ANALYSIS OF CONTINUOUS CIRCULAR BEAM LOADED OUT OF PLANE

By

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PREFACE

The writer, in completing the final phase of his work for his Master's Degree, wishes to express his gratitude to the following individuals:

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Stillwater, Oklahoma
June, 1962

J. A. P.

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NOMENCLATURE

f . . . . Flexibility of Basic Structure.

g . . . . Carry-Over Value of Basic Structure.

$x^{(L)}$ . . . . End-Moment in Isolated Span Due to Loads Only.

A, B, C . . . . . Influence Values of Applied End-Moments on a Basic Structure.

$BM_{SR}^{(L)}$ . . . . . Moment at a Section in the Radial Direction, Due to Loads and Reactions Alone.

$BM_{ST}^{(L)}$ . . . . . Moment at a Section in the Tangential Direction, Due to Loads and Reactions Alone.

E . . . . Modulus of Elasticity in Tension or Compression.

F . . . . Flexibility of Whole Structure.

G . . . . Modulus of Rigidity.

G_{o1} . . . . Carry-Over Value of Whole Structure.

I . . . . Moment of Inertia of Cross-Section of Member.

J . . . . Polar Moment of Inertia.

M_{iR} . . . . Moment About Radial Axis. Its Subscript Indicates Its Location.

M_{iT} . . . . Torque About Tangential Axis. Its Subscript Indicates Its Location.

R . . . . Radius of Curvature of Member.

R_{iZ} . . . . Reaction at the Support i Parallel to z-Axis.

T . . . . Load Function of Whole Structure.

U . . . . Strain Energy.

α . . . . Angular Distance of Any Section of the Member Counterclockwise from the Support.

θ, θ' . . . . . Angular Location of Load.

λ . . . . Elemental Angular Flexibility.

τ'. . . . . Load Function of Basic Structure.

ω . . . . Angular Distance of a Span.

CHAPTER I

INTRODUCTION

1-1. General

Continuous curved beams with general loading are of great importance in bridge and building structures. These may be basically classified in three groups:

(a) Continuous curved beams lying in a plane acted upon by a coplanar system of loading.

(b) Continuous curved beams lying in a plane with loading perpendicular to that plane.

(c) Continuous curved beams in space with general loading.

The second group is being investigated in this thesis by the flexibility method.

After developing the carry-over moment method applied to planar frames, Tuma[1, 2, 3] extended the application of this method to continuous beams and frames in space.

Also the analysis of girders curved in the plane and girders curved in space has been discussed by several other authors.

Bella Velutini[4] discussed the method of moment distribution applied to continuous circular beams. H. H. Fickel[5] developed influence lines for bending moments, torsional moments, and shearing forces in curved girders.

The symbols used in this thesis are explained wherever they occur first and rearranged under the title nomenclature.

The sign convention for loads, cross-sectional elements, reactions and deformations is shown in Fig. 1-1a-h. At each point in a member it is necessary to establish a right-hand system of orthogonal coordinates, a tangential axis x', a radial axis y', and a third axis perpendicular to the plane of these, z' (Fig. 1-1i).

Additional references are given in the bibliography.

1-2. Statement of the problem

A continuous circular beam lying in a plane xy, (fig. 1-2a) is acted upon by loads perpendicular to this plane. It has a constant cross-section and radius of curvature R. The beam is supported at points 0, 1, 2...i, j...n. The angle subtended at the center of curvature C, by each span is denoted by the symbol ω with a subscript corresponding to the particular span under consideration. The exterior ends o and n are fixed and the interior supports are assumed to be spherical hinges and not to deflect.

The analysis is carried out in terms of polar co-ordinates. M_{iR} denotes the bending moment at i which acts in the radial direction. M_{iT} denotes the twisting (torsional) moment at i and acts in the tangential direction. In Fib. 1-2a, the loads are acting in the negetive z-direction and the reactions are acting in the positive z-direction.

The n-span beam shown in Fig. 1-2a has (n + 5) reactive elements. For the analysis of this beam, three equations of static equilibrium are available and therefore (n + 2) deformation conditions are necessary. Thus, such a structure is statically indeterminate to (n + 2)th degree.

If the exterior ends of this beam are simply supported, (Fig. 1-2b), there are no end moments and the number of reactive elements is

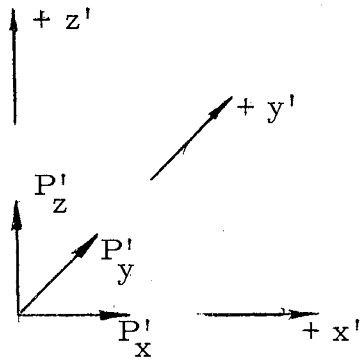

Fig. 1-1a
Loads
Positive Forces

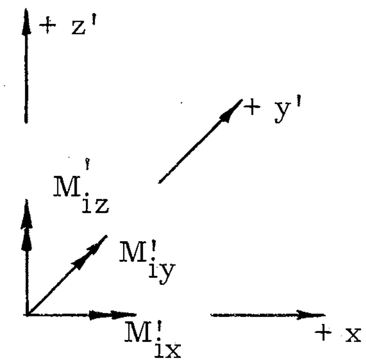

Fig. 1-1b
Loads
Positive Moments

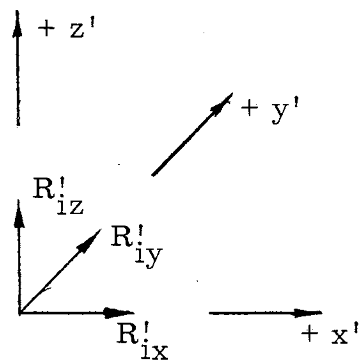

Fig. 1-1c
Reactions
Positive Forces

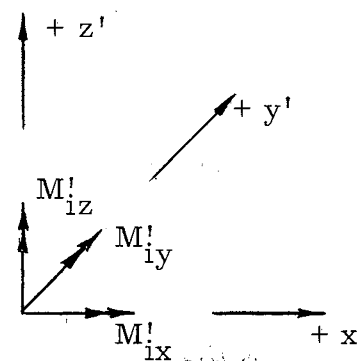

Fig. 1-1d
Reactions
Positive Moments

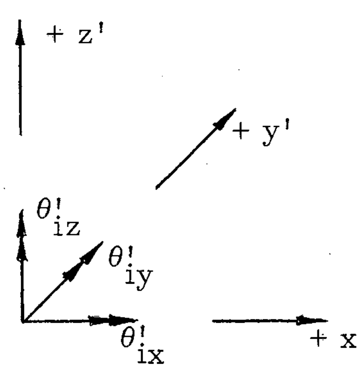

Fig. 1-1e
Deformations
Positive Rotations

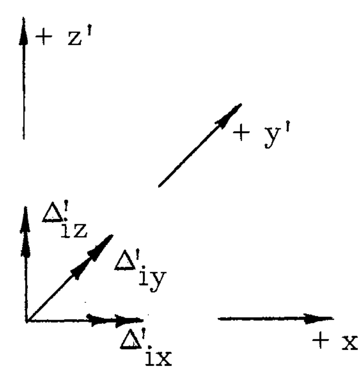

Fig. 1-1f
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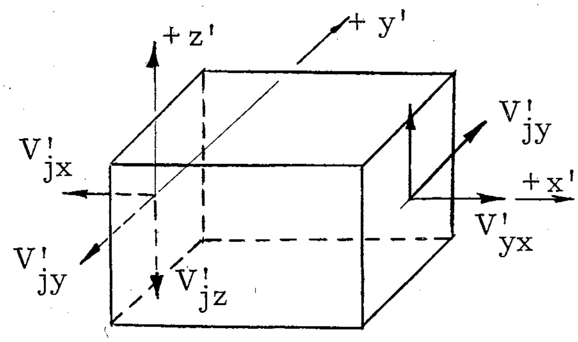


Fig. 1-1g
Cross-Sectional Elements
Positive
Normal and Shearing Forces

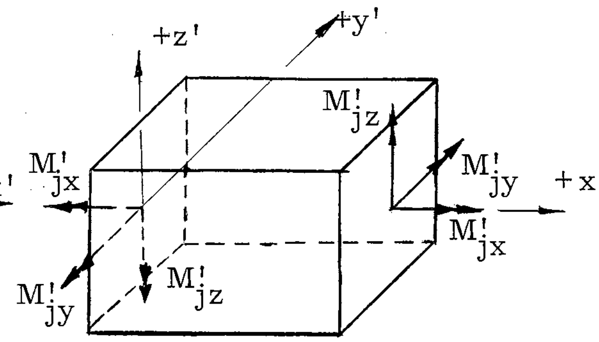


Fig. 1-1h
Cross-sectional Elements
Positive
Bending and Torsional Moments

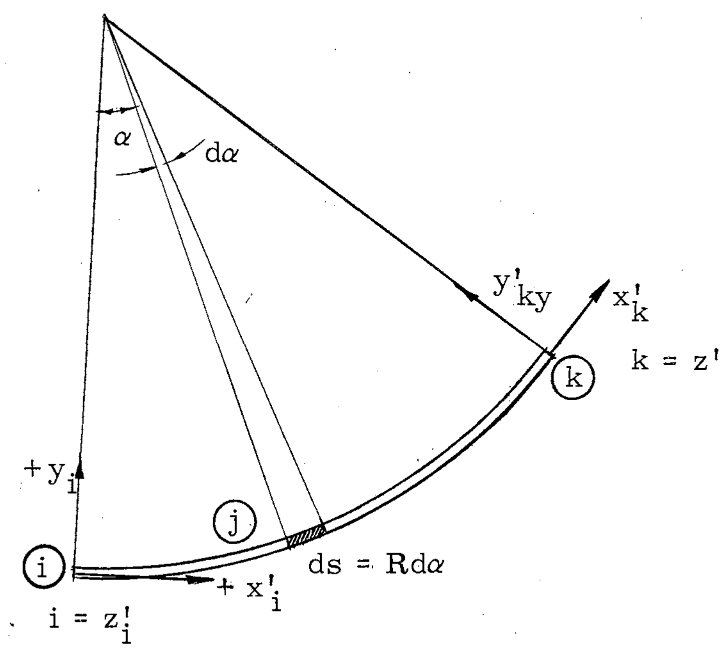


Fig. 1-1i
Circular Beam - Orthogonal Axes

reduced to (n + 1). In this case, the structure is statically indeterminate to (n - 2)nd degree.

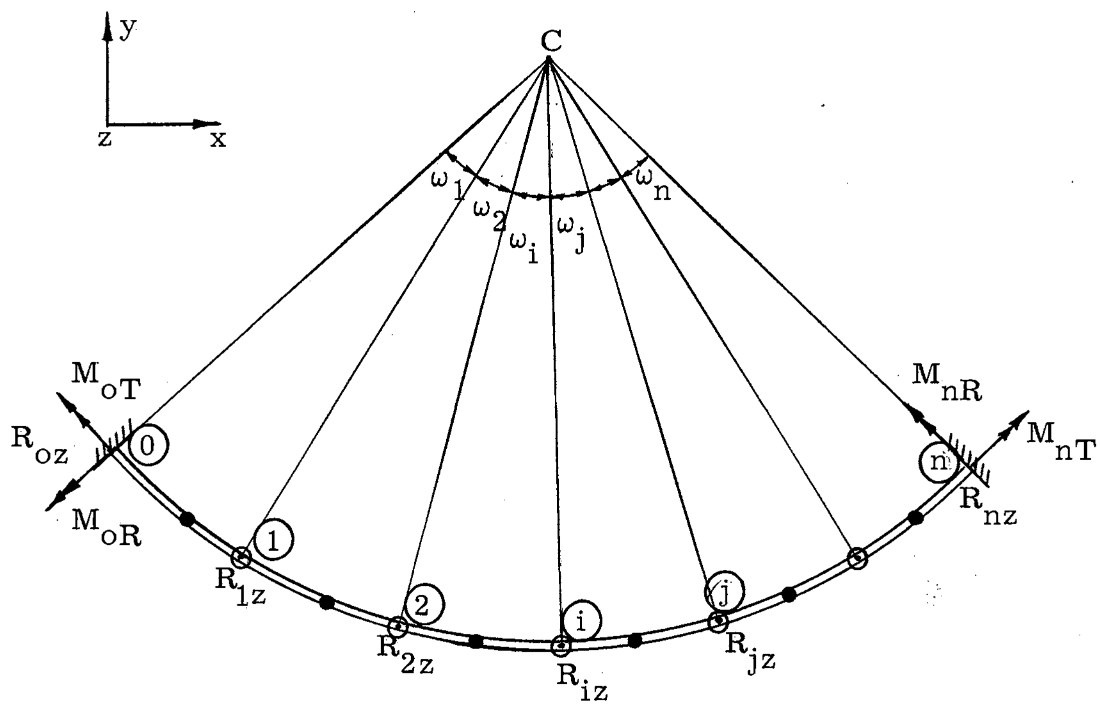


Fig. 1-2a. Continuous Circular Beam - Ends Fixed

The degree of indeterminancy depends on the end conditions of the structure and is equal to the number of redundants.

For the purpose of analysis, either the support moments or the support forces can be considered as redundants. In this thesis, the support moments are considered as redundants.

For the beams shown in Fig. 1-2a, 1-2b, the redundant support moments are shown in Figs. 1-2c, 1-2d respectively.

For this choice of redundants the basic structure is a one-span curved beam, restrained against torsional rotation.

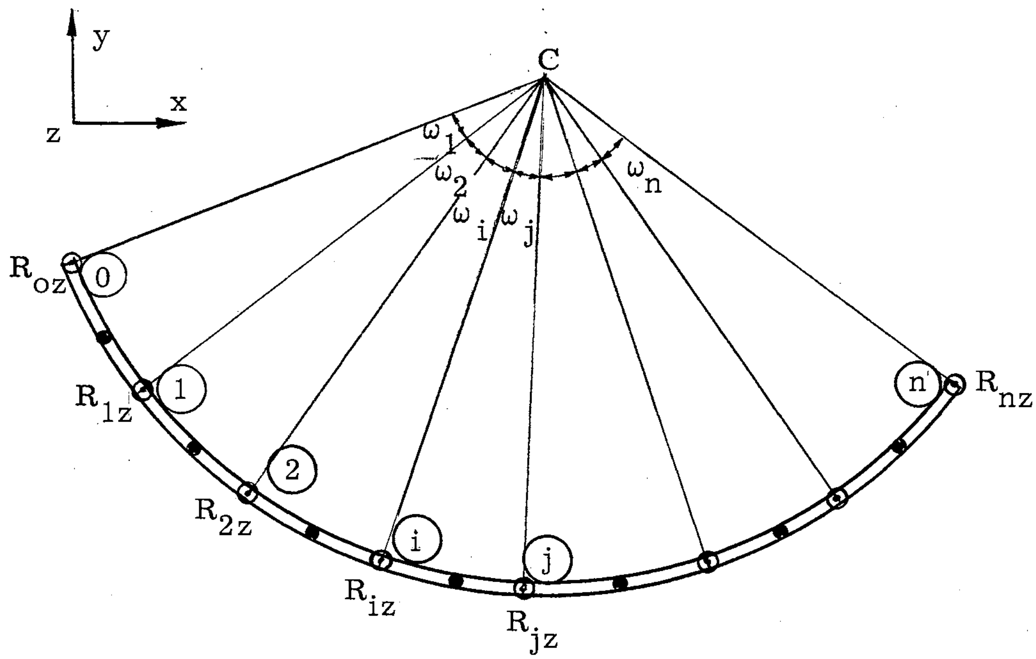


Fig. 1-2b. Continuous Circular Beam-Ends Simply Supported

Fig. 1-3 shows such a typical basic structure for span $\overline{ij}$. This beam is simply supported at i and j and in addition is restrained against tangential rotation at j. This structure is statically determinate and also stable. On this basic structure, moments M_{iT}, M_{iR} and M_{jR} are considered as externally applied moments.

For this one-span basic structure, angular functions, i.e., angular flexibilities, angular carry-over values, and angular load functions, are derived. They are denoted by symbols f, g, and τ respectively, with appropriate subscripts.

Angular functions for the entire continuous beam can be obtained in terms of angular functions for this one-span basic structure. Angular functions due to an applied unit external moment are derived for a specific case of a four span beam, with exterior ends fixed. These

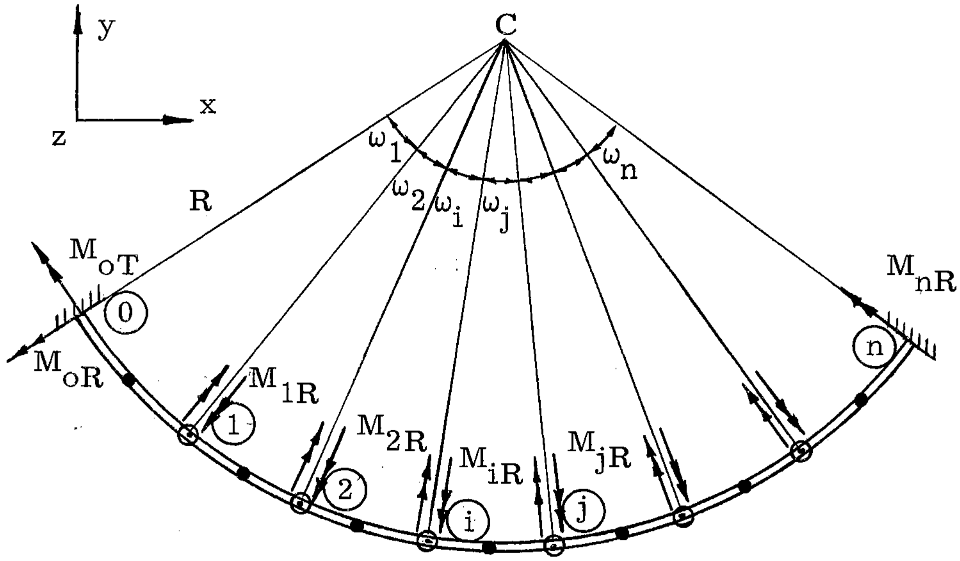


Fig. 1-2c

Redundants
Continuous Circular Beam - End Supports Fixed

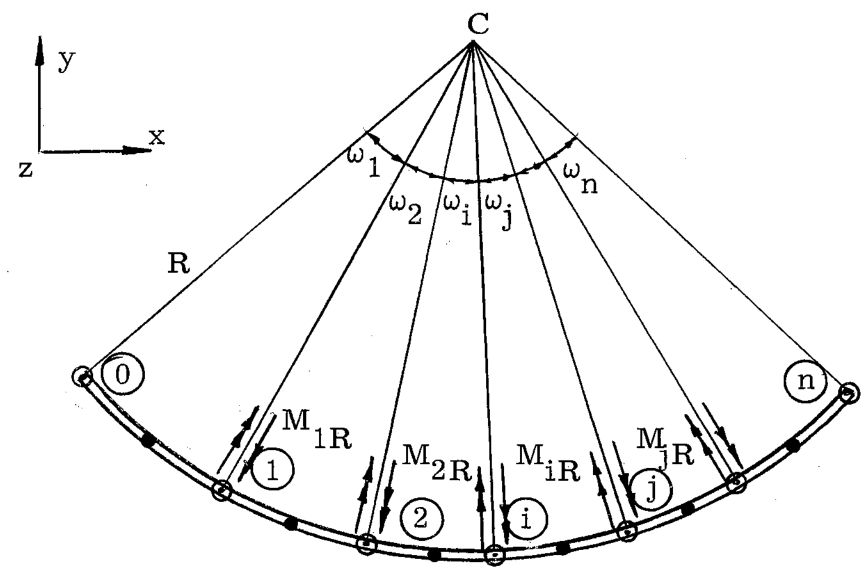


Fig. 1-2d

Redundants
Continuous Circular Beam - End Supports Simply Supported

angular functions for the whole beam are denoted by symbols F, G, and τ.

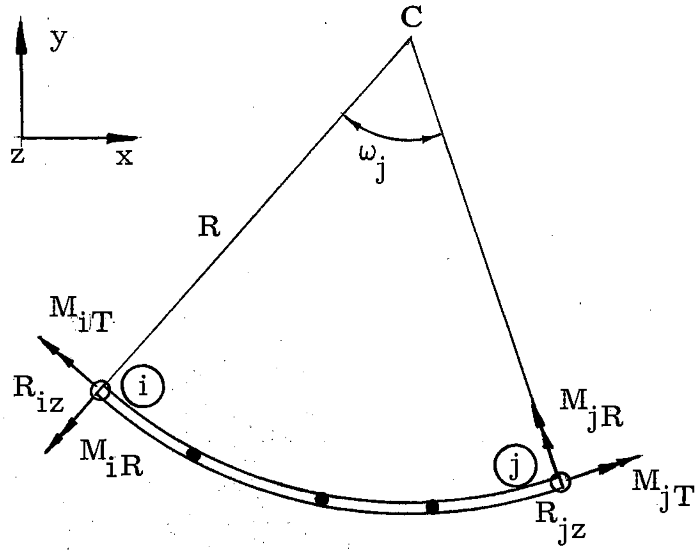


Fig. 1-3. Basic Structure $\overline{ij}$

The conditions of consistent deformations, expressed in terms of these angular functions F, G and τ, provide the necessary compatibility equations to solve for the redundant moments.

In the following chapters the above discussion is expanded fully according to the following phases:

Chapter II - Statics of the Basic Structure.

Chapter III - Deformation of the Basic Structure.

Chapter IV - Compatibility Equations.

CHAPTER II

STATICS OF THE BASIC STRUCTURE

An one-span basic structure $\overline{ij}$, described in the last chapter, is shown in Fig. 2-1a. The vertical reactions, R_{iz} and R_{jz}, and the restraining moment, M_{jT}, are the reactive elements at j. Other end moments (M_{iR}, M_{jR} and M_{iT}) and loads are considered to be applied moments and forces. In other words, the above structure can be considered as a one-span continuous beam separated from the whole beam as shown in Fig. 1-2b.

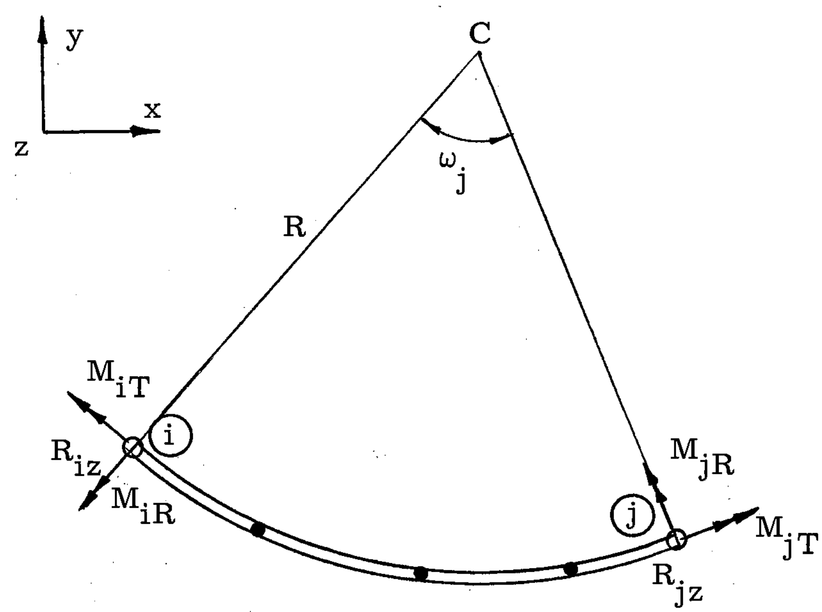


Fig. 2-1a. Basic Structure $\overline{ij}$

The reactive elements can be calculated using three equations of static equilibrium.

Summing all moments at j in the radial direction, it follows that

$$\Sigma M_{R[@j]} = 0$$

or,

$$M_{jR} - M_{iR} \cos \omega_j + M_{iT} \sin \omega_j + R_{iz} R \sin \omega_j + SM_{jR} = 0 \quad (2\text{-}1a)$$

Equilibrium of moments at j in the tangential direction is fulfilled by

$$\Sigma M_{T[@j]} = 0$$

or,

$$M_{jT} - M_{iR} \sin \omega_j - M_{iT} \cos \omega_j + R_{iz} R (1 - \cos \omega_j) + SM_{jT} = 0 \quad (2\text{-}1b)$$

The relationship provided by the fact that the forces on the structure in z-direction are in equilibrium,

$$\Sigma F_z = 0$$

or,

$$R_{iz} + R_{jz} + \Sigma P_z = 0 \quad (2\text{-}1c)$$

Solving simultaneously equations 2-1a, b, c, gives

$$R_{iz} = - \frac{M_{iT}}{R} + \frac{M_{iR} \cos \omega_j}{R \sin \omega_j} - \frac{M_{jR}}{R \sin \omega_j} - SM_{jR} \quad (2\text{-}2a)$$

$$R_{jz} = + \frac{M_{iT}}{R} - \frac{M_{iR} \cos \omega_j}{R \sin \omega_j} + \frac{M_{jR}}{R \sin \omega_j} + SM_{jR} - \Sigma P_z \quad (2\text{-}2b)$$

$$M_{jT} = + M_{iT} + \tan \frac{\omega_j}{2} (M_{iR} + M_{jR}) + M_{jR} \frac{1 - \cos \omega_j}{\sin \omega_j} - SM_{jT} , \quad (2\text{-}2c)$$

where SM_{jR} and SM_{jT} are the static moments due to loads only at j about the radial and the tangential directions respectively.

The cross-sectional elements M_{SR}, M_{ST} and V_{Sz}, i.e., the moments and the shearing force at any point s in the member $\overline{ij}$, are calculated by considering the free-body diagram of a part of the span $\overline{ij}$ (Fig. 2-1b). The arc length $\overline{is}$ subtends an angle α at the center C as shown in Fig. 2-1b.

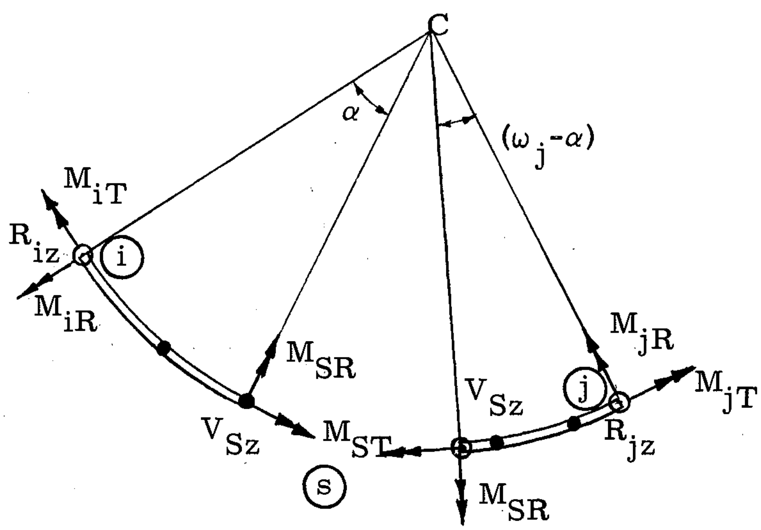


Fig. 2-1b. Free-Body Diagram

The cross-sectional elements can be calculated using three equations of static equilibrium.

Thus,

$$\Sigma M_{SR}^{(i)} = 0$$

$$M_{SR} + M_{iT} \sin\alpha - M_{iR} \cos\alpha + R_{iz} R \sin\alpha + SM_{SR}^{(i)} = 0 \quad \text{.... (2-3a)}$$

$$\Sigma M_{ST}^{(i)} = 0$$

$$M_{ST} - M_{iT} \cos \alpha - M_{iR} \sin \alpha + R_{iz} R(1 - \cos \alpha) + SM^{(i)}_{ST} = 0 \ldots . \tag{2-3b}$$

$$\Sigma F^{i}_{Sz} = 0$$

$$-V_{Sz} + R_{iz} - P^{(s)} = 0 \tag{2-3c}$$

Substituting the value of R_{iz} from equation 2-2a and simplifying, it is found that

$$M_{SR} = M_{iT}(0) + M_{iR} \frac{\sin(\omega_j - \alpha)}{\sin \omega_j} + M_{jR} \frac{\sin \alpha}{\sin \omega_j} + SM_{jR} R \sin \alpha$$

$$- SM^{(i)}_{SR}$$

$$= M_{iT}(0) + M_{iR} \frac{\sin(\omega_j - \alpha)}{\sin \omega_j} + M_{jR} \frac{\sin \alpha}{\sin \omega_j} + BM^{(i)}_{SR} \tag{2-4a}$$

$$M_{ST} = M_{iT} + M_{iR} \frac{\cos(\omega_j - \alpha) - \cos \omega_j}{\sin \omega_j} + M_{jR} \frac{1 - \cos \alpha}{\sin \omega_j}$$

$$+ SM_{jR} R(1 - \cos \alpha) - SM^{(i)}_{ST}$$

$$= M_{iT} + M_{iR} \frac{\cos(\omega_j - \alpha) - \cos \omega_j}{\sin \omega_j} + M_{jR} \frac{1 - \cos \alpha}{\sin \omega_j} + BM_{ST} \tag{2-4b}$$

$$V_{Sz} = + R_{iz} - P^{(s)}$$

$$= \frac{M_{iT}}{R} - M_{iR} \frac{\cos \omega_j}{R \sin \omega_j} + \frac{M_{jR}}{R \sin \omega_j} + SM_{jR} - P^{(s)}, \tag{2-4c}$$

where $BM^{(i)}_{SR}$ and $BM^{(i)}_{ST}$ represent the moments at the section considered in the radial and the tangential directions respectively, due to loads and reactions alone. $P^{(s)}$ is the load acting on the length $\overline{is}$.

$$BM_{SR}^{(i)} = SM_{jR}\, R \sin\alpha - SM_{SR}^{(i)} \qquad (2\text{-}5a)$$

$$BM_{ST}^{(i)} = SM_{jR}\, R\,(1 - \cos\alpha) - SM_{ST}^{(i)} \qquad (2\text{-}5b)$$

Denoting,

$$A_{Rij} = 0 \qquad\qquad A_{Tij} = +1$$

$$B_{Rij} = +\frac{\sin(\omega_j - \alpha)}{\sin\omega_j} \qquad\qquad B_{Tij} = +\frac{\cos(\omega_j - \alpha) - \cos\omega_j}{\sin\omega_j}$$

$$C_{Rij} = +\frac{\sin\alpha}{\sin\omega_j} \qquad\qquad C_{Tij} = +\frac{1 - \cos\alpha}{\sin\omega_j} \quad .$$

Equations (2-4a, 4b) can be written as:

$$M_{SR} = M_{iT} A_{Rij} + M_{iR} B_{Rij} + M_{jR} C_{Rij} + BM_{SR}^{(i)} \qquad (2\text{-}6a)$$

$$M_{ST} = M_{iT} A_{Tij} + M_{iR} B_{Tij} + M_{jR} C_{Tij} + BM_{ST}^{(i)} \qquad (2\text{-}6b)$$

CHAPTER III

DEFORMATION OF THE BASIC STRUCTURE

The end angular functions of the basic structure discussed earlier are studied in this chapter. Castigliano's principle is used to derive the analytical expressions for these end angular functions.

For the basic structure $\overline{ij}$ (Fig. 2-1a), the possible angular functions are:

1. The angular flexibility f_{ijTT}
2. The angular flexibility f_{ijRR}
3. The angular flexibility f_{jiRR}
4. The near-end angular carry-over function $f_{ijRT} (= f_{ijTR})$
5. The far-end angular carry-over function $g_{ijTR} (= g_{jiRT})$
6. The far-end angular carry-over function $g_{ijRR} (g_{jiRR})$
7. The angular load function τ'_{ijTT}
8. The angular load function τ'_{ijRR}
9. The angular load function τ'_{jiRR}

where an angular flexibility is the end-slope of the basic structure, due to a unit applied moment at that end and in the same direction as the moment.

An angular carry-over function is an end-slope of the basic structure due to a unit applied end moment, in a direction other than that of the moment, denoted by appropriate subscript, e.g., in expression g_{ijTR}, the first and third subscripts (i and T) indicate the location and direction of the slope respectively, and the second and fourth subscripts (j and R) indicate location and direction of an applied unit end-moment.

An angular load function is an end-slope of the basic structure due to applied loads.

It may be noted here that, by the Maxwell's Reciprocal Theorem, the angular carry-over functions f_{ijRT}, g_{ijRR}, and g_{ijTR} are equal to f_{ijTR}, g_{jiRR} and g_{jiRT} respectively.

For an elemental length in span $\overline{ij}$, the elemental angular flexibilities are:

$$\lambda_R = \frac{Rd\alpha}{EI_S} \qquad \text{and} \qquad \lambda_T = \frac{Rd\alpha}{GJ_S}$$

The strain energy of the basic structure $\overline{ij}$, due to applied loads and applied end moments is:

$$U_{ij} = U_{ijR} + U_{ijT}$$

$$= \int_i^j [M_{SR}]^2 \quad R + \int_i^j [M_{ST}]^2 \quad T \qquad (3\text{-}1)$$

Taking the first partial derivative of equation 3-1 with respect to M_{iT}, M_{iR} and M_{jR} respectively, gives the angular deformations in their respective directions, thus:"

$$\frac{\partial U_{ij}}{\partial M_{iT}} = \theta_{iT} = \int_i^j M_{SR} \frac{\partial M_{SR}}{\partial M_{iT}} \lambda_R + \int_i^j M_{ST} \frac{\partial M_{ST}}{\partial M_{iT}} \lambda_T \,. \qquad (3\text{-}2a)$$

$$\frac{\partial U_{ij}}{\partial M_{iR}} = \theta_{iR} = \int_i^j M_{SR} \frac{\partial M_{SR}}{\partial M_{iR}} \lambda_R + \int_i^j M_{ST} \frac{\partial M_{ST}}{\partial M_{iR}} \lambda_T \,. \qquad (3\text{-}2b)$$

$$\frac{\partial U_{ij}}{\partial M_{jR}} = \theta_{jR} = \int_i^j M_{SR} \frac{\partial M_{SR}}{\partial M_{jR}} \lambda_{RT} + \int_i^j M_{ST} \frac{\partial M_{ST}}{\partial M_{jR}} \lambda_T . \tag{3-2c}$$

The values of M_{SR} and M_{ST}, i.e., the moments at a general section S in span $\overline{ij}$, can be substituted from equations 2-4a, b, c, into the above equations. Table 3-1 shows the values of the first partial derivatives appearing in equations 3-2a, b, c.

The expressions for the required angular functions are obtained by substituting either unity or zero for appropriate moments, and/or loads in equations 3-2a, b, c. For example the expression for f_{ijTT} is

$$f_{ijTT} = \frac{\partial U_{ij}}{\partial M_{iT}} ,$$

for

$$M_{iT} = +1$$

and

$$M_{iR} = M_{jR} = \text{loads} = 0.$$

The expressions for all angular functions are obtained in this manner from equations 3-2a, b, c and are recorded in Table 3-2.

Using these angular functions the deformation equations 3-2a, b, c can be expressed as:

$$\frac{\partial U_{ij}}{\partial M_{iT}} = \theta_{iT} = M_{iT} f_{ijTT} + M_{iR} f_{ijRT} + M_{jR} g_{jiRT} + \tau'_{ijTT} \tag{3-3a}$$

$$\frac{\partial U_{ij}}{\partial M_{iR}} = \theta_{iR} = M_{iT} f_{ijTR} + M_{iR} f_{ijRR} + M_{jR} g_{jiRR} + \tau'_{ijRR} \tag{3-3b}$$

$$\frac{\partial U_{ij}}{\partial M_{jR}} = \theta_{jR} = M_{iT}\, g_{ijTR} + M_{iR}\, g_{ijRR} + M_{jR}\, f_{jiRR} + \tau'_{jiRR} \, . \tag{3-3c}$$

TABLE 3-1

FIRST PARTIAL DERIVATIVES, BASIC STRUCTURE

	Values of the First Partial Derivatives	Denoted By
$\frac{\partial M_{SR}}{\partial M_{iT}}$	0	$= A_{Rij}$
$\frac{\partial M_{ST}}{\partial M_{iT}}$	$+ 1$	$= A_{Tij}$
$\frac{\partial M_{SR}}{\partial M_{iR}}$	$+ \frac{\sin(\omega_j - \alpha)}{\sin \omega_j}$	$= B_{Rij}$
$\frac{\partial M_{ST}}{\partial M_{iR}}$	$+ \frac{\cos(\omega_j - \alpha) - \cos \omega_j}{\sin \omega_j}$	$= B_{Tij}$
$\frac{\partial M_{SR}}{\partial M_{jR}}$	$+ \frac{\sin \alpha}{\sin \omega_j}$	$= C_{Rij}$
$\frac{\partial M_{ST}}{\partial M_{jR}}$	$+ \frac{1 - \cos \alpha}{\sin \omega_j}$	$= C_{Tij}$

TABLE 3-2

ANGULAR FLEXIBILITIES, CARRY-OVER FUNCTIONS AND LOAD FUNCTIONS

BASIC STRUCTURE

f_{ijTT}	$\frac{\partial U_{ij}}{\partial M_{iT}}$	$M_{iT} = 1, \; M_{iR} = M_{jR} = \text{Loads} = 0$	$\int_i^j [1]^2 \lambda_T$
f_{ijRR}	$\frac{\partial U_{ij}}{\partial M_{iR}}$	$M_{iR} = 1, \; M_{iT} = M_{jR} = \text{Loads} = 0$	$\int_i^j \left[\frac{\sin(\omega_j - \alpha)}{\sin \omega_j}\right]^2 \cdot \lambda_R + \int_i^j \left[\frac{\cos(\omega_j - \alpha) - \cos \omega_j}{\sin \omega_j}\right]^2 \cdot \lambda_T$
f_{jiRR}	$\frac{\partial U_{ij}}{\partial M_{jR}}$	$M_{jR} = 1, \; M_{iT} = M_{iR} = \text{Loads} = 0$	$\int_i^j \left[\frac{\sin \alpha}{\sin \omega_j}\right]^2 \cdot \lambda_R + \int_i^j \left[\frac{1 - \cos \alpha}{\sin \omega_j}\right]^2 \cdot \lambda_T$
f_{ijRT} =	$\frac{\partial U_{ij}}{\partial M_{iT}}$	$M_{iR} = 1, \; M_{iT} = M_{jR} = \text{Loads} = 0$	$\int_i^j \frac{\cos(\omega_j - \alpha) - \cos \omega_j}{\sin \omega_j} \cdot \lambda_T$
f_{ijTR}	$\frac{\partial U_{ij}}{\partial M_{iR}}$	$M_{iT} = 1, \; M_{iR} = M_{jR} = \text{Loads} = 0$	
g_{ijRR} =	$\frac{\partial U_{ij}}{\partial M_{iR}}$	$M_{jR} = 1, \; M_{iT} = M_{iR} = \text{Loads} = 0$	$\int_i^j \frac{\sin(\omega_j - \alpha)}{\sin \omega_j} \cdot \frac{\sin \alpha}{\sin \omega_j} \cdot \lambda_R + \int_i^j \frac{\cos(\omega_j - \alpha) - \cos \omega_j}{\sin \omega_j} \cdot \frac{1 - \cos \alpha}{\sin \omega_j} \cdot \lambda_T$
g_{jiRR}	$\frac{\partial U_{ij}}{\partial M_{jR}}$	$M_{iR} = 1, \; M_{iT} = M_{jR} = \text{Loads} = 0$	
g_{ijTR} =	$\frac{\partial U_{ij}}{\partial M_{iT}}$	$M_{iT} = 1, \; M_{iT} = M_{iR} = \text{Loads} = 0$	$\int_i^j \frac{1 - \cos \alpha}{\sin \omega_j} \cdot \lambda_T$
g_{jiRT}	$\frac{\partial U_{ij}}{\partial M_{jR}}$	$M_{iT} = 1, \; M_{iR} = M_{jR} = \text{Loads} = 0$	
τ'_{ijTT}	$\frac{\partial U_{ij}}{\partial M_{iT}}$	$M_{iT} = M_{iR} = M_{jR} = 0$	$\int_i^j BM_{ST}^{(i)} \cdot (1) \cdot \lambda_T$
τ'_{ijRR}	$\frac{\partial U_{ij}}{\partial M_{iR}}$	$M_{iT} = M_{iR} = M_{jR} = 0$	$\int_i^j BM_{SR}^{(i)} \cdot \frac{\sin(\omega_j - \alpha)}{\sin \omega_j} \cdot \lambda_R + \int_i^j BM_{ST}^{(i)} \cdot \frac{\cos(\omega_j - \alpha) - \cos \omega_j}{\sin \omega_j} \cdot \lambda_T$
τ'_{jiRR}	$\frac{\partial U_{ij}}{\partial M_{jR}}$	$M_{iT} = M_{iR} = M_{jR} = 0$	$\int_i^j BM_{SR}^{(i)} \cdot \frac{\sin \alpha}{\sin \omega_j} \cdot \lambda_R + \int_i^j BM_{ST}^{i} \cdot \frac{1 - \cos \alpha}{\sin \omega_j} \cdot \lambda_T$

CHAPTER IV

COMPATIBILITY EQUATIONS

For a planar continuous curved beam acted upon by out of plane loads, the conditions of consistent deformation would provide compatibility equations. The number of available compatibility equations is equal to the number of total redundant support moments.

These compatibility equations for the four span continuous circular beam with exterior ends fixed (Fig. 4-1a) are now derived.

4-1. Derivation

A four-span continuous circular beam of radius of curvature R, lying in a plane xy, is subjected to loads acting perpendicular to that plane. The points of supports are 0, 1, 2, 3, and 4 (Fig. 4-1a). Exterior supports 0 and 4 are fixed against any rotation. The angle subtended at the centre C by each span is denoted by symbol ω with corresponding subscript.

This structure is indeterminate to the sixth degree. Thus, the structure has six redundant moments which are selected as shown in Fig. 4-1b. Using Castigliano's Theorem, it is possible to obtain six equations of consistent deformation in terms of the angular functions and these redundant moments (x_o, y_o, y_1, y_2, y_3 and y_4). Isolated single-spans with end moments, as shown in Fig. 4-1c, may be treated as the basic structures discussed in Chapter Two.

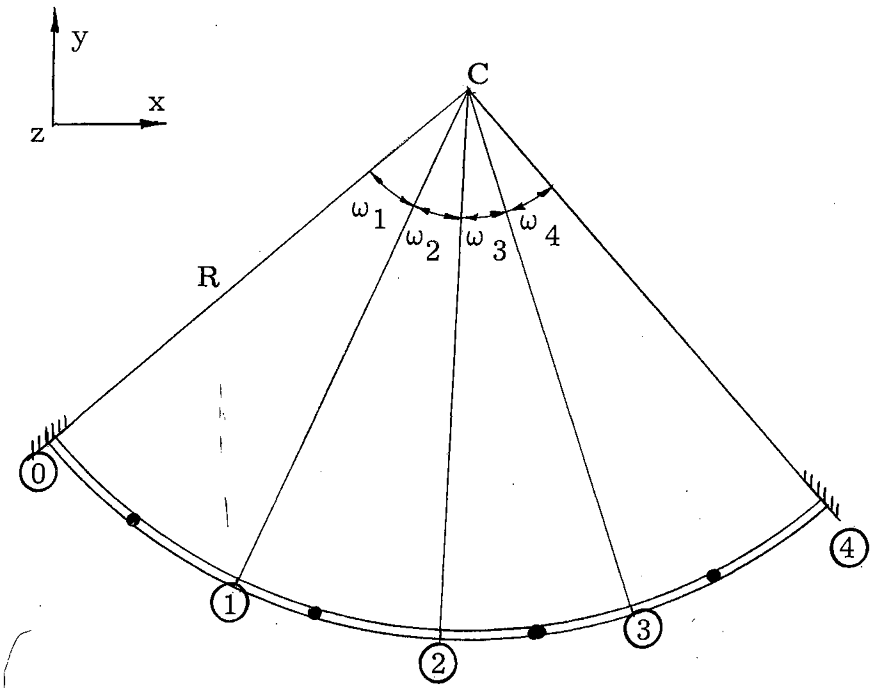


Fig. 4-1a
Four Span Continuous Circular Beam - Exterior Ends Fixed

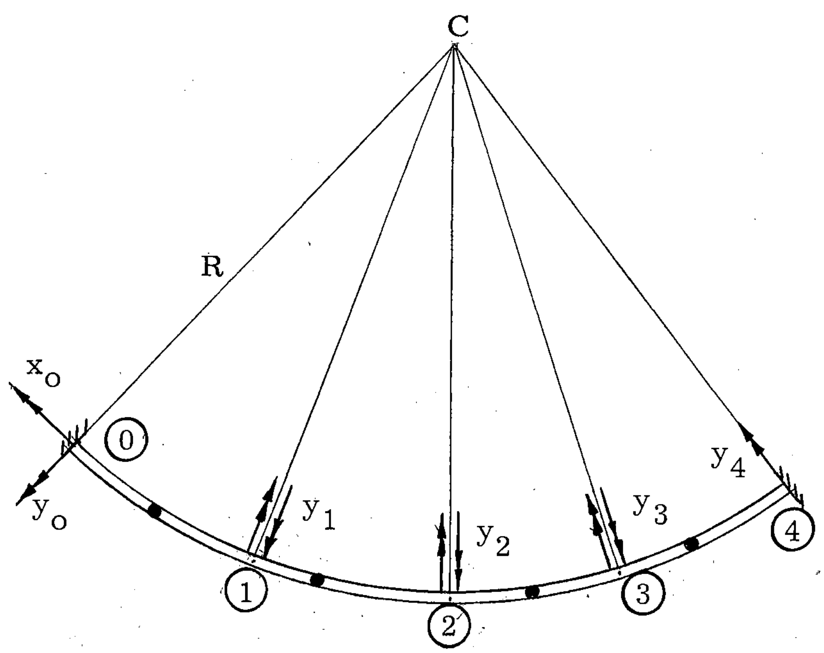


Fig. 4-1b
Four Span Continuous Circular Beam - With Redundants

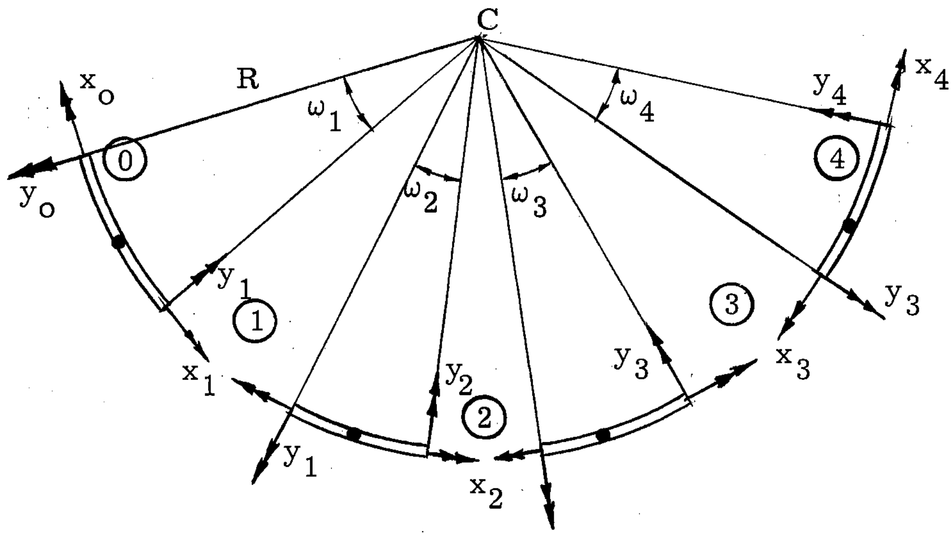


Fig. 4-1c. Four Span Continuous Circular Beam-Freebody Diagram

Consideration of the statics of the basic structure provides the relationships which follow.

The end moments are:

$$x_1 = x_o + y_o \tan\frac{\omega_1}{2} + y_1 \tan\frac{\omega_1}{2} + SM_{1R}\frac{1-\cos\omega_1}{\sin\omega_1} - SM_{1T} \tag{4-1a}$$

$$x_2 = x_1 + y_1 \tan\frac{\omega_2}{2} + y_2 \tan\frac{\omega_2}{2} + SM_{2R}\frac{1-\cos\omega_2}{\sin\omega_2} - SM_{2T} \tag{4-1b}$$

$$x_3 = x_2 + y_2 \tan\frac{\omega_3}{2} + y_3 \tan\frac{\omega_3}{2} + SM_{3R}\frac{1-\cos\omega_3}{\sin\omega_3} - SM_{3T} \tag{4-1c}$$

$$x_4 = x_3 + y_3 \tan\frac{\omega_4}{2} + y_4 \tan\frac{\omega_4}{2} + SM_{4R}\frac{1-\cos\omega_4}{\sin\omega_4} - SM_{4T} \tag{4-1d}$$

Denoting,

$$x_1^{(L)} = SM_{1R}\frac{1-\cos\omega_1}{\sin\omega_1} - SM_{1T}$$

$$x_2^{(L)} = SM_{2R} \frac{1 - \cos \omega_2}{\sin \omega_2} - SM_{2T}$$

$$x_4^{(L)} = SM_{4R} \frac{1 - \cos \omega_4}{\sin \omega_4} - SM_{4T}$$

where $x_1^{(L)}$, $x_2^{(L)}$, $x_3^{(L)}$, and $x_4^{(L)}$ are the end moments in isolated spans due to loads only. Rewriting equations 4-1a, b, c, d in terms of the redundant moments and loads,

$$x_1 = x_o + y_o \tan \frac{\omega_1}{2} + y_1 \tan \frac{\omega_1}{2} + X_1^{(L)} \qquad (4\text{-}2a)$$

$$x_2 = x_o + y_o \tan \frac{\omega_1}{2} + y_1 [\tan \frac{\omega_1}{2} + \tan \frac{\omega_2}{2}] + y_2 \tan \frac{\omega_2}{2} + X_2^{(L)} \qquad (4\text{-}2b)$$

$$x_3 = x_o + y_o \tan \frac{\omega_1}{2} + y_1 [\tan \frac{\omega_1}{2} + \tan \frac{\omega_2}{2}] + y_2 [\tan \frac{\omega_2}{2} + \tan \frac{\omega_3}{2}] + y_3 \tan \frac{\omega_3}{2} + X_3^{(L)} \qquad (4\text{-}2c)$$

$$x_4 = x_o + y_o \tan \frac{\omega_1}{2} + y_1 [\tan \frac{\omega_1}{2} + \tan \frac{\omega_2}{2}] + y_2 [\tan \frac{\omega_2}{2} + \tan \frac{\omega_3}{2}] + y_3 [\tan \frac{\omega_3}{2} + \tan \frac{\omega_4}{2}] + y_4 \tan \frac{\omega_4}{4} + X_4^{(L)} \qquad (4\text{-}2d)$$

where,

$$X_1^{(L)} = x_1^{(L)}$$

$$X_2^{(L)} = x_1^{(L)} + x_2^{(L)}$$

$$X_3^{(L)} = x_1^{(L)} + x_2^{(L)} + x_3^{(L)}$$

and

$$X_4^{(L)} = x_1^{(L)} + x_2^{(L)} + x_3^{(L)} + x_4^{(L)} .$$

The expressions for moments at any general section S in the spans are:

Span $\overline{01}$

$$M_{SR} = x_o A_{Ro1} + y_o B_{Ro1} + y_1 C_{Ro1} + BM_{SR}^{(0)}$$

$$M_{ST} = x_o A_{To1} + y_o B_{To1} + y_1 C_{To1} + BM_{ST}^{(0)}$$

Span $\overline{12}$

$$M_{SR} = x_1 A_{R12} + y_1 B_{R12} + y_2 C_{R12} + BM_{SR}^{(1)}$$

$$M_{ST} = x_1 A_{T12} + y_1 B_{T12} + y_2 C_{T12} + BM_{ST}^{(1)}$$

Span $\overline{23}$ (4-3)

$$M_{SR} = x_2 A_{R23} + y_2 B_{R23} + y_3 C_{R23} + BM_{SR}^{(2)}$$

$$M_{ST} = x_2 A_{T23} + y_2 B_{T23} + y_3 C_{T23} + BM_{ST}^{(2)}$$

Span $\overline{34}$

$$M_{SR} = x_3 A_{R34} + y_3 B_{R34} + y_4 C_{R34} + BM_{SR}^{(3)}$$

$$M_{ST} = x_3 A_{T34} + y_3 B_{T34} + y_4 C_{T34} + BM_{ST}^{(3)}$$

Substituting for x_1, x_2, and x_3 from equations 4-2a, b, c in equations 4-3,

Span $\overline{01}$

$$M_{SR} = x_o A_{Ro1} + y_o B_{Ro1} + y_1 C_{Ro1} + BM_{SR}^{(0)}$$

$$M_{ST} = x_o A_{To1} + y_o B_{To1} + y_1 C_{To1} + BM_{ST}^{(0)}$$

Span $\overline{12}$

$$M_{SR} = x_o A_{R12} + y_o A_{R12} \tan\frac{\omega_1}{2} + y_1 [A_{R12} \tan\frac{\omega_1}{2} + B_{R12}]$$

$$+ y_2 C_{R12} + BM_{SR}^{(1)} + x_1^{(L)} A_{R12}$$

$$M_{ST} = x_o A_{T12} + y_o A_{T12} \tan\frac{\omega_1}{2} + y_1 [A_{T12} \tan\frac{\omega_1}{2} + B_{T12}]$$

$$+ y_2 C_{T12} + BM_{ST}^{(1)} + x_1^{(L)} A_{T12}$$

Span $\overline{23}$

$$M_{SR} = x_o A_{R23} + y_o A_{R23} \tan\frac{\omega_2}{2} + y_1 A_{R23} [\tan\frac{\omega_1}{2} + \tan\frac{\omega_2}{2}]$$

$$+ y_2 [A_{R23} \tan\frac{\omega_2}{2} + B_{R23}] + y_3 C_{R23} + BM_{SR}^{(2)} + X_2^{(L)} A_{R23}$$

$$M_{ST} = x_o A_{T23} + y_o A_{T23} \tan\frac{\omega_2}{2} + y_1 A_{T23} [\tan\frac{\omega_1}{2} + \tan\frac{\omega_2}{2}]$$

$$+ y_2 [A_{T23} \tan\frac{\omega_2}{2} + B_{T23}] + y_3 C_{T23} + BM_{ST}^{(2)} + X_2^{(L)} A_{T23}$$

Span $\overline{34}$

$$M_{SR} = x_o A_{R34} + y_o A_{R34} \tan\frac{\omega_3}{2} + y_1 A_{R34} [\tan\frac{\omega_1}{2} + \tan\frac{\omega_2}{2}]$$

$$+ y_2 A_{R34} [\tan\frac{\omega_2}{2} + \tan\frac{\omega_3}{2}] + y_3 [A_{R34} \tan\frac{\omega_3}{2} + B_{R34}]$$

$$+ y_4 C_{R34} + BM_{SR}^{(3)} + X_3^{(L)} A_{R34}$$

$$M_{ST} = x_o A_{T34} + y_o A_{T34} \tan\frac{\omega_3}{2} + y_1 A_{T34} [\tan\frac{\omega_1}{2} + \tan\frac{\omega_2}{2}]$$

$$+ y_2 A_{T34} [\tan\frac{\omega_2}{2} + \tan\frac{\omega_3}{2}] + y_3 [A_{T34} \tan\frac{\omega_3}{2} + B_{T34}]$$

$$+ y_4 C_{T34} + BM_{ST}^{(3)} + X_3^{(L)} A_{T34} .$$

The total strain energy of the structure $\overline{01234}$ is:

$$U_S = U_{01} + U_{12} + U_{23} + U_{34}$$

$$= \int_0^1 [M_{SR}]^2 \lambda_R + \int_0^1 [M_{ST}]^2 \lambda_T + \int_1^2 [M_{SR}]^2 \lambda_R$$

$$+ \int_1^2 [M_{ST}]^2 \lambda_T + \int_2^3 [M_{SR}]^2 \lambda_R + \int_2^3 [M_{ST}]^2 \lambda_T$$

$$= \int_3^4 [M_{SR}]^2 \lambda_R + \int_3^4 [M_{ST}]^2 \lambda_T \qquad (4\text{-}5)$$

Partial differentiation of equation 4-5 with respect to x_o, y_o, y_1, y_2, y_3 and y_4 gives the six deformation equations:

$$\frac{\partial U_S}{\partial x_o} = \int_0^1 M_{SR} \frac{\partial M_{SR}}{\partial x_o} \lambda_R + \int_0^1 M_{ST} \frac{\partial M_{ST}}{\partial x_o} \lambda_T$$

$$+ \int_1^2 M_{SR} \frac{\partial M_{SR}}{\partial x_o} \lambda_R + \int_1^2 M_{ST} \frac{\partial M_{ST}}{\partial x_o} \lambda_T$$

$$+ \int_2^3 M_{SR} \frac{\partial M_{SR}}{\partial x_o} \lambda_R + \int_2^3 M_{ST} \frac{\partial M_{ST}}{\partial x_o} \lambda_T$$

$$+ \int_3^4 M_{SR} \frac{\partial M_{SR}}{\partial x_o} \lambda_R + \int_3^4 M_{ST} \frac{\partial M_{ST}}{\partial x_o} \lambda_T \qquad (4\text{-}6)$$

Similar expressions for $\frac{\partial U_S}{\partial y_o}$, $\frac{\partial U_S}{\partial y_1}$, $\frac{\partial U_S}{\partial y_2}$, $\frac{\partial U_S}{\partial y_3}$, and $\frac{\partial U_S}{\partial y_5}$ can be obtained. The values of the moments at a section in the span under consideration can be substituted in the above equations through the use of equations 4-4. Also, substituting the values of unity and zero for appropriate moments, analytical expressions for angular functions are derived. It is observed that these angular functions (F, G, τ) may be expressed in terms of angular functions (f, g, τ') of isolated spans. The first partial derivatives required in the above derivation are listed in Tables 4-1a, b, c, d. The final expressions obtained for the angular functions (F, G and τ) are recorded in Table 4-2a, b, c. Using these angular functions the deformation equations 4-6 can be expressed as:

$$\frac{\partial U_S}{\partial x_o} = x_o F_{ooTT} + y_o F_{ooTR} + y_1 G_{o1TR} + y_2 G_{o2TR}$$

$$+ y_3 G_{o3TR} + y_4 G_{o4TR} + \tau_{ooTT} = 0$$

$$\frac{\partial U_S}{\partial y_o} = x_o F_{4oRT} + y_o F_{o4RR} + y_1 G_{o1RR} + y_2 G_{o2RR}$$

$$+ y_3 G_{o3RR} + y_4 G_{o4RR} + \tau_{ooRR} = 0$$

$$\frac{\partial U_S}{\partial y_1} = x_o G_{10RT} + y_o G_{1oRR} + y_1 \Sigma F_{1RR} + y_2 G_{12RR}$$

$$+ y_3 G_{13RR} + y_4 G_{14RR} + \Sigma \tau_{1RR} = 0$$

$$\frac{\partial U_S}{\partial y_2} = x_o G_{2oRT} + y_o G_{2oRR} + y_1 G_{21RR} + y_2 \Sigma F_{2RR}$$

$$+ y_3 G_{23RR} + y_4 G_{24RR} + \Sigma \tau_{2RR} = 0$$

$$\frac{\partial U_S}{\partial y_3} = x_o G_{3oRT} + y_o G_{30RR} + y_1 G_{31RR} + y_2 G_{32RR}$$

$$+ y_3 \Sigma_{33RR} + y_4 G_{34RR} + \Sigma \tau_{3RR} = 0$$

$$\frac{\partial U_S}{\partial y_4} = x_o G_{4oRT} + y_o G_{4oRR} + y_1 G_{41RR} + y_2 G_{42RR}$$

$$+ y_3 G_{43RR} + y_4 F_{4oRR} + \tau_{44RR} = 0$$

TABLE 4-1a

FIRST PARTIAL DERIVATIVES - SPAN $\overline{01}$

First Partials	First Partial Values	First Partials	First Partial Values
$\frac{\partial M_{SR}}{\partial x_o}$	A_{Ro1}	$\frac{\partial M_{ST}}{\partial x_o}$	A_{To1}
$\frac{\partial M_{SR}}{\partial y_o}$	B_{Ro1}	$\frac{\partial M_{ST}}{\partial y_o}$	B_{To1}
$\frac{\partial M_{SR}}{\partial y_1}$	C_{Ro1}	$\frac{\partial M_{ST}}{\partial y_1}$	C_{To1}
$\frac{\partial M_{SR}}{\partial y_2}$	-	$\frac{\partial M_{ST}}{\partial y_2}$	-
$\frac{\partial M_{SR}}{\partial y_3}$	-	$\frac{\partial M_{ST}}{\partial y_3}$	-
$\frac{\partial M_{SR}}{\partial y_4}$	-	$\frac{\partial M_{ST}}{\partial y_4}$	-

TABLE 4-1b

FIRST PARTIAL DERIVATIVES - SPAN $\overline{12}$

First Partials	First Partial Values	First Partials	First Partial Values
$\frac{\partial M_{SR}}{\partial x_o}$	A_{R12}	$\frac{\partial M_{ST}}{\partial x_o}$	A_{T12}
$\frac{\partial M_{SR}}{\partial y_o}$	$A_{R12} \tan \frac{\omega_1}{2}$	$\frac{\partial M_{ST}}{\partial y_o}$	$A_{T12} \tan \frac{\omega_1}{2}$
$\frac{\partial M_{SR}}{\partial y_1}$	$A_{R12} \tan \frac{\omega_1}{2} + B_{R12}$	$\frac{\partial M_{ST}}{\partial y_1}$	$A_{T12} \tan \frac{\omega_1}{2} + B_{T12}$
$\frac{\partial M_{SR}}{\partial y_2}$	C_{R12}	$\frac{\partial M_{ST}}{\partial y_2}$	C_{T12}
$\frac{\partial M_{SR}}{\partial y_3}$	-	$\frac{\partial M_{ST}}{\partial y_3}$	-
$\frac{\partial M_{SR}}{\partial y_4}$	-	$\frac{\partial M_{ST}}{\partial y_4}$	-

TABLE 4-1c

FIRST PARTIAL DERIVATIVES - SPAN $\overline{23}$

First Partials	First Partial Values	First Partials	First Partial Values
$\frac{\partial M_{SR}}{\partial x_o}$	A_{R23}	$\frac{\partial M_{ST}}{\partial x_o}$	A_{T23}
$\frac{\partial M_{SR}}{\partial y_o}$	$A_{R23} \tan \frac{\omega_2}{2}$	$\frac{\partial M_{ST}}{\partial y_o}$	$A_{T23} \tan \frac{\omega_2}{2}$
$\frac{\partial M_{SR}}{\partial y_1}$	$A_{R23} [\tan \frac{\omega_1}{2} + \tan \frac{\omega_2}{2}]$	$\frac{\partial M_{ST}}{\partial y_1}$	$A_{T23} \tan \frac{\omega_1}{2} + \tan \frac{\omega_2}{2}$
$\frac{\partial M_{SR}}{\partial y_2}$	$A_{R23} \tan \frac{\omega_2}{2} + B_{R23}$	$\frac{\partial M_{ST}}{\partial y_2}$	$A_{T23} \tan \frac{\omega_2}{2} + B_{T23}$
$\frac{\partial M_{SR}}{\partial y_3}$	C_{R23}	$\frac{\partial M_{ST}}{\partial y_3}$	C_{T23}
$\frac{\partial M_{SR}}{\partial y_4}$	-	$\frac{\partial M_{ST}}{\partial y_4}$	-

TABLE 4-1d

FIRST PARTIAL DERIVATIVES - SPAN $\overline{34}$

First Partials	First Partial Values	First Partials	First Partial Values
$\frac{\partial M_{SR}}{\partial x_o}$	A_{R34}	$\frac{\partial M_{ST}}{\partial x_o}$	A_{T34}
$\frac{\partial M_{SR}}{\partial y_o}$	$A_{R34} \tan \frac{\omega_3}{2}$	$\frac{\partial M_{ST}}{\partial y_o}$	$A_{T34} \tan \frac{\omega_3}{2}$
$\frac{\partial M_{SR}}{\partial y_1}$	$A_{R34} \quad \tan \frac{\omega_1}{2} + \tan \frac{\omega_2}{2}$	$\frac{\partial M_{ST}}{\partial y_1}$	$A_{T34} \quad \tan \frac{\omega_1}{2} + \tan \frac{\omega_2}{2}$
$\frac{\partial M_{SR}}{\partial y_2}$	$A_{R34} \quad \tan \frac{\omega_2}{2} + \tan \frac{\omega_3}{2}$	$\frac{\partial M_{ST}}{\partial y_2}$	$A_{T34} \quad \tan \frac{\omega_2}{2} + \tan \frac{\omega_3}{2}$
$\frac{\partial M_{SR}}{\partial y_3}$	$A_{R34} \tan \frac{\omega_3}{2} + B_{R34}$	$\frac{\partial M_{ST}}{\partial y_3}$	$A_{T34} \tan \frac{\omega_3}{2} + B_{T34}$
$\frac{\partial M_{SR}}{\partial y_4}$	C_{R34}	$\frac{\partial M_{ST}}{\partial y_4}$	C_{T34}

4-2. Solution Matrix

The compatibility equations (eq. 4-7) are written in the matrix form as follows:

$$
\begin{bmatrix}
F_{ooTT} & F_{ooTR} & G_{o1TR} & G_{o2TR} & G_{o3TR} & G_{o4TR} \\
F_{4oRT} & F_{o4RR} & G_{o1RR} & G_{o2RR} & G_{o3RR} & G_{o4RR} \\
G_{1oRT} & G_{1oRR} & \Sigma F_{1RR} & G_{12RR} & G_{13RR} & G_{14RR} \\
G_{2oRT} & G_{2oRR} & G_{21RR} & \Sigma F_{2RR} & G_{23RR} & G_{24RR} \\
G_{3oRT} & G_{30RR} & G_{31RR} & G_{32RR} & \Sigma F_{3RR} & G_{34RR} \\
G_{4oRT} & G_{4oRR} & G_{41RR} & G_{42RR} & G_{43RR} & F_{4oRR}
\end{bmatrix}
\begin{bmatrix}
x_o \\ y_o \\ y_1 \\ y_2 \\ y_3 \\ y_4
\end{bmatrix}
= -
\begin{bmatrix}
\tau_{ooTT} \\ \tau_{ooRR} \\ \Sigma\tau_{1RR} \\ \Sigma\tau_{2RR} \\ \Sigma\tau_{3RR} \\ \tau_{44RR}
\end{bmatrix}
$$

TABLE 4-2a

ANGULAR FLEXIBILITIES AND LOAD FUNCTIONS

F_{ooTT}	$\frac{\partial U_S}{\partial x_o}$	$X_o = 1,\ y_o = y_1 = y_2 = y_3 = y_4 = \text{Loads} = 0$	$f_{o1TT} + f_{12TT} + f_{23TT} + f_{34TT}$
F_{ooRR}	$\frac{\partial U_S}{\partial y_o}$	$y_o = 1,\ x_o = y_1 = y_2 = y_3 = y_4 = \text{Loads} = 0$	$f_{o1RR} + \tan^2 \frac{\omega_1}{2} [f_{12TT} + f_{23TT} + f_{34TT}]$
ΣF_{1RR}	$\frac{\partial U_S}{\partial y_1}$	$y_1 = 1,\ x_o = y_o = y_2 = y_3 = y_4 = \text{Loads} = 0$	$f_{10RR} + f_{12RR} + 2 \tan \frac{\omega_1}{2} f_{12TR} + \tan^2 \frac{\omega_1}{2} + \tan \frac{\omega_1}{2} + \tan \frac{\omega_2}{2}^{\ 2} f_{23TT} + f_{34TT}$
ΣF_{2RR}	$\frac{\partial U_S}{\partial y_2}$	$y_2 = 1,\ x_o = y_o = y_1 = y_3 = y_4 = \text{Loads} = 0$	$f_{21RR} + f_{23RR} + 2 \tan \frac{\omega_2}{2} f_{23TR} + \tan^2 \frac{\omega_2}{2} f_{23TT} + \tan \frac{\omega_2}{2} + \tan \frac{\omega_3}{2}^{\ 2} f_{34TT}$
ΣF_{3RR}	$\frac{\partial U_S}{\partial y_3}$	$y_3 = 1,\ x_o = y_o = y_1 = y_2 = y_4 = \text{Loads} = 0$	$f_{32RR} + f_{34RR} + 2 \tan \frac{\omega_3}{2} f_{34RR} + \tan^2 \frac{\omega_3}{2} f_{34TT}$
F_{44RR}	$\frac{\partial U_S}{\partial y_4}$	$y_4 = 1,\ x_o = y_o = y_1 = y_2 = y_3 = \text{Loads} = 0$	f_{43RR}
τ_{ooTT}	$\frac{\partial U_S}{\partial x_o}$	$x_o = y_o = y_1 = y_2 = y_3 = y_4 = 0$	$\tau'_{o1TT} + \tau'_{12TT} + \tau'_{23TT} + \tau'_{34TT} + X_1^{(L)} f_{12TT} + X_2^{(L)} f_{23TT} + X_3^{(L)} f_{34TT}$
τ_{ooRR}	$\frac{\partial U_S}{\partial y_o}$	$x_o = y_o = y_1 = y_2 = y_3 = y_4 = 0$	$\tau'_{o1RR} + \tan \frac{\omega_1}{2}\ \tau'_{12TT} + \tau'_{23TT} + \tau'_{34TT} + X_1^{(L)} \tan \frac{\omega_1}{2} f_{12TT}$ $+ X_2^{(L)} \tan \frac{\omega_1}{2} f_{23TT} + X_3^{(L)} \tan \frac{\omega_1}{2} f_{34TT}$
$\Sigma\tau_{1RR}$	$\frac{\partial U_S}{\partial y_1}$	$x_o = y_o = y_1 = y_2 = y_3 = y_4 = 0$	$\tau'_{1oRR} + \tan \frac{\omega_1}{2}\ \tau'_{12TT} + \tau'_{23TT} + \tau'_{34TT} + \tan \frac{\omega_1}{2}\ X_1^{(L)} f_{12TT} + X_2^{(L)} f_{23TT} + X_3^{(L)} f_{34TT}$ $+ \tau'_{12RR} + \tan \frac{\omega_2}{2} \tau'_{23TT} + \tau'_{34TT} + X_1^{(L)} f_{12TR} + \tan \frac{\omega_2}{2}\ X_2^{(L)} f_{23TT} + X_3^{(L)} f_{34TT}$
$\Sigma\tau_{2RR}$	$\frac{\partial U_S}{\partial y_2}$	$x_o = y_o = y_1 = y_2 = y_3 = y_4 = 0$	$\tau'_{21RR} + \tan \frac{\omega_2}{2}\ \tau'_{23TT} + \tau'_{34TT} + \tan \frac{\omega_2}{2}\ X_2^{(L)} f_{23TT} + X_3^{(L)} f_{34TT} + X_1^{(L)} g_{12TR}$ $+ \tau'_{23RR} + \tan \frac{\omega_3}{2}\ \tau'_{34TT} + X_2^{(L)} f_{23TR} + \tan \frac{\omega_3}{2} X_3^{(L)} f_{34TT}$
$\Sigma\tau_{3RR}$	$\frac{\partial U_S}{\partial y_3}$	$x_o = y_o = y_1 = y_2 = y_3 = y_4 = 0$	$\tau'_{32RR} + \tan \frac{\omega_3}{2}\ \tau'_{34TT} + X_2^{(L)} g_{23TR} + \tan \frac{\omega_3}{2}\ X_3^{(L)} f_{34TT} + \tau'_{34RR} + X_3^{(L)} f_{34TR}$
τ_{44RR}	$\frac{\partial U_S}{\partial y_4}$	$x_o = y_o = y_1 = y_2 = y_3 = y_4 = 0$	$\tau'_{43RR} + X_3^{(L)} g_{34TR}$

TABLE 4-2b

ANGULAR CARRY-OVER FUNCTIONS

F_{ooRT} = F_{ooTR}	$\frac{\partial U_S}{\partial x_o}$	$y_o = 1,\ x_o = y_1 = y_2 = y_3 = y_4 =$ Loads $= 0$	$f_{o1RT} + \tan\frac{\omega_1}{2}\,[f_{12TT} + f_{23TT} + f_{34TT}]$
	$\frac{\partial U_S}{\partial y_o}$	$x_o = 1,\ y_o = y_1 = y_2 = y_3 = y_4 =$ Loads $= 0$	
G_{1oRT} = G_{o1TR}	$\frac{\partial U_S}{\partial x_o}$	$y_1 = 1,\ x_o = y_o = y_2 = y_3 = y_4 =$ Loads $= 0$	$g_{o1TR} + f_{12TR} + \tan\frac{\omega_1}{2}\,[f_{12TT} + f_{23TT} + f_{34TT}] + \tan\frac{\omega_2}{2}\,[f_{23TT} + f_{34TT}]$
	$\frac{\partial U_S}{\partial y_1}$	$x_o = 1,\ y_o = y_1 = y_2 = y_3 = y_4 =$ Loads $= 0$	
G_{2oRT} = G_{o2TR}	$\frac{\partial U_S}{\partial x_o}$	$x_o = 1,\ y_o = y_1 = y_2 = y_3 = y_4 =$ Loads $= 0$	$g_{12TR} + f_{23TR} + \tan\frac{\omega_3}{2}\,[f_{23TT} + f_{34TT}] + \tan\frac{\omega_3}{2}\,f_{34TT}$
	$\frac{\partial U_S}{\partial y_2}$	$x_o = 1,\ y_o = y_1 = y_2 = y_3 = y_4 =$ Loads $= 0$	
G_{3oRT} = G_{o3TR}	$\frac{\partial U_S}{\partial x_o}$	$y_3 = 1,\ x_o = y_o = y_1 = y_2 = y_4 =$ Loads $= 0$	$g_{23TR} + f_{34TR} + \tan\frac{\omega_3}{2}\,f_{34TT}$
	$\frac{\partial U_S}{\partial y_3}$	$x_o = 1,\ y_o = y_1 = y_2 = y_3 = y_4 =$ Loads $= 0$	
G_{4oRT} = G_{o4TR}	$\frac{\partial U_S}{\partial x_o}$	$x_o = 1,\ y_o = y_1 = y_2 = y_3 = y_4 =$ Loads $= 0$	g_{34TR}
	$\frac{\partial U_S}{\partial y_4}$	$x_o = 1,\ y_o = y_1 = y_2 = y_3 = y_4 =$ Loads $= 0$	
G_{1oRR} = G_{o1RR}	$\frac{\partial U_S}{\partial y_o}$	$y_1 = 1,\ x_o = y_o = y_2 = y_3 = y_4 =$ Loads $= 0$	$g_{o1RR} + \tan\frac{\omega_1}{2}\,f_{12TR} + \tan^2\frac{\omega_1}{2}\,[f_{12TT} + f_{23TT} + f_{34TT}]$
	$\frac{\partial U_S}{\partial y_1}$	$y_o = 1,\ x_o = y_1 = y_2 = y_3 = y_4 =$ Loads $= 0$	$+ \tan\frac{\omega_1}{2}\,\tan\frac{\omega_2}{2}\,[f_{23TT} + f_{34TT}]$
G_{2oRR} = G_{o2RR}	$\frac{\partial U_S}{\partial y_o}$	$y_2 = 1,\ x_o = y_o = y_1 = y_3 = y_4 =$ Loads $= 0$	$\tan\frac{\omega_1}{2}\,g_{12TR} + \tan\frac{\omega_1}{2}\,f_{23TR} + \tan\frac{\omega_1}{2}\,\tan\frac{\omega_2}{2}\,[f_{23TT} + f_{34FT}]$
	$\frac{\partial U_S}{\partial y_2}$	$y_o = 1,\ x_o = y_1 = y_2 = y_3 = y_4 =$ Loads $= 0$	$+ \tan\frac{\omega_1}{2}\,\tan\frac{\omega_3}{2}\,f_{34TT}$

TABLE 4-2c

ANGULAR CARRY-OVER FUNCTIONS

G_{3oRR} =	$\frac{\partial U_S}{\partial y_o}$	$y_3 = 1,\ x_o = y_o = y_1 = y_2 = y_4 = \text{Loads} = 0$	$\tan\frac{\omega_1}{2}\, g_{23TR} + \tan\frac{\omega_1}{2}\, f_{34TR} + \tan\frac{\omega_1}{2}\, \tan\frac{\omega_3}{2}\, f_{34TT}$
G_{o3RR}	$\frac{\partial U_S}{\partial y_3}$	$y_o = 1,\ x_o = y_1 = y_2 = y_3 = y_4 = \text{Loads} = 0$	
G_{4oRR} =	$\frac{\partial U_S}{\partial y_o}$	$y_4 = 1,\ x_o = y_o = y_1 = y_2 = y_3 = \text{Loads} = 0$	$\tan\frac{\omega_1}{2}\, g_{34TR}$
G_{o4RR}	$\frac{\partial U_S}{\partial y_4}$	$y_o = 1,\ x_o = y_1 = y_2 = y_3 = y_4 = \text{Loads} = 0$	
G_{21RR} =	$\frac{\partial U_S}{\partial y_1}$	$y_2 = 1,\ x_o = y_o = y_1 = y_3 = y_4 = \text{Loads} = 0$	$g_{12RR} + \tan\frac{\omega_1}{2}\, g_{12TR} + [\tan\frac{\omega_1}{2} + \tan\frac{\omega_2}{2}][f_{23TR} + \tan\frac{\omega_2}{2}\, f_{23TT}]$
G_{12RR}	$\frac{\partial U_S}{\partial y_2}$	$y_1 = 1,\ x_o = y_o = y_2 = y_3 = y_4 = \text{Loads} = 0$	$+ [\tan\frac{\omega_1}{2} + \tan\frac{\omega_2}{2}][\tan\frac{\omega_2}{2} + \tan\frac{\omega_3}{2}]\, f_{34TT}$
G_{31RR} =	$\frac{\partial U_S}{\partial y_1}$	$y_3 = 1,\ x_o = y_o = y_1 = y_2 = y_4 = \text{Loads} = 0$	$[\tan\frac{\omega_1}{2} + \tan\frac{\omega_2}{2}][g_{23TR} + f_{34TR} + \tan\frac{\omega_3}{2}\, f_{34TT}]$
G_{13RR}	$\frac{\partial U_S}{\partial y_3}$	$y_1 = 1,\ x_o = y_o = y_2 = y_3 = y_4 = \text{Loads} = 0$	
G_{41RR} =	$\frac{\partial U_S}{\partial y_1}$	$y_4 = 1,\ x_o = y_o = y_1 = y_2 = y_3 = \text{Loads} = 0$	$g_{34TR}\, [\tan\frac{\omega_1}{2} + \tan\frac{\omega_2}{2}]$
G_{14RR}	$\frac{\partial U_S}{\partial y_4}$	$y_1 = 1,\ x_o = y_o = y_2 = y_3 = y_4 = \text{Loads} = 0$	
G_{32RR} =	$\frac{\partial U_S}{\partial y_2}$	$y_3 = 1,\ x_o = y_o = y_1 = y_2 = y_4 = \text{Loads} = 0$	$g_{23RR} + \tan\frac{\omega_2}{2}\, g_{23TR} + [\tan\frac{\omega_2}{2} + \tan\frac{\omega_3}{2}][f_{34TR} + \tan\frac{\omega_3}{2}\, f_{34TT}]$
G_{23RR}	$\frac{\partial U_S}{\partial y_3}$	$y_2 = 1,\ x_o = y_o = y_1 = y_3 = y_4 = \text{Loads} = 0$	
G_{42RR} =	$\frac{\partial U_S}{\partial y_2}$	$y_4 = 1,\ x_o = y_o = y_1 = y_2 = y_3 = \text{Loads} = 0$	$g_{34TR}\, [\tan\frac{\omega_2}{2} + \tan\frac{\omega_3}{2}]$
G_{24RR}	$\frac{\partial U_S}{\partial y_4}$	$y_2 = 1,\ x_o = y_o = y_1 = y_3 = y_4 = \text{Loads} = 0$	
G_{43RR} =	$\frac{\partial U_S}{\partial y_3}$	$y_4 = 1,\ x_o = y_o = y_1 = y_2 = y_3 = \text{Loads} = 0$	$g_{34RR} + \tan\frac{\omega_3}{2}\, g_{34TR}$
G_{34RR}	$\frac{\partial U_S}{\partial y_4}$	$y_3 = 1,\ x_o = y_o = y_1 = y_2 = y_4 = \text{Loads} = 0$	

CHAPTER V

SPECIAL DERIVATIONS

The angular functions derived in the earlier chapters are applicable for beams of varying cross-section and any general loading. Calculations of the angular functions, (f, g, τ') of the one-span basic structure are now illustrated for the case of a constant cross-section and for specified load conditions . These values are used in numerical examples in Chapter VI.

5-1. Angular Flexibilities and Angular Carry-Over Functions (f, g) .

The expressions for the angular flexibilities and angular carry-over functions given in Table 3-1 are evaluated for constant EI and the results are summarised in Table 5-1.

5-2. Angular Load Functions (τ').

The following two loading conditions for a basic structure $\overline{ij}$ are considered.

1. Unit concentrated load (P = 1) at an angle θ from support i. (Fig. 5-1)
2. Uniformly distributed load (ω = 1). (Fig. 5-2)

The reactive and cross-sectional elements in the basic structure ij (Fig. 5-3 and 5-4), due to these two loading conditions, can be calculated by statics and are given in Table 5-2. Substituting the values of cross-sectional moments (BM_{SR}, BM_{ST}) from Table 5-2, into the expressions for load functions given in Table 3-2, and integrating gives the required load functions. The final values of these load functions are given in Tables 5-3a, b.

TABLE 5-1

FLEXIBILITIES AND CARRY-OVER FUNCTIONS

(BASIC SPAN $\overline{ij}$)

f_{ijTT}	$\frac{R\omega_j}{gJ}$
f_{ijRR}	$\frac{R}{4EI\sin^2\omega_j}[2\omega_j - \sin 2\omega_j] + \frac{R}{4gJ\sin^2\omega_j}[2\omega_j - 3\sin 2\omega_j + 4\omega_j\cos^2\omega_j]$
f_{jiRR}	$\frac{R}{4EI\sin^2\omega_j}[2\omega_j - \sin 2\omega_j] + \frac{R}{4gJ\sin^2\omega_j}[6\omega_j - 8\sin\omega_j + \sin 2\omega_j]$
f_{ijRT} = f_{ijTR}	$\frac{R}{gJ\sin\omega_j}[\sin\omega_j - \omega_j\cos\omega_j]$
g_{ijRR} = g_{jiRR}	$\frac{R}{2EI\sin^2\omega_j}[\sin\omega_j - \omega_j\cos\omega_j] + \frac{R}{2gJ\sin^2\omega_j}[\sin\omega_j + \sin 2\omega_j - 3\omega_j\cos\omega_j]$
g_{ijTR} = g_{jiRT}	$\frac{R}{gJ\sin\omega_j}[\omega_j - \sin\omega_j]$

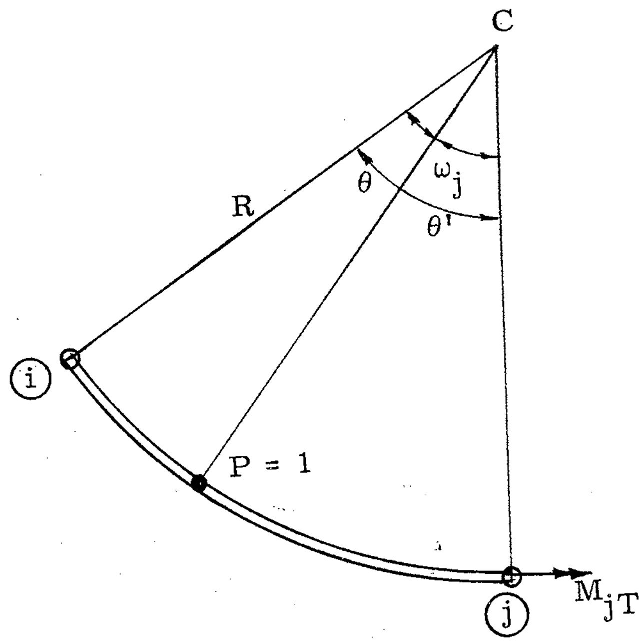


Fig. 5-1
Basic Structure - Loading Condition I

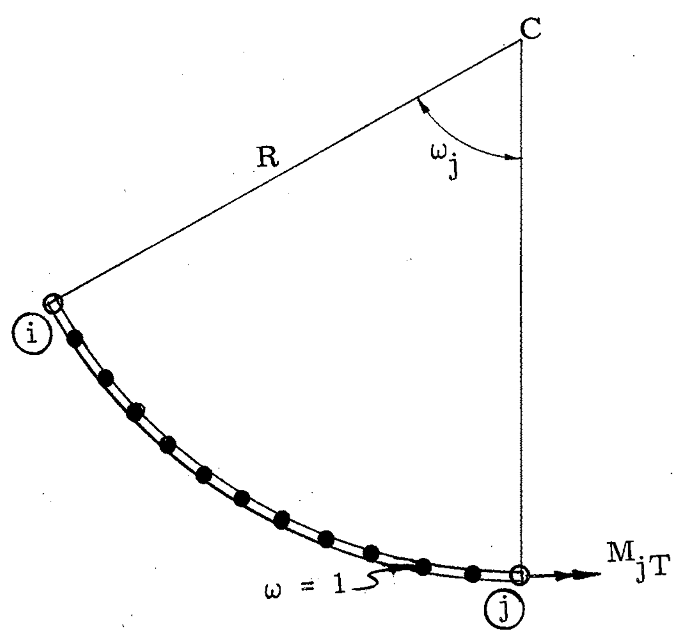


Fig. 5-2
Basic Structure - Loading Condition II

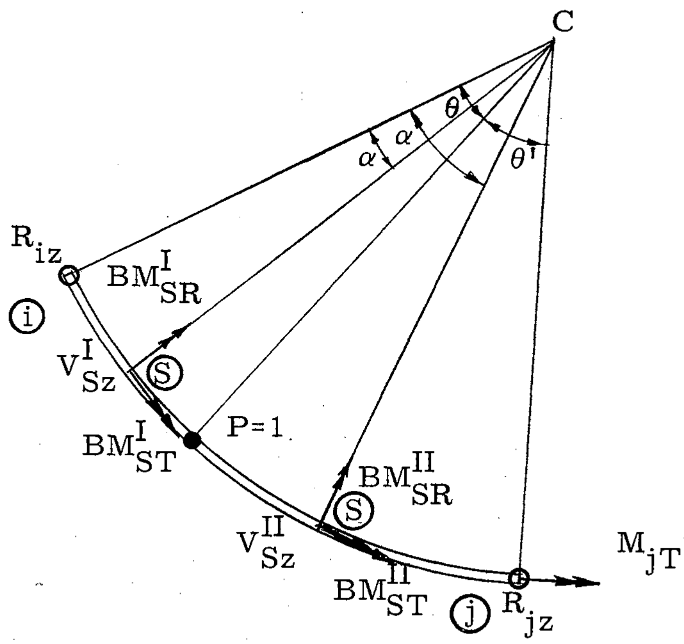


Fig. 5-3

Basic Structure - Loading Condition I,
Showing Reactive and Cross-Sectional Elements

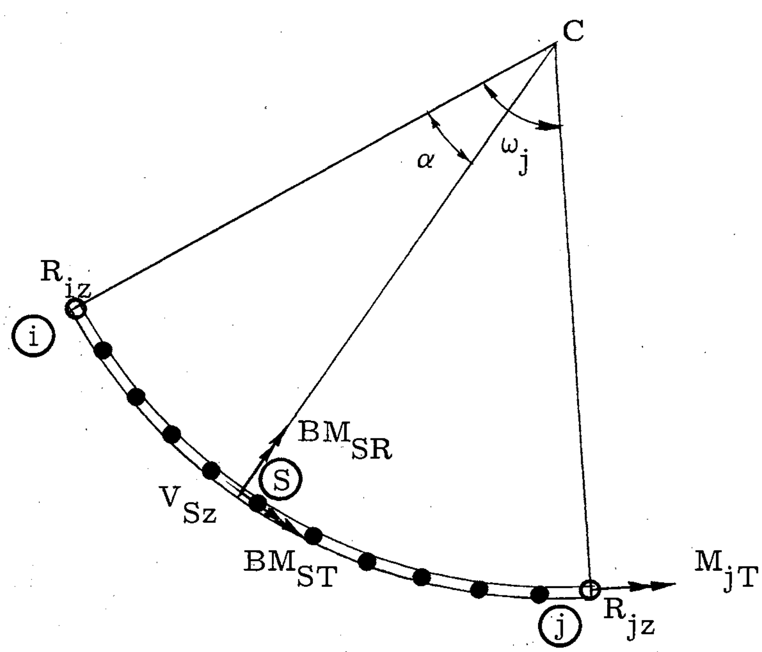


Fig. 5-4

The Basic Structure ij - Loading Condition II,
Showing Reactive and Cross-Sectional Elements

TABLE 5-2

REACTIVE ELEMENTS AND CROSS-SECTIONAL ELEMENTS IN A BASIC STRUCTURE ij, FOR LOAD CONDITIONS I AND II

	Element	Loading Condition I (P=1)	Element	Loading Condition II (ω=1)
Reactions	R_{iz}	$+\dfrac{\sin\theta'}{\sin\omega_j}$	R_{iz}	$+\dfrac{R(1-\cos\omega_j)}{\sin\omega_j}$
	R_{jz}	$-\dfrac{\sin\theta'}{\sin\omega_j}$	R_{jz}	$-\dfrac{R(1-\cos\omega_j)}{\sin\omega_j}$
	$M_{jT} = x_j^L$	$-\dfrac{R\sin\theta'}{\sin\omega_j}(1-\cos\omega_j)+R(1-\cos\theta')$	$M_{jT} = x_j^L$	$-\dfrac{R^2(1-\cos\omega_j)^2}{\sin\omega_j}+R^2(1+\omega_j-\cos\omega_j)$
Cross-Sectional Elements	BM_{SR}^{I}	$-\dfrac{R\sin\theta'\sin\alpha}{\sin\omega_j}$	BM_{SR}	$-\dfrac{R^2\sin\alpha(1-\cos\omega_j)}{\sin\omega_j}+R^2(1-\cos\alpha)$
	BM_{SR}^{II}	$-\dfrac{R\sin\theta'\sin\alpha}{\sin\omega_j}+R\sin(\alpha-\theta)$		
	BM_{ST}^{I}	$-\dfrac{R\sin\theta'(1-\cos\alpha)}{\sin\omega_j}$	BM_{ST}	$-\dfrac{R^2(1-\cos\alpha)(1-\cos\omega_j)}{\sin\omega_j}+R^2(\alpha-\sin\alpha)$
	BM_{ST}^{II}	$-\dfrac{R\sin\theta'(1-\cos\alpha)}{\sin\omega_j}+R[1-\cos(\alpha-\theta)]$		
	V_{Sz}^{I}	$-\dfrac{\sin\theta'}{\sin\omega_j}$	V_{Sz}	$-\dfrac{R(1-\cos\omega_j)}{\sin\omega_j}+R\alpha$
	V_{sz}^{II}	$-\dfrac{\sin\theta'}{\sin\omega_j}+1$		

TABLE 5-3

ANGULAR LOAD FUNCTIONS - BASIC STRUCTURE

Load Condition I Unit Concentrate Load	τ'_{ijTT}	$+\frac{R^2}{GJ}\left[\theta' - \frac{\omega_j \sin\theta'}{\sin\omega_j}\right]$
	τ'_{ijRR}	$+\frac{R^2}{2\,EI\sin\omega_j}\left[\frac{\omega_j\sin\theta'\cos\omega_j}{\sin\omega_j} - \theta'\cos\theta'\right] + \frac{R^2}{2\,GJ\sin\omega_j}\left[\frac{3\,\omega_j\sin\theta'\cos\omega_j}{\sin\omega_j} - 2\,\theta'\cos\omega_j - \theta'\cos\theta'\right]$
	τ'_{jiRR}	$+\frac{R^2}{4\,EI\sin\omega_j}\left[\frac{\sin\theta'}{\sin\omega_j}(-2\,\omega_j + \sin 2\,\omega_j) + \sin\theta + 2\,\theta'\cos\theta - \sin(\omega_j + \theta')\right]$ $+\frac{R^2}{4GJ\sin\omega_j}\left[\frac{\sin\theta'}{\sin\omega_j}(-6\omega_j + 8\sin\omega_j - \sin 2\,\omega_j) + 4\,\theta' - 4\sin\omega_j + 3\sin\theta + 2\theta'\cos\theta - 4\sin\theta'\right.$ $\left. + \sin(\omega_j + \theta')\right]$
Load Condition II Uniformly Distributed Load - $\omega = 1$	τ'_{ijTT}	$+\frac{R^3}{GJ\sin\omega_j}\left[-\omega_j + \frac{1}{2}\omega_j^2\sin\omega_j + \omega_j\cos\omega_j\right]$
	τ'_{ijRR}	$+\frac{R^3}{2\,EI\sin^2\omega_j}\left[\omega_j\cos\omega_j - \omega_j + \sin\omega_j - \frac{1}{2}\sin 2\,\omega_j\right]$ $+\frac{R^3}{2\,GJ\sin^2\omega_j}\left[3\,\omega_j\cos\omega_j - \frac{\omega_j^2}{2}\sin 2\,\omega_j + \sin\omega_j - 2\,\omega_j\cos^2\omega_j - \frac{1}{2}\sin 2\,\omega_j - \omega_j\right]$
	τ'_{jiRR}	$+\frac{R^3}{2\,EI\sin^2\omega_j}\left[\omega_j\cos\omega_j - \omega_j + \sin\omega_j - \frac{1}{2}\sin 2\,\omega_j\right]$ $+\frac{R^3}{2\,GJ\sin^2\omega_j}\left[3\,\omega_j\cos\omega_j + \omega_j^2\sin\omega_j - 2\,\omega_j\sin^2\omega_j + 5\sin\omega_j - \frac{5}{2}\sin 2\,\omega_j - 3\,\omega_j\right]$

CHAPTER VI

APPLICATION

The application of the theory developed in the previous chapters for the analysis of the curved beams is now illustrated by numerical examples. A four-span continuous circular beam, with exterior ends fixed, is analysed for various loading conditions.

6-1. Procedure of Analysis.

A systematic procedure of analysis for the class of continuous beams discussed in this thesis is presented in the following steps:

1. Break the structure into basic spans and determine the angular functions (f, g, τ') for these spans.
2. Select the redundant moments and determine the angular functions (F, G, and τ).
3. Formulate the compatibility equations in terms of these redundant moments and angular functions.
4. Solve the compatibility equations (Step 3) for the redundant moments.
5. Solve for the other unknowns by statics.

6-2. Numerical Examples.

A four-span continuous circular beam, with exterior ends fixed, and loaded as shown (Fig. 6-1, 2, 3) is analyzed by the method of flexibilities. The beam has a radius of curvature equal to 60 ft. and constant rectangular section. The $\frac{EI}{GJ}$ ratio is assumed equal to two.

[$\frac{EI}{GJ} = \frac{1+\mu}{2} \frac{d^2}{b^2}$, where Poisson's Ratio (μ) for a concrete is assumed to be 0.25; d and b are the depth and the width of the beam respectively]

In the solution of problems all values, unless stated otherwise, are in kips, feet and kip-feet. References are made in each example to the equations, and tables used.

The structure is analyzed for the following three load conditions.

1. A concentrated load of 10 kips at each mid-span. (Fig. 6-1)
2. A uniformly distributed load (ω = 1 k/ft.) on the entire structure. (Fig. 6-2)
3. A uniformly distributed load (ω = 1 k/ft.) on the spans $\overline{01}$ and $\overline{23}$. (Fig. 6-3)

As the geometry of the structure is same for all the three problems considered, the angular flexibilities and the angular carry-over flexibilities remains same for these problems.

The redundant moments are shown in Fig. 6-4 and freebody diagram is shown in Fig. 6-5.

A. Load Condition I

1. Flexibilities and Carry-over Values (f, g)

Since all the spans are of equal length, the values (f, g) are the same for each span

Flexibilities: From Table 5-1.

$$f_{o1TT} = f_{12TT} = f_{23TT} = f_{34TT} = +\frac{62.831856}{EI}$$

$$f_{o1RR} = f_{12RR} = f_{23RR} = f_{34RR} = +\frac{13.260468}{EI}$$

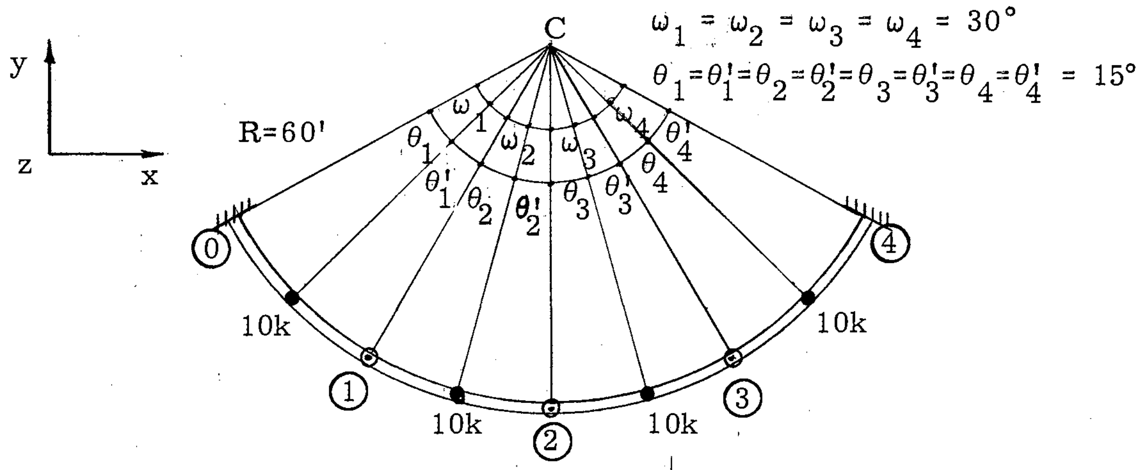


Fig. 6-1

Load Condition I

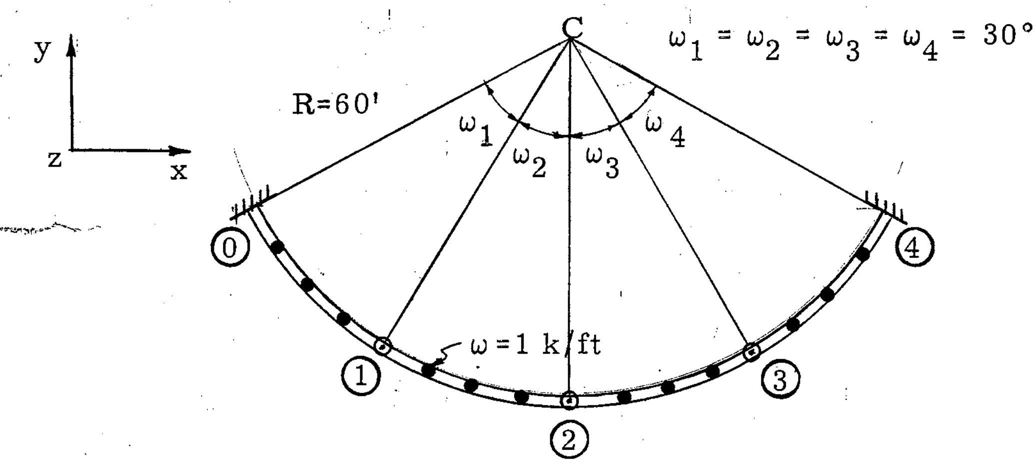


Fig. 6-2

Load Condition II

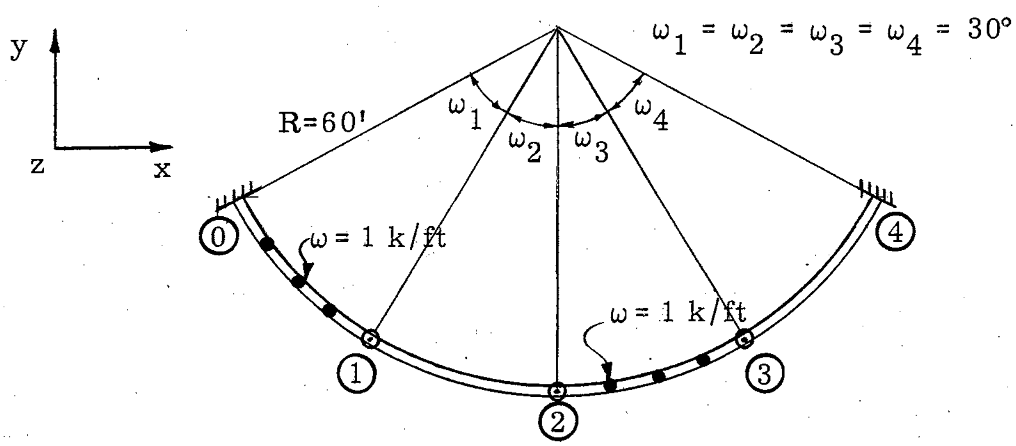


Fig. 6-3

Load Condition III

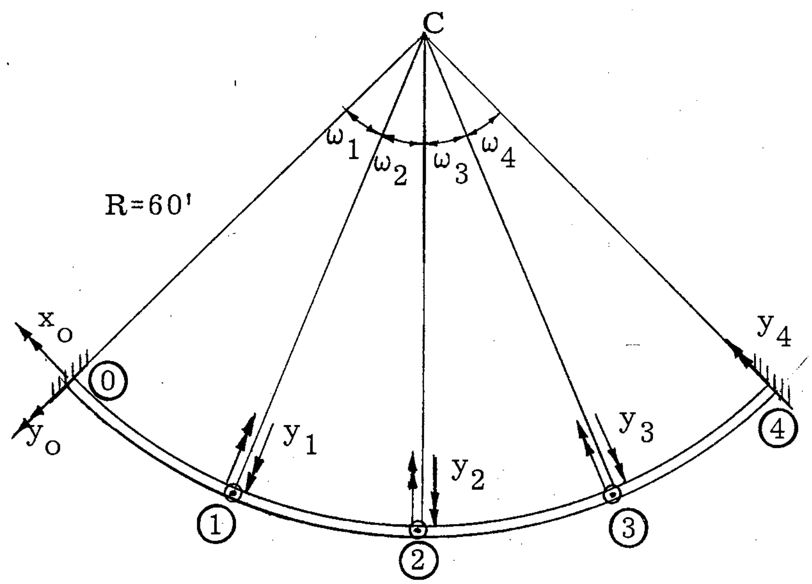


Fig. 6-4

Four-Span Continuous Circular Beam - With Redundants

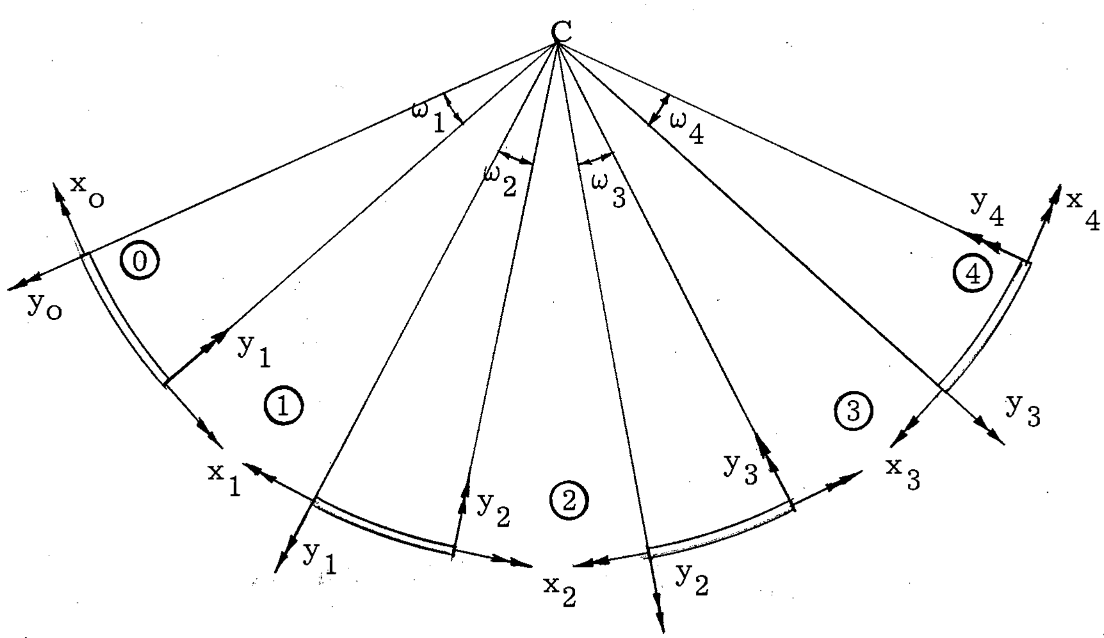


Fig. 6-5

Free-Body Diagram of a Four-Span Continuous Circular Beam

$$f_{1oRR} = f_{21RR} = f_{32RR} = f_{43RR} = + \frac{11.784516}{EI}$$

Carry-Over Values: From Table 5-1.

$$f_{o1RT} = f_{12RT} = f_{23RT} = f_{34RT} = f_{o1TR} = f_{12TR} = f_{23TR} = f_{34TR}$$
$$= + \frac{11.1720}{EI}$$

$$g_{o1TR} = g_{1oRT} = g_{12TR} = g_{21RT} \; g_{23TR} = g_{32RT} = g_{34TR} = g_{43RT}$$
$$= + \frac{5.663712}{EI}$$

$$g_{o1RR} = g_{1oRR} = g_{12RR} = g_{21RR} = g_{23RR} = g_{32RR} = g_{34RR}$$
$$= g_{43RR} = + \frac{6.94810}{EI}$$

2. Load Functions (τ)

Since all the spans are of equal length and are symmetrically loaded, the load functions (τ) are the same for each span.

Load Functions: From Table 5-3.

$$\tau'_{o1TT} = \tau'_{12TT} = \tau'_{23TT} = \tau'_{34TT} = - \frac{664.93700}{EI}$$

$$\tau'_{o1RR} = \tau'_{12RR} = \tau'_{23RR} = \tau'_{34RR} = - \frac{809.14320}{EI}$$

$$\tau'_{1oRR} = \tau'_{21RR} = \tau'_{32RR} = \tau'_{43RR} = - \frac{752.97852}{EI}$$

3. Angular Flexibilities and Carry-Over Flexibilities (F, G)

These values are obtained by substituting the values (f, g) in the expressions for flexibilities and carry-over flexibilities (Tables 4-2a, b, c).

$$F_{ooTT} = \frac{251.327424}{EI}$$

$$F_{ooTR} = F_{ooRT} = \frac{61.350728}{EI}$$

$$G_{o1TR} = G_{1oRT} = \frac{101.01444}{EI}$$

$$G_{o2TR} = G_{2oRT} = \frac{67.01444}{EI}$$

$$G_{o3TR} = G_{3oRT} = \frac{33.671424}{EI}$$

$$G_{o4TR} = G_{4oRT} = \frac{11.1720}{EI}$$

$$F_{o4RR} = \frac{26.793842}{EI}$$

$$G_{o1RR} = G_{1oRR} = \frac{32.497247}{EI}$$

$$G_{o2RR} = G_{20RR} = \frac{18.044490}{EI}$$

$$G_{o3RR} = G_{3oRR} = \frac{9.022401}{EI}$$

$$G_{o4RR} = G_{4oRR} = \frac{1.517587}{EI}$$

$$\Sigma F_{1RR} = \frac{71.632160}{EI}$$

$$G_{12RR} = G_{21RR} = \frac{41.519487}{EI}$$

$$G_{13RR} = G_{31RR} = \frac{18.04448}{EI}$$

$$G_{14RR} = G_{41RR} = \frac{3.035174}{EI}$$

$$\Sigma F_{2RR} = \frac{53.587664}{EI}$$

$$G_{23RR} = G_{32RR} = \frac{23.474989}{EI}$$

$$G_{24RR} = G_{42RR} = \frac{3.035174}{EI}$$

$$\Sigma F_{3RR} = \frac{35.543166}{EI}$$

$$G_{34RR} = G_{43RR} = \frac{8.465683}{EI}$$

$$F_{4oRR} = \frac{11.784516}{EI}$$

4. Load Functions (τ's)

Moments (x_1^L, x_2^L, x_3^L, and x_4^L) due to loads only: From Table 5-2

$$x_1^L = x_2^L = x_3^L = x_4^L = -10\left[\frac{60 \sin 15}{\sin 30}\ (1 - \cos 30) + 60(1 - \cos 15)\right]$$

$$= -21.165725$$

$$X_1^L = -21.165725$$

$$X_2^L = X_1^L + x_2^L = -42.33145$$

$$X_3^L = X_2^L + x_3^L = -63.497175$$

$$X_4^L = X_3^L + x_4^L = -84.66290$$

Substituting these values (X_1^L, X_2^L ...) and the calculated values (f, g, τ in the expressions for the load functions (Table 4-2a),

$$\tau_{ooTT} = -\frac{10639.0386}{EI}$$

$$\tau_{ooRR} = -\frac{3481.69576}{EI}$$

$$\Sigma\tau_{1RR} = -\frac{6609.18022}{EI}$$

$$\Sigma\tau_{2RR} = -\frac{5540.15933}{EI}$$

$$\Sigma\tau_{3RR} = -\frac{3758.45692}{EI}$$

$$\tau_{44RR} = -\frac{1112.6082}{EI}$$

5. Solution Matrix

Writing the compatibility equations (eq. 4-7), for all the elements in the structure the following matrix is obtained.

Because of the symmetry of the structure and loads acting on it, the solution matrix can be modified for moments $y_o = y_4$ and $y_1 = y_3$. The modified solution matrix is obtained and is solved for the redundant moments (x_o, y_o, y_1 and y_2).

The final moments are:

$x_o = -\ 3.01089$ kip-feet

$y_o = y_4 = +\ 43.204915$ kip-feet

$y_1 = y_3 = +\ 41.58187$ kip-feet

$y_2 = +\ 39.721744$ kip-feet

And from equations 4-2a, b, c, d,

$x_1 = -\ x_3 = -\ 1.45809$ kip-feet

$x_2 = 0$

$x_4 = +\ 3.010889$ kip-feet

Cross-sectional elements (M_{SR}, M_{ST}, V_{Sz}), at various sections in the structure are calculated by statics and are presented in Table 6-1. Table 6-1 shows these values of the left half portion of the structure and the rest is symmetrical.

SOLUTION MATRIX

$$\begin{bmatrix} 251.327424 & 61.350728 & 101.01444 & 67.01444 & 33.671424 & 11.1720 \\ 61.350728 & 26.793842 & 32.497247 & 18.04449 & 9.022401 & 1.517587 \\ 101.01444 & 32.497247 & 71.632160 & 41.519487 & 18.04448 & 3.035147 \\ 67.01444 & 18.04449 & 41.519487 & 53.587664 & 23.474989 & 3.035147 \\ 33.671424 & 9.022401 & 18.04448 & 23.474989 & 35.543166 & 8.465683 \\ 11.1720 & 1.517587 & 3.035147 & 3.035147 & 8.465683 & 11.784516 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = (-1) \begin{bmatrix} -10639.0386 \\ -3481.69576 \\ -6609.18022 \\ -5540.15933 \\ -3758.45692 \\ -1117.6082 \end{bmatrix}$$

MODIFIED SOLUTION MATRIX

$$\begin{bmatrix} 251.327424 & 72.522728 & 134.685864 & 67.01444 \\ 61.350728 & 28.311429 & 41.519648 & 18.04449 \\ 101.01444 & 35.532421 & 89.67664 & 41.519487 \\ 67.01444 & 21.079663 & 64.994476 & 53.587664 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ y_1 \\ y_2 \end{bmatrix} = (-1) \begin{bmatrix} -10639.0386 \\ -3481.69576 \\ -6609.18022 \\ -5540.15933 \end{bmatrix}$$

TABLE 6-1

CROSS-SECTIONAL ELEMENTS - LOAD CONDITION I

		M_{SR} (Kip-Feet)	M_{ST} (Kip-Feet)	V_{Sz} (Kips)
Span $\overline{01}$	Support 0	+ 43.20492	- 3.01089	+ 5.26604
	5°	+ 17.39396	- 0.36627	+ 5.26604
	10°	- 8.54823	- 0.42104	+ 5.26604
	15°	- 34.42541	- 1.8542	+5.26604 - 4.73496
	20°	- 7.74781	- 3.69424	- 4.73496
	25°	+ 18.98870	- 3.207160	- 4.73496
	Support 1	+ 41.58187	- 1.45809	- 4.73496 + 5.27577
Span $\overline{12}$	5°	+ 18.38151	+ 1.01113	+ 5.27577
	10°	- 8.95752	+ 1.15670	+ 5.27577
	15°	- 36.22844	- 1.02499	+ 5.27577 - 4.72423
	20°	- 10.93086	- 3.23386	- 4.72423
	25°	+ 14.44984	- 3.17494	- 4.72423
	Support 2	+ 39.72174	0	- 4.72423 + 4.72423

B. Load Condition II

Flexibility matrix will be the same as calculated in problem (A). The load functions are now calculated.

1. Load Functions (τ')

Since all the spans are of same length and have the same load acting on them, the load functions (τ's) are same for each span.

Load Functions: From Table 5-3

$$\tau'_{o1TT} = \tau'_{12TT} = \tau'_{23TT} = \tau'_{34TT} = -\frac{1390.9536}{EI}$$

$$\tau'_{o1RR} = \tau'_{12RR} = \tau'_{23RR} = \tau'_{34RR} = -\frac{1687.6944}{EI}$$

$$\tau'_{1oRR} = \tau'_{21RR} = \tau'_{32RR} = \tau'_{43RR} = -\frac{1565.7408}{EI}$$

2. Load Functions (τ's)

Moments (x_1^L, x_2^L, x_3^L and x_4^L) due to loads only:

From Table 5-2

$$x_1^L = x_2^L = x_3^L = x_4^L = -(60)^2 \left[\frac{(1-\cos 30)^2}{\sin 30} + 1 + \frac{\pi}{6} - \cos 30\right]$$

$$= -44.278184$$

$$X_1^L = -44.278184$$

$$X_2^L = X_1^L + x_2^L = -88.556368$$

$$X_3^L = X_2^L + x_3^L = -132.834552$$

$$X_4^L = X_3^L + x_4^L = -177.112736$$

Substituting these values (X_1^L, X_2^L ...) and the values (f, g, τ') in the expressions for load functions (Table 4-2a),

$$\tau_{ooTT} = -\frac{22256.29714}{EI}$$

$$\tau_{ooRR} = -\frac{7278.54651}{EI}$$

$$\Sigma\tau_{1RR} = -\frac{13811.65408}{EI}$$

$$\Sigma\tau_{2RR} = -\frac{11575.33038}{EI}$$

$$\Sigma\tau_{3RR} = -\frac{7848.09420}{EI}$$

$$\tau_{44RR} = -\frac{2318.07744}{EI}$$

3. Solution Matrix

The final solution matrix is obtained in this case as it is done in the previous problem. This solution matrix is modified for the symmetry of the structure and the loads acting on it. ($y_o = y_4$ and $y_1 = y_3$)

The final moments are:

$X_o = -6.03838$ kip-feet

$y_o = y_4 = +90.191478$ kip-feet

$y_1 = y_3 = +86.672578$ kip-feet

$y_2 = +82.95835$ kip-feet

And from equations 4-2a, b, c, d,

$X_1 = -X_3 = -2.92598$ kip-feet

$X_2 = 0$

$X_4 = +6.03838$ kip-feet

Cross-sectional elements (M_{SR}, M_{ST} and V_{Sz}) at the various sections in the structure are calculated by statics and are presented in Table 6-2.

SOLUTION MATRIX

$$
\begin{bmatrix}
251.327424 & 61.350728 & 101.01444 & 67.01444 & 33.671424 & 11.17200 \\
61.350728 & 26.793842 & 32.497247 & 18.04449 & 9.022401 & 1.517487 \\
101.01444 & 32.497247 & 71.632160 & 41.519487 & 18.04448 & 3.035147 \\
67.01444 & 18.04449 & 41.519487 & 53.587664 & 23.474989 & 3.035147 \\
33.671424 & 9.022401 & 18.04448 & 23.474989 & 35.543166 & 8.465683 \\
11.17200 & 1.517587 & 3.035147 & 3.035147 & 8.465683 & 11.784516
\end{bmatrix}
\begin{bmatrix} x_o \\ y_o \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}
= (-1)
\begin{bmatrix} -22256.29714 \\ -\ 7278.54651 \\ -13811.65408 \\ -11575.33038 \\ -\ 7848.0946 \\ -\ 2318.07744 \end{bmatrix}
$$

MODIFIED SOLUTION MATRIX

$$
\begin{bmatrix}
251.327424 & 72.522728 & 134.685864 & 67.01444 \\
61.350728 & 28.311429 & 41.519648 & 18.04449 \\
101.01444 & 35.532421 & 89.67664 & 41.519487 \\
67.01444 & 21.079663 & 64.994476 & 53.587664
\end{bmatrix}
\begin{bmatrix} x_o \\ y_o \\ y_1 \\ y_2 \end{bmatrix}
= (-1)
\begin{bmatrix} -22256.29714 \\ -\ 7278.54651 \\ -13811.65408 \\ -11575.33038 \end{bmatrix}
$$

TABLE 6-2

CROSS-SECTIONAL ELEMENTS - LOAD CONDITION II

	M_{SR} (Kip-Feet)	M_{ST} (Kip-Feet)	V_{Sz} (Kip)
Support 0	+90.19148	-6.03838	+16.26184
5°	+20.98193	-1.40570	+11.02585
10°	-21.02629	-1.59276	+ 5.78987
15°	-35.45904	-4.25175	+ 0.55388
20°	-22.21125	-6.97014	- 4.68210
25°	+18.62100	-7.33112	- 9.9161
Support 1	+86.67258	-2.92598	-15.15408 +16.29149
5°	+17.35972	+1.39502	+11.0555
10°	-24.72332	+0.88839	+ 5.81952
15°	-39.20320	-3.09552	+ 0.58353
20°	-25.97404	-5.14170	- 4.65245
25°	+14.86820	-5.83073	- 9.88845
Support 2	+82.95835	-	-15.12443 +15.12443

C. Load Condition III

In this case, the analysis of the structure is simplified by analysing it for the symmetrical loading and the antisymmetrical loading (Fig. 6-6a, b). The superimposition of these two cases would give the required results for the given problem.

1. Case I - Symmetrical Loading

For this case of symmetrical loading, the structure (Fig. 6-6a) is loaded by a uniformly distributed load of 0.5 kip per feet, acting down and perpendicular to the plane of the structure. The results for this case are obtained from the problem (A) and these are shown in the Table 6-3.

2. Case II - Antisymmetrical Loading

In this case, the spans $\overline{01}$ and $\overline{23}$ are acted upon by a uniformly distributed load of 0.5 kip per foot, acting downward and the spans $\overline{12}$ and $\overline{34}$ are loaded by a uniformly distributed load of 0.5 kip per foot, acting upward (Fig. 6-6b). These loads are acting perpendicular to the plane of the structure.

(a) Load Functions (τ's) : From Table 5-3

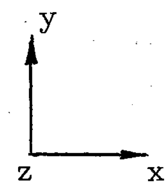
$$\tau'_{o1TT} = -\tau'_{12TT} = \tau'_{23TT} = -\tau'_{34TT} = -\frac{695.4768}{EI}$$

$$\tau'_{1oRR} = -\tau'_{12RR} = \tau'_{23RR} = -\tau'_{34RR} = -\frac{843.8472}{EI}$$

$$\tau'_{1oRR} = -\tau'_{21RR} = \tau'_{32RR} = -\tau'_{43RR} = -\frac{782.8704}{EI}$$

(b) Load Functions (τ's) = From Tables 4-2a, 5-2

$$x_1^L = -x_2^L = x_3^L = -x_4^L = -22.139092$$



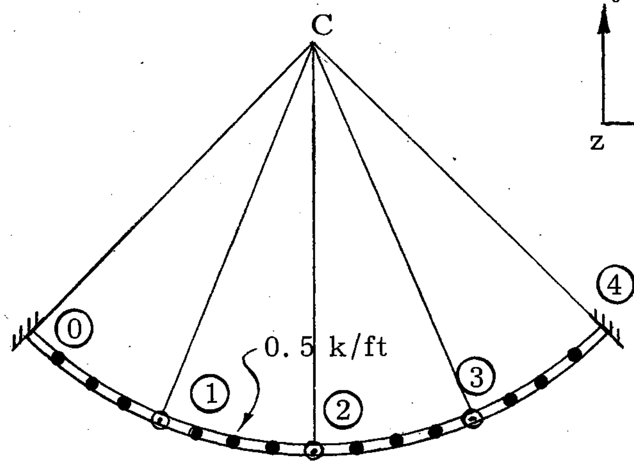


Fig. 6-6a

Symmetrical Loading - Case I

+

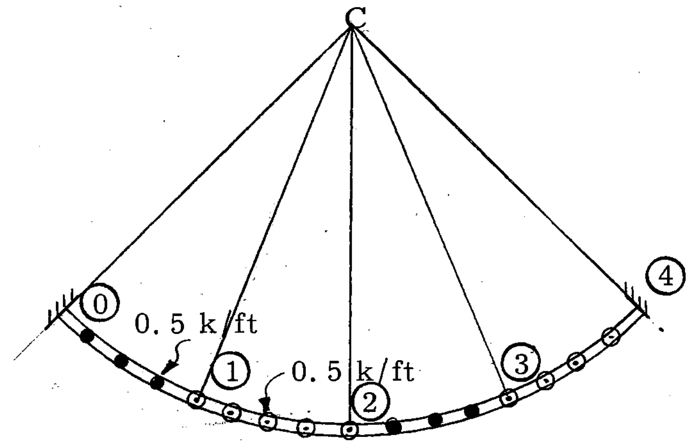


Fig. 6-6b

Antisymmetrical Loading - Case II

=

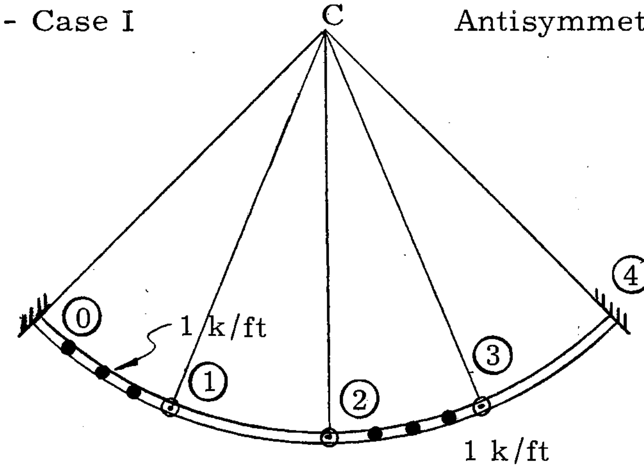


Fig. 6-6c

Superimposition of Case I and Case II

$X_1^L = -22.139092$

$X_2^L = 0$

$X_3^L = -22.139092$

$X_4^L = 0$

Substituting these values (X_1^L, X_3^L . . .) and the values (f, g, τ), in the expressions for the load functions (Table 4-2a),

$$\tau_{ooTT} = -\frac{1391.04023}{EI}$$

$$\tau_{ooRR} = -\frac{701.47549}{EI}$$

$$\Sigma\tau_{1RR} = -\frac{559.09652}{EI}$$

$$\Sigma\tau_{2RR} = -\frac{372.72601}{EI}$$

$$\Sigma\tau_{3RR} = -\frac{186.36840}{EI}$$

$$\tau_{44RR} = +\frac{328.7404}{EI}$$

(c) Solution Matrix

Solution matrix is written for the compatibility equations (eq. 4-7). For an antisymmetrical loading on the symmetrical structure, this solution matrix is modified for the moments $y_o = -y_4$, $y_1 = -y_3$ and $y_2 = 0$. The modified solution matrix is given and is solved for the redundants.

SOLUTION MATRIX

$$\begin{bmatrix} 251.327424 & 61.350728 & 101.01444 & 67.01444 & 33.671424 & 11.17200 \\ 61.350728 & 26.793842 & 32.497247 & 18.04449 & 9.022401 & 1.517587 \\ 101.01444 & 32.497247 & 71.632160 & 41.519487 & 18.04448 & 3.035147 \\ 67.01444 & 18.04449 & 41.519487 & 53.587664 & 23.474989 & 3.035147 \\ 33.671424 & 9.022401 & 18.04448 & 23.474989 & 35.543166 & 8.465683 \\ 11.17200 & 1.517587 & 3.035147 & 3.035147 & 8.465683 & 11.784516 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = (-1) \begin{bmatrix} -1391.04023 \\ -601.47549 \\ -559.09652 \\ -372.72601 \\ -186.36840 \\ +328.7404 \end{bmatrix}$$

MODIFIED SOLUTION MATRIX

$$\begin{bmatrix} 251.327424 & 50.178728 & 67.343016 \\ 61.350728 & 25.276227 & 23.474846 \\ 101.01444 & 29.462073 & 53.587680 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ y_1 \end{bmatrix} = (-1) \begin{bmatrix} -1391.04023 \\ -701.47549 \\ -559.09652 \end{bmatrix}$$

The final moments for an antisymmetrical loading are recorded in Table 6-3. The final moments for the loading condition III are obtained by superimposing the results of the symmetrical case and the results of an antisymmetrical case. These moments are shown in the Table 6-3.

TABLE 6-3

FINAL MOMENTS - LOAD CONDITION III

Moments	Case I Symmetrical Load	Case II Antisymmetrical Load	Final Moments (Kip-Feet)
x_o	- 3.01919	+ 3.82179	+ 0.80260
y_o	+45.09574	+68.53403	+113.62977
y_1	+43.33629	-24.01722	+ 19.31907
y_2	+42.47918	-	+ 42.47918
y_3	+43.33629	+24.01722	+ 67.35351
y_4	+45.09574	-68.53403	- 23.43829
x_1	- 1.46299	- 6.08913	- 7.55212
x_2	-	+ 9.30667	+ 9.30667
x_3	+ 1.46299	- 6.08913	- 5.6261
x_4	+ 3.01919	+ 3.82179	+ 6.84098

TABLE 6-4

CROSS-SECTIONAL ELEMENTS - LOAD CONDITION III

Span $\overline{01}$	Support 0	+113.62977	+0.80253	+13.45414
	5°	+ 29.05185	+6.81015	+ 8.21815
	10°	- 28.38544	+6.65488	+ 2.98216
	15°	- 58.19132	+2.68266	- 2.25383
	20°	- 60.14362	-2.68545	- 7.48982
	25°	- 34.22309	-7.00946	-12.72581
Span $\overline{12}$	Support 1	+ 19.31907	-7.55212	-18.69986 + 1.47038
	5°	+ 23.73422	-5.67209	+ 1.47038
	10°	+ 27.96804	-3.41551	+ 1.47038
	15°	+ 31.98924	-0.79812	+ 1.47038
	20°	+ 35.76697	+2.16116	+ 1.47038
	25°	+ 39.27280	+5.43677	+ 1.47038
Span $\overline{23}$	Support 2	+ 42.47918	+9.30667	+ 1.47038 +17.92226
	5°	- 22.71412	+9.94842	+12.68627
	10°	- 60.37293	+6.13640	+ 7.45028
	15°	- 70.15716	+0.24398	+ 2.21429
	20°	- 51.99675	-5.28990	- 3.02170
	25°	- 6.02543	-8.02655	- 8.25769
Span $\overline{34}$	Support 3	+ 67.35351	-5.62614	-14.23194 + 1.33703
	5°	+ 52.84412	+0.00547	+ 1.33703
	10°	+ 37.93237	+3.96853	+ 1.33703
	15°	+ 22.73227	+6.61821	+ 1.33703
	20°	+ 7.35914	+7.93144	+ 1.33703
	25°	- 8.06992	+7.89967	+ 1.33703
	Support 4	- 23.43829	+6.82512	+ 1.33703

CHAPTER VII

SUMMARY AND CONCLUSIONS

7-1. Summary

The application of the flexibility method for the analysis of a planar continuous circular beam, laterally loaded, is outlined in the previous chapters. The statics of a one-span basic structure is studied. Using Castigliano's Theorem, the angular functions (f, g, τ') for this basic structure are derived.

The analysis of a four-span continuous circular beam, with exterior ends fixed, is considered in this thesis. The angular functions (F, G, and τ) for this structure are derived in terms of angular functions (f, g, τ') of an isolated one-span basic structure. The condition of consistent deformations, expressed in terms of these angular functions and redundant moments, provides the necessary compatibility equations to solve for the redundant moments.

The theory presented in this thesis is illustrated by a numerical example.

7-2. Conclusions

The flexibility method provides an adequate solution for the analysis of a planar continuous circular beam, loaded out of plane. However, it requires considerable amount of computation and accuracy. In problems involving a small number of spans, this method can be advantageously used.

The application of this theory derived is limited to a four-span continuous circular beam loaded laterally. However, the study can be extended to continuous circular beams with any number of spans.

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