of ϕ if and only if

$$R(P_n) = \{x/x \subseteq \bigcup_{i=1}^{n} P_i \text{ and } x = \{p_1 \in P_1, \ldots p_n \in P_n\}$$

and $\forall$ ordering (i) of x, $\phi(x)\}$.

An element of $R(P_n)$ will be referred to as an unordered n-tuple $\pi_n p$[1]. The graph $R(P_n) = \Pi_n P$ corresponding to, $\phi(x) \rightarrow x \in \Pi_{ni}\, P$ represents the totality of distinct unordered n-tuples which can be formed from n disjoint parameters.

From D2-9, and the above remarks, each element of $R(P_n)$ contains those and just those scalars which satisfy ϕ. Also, each element of $R(P_n)$ has one and only one scalar from each parameter of $\{P_n\}$. The next classification is to distinguish the unordered projections which can be formed from $R(P_n)$. Here $\{P_{n'}\}$ will be used to denote a subset of $\{P_n\}$.

D2-10: Given the relation ϕ on $\{P_n\}$ with graph $R(P_n)$, the set $R_n(P_{n'})$ is called the unordered projection relative to ϕ, if and only if

$$R_n(P_{n'}) = \{x/x \in \Pi_{n'} P \text{ and } \exists y \in R(P_n) \text{ of}$$

$$\phi \ni x \cap y = x\}.$$

T2-2: (1) $R_n(P_{n'}) = R_n(P_{v'})$ if and only if $\{P_{n'}\} = \{P_{v'}\}$.

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[1]The word "unordered" will be attached only when confusion might arise between the ordinary concept of graph and the unordered graph.

(2) $R_n (P_n) = R (P_n)$.

Both (1) and (2) follow from D2-9, D2-10, and Π_n P. The justification for calling $R_n (P_{n'})$ a projection relative to ϕ is seen by considering the relation ϕ_s on the disjoint set $\{P_{n'}\}$ as follows:

$R_n (P_{n'}) = \{x/x \in \Pi_n, P \text{ and } \phi_s (x)$

where $\phi_s (x) \leftrightarrow \exists \phi (y) \ni y \in R (P_n) \text{ and } x \cap y = x$.

Using this, if ϕ is known, from the unordered graphs of ϕ the unordered graph $R_n' (P_{n'})$ of ϕ_s can be constructed. This graph is the same as that obtained by using the unordered graph of ϕ and D2-10. Also note that ϕ_s must be consistent, algebraic, and independent if ϕ has these properties. From (1) of T2-2 there are 2^n distinct projections.

T2-3: Given the disjoint set $\{P_n\}$ and the n parameter relation ϕ, there are exactly 2^n unordered projections relative to ϕ. (This includes no parameters as the $\emptyset$ graph.)

It is noted that a projection, as given in D2-10, is not the same thing as the subrelations which are used to denote subsets of graph sets (i.e., $R' (P_n) \subseteq R (P_n)$ is normally called a subrelation). The projections of ϕ as given here are the quantities which denote "specifications" in engineering design. Also, the complement of a projection set can be thought of as the "solution" set. These ideas are made clearer in discussing the following question:

Given ϕ on $\{P_n\} \ni R (P_n) \neq \emptyset$ and the set

$R_p\ (P_{n'}) \subseteq \Pi_{n'}\ P \ni \{P_{n'}\} \subseteq \{P_n\}$. Is there a set $R_s\ (P_{\bar{n}'}) \subseteq \Pi_{\bar{n}'}\ P \ni \forall\ x\ \varepsilon\ R_p\ (P_{n'})\ \exists$ $y\ \varepsilon\ R_s\ (P_{\bar{n}'}) \ni x \cup y\ \varepsilon\ R\ (P_n)$?

The above question indicates the need for several conventions before discussing an answer. These are noted in D2-11 and T2-4.

D2-11: $\{P_{\bar{n}'}\}$ will denote the complement of the subset $\{P_{n'}\}$ of $\{P_n\}$.

Using D2-11 and the definition of $\Pi_n\ P$ gives

T2-4: (1) $(x\ \varepsilon\ \Pi_{n'}\ P) \cap (y\ \varepsilon\ \Pi_{\bar{n}'}\ P) = \emptyset$.

(2) $(x\ \varepsilon\ \Pi_{n'}\ P) \cup (y\ \varepsilon\ \Pi_{\bar{n}'}\ P) = z\ \varepsilon\ \Pi_n\ P$.

The above question is partly answered in the next theorem.

T2-5: Given ϕ on $\{P_n\} \ni R\ (P_n) \neq \emptyset$ and $x\ \varepsilon\ \Pi_{n'}\ P$,

then (1) $\nexists\ y\ \varepsilon\ \Pi_{\bar{n}'}\ P \ni x \cup y\ \varepsilon\ R\ (P_n)$ if $\{P_{n'}\} \cap \{P_n\} \neq \{P_{n'}\}$.

(2) $x\ \varepsilon\ R_n\ (P_{n'}) \rightarrow \exists\ y\ \varepsilon\ R_n\ (P_{\bar{n}'}) \ni x \cup y\ \varepsilon\ R\ (P_n)$.

(3) $\{P_{n'}\} \subset \{P_n\}$, $R_n\ (P_{n'}) = \Pi_{n'}\ P \rightarrow \exists\ y\ \varepsilon\ \Pi_{\bar{n}'}\ P \ni$ $x \cup y\ \varepsilon\ R\ (P_n)$, $x\ \varepsilon\ \Pi_{n'}\ P$.

Proof: (1) All that (1) states is that the "specification must involve parameters for which ϕ is defined.

(2) This follows from D2-10 and D2-11. Note that this result holds even when n = n'. That is, it certainly is possible to specify an element

of ΠP_n which lies in $R(P_n)$. The "chances" of doing this are usually considered small. (However, a notable exception is when $R(P_n) = \Pi P_n$.)

(3) From (2) there is a contradiction if it is assumed $\nexists\ y \in \Pi_{\bar{n}'} P \ni x \cup y \in R(P_n)$ given $x \in \Pi_{n'} P$.

The above conditions are formal facts which are usually taken for granted when working with the more common algebraic equations. For most design problems where the "system" is assumed, it is convenient to assume (3) when seeking "compatible specifications". These are discussed in Chapter IV.

The previous classifications have involved looking at relations (projections) generated from "higher dimensional" relations. The classifications so far are not unlike those discussed in an ordinary algebraic expression of n variables.

The real advantage of the unordered relation approach is in looking at "higher dimensional" relations "generated" from two or more "lower dimensional" relations. These are the situations which are common in engineering design.

The projections which have been studied are not defined in a manner which, in general, allows the original relation to be deduced.

The fact that a proper projection does not characterize $R(P_n)$ is indicated by T2-5 (3) and can be seen from D2-10 as follows:

Let $\{P_{n'}\} \subset \{P_n\}$. Then $R_n(P_{n'})$ is called a proper projection of $R(P_n)$. Let $x_1 = x_2 \in R_n(P_{n'})$. Then if this does not indicate $y_1 = y_2 \in R\ (P_n)$, $x_1 \subsetneq y_1$, $x_2 \subseteq y_2$, $R_n\ (P_{n'})$ cannot completely characterize $R\ (P_n)$. From D2-10 $x_1 \cap y_1 = x_1$ and $x_2 \cap y_2 = x_2 \rightarrow x_1 \cap y_1 = x_1 \cap y_2$, but does not imply $y_1 = y_2$.

The above indicates that the graph $R\ (P_n)$ cannot be represented directly by the projections. The following classification involves a "weak" connection of two arbitrary relations which hints of a method for generating "higher dimensional" relations.

D2-12: Two relations ϕ_1 and ϕ_2 are said to be connected if and only if $\exists\ \{P_{n',1}\} \subsetneq \{P_{n1}\}$ and $\{P_{n',2}\} \subseteq \{P_{n2}\}$ $\ni R_{n1}\ (P_{n',1}) \cap R_{n2}\ (P_{n',2}) \neq \emptyset$.

The conditions placed on two relations for them to be connected are very weak. In fact, from a practical point of view, the above is equivalent to:

If $\{P_{n',1}\} \cap \{P_{n',2}\} \neq \emptyset \rightarrow \phi_1$ and ϕ_2 are connected. The notation $\phi_1\ c\ \phi_2$ will be used to symbolize that ϕ_1 and ϕ_2 are connected.

The above condition suggests an unordered relation graph which can be constructed in the space $\Pi_n\ P$, where $\{P_n\} = \{P_{n1}\} \cup \{P_{n2}\}$. Namely, a graph consisting of those scalars of $\Pi_n\ P$ which are the union of a scalar of $R\ (P_{n1})$ and one from $R\ (P_{n2})$. Now this graph is well defined, since $x \in \Pi_n\ P$ has one and only one element from each of the parameters of $\{P_n\}$. The cases when $\{P_{n1}\} \cap \{P_{n2}\} = \emptyset$ are of no interest since from D2-12 there is no connection between ϕ_1 and ϕ_2. However, even if $\phi_1\ c\ \phi_2$, the suggested graph might be empty.

D2-13: Assume a collection $\{\phi_i\}$ on the parameter sets $\{P_{ni}\}$. Let $\{\phi_j\} \subset \{\phi_i\}$ have the parameter sets $\{P_{nj}\}$. Let $\{\phi_{\bar{j}}\} = \{\phi_i\} - \{\phi_j\}$, and $\{P_{n\bar{j}}\}$ denote the parameter sets of $\{\phi_{\bar{j}}\}$. If there is no proper subset $\{\phi_j\} \ni \left(\bigcup \{P_{nj}\}\right) \bigcap \left(\bigcup \{P_{n\bar{j}}\}\right) = \emptyset$. Then the set $\Pi_n\ P/\{\phi_i\} = \{x/x \in \Pi_n\ P \text{ and } x = \bigcup y_i \text{ and } y_i \in R\ (P_{ni})\}$ are called the <u>natural points</u> of $\{\phi_i\}$. If there is a proper subset such that $\left(\bigcup \{P_{nj}\}\right) \bigcap \left(\bigcup \{P_{n\bar{j}}\}\right) = \emptyset$, then $\Pi_n\ P/\{\phi_i\}$ is not defined.

The idea of the natural point is the generalized concept of graph when dealing with "systems". In terms of one relation, the natural points are simply the graph $R\ (P_n)$. The set of natural points, for any arbitrary collection of relations, is the main concern of the engineer in the area of design. That is, given an abstract model which has natural points, what happens to the number of natural points when this "mathematical model" is combined with an arbitrary collection of relations? Also, how "deductive" is a given "model"? (In Chapter III these are called physical systems.) Facts concerning both of these questions are presented in the next theorem.

T2-6: (1) $\{\phi_i\}$ $i = 1, 2, \ldots V$ has a natural point only if $\phi_i \subset \phi_j \rightarrow \exists\, y_i \in R\ (P_{ni})$ and $y_j \in R\ (P_{nj}) \ni y_i \bigcap y_j \in R_{ni}\ (P_{nij})$ and $R_{nj}\ (P_{nij})$. Here $\{P_{nij}\} = \{P_{ni}\} \bigcap \{P_{nj}\}$.

(2) If $\{\phi_i\}$, $i = 1, 2, \ldots V$ are the relations

$\phi_i(x) \rightarrow x \in \Pi_{ni} P$ for the graphs $\Pi_{n1} P \ldots \Pi_{nv} P$, and $\{\phi_i\}$ has one natural point, then $\Pi_n P/\{\phi_i\} = \Pi_n P$, $P_n = \bigcup_{i=1}^{v} \{P_{ni}\}$.

Proof: (1) Assume $x \in \Pi_n P/\{\phi_i\}$ and $\nexists\, y_i$ and y_j of $R(P_{ni})$ and $R(P_{nj}) \ni y_i \cap y_j \notin R_{ni}(P_{nij})$ when $\phi_i \not\subset \phi_j$. Hence $y_j \cup y_j \not\subseteq x \in \Pi_n P$. Hence there is no x which can satisfy D2-13.

(2) Let $x \in \Pi_n P$ and assume $x \notin \Pi_n P/\{\phi_i\}$. However $\exists\, y_1 \in \Pi_{n1} P$, $y_2 \in \Pi_{n2} P \ldots y_v \in \Pi_{nv} P \ni x \cap y_i = y_i$, $i = 1, 2, \ldots v$, by definition of $\Pi_n P$. Hence choose these y_i and apply D2-13 to obtain $x \in \Pi_n P/\{\phi_i\}$ which contradicts the above. Hence $x \in \Pi_n P \rightarrow x \in \Pi_n P/\{\phi_i\}$. The converse follows from D2-13.

T2-6 gives a simple, necessary condition for a natural point. Also a trivial, but interesting from the design point of view, sufficient condition is stated.

The next classification distinguishes relations which have a "stronger" connection than just being connected. First, the relations $\{\phi_i\}$ for which $\Pi_n P/\{\phi_i\} \neq \emptyset$ are called consistent relations. This classification follows from the "consistent" equations studied in any of the operational settings of mathematics.

D2-14: The relations $\{\phi_i\}$ are said to be consistent if and only if $\{\phi_i\}$ has a natural point.

D2-15: Assume the relations $\{\phi_i\}$ with graphs $R\ (P_{ni}) \ni$ if $\{P_{nij}\} = \{P_{ni}\} \bigcap \{P_{nj}\} \neq \emptyset$. Then either $R_{ni}\ (P_{nij}) \subsetneq R_{nj}\ (P_{nij})$ or the reverse.

A consistent set $\{\phi_i\}$ with the above properties is said to be naturally connected. Any two which are naturally connected will be denoted by $(\phi_i \text{ nc } \phi_j)$ where i before j indicates the above inclusion.

The above properties are some of the more general restrictive properties of a useful "model". These properties are present in all the "physical systems" as classified in Chapter III.

The conditions placed on the relations in D2-15 are still quite general from a "consistent model" point of view. That is, abstract models, generated for the purpose of paralleling "natural phenomena" seem to have an inherent consistency. It is a useful characterization of this consistency which is being sought. The following theorem indicates some of the properties possessed by naturally connected sets.

T2-7: (1) If $\{P_{nij}\} = \{P_{ni}\} = \{P_{nj}\}$ and $(\phi_i \text{ nc } \phi_j)$, then $R\ (P_{ni})_i \subsetneq R\ (P_{nj})_j$.

Proof: (1) From D2-15 $(\phi_i \text{ nc } \phi_i) \rightarrow R_{ni}\ (P_{nij}) \subsetneq R_{nj}\ (P_{nij})$.
From T2-3 $R_{ni}\ (P_{ni}) = R\ (P_{ni})$.

Note that the conditions of T2-7 result in the possibilities for projections from "higher dimensional" relations being contained in a "lower dimensional" relation. This, in general, is not the case. The common situation is when the reverse prevails. Even more common is the conditions of the following theorem.

T2-8: Given $\{\phi_i\}$ $i = 1, 2, \ldots v$ with $\{R(P_{ni})\}$. Assume $\{P_{nik}\}$ has exactly one parameter for each $i \neq k$ and $\bigcup_{i \neq k} \{P_{ni}\} \subset \{P_{nk}\}$. Let ϕ_k be an onto relation relative to $\bigcup_{i \neq k} \{P_{ni}\}$ (i.e., $R_{nk}(P_{nk'}) = \Pi_{nk'} P$ where $P_{nk'} = \bigcup_{i \neq k} P_{ni}$). Then ($\phi_i$ nc ϕ_k) and $\forall y_i \in R(P_{ni})$ $i \neq k$ $\exists x \in \Pi_n P/\{\phi_i\} \ni x = (\bigcup_{i \neq k} y_i) \cup y_k$ where $y_k \in R_{nk}(P_{\overline{nk'}})$.

Proof: The fact that $\{P_{nk'}\} \cap \{P_{nk}\} \neq \emptyset$, combined with the fact that $R_{nk}(P_{nik}) = P_{ni}$ when $i \neq k$, gives (ϕ_i nc ϕ_k). For the second part the fact that ϕ_k is onto relative to $\{P_{nk'}\}$ gives the desired result.

The situation outlined in T2-8 is used in connection with "compatible specifications", which are related to design "solutions". These are discussed in Chapter IV. When thinking of ordinary "functions" of n variables (i.e., $y = f(x_1 \ldots x_n)$, the above says you can choose $x_1, \ldots x_n$ so there is $\overline{y} \subseteq \overline{Y}$ such that the above equation is satisfied. Hence, T2-8 shows that single formula problems can be thought of as a function of n variable problem or an n + 1 relation problem. In fact the situation of T2-8 is usually thought of as an n variable problem.

However, the use of the natural point leads to simpler classifications in general. One of the important concepts involved in "simultaneous equations" is that of independence and dependence. Conditions for these properties are usually expressed in terms of the algebraic operations associated with the variables of interest (e.g., these are evident in vector spaces, real and complex numbers, etc.). This problem is looked at differently by the unordered relation approach, that is, the logical concept involved can be expressed as follows:

D2-16: The consistent sets $\{\phi_{i1}\}$ and $\{\phi_{i2}\}$ are said to be dependent if and only if

(1) $\exists\ \{\phi_{i'1}\} \subseteq \{\phi_{i1}\}$ and $\{\phi_{i'2}\} \subseteq \{\phi_{i2}\}$ $\ni\ \forall\ x \in \Pi_{ni1}\ P/\{\phi_{i'1}\} \exists\ y \in \Pi_{ni2}\ P/\{\phi_{i'2}\} \ni$ $x \subseteq y$.

(2) $\forall\ y' \in \Pi_{ni2}\ P/\{\phi_{i'2}\} \exists\ z \in \Pi_{ni12}\ P/\{\phi_{i'1}\} \cup \{\phi_{i'2}\} \ni y' \subseteq z$.

The above definition requires two consistent sets of relations which may seem undue restrictive. However, "logical dependence" can be thought of as a "measure" of how many logical facts one theory contains of another. The above restriction is only for the purpose of requiring some facts to start with. The following theorems are noted in this connection:

T2-9: (1) If $\{\phi_i\}$ has a natural point, then every proper subset $\{\phi_{i'}\}$ of $\{\phi_i\}$ for which $\Pi_{ni'}\ P$ is defined has a natural point.

(2) $\Pi_n\ P/\{\phi_i\} = \emptyset \rightarrow \Pi_{n'}\ P/\{\phi_i'\} = \emptyset\ \forall\ \{\phi_i'\} = \{\phi_i\} \cup \{\phi_j\}$.

Proof: These follow directly from D2-13 and T2-6.

With T2-9 the definition D2-16 is seen to be sufficiently general for the logically consistent algebraic formulas normally used in engineering.

One of the simplest facts to obtain and observe is as follows:

T2-10: $\{\phi_i\}$ dependent $\rightarrow \exists \{\phi_1\}$ and $\{\phi_2\}$ contained in $\{\phi_i\} \ni \{ \bigcup\{P_{ni1}\} \} \bigcap \{ \bigcup\{P_{ni2}\} \} \supseteq \{ \bigcup\{P_{ni1}\} \}$ or $\supseteq \{\bigcup\{P_{ni2}\} \}$.

Proof: Assume the opposite of the implied conditions and let $\{\phi_i\}$ be dependent. From D2-16 and definition of Π_n P a contradiction is obtained.

Also it is easy to see that $\{\phi_i\}$ and $\{\phi_i\}$ are not "independent".

D2-17: Sets $\{\phi_{i1}\}$ and $\{\phi_{i2}\}$ are said to be independent if they are not dependent.

Information concerning dependence among relations is of considerable importance in design theory. The main concern in this chapter is to look for conditions relative to dependence which can be assigned to parameter "models" representing "physical systems". Natural connectedness is one property already attributed to these "models". Even stronger, the condition between the graphs of naturally connected relations is, in this thesis, made an equality when either $\{P_{ni}\}$ or $\{P_{nj}\}$ is equal to $\{P_{nij}\}$. A useful theorem is obtained by using these conditions along with two others. Involved in this theorem is the following set decomposition process:

D2-18: A set of relations $\{\phi_i\}$ are said to be decomposable if the following process yields the empty set:

Consider $\{\phi_{i1}\} \subseteq \{\phi_i\} \ni \phi' \in \{\phi_{i1}\} \leftrightarrow \exists P' \in \{P'_1\}$

the parameter set of $\phi' \ni P' \notin$ to any of the parameters sets of $\{\phi_i\} - \{\phi'\}$. Now either $\{\phi_i\} - \{\phi_{i1}\} = \emptyset$ or not. If not, repeat the process using $\{\phi_i\} - \{\phi_{i1}\}$ in place of $\{\phi_i\}$. If so, the process is complete. If this process yields the empty set in a finite number of steps, let $\{\phi_{in}\}$ denote the last non-empty set. The set $\{\phi_i\}$ is called decomposable with the n decomposition sets $\{\phi_{in}\}$. A set is said to be non-decomposable if it is not decomposable.

D2-19: A naturally connected set such that ϕ_i nc $\phi_j \leftrightarrow \phi_j$ nc ϕ_i is said to be normally connected. (ϕ_i nc ϕ_j will be taken for normally connected unless otherwise stated.)

Some facts concerning decomposable sets are needed to look at the more important independence conditions.

T2-11: Let $\{\phi_i\}$ be a decomposable normally connected independent set. Then if $\{\phi_i\}$ has a one element decomposition class, it is the last.

Proof: Assume there is a one element class not the last. Let $\{\phi_{ij}\}$ represent this class. Then $\{P_{nij}\} \supsetneqq \{P'_{ni,\ j+1}\}$ where $\{P'_{ni,\ j+1}\}$ is a parameter set of $\phi'_{ij+1} \in \{\phi_{i,\ j+1}\}$. If not, then $\phi'_{ij+1} \in \{\phi_{ij}\}$ by D2-18. Now ϕ_{ij} nc $\phi'_{i,\ j+1}$ by hyp. Hence they are dependent by D2-16.

T2-12: Let $\{\phi_i\}$ be a non-decomposable set. Then

$$\exists \{\phi_i'\} \subseteq \{\phi_i\} \ni \left(\bigcup \{P'_{ni1}\} \right) = \left(\bigcup \{P'_{ni2}\} \right)$$

where $\{P'_{ni1}\}$ refers to $\{\phi'_{i1}\} \subset \{\phi'_i\}$, $\{P'_{ni2}\}$ refers to $\{\phi'_{i2}\} \subset \{\phi'_i\}$.

Proof: Assume the above results are false. Then

$$\exists P \in \left(\bigcup \{P'_{ni_1}\} \right) \ni P \notin \left(\bigcup \{P'_{ni_2}\} \right)$$

which gives a contradiction by D2-18. (Note that actually the other side of the set equality might need be used.)

T2-13: Let $\{\phi_i\}$ be a non-decomposable set $\ni \{P_{ni}\} \not\subseteq \{P_{nj}\}$ $\forall\, i \neq j$. Then there is one and only one $\{\phi'_i\}$ with the property that;

(1) $\{\phi_i\} - \{\phi'_i\}$ is either $\emptyset$ or a decomposable set.

(2) There are at least two distinct subsets of $\{\phi'_i\} \ni \left\{ \bigcup \{P'_{ni1}\} \right\} = \left(\bigcup \{P'_{ni2}\} \right) = \left(\bigcup \{P'_{ni}\} \right)$.

Proof: Consider the set $\{\phi'_i\} \subseteq \{\phi_i\}$ where $\{\phi'_i\}$ is the set which remains when the procedure in D2-18 fails to yield an empty set. The fact that it must fail is given by hyp. Now if the process fails on the first try, then $\{\phi'_i\} = \{\phi_i\}$ and $\{\phi_i\} - \{\phi'_i\} = \emptyset$. If not, then the other condition of (1). Now it is clear from D2-18 that $\{\phi'_i\}$ constructed as above is unique. Hence, if (2) is satisfied by $\{\phi'_i\}$, then the proof is complete.

If $\{\phi_i'\} = \{\phi_i\}$, then by T2-12 and the condition $\{P_{ni}\} \not\subseteq \{P_{nj}\} \; \forall \; i \neq j$ the set equality of (2) must hold for at least two distinct subsets.

Another fact concerns the "covering" properties of the decomposition sets.

T2-14: If $\{\phi_i\}$ is an N decomposable set of V relations with decompositions classes $\{\phi_{in}\}$, then

$$\bigcup_{i=1}^{v} \{P_{ni}\} = \bigcup \{P_{ni1}\} \supset \bigcup \{P_{ni2}\} \ldots \supset \bigcup \{P_{nin}\}.$$

Proof: Assume $\exists$ P ε ith decomposition set $\ni$ P $\not\in$ i - 1 decomposition set. Hence $\exists \, \phi' \; \varepsilon \; \{\phi_{ii}\} \ni \phi' \not\in \{\phi_{i,\,i-1}\}$. Also $\exists$ a P' ε $\{P_{ni}'\} \ni$ P' $\not\in$ to any other parameter relation set for j = i, i + 1, ... n by D2-18. Hence, by the construction process of D2-18 ϕ' can ε $\{\phi_{i,\,i-1}\}$. This contradicts the assumption above. Hence at least $\supseteq$ holds. To see that only $\supset$ can hold, D2-18 again is used to get non-decomposability which contradicts the hypothesis. That $\bigcup_{i=1}^{v} \{P_{ni}\} = \{P_{ni1}\}$ must follow from starting with a finite set of finite objects and D2-18.

From T2-13 and T2-14, it can be said that there is one and only one decomposition class for a decomposable set which "covers" the parameters. This fact along with the property of being consistent leads to interesting properties when more than one decomposition class is obtained. It is first noted that:

T2-15: For a consistent set $\{\phi_i\}$ with decomposition classes $\{\phi_{in}\}$ then when Π_{nij} $P/\{\phi_{ij}\}$ is defined
$\sigma_n (\Pi_{nin} P/\{\phi_{in}\}) \gtrsim \cdots \gtrsim \sigma_1 (\Pi_n P/\{\phi_{i1}\})$.

Proof: This is an extension of T2-9.

Hence the "number" of natural points tend to increase as "restrictions" (relations) are removed. Also, for decomposable sets to be dependent there must be a covering as by T2-10.

T2-16: Let $\{\phi_i\}$ be an N decomposable dependent set such that $\{P_{ni}\} \not\subseteq \{P_{nj}\}$ $\forall$ $i \neq j$. Then $n > 1$.

Proof: Assume $n = 1$. From T2-10 and D2-18, there is contradiction when the special conditions of the hyp. are used.

From T2-16 any set of relations which is first decomposable is independent. Also it is noted that one relation sets are independent, since D2-16 cannot be satisfied with just one set. From T2-16 and D2-18, along with T2-15, each decomposition class of a consistent decomposable set is independent.

T2-17: Given a consistent N decomposable set $\{\phi_i\}$ with $\{\phi_{in}\}$, then each $\{\phi_{ij}\}$ is an independent set.

So far most of the independence and dependence conditions have been quite general and only required consistency of the original set. The next theorem deals directly with dependence of normally corrected relation sets and indicates why the condition $\{P_{ni}\} \not\subseteq \{P_{nj}\}$ $\forall$ $i \neq j$ is not unusual.

T2-18: If for the normally connected set $\{\phi_k\} \exists \phi_i$ and ϕ_j, $i \neq j \ni \{P_{ni}\} \subsetneq \{P_{nj}\}$, then $\{\phi_k\}$ is dependent.

Proof: By D2-19 and D2-18.

The next definition is a key factor when showing that it is necessary to be able to decompose a set of independent relations which represent a "physical system".

D2-20: A consistent set $\{\phi_i\}$ with properties (1), (2), and (3) is said to have property c.

(1) $\{P_{ni}\} \not\subseteq \{P_{nj}\}$ $\forall$ $i \neq j$.

(2) If $\{\phi_{i_1}\}$ and $\{\phi_{i_2}\}$ such that $(\bigcup\{P_{ni_1}\}) = (\bigcup\{P_{ni_2}\}) \rightarrow \Pi_{ni_1} P/\{\phi_{i_1}\} \subsetneq \Pi_{ni_2} P/\{\phi_{i_2}\}$.

(3) $\{\phi_i\}$ is normally connected.

Using the above along with T2-13 and D2-19 gives:

T2-19: A set $\{\phi_i\}$ with property c is independent only if it is decomposable.

Reviewing the conditions for independence and dependence of consistent sets, the following is noted: Sets of relations with property c can be checked for independence and dependence in many cases by looking at their decomposition classes. The main cases for which conditions have not been given are the decomposable sets with more than one decomposition class. This involves removing the equality for set inclusion of condition (2) in D2-20. Before discussing this possibility, the essence of giving a model property c is discussed.

The restriction imposed on a model by D2-20 can be thought of in terms of the undefined notations called principles. A principle will be denoted by Φ. It is assumed that each set $\{\phi_i\}$ is obtained from a collection of principles $\{\Phi_i\}$. The restrictions imposed by D2-20 on a set $\{\phi_i\}$ are in effect restricting the set of principles which can be used to generate the set $\{\phi_i\}$. The primary restrictions are those of (1) and (2). In effect (1) says that the principles used disallow generating two independent relations ϕ_i and ϕ_j such that $\{P_{ni}\} \subseteq \{P_{nj}\}$. Property (2) is the extension of (1). Number (3) requires the "overlapping" relations to have identical projections. It is noted from (2) that the case of "n equations and n unknowns" is ruled out for models with property c.

The above restrictions placed on principles used for generating models do not seem to conflict with the general principles presently used in engineering. In addition, the following property appears to fit the class of general principles used to obtain deductive models.

D2-21: A set of relations $\{\phi_i\}$ which has the following properties is said to have property cc.

(1) $\{\phi_i\}$ has property c.

(2) If $\{\phi_{ij}\}$ and $\{\phi_{ik}\}$ are decomposition classes of $\{\phi_i\} \ni \bigcup\{P_{nij}\} \subset \bigcup\{P_{nik}\}$, then $\exists\, y' \in \Pi_{nik}\, P/\{\phi_{ik}\} \ni \forall\, z \in \Pi_{nijk}\, P/\{\phi_{ij}\} \cup \{\phi_{ik}\}, y' \not\subset z$.

The essence of (2) in D2-21 can be stated as follows. If the principles $\{\Phi_i\}$ generate a model which decomposes, any two independent decomposition classes are also independent. (The fact that a decomposition class is independent comes from T2-17.) It is noted that (2) is just the opposite of condition (2) in D2-16, the second requirement for dependence. Hence for sets with property cc, the necessary and sufficient condition for independence is decomposability.

T2-20: A set of relations $\{\phi_i\}$ with property cc is independent if and only if $\{\phi_i\}$ is decomposable.

In terms of the "number" of natural points, the models with property cc are required to have a strictly increasing "number" of natural points as the number of decomposition classes increase (see T2-15).

Principles $\{\Phi_i\}$ applied to the parameters $\{P_n\}$ generating a model with property cc is said to form a parameter model. This is formalized in the following definition.

D2-22: A set of v relations $\{\phi_i\}$ generated by the principles $\{\Phi_i\}$ with the properties (1), (2), and (3) is said to be a parameter model.

(1) Each $P \in \bigcup_{i=1}^{v} \{P_{ni}\}$ is indexed by a subset of the set A.

(2) $\{\phi_i\}$ has property cc.

(3) Let P'_{ni} be a single parameter of $\{P_{ni}\}$, then $R_{ni}\ (P'_{\overline{n}i}) = \Pi_{\overline{n}i}\ P$.

It is noted that the principles $\{\Phi_i\}$ indicated in D2-22 are not formally involved in the definition of a parameter model. However, it seems appropriate to include the phrase, "generated by the principles $\{\Phi_i\}$". The idea of the principle is used in the following example:

Example: Let $\{e_n\}$ be a set of n distinct parameters called voltages. Assume the one principle Φ, called Kirchhoff's voltage law, relating the voltages about any closed path in a network. Then if $\{\phi_i\}$ is a parameter model generated by Φ when applied to the network with voltages $\{e_n\}$, $\{\phi_i\}$ has property cc and hence is a set of independent relations if and only if $\{\phi_i\}$ is decomposable.

The parameter models are used in this thesis as if they were the class of models used to deductively solve engineering design problems. In this connection, parameters and relations, in more generality than parameter models, are introduced in the next chapter.

CHAPTER III

SYSTEMS

The purpose of this chapter is to define and characterize systems which can be utilized in the area of engineering design.

The idea of a system in this thesis is as follows:

D3-1: A collection of n parameters $\{P_n\}$ and v relations $\{\phi_v\}$ are said to form a system S_{nv} if and only if:

(1) $\forall P \in \{P_n\} \exists \phi_k \in \{\phi_v\} \ni P \in \{P_{nk}\}$.

(2) $\forall \phi_i \in \{\phi_v\}$ and $\forall P \in \{P_{ni}\}$, $P \in \{P_n\}$.

(3) $\forall \phi_i \in \{\phi_v\}$, ϕ_i is algebraic (D2-8).

(4) $\{P_{nj}\} \bigcap \{P_{ni}\} \neq \emptyset \rightarrow \phi_j \text{ c } \phi_i$ (D2-12).

(5) $\forall \{\phi_{i1}\} \subset \{\phi_v\}$, $\{P_{ni1}\} \bigcap \{P_{ni2}\} \neq \emptyset$ where $\{P_{ni2}\}$ are the parameter sets of $\{\phi_v\} - \{\phi_{i1}\}$.

(6) $S_{oo} \leftrightarrow n = v = 0$ is called the "empty" system.

Although a system S_{nv} is quite general, it is also restrictive. The definition above includes the usual sets of v equations and n unknowns which are common in engineering. Also, it requires that any postulated system have certain basic properties. Mainly, that the quantities of interest be identified <u>and</u> some abstract connection be centered about these quantities (1), (2). In addition, there must be a "logical" path established between the parameters. This is the requirement of (4) and (5). Number (4) requires that if two relations are set connected, then their common projections are "overlapping". Number (5) requires the necessary set conditions for having the natural points of a system defined. The property (3) is included based on the fact that the usual mathematical formulas of the basic sciences are defined on a single set of elements. Hence, the above requirements are seen to be primitive, yet not at all trivial.

One of the more interesting facts, and desirable one from a practical point of view, is the following:

T3-1: If $\exists$ a set of parameters $\{P_n\} \ni n > 0$ and each $P_i \in \{P_n\}$ is indexed by the same set A, then $\exists$ a system $S_{nv} \neq$ Soo.

Proof: Consider the relation $\phi(x) \rightarrow x \in \Pi_n P$. It is clear that properties 1-6 are satisfied.

The above theorem is important in a design theory. When considering design techniques from a logical point of view, it is indeed practical to have at least one system.

When dealing with system classifications and properties directed towards design, it is convenient to consider sets of parameters $\{P_n\}$ and relations $\{\phi_v\}$ which are not necessarily a system but do have some of the properties of D3-1. The sets will be denoted by A_{nv} and are given as follows:

D3-2: A collection of parameters $\{P_n\}$ and relations $\{\phi_v\}$ are said to form a partial system A_{nv} if and only if the two collections satisfy properties 1-3 and 6 of D3-1.

The properties 1-3 still require the identification of parameters and algebraic relations. However, these relations may not be connected (6). In fact, their parameters sets may not form a path (5). From T3-1, it is clear that a non-empty partial system exists. The following facts are noted concerning A and S which are directly from D3-1 and D3-2:

T3-2: (1) $\forall S_{nv} \in S, S_{nv} \in A$.

(2) $\exists A_{nv} \in A \ni A_{nv} \notin S$.

Just as in Chapter II, not much can be said about the elements of A and S until more properties are assigned. Most of the basic properties which are used to classify systems are a direct carry over from the chapter on unordered relations. When possible, the properties will be attributed to elements of A, understanding that if properties (4) and (5) are involved, the definition only applies to the subsets which belong to S.

D3-3: A partial system A_{nv} is said to be an algebraic system if and only if for each P of A_{nv}, P is indexed by a subset of the same indexing set.

D3-4: A partial system A_{nv} is said to be connected if $\{P_{ni}\} \bigcap \{P_{nj}\} \neq \emptyset \rightarrow \phi_i \subset \phi_j$.

D3-5: The set $\Pi_n P/\{\phi_i\}$, if it is defined, is said to be the natural points of A_{nv}. (When discussing natural points, it will be more convenient to use $\Pi_n \; P/A_{nv}$.)

D3-6: A_{nv} is said to be consistent if $\Pi_n \; P/A_{nv} \neq \emptyset$.

D3-7: Given a partial system A_{nv} with $\{\phi_i\}$, then A^i_{nv} is called a subsystem of A_{nv} if and only if $\forall \; \phi_i$ of A^i_{nv}, $\phi_i \in \{\phi_i\}$ of A_{nv}.

T3-3: If A_{nv} is consistent, then each A^i_{nv} which has $\Pi_{ni} \; P/A^i_{nv}$ defined is consistent.

Proof: From T2-9.

T3-4: If $\exists \; A^i_{nv}$ of A_{nv} which is not consistent, then A_{nv} is not consistent.

Proof: From T2-9.

T3-5: If A_{nv} is a connected partial system, then

(1) $\exists \; A^1_{nv}$ and A^2_{nv} proper subsystems (i.e., $A^1_{nv} \neq A_{nv}$ and $A^2_{nv} \neq A_{nv}$) $\ni A^1_{nv}$ and $A^2_{nv} \in S$.

(2) $A_{nv} \notin S \rightarrow \exists \; A^1_{nv}$ and $A^2_{nv} \ni (\bigcup \{P_{ni_1}\}) \cap (\bigcup \{P_{ni_2}\}) = \emptyset$.

D3-8: A partial system A_{nv} is said to be dependent if and only if $\{\phi_{i_1}\}$ and $\{\phi_{i_2}\}$ of $\{\phi_v\}$ of A_{nv} are dependent.

D3-9: A partial system A_{nv} is said to be independent if it is not dependent.

In looking for useful ways to classify the elements of A and S a simple equivalence relation is observed in the ordered pair (n, v) of A_{nv}.

T3-6: Let IXI denote the ordered pairs of integers. If a pair of integers is denoted by (n, v), then the rule $A_{n_1v_1} \overset{B}{\sim} A_{n_2v_2} \leftrightarrow (n_1, v_1) = (n_2, v_2)$ is an equivalence relation $\overset{B}{\sim}$ on the set A.

Proof: $A_{n_1v_1} \overset{B}{\sim} A_{n_1v_1} \rightarrow$ reflexive since (n, v) = (n, v)

$A_{n_1v_1} \overset{B}{\sim} A_{n_2v_2} \rightarrow A_{n_2v_2} \sim A_{n_1v_1} \rightarrow$ symmetric.

Since $(n_1, v_1) = (n_2, v_2) \leftrightarrow n_1 = n_2$ and $v_1 = v_2$, $A_{n_1v_1} \sim A_{n_2v_2}$, $A_{n2v2} \sim A_{n_3v_3} \rightarrow A_{n_1v_1} \sim A_{n_3v_3} \rightarrow$ transitive. This follows as in the proof of symmetric.

Note that by T3-2 the same relation also reduces S to equivalence classes. Because the above "basic size" relation is produced by (n, v), these integers are given the names "order" (n) and "parts" (v).

D3-11: The order n of a partial system A_{nv} denotes the number of parameters in $\{P_n\}$ of A_{nv}. The "parts" v of a system denotes the number of distinct relations $\{\phi_i\}$. (Where distinct means that if ϕ_i and ϕ_j have the same parameters then $R\,(P_{ni})_j \neq R\,(P_{nj})_j$.)

In the above classification it is noted that only distinct relations are considered when discussing the number of parts. In the remainder of this thesis, it is assumed that for A_{nv}, v denotes the number of parts.

Probably the most important classifications from a design aspect are those of "physical system", "specification", and "solution". The "physical system" corresponds to a parameter model.

D3-12: A_{nv} is said to be a physical system if and only if $\{\phi_i\}$ has the properties of a parameter model.

The class of physical systems will be denoted by PS. From D2-21 the elements of PS are also elements of S. Also, from T2-20 the elements of PS have independent relations. This is stated after decomposable systems are introduced.

D3-13: A_{nv} is said to be decomposable if and only if $\{\phi_v\}$ of A_{nv} is decomposable.

D3-14: A_{nv} is said to be a fixed system if and only if A_{nv} is not decomposable.

The term, fixed system, is used because of the properties of decomposability. (That is, when relations are added to a decomposable system until the system is no longer decomposable, the number of natural points in most cases decreases. In any case it cannot increase by T2-9. Hence the term, fixed, refers to the possibility that there is only one natural point. This parallels the idea of possibly having a unique solution in ordinary algebraic systems of n equations and n unknowns.) However, it is noted, that a fixed system does not necessarily have n parameters and n relations or a single natural point.

For decomposable systems, the subsystems corresponding to the decomposition classes of $\{\phi_V\}$ are called characteristic subsystems. These characteristic subsystems are ordered by the decomposition property of T2-14. When the system is a parameter model and if the relations are related to some physical interconnection property (e.g., some electrical or mechanical network of components), then the characteristic subsystems indicate the physical interconnection. This is illustrated below:

Example: Consider the relations $\{\phi_i\}$ of $S_{9,4} \varepsilon$ PS induced by the Kirchhoff principle of voltages summing to zero in a closed loop.

$$\phi_1 (e_1, e_2, e_3)$$

$$\phi_2 (e_4, e_5, e_6)$$

$$\phi_3 (e_7, e_8, e_9)$$

$$\phi_4 (e_1, e_5, e_9)$$

Now $S_{9,4}$ is decomposable with characteristic subsystems $(\{\phi_1, \phi_2, \phi_3\}, S_{9,3})$ and (ϕ_4, S_{31}). Assume that each ϕ corresponds to traversing a geometrical closed path of components. (One component for each parameter is assumed.) If this is the case, then it must be true that the components of ϕ_4 are connected in a loop which is imbedded in the loops of the system $S_{9,3}$. A network with this property is shown in Figure 3-1.

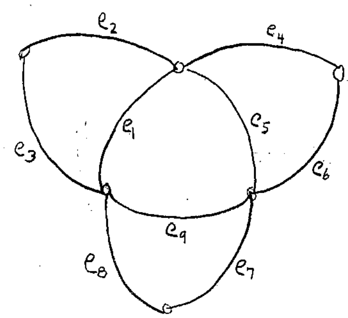

Figure 3-1. Four Loop Network

D3-15: If A_{nv} is decomposable, the subsystem A^i_{nv} which corresponds to the ith decomposition class of $\{\phi_v\}$ is called the ith characteristic subsystem.

As mentioned earlier, the properties of T2-20 as extended to systems are given to the physical systems of this thesis.

T3-7: If S_{nv} ε PS, then

(1) S_{nv} is decomposable.

(2) $S_{nv} \rightarrow \{\phi_v\}$ where each ϕ_i is the product relation $(\Pi_{ni}\ P)$ only if S_{nv} is first decomposable.

(3) S_{nv} is independent.

Proof: (1) and (3) come from D2-21 and T2-20. Number (2) comes from D2-21 and definition of Π_n P. From D2-21 the principle for $\{\phi_v\}$ is that used to define the product relation. However, if $\{\phi_v\}$ is not first decomposable, then property cc cannot hold since any covering of a product relation by product relations would not be independent. Also, from property c any product relation with parameters contained by some other product relation is ruled out. Hence, the product relation can only be used as a physical model when certain "restraints" are fulfilled.

The class of physical systems have been developed for the purpose of having an abstract mathematical model with properties in common with the ordinary systems studied in engineering. The systems which are commonly analyzed in engineering are formulated in one of the ordinary number systems based on several principles which appear to give the parameter model properties (e.g., amplifiers, motors, servo-systems, etc.). All of these common systems can be thought of as elements of PS as looked at in this thesis.

The class of physical systems can be thought of as the general solution space for an engineering problem. In practice, it is known that a complete analytical model is seldom to be found which allows the given problem to be solved numerically. However, it is often the case that a "theoretical" model can be found, although a complete set of operational formulas is not available. This being the case, the necessary ingredients for assuming the properties of PS are sometimes available. Also, it is often the case that more restrictive properties on the relations are available.

Probably the most powerful deductive property that an abstract model can have is that of the being "1 - 1". In terms of the ordered (binary) relations, this is expressed in terms of the first and second elements of the ordered pairs. This requires the order of the relation to be considered in formal discussions. In the unordered approach there is no formal problem of this nature. The classifications concerning uniqueness of related scalars parallels those of ordered relations.

D3-16: ϕ on $\{P_n\}$ is said to be a partial function relative to $\{P_{n-1}\}$, some n - 1 subset of $\{P_n\}$ if and only if $\exists\ \emptyset \neq \{P_{n-1}\} \subset \{P_n\} \ni \forall\ \Pi_{n-1} P \in R_{n-1}(P_{n-1})$. There is exactly one $\pi_n\ p \in \Pi_n\ P/\phi \ni \pi_n\ p \cap \pi_{n-1}\ p = \pi_{n-1}\ p$.

In terms of an ordered binary relation, R, the above implies that either R or R converse is a function. In terms of "higher dimensional" relations, the partial function defines a mapping from the ordered set $\Pi_{(n-1)_i} P$ into $P_{\overline{n-1}}$. This is usually denoted by ϕ: $\Pi_{(n-1)_i} P \rightarrow P_{\overline{n-1}}$ or $f(P_1, \ldots P_{n-1}) = P_{n-1}$.[1] The important thing is that the partial function parallels the function concepts of ordered relations. A complete function will denote the case when ϕ is a partial function relative to each n - 1 parameter subset.

D3-17: ϕ is said to be a complete function on $\{P_n\}$ if ϕ is a partial function on every distinct n - 1 subset of $\{P_n\}$.

In the case of n = 2, the ordered relation name for a complete function is one-to-one (1 - 1). It is clear that a complete function

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[1] More correctly the partial function defines a mapping only when $R_{n-1}(P_{n-1}) = \Pi_{n-1} P$.

is a partial function but not necessarily the converse.

The partial function is very common in the parameter models of engineering. Also, the often used linear models involve only complete functions (i.e., models based on linear algebraic equations). These are the two basic function classifications of unordered relations of this thesis. However, it is noted that this approach suggests other classifications which might be useful in constructing model properties. Some of these are indicated in Chapter IV when discussing general design solutions.

The idea of a "solution" is inherent in the concept of a relation. Also, the idea of a function allows the idea of "unique solution". In design, it is a "solution" which is of primary interest. The concern for the "unique solution" is usually small. However, the properties of partial functions can become useful in design to indicate the non-existence of "solutions". Using the definitions D2-10 and D2-11, a "specification" and "solution" are defined.

D3-18: Given ϕ on $\{P_n\}$ and $\overline{R_{n'}(P_{n'})} \subseteq R_{n'}(P_{n'})$ of ϕ. Then $\overline{R_{\bar{n}'}(P_{\bar{n}'})} = \{x/x \in R_{\bar{n}'}(P_{\bar{n}'})$ and $x \cup y \in \Pi_n P/\phi$ and $y \in \overline{R_{n'}(P_{n'})}\}$ is called the solution for the specification $\overline{R_{n'}(P_{n'})}$.

It is again noted that the unordered concept of solution is the same as that of the ordered relation (i.e., basically a solution is relative to some "problem" where ϕ can be thought of as the "problem domain" and the specifications as the "problem generator").

The solution as given by D3-18 in general gives no information about existence or uniqueness. Some of the simple existence facts were stated in T2-5. Some of the simple uniqueness facts are listed in the next theorem.

T3-8: Given ϕ and the specification $\overline{R_{n'} (P_{n'})}$:

(1) ϕ a partial function relative to $P_{n'} \rightarrow$

$\forall x \in \overline{R_{n'} (P_{n'})} \exists$ a unique $y \in P_{\overline{n'}} \ni y \in \overline{R_{\overline{n'}} (P_{\overline{n'}})}$.

(2) If ϕ is the product relation, then $\overline{R_{\overline{n'}} (P_{\overline{n'}})} \neq \emptyset$ is unique if and only if (1) $\{P_{n'}\} = \{P_n\}$ and (2) $R_{n'} (P_{n'})$ has exactly one element.

The classifications involving partial functions, complete functions, specifications, and solutions can be naturally extended to systems. The extension of the partial function and complete functions are delayed until Chapter IV when discussing types of "real" physical systems. The "system" specification and solution are given in this chapter after the concepts of the measurably rational, measurably irrational, and sum of two systems are introduced.

D3-19: $A_{nv} \varepsilon A$ is said to be measurably rational (MRS) if and only if

(1) A_{nv} is algebraic.

(2) $\forall A^i_{nv}$ of A_{nv} then either $\Pi^i_n P/A^i_{nv} \neq \emptyset$ or not defined.

T3-9: (1) A partial system $A_{nv} \ni \Pi_n P/A_{nv} \neq \emptyset$ is measurably rational and connected.

(2) $\forall S_{nv} \varepsilon PS, S_{nv}$ is measurably rational.

Proof: These come directly from the definitions of PS and D3-20.

The role of the measurably rational partial system in design is quite important. The name measurably refers to the fact that A_{nv} is algebraic; hence, a common reference frame. The name, rational, refers to the fact that nothing is inferred to be impossible by this class of A. This class represents the "specifications" for a design problem which have a possibility of being "satisfied" by some element of PS. An element of A which can satisfy (1) but not satisfy (2) of D3-19 is said to be irrational.

D3-20: $A_{nv} \in A$ is said to be measurably irrational (IRS) if and only if

(1) A_{nv} is algebraic.

(2) $\exists\ A^{i}_{nv} \in A_{nv} \ni \Pi_n P'/A'_{nv} = \emptyset$.

The class IRS is at the other end of the spectrum when considering specifications. That is, no element of PS has the properties of IRS. Hence, from T3-4, T3-10 is obtained. First, the sum of two systems is defined.

D3-21: Let $A_{n_1v_1}$, $A_{n_2v_2} \in A$. The sum of the partial systems, $A_{n_3v_3}$, is denoted by $A_{n_1v_1} + A_{n_2v_2} = A_{n_3v_3}$. Where the set $\{P_{ni_3}\} = (\bigcup\{P_{ni_1}\}) \bigcup (\bigcup\{P_{ni_2}\})$ and $\{\phi_{vi_3}\} = \{\phi_{vi_1}\} \bigcup \{\phi_{vi_2}\}$ is the parameter set and relation set of $A_{n_3v_3}$.

T3-10: If $A_{n_1v_1} \in$ IRS, then $\not\exists\ S_{n_2v_2} \in PS \ni \Pi_{n_3} P/A_{n_3v_3} \neq \emptyset$.

D3-22: $A_{nv} \varepsilon A$ is called a system specification if and only if A_{nv} is algebraic. This class is denoted by SA. (This is an equivalent definition of D3-3.) When (A_{nv} of SA) ε MRS, A_{nv} is said to be feasible. When (A_{nv} of SA) ε IRS, A_{nv} is said to be impossible.

D3-23: Assume $A_{n_1v_1}$ ε SA, $A_{n_2v_2}$ ε PS, $A_{n_3v_3}$ of A. Then if

(1) $A_{n_1v_1} + A_{n_2v_2} = A_{n_3v_3}$.

(2) $A_{n_3v_3}$ ε SA.

(3) $A_{n_3v_3} \not\varepsilon$ IRS.

$A_{n_2v_2}$ is called a system solution for the system specification $A_{n_1v_1}$.

Note that system solutions are relative to some system specification $A_{n_1v_1}$. Yet, as in other functions of more than two variables, the "solution variable" can be held constant while the "specification variable" is changed. This fact is very necessary in a "correct" theory for design, since this "design process" is commonly used (i.e., very seldom is the model changed). Several facts which are quite simple to obtain yet useful in design are given in the next theorem.[2]

T3-11: Given: $A_{n_1v_1}$ ε SA, $A_{n_2v_2}$ ε PS such that $A_{n_1v_1} + A_{n_2v_2} = A_{n_3v_3}$.

(1) If $A_{n_2v_2}$ is a solution of $A_{n_1v_1} \to A_{n_2v_2}$ and

[2]The distinction between <u>system specification</u> and <u>specification</u> will be omitted unless confusion might arise.

$A_{n_2 v_2}$ are indexed by the same set.

(2) $\exists$ a solution for $A_{n_1 v_1}$ if and only if $A_{n_1 v_1} \varepsilon$ MRS.

Proof: (1) This follows from D3-23, D3-3, D3-22, D3-24.

(2) From T2-9 and D3-22 the necessary part is obtained. From T3-1 the relation $\phi\ (n) \rightarrow x \ \varepsilon\ \Pi_n\ P$ is always available. Also, given any A_{nv} of MRS, then $\Pi_n\ P + A_{nv}\ \varepsilon$ MRS by properties of $\Pi_n\ P$, D3-19 and D3-17.

The value of a system is defined in order to have another way of looking at the logical information of a system.

D3-24: Given an element A_{nv} of SA, the value of A_{nv} is the set of natural points $\Pi_n\ P/A_{nv}$ if it is defined. If empty or not defined, A_{nv} is said to have a null value.

The above definition closely parallels the ordered version for an n variable formula. There is a distinct difference, however, which is, conceptually at least, important. There is no concept like the "picture graph" found with relations on a set which is ordered (i.e., the concept "is $P_1 > P_2$" is "meaningless" in unordered relation). This, in fact, characterizes the "unordered" from the "ordered". However, it suggests a natural way to "project" a system onto a "calibrated" scale.

D3-25: The ordered value of an algebraic element A_{nv} is either empty when $\Pi_n P/A_{nv} = \emptyset$ or the order of the scalar values of the elements of $\Pi_n P/\Pi_{no} P$. Here $\Pi_n P/\Pi_{no} P$ denotes that each element of the unordered set $\Pi_n P$ is ordered by the ordering structure of the indexing set. (Also, if the indexing set is not ordered, the ordered value is said to be empty.)

D3-26 indicates that $\Pi_n P/\Pi_{no} P$ represents the ordinary graphs which are used in engineering to obtain a "measurable feel" for a particular system. Also, it clearly indicates why the ordinary graphs of some indexing sets are not used nearly as often as others. For example, Boolean algebra is defined on a set of two elements, say 0, 1. Yet the graphs of formulas of a Boolean algebra are seldom displayed. Note also that the ordering properties of 0, 1 are limited indeed. That is, $0 < 1$, $1 > 0$, or $1 = 0$ (trivial). Yet, the Boolean algebra gives very useful formulas, since many parameters can be usefully indexed by only two elements.

The above discussion indicates the possibility that algebraic systems can be projected into many indexing sets if they can be projected into one. Also, from D3-24 the value of a system is its natural points. However, the common way of discussing a system is in terms of its ordered value in some indexing set and the formulas so defined in these sets. This procedure is useful in particular cases but conceptually is quite limited. The natural points indicate the "logical information" of a system. This being the case, any two systems which have a 1 - 1 correspondence between natural points are in a logical sense equivalent.

Using the equivalence classes $\overset{B}{\sim}$ and the "number" of system values, establishes an equivalence relation on the Class A.

T3-12: $A_{n_1v_1} \overset{N}{\sim} A_{n_2v_2} \leftrightarrow$

(1) $\sigma(\Pi_n\ P/A_{n_1v_1}) = \sigma(\Pi_n\ P/A_{n_2v_2})$.

(2) $A_{n_1v_1}$ and $A_{n_2v_2}$ ε $(\overline{A_{nv}})$ an equivalence class of $\overset{B}{\sim}$.

From the above it is clear that (1) the "amount" of logical information (natural points) must be the same in two systems for them to be equivalent and (2) the number of parameters and relations must be the same. The next theorem shows that when there is a 1 - 1 association between scalars of each parameter of $\{P_n\}$ and $\{P'_n\}$, then there is a system $A'_{nv} \overset{N}{\sim} A_{nv}$.

T3-13: Assume a system A_{nv} with a natural point and subsystems $A^i_{nv} \ni (\sum_{i=1}^{v} A^i_{nv} = A_{nv})$. If T has the property that $T\ P_{ni} \leftrightarrow P'_{ni}$ for every P_{ni} ε $\{P_n\}$of A_{nv}, then $A'_{nv} = \sum_{i=1}^{v} A^{i'}_{nv}$ is equivalent to A_{nv}.

Proof: First, it is noted that if $(\pi_{ni}\ p \bigcup \pi_{nk}\ p) \varepsilon$ $\Pi_{nik}\ (P_{ni} \bigcup P_{nk})$, then $(T\ \pi_{ni}\ p \bigcup T\ \pi_{nk}\ p) \varepsilon$ $\Pi_{n'ik}\ (T\ P_{ni} \bigcup T\ P_{nk})$, where $\pi_{ni}\ p\ \varepsilon\ \pi_{ni}\ P$ of A^i_{nv}. Now unless $\{P_{ni}\} \bigcap \{P_{nk}\} \neq \emptyset$, then $\pi_{ni}\ p \bigcup \pi_{nk}\ p\ \varepsilon\ \Pi_{nik}\ (P_{ni} \bigcup P_{nk})$. Hence, the disjoint case is of no concern. Assume that $\pi_{ni}\ p \bigcup \pi_{nk}\ p \not\varepsilon\ \Pi_{nik}\ (P_{ni} \bigcup P_{nk})$ and

$T(\pi_{ni} p) \cup T(\pi_{nk} p) \in \Pi_{n'ik} (T P_{ni} \cup T P_{nk})$. (Here $T P_{ni}$ refers to the P'_{ni} which each P_{ni} is associated with in a 1 - 1 manner.) But if $\pi_{ni} p \cup \pi_{nk} p \notin \Pi_{nik} (P_{ni} \cup P_{nk}) \rightarrow \exists\ p_1 \in \pi_{ni} p$ and $p'_1 \in \pi_{nk} p \ni p_1$ and $p'_1 \in P_i \in (\{P_{ni}\} \cap \{P_{nk}\})$ $\ni p_1 \neq p'_1$. Therefore, by hypothesis $Tp_1 \neq Tp'_1$. Hence, $(T\pi_{ni} p) \cup (T\pi_{nk} p) \notin \Pi_{n'ik} (TP_{ni} \cup TP_{nk})$. The above fact is used to show the 1 - 1 mapping T' between $\Pi_n P/A_{nv}$ and $\Pi_n P'/A'_{nv}$ exists. Consider $T' (\Pi_n P/A_{nv}) = \{x/x \in \Pi_{n'} P$ and $x = \bigcup_{i=1}^{v} T(\pi_{ni} p)$ and $\bigcup \pi_{ni} p \in \pi_n P/A_{nv}\}$. Now $T' (\Pi_n P/A_{nv})$ is a 1 - 1 mapping between $\Pi_n P/A_{nv}$ and $T' (\Pi_n P/A_{nv}) \subseteq \Pi_n P'/A'_{nv}$. The proof is then complete if it can be shown that $\Pi_n P'/A'_{nv} \subseteq T'$. (The fact that $T' \subseteq \Pi_n P'/A'_{nv}$ comes directly from the properties of T and T'.) Assume $x \in \Pi_n P'/A'_{nv}$ and $x \notin T' (\Pi_n P/A_{nv})$. Therefore, $\exists\ (T\pi_{ni} p) \cup (T\pi_{nk} p) \in \Pi_{n'ik} (TP_{ni} \cup TP_{nk}) \ni (\pi_{ni} p) \cup (\pi_{nk} p) \notin \Pi_{nik} (P_{ni} \cup P_{nk})$. But by the previous result, this assumption gives a contradiction. Hence, $T' (\Pi_n P/A_{nv}) \supseteq \Pi_n P'/A'_{nv}$ and $A_{nv} \overset{N}{\sim} A'_{nv}$.

The above theorem is quite unrestrictive when the normal systems studied in engineering are considered. An example is given below which commonly makes use of the above fact.

Example: Consider ϕ_1 of S_{n_1} an algebraic system indexed by the reals generated by the real number formula $x_1 = f(x_2, \ldots x_n)$. Consider a transformation of the type $x \to ax$ on the reals where a is some positive integer. This is 1 - 1 of each P_i onto P'_i. Hence, for each π_{n1} p of Π_n P/ϕ there is a unique π_n p' which can be associated with π_{n1} p. Hence, any transformation of the above type gives an equivalent system.

In the actual situations, care is exercised in changing the information between equivalent systems (i.e., $y = \sin x$ and $ay = \sin ax$ are normally considered to be different when $a \neq 1$). The above discussion indicates that this has nothing to do with the logical value of the two systems.

The main classifications given in this chapter are those which can be used in engineering design. In leaving this chapter the connection between design and systems can be indicated as follows:

Given a feasible specification $A_{n_1v_1}$ (i.e., $A_{n_1v_1} \in$ MRS), then <u>any</u> system $S_{n_2v_2}$, such that $A_{n_1v_1} + S_{n_2v_2}$ is MRS, can be thought of as a theoretical design solution.

CHAPTER IV

FUNDAMENTALS FOR ENGINEERING DESIGN

The "general design" problem is formally indicated in D4-1. That is, the ingredients for a problem are presented in this definition. Conceptually, the problem is no different from the following algebraic problem:

Given: $\phi \leftrightarrow b + x \geq y \quad x, y, b \in R.$

Find: $X = \{x/x \in R, \text{ and } \phi\ (x) \text{ and } y \in I\}.$

Now there are other possibilities which could be used in the above example. The point of the analogy is to discuss the "solution". Unless each of the above terms was well defined, no procedure could be unambiguously outlined (not necessarily carried out) to give the answer or answers.

This last fact has been the main problem in developing a useful design theory. It was for this reason that Chapters II and III of this thesis were essential. The design problem, process, etc. is not unique to this thesis. It is the precise definitions of the elements used in this chapter which are unique.

D4-1: The set DS is called the general solution of a design problem by the specifications of (1) if and only if:

(1) The set $\{A_{n_1 v_1}, A_{n_2 v_2} \ldots A_{n_r v_r}\} \ni$ each $A_{n_i v_i} \in SA$.

(2) $DS = \{x/x \in PS \text{ and } x + \sum_{i=1}^{v} A_{n_i v_i} \notin IRS\}$.

The design problem as stated above is quite general, yet quite well defined. That is, each of the elements involved in the above definition can be traced back to the primitive concepts of a "set" and "unordered relations". It is also noted that these unordered relations are directly connected to the ordered relations by the concept of indexing sets with ordering properties (value concept) and algebraic operations (useful formulas). Thus, the above definition relates a general system concept (unordered relations) to particular system concepts of ordered relations. This allows the elements of the class of all possible MRS systems to be used as a theoretical solution space for design problems.

Some results of the previous classifications are seen in the set DS. From (1) it is observed that more than one specification might be involved in finding a solution DS for the design problem. Note also that DS might be empty (i.e., from the results of Chapter II if $A_{n_i v_i} \in IRS$, for some i, then DS is empty).

T4-1: Given a specification set $\{A_{n_r v_r}\}$, then $DS = \emptyset$ if $\exists$ a subset $\{A_{n_i v_i}\}$ of $\{A_{n_r v_r}\} \ni \sum_{i'}^{i} A_{n_i v_i} \in IRS$.

Proof: First, $\sum_{i'}^{i} A_{n_{i'} v_{i'}}$ must be algebraic or $DS = \emptyset$.

Also, when $\sum_{i'}^{i} A_{n_{i'} v_{i'}} \in SA$ by T3-8 and T3-9, the conclusion follows.

The extended version of T2-10 also follows when $\sum_{i=1}^{r} A_{n_i v_i} \in MRS$.

T4-2: Given a specification set $\{A_{n_r v_r}\}$, then DS $\neq \emptyset$ if and only if $\sum A_{n_i v_i} \in$ MRS.

From T4-1 and T4-2 it is seen that definite boundaries can be placed on the existence of any general solution for the class of unordered specifications. In light of the above, a collection of specifications can be considered as <u>one</u> partial system. That is, from T4-1 if there is any single A_{nv} which $\in$ IRS, then DS is empty (no physical system exists). Also, when the sum of all the specifications does not belong to IRS, at least one physical model exists. The one model which is known to exist is Π_n P. In the remainder of this thesis a specification will be represented by a single element of SA.

While T4-1 and T4-2 are positive boundaries for a theoretical design solution, only T4-1 is immediately applicable to engineering design. If the conditions of T4-1 can be shown to hold, then the search for a solution can be avoided. Unfortunately, the conditions of T4-2 are not as positive, in a practical sense. The existence of a model such as Π_n P does not give any method of inferring additional information. The model Π_n P only indicates that anything is theoretically possible concerning scalars of the parameters $\{P_n\}$ if (1) the set $\{P_n\}$ is assumed and (2) impossible conditions have not already been imposed (T4-1). Hence, T4-2 only says that the conditions (1) and (2) hold.

In engineering design the results of T4-1 and T4-2 are probably the only generally-accepted facts. Hence, the fact that these are the boundaries of the theoretical solution to D4-1 indicates the logical methods of engineering design are paralleled by the methods for obtaining the set DS.

To outline a single process which would yield positive results for any design problem is beyond the scope of this thesis. The reference to positive results refers to the existence or non-existence of an element of PS other than Π_n P which has a set of natural points containing those of the specifications (i.e., given A_{nv} does $S_{n'v'} \varepsilon$ PS, $\ni \forall x \varepsilon \Pi_n P/A_{nv}$, $\exists$ $y \varepsilon \Pi_{n'} P/S_{n'v'} \ni x \subseteq y$ and $\Pi_{n'} P/S_{n'v'} \neq \Pi_{n'} P$).

This theory for design only claims positive results when the conditions for T4-1 are met (e.g., if a set of specifications $\phi_1(P_1)$ and $\phi_2(P_1) \ni \phi_1 \leftrightarrow p_1 \varepsilon P_1$ and $\phi_2 \leftrightarrow p_1' \varepsilon P_1 \ni p_1 \neq p_1'$, then this theory claims that no solution exists).

From a practical point of view, a useful physical system would be an element of PS which indicated a "definite" interconnection of the parameters involved. A particular logical interconnection would be observed by constructing order relations in the indexing set of the parameters where these constructions are based on principles Φ_i. These principles can be thought of as order-determining postulates or order-determining experiments, whichever is the most convenient. From a design theory point of view they are considered as the necessary postulates to have the ingredients of a useful physical system.

To obtain more deductive power for seeking solutions to the design equation, additional restrictions on PS and the specifications are needed (i.e., the boundaries are too broad). The most simple type of specifications are those which each $A_{n_i v_i}$ involves only one scalar of a parameter.

D4-2: A specification $A_{nv} \ni \sigma(\{P_{ni}\}) = 1$, i = 1, 2, ... v, $\sigma(\Pi_n P/\phi_i) = 1$, i = 1, 2, ... V is called a simple specification.

A simple specification certainly belongs to MRS. Hence, from T4-2 DS is not empty. However, as indicated above, the "real" physical systems are the only ones of practical interest to the engineer. These are the physical systems which are different from the relation $\phi(x) \rightarrow x \in \Pi_n P$. The real system axiom is a formal way of introducing these models. First the definition of a real physical system is given.

D4-3: A system $S_{nv} \in PS$ is called a real physical system RPS if and only if for each parameter set $\{P_{ni}\}$ of ϕ_i

(1) $\Pi_{ni} P/\phi_i \subset \Pi_{ni} P$ unless $\sigma \{P_{ni}\} = 1$, then $\Pi_{ni} P/\phi_i = P_{ni}$.

(2) Every proper projection of ϕ_i is the product relation.

(3) If $S_{n_1v_1}$ and $S_{n_2v_2} \in RPS \ni \Pi_{n_1} P/S_{n_1v_1} = \Pi_{n_2} P/S_{n_2v_2} \rightarrow v_1 = v_2$.

D4-4: If $S_{n_1v_1}$ is a real system, then $\Pi_n P/S_{n_1v_1}$ is called the standard points of $S_{n_1v_1}$. (Note, not all systems have standard points.)

Real System Axiom: For every finite set of parameters $\{P_{n_1}\}$ indexed by a common indexing set $\exists$ a real physical system $S_{nv} \ni \{P_{n_1}\} \subseteq \{P_n\}$.

The class of real physical systems plays the role of the best available mathematical models for a given set of parameters. These models usually become more reliable in a manner "proportional" to the use of the model. However, design theory as developed in this thesis is considered to be independent of the "state of the art" in engineering science. (That is, this design theory only lists properties which a real physical system has; the conclusions are based on these properties. Whether or not a particular model has these properties is the problem commonly referred to as analysis.)

By eliminating the formal barriers to a practical design theory, other problems are introduced. The one directly introduced by the real system axiom is that of uniqueness. If there is always one solution, there may be many. In general there is. However, there is also a minimal real physical system for each set of physical parameters. This fact allows a practical method of standardization for the real physical systems which are considered worthy of preservation (e.g., motors, amplifiers, relays, transistors, etc.).

To approach the problem of standardization, a special type of equivalence between systems is useful. From T3-12 two systems are equivalent if they come from the same equivalence class of the size relation and the number of natural points is the same. Although this type of equivalence insures that two systems have an equivalent basic logic structure, T3-13 showed that this is not very restrictive. To have a useful standard approach, a stronger equivalence than T3-12 is needed. This is obtained by letting the natural points of two systems be equal. When this is the case, the systems can, from a practical point of view, be considered equal. To strengthen the "equality" even more, the number of parts is made equal.

T4-3: Let $A_{n_1v_1}$ and $A_{n_2v_2}$ belong to A. Then $A_{n_1v_1}$ and

$A_{n_2 v_2}$ are said to be equal under the equivalence relation ($A_{n_1 v_1} \overset{E}{\sim} A_{n_2 v_2}$) if and only if

(1) $\Pi_{n_1} P/A_{n_1 v_1} = \Pi_{n_2} P/A_{n_2 v_2}$.

(2) $v_1 = v_2$.

Equal systems play the role of equating all of the possible ways of writing a set of equations which describe the same "quantity". Although the information is not changed, the rules for finding the information changes. However, all operational rules are necessarily assumed in an approach of this nature. Hence, the new rules for formula manipulation are known in a practical case. For this reason, a standard form for the system relations can be chosen from any of the many possibilities. It is noted that the rules of some trivial systems (Π_n P) are so easy that the natural points can be obtained from any "form", independent of the parts v. It is for these cases and the dependent cases that the extra condition $v_1 = v_2$ is included.

The dependent cases have a "null" set of formulas that are included in the equivalence classes of $\overset{E}{\sim}$. These different null classes are obtained by letting the number of parts vary but keeping (1) of T4-3. The number of dependent equations in an E class relative to some independent set V_o can be thought of as the redundancy of an E class. The non-redundant E classes are those which are of primary interest in the solution of the design equation. The class of RPS are of this type.

T4-4: $S_{n_1 v_1}, S_{n_2 v_2} \in \text{RPS} \ni \Pi P_{n_1}/S_{n_1 v_1} = \Pi P_{n_2}/S_{n_2 v_2} \rightarrow S_{n_1 v_1} \overset{E}{\sim} S_{n_2 v_2}$.

Proof: From D4-3 and T4-3.

From the above results a standard representation for a real physical system can be adopted. This representation does not have to be changed unless new standard points change the elements of the parameter sets. Hence, it gives, abstractly at least, the standard points of the parameters involved.

The most restrictive elements of RPS are the exact systems. These are paralleled by the most restrictive equations which are studied in mathematics. A complete function ϕ on $\{P_n\}$ (D3-17) has the ability to yield a unique scalar for each unordered π_{n-1} p scalar tuple (e.g., $\Sigma x_i = 0$ each x_i a real variable would generate a complete function if used to represent ϕ of $\{P_i\}$).

D4-4: A system $A_{nv} \ni$ each ϕ_i is a complete function is said to be an exact system.

Many of the models of engineering will not be exact systems. However, they usually can be considered as "partially exact systems".

D4-5: A system $A_{nv} \ni$ each ϕ_i is a partial function is called a partially exact system.

Although there are many classifications which could be placed in between D4-4 and D4-5, these are the typical cases considered. This is illustrated by considering a system of PS which is not of D4-5. This being the case then, there is a relation ϕ which is not a partial function. Hence, there is no unique correspondence between any of the n - 1 parameters and the complement parameter. Hence, there is no strict "dependence" between any of the parameters. Although these systems are becoming common in advanced engineering, the systems based on equations from basic sciences are usually composed of partial functions (i.e., some two sets of parameters in each equation play the role of the "independent" and "dependent" variables. For instance, $e = \sin \omega t$, where e indexes voltage, ω frequency, and t time gives a partial function but not a complete function. It is usually considered that for each time and frequency there is unique voltage. Thus, the voltage is "dependent" on time and frequency.

The above discussion shows that, in general, a theory (system) which satisfies D4-5 has the ingredients of the theories developed in the basic sciences.

Example: Consider $S_{nv} \varepsilon$ RPS such that S_{nv} is an exact system. Then if P_1 is "dependent" on P_2, the converse holds. This follows from D4-4 and RPS. Hence, in an exact system the choosing of "units" to be used in the scalar value indexing set is quite arbitrary.

The next classification is for the purpose of finding useful conditions which allow determination of specifications which can be satisfied.

D4-6: A system A_{nv} with parameters $\{P_n\}$ is said to be compatible with $S_{n_1 v_1}$ if and only if $\forall\ x \ \varepsilon \ \Pi_n\ P$ $x + S_{n_1 v_1}$ ε MRS, where x is the natural point of S_{n_1} given by the formula ϕ and $\{P_n\}$ which gives $S_{n1} \ni$ $\Pi_n\ P/S_{n1} = x \leftrightarrow x \ \varepsilon \ \Pi_n\ P/S_{n_1 v_1}$.

The compatible specifications are the abstract classifications of systems denoting the parameters which can be arbitrarily restricted relative to the theory.

The conditions of D4-6 might seem severe. However, to strengthen conclusions based on the standard points of a system, the above type definition is useful. For all standard point models, unless the basic principles change, compatible specifications can be thought of as design problems which can always be solved by a particular model. Hence, a primary process of theoretical design, as imbedded in this design theory, is the process of testing two systems of compatibility. In practice there are cases where a real physical system has to be constructed; in others they are available.

In engineering the "device" systems are for the purpose of satisfying a limited number of arbitrary specifications. The more general systems (linear circuit systems, etc.) have a quite large application. In advanced engineering, systems are derived from less tested principles than those of the basic sciences. Thus, the need for a specification "test" independent of form becomes more important. (This is another way of looking at compatible specifications.)

The interesting conclusion which is available with compatible specifications is that the standard points are relatively immaterial. Hence, only the fundamental principles are involved.

That specifications are not compatible, in general, is well known. In fact, it can be done for real systems only if certain requirements are met. However, there are specifications which are compatible in any RPS, namely, those which "map" through the "system structure". In terms of functions of n variables, this idea is like picking the sets of variables which cannot possibly contradict the set of function statements. (It was a technique for rapidly performing this process which led the author to investigate a theory for design.) (11, 12).

The "system structure" refers to the "set interconnection" of the system relation parameter sets. This is a natural concept coming from the idea of systems being characterized by natural points which are logical sums of the natural points of subsystems. The "set structure" suggests geometrical representations of systems. For example, when a system has a natural point, it can be visualized quite easily as a network of loops and branches. Letting a parameter be a "branch" and an unordered relation a "closed loop" of branches gives a natural representation for a system.[1] This idea is used often to denote physical interconnection of "components". In fact, this theory gives a natural way to characterize the "interconnection relation" for any set of "components" which has a physical interconnection.[2] An example of such a network was given in Chapter III. Other ways of visualizing set interconnections geometrically are available. Probably the most notable is the Venn Diagram. In fact, these diagrams are used extensively in solving system problems where the parameters are indexed by only two values. A more useful and systematic way of looking at a system is by forming a matrix using the relations and parameters. This is illustrated in Chapter V. This form is analogous to the ordinary ordered n-tuple approach to n - 1 variable equations. Each relation is a column

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[1]This method of visualizing abstract systems was suggested by John C. Paul.

[2]See D4-16.

(n-tuple) and the rows are the parameters. Also, it is analogous to the linear graph theory concept of an incident matrix. (4).

For arbitrary systems these representations are only conceptual crutches. For the systems with more structure, such as the class RPS, they become deductively useful. The property of the RPS systems which is most advantageous is their E equivalence. That is, any form contains the same information relative to a set of standard points. Hence, any "path" which establishes compatibility in one system automatically establishes compatibility in the whole E equivalence class. The only paths that cannot be readily established are those involving "indeterminate" specifications. These are specifications which give rise to "overlapping" relations. The above ideas are made more definite in the following definitions and theorems:

D4-7: Let $\{P_{ni}\}$ be the parameter sets of $A_{nv} \varepsilon$ MRS. The set structure of A_{nv} is the collection of parameter sets. Each $\{P_{ni}\}$ will be denoted by (P_{ni}) throughout the remainder of this thesis for the purpose of convenience.

From D4-7 it is clear that the set structure of a system is set interconnected if and only if A_{nv} has a value other than the null value. Some set properties of systems involving n and v are given below:

T4-5: If $A_{nv} \varepsilon S$, then $n \leq \sum_{i=1}^{v} \sigma(P_{ni}) - (v - 1)$.

Proof: First note that $n = \sigma\left(\bigcup_{i=1}^{v} (P_{ni})\right) =$

$$\sum_{i=1}^{v} \sigma(P_{ni}) - \sum_{i=1}^{v-1} \sigma\left(\bigcup_{j=i+1}^{v} [(P_{nj}) \bigcap (P_{ni})]\right).$$

This is obtained by starting with any parameter set and removing the number of parameter appearances which occur in each of the remaining sets. Continuing this process gives the above identity. Since n and $\sum_{i=1}^{v} \sigma(P_{ni})$ are constant, the above is clearly independent of order.

This being the case, all that needs to be shown is that there is an order such that

$$\sum_{i=1}^{v-1} \sigma\left(\bigcup_{j=i+1}^{v} [(P_{nj}) \bigcap (P_{ni})]\right) \geq v - 1.$$

If v = 1, the above is satisfied. Hence, consider the cases v > 1.

Consider the $\{(P_{ni'})\}$ constructed in the following manner:

Choose

$$(P_{ni}) \text{ and } (P_{ni+1'}) \ni (P_{ni}) \bigcap (P_{ni+1'}) \neq \emptyset$$

if v > 1. These must exist by (5) of D3-1.

Choose

$$(P_{ni+2'}) \ni \text{, either}$$

$$(P_{ni+2'}) \bigcap (P_{ni+1'}) \neq \emptyset \text{ or}$$

$$(P_{ni}) \bigcap (P_{ni+2'}) \neq \emptyset.$$

Continuing in this manner a set $\{(P_{ni'})\}$ can be found such that

$$(P_{ni'}) \bigcap (P_{ni'+1}) \neq \emptyset,\ i' = 1, 2, \ldots v - 1.$$

Here the i' is not necessarily the same as used in the construction demonstration. Using the same order as i' in the above expression gives the desired result. (It is noted that when the equality holds, the system structure is necessarily of a particularly simple "geometrical" pattern.)

D4-8: Let $A_{n_1 v_1}$ be a specification for S_{nv}. $\{\Omega_i\} = \{\{P_{n_1}\} \bigcap (P_{ni})\}$ $i = 1, 2, \ldots v$ is called the restricted sets of S_{nv} relative to $A_{n_1 v_1}$. (The relative reference will only be used if confusion might arise.)

From D4-6 and D4-3 the next theorem gives the simple facts about when a compatible solution can and cannot exist. These are paralleled in ordinary n variable functions by the same conditions.

T4-6: Let $S_{nv} \varepsilon$ RPS and $\{\Omega_i\}$ be the restricted sets of $A_{n_1 v_1}$. Then:

(1) $\sigma(\Omega_i) = \sigma(P_{ni}) \rightarrow A_{n_1 v_1}$ is not compatible.

(2) $\left.\begin{array}{ll} 0 < \sigma(\Omega_i) < \sigma(P_{ni}) & i = k \\ \sigma(\Omega_i) = 0 \text{ otherwise} & \end{array}\right\} \rightarrow A_{n_1 v_1}$ is compatible.

T4-6 says that arbitrary specifications cannot be imposed on a system with the restrictive structure of RPS. (In fact, this will exclude all relations except the product relation.)

For the class S_{n1} of RPS, there is still the freedom given by (2). This allows all $2^n - 2$ distinct proper subsets of the parameters to be chosen arbitrarily.

The first condition of T4-6 gives a condition for which compatibility is easily observed in any system of RPS. To extend the second condition the set structure can be used to obtain sets of specifications which are compatible when v is greater than 1. The complement of a restricted set is useful in this connection.

D4-9: Let Ω_i be a restricted set of $A_{n_1 v_1}$ and S_{nv}. Then $\bar{\Omega}_i = (P_{ni}) - \Omega_i$ denotes the complement of the restricted set.

Also the following parameter set decomposition is useful when discussing compatible specifications.

D4-10: Given S_{nv} with a parameter set (P_{ni}). The following decomposition of (P_{ni}) into the sets $\{P_{D_{i+1}}\}$ is called the parameter set decomposition relative to S_{nv}. Let P_{D_i} be the elements of (P_{ni}) which do not appear in any of the remaining parameter sets. Let $P_{D_{i+1}}$ be the elements of (P_{ni}) which occur in only one other parameter set. This process can be continued out to the v - 1 other parameter sets. Hence,

$$(P_{ni}) = \bigcup_{j=0}^{v-1} P_{D_{i+j}}.$$

The above decomposition gives (P_{ni}) in terms of disjoint sets. Hence, the number of parameters in a parameter set can be written as a sum of the number of elements in the decomposition classes.

T4-7: (1) $\sigma(P_{ni}) = \sum_{j=0}^{v-1} \sigma(P_{D_{i+j}})$.

$$(2) \quad \sum_{i=1}^{v} \sigma(P_{ni}) = \sum_{i=1}^{v} \sigma(P_{D_i}) + \sum_{i=1}^{v} \sigma(P_{D_i + 1}) + \ldots$$

$$\sum_{i=1}^{v} \sigma(P_{D_i + v - 1}).$$

It seems that (2) above could be a convenient way for studying specification compatibility and systems. For example, a system is first decomposable if and only if the first sum on the right side of (2) is non zero for each i. This follows directly from the definition of system decomposition and D4-10. Also, the higher-order parameter decomposition classes indicate strong possibilities for "overlapping" equations. This is also indicated by elements of S_{nv} which have more than one characteristic subsystem.

T4-8: Let $S_{nv} \varepsilon$ RPS and S^k_{nv} denote the kth characteristic subsystem, $k > 1$. Then if (P_{ni}) is a parameter set of S^k_{nv}, $P_{D_i + j} = \emptyset \rightarrow j < k$.

This follows directly from the properties of decomposable systems and parameter decompositions. Thus, it is noted that the two types of decompositions are connected as in T4-8. The next theorem which follows directly from D4-10 gives a simple condition for "overlapping" relations.

T4-9: Given A_{nv} with a non-empty parameter decomposition class $P_{D_i + k}$, then if v' is the number of elements of this class, there are k distinct relations on v' distinct parameters.

It is noted that if $v' < k$, then there are more "equations than variables".

To choose a set of compatible specifications, the important criteria is to leave one "degree of freedom" at all times (i.e., keep the condition of T4-6 (2) satisfied). The following process is for the purpose of choosing specifications which are compatible and, in addition, choosing a set which "fixes" the system.

T4-10: Given $S_{nv} \varepsilon$ RPS, a specification $A_{n_1 v_1}$ on the parameters $\{P_{n_1}\}$ which is equal to the set of parameters determined below is a compatible specification.

Proof: Let $P_{n_{11}}$ be an arbitrary element of $\{P_n\}$ of S_{nv}. Let $\{\overline{\Omega}_{k_1}\}_1$ be the restricted complements of the sets $\Omega_{k_1} \ni \sigma(\overline{\Omega}_{k_1})_1 = 1$ determined by $P_{n_{11}}$. (Note that the classes S_{11} will be excluded from the discussion.) Now, either $\{\Omega_{k_1}\}_1$ is empty or it is not. Assume it is not empty. Let $\{\phi_{k_1}\}_1$ be the relations of S_{nv} which correspond to the restricted complements $\{\overline{\Omega}_{k_1}\}_1$. Consider $S_{n_{11}v_{11}}$ to be the new system which is formed by removing the relations $\{\phi_{k_1}\}_1$ from S_{nv} (i.e., $S_{nv} = S_{n_{11}v_{11}} + \{\phi_{k_1}\}_1$). Now $v_{11} = v - \sigma\{\phi_{k_1}\}_1$ and $n_{11} = n - \sigma\{\phi_{k_1}\}_1$ by properties of RPS. Consider the sets $\{\Omega_{k_2}\}_1$ determined by $P_{n_{11}} \bigcup \{\overline{\Omega}_{k_1}\}_1$ and $S_{n_{11}v_{11}} \ni \sigma(\overline{\Omega}_{k_2})_1 = 1$. Repeat the process as done above. This gives $S_{n_{12}v_{12}}$ such that $n_{12} = n - \sigma\{\phi_{k_1}\}_1 - \sigma\{\phi_{k_2}\}_1$ and $v_{12} = v - \sigma\{\phi_{k_1}\}_1 - \sigma\{\phi_{k_2}\}_1$. Let $S_{n_{1j}v_{1j}}$ denote the system obtained after repeating the process j number of times. j is associated with each $P_{n_{1i}}$ which gives a set $\{\bar{\Omega}_{k_j}\}_i$ with the above properties.

Now the process above must terminate due to the finite properties of RPS. Also, it can only terminate with the conditions that some $\{\bar{\Omega}_{k_j}\}_1 = \emptyset$ or $S_{n_{1j}}\, v_{1j} = Soo$. Assume it terminates with Soo. Hence, $v = \sum_{i=1}^{j} \sigma\{\phi_{k_i}\}$, $n = \sum_{i=1}^{j} \sigma\{\phi_{k_i}\} + 1$, or$\left(\bigcup_{i=1}^{j} \{\bar{\Omega}_{k_i}\}_1 \bigcup P_{n_{11}} = \{P_n\}\right)$which gives, since each $\bar{\Omega}_{k_j}$ must be distinct, by RPS $v + 1 = n$. Hence, there is only one more parameter than relations. Assume $\{\bar{\Omega}_{k_j}\}_1 = \emptyset$ and $S_{n_{1j}}\, v_{1j} \neq Soo$. This gives $S_{n_{1j}}\, v_{1j}$ and $P_{n_{11}} \bigcup \left(\bigcup_{i=1}^{j} \{\Omega_{k_i}\}_1 \right)$ which does not give restricted complements as indicated. Choose a $P_{n_{12}}$ from the parameters of $S_{n_{1j}}\, v_{1j}$ which will give the desired set of restricted complements. Also, $P_{n_{12}} \not\in [\bigcup_{i=1}^{j} \{\Omega_{k_i}\}_1$. If this is not possible, choose $P_{n_{13}}$, etc. This process must yield results since $S_{n_{1j}}\, v_{1j} \neq Soo$ and is finite. Let $P_{n_{1\ell}}$ be the parameter required to obtain the above sets. Hence, $\{P_{n_{11}} \ldots P_{n_{1\ell}}\} \bigcup \{\bar{\Omega}_{k_j}\}_1 \bigcup \{\bar{\Omega}_{k_j}\}_\ell$ is the parameter set used to find $S_{n_{1\ell+1}}\, v_{1\ell+1}$. Also, it is noted that each element of $\{P_{n_{11}} \ldots P_{n_j}\} \bigcup \{\bigcup \bar{\Omega}_{k_j}\}_1 \bigcup \{\bigcup \bar{\Omega}_{k_j}\}_\ell$

must be distinct by the construction process and the properties of RPS.

Hence, the number of elements in the above set must always remain less than n. Also each element $\bar{\bar{\Omega}} \varepsilon \bigcup_{i=1}^{v} (\{\bar{\bar{\Omega}}_{k_j}\}_i)$ must remove one and only one relation from S_{nv} by construction and RPS. Hence, when Soo is reached, this gives $\sigma \left[\bigcup_{i=1}^{v} (\{\bar{\bar{\Omega}}_{k_j}\}_i) \right] = v$. Also, each parameter of $\{P_n\}$ belongs to either $\{P_{n_{1i}}\}$ or the restricted complements, but not both. Hence, $n = \sigma \{P_{n_{1i}}\} + v$. This is the generalization of the case when i = 1. Hence, the process needs n - v parameters to reduce S_{nv} to Soo. A set of parameters $\{P_{n_1}\}$ and S_{nv} which corresponds to a set with the above properties is said to fix the system. The fact that a set which fixes a system is compatible comes directly from the construction process and the properties of RPS. That is, the construction process insures that the set $\{P_{n_1}\}$ only determines projections of the relations of S_{nv}. Also each relation is used only once to check every π_{ni} p. This along with RPS forces compatibility.

The above process is easier to actually perform than to theoretically demonstrate. The performance is simple when using a matrix to represent a system (e.g., see Figure 5-1). Also, it is noted that the above process always yields a compatible set of specifications which fix the system independent of which parameter is used to start the process. Hence, there is at least n distinct sets possible by the above process. Actually the upper bound for the number of distinct fixed specification sets is given by

$$\frac{n!}{(n - v)!\ v!} .$$

This is simply the number of distinct ways that n things can be chosen (n - v) at a time. The size of this number can be large with relatively few parameters. This indicates the need for having a method for looking at systems and specifications in a rapid manner.

The fact that there is no "simple" formula for finding the number of compatible specifications which fix a system is suggested by the systems studied using linear graph theory. (13). Here the number of "trees" which uniquely specify a graph corresponds to the number of distinct compatible specifications which fix a system. The correspondence is obtained by letting the "circuits" be represented by unordered relations and the "branches" by parameters. The systems characterized by linear graphs are relatively simple (referring to the linear algebra which can be used to describe these systems), and yet a formula in terms of n, v, and n - v has not been developed to give the number of distinct "trees". (To the author's knowledge this has not been done.)

The process in T4-10 suggests that any specification set with more than n - v parameters will not have the compatible properties of the fixed-type specifications. The process of T4-10 can be utilized to indicate the validity of this idea.

T4-11: Given $S_{nv} \varepsilon$ RPS and $\{P_{n_1}\} \subseteq \{P_n\}$ of $S_{nv} \ni \sigma \{P_{n_1}\} > n - v$. Then either

(1) $\exists\ \overline{\Omega}_i = \emptyset$ or $(\overline{\Omega}_i) = (\overline{\Omega}_j) \ni \sigma (\overline{\Omega}_j) = 1$ (and $\Omega_i \neq \Omega_j$) or $\overline{\Omega}_i = P\ \varepsilon\ \{P_{n_1}\}$ or

(2) $\exists\ \{\overline{\Omega}_i\}_1, \{\overline{\Omega}_j\}_2 \ni \{\overline{\Omega}_i\}_1 = \{\overline{\Omega}_i\}_2$ and $\sigma (\overline{\Omega}) > 1\ \forall$ $\overline{\Omega}\ \varepsilon\ \{\overline{\Omega}_i\}_1$ or $\{\overline{\Omega}_j\}_2$ and $i + j < \sigma (\{\overline{\Omega}_i\}_1 \bigcup \{\overline{\Omega}_j\}_2)$.

Assume that none of the conditions of (1) hold. Let $P_{n_{11}}\ \varepsilon\ \{P_{n_1}\}$ be used to start the process of T4-10. Now by the above assumption, there are no elements of $\{\overline{\Omega}_{kj}\}_1$ which are members of $\{P_{n_1}\} - \{P_{n_{11}}\}$. Instead of choosing element $P_{n_{12}}$ as in T4-10, choose $P_{n_{12}}$ from $\{P_{n_1}\} - \{P_{n_{11}}\}$. Now if none satisfy the restricted complement criteria, then choose another and test for restricted complement set with desired properties. Now if the set $\{P_{n_1}\}$ is exhausted before the restricted complement criteria is satisfied, this gives the distinct sets $\{P_{n_1}\} \bigcup (\{\overline{\Omega}_{kj}\}_1)$. However, the following is noted about $\sigma \{P_{n_1}\} > n - v$ when the conditions of (1) do not hold. Assume $\{\Omega_i\}$ relative to $\{P_{n_1}\}$ is less in number then v. Let $\{\phi_i\}$ be the relations whose restricted sets are empty relative to $\{P_n\}$. (The argument is carried out based on first decomposable systems but is shown to be no weaker for any system of RPS.) Let $S_{n'v'}$ be the system obtained by removing $\{\phi_j\}$. Now $n' \leq n - r$ $(r = v - v')$ by properties of RPS and decomposability. Hence $\sigma\{P_n\} > n' - v'$. Also, note that $\sigma\{\overline{P}_{n_1}\} + \sigma\{P_{n_1}\} = \sigma \{P_{n'}\}$, where $\{\overline{P}_{n_1}\} = \{P_{n'}\} - \{P_{n_1}\}$. Thus from the hypothesis $\sigma\{P_{n_1}\} > n - v$, the above shows that $\sigma\{\overline{\overline{P}}_{n_1}\} < v'$. Now let $\{\overline{\overline{\Omega}}_i\}$ be the restricted complements of $\{\Omega_i\}$. Then $\overline{\overline{\Omega}}_i \subseteq \{\overline{\overline{P}}_{ni}\} \; \forall \; i$ by definition of $\overline{\overline{\Omega}}_i$ and $\{\overline{\overline{P}}_{n_1}\}$. Hence, there are more relations than there are parameters for $\{\overline{\overline{\Omega}}_i\}$. Also, there is $\overline{\overline{\Omega}}'_i \ni (\bigcup \overline{\overline{\Omega}}_i - \overline{\overline{\Omega}}'_i) = (\bigcup \overline{\overline{\Omega}}_i)$, while $\sigma\{\overline{\overline{\Omega}}_i\} < v'$. In case S_{nv} is not first decomposable, an additional fact must be considered. When any of the elements of $\{\phi_j\}$ discussed above belong to decomposable classes (say the kth), then its removal will not change n. However, it also was not considered in the sets $\{\overline{\overline{\Omega}}_i\}$. Hence, these factors have "equal" and "opposite" influence on the relation $\sigma\{\overline{\overline{P}}_{n_1}\} < v'$ if these sets are included in $\{\overline{\overline{\Omega}}_i\}$. This result along with properties of T4-10 gives the desired result (i.e., if it is assumed some $P_{n_{1_\ell}} \in \{P_{n_1}\}$ satisfies the criteria of the construction process, then eventually a condition of (1) arises or the requirements for (2) as just discussed).

The cases of (2) in T4-11 indicate the conditions for two sets of "overlapping equations" defined on the same variables. Detailed investigation of compatible solutions and "non-compatible" solutions under more specialized conditions on S_{nv} have not been carried out by the author. However, a main characterization which can be used in further investigations is that of the fixed type specification. These can exist if and only if the specification set is of size n - v. This follows from the result of T4-10 and T4-11. Also T4-11 gives three simple conditions for which a set of specifications are not compatible.

D4-12: A specification $A_{n_1v_1}$ is said to be incompatible with S_{nv} if any of the following conditions prevail for the restricted complement sets determined by the procedure of T4-11 using $\{P_{n_1}\}$ of $A_{n_1v_1}$.

(1) $\exists\ \bar{\bar{\Omega}}_i = \emptyset$.

(2) $\bar{\bar{\Omega}}_i = \bar{\bar{\Omega}}_j \ni \Omega_i \neq \Omega_j$ and $\sigma(\bar{\bar{\Omega}}_i) = 1$.

(3) $\bar{\bar{\Omega}}_i = P_{n_1i} \in \{P_{n_1}\}$.

T4-12: Let $S_{nv} \in RPS$ and $A_{n_1v_1}$ be specified for S_{nv}. If $\sigma\{P_{n_1}\} > n - v$ and (2) of T4-11 do not hold, then A_{nv} is incompatible with S_{nv}.

A specification and system which create the condition (2) of T4-11 will be called an indeterminate specification. Strictly speaking, the condition is relaxed to include the "overlapping relations" which can occur for $\sigma\{P_{n_1}\} = n - v$.

D4-13: Let $S_{nv} \in RPS$ and $A_{n_1v_1}$ be a specification which is not incompatible. Then if $\{\bar{\bar{\Omega}}\}$ is the set determined by the procedure of T4-11 and $\exists$ a subset $\{\bar{\bar{\Omega}}_i\} \subseteq \{\bar{\bar{\Omega}}\} \ni i \leq \sigma(\bigcup \bar{\bar{\Omega}}_i)$ A_{nv} is said to be an indeterminate specification.

In the classification of types of solutions to the design problem the restrictions can be made finer and finer in order to increase the deductive properties. In this thesis three main classifications have been given. These seem to be a natural division of solutions for all types of deductive models.

A classification in the direction of system standardization would involve partitioning into what might be called applied and design parameters. These are parameters which, conceptually at least, separate the "application" parameters from the "component" parameters. This is done to a large extent by the analytical studies which are performed on a particular model [e.g., the amplifier system is usually analyzed relative to gain, bandwidth (applied) vs. several parameters which are used in the amplifier circuit (design)]. The author does not see a method of separating these two types of parameters relative to a system except by axiom. That is, these are partitions of (S) by parameters. Similarly, the real physical system was a partition largely by the logic properties of the unordered relations. The problem with the design and applied classifications is that they may not be different. Hence, an additional assumption is needed in order to partition (i.e., many "theoretical" problems must be looked at from every "angle". Hence, any "abstract" system relating the parameters of the problem should not be partitioned by the applied and design classifications. On the other hand, a system representing a "physical device" would seldom not be partitioned by the applied and design classifications).

Axiom of parameter classification:

If $S_{nv} \in RPS$, then $\exists$ two subsets $\{P_{n_a}\} \neq \emptyset$ and $\{P_{nd}\}$ of $\{P_n\}$ called the applied parameters and the design parameters $\ni \{P_{n_a}\} \cup \{P_{n_d}\} = \{P_n\}$. In addition, there exists some elements of RPS, $S_{nv} \ni \{P_{nd}\} \neq \emptyset$ and $\{P_{n_a}\} \neq \{P_{nd}\} \cap \{P_{n_a}\} \neq \emptyset$ $\{P_{nd}\} \cap \{P_{na}\} \neq \{P_{nd}\}$.

When a system is partitioned by $\{P_{n_a}\}$ and $\{P_{n_d}\}$, it will be called a device. (Partitioned implies the restrictive conditions of D4-13 hold.)

D4-14: The class of PD $\subset$ RPS such that each element of PD is partitioned is called the class of device systems.

Now it is noted that PD is properly contained in RPS. This is because the class S_{1v} cannot possibly belong to PD. In fact, the above axiom of parameter classification says that a device system (theory) must have at least three parameters involved.

T4-12: $\forall S_{nv} \in PD \quad n \geq 3$. This follows from the conditions of the axiom of parameter classification.

This agrees with the well-known fact that to "measure" a component (describe) there must be at least two different (disjoint) parameters which can be related to the component. (1). Also, this is evident in the ingredients for a mathematical operation. (9). Here at least three elements are involved in order to have something more than a logical identity. In engineering the number of classes for the systems of interest are usually much greater than three. The "simple" relay system indicated in Chapter V is an illustration.

There are many systems which have a convenient relationship between the set structures of equivalent systems. The parts of these systems are already known to be equal as well as the parameters. In many systems an additional feature is present. In terms of ordinary mathematical equations, this feature is commonly noticed when solving for a variable in one equation and placing that variable in another equation. During this process, some of the variables other than the one being replaced might be eliminated. For example, given

(1) $a\ b^2 = c\ e$ and

(2) $d\ b\ c = a$,

in ordinary algebra if (2) is used to eliminate (a) from (1), then (c) is also eliminated. Hence,

(3) $a = d\ b\ c$ and

(4) $d\ b^3 = e$

represent the same equations as (1) and (2).

The above implies that there is only one independent relation between (a) and (c). If it was known otherwise, then the conditions of the next definition would apply; and the above operation could be done abstractly by using the set structure between (1), (2) and (3), (4).

D4-15: Assume $S_{nv} \varepsilon$ RPS. Let (P_{ni}) and (P_{nj}) be elements of the set structure of $S_{nv} \ni (P_{ni}) \bigcap (P_{nj}) \neq \emptyset$ and $i \neq j$. Let $P' \varepsilon (P_{ni}) \bigcap (P_{nj})$. Then if $\exists S'_{nv} \ni S'_{nv} \overset{E}{\sim} S_{nv}$ and $\exists (P'_{ni})$ of $S'_{nv} \ni (P'_{ni}) = (P_{ni}) \bigcup (P_{nj}) - P'$, then S_{nv} is said to be completely independent.

Hence, for the class of systems which are completely independent, "equation solving" can be done using the set structure of the system. From the example and D4-15 when the conditions of D4-15 are paralleled except p' is more than one parameter, S_{nv} is not completely independent. This type of condition appears to be a characterization of "physical interconnection". In a sense these systems seem to be very "dependent" on their set structure. In terms of equation solving this can be thought of as the strong form of D4-15.

D4-16: Assume $S_{nv} \varepsilon$ RPS. Let (P_{ni}) and (P_{nj}) be elements of the set structure of $S_{nv} \ni (P_{ni}) \bigcap (P_{nj}) \neq \emptyset$, $i \neq j$. Then if $\exists\, S'_{nv} \ni S'_{nv} \overset{E}{\sim} S_{nv}$ and $\exists (P'_{ni})$ of $S'_{nv} \ni (P'_{ni}) = (P_{ni}) \bigcup (P_{nj}) - (P_{ni}) \bigcap (P_{nj})$, S_{nv} is said to be physically interconnected.

The physically interconnected systems have nice properties as in D4-15. They are paralleled by the sets of equations which involve only addition or multiplication (i.e., only one fundamental operation is involved). These types of systems appear to be similar to those studied in the linear graph theory.

This chapter has included the basic structure of a design theory. A finer subdivision of this theory is not extended in this thesis because of the newness of the theory. Before extending the classifications, the underlying principles and the "best" direction of investigation should be studied. Many of the classifications given in this thesis have not been explored with any detail. Some of the classifications, such as in D4-16, seem to be characterizations of systems presently studied in terms of particular algebraic operations. This approach to systems offers a way to eliminate much of the duplication between analysis theory and design. The analysis phase of engineering deals with the particular number systems and formulas. That these studies are for the purpose of indexing a set of scalars is known. The design theory developed in this thesis assumes that the properties of these mathematical formulas are known relative to the classifications given for unordered relations and parameters. This being the case, the large systems can be analyzed with much of the deductive power and simplicity of a single formula.

CHAPTER V

SUMMARY

The body of this thesis was split into three parts as introduced in Chapter I. Before discussing applications and further investigations, a review of the body is given.

In Chapter II the concept of the parameter is introduced. The parameter is a logic class of elements which is indexed by some indexing set A. The parameter (set of scalars) and the indexing set (set of scalar values) are disjoint sets. Once this concept is established, the idea of collections of disjoint parameters leads to the unordered relation concept. This is naturally connected to the ordered theory of mathematics by use of the indexing set. This is done by letting indexing sets of related classes be subsets of the same set (e.g., let each parameter be indexed by a subset of the reals).

The unordered relations are introduced through the classical idea of the ordered n-tuple used in mathematics. This was unnecessary, but seemed appropriate to indicate the problems which would be involved in trying to develop polyadic (higher dimensional) relations by using ordered sets. Also, it allowed the ordered terminology to be naturally adapted to unordered relations.

The unordered relations can be thought of as axioms which relate the elements from disjoint classes. The logic formula needed to indicate which elements are related and which are not is assumed. This is assumed because of the preciseness of the mathematical operations used by the engineer.

The unordered relation is characterized by the unordered n-tuple. This is the ordered n-tuple with the order being removed by the use of the formula and disjoint parameters. This allows n variables (parameters) to be formally classified with the same ease as used in two variables (ordered pairs). The main concept in unordered relations is that of the natural point. These can be thought of as n elements, one from each parameter, which do not contradict any of the unordered relations (axioms). These are well defined in terms of the unordered n-tuples of the distinct relations.

Most of the terminology in Chapter II, as in Chapter III, is directed towards Chapter IV on design. One of the important concepts in engineering design is the mathematical model. The basic properties of these models are formulated in Chapter II. These were called property c and property cc. These properties are automatically given to an engineering model which abstractly relates certain classes of "observable" (the idea of a principle was used here) parameters. In terms of assumptions these are not unduly restrictive. The only requirements are (1) the observables must satisfy the universal law of "x is x", and (2) the model must be formed by principles (postulates, rules, etc.) which "restrict" natural points when applied. (The "restriction" applies only in certain cases given by property cc.) These properties, along with a decomposition process, give a simple method for testing a physical model for independence. The model is independent if and only if it is decomposable. This is a set operation on the parameters and is quite easily applied.

Several other properties are given to these parameter models such as being algebraic. An algebraic unordered relation is one whose parameters are indexed by a common indexing set. This implies that the unordered relation has its counterpart in some ordered relation on a set. This keeps the ordinary concept of value connected to the relations. However, the normal concept of value is not directly attached to the unordered relations. (There does not seem to be a real problem in this respect. The formulas in most actual engineering problems will be defined on the elements of some ordered indexing set.)

Chapter III introduces the system. The system is the "elements" with which the engineer works. These are collections of unordered relations and parameters with certain properties. The properties are quite unrestrictive, yet restrictive. There are two main classifications of systems: the partial system (A), and the system (S).

The main property of a system is to have the basic structure of set connectedness. This is for the purpose of having natural points. Systems which have natural points represent theories which are logically consistent. Those which do not, represent theories which are inconsistent. The system can be "naturally" classified in several ways. The first classification used in this thesis is by the "parts" (v) and the parameters (n). The "parts" represent the number of unordered relations (axioms). The ordered pairs of integers give an equivalence relation on (A).

Another equivalence relation given is that involving the natural points of the systems. This equivalence is basically like that of set equivalence (i.e., same cardinality of natural points). The additional restraint of being in the same "size class" (the equivalence relation above) is also required. This equivalence relation shows that logic systems can be "interchanged" at will if their natural points and class size are in 1-1 correspondence.

Many of the classifications of systems are directly from those of unordered relations. The parameter model shows up as a physical system in Chapter III. The decomposition classes become characteristic subsystems. Two interesting classifications are the partially exact and exact systems. The partially exact system is one in which each unordered relation is a partial function relative to some n-1 subset of its n parameters. This shows the ease of describing ordinary n variable ideas in terms of unordered relations. The exact system is obtained by having each unordered relation be a complete function. The complete function is an unordered relation which is a partial function relative to each n-1 subset. These play the role of the highly restrictive systems. The functions of n variables which have these properties play the same role in ordered theory (e.g., linear algebraic equations have these properties).

A single unordered relation could have been defined for a system by logically adding the parts of a system. This was not done since it does not appear to be practical (i.e., the parts are given by formulas which are known; hence, the development of system theory should give the important relations between parts through their properties). However, this idea does support the viewpoint that equivalence of abstract systems basically involves only the number of classes and the number of natural points.

The idea of a specification and solution is introduced by the use of natural points and the concept of system addition. The addition of two systems is defined as the union of their respective parts and parameters.

The physical system is defined to have the basic properties of a parameter model. It is this class of system which is of primary interest in an abstract design theory. These play a role in the general solution of the design problem.

Chapter IV deals with the formulation of a theory for design. Although there is no conceptual limitation to engineering design, this was the motivating area. Also, this is an area where many "definite" formulas are available. The design problem is formulated in a concise manner. Definite boundaries on the existence of general solutions are established which coincide with accepted principles. Unfortunately, only one of these boundaries gives useful information in a direct manner. This is the nonexistence of solution. The other boundary gives at least one theoretical model as long as nonexistence has not been shown. Unfortunately, this model is the "least" restrictive model. This being the product relation which in essence says, "You can assume it can be done if it has not been shown otherwise". The nonexistence criteria comes from contradictory specifications. These are those specifications for which it can be shown (without a model) there are no natural points. This fact gives an immediate nonexistence under the properties of systems.

In order to obtain more deductive power on the positive (existence) side of the design solution, more properties were established. These had to be assumed.

To choose properties in any "absolute" sense was not thought to be realistic. (That is, the physical principles seem to stay fairly stable, but the application of these principles (projection into some indexing set) seems to result in an "unstable" set of natural points. This could be caused by a number of factors. One which is commonly accepted is that "certain assumptions have to be made". The stability of the natural points seems to be "proportional" to these assumptions. Also, the indexing set which is used might not be "naturally" suited for the parameters (classes) of interest (i.e., to partition a parameter finer than it can be "observed" (measured) can be thought of as an unnatural indexing set). The basic sciences offer a good example in the case of the indexing set of reals. There are no known meters which can measure this fine a partition.

What is important is that there does appear to be indexing sets which allow the projection of these principles to the extend of being restrictive. These are the models which the engineer has available to solve design problems in a logical manner. This was the basic assumption used to formulate the axiom of real physical systems. This axiom also assumed a "uniqueness" of parts relative to the natural points of two systems. This "uniqueness" comes from making the number of parts the same when two systems have equal natural points. Also, it was assumed that each part (unordered relation) had some deductive power. In addition, it was required that each n - 1 projection of a part was onto.

Although the real physical systems are not necessarily the most restrictive, they do have many restrictive properties important in design. Probably the most useful property is that of $\overset{\varepsilon}{\sim}$ equivalence. The real physical systems (RPS) which have equal standard points have the same number of parts. (The standard points refer to the natural points of a particular real physical system.) This allows a standard set structure for a system relative to a set of standard points. In fact, the same parts can be used for all the standard point models which have these parts even though they might not be equal. This is the power of being able to formally separate the indexing set from the parameter set. This approach gives the restrictive properties to the system value (natural points) rather than to some particular operations in a particular indexing set. That is, properties can be assigned to the natural points of the unordered relations and the deductive results shown independent of a particular mathematical model. (This is similar to the mathematical studies in abstract algebra. (9). Here the operations defined in a set are classified independent of the set. This is advantageous because of the deductive power of some of the common properties, for example, identity, inverse, associativity, etc.)

The type of solution to the design problem considered the most important by this thesis is that associated with the compatible specifications. These are specification systems whose values can be added to some real physical system and still be a measurably rational (logically consistent) system. This type of specification can be thought of as those for which the real physical system (RPS) is naturally compatible. The compatible specifications are those which can be satisfied by any RPS having the same number of parts and equal parameters in these parts (i.e., the compatible specifications are "independent" of the functional form).

Some of the set conditions which can be used to check rapidly for compatibility are given. Several of these are facts which are commonly used. These show up in "simultaneous equation" studies of mathematics (e.g., in general the number of variables must be greater than or equal to the number of equations, etc.). The set structure of a system is defined as the collection of parameter sets of the unordered relations. This gives an easy way to visualize some of the basic "interconnection" properties of systems. These become more realistic as the properties of RPS become more restrictive. In the case of parameters indexed by two elements, the RPS systems can be thought of as logically consistent formulas on variables with two values. Hence, the tools of Boolean algebra are available for problem solving.

The set structure can also be used for the class of systems in which "equation solving" can be paralleled by set operations. This allows the different equivalence classes to be obtained abstractly. This "by-passes" the manipulations of the particular formulas.

Classifications for the purpose of choosing systems which can satisfy many specifications are needed in engineering. This is for the purpose of being able to standardize useful engineering models by some practical, as well as stable, criteria. Classifications for this purpose in terms of applied and design parameters are given. These can play the role of the specifications which are given to the engineer, and those which he must choose relative to a given system. It is the sum of these two parameter sets which should be included in a practical model.

Applications for the design theory of this thesis seem to be in abundance. However, the foundations work excluded detailed investigations for applications. For this reason, the indicated applications in the body were in the form of examples and conjectures. Most of these suggest broad areas of investigations (e.g., synthesis procedures, "interconnection" algebra, etc.).

An example of the matrix form of representing a system is given in Figure 5-1. This design "map" represents the system of a relay. The parameters are indexed by the reals in the formula below. (11).

$$\phi_1 \rightarrow \quad \eta = \frac{(157.5)\ R_t\ (x_o + \alpha)\ \sqrt{2P_o}}{EN\ \sqrt{\mu_o\ A}}$$

$$\phi_2 \rightarrow \quad t_p = (10^{-8})\ \frac{\eta^2\ \mu_o\ A}{(x_o + \alpha)R_t}\ \ln\left[\frac{1}{1 - \eta}\right]$$

$$\phi_3 \rightarrow \quad k = (8.66 \times 10^{-3})\left[\frac{36\ M\ x_o^2\ R_t}{E^2\ \eta\ (1 - \eta)}\right]^{1/3}$$

$$\phi_4 \rightarrow \quad P = \frac{E^2}{R_t}$$

$$\phi_5 \rightarrow \quad R_t = R_s + R_c$$

$$\phi_6 \rightarrow \quad R_c = \frac{(.865 \times 10^{-6})\ s^2\ (1 - \beta - \sigma)\ (1 + \beta + \sigma)\ \ell g_r}{\delta^4}$$

$$\phi_7 \rightarrow \quad N = \frac{(.637)\ \ell s\ (1 - \beta - \sigma)\ g_n}{\delta^2}$$

$$\phi_8 \rightarrow \quad \mathcal{R}_p = \frac{(x_o + \alpha)}{\mu_o\ A}$$

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}	
η	η	η	η								
$\mathcal{R}_p$								$\mathcal{R}_p$			
E	E		E	E							
k			k							k	
ℓ						ℓ	ℓ				
M			M								
N	N	N					N				
P				P							
P_o	P_o										
R_c					R_c	R_c					
δ						δ	δ				
R_t	R_t	R_t	R_t	R_t	R_t						
R_s					R_s						
S						S	S		S		
T_p		T_p								T_p	
T_s										T_s	
X_o	X_o	X_o	X_o					X_o			
μA	μA	μA						μA	μA		

Figure 5-1. Design Map of Relay System

$$\phi_9 \rightarrow \mu_o A = \frac{\mu_o \pi a^2 s^2 \beta^2}{4}$$

$$\phi_{10} \rightarrow t_s = t_p + k$$

The above formulas can be put in many forms by using the indicated operations in a "correct" manner. In general, the "correct" manner for manipulation of formulas requires more knowledge than just the mathematical operations involved.

The above refers to the onto aspect of the formulas and the "range" of the indexing set. The class of RPS is indexed by the same set and is, in addition, an onto relation. Hence, the range of each parameter must be preserved in any mathematical operations.

Example: $\delta = \sqrt{\frac{.637 \text{ ls } (1 - \beta - \alpha)\, g_n}{N}}$ not $\delta = \pm\sqrt{\frac{.637 \text{ ls } (1 - \beta - \alpha)\, g_n}{N}}$

The additional rules are usually necessary because the indicated operations are defined over the whole indexing set while the parameters are only indexed by a subset (i.e., algebraic does not require the indexing sets to be identical, just a subset relation must hold).

It is noted that there are variables in the formulas which do not appear in the system map. This points out one of the "conveniences" of design by theory. The variables β, σ, α, g_r, g_n, a, can be thought of as "restricted" variables. Their ranges are very limited; however, they are theoretically parameters. This allows them to be used to satisfy specifications. Also, they are definitely not applied parameters. Hence, they can be used to "change" the standard points in order to satisfy incompatible or indeterminate specifications when the need arises.

It is noted that the upper bound for the number of compatible specifications in the above system is given by

$$\frac{18!}{(8)!\ 10!},$$

which is almost a maximum for a fixed number of parameters. The above type of representation for systems offers a general method for investigating engineering design problems. The theory behind the representation is that which can be developed by placing properties on the class of RPS as done in Chapter IV.

The actual processes of design are paralleled by the steps involved in solving the general design equation. For these reasons the most immediate application of the design theory would be that of supplementing engineering analysis with design theory. This would allow a philosophy of engineering which could systematically correlate particular model studies in an area. The particular model studies will be used in a particular design when possible (i.e., actual numerical relations are needed and available). Also, in this thesis the requirements for finding a "new" system are seen to be quite definite. Hence, investigation of new models might analytically become a part of the engineering. These requirements are already imposed on the engineer, but principles seem to be lacking.

This last suggestion for application is connected with synthesis studies for design problems. Assuming the properties of RPS, there are definite relations among the set structure which must hold in general. For specifications which pertain to certain types of physical interconnections (these are the class of RPS which have the simple set operations paralleling the algebraic operations in the indexing set), these conditions indicate some useful configuration synthesis techniques might be obtained.

The natural point equivalence between systems suggests interesting possibilities of changing systems in order to work problems faster. This is presently done in linear systems through the use of the Laplace transform. Here the system is converted to the complex domain in order to solve many problems faster. Also, the transforming of a system index set into two values automatically transforms a system into a simple type (Boolean) <u>if</u> the new standard points do not change the parts of the system. These areas suggest design procedures which may be very efficient.

The technical areas for future investigation should involve the foundations of this thesis. The unordered relation appears to offer a practical method for studying problems with a large number of variables and relations. If this is the case, the classifications of this thesis should be examined for inconsistencies which might have been overlooked. Also, more useful classifications and facts might be developed using parameters and unordered relations.

APPENDIX

FUNDAMENTALS OF SETS AND RELATIONS

The undefined concepts and terminology used in the body of this thesis, which might be unfamiliar, are presented below. Listed in Table I at the end of this section are the common logic symbols and their definition.

The notation of a set and the associated ideas and terminology are the first to be presented. For a more extensive coverage of these topics the references 6, 7, 9, and 14 are suggested.

DA-1: P is called a set if and only if ($\leftrightarrow$) there exists ($\exists$) an element x such that ($\ni$)x belongs to (ε)P or P is empty ($\emptyset$). This is formalized by the symbols denoted above as:

(a) P is a set $\leftrightarrow$ $\exists$ x $\ni$ x ε P or P = $\emptyset$, also

(b) P = $\{x_1 \ldots\}$ or P = $\emptyset$ is used to denote the set P.

The empty set $\emptyset$ is sometimes called the null set. Note that if P has any element, then P is <u>not</u> empty. The notation used in (b) is common and suggests that the elements must be listed in order to confirm if P is not empty. This is not practical when many elements are involved. For this reason a particular set is postulated by a schema for separation using a formula ϕ. This method separates the distinct elements under consideration by using a common property which they, and only they, enjoy. This Axiom Schema Method is given as:

DA-2: When there is a set A and property ϕ then,

$$\exists \; P \ni \forall(x)\big(x \in P \leftrightarrow x \in A \text{ and } \phi(x)\big)$$

where ϕ (x) does not involve P, which is read, "There is a set P such that x is an element of P if and only if x is a member of A, and x has property ϕ. This is also written in a more compact notation as:

$$P = \{x \,/ x \in A;\ \phi(x).$$

Note that the above Axiom Schema requires that a set A be known such that <u>if</u> $x \in P$, then $x \in A$. Also, note that the formula ϕ (property ϕ) cannot involve P. This is to avoid contradictory conditions which could be placed on P without this restriction. (6).

The common operations involving sets are those of union intersection and complement. These are defined below along with the set inclusion concepts.

DA-3: Set P_1 is said to be contained in ($\subseteq$) set P_2 if and only if for every $x \in P_1$, $x \in P_2$. If $P_1 \subseteq P_2$, P_1 is said to be a subset of P_2. Formally, the above can be written as:

$$P_1 \subseteq P_2 \leftrightarrow \forall\, x \in P_1 \rightarrow x \in P_2 .$$

DA-4: $P_1 = P_2 \leftrightarrow P_1 \subseteq P_2$ and $P_2 \subseteq P_1$ read, "set P_1 equals set P_2 if and only if $P_1 \subseteq P_2$ and $P_2 \subseteq P_1$".

DA-5: $P_1 \subset P_2 \leftrightarrow P_1 \subseteq P_2$, and $P_2 \nsubseteq P_1$.

If, in addition to the above, $P_1 \neq \emptyset$, then P_1 is said to be a proper subset of P_2.

The union of P_1 and P_2 is the set $P_1 \bigcup P_2$ of elements each of which belongs to either P_1 or P_2.

DA-6: $P_1 \bigcup P_2 = \{x/x \in P_1 \text{ or } x \in P_2\}$.

The union is easily extended to any number of sets as:

DA-7: $\bigcup P_{ni} = \{\{x/x \in P_{ni} \text{ for some } P_{ni} \in \{P_{ni}\}\}$,

The intersection of two sets P_1 and P_2 is the set $P_1 \bigcap P_2$ of elements each of which belongs to both P_1 and P_2.

DA-8: $P_1 \bigcap P_2 = \{x/x \in P_1 \text{ and } x \in P_2\}$.

Again, the above can be extended to any number of sets.

DA-9: $P_{ni} = \{x/x \in P_{ni} \text{ for all } P_{ni} \in \{P_{ni}\}\}$.

The complement (difference) of two sets, P_1 relative to P_2, is denoted by $P_2 - P_1$. This set represents the elements of P_2 which are not in P_1.

DA-10: $P_2 - P_1 = \{x/x \in P_2, x \notin P_1\}$.

For a set P the distinct subsets of P is denoted by 2^P. The reason for this notation is in the fact that for a finite set of elements (number n of distinct elements is finite), the number of distinct subsets is 2^N.

DA-11: $2^P = \{x/x \subseteq P\}$.

Some of the more commonly-used facts, not already mentioned, are listed below:

TA-1: $\emptyset \subseteq P$ for all sets P. That is, the empty set is a subset of every set.

TA-2: $P \subseteq P$ for all sets P. That is, every set is a subset of itself (not a member of itself).

TA-3: $\emptyset$ and $P \varepsilon 2^P$. This follows from DA-11, TA-1, and TA-2.

TA-4: $P_1 \cup (P_1 - P_2) = P_1$.

TA-5: $P_1 \cap (P_2 - P_1) = \emptyset$.

TA-6: $P_1 \cup (P_2 \cap P_3) = (P_1 \cup P_2) \cap (P_1 \cup P_3)$.

TA-7: $P_1 \cap (P_2 \cup P_3) = (P_1 \cap P_2) \cup (P_1 \cap P_3)$.

The "number" of distinct elements in a set is referred to as the cardinality of the set. For finite sets "number" has a unique meaning in terms of a natural number (i.e., 1, 2, ... n). However, for non-finite cardinals the intuitive idea of number is somewhat less clear. However, even in these cases it is still clear that two sets can have the same number. Hence, two sets are said to be equivalent if there is a one-to-one correspondence between the elements. All the sets which are equivalent are said to have the same cardinality. In this thesis the cardinality of a set, P, will be denoted by $\sigma(P)$.

In many cases it is convenient to denote the totality of elements involved in an abstraction (domain of discourse). Some of the more familiar notations which are used in this thesis are:

R = Set of real numbers

C = Set of complex numbers

I = Set of integers

$P_1 \times P_2$ = Set of ordered pairs (Cartesian product)

The notion of order is also a fundamental concept of mathematics. The formal definition of an ordered pair in terms of sets is:

DA-12: An ordered pair of elements $p_1 \varepsilon P_1$ and $p_2 \varepsilon P_2$ denoted by (p_1, p_2) is the set $\{\{p_1\}, \{p_1, p_2\}\}$.
$(p_1, p_2) = \{x/x \subseteq P_1 \cup P_2$ and $x = \{p_1\}$ or $x = \{p_1, p_2\}\}$.

Using this definition, it is easy to show that $(p_1, p_2) = (p'_1, p'_2)$ if and only if $p_1 = p'_1$ and $p_2 = p'_2$, using the definition of set equality.

Chapter II of this thesis uses an extension of the above idea to define ordering of elements from more than two sets. For two sets P_1 and P_2 the totality of ordered pairs is denoted by $P_1 \times P_2$ called the Cartesian product of P_1 and P_2.

DA-13: $P_1 \times P_2 = \{x/x = (p_1, p_2)$ and $p_1 \varepsilon P_1$ and $p_2 \varepsilon P_2\}$.

Using the above set, relationships between two sets other than those of DA-12 and 13 can be defined. A subset R of $P_1 \times P_2$ is called the graph of a relation ϕ. This is defined as:

DA-14: $R = \{x/x \varepsilon P_1 \times P_2$ and $\phi(x)\}$.

The graph R of ϕ is always a subset of $P_1 \times P_2$. Also, note that in the above form R is the result of applying the axiom schema of separation, with the assumption that $P_1 \times P_2$ exists. Hence, every abstract postulate, ϕ, relating the sets P_1 and P_2 can be, theoretically at least, exhibited in the form of a graph R. In fact, the graph of a relation is formally defined as the relation. (6). The above discussion indicates that the graph and the formula are equivalent in a logical sense. The only logical problem which seems to appear while using DA-14 is the fact that ϕ_1 (x) and ϕ_2 (x) as applied to $P_1 \times P_2$ can yield the same graph R. Hence, to remove this apparent ambiguity from formal logic, it appears that the graph of a relation is called the relation. However, in the engineering application of mathematical relations, it is the relation, ϕ, (usually called a formula) which is of interest. For this reason and because of the role of the relation in this thesis, the above discussion is thought necessary.

In order to remove the logical ambiguity of using DA-14, the "equivalence" of two relations, ϕ_1 and ϕ_2, is defined after an "equivalence relation" is defined.

In many instances it is desirable to define a relation among the elements of a set P. Hence, in DA-14 P_1 and P_2 are taken to be equal. When this is the case, the concept of an equivalence relation arises quite naturally. Following the standard notation for binary relations on a set, the following definition is given:

DA-15: $p_1 \mathcal{R} p_2 \leftrightarrow (p_1, p_2) \in R$. Read "$p_1$ is in the relation R to p_2 if and only if the ordered pair $(p_1, p_2) \in R$".

DA-16: A relation on P x P is called an equivalence relation denoted by (RST) or $\sim$ if and only if

(a) $p_1 \mathcal{R} p_1$. (Reflexive)

(b) $p_1 \mathcal{R} p_2 \rightarrow p_2 \mathcal{R} p_1$. (Symmetric)

(c) $p_1 \mathcal{R} p_2$ and $p_2 \mathcal{R} p_3 \rightarrow p_1 \mathcal{R} p_3$. (Transitive)

Probably the most commonly used and strongest RST relation is that of equality.

Using the above definition, a one-to-one correspondence between a relation and its graph can be achieved. This is done after an example of an equivalence relation is given.

Example: Consider the set of formulas Φ which relate the elements of the sets P_1 and P_2. Φ is not empty, since the formula for obtaining $P_1 \times P_2$ is known. Consider the set $\Phi \times \Phi$ and the graph R given by $R = \{x/x \in \Phi \times \Phi$ and $x = (\phi_1, \phi_2) \leftrightarrow (\phi_1 \rightarrow A$ and $\phi_{21} \rightarrow A$ and $A \subseteq P_1 \times P_2\}$. That is, R is the set of ordered pairs each of which represents two formulas which yield the same graph when applied to $P_1 \times P_2$. Now $\phi_1 R \phi_1$ by definition of R and Φ. Also, if $\phi_1 R \phi_2$, then $\phi_2 R \phi_1$ since $A = B = C$ implies $C = B = A$. Finally $\phi_1 R \phi_2$ and $\phi_2 R \phi_3 \rightarrow \phi_1 R \phi_3$, since $A = B = C$ and $C = D = E \rightarrow A = B = E$. Hence, the above relation is an equivalence relation on the set Φ.

The elements which are equivalent under an equivalence relation $\sim$ are said to form an equivalence class. These are defined as:

DA-17: The set $\overline{P}_1 = \{x/x \in P$ and $(x, P_1) \in \sim \subseteq P \times P\}$ is called an equivalence class of the equivalence relation $\sim$. Note that the condition "$\sim \subseteq P \times P$" signifies the above definition only has meaning when an RST relation is involved.

One of the fundamental theorems of binary relations shows that the totality of equivalence classes generated by $\sim$ are disjoint and that the union of the totality is the original set. (9).

Considering the example given and the above discussion, let $\bar{\Phi}$ be the set whose elements are the equivalences classes $\bar{\phi}$. Now consider the set $2^{P_1 \times P_2}$ which is the collection of distinct possible graphs of $P_1 \times P_2$ by DA-14. Hence, if $\alpha \varepsilon 2^{P_1 \times P_2}$, there is one and only one element in $\bar{\Phi}$ which corresponds to α. Conversely, for every element $\bar{\phi}$ of $\bar{\Phi}$ there is one and only one element of $2^{P_1 \times P_2}$ which corresponds to $\bar{\phi}$. Hence, these sets are equivalent.

If the elements $\bar{\phi}$ of $\bar{\Phi}$ are used in the definition of a relation instead of ϕ, then each graph (subset of $P_1 \times P_2$) can be associated with a single "equivalence relation" $\bar{\phi}$. In this manner a 1 - 1 correspondence between a relation and the graph of a relation can be obtained.

Returning to the classification of binary relations, probably the most celebrated classification is that of the function.

DA-18: A relation ϕ on $P_1 \times P_2$ is said to be a function if and only if (p_1, p_2) and $(p_1, p_2') \varepsilon R \rightarrow p_2 = p_2'$.

A function is called a single-valued mapping of P_1 into P_2 written ϕ: $P_1 \rightarrow P_2$ if $\forall p_1 \varepsilon P_1 \exists (p_1', p_2) \varepsilon R \ni p_1 = p_1'$. It is important to note that the uniqueness property expressed in DA-18 is one way only. For functional relations the notation $f(P_1) = P_2$ is often used to denote the fact that for each first element of the graph set there is a single second element. When the above uniqueness property holds in both directions, the functions are called one-to-one (1 - 1). This and several other common classifications are given below:

DA-19: A function f is said to be one-to-one (1 - 1) when (p_1', p_2) and $(p_1, p_2) \in R$, the graph of f $\rightarrow$ $p_1' = p_1$.

DA-20: A single-valued mapping $\phi: P_1 \rightarrow P_2$ is said to be onto if and only if $\forall\, p \in P_2\ \exists\, (P_1, P_2) \in R$, the graph of ϕ, $\ni p = p_2$.

DA-21: A 1 - 1 mapping $\phi: P_1 \rightarrow P_2$ is a 1 - 1 function which is a single-valued mapping.

DA-22: A 1 - 1 onto mapping $\phi: P_1 \rightarrow P_2$ is a 1 - 1 mapping which is onto mapping.

The above classifications are logically related in the inclusion diagram of Figure A-1.

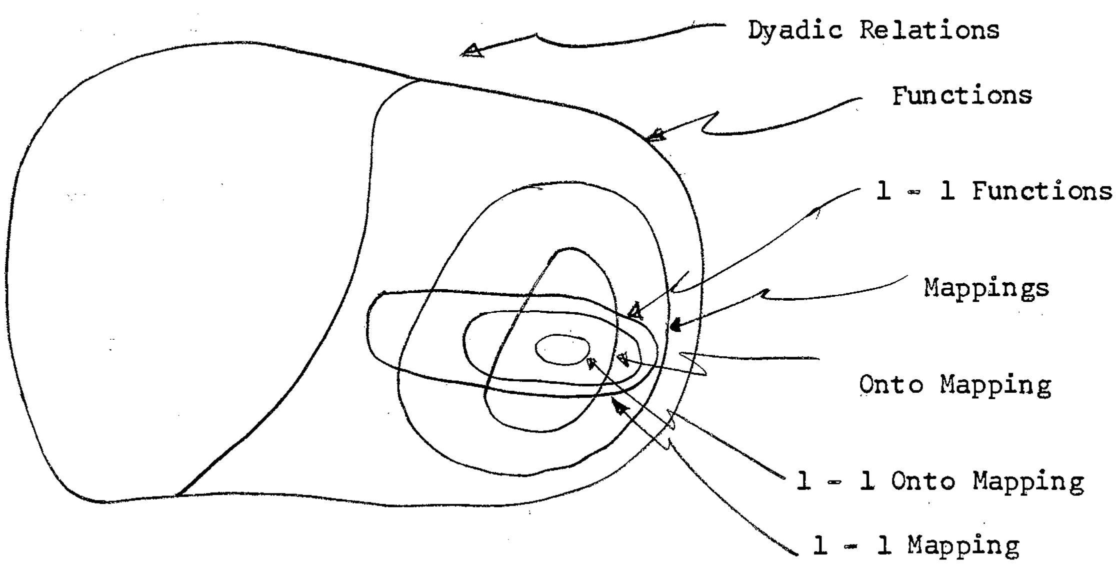

Figure A-1. Inclusion Diagram for Dyadic Relations

The idea of a binary relation between two sets is made precise by using the concept of the ordered pair. It is obvious that given any subset of $P_1 \times P_2$ this defines a unique subset in $P_2 \times P_1$. Hence, for each relation ϕ there is a converse relation ϕ_c. Also, it is apparent that $\phi_{cc} = \phi$.

DA-23: The relation ϕ_c is said to be the converse relation of the relation ϕ if and only if the graph R_c of ϕ_c and R of ϕ satisfy:

(a) $\forall\ (p_1, p_2) \in R\ \exists\ (p_2, p_1) \in R_c$.

(b) $\forall\ (p_2, p_1) \in R_c\ \exists\ (p_1, p_2) \in R$.

TA-8: From DA-23 it is immediate that $\phi_{cc} = \phi$.

Notice the relation classification definitions where given in terms of the "first" element and the "second" element of the ordered pair. This being the case the conditions of a definition involving ϕ and ϕ_c are not necessarily satisfied. That is, ϕ might be a function, but ϕ_c might not. These and related facts which arise from the relative classifications are a part of mathematical analysis studies. (14). However, as discussed in Chapter II, the concern with relations in this thesis is slanted in a different direction.

TABLE I

DEFINITION OF LOGIC SYMBOLS

LOGIC SYMBOL	DEFINITION
($\exists$ (x) ...)	(There exists an (x) ...)
($\nexists$ (x) ...)	(There does not exist an (x) ...)
($\forall$(x) ...)	(For all (x) ...)
(... $\rightarrow$ ...)	(If ... Then ...)
(... $\leftrightarrow$ ...)	(If ... Then ... , and conversely)
(... and ...)	(... and ...)
(... or ...)	(... or ...)
(... $\ni$ ...)	(... such that ...)

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