ANALYSIS OF
CONTINUOUS RECTANGULAR PLATES ON RIGID SUPPORTS
BY FLEXIBILITY METHODS

By

Hanan Lechter

Bachelor of Science

The University of Kansas

Lawrence, Kansas

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Thesis Approved:

Kenny Hanna
Thesis Adviser

Roger L. Flanders

Dean of the Graduate School

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NOMENCLATURE

g	Cofactor of Angular Carry-Over
h	Thickness of Plate
i, j, k	Letters Designating Points of Plate
l, m	Panels of Continuous Plate
n	Summation Index
o	Origin of Coordinate System
q	Intensity of Distributed Load
r	Carry-Over Factor
t_{ij}	$\Delta x/\Delta y$, $\Delta y/\Delta x$, or 1 Depending upon Position of i and j
w	Vertical Deflection
x, y, z	Rectangular Coordinates, Coordinate Axis
$D = \frac{Eh^3}{12(1-\mu^2)}$	Flexural Rigidity of Plate
E	Young's Modulus of Elasticity
F	Angular Flexibility
G	Angular Carry-Over
L_x	Length of Plate (Parallel to x-axis)
L_y	Length of Plate (Parallel to y-axis)
M	Bending Moment
M^*	Starting Moment
M'	Concentrated Moment

M''	Moment per Unit Length
P	Concentrated Load
α_n	$\frac{n\pi L_y}{2L_x}$
η	Deflection Influence Coefficient
θ	Rotation of Plate
μ	Poisson's Ratio
ξ	Distance of Concentrated Load or Edge Couple from y-axis
τ	Angular Load Function
Δx, Δy	Lengths of Plate Element
Δ', Δ'', Δ'''	Δx or Δy Depending upon Position of i and j
$\supset$	Containing
$\parallel$	Parallel to
$\perp$	Perpendicular to

CHAPTER I

INTRODUCTION

1.1. Historical Review. The problem of continuous rectangular plates has been treated in several different ways. Solutions for rectangular plates that are continuous in a single direction over rigid supports have been presented by Marcus (1)[+], Galerkin (2), and Jensen (3). A distribution procedure for the analysis of plates, continuous in a single direction over rigid or flexible beams with the two other sides simply supported, was developed by Newmark (4). Hawk (5) applied the carry-over moment method to plates continuous over rigid simple supports which lie transverse to simply supported edges, the angular functions being developed to describe the equilibrium state of a simple plate acted upon by lateral loads and bending moments distributed along its edges.

Solutions to the problem of a plate consisting of an infinite number of identical panels supported by point columns and subjected to a uniformly distributed load were treated by Nadai (6) and Galerkin (2). Sutherland, et al., (7) treated the same problem

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[+]Note: Numbers in parentheses, after names, refer to numbered references in Selected Bibliography.

with beams of equal flexural stiffness framed between the point columns. Nielsen (8) made analyses of the same problems by finite differences. Bittner (9), and Maugh and Pan (10) presented algebraic solutions for plates continuous over rigid beams by assuming certain approximate continuity conditions between panels and expressing the rotations on each edge of a panel in terms of the moments on all edges. Engelbreth (11), and Siess and Newmark (12) developed approximate distribution procedures for determining the total moments across any section for plates continuous over rigid beams. A combination of finite differences to get single panel solutions and an extension of the moment distribution procedure to get the boundary deformations was described by Ang and Newmark (13) to handle problems involving plates continuous over supports consisting of columns and beams with flexural and torsional stiffnesses.

1.2. Scope of Study. A method for analyzing thin rectangular plates of constant thickness, continuous in two directions over rigid supports, is described herein.

The flexibility approach is used, the basic structure being a simply supported rectangular plate. The essentials of this method have been discussed by Tuma (17) in a graduate course in plate structures. The angular functions are defined in terms of the influence coefficients for deflection of the simple plate. These coefficients are obtained from a set of comprehensive tables prepared by Tuma, Havner, and French (16). It is assumed that the ordinary theory of flexure of plates is valid and that no horizontal direct stress exists on any vertical cross section of the plate.

A general moment equation is presented in both matrix and carry-over form. The solution of the continuous plate is derived from single panel solutions by accounting for the conditions of continuity and equilibrium with adjoining panels. Since the solutions for the simple plate are based on the calculus of finite differences, these solutions are necessarily approximate and therefore the angular functions obtained are also approximate. A comparison with angular functions obtained by classical methods is presented. Finally, a general moment equation directly in terms of influence coefficients for deflection of a simple plate is given.

CHAPTER II

GENERAL MOMENT EQUATION

2.1. Derivation of Moment Equation. A continuous rectangular plate that is subjected to loads normal to the middle plane of the plate is considered (Figure 2.1). The flexural rigidity in any panel is constant. The supports are rigid.

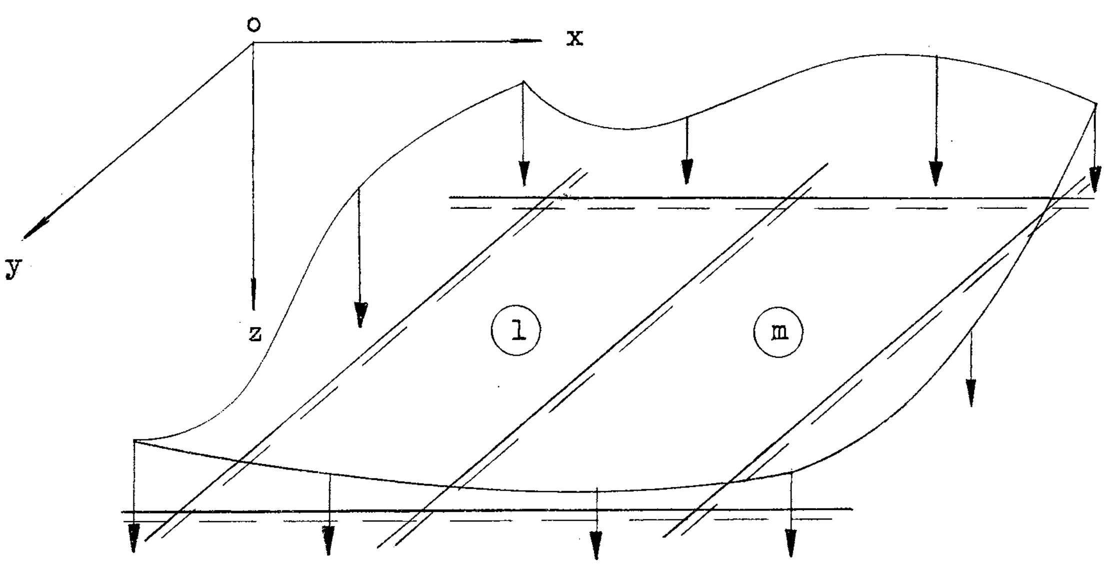

Figure 2.1. General Structure

The equilibrium states of panels l and m (any two adjacent panels) are now combined so that geometrical compatibility is satisfied at the intersection of the two panels. This condition requires that

$$\sum \theta_i = 0 \qquad (2.1)$$

where θ_i denotes rotation at point i, i being a particular point between panels.

Equation (2.1) can be written (see Figure 2.2):

$$(\theta_i)_l + (\theta_i)_m = 0 \qquad (2.2)$$

where $(\theta_i)_l$ and $(\theta_i)_m$ are the rotations at i of panels l and m respectively.

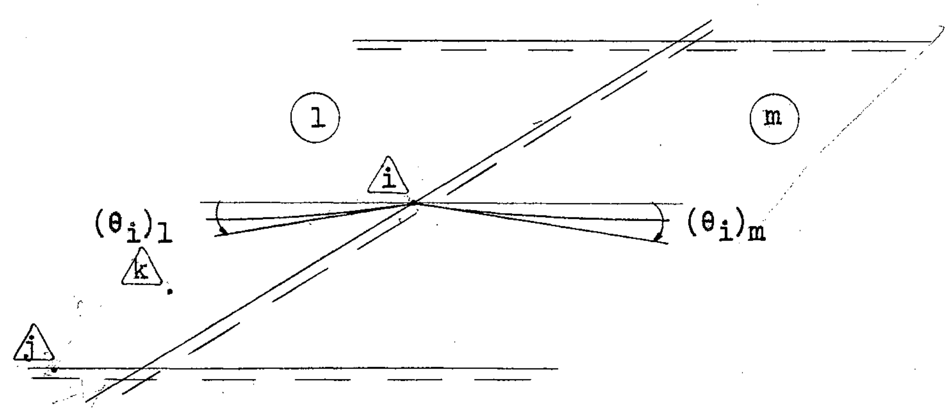

Figure 2.2. Slope Compatibility of Adjacent Panels

The algebraic expressions for the slopes are:

$$(\theta_i)_l = \sum_l \tau_{ik} P_k + (F_i)_l M_i + \sum_l G_{ij} M_j \qquad (2.3)$$

$$(\theta_i)_m = \sum_m \tau_{ik} P_k + (F_i)_m M_i + \sum_m G_{ij} M_j \qquad (2.4)$$

where:

j is any point, other than i, on the boundary of panels.

k is a typical interior point of panels.

The angular load function τ_{ik} is the edge slope at i due to a unit load at k, considering the plate simply supported.

The angular flexibility $(F_i)_{l,\,m}$ is the edge slope at i, of plate l or m respectively, due to a unit moment at i, considering the plate simply supported.

The angular carry-over G_{ij} is the edge slope at i due to a unit moment at j, considering the plate simply supported.

P_k is any load applied at k.

M_i is the bending moment at i.

M_j is the bending moment at j.

An equation of the form of equations (2.3) and (2.4) was presented by Tuma (14).

Using equations (2.3) and (2.4) in equation (2.2),

$$\sum_l \tau_{ik} P_k + (F_i)_l M_i + \sum_l G_{ij} M_j + \sum_m \tau_{ik} P_k + (F_i)_m M_i + \sum_m G_{ij} M_j = 0 \qquad (2.5)$$

Rearranging terms in equation (2.5), the general moment equation becomes:

$$\sum_{1,m} P_k \tau_{ik} + M_i \sum_{1,m} F_i + \sum_{1,m} M_j G_{ij} = 0 \qquad (2.6)$$

where 1,m beneath the summation sign indicates a summation over all boundary points of both panels 1 and m.

2.2. Moment Equation in Carry-Over Form. The carry-over method was originated by Tuma (15) for the analysis of continuous beams. This method can be used as a very valuable tool for the solution of the general moment equation of continuous plates derived in article 2.1.

From equation (2.6), the following identity is obtained:

$$M_i \sum_{1,m} F_i = - \sum_{1,m} G_{ij} M_j - \sum_{1,m} \tau_{ik} P_k \qquad (2.7)$$

which leads to the relation

$$M_i = - \frac{\sum_{1,m} G_{ij} M_j}{\sum_{1,m} F_i} - \frac{\sum_{1,m} \tau_{ik} P_k}{\sum_{1,m} F_i} \qquad (2.8)$$

Substituting:

$$- \frac{G_{ij}}{\sum_{1,m} F_i} = r_{ij} \qquad (2.9)$$

$$- \frac{\sum_{1,m} \tau_{ik} P_k}{\sum_{1,m} F_i} = M_i^* \qquad (2.10)$$

in equation (2.8), the moment equation in its carry-over form is obtained:

$$M_i = \sum_{l,m} r_{ij} M_j + M_i^* \tag{2.11}$$

The constants involved in equation (2.11) can be interpreted in the carry-over procedure as follows:

The carry-over factor r_{ij} is the moment at i due to a unit moment applied at j if continuity with adjacent panels is developed only at point i.

The starting moment M_i^* is the moment at i due to loads in panels l and m if continuity is developed only at point i.

2.3. Matrix Formulation of Moment Equation. When the number of panels in the continuous plate becomes large the applicability of the carry-over moment method decreases because of the complexity of the carry-over process. A matrix formulation and computer solution of the problem becomes desirable in such cases.

The general moment equation [Eq. (2.6)] can be expressed in matrix form as follows:

$$\begin{bmatrix} \tau_1 k \\ \tau_2 k \\ \tau_3 k \\ \cdot \\ \cdot \\ \cdot \\ \tau_p k \end{bmatrix} + \begin{bmatrix} \sum F_1 & G_{12} & G_{13} & \cdot & \cdot & \cdot & G_{1p} \\ G_{21} & \sum F_2 & G_{23} & \cdot & \cdot & \cdot & G_{2p} \\ G_{31} & G_{32} & \sum F_3 & \cdot & \cdot & \cdot & G_{3p} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ G_{p_1} & G_{p_2} & G_{p_3} & \cdot & \cdot & \cdot & \sum F_p \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \cdot \\ \cdot \\ \cdot \\ M_p \end{bmatrix} = 0 \tag{2.12}$$

where the loading has been reduced to a unit concentrated load at a single point k in order to obtain an influence coefficient matrix, and the subscript p corresponds to the total number of boundary points.

The solution of equation (2.12) can be obtained by using the inverse matrix. The solution procedure is as follows.
Using an abbreviated notation, equation (2.12) becomes

$$\left[\tau\right] + \left[G\right]\left[M\right] = 0$$

from which

$$\left[G\right]\left[M\right] = -\left[\tau\right].$$

Premultiplying both sides by the inverse of matrix G,

$$\left[G\right]^{-1}\left[G\right]\left[M\right] = -\left[G\right]^{-1}\left[\tau\right]$$

or, since $\left[G\right]^{-1}\left[G\right] = \left[I\right] =$ unit matrix of order p,

$$\left[M\right] = -\left[G\right]^{-1}\left[\tau\right].$$

The inverse of $\left[G\right]$ is the adjoint of $\left[G\right]$ divided by the determinant of $\left[G\right]$.

Then,
$$\left[M\right] = -\frac{\text{Adj } G}{|G|}\left[\tau\right]$$

where: Adj G is the adjoint of the square matrix G (the matrix obtained by replacing each element in $\left[G\right]$ by its cofactor, then interchanging rows and columns).

$|G|$ is the determinant of the square matrix G.

Denoting

$$g_{ij} = \text{cofactor } G_{ij} = (-1)^{i+j} (\text{minor of } G_{ij}),$$

the solution of the moment equation in its expanded form becomes:

$$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \cdot \\ \cdot \\ \cdot \\ M_p \end{bmatrix} = -\frac{1}{|G|} \begin{bmatrix} g_{11} & g_{21} & g_{31} & \cdot & \cdot & \cdot & g_{p1} \\ g_{12} & g_{22} & g_{32} & \cdot & \cdot & \cdot & g_{p2} \\ g_{13} & g_{23} & g_{33} & \cdot & \cdot & \cdot & g_{p3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ g_{1p} & g_{2p} & g_{3p} & & & & g_{pp} \end{bmatrix} \begin{bmatrix} \tau_1 k \\ \tau_2 k \\ \tau_3 k \\ \cdot \\ \cdot \\ \cdot \\ \tau_{pk} \end{bmatrix} \quad (2.13a)$$

or,

$$M_i = -\frac{1}{|G|} (g_{1i}\, \tau_1 k + g_{2i}\, \tau_2 k + g_{3i}\, \tau_3 k + \ . \ . \ . + g_{pi} \tau_{pk})$$

$$(i = 1, 2, . . ., p) \quad (2.13b)$$

where: g_{ii} = cofactor $\sum F_i$.

The minor of G_{ij} is the determinant of the matrix that remains when the row and column containing the element G_{ij} in the square matrix G are deleted.

CHAPTER III

ANGULAR FUNCTIONS

3.1. Angular Load Function.

(a) Derivation of Formula. Consider a simply supported rectangular plate to be acted upon by a load $P = 1$ at point k (Figure 3.1). From the definition given in Chapter II, the rotation (slope of the deflection curve) at i due to $P_k = 1$ is the angular load function τ_{ik}.

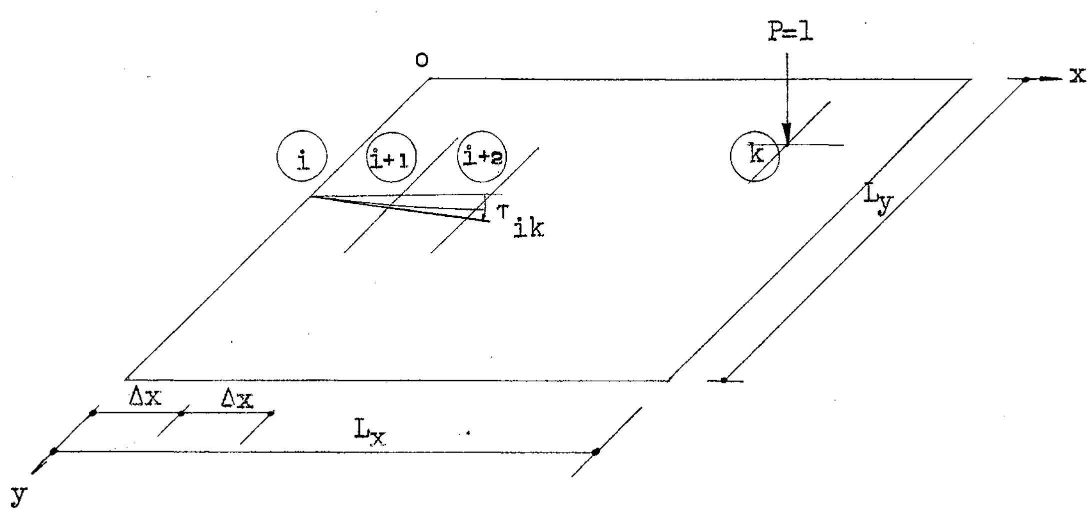

Figure 3.1. Angular Load Function in Simple Plate

If the plate is divided into an arbitrary number of equally sized rectangular elements with sides Δx and Δy in the directions x and y, respectively, the slope of the deflection curve can be approximated as:

$$\theta_i = \frac{w_{i+1}}{\Delta x} \tag{3.1}$$

where w_{i+1} is the displacement of point $i + 1$.

From equation (7-3), reference 19, the displacement of point $i + 1$ due to a unit load at k is:

$$w_{(i+1)k} = \frac{1}{D} \Delta x \; \Delta y \; \eta_{(i+1)k} \tag{3.2}$$

where: D is the flexural rigidity of the plate.

$\eta_{(i+1)k}$ is the influence coefficient for displacement at $i + 1$ due to a unit load at k.

Using equation (3.2) in equation (3.1), the angular load function is found to be:

$$\tau_{ik} = \frac{\Delta y}{D} \eta_{(i+1)k} \tag{3.3a}$$

If i is on an edge parallel to the x direction, the expression for the angular load function becomes:

$$\tau_{ik} = \frac{\Delta x}{D} \eta_{(i+1)k} \tag{3.3b}$$

(b) Alternate Derivation. Equations (3.3) can be derived using the reciprocal theorem of Maxwell. This theorem (generalized by Betti and Rayleigh) states that: "if an elastic body is subjected to two systems of body and surface forces, then the work that would be done by the first system of external forces in acting through the displacements due to the second system of forces is equal to the work that would be done by the second system of forces in acting through the displacements due to the first system of forces" (page 169, reference 20). Thus, for the two load systems given in Figure 3.2:

$$M_i \quad (\theta_i)_{P_k} \quad = \quad P_k \quad (w_k)_{M_i} \tag{3.4}$$

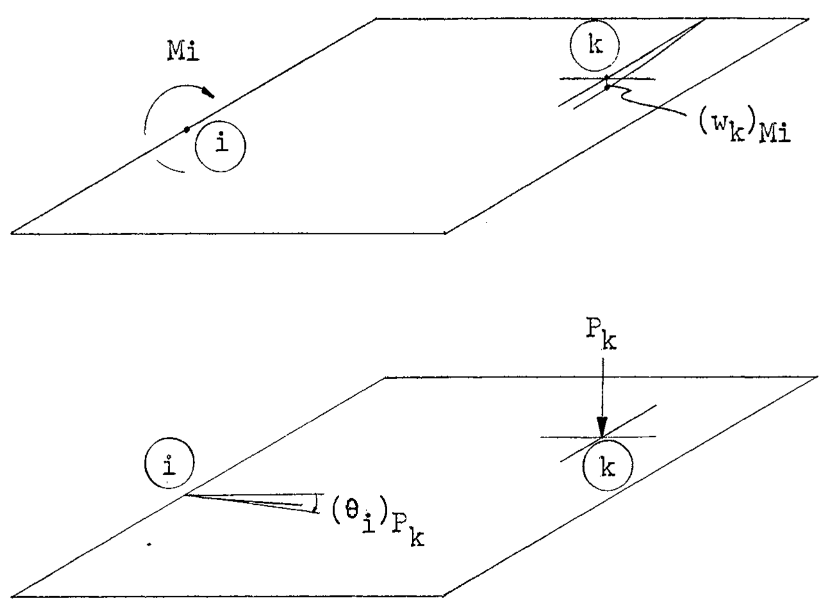

Figure 3.2. Load Systems in Simple Plate

When $P_k = 1$, $(\theta_i)_{P_k} = \tau_{ik}$, and equation (3.4) becomes,

$$M_i \ (\tau_{ik}) = (1)\ (w_k)_{M_i} \tag{3.5}$$

from which

$$\tau_{ik} = \frac{(w_k)_{M_i}}{M_i} \tag{3.6}$$

To solve equation (3.6) for the angular load function, it is necessary to find the deflection at k due to a moment at i. Applying a unit moment at i (a couple with forces $\frac{1}{\Delta x}$ at i and i + 1 in Figure 3.3), the deflection at k is:

$$w_{k(i+1)} = \frac{\frac{1}{\Delta x}}{D} \ \Delta x \ \Delta y \ \eta_{k(i+1)} \tag{3.7}$$

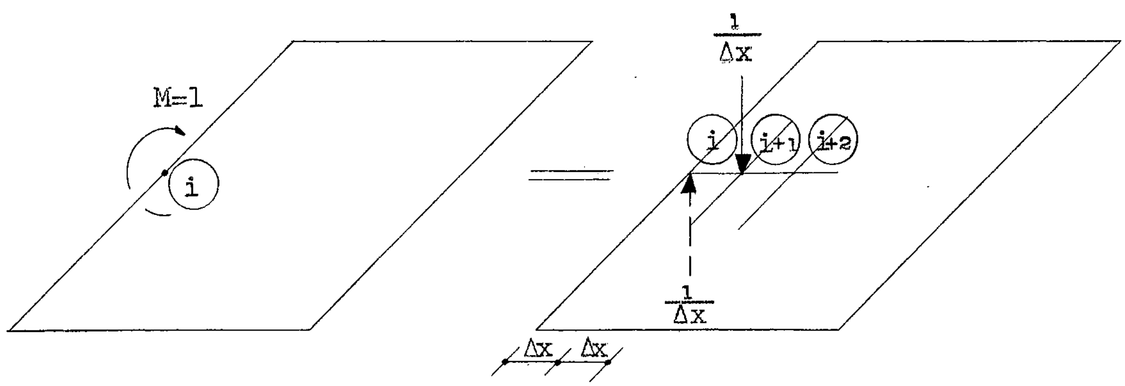

Figure 3.3. Moment Equivalence in Simple Plate

Using now equation (3.7) in equation (3.6),

$$\tau_{ik} = \frac{\frac{\frac{1}{\Delta x}}{D} \Delta x \ \Delta y \ \eta_{k(i+1)}}{(1)}$$

or,

$$\tau_{ik} = \frac{\Delta y}{D} \eta_{k(i+1)} \tag{3.8}$$

From the reciprocal relation between influence coefficients in the plate,

$$\eta_{k(i+1)} = \eta_{(i+1)k}$$

Thus, equation (3.8) becomes identical to equation (3.3a).

(c) Non-Nodal Load. Equations (3.4a) and (3.4b) can be evaluated numerically for a particular problem with the aid of the tables in reference 16 if P_k is a nodal load. If P_k is a non-nodal load, the tables cannot be used directly and a formula for the angular load function in terms of the influence coefficients for deflection due to equivalent nodal loads must be derived.

In Figure 3.4, a non-nodal load is resolved into four adjacent nodal loads.

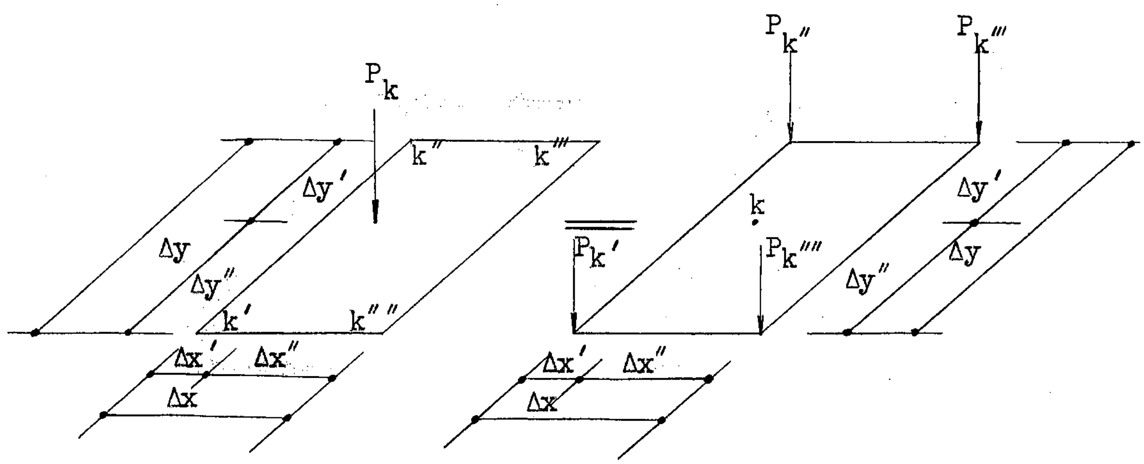

Figure 3.4. Equivalents of Non-Nodal Load

The formulas for these nodal loads are:

$$P_{k'} = P_k \frac{\Delta x'' \ \Delta y'}{\Delta x \ \Delta y} \tag{3.9a}$$

$$P_{k''} = P_k \frac{\Delta x'' \ \Delta y''}{\Delta x \ \Delta y} \tag{3.9b}$$

$$P_{k'''} = P_k \frac{\Delta x' \ \Delta y''}{\Delta x \ \Delta y} \tag{3.9c}$$

$$P_{k''''} = P_k \frac{\Delta x' \ \Delta y'}{\Delta x \ \Delta y} \tag{3.9d}$$

$$P_{k'} + P_{k''} + P_{k'''} + P_{k''''} = P_k \tag{3.9e}$$

The displacement of a point $i + 1$ due to P_k will be equal to the sum of the displacements developed at $i + 1$ by $P_{k'}$, $P_{k''}$, $P_{k'''}$, and $P_{k''''}$, thus:

$$w_{(i+1)k} = w_{(i+1)k'} + w_{(i+1)k''} + w_{(i+1)k'''} + w_{(i+1)k''''} \tag{3.10a}$$

and, from equations (3.9) and the equation for deflections,

$$w_{(i+1)k} = \frac{P_k}{\Delta x \, \Delta y \, D} \Delta x \, \Delta y \left[\Delta x'' \, \Delta y' \, \eta_{(i+1)k'} + \Delta x'' \, \Delta y'' \, \eta_{(i+1)k''} + \Delta x' \, \Delta y'' \, \eta_{(i+1)k'''} + \Delta x' \, \Delta y' \, \eta_{(i+1)k''''} \right] \tag{3.10b}$$

Combining equations (3.10) and (3.1) (with $P_k = 1$), the angular load function becomes:

$$\tau_{ik} = \frac{1}{D \, \Delta x} \left[\Delta x' (\Delta y' \, \eta_{(i+1)k''''} + \Delta y'' \, \eta_{(i+1)k'''}) + \Delta x'' \, (\Delta y' \, \eta_{(i+1)k'} + \Delta y'' \, \eta_{(i+1)k''}) \right] \tag{3.11a}$$

The above equation is valid when i is on an edge parallel to the y direction. When i is on an edge parallel to the x direction, the angular load function is:

$$\tau_{ik} = \frac{1}{D \, \Delta y} \left[\Delta x' \, (\Delta y' \, \eta_{(i+1)k''''} + \Delta y'' \, \eta_{(i+1)k'''}) + \Delta x'' \, (\Delta y' \, \eta_{(i+1)k'} + \Delta y'' \, \eta_{(i+1)k''}) \right] \tag{3.11b}$$

3.2. Angular Flexibility. Consider a simply supported rectangular plate to be acted upon by a unit moment at point i (Figure 3.5). From the definition given in Chapter II, the rotation at i due to $M_i = 1$ is the angular flexibility F_i.

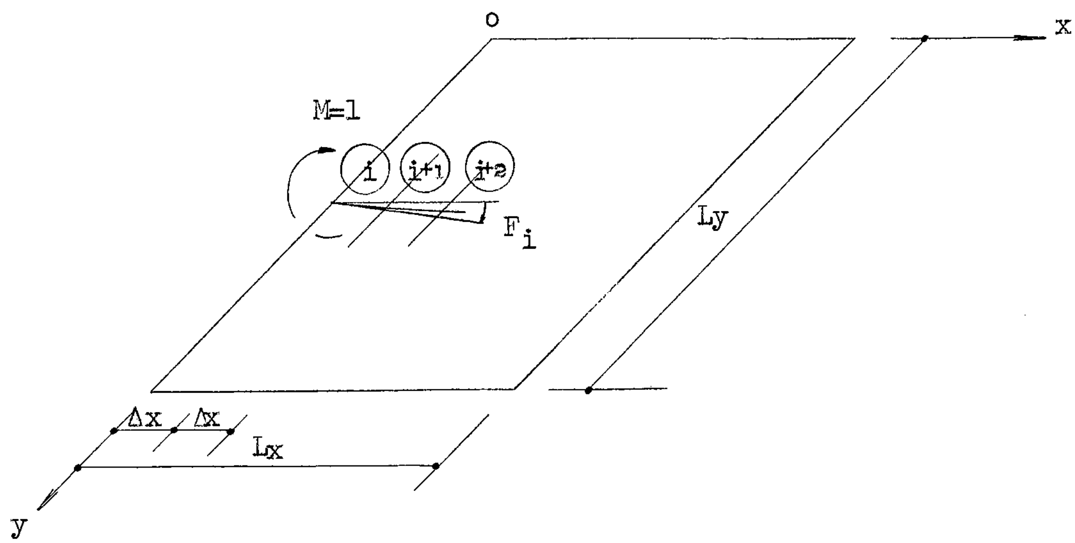

Figure 3.5. Angular Flexibility in Simple Plate

Using the approach illustrated in Figure 3.2,

$$(w_{i+1})_{M_i=1} = W_{(i+1)(i+1)} = \frac{\frac{1}{\Delta x}}{D} \Delta x \, \Delta y \, \eta_{(i+1)(i+1)} \tag{3.12}$$

The angular flexibility can be approximated as:

$$F_i = \frac{w_{(i+1)(i+1)}}{\Delta x}$$

that is, from the value for $w_{(i+1)(i+1)}$ given in equation (3.12),

$$F_i = \frac{\frac{\frac{1}{\Delta x}}{D} \Delta x \, \Delta y \, \eta_{(i+1)(i+1)}}{\Delta x}$$

or,

$$F_i = \frac{\Delta y}{\Delta x} \frac{1}{D} \eta_{(i+1)(i+1)} \qquad (3.13a)$$

If i is on an edge parallel to the x direction,

$$F_i = \frac{\Delta x}{\Delta y} \frac{1}{D} \eta_{(i+1)(i+1)} \qquad (3.13b)$$

3.3. Angular Carry-Over. Consider a simply supported rectangular plate to be acted upon by a unit moment at j (Figure 3.6). From the definition given in Chapter II, the rotation at i due to $M_j = 1$ is the angular carry-over G_{ij}.

Using the approach illustrated in Figure 3.2 and the reciprocal relation between coefficients,

$$(w_{i+1})_{M_j = 1} = w_{(i+1)(j+1)} = w_{(j+1)(i+1)} =$$

$$\frac{\frac{1}{\Delta x}}{D} \Delta x \, \Delta y \, \eta_{(j+1)(i+1)} \qquad (3.14)$$

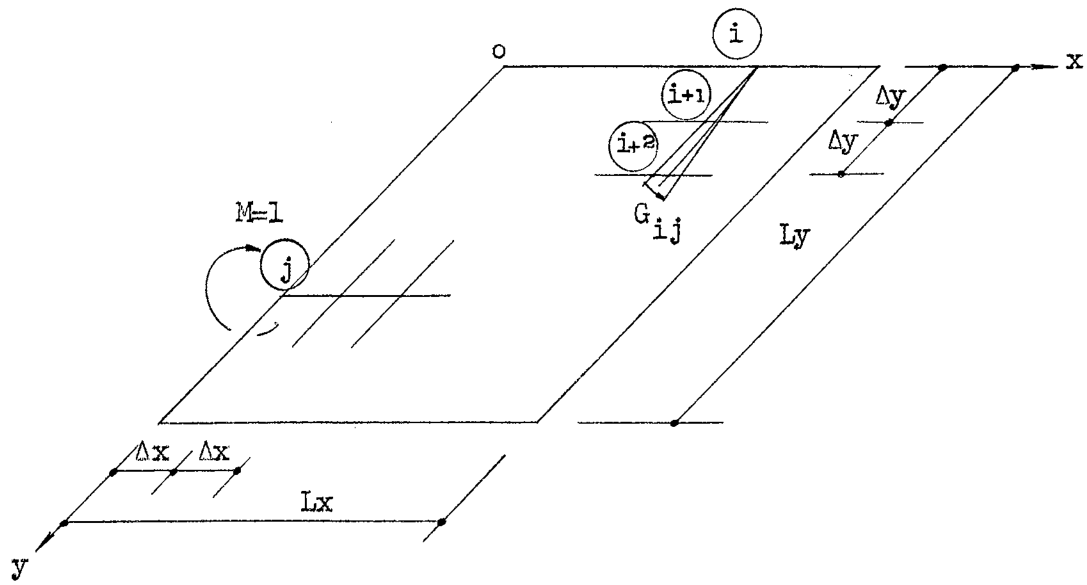

Figure 3.6. Angular Carry-Over in Simple Plate

The angular carry-over can be approximated as:

$$G_{ij} = \frac{W_{(j+1)(i+1)}}{\Delta y}$$

thus,

$$G_{ij} = \frac{1}{D} \; \eta_{(j+1)(i+1)} \tag{3.15a}$$

If j and i are on parallel edges,

(normal to the x direction)

$$G_{ij} = \frac{\Delta y}{\Delta x} \; \frac{1}{D} \; \eta_{(j+1)(i+1)} \tag{3.15b}$$

(normal to the y direction)

$$G_{ij} = \frac{\Delta x}{\Delta y} \; \frac{1}{D} \; \eta_{(j+1)(i+1)} \tag{3.15c}$$

3.4. Modification of Formulae for τ, F, and G. In the preceding articles, the derivation of the formulas for the angular functions was made by assuming that the deflection surface is composed of a series of string polygons. If now the deflection surface is considered to be composed of a series of second degree parabolas passing through the vertices of the assumed string polygons (Figure 3.7), another approximate formula for the slope of the deflection curve will be obtained and a new set of formulas for the angular functions can be derived.

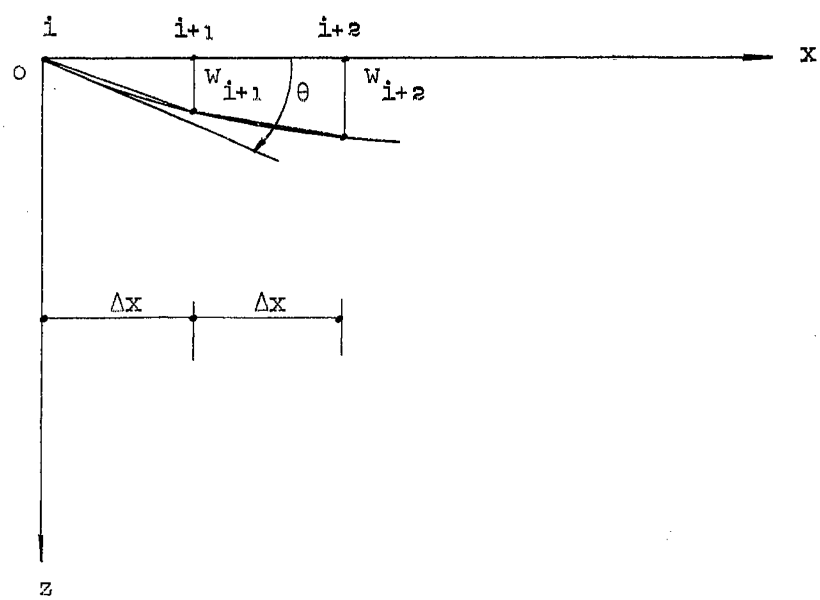

Figure 3.7. Deflection Curve, Second Degree Parabola

For the parabolic segment shown in Figure 3.7, the slope at i can be obtained using forward difference equations and considering points i, i + 1, and i + 2. The general equation yields,

$$\left(\frac{dz}{dx}\right)_i = \frac{w_{i+1} - w_i}{\Delta x} - \frac{w_{i+2} - 2w_{i+1} + w_i}{2\Delta x} = \frac{-3w_i + 4w_{i+1} - w_{i+2}}{2\Delta x}$$

and, since w_i is equal to zero,

$$\left(\frac{dz}{dx}\right)_i = \frac{4w_{i+1} - w_{i+2}}{2\Delta x} \qquad (3.16)$$

Using equation (3.16), and the equations for deflection found in articles 3.1, 3.2, and 3.3 for the angular load function, the angular flexibility, and the angular carry-over, respectively, the following equations for the angular functions are obtained:

$$(\text{edge} \,)\, i \,||\, x \text{ axis}) \quad \tau_{ik} = \frac{\Delta x}{D} \frac{4\eta_{(i+1)k} - \eta_{(i+2)k}}{2} \qquad (3.17a)$$

$$(\text{edge} \,)\, i \,||\, y \text{ axis}) \quad \tau_{ik} = \frac{\Delta y}{D} \frac{4\eta_{(i+1)k} - \eta_{(i+2)k}}{2} \qquad (3.17b)$$

$$(\text{edge} \,)\, i \,||\, x \text{ axis}) \quad F_i = \frac{\Delta x}{\Delta y} \frac{1}{D} \frac{4\eta_{(i+1)(i+1)} - \eta_{(i+2)(i+1)}}{2} \qquad (3.18a)$$

(edge $\supset$ i $\|$ y axis) $$F_i = \frac{\Delta y}{\Delta x}\,\frac{1}{D}\,\frac{4\eta_{(i+1)(i+1)} - \eta_{(i+2)(i+1)}}{2} \tag{3.18b}$$

(edge $\supset$ i,j $\|$ x axis) $$G_{ij} = \frac{\Delta x}{\Delta y}\,\frac{1}{D}\,\frac{4\eta_{(j+1)(i+1)} - \eta_{(j+1)(i+2)}}{2} \tag{3.19a}$$

(edge $\supset$ i,j $\|$ y axis) $$G_{ij} = \frac{\Delta y}{\Delta x}\,\frac{1}{D}\,\frac{4\eta_{(j+1)(i+1)} - \eta_{(j+1)(i+2)}}{2} \tag{3.19b}$$

(edge $\supset$ i $\perp$ edge $\supset$ j) $$G_{ij} = \frac{1}{D}\,\frac{4\eta_{(j+1)(i+1)} - \eta_{(j+1)(i+2)}}{2} \tag{3.19c}$$

where the following symbolism is used:

$\supset$ . . . containing.

$\|$ . . . parallel to.

$\perp$ . . . perpendicular to.

CHAPTER IV

COMPARISON OF RESULTS

4.1. <u>General</u>. The algebraic procedure given in reference 19 yields exact solutions of the single panel finite difference equations, and the numerical procedure used to solve the continuous plate moment equations can be carried out to a desired degree of accuracy (either by matrix or iteration methods). Thus the values for the angular load function, the angular flexibility, and the angular carry-over, and consequently the solution of the whole problem, depend upon two factors: the grid chosen to obtain the influence coefficients for deflection in a simple plate, and the resemblance of the deflection curve chosen to the true one. A comparison of the results obtained by applying the formulae in Chapter III with those obtained by classical methods thus becomes necessary and will be the object of this chapter.

4.2. <u>Classical Solutions for Angular Functions</u>. Expressions for deflections of a simply supported rectangular plate under transverse loads or edge moments can be found in Timoshenko and Woinowsky-Krieger (18). From these classical expressions for deflections, the slopes (angular functions) of the simple plates are computed.

Some changes in notation have been introduced in the classical expressions in order to be consistent with the general notation adopted in this thesis.

(a) Simple Plate Under a Concentrated Load. Consider a concentrated load P acting on a simple plate at $y = \frac{L_y}{2}$, $x = \xi$ (Figure 4.1).

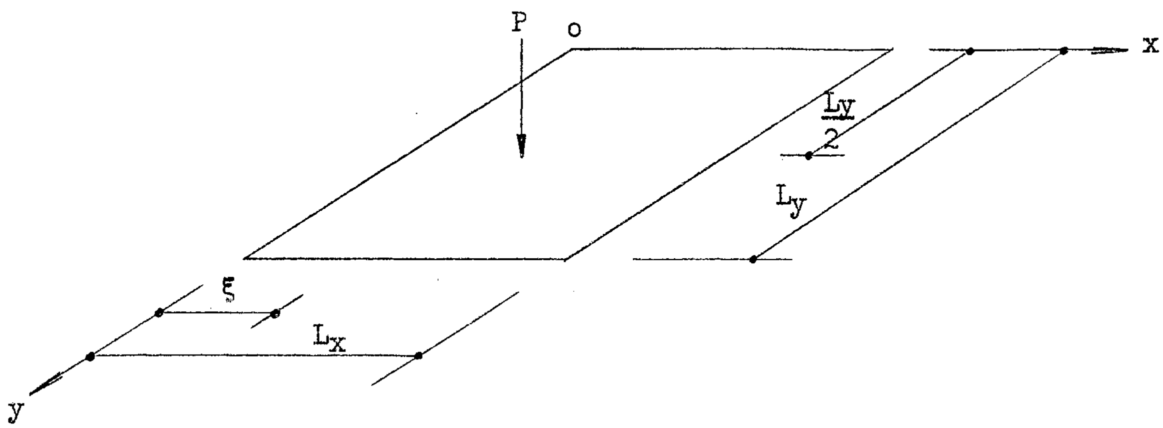

Figure 4.1. Simple Plate Under a Concentrated Load

From the Navier solution, the deflection is given as (reference 18):

$$(w)_{y = \frac{L_y}{2}} = \frac{PL_x^2}{2\pi^3 D} \sum_{n=1}^{\infty} \left(\tanh \alpha_n - \frac{\alpha_n}{\cosh^2 \alpha_n}\right) \frac{\sin \frac{n\pi\xi}{L_x} \sin \frac{n\pi x}{L_x}}{n^3} \tag{4.1}$$

where: $\alpha_n = \frac{n\pi L_y}{2L_x}$

From equation (4.1),

$$\left(\frac{\partial w}{\partial x}\right)_{y = \frac{L_y}{2}} = \frac{PL_x}{2\pi^2 D} \sum_{n=1}^{\infty} \left(\tanh\alpha_n - \frac{\alpha_n}{\cosh^2\alpha_n}\right) \frac{\sin\frac{n\pi\xi}{L_x} \cos\frac{n\pi x}{L_x}}{n^2} \tag{4.2}$$

(b) Simple Plate Under a Uniform Load. Consider a uniform load q acting over the entire surface of a simple plate (Figure 4.2).

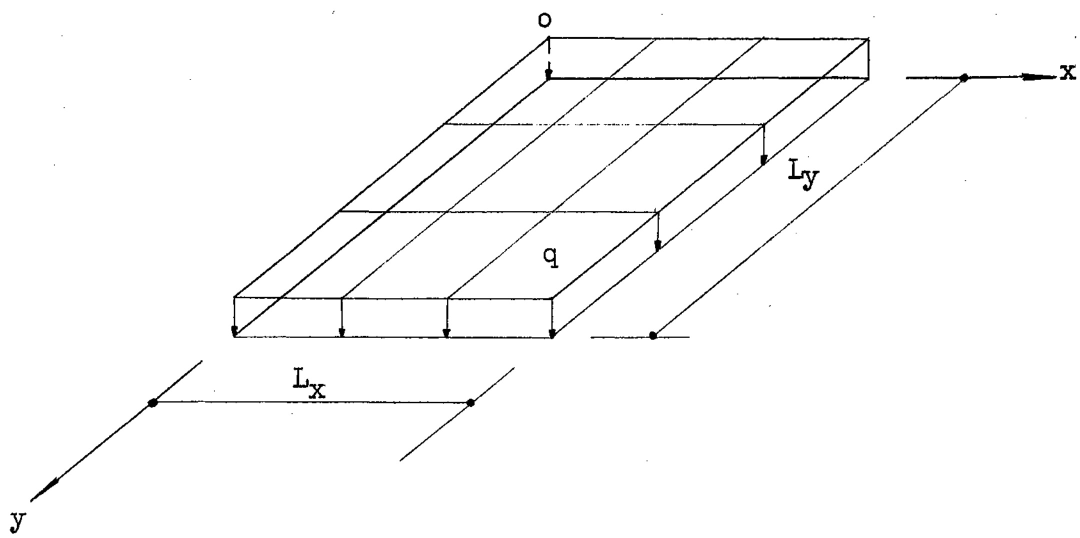

Figure 4.2. Simple Plate Under a Uniform Load

From the Levy solution, the deflection is given as (reference 18):

$$w = \frac{4\, q\, L_x^4}{\pi^5 D} \sum_{n=1,3,5,\ldots}^{\infty} \left[1 - \frac{\alpha_n \tanh \alpha_n + 2}{2 \cosh \alpha_n} \cosh \frac{2\, \alpha_n (y - \frac{L_y}{2})}{L_y} + \frac{\alpha_n (y - \frac{L_y}{2})}{L_y \cosh \alpha_n} \sinh \frac{2\, \alpha_n (y - \frac{L_y}{2})}{L_y}\right] \frac{\sin \frac{n\pi x}{L_x}}{n^5} \tag{4.3}$$

where: $\alpha_n = \frac{n\pi L_y}{2L_x}$

From equation (4.3),

$$\left(\frac{\partial w}{\partial x}\right)_{y = \frac{L_y}{2}} = \frac{4\, q\, L_x^3}{\pi^4 D} \sum_{n=1,3,5,\ldots}^{\infty} \left(1 - \frac{\alpha_n \tanh \alpha_n + 2}{2 \cosh \alpha_n}\right) \frac{\cos \frac{n\pi x}{L_x}}{n^4} \tag{4.4}$$

(c) Simple Plate Under Moments Distributed Along the Edges. Consider distributed moments, represented by a trigonometric series

$$M_y = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi x}{L_x} \quad ,$$

acting along the edge $y = L_y$ of a simple plate (Figure 4.3).

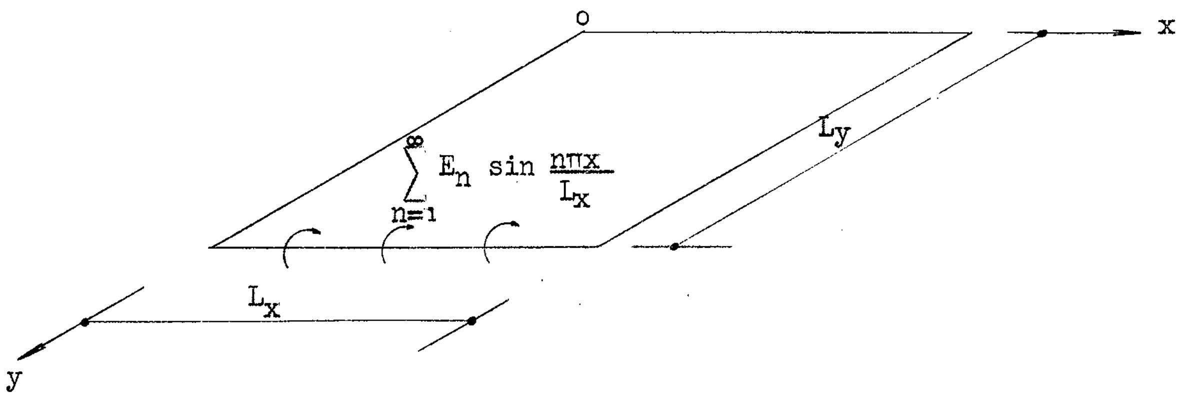

Figure 4.3. Bending Moments Along Edge of Simple Plate

The deflection is given as (reference 18):

$$w = \frac{L_x^2}{4\pi^2 D} \sum_{n=1}^{\infty} \Bigg\{ \frac{1}{\cosh \alpha_n} \left[\alpha_n \tanh \alpha_n \cosh \frac{n\pi(y - \frac{L_y}{2})}{L_x} + (-) \frac{n\pi(y - \frac{L_y}{2})}{L_x} \sinh \frac{n\pi(y - \frac{L_y}{2})}{L_x} \right] + \frac{1}{\sinh \alpha_n} \left[\alpha_n \coth \alpha_n \sinh \frac{n\pi(y - \frac{L_y}{2})}{L_x} + (-) \frac{n\pi(y - \frac{L_y}{2})}{L_x} \cosh \frac{n\pi(y - \frac{L_y}{2})}{L_x} \right] \Bigg\} \frac{E_n \sin \frac{n\pi x}{L_x}}{n^2} \tag{4.5}$$

where: $\alpha_n = \frac{n\pi L_y}{2 L_x}$

From equation (4.5) the three following equations for slopes are derived:

$$\left(\frac{\partial w}{\partial x}\right)_{y=\frac{L_y}{2}} = \frac{L_x}{4\pi D}\sum_{n=1}^{\infty}\frac{\alpha_n \tanh \alpha_n}{n \cosh \alpha_n} E_n \cos \frac{n\pi x}{L_x} \qquad (4.6)$$

$$\left(\frac{\partial w}{\partial y}\right)_{y=0} = \frac{L_x}{4\pi D}\sum_{n=1}^{\infty}\Big[\alpha_n (\coth^2\alpha_n - \tanh^2\alpha_n) + (\tanh\alpha_n - \coth\alpha_n)\Big]\frac{E_n \sin\frac{n\pi x}{L_x}}{n} \qquad (4.7)$$

$$\left(\frac{\partial w}{\partial y}\right)_{y=L_y} = \frac{L_x}{2\pi D}\sum_{n=1}^{\infty}\left(\frac{2\alpha_n}{\sinh^2 2\alpha_n} - \coth 2\alpha_n\right)\frac{E_n \sin\frac{n\pi x}{L_x}}{n} \qquad (4.8)$$

In the case of concentrated moments (the group of concentrated moments must be symmetrical with respect to the line $x = \frac{L_x}{2}$),

$$\sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{L_x} = \frac{2}{L_x} M' \sum_{n=1,3,5\ldots}^{\infty} \sin\frac{n\pi\xi}{L_x} \sin\frac{n\pi x}{L_x} \qquad (4.9)$$

where M' is the magnitude of the moment at a distance $x = \xi$.

In the case of uniform distribution of the bending moments,

$$\sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{L_x} = \frac{4}{\pi} M'' \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{L_x} \qquad (4.10)$$

where M'' is the intensity of the moment per unit length.

Negative signs will be obtained for some angular functions when classical solutions are used because of the convention adopted (see Figure 4.4).

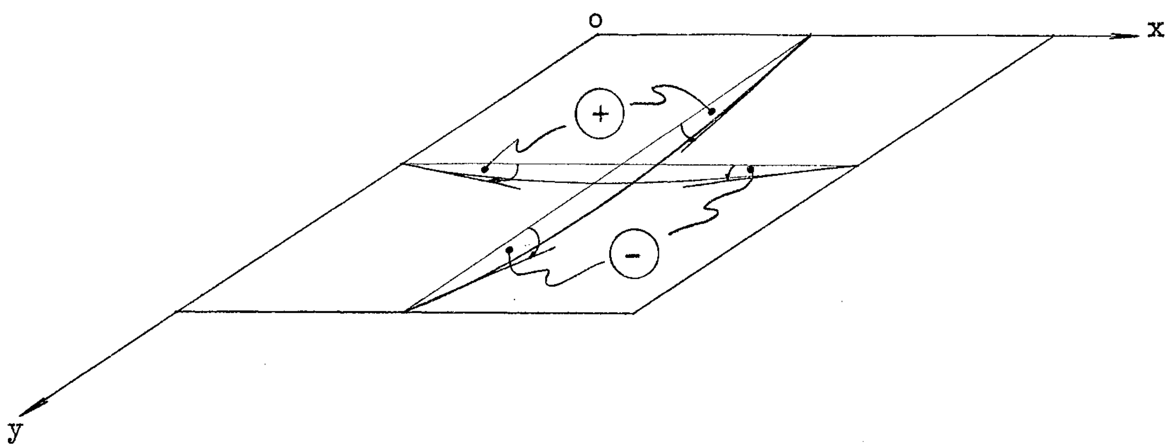

Figure 4.4. Sign Convention for Rotations of Plates. Classical Solutions

4.3. Numerical Angular Load Functions by Classical Methods. From equation (4.2), angular load functions due to a concentrated load at different points on the plate are calculated. From equation (4.4), angular load functions due to a uniform load over the entire surface of the plate are calculated. Results are tabulated for three different ratios of Ly/Lx (Tables 1 and 2).

TABLE 1

ANGULAR LOAD FUNCTIONS FOR UNIFORM LOAD q = 1 OVER ENTIRE SURFACE OF PLATE. CLASSICAL SOLUTION.

$\frac{L_y}{L_x}$	$\frac{D}{L^3}\tau_{x=o,\ y=\frac{L_y}{2}}$	$\frac{D}{L^3}\tau_{x=L_x,\ y=\frac{L_y}{2}}$
$\frac{L}{2L}$	.0195	-.0195
$\frac{L}{L}$	.0135	-.0135
$\frac{2L}{L}$	.0325	-.0325

TABLE 2

ANGULAR LOAD FUNCTIONS FOR CONCENTRATED LOAD. CLASSICAL SOLUTION

x coordinate of unit load ($y = L_y/2$)	$\frac{L_y}{L_x}$	$\frac{D}{L}\tau_{x=o,\ y=\frac{L_y}{2}}$	$\frac{D}{L}\tau_{x=L_x,\ y=\frac{L_y}{2}}$
$\frac{L_x}{2}$	$\frac{L}{2L}$	.0135	-.0135
	$\frac{L}{L}$	.0295	-.0295
	$\frac{2L}{L}$	.0450	-.0450
$\frac{L_x}{3}$	$\frac{L}{2L}$	.0264	-.0061
	$\frac{L}{L}$	.0366	-.0200
	$\frac{2L}{L}$	.0503	-.0330
$\frac{L_x}{6}$	$\frac{L}{2L}$	.0390	-.0024
	$\frac{L}{L}$	.0352	-.0100
	$\frac{2L}{L}$	.0432	-.0175

4.4. Numerical Angular Flexibilities and Carry-Overs by Classical Methods. It is possible to find a classical solution for the slope of a simple plate acted upon by an edge moment at a particular point, but this solution is rather complex. To simplify the problem, since only a few comparisons are intended, the solutions obtained by using equations (4.9) and (4.10) with equations (4.6), (4.7), and (4.8) are selected. By substituting equation (4.9), when $\xi = \frac{L_x}{2}$ and $M' = 1$, in equations (4.6) and (4.7), angular carry-overs are obtained. When equation (4.9) is substitued in equation (4.8), however, the angular flexibility is expressed as a series which is practically non-computable. Consequently, this solution is discarded and equation (4.10) is used in equation (4.8) to obtain a quantity which is called the total angular flexibility under uniform moment. Results are presented in tabular form in Tables 3 and 4. The product $M'' L_x$ is taken equal to five in equation (4.10), believeing it to be more realistic when comparing the results obtained with those given by approximate methods.

4.5. Numerical Angular Load Functions by Deflection Coefficients Formulae. Two sets of formulas, one for the straight line approximation and the other for the parabolic approximation to the deflection curve, are used. The influence coefficients for deflection are taken from Tables C, reference 16, and the calculated angular load functions are shown in tabular form in Tables 5 and 6.

TABLE 3

ANGULAR CARRY-OVERS DUE TO UNIT MOMENT AT

$x = \frac{L_x}{2}$, $y = L_y$. CLASSICAL SOLUTION

$\frac{L_y}{L_x}$	$DG_{x=o,\ y=\frac{L_y}{2}}$	$DG_{x=\frac{L_x}{6},\ y=o}$	$DG_{x=\frac{L_x}{3},\ y=o}$	$DG_{x=\frac{L_x}{2},\ y=o}$	$DG_{x=\frac{2L_x}{3},\ y=o}$	$DG_{x=\frac{5L_x}{6},\ y=o}$	$DG_{x=L_x,\ y=\frac{L_y}{2}}$
$\frac{L}{2L}$	.0431	.0574	.1108	.1359	.1108	.0574	-.0431
$\frac{L}{L}$	.0870	.0216	.0378	.0438	.0378	.0216	-.0870
$\frac{2L}{L}$	.0428	.0032	.0056	.0064	.0056	.0032	-.0428

TABLE 4

TOTAL ANGULAR FLEXIBILITIES DUE TO UNIFORM MOMENT ALONG $y = L_y$. CLASSICAL SOLUTION

$\frac{L_y}{L_x}$	$D(\text{Total } F_x = \frac{L_x}{2}, y = L_y)$
$\frac{L}{2L}$	-.7196
$\frac{L}{L}$	-.9083
$\frac{2L}{L}$	-.9284

TABLE 5

ANGULAR LOAD FUNCTIONS FOR UNIFORM LOAD $q = 1$ OVER ENTIRE SURFACE OF PLATE. APPROXIMATE SOLUTIONS.

$\frac{L_y}{L_x}$	Approximation	$\frac{D}{L^3}\tau_{x=o,\ y=\frac{L_y}{2}}$	$\frac{D}{L^3}\tau_{x=L_x,\ y=\frac{L_y}{2}}$
$\frac{L}{2L}$	Straight Line	.0175	.0175
	Parabola	.0214	.0214
$\frac{L}{L}$	Straight Line	.0127	.0127
	Parabola	.0147	.0147
$\frac{2L}{L}$	Straight Line	.0309	.0309
	Parabola	.0354	.0354

TABLE 6

ANGULAR LOAD FUNCTIONS FOR CONCENTRATED LOAD. APPROXIMATE SOLUTIONS.

x coordinate of unit load ($y = L_y/2$)	$\frac{L_y}{L_x}$	Approximation	$\frac{D}{L}\tau_{x=o,\ y=\frac{L_y}{2}}$	$\frac{D}{L}\tau_{x=L_x,\ y=\frac{L_y}{2}}$
$\frac{L_x}{2}$	$\frac{L}{2L}$	Straight Line Parabola	.0145 .0122	.0145 .0122
	$\frac{L}{L}$	Straight Line Parabola	.0303 .0316	.0303 .0316
	$\frac{2L}{L}$	Straight Line Parabola	.0484 .0522	.0484 .0522
$\frac{L_x}{3}$	$\frac{L}{2L}$	Straight Line Parabola	.0266 .0268	.0069 .0052
	$\frac{L}{L}$	Straight Line Parabola	.0377 .0422	.0204 .0206
	$\frac{2L}{L}$	Straight Line Parabola	.0539 .0616	.0353 .0372
$\frac{L_x}{6}$	$\frac{L}{2L}$	Straight Line Parabola	.0383 .0634	.0027 .0019
	$\frac{L}{L}$	Straight Line Parabola	.0361 .0534	.0102 .0101
	$\frac{2L}{L}$	Straight Line Parabola	.0441 .0612	.0185 .0193

4.6. Numerical Angular Flexibilities and Carry-Overs by Deflection Coefficients Formulae. Angular carry-overs are tabulated in Table 7. Functions necessary to calculate the total flexibilities are shown in Table 8 and the evaluated total F's are tabulated in Table 9.

4.7. Errors in Angular Load Functions. A tabulation of the errors in angular load functions, obtained by using the approximate procedures suggested, is presented in Table 10. The classical solution is considered the exact one.

4.8. Errors in Angular Flexibilities and Carry-Overs. A tabulation of the errors in G's and total F's, obtained by using the approximate procedures suggested, is presented in Table 11. The classical solution is considered the exact one.

TABLE 7

ANGULAR CARRY-OVERS DUE TO UNIT MOMENT AT $x = \frac{L_x}{2}$, $y = L_y$. APPROXIMATE SOLUTIONS

$\frac{L_y}{L_x}$	Approximation	$DG_{x=o,y=\frac{L_y}{2}}$	$DG_{x=\frac{L_x}{6},y=o}$	$DG_{x=\frac{L_x}{3},y=o}$	$DG_{x=\frac{L_x}{2},y=o}$	$DG_{x=\frac{2L_x}{3},y=o}$	$DG_{x=\frac{5L_x}{6},y=o}$	$DG_{x=L_x,y=\frac{L_y}{2}}$
$\frac{L}{2L}$	Straight Line	.0428	.0385	.0813	.1109	.0813	.0385	.0428
	Parabola	.0374	.0423	.0874	.1158	.0874	.0423	.0374
$\frac{L}{L}$	Straight Line	.0827	.0296	.0523	.0611	.0523	.0296	.0827
	Parabola	.0899	.0301	.0525	.0608	.0525	.0301	.0899
$\frac{2L}{L}$	Straight Line	.0428	.0040	.0070	.0080	.0070	.0040	.0428
	Parabola	.0482	.0029	.0050	.0057	.0050	.0029	.0482

TABLE 8

ANGULAR CARRY-OVERS AND FLEXIBILITIES FOR CALCULATING TOTAL FLEXIBILITIES. APPROXIMATE SOLUTIONS.

$\frac{L_y}{L_x}$	Function	Straight Line	Parabola
$\frac{L}{2L}$	$G_{x=\frac{L_x}{2},\ y=L_y}$ $x=\frac{L_x}{6},\ y=L_y$	.0496	.0591
	$G_{x=\frac{L_x}{2},\ y=L_y}$ $x=\frac{L_x}{3},\ y=L_y$	.1284	.1611
	$F_{x=\frac{L_x}{2},\ y=L_y}$	.2645	.3672
$\frac{L}{L}$	$G_{x=\frac{L_x}{2},\ y=L_y}$ $x=\frac{L_x}{6},\ y=L_y$	.0696	.0931
	$G_{x=\frac{L_x}{2},\ y=L_y}$ $x=\frac{L_x}{3},\ y=L_y$	.1459	.2033
	$F_{x=\frac{L_x}{2},\ y=L_y}$	.2167	.3201
$\frac{2L}{L}$	$G_{x=\frac{L_x}{2},\ y=L_y}$ $x=\frac{L_x}{6},\ y=L_y$	.0472	.0755
	$G_{x=\frac{L_x}{2},\ y=L_y}$ $x=\frac{L_x}{3},\ y=L_y$	.0896	.1454
	$F_{x=\frac{L_x}{2},\ y=L_y}$	.1150	.1901

TABLE 9

TOTAL ANGULAR FLEXIBILITIES DUE TO UNITARY MOMENTS ALONG $y = L_y$. APPROXIMATE SOLUTIONS

$\frac{L_y}{L_x}$	Approximation	D(Total $F_x = \frac{L_x}{2}, y = L_y$)
$\frac{L}{2L}$	Straight Line Parabola	.6203 .8077
$\frac{L}{L}$	Straight Line Parabola	.6479 .9129
$\frac{2L}{L}$	Straight Line Parabola	.3886 .6318

TABLE 10

ERRORS IN ANGULAR LOAD FUNCTIONS

$\frac{L_y}{L_x}$	Approximation	Location	% ERROR when $P_{x=\frac{L_x}{2},y=\frac{L_y}{2}}$	$P_{x=\frac{L_x}{3},y=\frac{L_y}{2}}$	$P_{x=\frac{L_x}{6},y=\frac{L_y}{2}}$	q_{xy}
$\frac{L}{2L}$	Straight Line	$x = 0$, $y = \frac{L_y}{2}$	+ 7	+ 1	-2	-10
		$x = L_x$, $y = \frac{L_y}{2}$	+ 7	+13	+12	-10
	Parabola	$x = 0$, $y = \frac{L_y}{2}$	-10	+ 1	+62	+10
		$x = L_x$, $y = \frac{L_y}{2}$	-10	-15	-21	+10
$\frac{L}{L}$	Straight Line	$x = 0$, $y = \frac{L_y}{2}$	+ 3	+ 3	+ 3	- 6
		$x = L_x$, $y = \frac{L_y}{2}$	+ 3	+ 2	+ 2	- 6
	Parabola	$x = 0$, $y = \frac{L_y}{2}$	+ 7	+15	+ 52	+ 9
		$x = L_x$, $y = \frac{L_y}{2}$	+ 7	+ 3	+ 1	+ 9
$\frac{2L}{L}$	Straight Line	$x = 0$, $y = \frac{L_y}{2}$	+ 8	+ 7	+ 2	- 5
		$x = L_x$, $y = \frac{L_y}{2}$	+ 8	+ 7	+ 6	- 5
	Parabola	$x = 0$, $y = \frac{L_y}{2}$	+16	+ 22	+ 42	+ 9
		$x = L_x$, $y = \frac{L_y}{2}$	+16	+13	+10	+ 9

TABLE 11

ERROR IN ANGULAR FLEXIBILITIES AND CARRY-OVERS

$\frac{L_y}{L_x}$	Approximation	% ERROR in Total $F_{x=\frac{L_x}{2},y=L_y}$	$G_{x=o,y=\frac{L_y}{2}}$	$G_{x=\frac{L_x}{6},y=o}$	$G_{x=\frac{L_x}{3},y=o}$	$G_{x=\frac{L_x}{2},y=o}$	$G_{x=\frac{2L_x}{3},y=o}$	$G_{x=\frac{5L_x}{6},y=o}$	$G_{x=L_x,y=\frac{L_y}{2}}$
$\frac{L}{2L}$	Straight Line	-14	- 1	-33	-27	-18	-27	-33	- 1
	Parabola	+12	-13	-26	-21	-15	-21	-26	-13
$\frac{L}{L}$	Straight Line	-29	- 5	+37	+38	+39	+38	+37	- 5
	Parabola	+ 1	+ 3	+39	+39	+39	+39	+39	+ 3
$\frac{2L}{L}$	Straight Line	-58	0	+25	+25	+25	+25	+25	0
	Parabola	-32	+13	- 9	-11	-11	-11	- 9	+13

4.9. Analysis of Errors. From the tabulation (Tables 10 and 11) of the errors obtained in the Angular Functions by using the formulae in terms of deflection coefficients, several conclusions can be drawn:

1. For the angular load functions better results are obtained by using the string polygon as the deflection curve. When this straight line approximation is used, most of the errors are positive if a concentrated load is applied and negative if a uniform load is applied. In practice, when a combination of uniform and concentrated loads is encountered, much better results can be expected. The results found here are nevertheless considered satisfactory since a great majority of them yield errors within ten percent.

2. For the angular flexibilities and carry-overs fair results are obtained by both approximate methods. It is difficult to decide in general which of the two approximations would give better solutions because of the lack of adequate classical results, but the parabolic approximation seems to be somewhat more advantageous.

3. Using a finer grid some improvement undoubtedly would be obtained in the results, and the change in the curvature of the deflection curve observed for some loading conditions might be partly avoided (the angular functions obtained by the parabolic approximation being smaller than those obtained by the straight line approximation in these cases).

4. It is doubtful that an investigation of the use of other curves to fit the deflection surface would prove worth while.

5. Finally, it should be remarked that the classical solutions are not exact. They are considered exact solutions here because the assumptions made in the ordinary theory of bending of elastic plates are made for both the classical and the approximate methods. It may also be added that in practice some approximate results may be as good as some classical results since there is no exact knowledge in loading, actual edge conditions, and rigidity of foundations.

CHAPTER V

MOMENT EQUATION IN TERMS OF DEFLECTION COEFFICIENTS

5.1. General Moment Equation. Combining the formulas developed for the angular functions in Chapter III with the formulas derived for the moments in Chapter II, the moment equation in its different forms can be expressed directly in terms of the influence coefficients for deflection. The equations developed for the angular load function corresponding to the string polygon approximation and the equations developed for the angular flexibility and the angular carry-over corresponding to the parabolic approximation are applied here (based on the results obtained in Chapter IV). It should be noted that the moments computed from the final moment equations are actually the edge bending moments divided by $1 + \mu$, where μ is Poisson's ratio.

The general moment equation for the particular case in which i is on an edge perpendicular to the x direction and j is on an edge parallel to the x direction is developed first and then equations for the other three particular cases are shown. Using equations (3.3a), (3.13a), and (3.15a) in equation (2.6), the following expression is obtained:

$$\frac{\Delta y}{D} \sum \eta_{(i+1)k} P_k + \left(\sum \frac{\Delta y}{\Delta x} \frac{1}{D} \frac{4\eta_{(i+1)(i+1)} - \eta_{(i+2)(i+1)}}{2} \right) M_i + $$
$$+ \sum \frac{1}{D} \frac{4\eta_{(j+1)(i+1)} - \eta_{(j+1)(i+2)}}{2} M_j = 0 \qquad (5.1)$$

Multiplying equation (5.1) by 2D and noting that $\frac{\Delta y}{\Delta x} = \frac{L_y}{L_x}$,

$$2\Delta y \sum \eta_{(i+1)k} P_k + \frac{L_y}{L_x} \left[\sum \left(4\eta_{(i+1)(i+1)} - \eta_{(i+2)(i+1)} \right) \right] M_i + $$
$$+ \sum \left(4\eta_{(j+1)(i+1)} - \eta_{(j+1)(i+2)} \right) M_j = 0 \qquad (5.2)$$

If the moments are expressed dimensionally as force times length per length instead of force times length, a useful simplification can be made, the moments in equation (5.2) then becoming moments multiplied by Δy. Cancelling Δy, the general moment equation reduces to:

$$2 \sum \eta_{(i+1)k} P_k + \frac{L_y}{L_x} \left[\sum \left(4\eta_{(i+1)(i+1)} - \eta_{(i+2)(i+1)} \right) \right] M_i + $$
$$+ \sum \left(4\eta_{(j+1)(i+1)} - \eta_{(j+1)(i+2)} \right) M_j = 0 \qquad (5.3)$$

where an interesting and important observation can be made: the equation depends on the relative lengths of the sides of the plates and not on the separate size of the network elements chosen.

From the relative position of i and j, three additional equations similar to equation (5.3) can be developed. These equations are:

(edge $\supset$ i $\parallel$ x axis $\perp$ edge $\supset$ j)

$$2\sum \eta_{(i+1)k} P_k + \frac{L_x}{L_y} \left[\sum \left(4\eta_{(i+1)(i+1)} - \eta_{(i+2)(i+1)} \right) \right] M_i + \sum \left(4\eta_{(j+1)(i+1)} - \eta_{(j+1)(i+2)} \right) M_j = 0 \tag{5.4}$$

(edge $\supset$ i, j $\parallel$ x axis)

$$2\sum \eta_{(i+1)k} P_k + \frac{L_x}{L_y} \left[\sum \left(4\eta_{(i+1)(i+1)} - \eta_{(i+2)(i+1)} \right) \right] M_i + \frac{L_x}{L_y} \sum \left(4\eta_{(j+1)(i+1)} - \eta_{(j+1)(i+2)} \right) M_j = 0 \tag{5.5}$$

(edge $\supset$ i, j $\perp$ x axis)

$$2\sum \eta_{(i+1)k} P_k + \frac{L_y}{L_x} \left[\sum \left(4\eta_{(i+1)(i+1)} - \eta_{(i+2)(i+1)} \right) \right] M_i + \frac{L_y}{L_x} \sum \left(4\eta_{(j+1)(i+1)} - \eta_{(j+1)(i+2)} \right) M_j = 0 \tag{5.6}$$

5.2. Carry-Over Moment Equation. To define the moment equation in its carry-over form in terms of the influence coefficients for deflection, it is necessary to express only the starting moment and the carry-over factor as functions of these coefficients.

Using equations (2.10) and (2.9) with their corresponding angular functions for the four particular cases, the following formulas are obtained:

$$M_i^* = \frac{2\Delta' \sum \eta_{(i+1)k} P_k}{\sum \left[4\eta_{(i+1)(i+1)} - \eta_{(i+2)(i+1)} \right]} \tag{5.7}$$

$$r_{ij} = \frac{\Delta' \left[4\eta_{(j+1)(i+1)} - \eta_{(j+1)(i+2)} \right]}{\Delta'' \sum \left[4\eta_{(i+1)(i+1)} - \eta_{(i+2)(i+1)} \right]} \tag{5.8}$$

where the primed Δ's can be either Δx or Δy, depending upon the position of points i and j. Thus

$\Delta' = \Delta x$, $\Delta'' = \Delta y$, when edge $\supset$ i $\perp$ x axis

$\perp$ edge $\supset$ j;

$\Delta' = \Delta y$, $\Delta'' = \Delta x$, when edge $\supset$ i $\parallel$ x axis

$\perp$ edge $\supset$ j;

$\Delta' = \Delta'' = \Delta y$, when edge $\supset$ i, j $\parallel$ x axis;

$\Delta' = \Delta'' = \Delta x$, when edge $\supset$ i, j $\perp$ x axis.

5.3. Matrix Moment Equation. The expressions of angular functions in matrix form in terms of influence coefficients for deflection are:

$$\begin{bmatrix} \tau_{1k} \\ \tau_{2k} \\ \tau_{3k} \\ \cdot \\ \cdot \\ \cdot \\ \tau_{pk} \end{bmatrix} = \frac{1}{D} \begin{bmatrix} \Delta_1' \eta_{(1+1)k} \\ \Delta_2' \eta_{(2+1)k} \\ \Delta_3' \eta_{(3+1)k} \\ \cdot \\ \cdot \\ \cdot \\ \Delta_p' \eta_{(p+1)k} \end{bmatrix} \tag{5.9}$$

and

$$\begin{bmatrix} G_{1j} \\ G_{2j} \\ G_{3j} \\ \cdot \\ \cdot \\ \cdot \\ G_{pj} \end{bmatrix} = \frac{1}{2D} \begin{bmatrix} \frac{\Delta_1''}{\Delta_1'''} (4\eta_{(1+1)j} - \eta_{(1+2)j}) \\ \frac{\Delta_2''}{\Delta_2'''} (4\eta_{(2+1)j} - \eta_{(2+2)j}) \\ \frac{\Delta_3''}{\Delta_3'''} (4\eta_{(3+1)j} - \eta_{(3+2)j}) \\ \cdot \\ \cdot \\ \cdot \\ \frac{\Delta_p''}{\Delta_p'''} (4\eta_{(p+1)j} - \eta_{(p+2)j}) \end{bmatrix} \tag{5.10}$$

where each primed Δ can take on either of the values Δx and Δy, depending upon the position of points i, j.

Using equations (5.9) and (5.10) in equation (2.12), the moment equation in matrix form in terms of deflection coefficients is obtained:

$$
2\begin{bmatrix} \eta_{(1+1)k} \\ \eta_{(2+1)k} \\ \eta_{(3+1)k} \\ \cdot \\ \cdot \\ \cdot \\ \eta_{(p+1)k} \end{bmatrix}
+
\begin{bmatrix}
\sum(4\eta_{(1+1)1}-\eta_{(1+2)1})t_{11} & (4\eta_{(1+1)2}-\eta_{(1+2)2})t_{12} & (4\eta_{(1+1)3}-\eta_{(1+2)3})t_{13} & \cdots & (4\eta_{(1+1)p}-\eta_{(1+2)p})t_{1p} \\
(4\eta_{(2+1)1}-\eta_{(2+2)1})t_{21} & \sum(4\eta_{(2+1)2}-\eta_{(2+2)2})t_{22} & (4\eta_{(2+1)3}-\eta_{(2+2)3})t_{23} & \cdots & (4\eta_{(2+1)p}-\eta_{(2+2)p})t_{2p} \\
(4\eta_{(3+1)1}-\eta_{(3+2)1})t_{31} & (4\eta_{(3+1)2}-\eta_{(3+2)2})t_{32} & \sum(4\eta_{(3+1)3}-\eta_{(3+2)3})t_{33} & \cdots & (4\eta_{(3+1)p}-\eta_{(3+2)p})t_{3p} \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
(4\eta_{(p+1)1}-\eta_{(p+2)1})t_{p1} & (4\eta_{(p+1)2}-\eta_{(p+2)2})t_{p2} & (4\eta_{(p+1)3}-\eta_{(p+2)3})t_{p3} & \cdots & \sum(4\eta_{(p+1)p}-\eta_{(p+2)p})t_{pp}
\end{bmatrix}
\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ \cdot \\ \cdot \\ \cdot \\ M_p \end{bmatrix} = 0 \qquad (5.11)
$$

where: moments are given in force times length per length, and

t_{ij} $(i,j = 1, 2, 3, \ldots, p) = 1$ if edge $\supset i \perp$ edge $\supset j$,

$\frac{\Delta x}{\Delta y}$ if edge $\supset i, j \parallel x$ axis, and $\frac{\Delta y}{\Delta x}$ if edge $\supset i, j \parallel y$ axis.

From equation (5.11) the complete solution of the continuous plate problem is obtained. It should be noted that t_{ij} can be different in each row and/or column and is a function of the relative lengths of the sides of the plate containing i and j. An illustration of the use of the derived moment equations in a numerical problem is given in the next chapter.

CHAPTER VI

NUMERICAL EXAMPLE

6.1. Statement of Problem. A uniformly loaded rectangular plate of constant thickness, continuous in two directions over rigid supporting beams, is considered (Figure 6.1). The uniform load q over the entire area of the plate is taken as 100 lb per sq ft, and the Poisson's ratio is taken equal to zero. Moments in pounds are to be calculated.

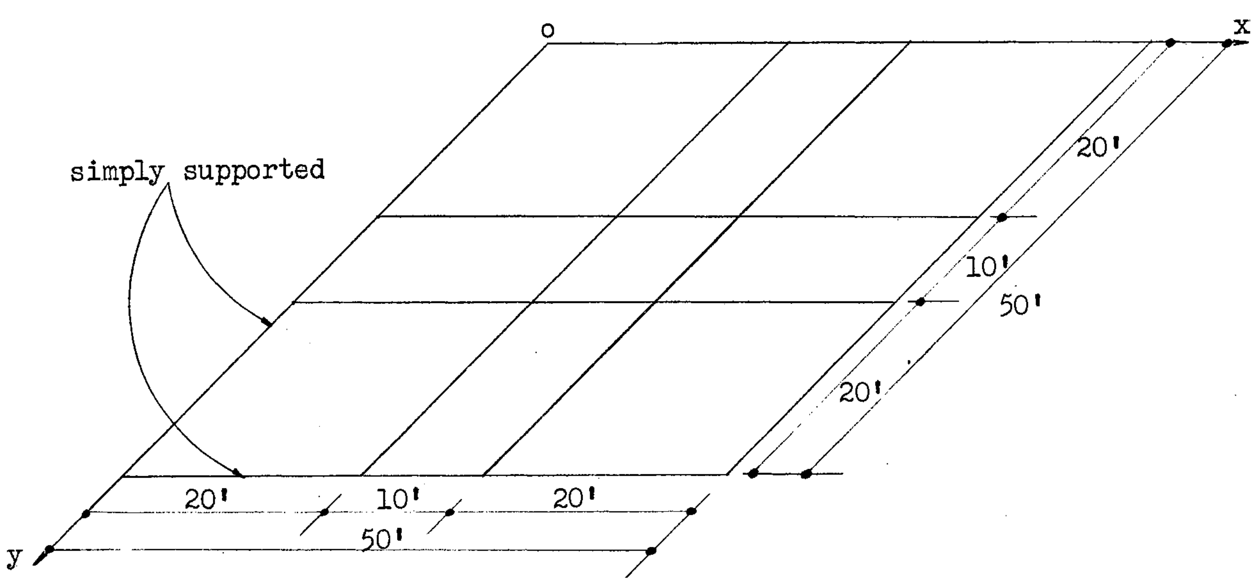

Figure 6.1. Continuous Rectangular Plate

6.2. Modified Moment Equations. Since both the plate and loading are symmetrical, the general moment equation (Eq. 2.6) can be modified in the following manner:

$$\sum \tau_{ik} P_k + M_i \left(\sum F_i + G_{ii'} + G_{ii''} + \dots \right) + \sum \left(G_{ij} + G_{ij'} + G_{ij''} + \dots \right) M_j = 0 \qquad (6.1)$$

In particular (see Figure 6.2):

$$\begin{aligned} q \left[\frac{20}{6} \frac{20}{6} (\tau_1)_l + \frac{20}{6} \frac{10}{6} (\tau_1)_m \right] &+ M_1 \left(\sum F_1 + G_{11'} + G_{11''} \right) + \\ &+ M_2 \left(\sum G_{12} + G_{12'} + G_{12''} \right) + \\ &+ M_3 \left(\sum G_{13} + G_{13'} + G_{13''} \right) + \\ &+ M_4 \left(\sum G_{14} + G_{14'} + G_{14''} \right) + \\ &+ M_5 \left(\sum G_{15} + G_{15'} + G_{15''} \right) + \\ &+ M_6 \left(G_{16} + G_{16'} \right) + \\ &+ M_7 \left(G_{17} + G_{17'} \right) + \\ &+ M_8 \left(G_{18} \right) = 0 \end{aligned}$$

Similarly, the other seven moment equations are obtained.

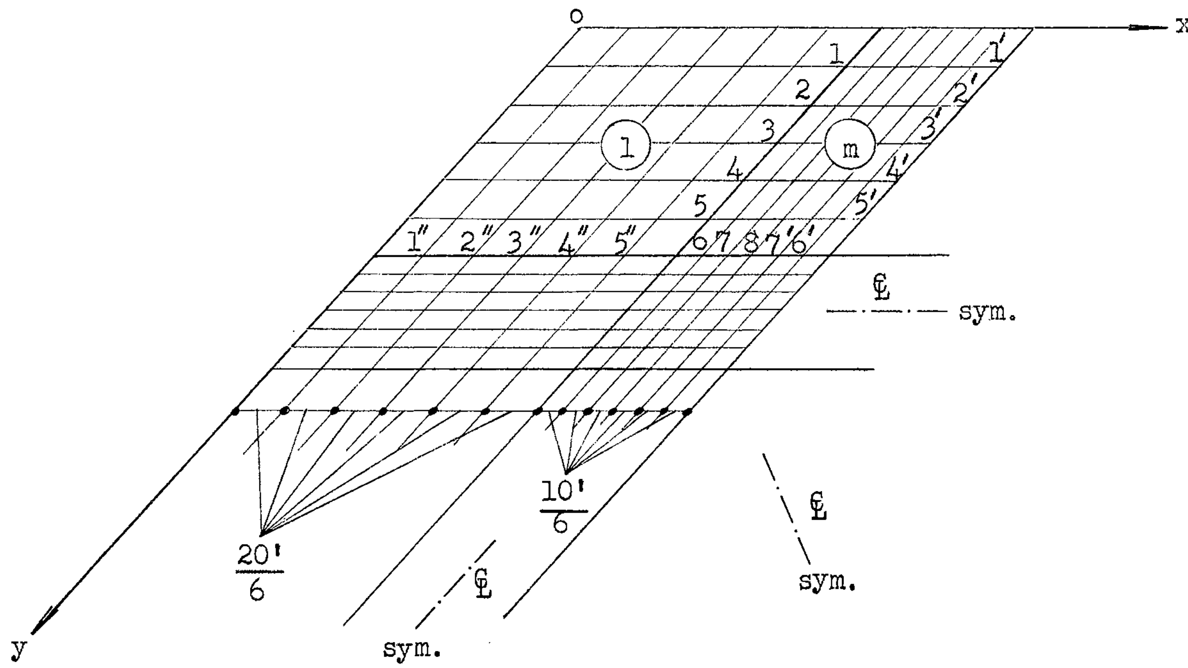

Figure 6.2. Modified Continuous Rectangular Plate

6.3. Matrix Moment Equation. An equation of the form of equation (5.11) can be developed by using the modified moment equations. Using Tables C in reference 16, the coefficients are evaluated and the matrix moment equation becomes:

$$
\begin{bmatrix} 2139.420 \\ 3506.014 \\ 3965.576 \\ 3506.014 \\ 2139.420 \\ 940.672 \\ 1585.498 \\ 1812.128 \end{bmatrix}
+
\begin{bmatrix}
.5790711 & .3759861 & .2276272 & .1350766 & .0663538 & .0091323 & .0157675 & .0090901 \\
.3751273 & .8070671 & .5166264 & .3082739 & .1532807 & .0236643 & .0405758 & .0233243 \\
.2245610 & .5125393 & .8930432 & .5606568 & .2875547 & .0507051 & .0846619 & .0481799 \\
.1292341 & .2972546 & .5482486 & .8952217 & .5019071 & .1006040 & .1522410 & .0839392 \\
.0603421 & .1391171 & .2608198 & .4614773 & .7693866 & .1816912 & .1961134 & .0992082 \\
.0056954 & .0155197 & .0356532 & .0782648 & .2292477 & .6301563 & .6091731 & .2980986 \\
.0099165 & .0272583 & .0640606 & .1430393 & .2908306 & .6570652 & 1.0821229 & .5623904 \\
.0114764 & .0316432 & .0747584 & .1657040 & .3018622 & .6435344 & 1.1414230 & .7507406
\end{bmatrix}
\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \\ M_7 \\ M_8 \end{bmatrix}
= 0
$$

6.4. Solution Matrix. The task of calculating the moments from equation (6.2) was assigned to the IBM 650 electronic computer at the Oklahoma State University computing center. Use was made of an available program for the inversion of a matrix. The results given are:

$$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \\ M_7 \\ M_8 \end{bmatrix} = - \begin{bmatrix} 1357.7687 \\ 1799.6032 \\ 1842.3855 \\ 1508.0989 \\ 572.81715 \\ 102.44873 \\ 534.95294 \\ 669.36579 \end{bmatrix} = \begin{bmatrix} M_{x_1} \\ M_{x_2} \\ M_{x_3} \\ M_{x_4} \\ M_{x_5} \\ M_{y_6} \\ M_{y_7} \\ M_{y_8} \end{bmatrix} \tag{6.3}$$

6.5. Analysis of Results. The results given in equation (6.3) are compared with solutions of the same problem given in reference 12 (see Table 12).

TABLE 12

COMPARISON OF MOMENTS

Description	$-M_{x_1}$	$-M_{x_2}$	$-M_{x_3}$	$-M_{x_4}$	$-M_{x_5}$	$-M_{x_{ave}}$	$-M_{y_6}$	$-M_{y_7}$	$-M_{y_8}$	$-M_{y_{ave}}$
Thesis	1358	1800	1842	1508	573	1416	102	535	669	435
Reference 12	1209	1732	1813	1521	821	1419	55	232	304	197

The comparison of moments shown in Table 12 indicates a fairly good agreement between the two methods of computation along the long edge of panel m, the average moments being almost equal. Along the short edge of panel m, however, the moments calculated by finite differences are approximately twice as large as the ones given in reference 12. These results could be expected from the respective errors in the angular functions corresponding to these two edges and because the moments along the short edge are much smaller than the moments along the longer one.

CHAPTER VII

SUMMARY AND CONCLUSIONS

The flexibility method, as applied to the analysis of continuous rectangular plates rigidly supported, is presented in this thesis. The major steps in the discussion are summarized as follows.

1. The moment equation in terms of angular functions for simple rectangular plates is derived and methods for solving this equation are given.
2. The angular functions are expressed in terms of deflection coefficients that are obtained by solving a finite difference network.
3. A comparison of the results obtained for the angular functions by using the formulas developed with those obtained by using classical solutions is presented.
4. Moment equations in terms of deflection coefficients are developed, including a matrix moment equation which yields the complete solution of the continuous plate problem.
5. The use of the method in the analysis of a plate continuous in two directions is illustrated by an example.

The most significant conclusions drawn from this study can be described in the following manner.

a. The angular functions evaluated from deflection coefficients that are obtained by solving a thirty-six unit finite difference net are in many cases sufficiently accurate.

b. The use of the angular functions developed in the moment equations derived yields satisfactory results along the longer edges of rectangular plates.

c. The formulated solution for the continuous plate problem is relatively simple to apply (in the illustrative problem only eight equations had to be solved against seventeen equations that had to be solved to obtain the results listed in reference 12). By increasing the gridwork points a greater accuracy can be obtained if desired (the gridwork used actually being somewhat coarse), while the number of equations to be solved still remains relatively low.

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VITA

Hanan Lechter

Candidate for the Degree of

Master of Science

Thesis: ANALYSIS OF CONTINUOUS RECTANGULAR PLATES ON RIGID SUPPORTS BY FLEXIBILITY METHODS.

Major Field: Civil Engineering

Biographical:

Personal Data: Born January 29, 1940, in Cali, Colombia, the son of Jose and Ita Lechter.

Education: Graduated from Colegio San Luis Gonzaga, Cali, Colombia, in July, 1956. Joined La Universidad de los Andes, Bogota, Colombia, in January, 1957. Transferred to The University of Kansas, Lawrence, Kansas, in January, 1960. Received the degree of Bachelor of Science in Civil Engineering from The University of Kansas, June, 1961. Received the degree of Ingeniero Civil from La Universidad de los Andes, June, 1961. Completed the requirements for the Master of Science degree in May, 1962, at Oklahoma State University.