APPLICATION OF THE LAPLACE TRANSFORM METHOD TO THE ANALYSIS OF LOAD CARRYING MEMBERS

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PREFACE

The Laplace transformation has not enjoyed the same popularity in some areas of engineering analysis as it has in others. In particular, it is not commonly known that it affords a simple and efficient approach to the elementary beam problem. In recent years there has been an increasing number of engineers who have become adept at using the Laplace transformation in the fields of automation, process controls, servomechanisms, etc.

In view of these circumstances, it seems desirable that a procedure, utilizing the Laplace transformation, should be developed for the analysis of elementary beam systems. The development of such a procedure is presented in this study as the major objective.

The writer wishes to express his indebtedness and sincere appreciation to the following individuals:

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Kenneth H. Koerner Jr.

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NOTATION

Notations employed only in a single article are not, as a rule, listed below.

a, b, c, d	Position coordinates of load system
E	Young's Modulus of Elasticity
$F(x)$	Load function
$f(s)$	Laplace transform of the load function, $F(x)$
i, j, k	General subscripts
I	Moment of inertia of area
ℓ	Over-all length
$M_b(x)$	Bending moment at x
M_i	Applied moment
M_1, M_2 etc.	Reactive moments
P_i	Concentrated load
q_i	Arbitrarily distributed load per unit of length
Q_i	Total distributed load
R_1, R_2, etc.	Reactions
$S_k(x)$	Unit step function at k
$S'_k(x)$	Unit impulse function at k
$S''_k(x)$	Unit doublet function at k
$V(x)$	Vertical shearing force at x

$\bar{x}_i$ . . . . Centroidal distance

$y(x)$ . . . . Transverse deflection of a point on the elastic curve of a beam at a distance x from one end.

$Y(s)$ . . . . Laplace transform of deflection, $Y(x)$

β . . . . Symbol for $\sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right]$

$\phi(x)$ . . . . Slope at x

$\sum$ . . . . Summation where $i = 1, 2, 3, \ldots\ldots, m, n, p, \ldots\ldots$

CHAPTER I

INTRODUCTION

The Laplace transformation was introduced by P. S. Laplace in 1779 (4). It is a linear integral transformation which enables one to solve many ordinary and partial linear differential equations. The solution is easily obtained without finding the general solution and then evaluating the arbitrary constants, as required by the classical method. This results in a savings of time and labor.

The Laplace transformation is the best known form of operational mathematics. The form as it is known today is the result of extensive research and development by Doetsch (3) and others.

Beginning in the late 1930's, the Laplace transformation has been a powerful tool in the solution of linear circuit problems in the electrical engineering field. Only in the last ten to fifteen years has it gained usage in the dynamics of mechanical and fluid systems. An area in which it has not been exploited to the same degree is the analysis of statically loaded structural members. It is this area with which the major portion of this thesis will be concerned.

CHAPTER II

PREVIOUS APPLICATIONS

The more prominent American textbooks on the Laplace transformation, e.g. Churchill (2) and Thomson (13), apply the transform method to the static deflection of beams and columns. These applications are simple and few in number. Their objectives are to aid in teaching the mechanics of the transform and to illustrate its potential uses.

Other applications of the Laplace transformation to general static beam and column theory can be found in the engineering literature. The following is a synopsis of these articles. For brevity, only the structures and load systems will be listed. Unless otherwise noted, all structures have a constant cross section and are loaded transversely.

Strandhagen (12) applied the transform to the deflection of "beam columns", i.e. beams subjected to axial loads as well as transverse loads. The general cases were:

1. Simple beam with unequal end moments and no transverse loads
2. Propped cantilever beam with a uniformly distributed load

3. Propped cantilever beam with a triangularly distributed load
4. Fixed beam with a triangularly distributed load
5. Fixed beam with a parabolically distributed load.

Pipes (9) dealt with the deflection of:

1. Fixed beams with
   a. A uniformly distributed load
   b. A concentrated load
   c. An applied moment
   d. A concentrated load and on an elastic foundation.
2. Cantilever beam with
   a. A concentrated load and on an elastic foundation.

Gardner and Barnes (4) solved for the deflections of a simple beam with:

1. Two overhangs loaded by a uniformly distributed and concentrated load
2. A uniformly distributed and concentrated load
3. Two unequal spans loaded by a uniformly distributed load.

Iwinski (5) determined the elastic curves for:

1. Single span beams
   a. Simple beam
   b. Simple beam with an overhang
   c. Simple beam with terminal forces and moments
   d. Cantilever beam
   e. Propped cantilever beam

f. Propped cantilever beam with an overhang

g. Fixed beam

h. On elastic supports

2. Two span beams

3. Continuous beams

4. Continuous beam on elastic supports

5. Beams with variable flexural rigidity.

In each of the above cases the beams are analyzed for a generalized system of uniformly distributed loads, concentrated loads, and applied moments.

Wagner (16) investigated the stability of buckling members. The types of columns covered were:

1. Hinged on both ends

2. One end fixed, and the other free

3. One end hinged, the other guided and hinged

4. One end fixed, the other guided and hinged

5. One end fixed, the other guided and fixed

6. Multisection.

Thomson (14) solved for the deflections of:

1. Simple beams with

   a. A concentrated load

   b. An applied moment

   c. A partial uniformly distributed load

d. An abrupt cross sectional change loaded by a uniformly distributed load.

2. Cantilever beams with

   a. A triangularly distributed load

   b. A narrow slot loaded by a concentrated load at the end of the beam.

Blanco (1) used the Laplace transform to determine the deflections for:

1. Simple beams with

   a. A uniformly distributed load

   b. A triangularly distributed load.

2. Propped cantilever beam with a partial uniformly distributed load.

CHAPTER III

OBJECTIVES

It can be ascertained from Chapter II that the Laplace transform has been employed either as an instruction aid or applied in a random fashion to the analysis of static structures.

With the notable exception of Iwinski (5), there have been no attempts (particularly in this country) to develop and compile into one text, the basic elements of static structures as analyzed by the Laplace transformation. Therefore, the objectives of this thesis are as follows:

<u>Part I</u> (a) The development of a systematic procedure of analysis using the Laplace transform as applied to elementary beams. (Note: Elementary beams will be taken to include only single span beams with constant cross sections.) Symbolism and terminology compatible with backgrounds in systems engineering, i.e., automation, process controls, servomechanisms, advanced dynamics, etc., will be used.

(b) The development of general solutions to elementary beams that are subjected to any transverse system of distributed loads, concentrated loads, and applied moments. (The load systems are assumed to have Laplace transforms.)

Part II The investigation, in a general nature, of other miscellaneous topics in the static structures field for which the Laplace transform method could be utilized.

Part III The use of the Laplace transform method as a basic tool of analysis in impact investigations.

CHAPTER IV

PART I - ANALYSIS OF ELEMENTARY BEAMS BY THE LAPLACE TRANSFORM METHOD

4-1 General

Strength of materials is that science which establishes the relationships between the external forces, acting on an elastic body, and the internal forces and deformations which result from these external forces.

A large portion of any elementary strength text is devoted to the analysis of beams and columns. The fundamental basis for the analysis of these members is the equation of their elastic curves. When these equations are known, the other pertinent design data can be readily obtained, i.e.:

a. The maximum deflection

b. The support reactions

c. The restraining moments (if any)

d. The slopes at the supports

e. The distribution of the bending moment and shearing force along the member

f. The maximum bending moment.

In normal design practice, the bending moment and shearing force distribution are perhaps the most important of the above data.

4-2 Beam Equation

The equation of the elastic curve can be obtained from the basic beam equation

$$EI\, Y^{(4)} = F(x) \qquad (4\text{-}1)$$

where

E = Young's modulus of elasticity,

I = Moment of inertia of area,

$Y^{(4)}$ = Fourth derivative, with respect to x, of the transverse deflection,

$F(x)$ = Load function.

This equation may be found in any elementary strength text. (15).

4-3 Transformation of the Beam Equation

If $y(s)$ and $f(s)$ denote the Laplace transforms of $Y(x)$ and $F(x)$ respectively, then the Laplace transformation of Eq. (4-1) gives

$$EI\left[s^4 y(s) - s^3 Y(0) - s^2 Y'(0) - sY''(0) - Y'''(0)\right] = f(s) \quad .$$

This expression is then solved for the subsidiary equation, $y(s)$,

$$y(s) = \frac{Y(0)}{s} + \frac{Y'(0)}{s^2} + \frac{Y''(0)}{s^3} + \frac{Y'''(0)}{s^4} + \frac{1}{EI}\,\frac{f(s)}{s^4} \quad . \qquad (4\text{-}2a)$$

Performing the inverse transformation on the subsidiary equation results in

$$L^{-1}\left[y(s)\right] = Y(x) = Y(0) + Y'(0)x + \frac{Y''(0)x^2}{2!} + \frac{Y'''(0)x^3}{3!} + \frac{1}{EI}\, L^{-1}\left[\frac{f(s)}{s^4}\right] \qquad (4\text{-}2b)$$

where the initial boundary conditions are

$$Y(0) = \text{Deflection at } x = 0,$$

$$Y'(0) = \text{Slope at } x = 0,$$

$$Y''(0) = \frac{1}{EI} \text{ times moment at } x = 0,$$

$$Y'''(0) = \frac{1}{EI} \text{ times shear at } x = 0\ .$$

The boundary conditions which are unknown at $x = 0$ can be evaluated from known conditions of deflection, slope, moment, or shear existing at other positions along the beam.

The general expression (4-2b) yields the equation of the elastic/ deflection curve for any arbitrary elementary beam subjected to any system of transverse loads and/or applied moments. The solution to a particular beam will involve the evaluation of its respective boundary conditions and load function.

4-4 Load Function

The load function $F(x)$ is defined as the system of loads acting on the beam. Distributed loads will be represented by the unit step function $S_k(x)$, concentrated loads by the unit impulse function $S_k'(x)$, and applied moments by the unit doublet function $S_k''(x)$.

The load function is formulated by use of the following canonical set of rules:

a. The load function is formed for the region $0 \leq x \leq \ell$.

b. Concentrated loads, applied moments, or support reactions occurring at $x = 0$ are included in the load function. (Note:

Those occurring at $x = \ell$ could be included in the load function, however, beyond their use in calculating the support reactions they do not affect the solution for the region $0 \leq x \leq \ell$.)

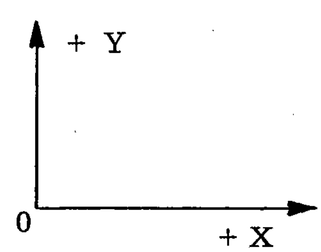
c. The remaining portion of the load function is comprised of arbitrarily distributed loads, concentrated loads, applied moments, and support reactions occurring in the region $0 < x < \ell$.

4-5 Sign Convention

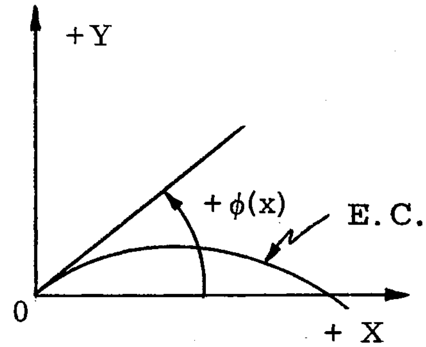
In a systematic analysis, a sign convention is necessary for a uniform interpretation of data. The following sign convention, which is compatible with most strength of materials texts, will be used.

a. A right-handed system of rectangular co-ordinates X, Y, Z will be used. The origin will be taken at the left end of the beam with the X-axis coinciding with the neutral surface and the Y- and Z- axes taken along the centroidal principal axes of the cross section. (Note: The origin could be taken at any cross section of the beam but it is usually most convenient to take it at the left end.)

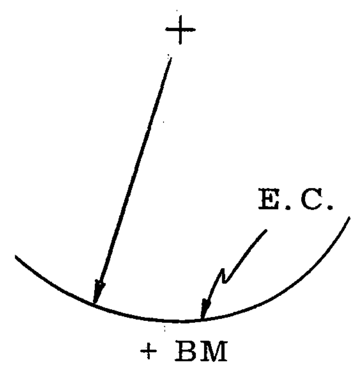
b. The positive direction of the Y-axis (deflection) will be taken vertically upward.



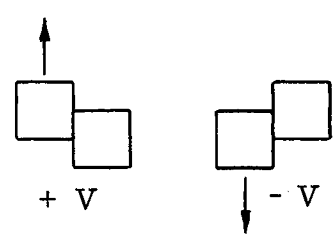
c. The slope $\phi(x)$ of the elastic curve at a given point will be taken as positive if the rotation of a tangent at that point is measured in a CCW direction with respect to the origin and X-axis.



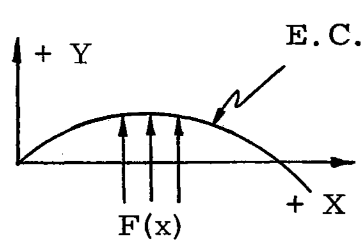
d. The bending moment $M_b(x)$ at a section will be taken positive if the center of curvature of the elastic curve in that region lies above the curve. (The moments of upward directed loads will result in positive bending moments.)



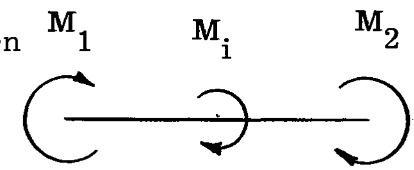
e. The vertical shear $V(x)$ at a section will be taken positive if the resultant of the vertical loads acting on the portion of the beam to the left of the section is upward.



f. The loads $F(x)$ directed upward will be taken as positive.



g. Reactive moments M_1, M_2 etc. and applied moments M_i, will be taken as positive when acting in a CW direction.



4-6 Analytical Procedure

As the result of the previous sections of this chapter, a simple yet powerful tool has been developed for use in analyzing elementary static structures.

Before continuing further with the formulation of a step by step procedure of analysis, a few of the possible pitfalls and misinterpretations of sections 4-3, 4-4, and 4-5 will be considered.

The region $0 \leq x \leq \ell$ will always be defined as the over-all length of the beam. This will be true even in the case of a multi-span beam.

In formulating the load function it cannot be emphasized too strongly the importance of placing the origin at the left end of the beam; that of including all concentrated loads, applied moments, and support reactions occurring at $x = 0$; and the omission of all concentrated loads, applied moments, and support reactions occurring at $x = \ell$. By complying with these rules a considerable amount of time, labor, and confusion can be saved.

An area in which possible uncertainty may arise is that of determining boundary conditions. Since the reactions at the left support ($x = 0$) have been included in the load function, the initial moment $Y''(0)$ and shear $Y'''(0)$ will always be equal to zero in Eq. (4-2b). The value of the initial deflection $Y(0)$ and slope $Y'(0)$ will depend on the physical situation at $x = 0$, i.e., whether it has a support or is an overhang. If supported, the values of $Y(0)$ and $Y'(0)$ may be known depending upon the type of support. If they are not known, as in the case of an overhang, then they must be evaluated from other known conditions of deflection, slope, moment, or shear existing along the beam.

Recognizing the correct moment and shear conditions at $x = \ell$ can sometimes be difficult, especially if the person isn't as familiar with strength of materials as he once was due to the lack of usage. It has been found during the development of this thesis that a simple beam with one overhang, a cantilever beam, and a propped cantilever beam with or without an overhang are the types with which one might experience difficulty in determining the conditions at $x = \ell$. As a solution to this problem, the procedure which has been used in the development of general solutions to elementary beams (section 4-7) is recommended. It consists of always placing the overhang of the simple beam at $x = 0$ and the fixed ends of the cantilever and propped cantilever beams at $x = \ell$. By doing this, the moment and shear at $x = \ell$ is eliminated by the rule of neglecting concentrated loads, applied moments, and support reactions occurring at $x = \ell$. The values of the deflection and slope can then be easily determined by inspection.

Continuing with the development of a systematic procedure of analysis, it is suggested that the following list of steps be used in arriving at a solution of an elementary beam problem.

A. List known boundary conditions

B. Formulate the load function

C. Transform the load function

D. Inverse transform $\frac{1}{EI}\left[\frac{f(s)}{s^4}\right]$

E. Substitute (A) and (D) into Eq. (4-2b)

F. Evaluate unknown initial boundary conditions and reactions appearing in (E) from known conditions at other positions along the beam

G. Solve for the final expression of the elastic/deflection curve, $Y(x)$

H. Differentiate (G) successively with respect to x to obtain other desired data, i.e., slope, $\phi(x) = Y'(x)$; bending moment, $M_b(x) = EIY''(x)$; and shearing force, $V(x) = EIY'''(x)$.

In the next section the above steps and rules will be used to develop a set of general solutions to elementary beams that are subjected to any transverse system of distributed loads, concentrated loads, and applied moments. The only restriction imposed is that the load systems be Laplace transformable.

4-7 General Solutions to Elementary Beams

4-7-1 General Solution of a Simple Beam

A. Boundary Conditions

$Y(0) = Y''(0) = Y'''(0) = 0$

$Y(\ell) = 0$

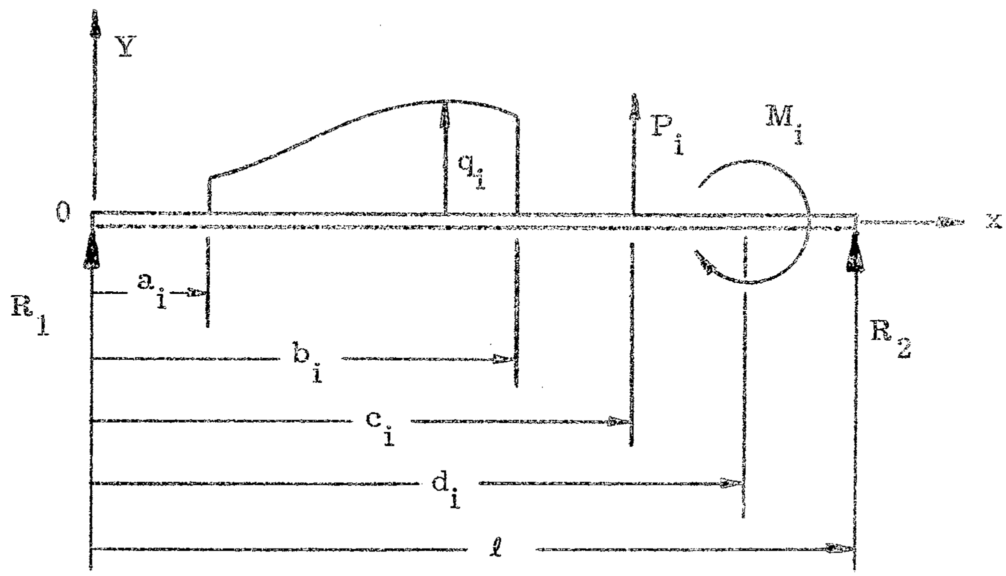


Fig. 4-7-1. Simple Beam

B. Load Function

$$F(x) = R_1 S_o'(x) + \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right] + \sum P_i S_{c_i}'(x) +$$
$$\sum M_i S_{d_i}''(x)$$

Letting $\beta = \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right]$,

$$F(x) = R_1 S_o'(x) + \beta + \sum P_i S_{c_i}'(x) + \sum M_i S_{d_i}''(x) .$$

C. Laplace Transform of Load Function

$$f(s) = R_1 + L\beta + \sum P_i e^{-c_i s} + \sum M_i s e^{-d_i s}$$

D. Inverse Transform of $\frac{1}{EI}\left[\frac{f(s)}{s^4}\right]$

$$\frac{1}{EI} \; L^{-1}\left[\frac{f(s)}{s^4}\right] = \frac{1}{EI}\left\{\frac{R_1 x^3}{3!} + L^{-1}\left[\frac{L\beta}{s^4}\right] + \right.$$

$$\left.\sum \frac{P_i (x - c_i)^3}{3!} S_{c_i}(x) + \sum \frac{M_i (x - d_i)^2}{2!} S_{d_i}(x)\right\}$$

E. Substituting (A) and (D) into Eq. (4-2b)

$$Y(x) = Y'(0)x + \frac{1}{EI}\left\{\frac{R_1 x^3}{3!} + L^{-1}\left[\frac{L\beta}{s^4}\right] + \right.$$

$$\left.\sum \frac{P_i(x - c_i)^3}{3!} S_{c_i}(x) + \sum \frac{M_i(x - d_i)^2}{2!} S_{d_i}(x)\right\}$$

F. Evaluation of Unknown Boundary Conditions and Reactions

$$Y(\ell) = 0 = Y'(0)\ell + \frac{1}{EI}\left\{\frac{R_1 \ell^3}{3!} + L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \right.$$

$$\left.\sum \frac{P_i(\ell - c_i)^3}{3!} + \sum \frac{M_i(\ell - d_i)^2}{2!}\right\}$$

$\therefore$

$$Y'(0) = -\frac{1}{EI}\left\{\frac{R_1 \ell^2}{3!} + \frac{1}{\ell} L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \right.$$

$$\left.\sum \frac{P_i(\ell - c_i)^3}{3!\,\ell} + \sum \frac{M_i(\ell - d_i)^2}{2!\,\ell}\right\}$$

The reaction R_1 can be determined by statics.

G. General Elastic/Deflection Curve Equation

$$Y(x) = -\frac{1}{EI}\left\{\frac{R_1\ell^2}{3!} + \frac{1}{\ell}\, L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \sum \frac{P_i(\ell - c_i)^3}{3!\,\ell} + \sum \frac{M_i(\ell - d_i)^2}{2!\,\ell}\right\} x +$$

$$\frac{1}{EI}\left\{\frac{R_1 x^3}{3!} + L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum \frac{P_i(x - c_i)^3}{3!}\, S_{c_i}(x) + \sum \frac{M_i(x - d_i)^2}{2!}\, S_{d_i}(x)\right\}$$

H. Slope, Bending Moment, and Shearing Force Equations

a. Slope

$$\phi(x) = -\frac{1}{EI}\left\{\frac{R_1\ell^2}{3!} + \frac{1}{\ell}\, L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \sum \frac{P_i(\ell - c_i)^3}{3!\,\ell} + \sum \frac{M_i(\ell - d_i)^2}{2!\,\ell}\right\} +$$

$$\frac{1}{EI}\left\{\frac{R_1 x^2}{2} + \frac{d}{dx}\, L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum \frac{P_i(x - c_i)^2}{2}\, S_{c_i}(x) + \sum M_i(x - d_i) S_{d_i}(x)\right\}$$

b. Bending Moment

$$M_b(x) = R_1 x + \frac{d^2}{dx^2} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i(x - c_i)S_{c_i}(x) +$$

$$\sum M_i S_{d_i}(x)$$

c. Shearing Force

$$V(x) = R_1 + \frac{d^3}{dx^3} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i S_{c_i}(x)$$

4-7-2 General Solution of a Simple Beam with Two Overhangs

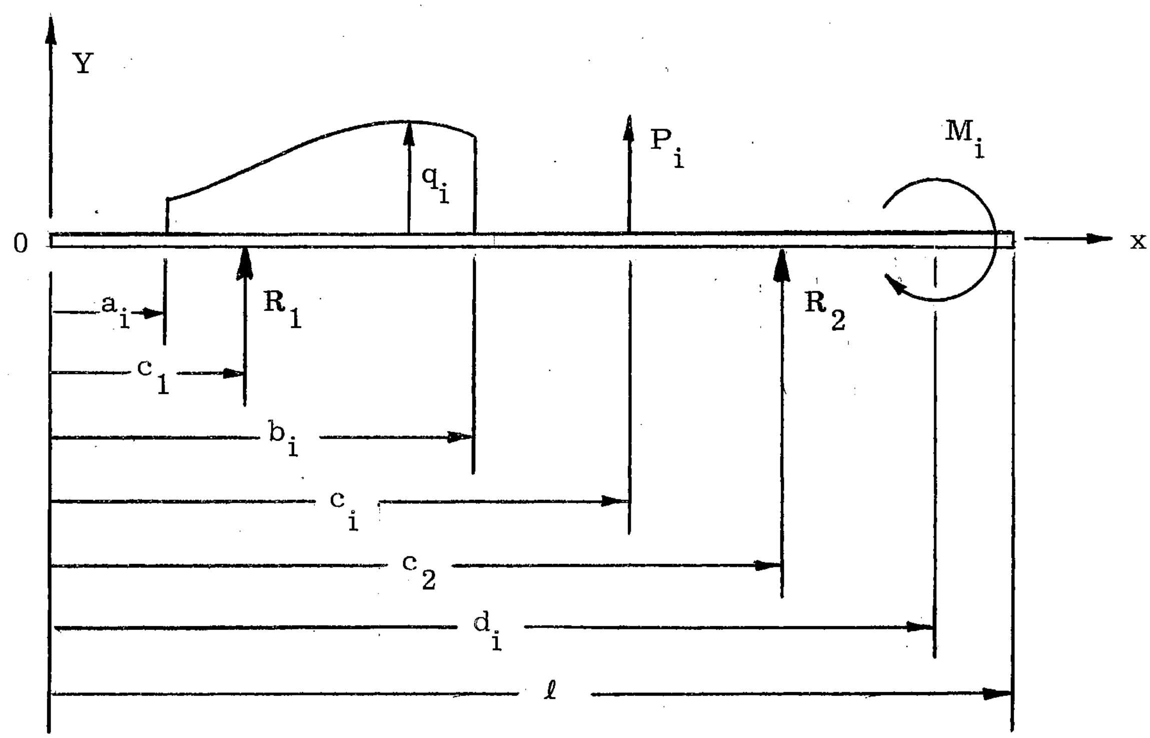


Fig. 4-7-2. Simple Beam with Two Overhangs

A. Boundary Conditions

$$Y''(0) = Y'''(0) = 0$$

$$Y(c_1) = Y(c_2) = 0$$

B. Load Function

$$F(x) = R_1 S'_{c_1}(x) + R_2 S'_{c_2}(x) + \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right] +$$

$$\sum P_i S'_{c_i}(x) + \sum M_i S''_{d_i}(x)$$

Letting $\beta = \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right]$,

$$F(x) = R_1 S'_{c_i}(x) + R_2 S'_{c_2}(x) + \beta + \sum P_i S'_{c_i}(x) + \sum M_i S''_{d_i}(x) .$$

C. Laplace Transform of Load Function

$$f(s) = R_1 e^{-c_1 s} + R_2 e^{-c_2 s} + L\beta + \sum P_i e^{-c_i s} + \sum M_i s e^{-d_i s}$$

D. Inverse Transform of $\frac{1}{EI} \left[\frac{f(s)}{s^4} \right]$

$$\frac{1}{EI} L^{-1} \left[\frac{f(s)}{s^4} \right] = \frac{1}{EI} \left\{ R_1 \frac{(x - c_1)^3}{3!} S_{c_1}(x) + R_2 \frac{(x - c_2)^3}{3!} S_{c_2}(x) + \right.$$

$$L^{-1} \left[\frac{L\beta}{s^4} \right] + \sum P_i \frac{(x - c_i)^3}{3!} S_{c_i}(x) +$$

$$\left. \sum M_i \frac{(x - d_i)^2}{2!} S_{d_i}(x) \right\}$$

E. Substituting (A) and (D) into Eq. (4-2b)

$$Y(x) = Y(0) + Y'(0)x + \frac{1}{EI}\left\{ R_1 \frac{(x-c_1)^3}{3!} S_{c_1}(x) + \right.$$

$$R_2 \frac{(x-c_2)^3}{3!} S_{c_2}(x) + L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i \frac{(x-c_i)^3}{3!} S_{c_i}(x) +$$

$$\left. \sum M_i \frac{(x-d_i)^2}{2!} S_{d_i}(x) \right\}$$

F. Evaluation of Unknown Boundary Conditions and Reactions

$$Y(c_1) = 0 = Y(0) + Y'(0)c_1 + \frac{1}{EI}\left\{ L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_1} + \right.$$

$$\left. \sum P_i \frac{(c_1-c_i)^3}{3!} S_{c_i}(c_1) + \sum M_i \frac{(c_1-d_i)^2}{2!} S_{d_i}(c_1) \right\}$$

$$Y(c_2) = 0 = Y(0) + Y'(0)c_2 + \frac{1}{EI}\left\{ R_1 \frac{(c_2-c_1)^3}{3!} + \right.$$

$$L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_2} + \sum P_i \frac{(c_2-c_i)^3}{3!} S_{c_i}(c_2) +$$

$$\left. \sum M_i \frac{(c_2-c_i)^2}{2!} S_{d_i}(c_2) \right\}$$

Solving simultaneously, the boundary conditions are:

$$Y(0) = \frac{1}{(c_2 - c_1)EI}\left\{ c_1 R_1 \frac{(c_2 - c_1)^3}{3!} + c_1 L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_2} - \right.$$

$$c_2 L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_1} + c_1 \sum P_i \frac{(c_2 - c_i)^3}{3!} S_{c_i}(c_2) -$$

$$c_2 \sum P_i \frac{(c_1 - c_i)^3}{3!} S_{c_i}(c_1) + c_1 \sum M_i \frac{(c_2 - d_i)^2}{2!} S_{d_i}(c_2) -$$

$$\left. c_2 \sum M_i \frac{(c_1 - d_i)^2}{2!} S_{d_i}(c_1) \right\}$$

$$Y'(0) = \frac{1}{(c_1 - c_2)EI}\left\{ R_1 \frac{(c_2 - c_1)^3}{3!} + L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_2} - \right.$$

$$L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_1} + \sum P_i \frac{(c_2 - c_i)^3}{3!} S_{c_i}(c_2) -$$

$$\sum P_i \frac{(c_1 - c_i)^3}{3!} S_{c_i}(c_1) + \sum M_i \frac{(c_2 - d_i)^2}{2!} S_{d_i}(c_2) -$$

$$\left. \sum M_i \frac{(c_1 - d_i)^2}{2!} S_{d_i}(c_1) \right\}$$

The reactions $\mathbf{R}_1$ and $\mathbf{R}_2$ can be determined by statics.

G. General Elastic/Deflection Curve Equation

The same as (E) where R_1, R_2, Y(0), and Y'(0) are the values found in (F).

H. Slope, Bending Moment, and Shearing Force Equations

a. Slope

$$\phi(x) = \frac{1}{(c_1 - c_2)EI} \left\{ R_1 \frac{(c_2 - c_1)^3}{3!} + L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_2} - L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_1} + \sum P_i \frac{(c_2 - c_i)^3}{3!} S_{c_i}(c_2) - \sum P_i \frac{(c_1 - c_i)^3}{3!} S_{c_i}(c_1) + \sum M_i \frac{(c_2 - d_i)^2}{2!} S_{d_i}(c_2) - \sum M_i \frac{(c_1 - d_i)^2}{2!} S_{d_i}(c_1) \right\} + \frac{1}{EI} \left\{ R_1 \frac{(x - c_1)^2}{2} S_{c_1}(x) + R_2 \frac{(x - c_2)^2}{2} S_{c_2}(x) + \frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i \frac{(x - c_i)^2}{2} S_{c_i}(x) + \sum M_i (x - d_i) S_{d_i}(x) \right\}$$

b. Bending Moment

$$M_b(x) = R_1(x - c_1)S_{c_1}(x) + R_2(x - c_2)S_{c_2}(x) +$$

$$\frac{d^2}{dx^2} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i(x - c_i)S_{c_i}(x) +$$

$$\sum M_i S_{d_i}(x)$$

c. Shearing Force

$$V(x) = R_1 S_{c_1}(x) + R_2 S_{c_2}(x) + \frac{d^3}{dx^3} L^{-1}\left[\frac{L\beta}{s^4}\right] +$$

$$\sum P_i S_{c_i}(x)$$

4-7-3 General Solution of a Simple Beam with One Overhang

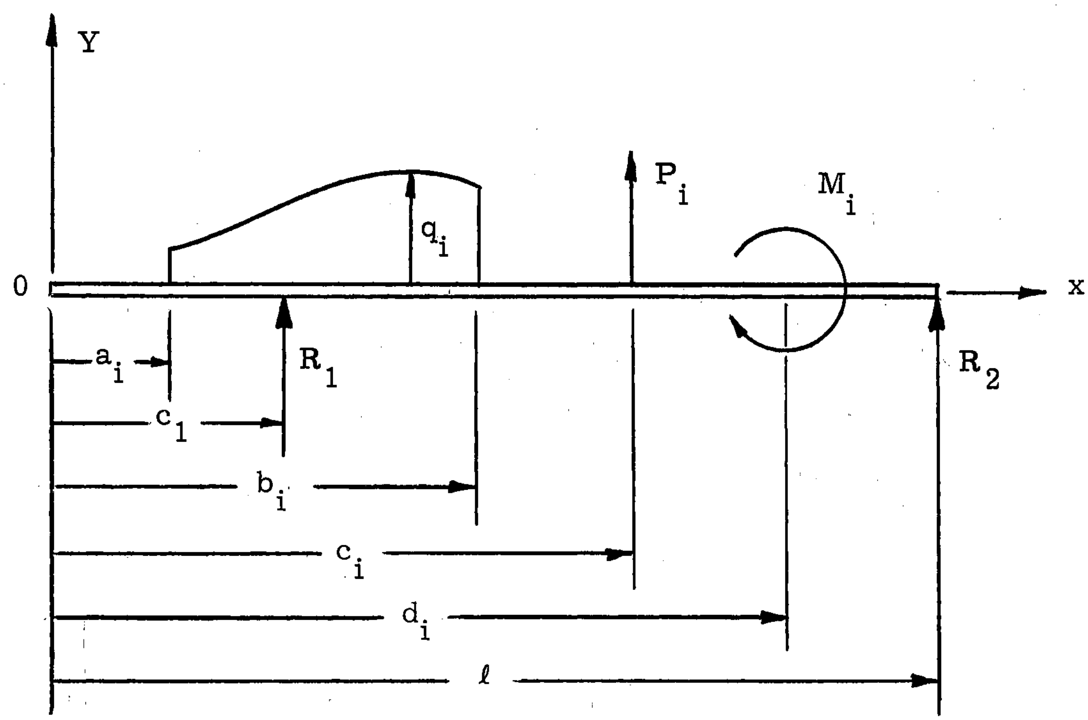


Fig. 4-7-3. Simple Beam with One Overhang

A. Boundary Conditions

$$Y''(0) = Y'''(0) = 0$$

$$Y(c_1) = Y(\ell) = 0$$

B. Load Function

$$F(x) = R_1 S'_{c_1}(x) + \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right] + \sum P_i S'_{c_i}(x) +$$

$$\sum M_i S''_{d_i}(x)$$

Letting $\beta = \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right]$,

$$F(x) = R_1 S'_{c_i}(x) + \beta + \sum P_i S'_{c_i}(x) + \sum M_i S''_{d_i}(x) \; .$$

C. Laplace Transform of Load Function

$$f(s) = R_1 e^{-c_1 s} + L\beta + \sum P_i e^{-c_i s} + \sum M_i s e^{-d_i s}$$

D. Inverse Transform of $\frac{1}{EI}\left[\frac{f(s)}{s^4}\right]$

$$\frac{1}{EI} L^{-1}\left[\frac{f(s)}{s^4}\right] = \frac{1}{EI}\left\{ R_1 \frac{(x - c_1)^3}{3!} S_{c_1}(x) + L^{-1}\left[\frac{L\beta}{s^4}\right] + \right.$$

$$\left. \sum P_i \frac{(x - c_i)^3}{3!} S_{c_i}(x) + \sum M_i \frac{(x - d_i)^2}{2!} S_{d_i}(x) \right\}$$

E. Substituting (A) and (D) into Eq. (4-2b)

$$Y(x) = Y(0) + Y'(0)x + \frac{1}{EI}\left\{ R_1 \frac{(x - c_1)^3}{3!} S_{c_1}(x) + \right.$$

$$L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i \frac{(x - c_i)^3}{3!} S_{c_i}(x) + \sum M_i \frac{(x - d_i)^2}{2!} S_{d_i}(x) \Bigg\}$$

F. Evaluation of Unknown Boundary Conditions and Reactions

$$Y(c_1) = 0 = Y(0) + Y'(0)c_1 + \frac{1}{EI}\Bigg\{ L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_1} +$$

$$\sum P_i \frac{(c_1 - c_i)^3}{3!} S_{c_i}(c_1) + \sum M_i \frac{(c_1 - d_i)^2}{2!} S_{d_i}(c_1) \Bigg\}$$

$$Y(\ell) = 0 = Y(0) + Y'(0)\,\ell + \frac{1}{EI}\Bigg\{ R_1 \frac{(\ell - c_1)^3}{3!} + L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} +$$

$$\sum P_i \frac{(\ell - c_i)^3}{3!} + \sum M_i \frac{(\ell - d_i)^2}{2!} \Bigg\}$$

Solving simultaneously, the boundary conditions are:

$$Y(0) = \frac{1}{(\ell - c_1)\,EI}\Bigg\{ c_1 R_1 \frac{(\ell - c_1)^3}{3!} - \ell\, L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_1} +$$

$$c_1 L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} - \ell \sum P_i \frac{(c_1 - c_i)^3}{3!} S_{c_i}(c_1) +$$

$$c_1 \sum P_i \frac{(\ell - c_i)^3}{3!} - \ell \sum M_i \frac{(c_1 - d_i)^2}{2!} S_{d_i}(c_1) +$$

$$c_1 \sum M_i \frac{(\ell - d_i)^2}{2!} \Bigg\}$$

$$Y'(0) = \frac{1}{(c_1 - \ell)EI}\left\{R_1 \frac{(\ell - c_1)^3}{3!} - L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_1} + \right.$$

$$L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} - \sum P_i \frac{(c_1 - c_i)^3}{3!} S_{c_i}(c_1) +$$

$$\sum P_i \frac{(\ell - c_i)^3}{3!} - \sum M_i \frac{(c_1 - d_i)^2}{2!} S_{d_i}(c_1) +$$

$$\left. \sum M_i \frac{(\ell - d_i)^2}{2!} \right\}$$

The reaction R_1 can be determined by statics.

G. General Elastic/Deflection Curve Equation

The same as (E) where R_1, $Y(0)$, and $Y'(0)$ are the values found in (F).

H. Slope, Bending Moment and Shearing Force Equations

a) Slope

$$\phi(x) = Y'(0) + \frac{1}{EI}\left\{R_1 \frac{(x - c_1)^2}{2} S_{c_1}(x) + \frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right] + \right.$$

$$\left. \sum P_i \frac{(x - c_i)^2}{2} S_{c_i}(x) + \sum M_i (x - d_i) S_{d_i}(x) \right\}$$

b) Bending Moment

$$M_b(x) = R_1 (x - c_1) S_{c_1}(x) + \frac{d^2}{dx^2} L^{-1}\left[\frac{L\beta}{s^4}\right] +$$

$$\sum P_i (x - c_i) S_{c_i}(x) + \sum M_i S_{d_i}(x)$$

c) Shearing Force

$$V(x) = R_1 S_{c_1}(x) + \frac{d^3}{dx^3} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i S_{c_i}(x)$$

4-7-4. General Solution of a Cantilever Beam

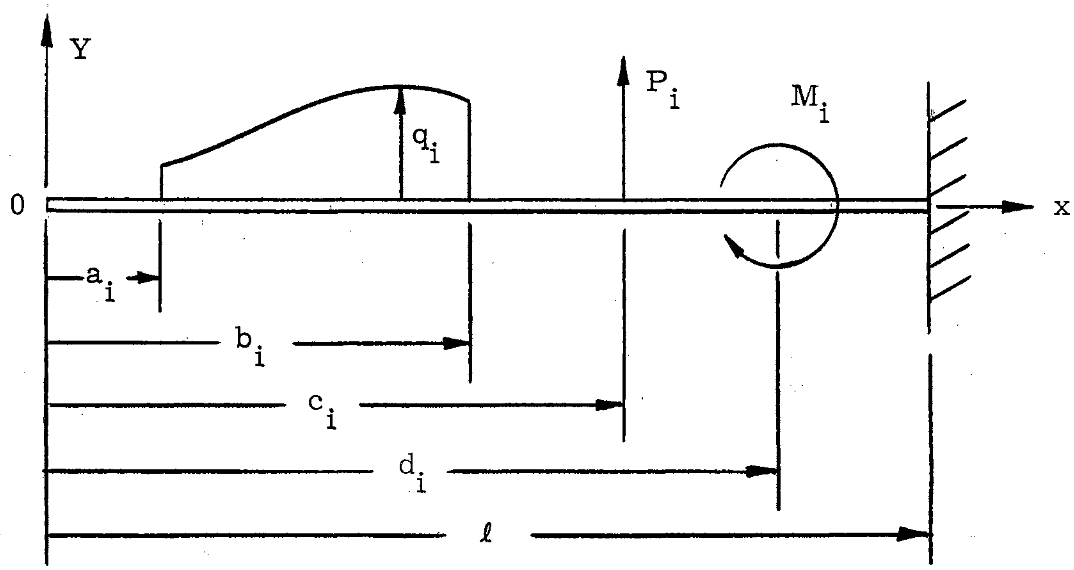


Fig. 4-7-4. Cantilever Beam

A. Boundary Conditions

$$Y''(0) = Y'''(0) = 0$$
$$Y(\ell) = Y'(\ell) = 0$$

B. Load Function

$$F(x) = \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x)\right] + \sum P_i S'_{c_i}(x) + \sum M_i S''_{d_i}(x)$$

Letting $\beta = \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right]$

$$F(x) = \beta + \sum P_i S'_{c_i}(x) + \sum M_i S''_{d_i}(x)$$

C. Laplace Transform of Load Function

$$f(s) = L\beta + \sum P_i e^{-c_i s} + \sum M_i s e^{-d_i s}$$

D. Inverse Transform of $\frac{1}{EI}\left[\frac{f(s)}{s^4}\right]$

$$\frac{1}{EI} L^{-1}\left[\frac{f(s)}{s^4}\right] = \frac{1}{EI}\left\{ L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i \frac{(x - c_i)^3}{3!} S_{c_i}(x) + \sum M_i \frac{(x - d_i)^2}{2!} S_{d_i}(x) \right\}$$

E. Substituting (A) and (D) into Eq. (4-2b)

$$Y(x) = Y(0) + Y'(0)x + \frac{1}{EI}\left\{ L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i \frac{(x - c_i)^3}{3!} S_{c_i}(x) + \sum M_i \frac{(x - d_i)^2}{2!} S_{d_i}(x) \right\}$$

F. Evaluation of Unknown Boundary Conditions and Reactions

$$Y(\ell) = 0 = Y(0) + Y'(0)\ell + \frac{1}{EI}\left\{L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \right.$$

$$\left. \sum P_i \frac{(\ell - c_i)^3}{3!} + \sum M_i \frac{(\ell - d_i)^2}{2!}\right\}$$

$$Y'(\ell) = 0 = Y'(0) + \frac{1}{EI}\left\{\left(\frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} + \right.$$

$$\left. \sum P_i \frac{(\ell - c_i)^2}{2} + \sum M_i (\ell - d_i)\right\}$$

Solving simultaneously, the boundary conditions are:

$$Y(0) = \frac{1}{EI}\left\{\left(\frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} + \sum P_i \frac{(\ell - c_i)^2}{2} + \sum M_i (\ell - d_i)\right\}\ell -$$

$$\frac{1}{EI}\left\{L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \sum P_i \frac{(\ell - c_i)^3}{3!} + \sum M_i \frac{(\ell - d_i)^2}{2!}\right\}$$

$$Y'(0) = -\frac{1}{EI}\left\{\left(\frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} + \sum P_i \frac{(\ell - c_i)^2}{2} + \sum M_i (\ell - d_i)\right\}$$

G. General Elastic/Deflection Curve Equation

The same as (E) where $Y(0)$ and $Y'(0)$ are the values found in (F).

H. Slope, Bending Moment and Shearing Force Equations

a) Slope

$$\phi(x) = Y'(0) + \frac{1}{EI}\left\{\frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i \frac{(x - c_i)^2}{2} S_{c_i}(x) + \sum M_i (x - d_i) S_{d_i}(x)\right\}$$

b) Bending Moment

$$M_b(x) = \frac{d^2}{dx^2} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i (x - c_i) S_{c_i}(x) + \sum M_i S_{d_i}(x)$$

c) Shearing Force

$$V(x) = \frac{d^3}{dx^3} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i S_{c_i}(x)$$

4-7-5. General Solution of a Propped Cantilever Beam

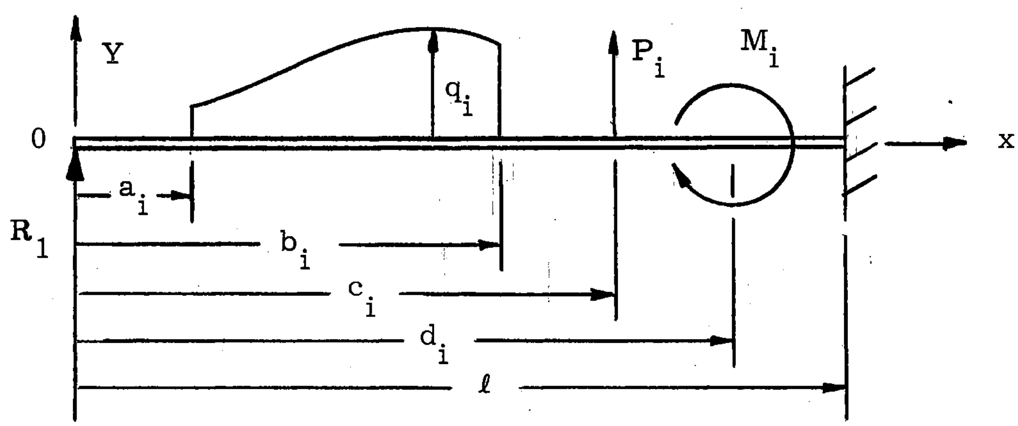


Fig. 4-7-5. Propped Cantilever Beam

A. Boundary Conditions

$$Y(0) = Y''(0) = Y'''(0) = 0$$
$$Y(\ell) = Y'(\ell) = 0$$

B. Load Function

$$F(x) = R_1 S'_o(x) + \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x)\right] + \sum P_i S'_{c_i}(x) + \sum M_i S''_{d_i}(x)$$

$$\text{Letting } \beta = \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x)\right]$$

$$F(x) = R_1 S'_o(x) + \beta + \sum P_i S'_{c_i}(x) + \sum M_i S''_{d_i}(x)$$

C. Laplace Transform of Load Function

$$f(s) = R_1 + L\beta + \sum P_i e^{-c_i s} + \sum M_i s e^{-d_i s}$$

D. Inverse Transform of $\frac{1}{EI}\left[\frac{f(s)}{s^4}\right]$

$$\frac{1}{EI} L^{-1}\left[\frac{f(s)}{s^4}\right] = \frac{1}{EI}\left\{\frac{R_1 x^3}{3!} + L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i \frac{(x - c_i)^3}{3!} S_{c_i}(x) + \sum M_i \frac{(x - d_i)^2}{2!} S_{d_i}(x)\right\}$$

E. Substituting (A) and (D) into Eq. (4-2b)

$$Y(x) = Y'(0)\,x + \frac{1}{EI}\left\{\frac{R_1 x^3}{3!} + L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i \frac{(x - c_i)^3}{3!} S_{c_i}(x) + \sum M_i \frac{(x - d_i)^2}{2!} S_{d_i}(x)\right\}$$

F. Evaluation of Unknown Boundary Conditions and Reactions

$$Y(\ell) = 0 = Y'(0)\,\ell + \frac{1}{EI}\left\{\frac{R_1 \ell^3}{3!} + L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \sum P_i \frac{(\ell - c_i)^3}{3!} + \sum M_i \frac{(\ell - d_i)^2}{2!}\right\}$$

$$Y'(\ell) = 0 = Y'(0) + \frac{1}{EI}\left\{\frac{R_1 \ell^2}{2} + \left(\frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} + \sum P_i \frac{(\ell - c_i)^2}{2} + \sum M_i (\ell - d_i)\right\}$$

Solving simultaneously, R_1 and $Y'(0)$ are:

$$R_1 = -\frac{3}{\ell^3}\left\{\ell\left(\frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} - L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \ell\sum P_i \frac{(\ell - c_i)^2}{2} - \sum P_i \frac{(\ell - c_i)^3}{3!} + \ell\sum M_i (\ell - d_i) - \sum M_i \frac{(\ell - d_i)^2}{2!}\right\}$$

$$Y'(0) = \frac{3}{2EI}\left\{\frac{1}{3}\left(\frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} - \frac{1}{\ell} L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \right.$$

$$\sum P_i \frac{(\ell - c_i)^2}{3!} - \sum P_i \frac{(\ell - c_i)^3}{3!\ell} + \sum M_i \frac{(\ell - d_i)}{3} -$$

$$\left. \sum M_i \frac{(\ell - d_i)^2}{2!\ \ell} \right\}$$

G. General Elastic/Deflection Curve Equation

The same as (E) where R_1 and $Y'(0)$ are the values found in (F).

H. Slope, Bending Moment and Shearing Force Equations

a) Slope

$$\phi(x) = Y'(0) + \frac{1}{EI}\left\{\frac{R_1 x^2}{2} + \frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right] + \right.$$

$$\left. \sum P_i \frac{(x - c_i)^2}{2} S_{c_i}(x) + \sum M_i (x - d_i) S_{d_i}(x) \right\}$$

b) Bending Moment

$$M_b(x) = R_1 x + \frac{d^2}{dx^2} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i (x - c_i) S_{c_i}(x) +$$

$$\sum M_i S_{d_i}(x)$$

c) Shearing Force

$$V(x) = R_1 + \frac{d^3}{dx^3} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i S_{c_i}(x)$$

4-7-6. General Solution of a Propped Cantilever Beam with an Overhang

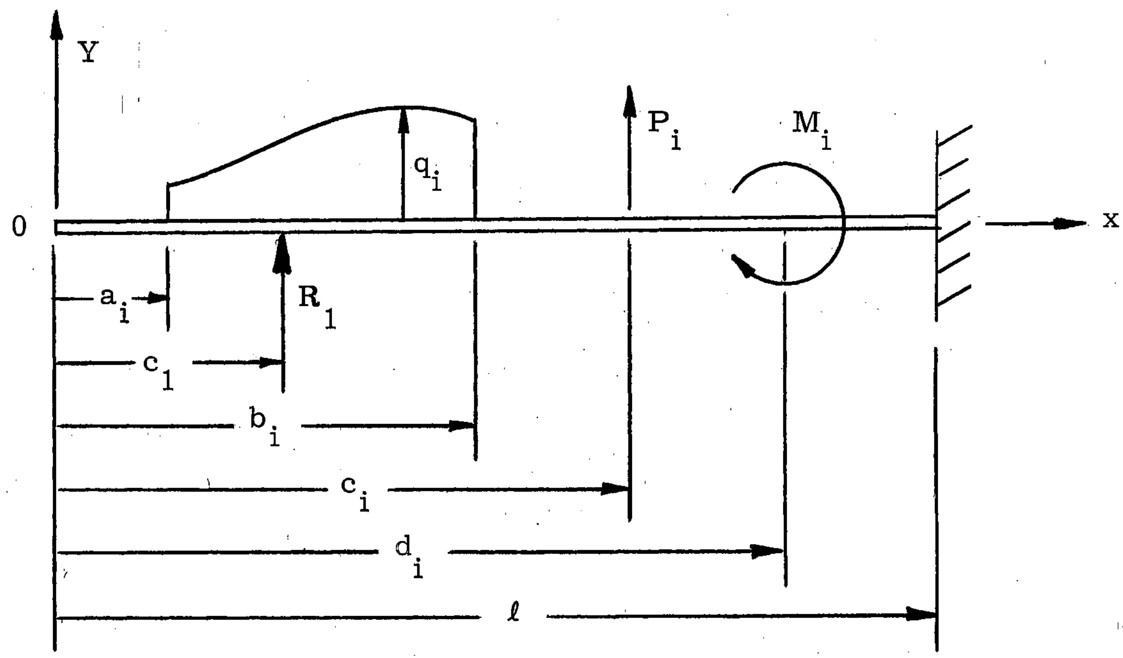


Fig. 4-7-6. Propped Cantilever Beam with an Overhang

A. Boundary Conditions

$Y''(0) = Y'''(0) = 0$
$Y(c_1) = Y(\ell) = Y'(\ell) = 0$

B. Load Function

$$F(x) = R_1 S'_{c_1}(x) + \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x)\right] + \sum P_i S'_{c_i}(x) + \sum M_i S''_{d_i}(x)$$

Letting $\beta = \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right]$

$$F(x) = R_1 S'_{c_1}(x) + \beta + \sum P_i S'_{c_i}(x) + \sum M_i S''_{d_i}(x)$$

C. Laplace Transform of Load Function

$$f(s) = R_1 e^{-c_1 s} + L\beta + \sum P_i e^{-c_1 s} + \sum M_i s e^{-d_i s}$$

D. Inverse Transform of $\frac{1}{EI}\left[\frac{f(s)}{s^4}\right]$

$$\frac{1}{EI} L^{-1}\left[\frac{f(s)}{s^4}\right] = \frac{1}{EI}\left\{ R_1 \frac{(x - c_1)^3}{3!} S_{c_1}(x) + L^{-1}\left[\frac{L\beta}{s^4}\right] + \right.$$

$$\left. \sum P_i \frac{(x - c_i)^3}{3!} S_{c_i}(x) + \sum M_i \frac{(x - d_i)^2}{2!} S_{d_i}(x) \right\}$$

E. Substituting (A) and (D) into Eq. (4-2b)

$$Y(x) = Y(0) + Y'(0)\,x + \frac{1}{EI}\left\{ R_1 \frac{(x - c_1)^3}{3!} S_{c_1}(x) + L^{-1}\left[\frac{L\beta}{s^4}\right] + \right.$$

$$\left. \sum P_i \frac{(x - c_i)^3}{3!} S_{c_i}(x) + \sum M_i \frac{(x - d_i)^2}{2!} S_{d_i}(x) \right\}$$

F. Evaluation of Unknown Boundary Conditions and Reactions

$$Y(c_1) = 0 = Y(0) + Y'(0)\, c_1 + \frac{1}{EI}\left\{ L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_1} + \right.$$

$$\left. \sum P_i \frac{(c_1 - c_i)^3}{3!} S_{c_i}(c_1) + \sum M_i \frac{(c_1 - d_i)^2}{2!} S_{d_i}(c_1) \right\}$$

$$Y(\ell) = 0 = Y(0) + Y'(0)\, \ell + \frac{1}{EI}\left\{ R_1 \frac{(\ell - c_1)^3}{3!} + L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \right.$$

$$\left. \sum P_i \frac{(\ell - c_i)^3}{3!} + \sum M_i \frac{(\ell - d_i)^2}{2!} \right\}$$

$$Y'(\ell) = 0 = Y'(0) + \frac{1}{EI}\left\{ R_1 \frac{(\ell - c_1)^2}{2} + \left(\frac{d}{dx}\, L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} + \right.$$

$$\left. \sum P_i \frac{(\ell - c_i)^2}{2} + \sum M_i (\ell - d_i) \right\}$$

Letting

$$K_1 = \frac{1}{EI}\left\{ L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_1} + \sum P_i \frac{(c_1 - c_i)^3}{3!} S_{c_i}(c_1) + \right.$$

$$\left. \sum M_i \frac{(c_1 - d_i)^2}{2!} S_{d_i}(c_1) \right\}$$

$$K_2 = \frac{1}{EI}\left\{ L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \sum P_i \frac{(\ell - c_i)^3}{3!} + \right.$$

$$\left. \sum M_i \frac{(\ell - d_i)^2}{2!} \right\}$$

$$K_3 = \frac{1}{EI}\left\{\left(\frac{d}{dx} \; L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} + \sum P_i \frac{(\ell - c_i)^2}{2} + \right.$$

$$\left. \sum M_i (\ell - d_i) \right\}$$

Then

$$Y(0) + Y'(0)\, c_1 = -K_1$$

$$Y(0) + Y'(0)\,\ell + R_1 \frac{(\ell - c_1)^3}{3!\, EI} = -K_2$$

$$Y'(0) + R_1 \frac{(\ell - c_1)^2}{2\, EI} = -K_3$$

Solving by Determinates; $Y(0)$, $Y'(0)$, and R_1 are:

$$
Y(0) = \frac{\begin{vmatrix} -K_1 & c_1 & 0 \\ -K_2 & \ell & \dfrac{(\ell - c_1)^3}{3!\,EI} \\ -K_3 & 1 & \dfrac{(\ell - c_1)^2}{2\,EI} \end{vmatrix}}{\begin{vmatrix} 1 & c_1 & 0 \\ 1 & \ell & \dfrac{(\ell - c_1)^3}{3!\,EI} \\ 0 & 1 & \dfrac{(\ell - c_1)^2}{2\,EI} \end{vmatrix}}
$$

$$
= \frac{1}{2(\ell - c_1)} \left[3\,K_2 c_1 + K_3 c_1 \,(c_1 - \ell) - K_1\,(2\ell + c_1) \right]
$$

$$
= \frac{1}{2\,(\ell - c_1)\,EI} \left[3 c_1 \left\{ L^{-1} \left[\frac{L\beta}{s^4} \right]_{x=\ell} + \sum P_i \frac{(\ell - c_i)^3}{3!} + \right. \right.
$$

$$
\left. \sum M_i \frac{(\ell - d_i)^2}{2!} \right\} +
$$

$$
c_1\,(c_1 - \ell) \left\{ \left(\frac{d}{dx}\; L^{-1} \left[\frac{L\beta}{s^4} \right] \right)_{x=\ell} + \sum P_i \frac{(\ell - c_i)^2}{2} + \right.
$$

$$\sum M_i \, (\ell - d_i) \Bigg\} \quad -$$

$$(2\ell + c_1) \left\{ L^{-1} \left[\frac{L\beta}{s^4} \right]_{x=c_1} + \sum P_i \, \frac{(c_1 - c_i)^3}{3!} \, S_{c_i}(c_1) + \right.$$

$$\left. \sum M_i \, \frac{(c_1 - d_i)^2}{2!} \, S_{d_i}(c_1) \right\} \Bigg]$$

$$Y'(0) = \frac{\begin{vmatrix} 1 & -K_1 & 0 \\ 1 & -K_2 & \dfrac{(\ell - c_1)^3}{3!\,EI} \\ 0 & -K_3 & \dfrac{(\ell - c_1)^2}{2\,EI} \end{vmatrix}}{\dfrac{(\ell - c_1)^3}{3\,EI}}$$

$$= \frac{1}{2\,(\ell - c_1)} \left[3\,K_1 - 3\,K_2 + K_3\,(\ell - c_1) \right]$$

$$= \frac{1}{2(\ell - c_1)\,EI} \left[3 \left\{ L^{-1} \left[\frac{L\beta}{s^4} \right]_{x=c_1} + \sum P_i \, \frac{(c_1 - c_i)^3}{3!} \, S_{c_i}(c_1) + \right. \right.$$

$$\left. \sum M_i \, \frac{(c_1 - d_i)^2}{2!} \, S_{d_i}(c_1) \right\} \quad -$$

$$3\left\{L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell}+\sum P_i\frac{(\ell-c_i)^3}{3!}+\sum M_i\frac{(\ell-d_i)^2}{2!}\right\}+$$

$$\left.(\ell-c_1)\left\{\left(\frac{d}{dx}\;L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell}+\sum P_i\frac{(\ell-c_i)^2}{2}+\sum M_i(\ell-d_i)\right\}\right]$$

$$R_1=\frac{\begin{vmatrix}1 & c_1 & -K_1\\ 1 & \ell & -K_2\\ 0 & 1 & -K_3\end{vmatrix}}{\dfrac{(\ell-c_1)^3}{3\,EI}}=\frac{-K_1+K_2+K_3(c_1-\ell)}{\dfrac{(\ell-c_1)^3}{3\,EI}}$$

$$=\frac{3}{(\ell-c_1)^3}\left[-\left\{L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_1}+\sum P_i\frac{(c_1-c_i)^3}{3!}S_{c_i}(c_1)+\right.\right.$$

$$\left.\sum M_i\frac{(c_1-d_i)^2}{2!}S_{d_i}(c_1)\right\}+$$

$$\left\{L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell}+\sum P_i\frac{(\ell-c_i)^3}{3!}+\sum M_i\frac{(\ell-d_i)^2}{2!}\right\}+$$

$$\left.(c_1-\ell)\left\{\left(\frac{d}{dx}\;L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell}+\sum P_i\frac{(\ell-c_i)^2}{2}+\sum M_i(\ell-d_i)\right\}\right]$$

G. General Elastic/Deflection Curve Equation

The same as (E) where Y(0), Y'(0), and R_1 are the values found in (F).

H. Slope, Bending Moment and Shearing Force Equation

a) Slope

$$\phi(x) = Y'(0) + \frac{1}{EI}\left\{ R_1 \frac{(x - c_1)^2}{2} S_{c_1}(x) + \frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right] + \right.$$

$$\left. \sum P_i \frac{(x - c_i)^2}{2} S_{c_i}(x) + \sum M_i (x - d_i) S_{d_i}(x) \right\}$$

b) Bending Moment

$$M_b(x) = R_1 (x - c_1) S_{c_1}(x) + \frac{d^2}{dx^2} L^{-1}\left[\frac{L\beta}{s^4}\right] +$$

$$\sum P_i (x - c_i) S_{c_i}(x) + \sum M_i S_{d_i}(x)$$

c) Shearing Force

$$V(x) = R_1 S_{c_1}(x) + \frac{d^3}{dx^3} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i S_{c_i}(x)$$

4-7-7. General Solution of a Fixed Beam

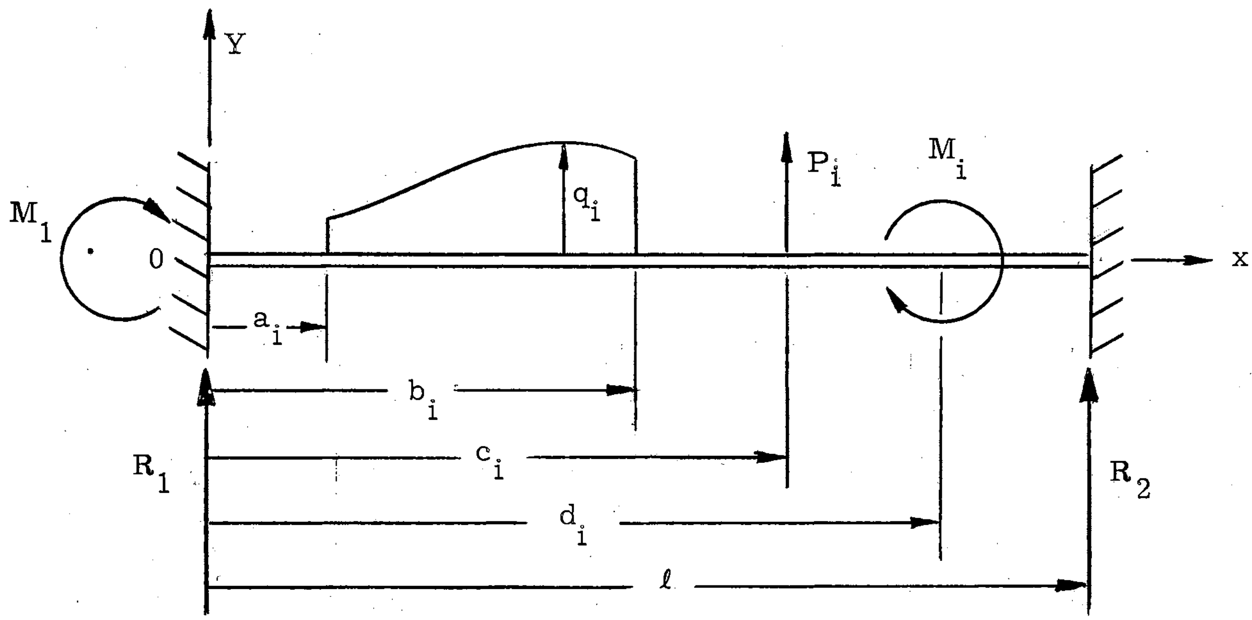


Fig. 4-7-7. Fixed Beam

A. Boundary Conditions

$$Y(0) = Y'(0) = Y''(0) = Y'''(0) = 0$$
$$Y(\ell) = Y'(\ell) = 0$$

B. Load Function

$$F(x) = R_1 S_o'(x) + M_1 S_o''(x) + \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right] +$$

$$\sum P_i S_{c_i}'(x) + \sum M_i S_{d_i}''(x)$$

Letting $\beta = \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right]$

$$F(x) = R_1 S'_o(x) + M_1 S''_o(x) + \beta + \sum P_i S'_{c_i}(x) + \sum M_i S''_{d_i}(x)$$

C. Laplace Transform of Load Function

$$f(s) = R_1 + M_1 s + L\beta + \sum P_i e^{-c_i s} + \sum M_i s e^{-d_i s}$$

D. Inverse Transform of $\frac{1}{EI}\left[\frac{f(s)}{s^4}\right]$

$$\frac{1}{EI} L^{-1}\left[\frac{f(s)}{s^4}\right] = \frac{1}{EI}\left\{\frac{R_1 x^3}{3!} + \frac{M_1 x^2}{2!} + L^{-1}\left[\frac{L\beta}{s^4}\right] + \right.$$

$$\left.\sum P_i \frac{(x - c_i)^3}{3!} S_{c_i}(x) + \sum M_i \frac{(x - d_i)^2}{2!} S_{d_i}(x)\right\}$$

E. Substituting (A) and (D) into Eq. (4-2b)

$$Y(x) = \frac{1}{EI}\left\{\frac{R_1 x^3}{3!} + \frac{M_1 x^2}{2!} + L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i \frac{(x - c_i)^3}{3!} S_{c_i}(x) + \right.$$

$$\left.\sum M_i \frac{(x - d_i)^2}{2!} S_{d_i}(x)\right\}$$

F. Evaluation of Unknown Boundary Conditions and Reactions

$$Y(\ell) = 0 = \frac{1}{EI}\left\{\frac{R_1\ell^3}{3!} + \frac{M_1\ell^2}{2!} + L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \right.$$

$$\left.\sum P_i \frac{(\ell - c_i)^3}{3!} + \sum M_i \frac{(\ell - d_i)^2}{2!}\right\}$$

$$Y'(\ell) = 0 = \frac{1}{EI}\left\{\frac{R_1\ell^2}{2} + M_1\ell + \left(\frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} + \right.$$

$$\left.\sum P_i \frac{(\ell - c_i)^2}{2} + \sum M_i (\ell - d_i)\right\}$$

Solving simultaneously, R_1 and M_1 are:

$$R_1 = -\frac{12}{\ell^3}\left\{\frac{\ell}{2}\left(\frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} - L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \frac{\ell}{2}\right.$$

$$\sum P_i \frac{(\ell - c_i)^2}{2!} - \sum P_i \frac{(\ell - c_i)^3}{3!} + \frac{\ell}{2}$$

$$\left.\sum M_i (\ell - d_i) - \sum M_i \frac{(\ell - d_i)^2}{2!}\right\}$$

$$M_1 = \frac{6}{\ell^2}\left\{\frac{\ell}{3}\left(\frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} - L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \frac{\ell}{3}\right.$$

$$\sum P_i \frac{(\ell - c_i)^2}{2} - \sum P_i \frac{(\ell - c_i)^3}{3!} + \frac{\ell}{3}$$

$$\sum M_i (\ell - d_i) - \sum M_i \frac{(\ell - d_i)^2}{2!} \Bigg\}$$

G. General Elastic/Deflection Curve Equation

The same as (E) where R_1 and M_1 are the values found in (F).

H. Slope, Bending Moment and Shearing Force Equations

a) Slope

$$\phi (x) = \frac{1}{EI} \Bigg\{ \frac{R_1 x^2}{2} + M_1 x + \frac{d}{dx} L^{-1} \left[\frac{L\beta}{s^4} \right] +$$

$$\sum P_i \frac{(x - c_i)^2}{2} S_{c_i} (x) + \sum M_i (x - d_i) S_{d_i} (x) \Bigg\}$$

b) Bending Moment

$$M_b = R_1 x + M_1 + \frac{d^2}{dx^2} L^{-1} \left[\frac{L\beta}{s^4} \right] +$$

$$\sum P_i (x - c_i) S_{c_i} (x) + \sum M_i S_{d_i} (x)$$

c) Shearing Force

$$V(x) = R_1 + \frac{d^3}{dx^3} L^{-1} \left[\frac{L\beta}{s^4} \right] + \sum P_i S_{c_i} (x)$$

For convenience in solving elementary beam problems, the generalized equations of deflection, slope, moment, and shear for the preceding seven elementary cases have been summarized into handbook form and placed in Appendix A.

As a companion aid, Appendix B contains the Laplace transforms and inverse transforms for several of the more common distributed load systems. When the β factors are more complex than those considered, the principle of convolution can be used to obtain the inverse transformation of $\left[\frac{L\beta}{s^4}\right]$. This method will hold even when the functions are so complex that no formula is available for the analytical integration of the convolution integral, since the convolution of two functions can be evaluated graphically or numerically. In this form, the electronic computer can be used to facilitate the solution.

In addition, a numerical example has been worked out for a simple overhanging beam and is found in Appendix C. This example has been solved by three methods. A classical method (such as area-moment), the procedure of section 4-7-2, and the summarized results of Appendices A and B. The object was to illustrate the use of the formulated procedures and to determine the time advantage, if any, over classical methods.

The times involved to compute the deflection at $x = 11$ ft. were: 35 minutes by area-moment; 29 minutes by section 4-7-2; and 25 minutes by Appendices A and B. The additional time required to find the deflection at another point was found to be an average of 5 minutes by either section 4-7-2 or Appendices A and B, whereas the area-moment method required approximately the same amount of time (35 minutes) as the first point. From these results, the time advantage over classical methods is apparent, especially when additional values are required.

Using the procedures of section 4-7-2 and Appendices A and B, the times required for a complete general analysis (i.e., deflection, slope, moment, and shear equations) were also determined for the example problem. The times were 28 and 23 minutes, respectively. In addition, an average of five minutes each was required to evaluate the deflection, slope, moment, and shear at a point on the beam.

In summary, the specific advantages of the Laplace transform method over classical methods in the solution of elementary beams are:

a. The ease with which complex load systems may be dealt with. The more complex the function is, the greater the advantage becomes.

b. The capability of being able to write the solution as one equation for the entire span, thereby reducing the number of arbitrary constants to be determined.

c. The reduced solution times involved for specific values of deflection, slope, moment, and shear at a particular point. As the number of points increases, the time saved increases considerably over classical methods.

CHAPTER V

PART II - ANALYSIS OF CONTINUOUS BEAMS BY THE LAPLACE TRANSFORM METHOD

5-1 General

The general procedures for the analysis of elementary beams were developed in Chapter IV. It will now be shown that these same rules and procedures apply directly to the analysis of continuous (multiple span) beams.

5-2 General Solution of a Two Span Simple Beam

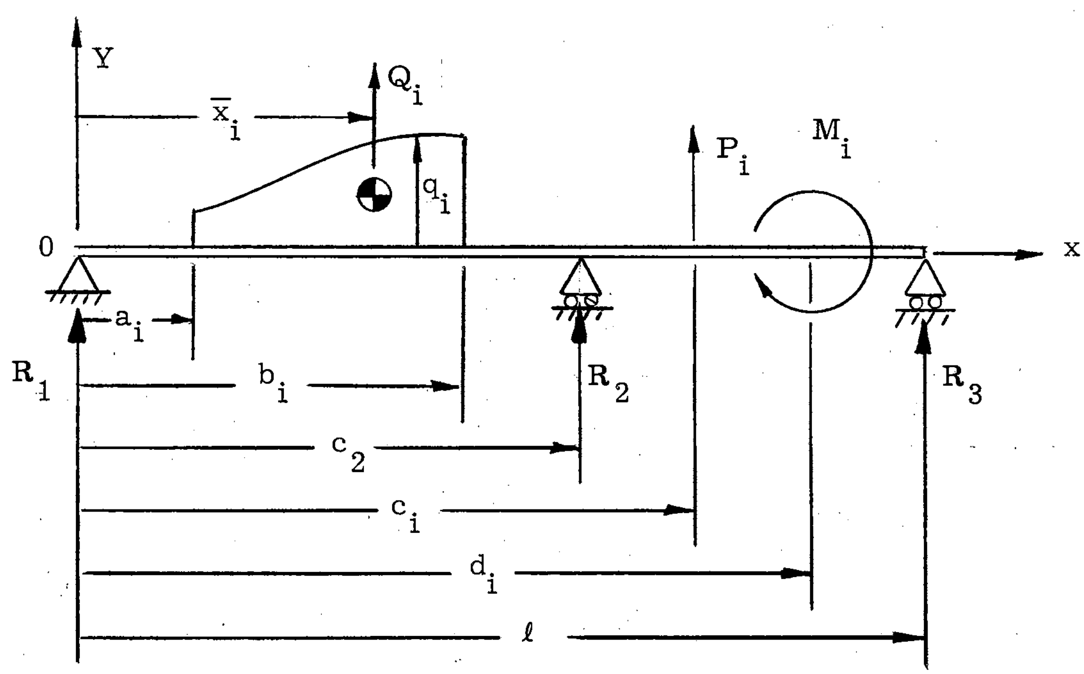


Fig. 5-2. Two Span Simple Beam

A. Boundary Conditions

$$Y(0) = Y''(0) = Y'''(0) = 0$$
$$Y(c_2) = Y(\ell) = 0$$

B. Load Function

$$F(x) = R_1 S'_o(x) + R_2 S'_{c_2}(x) + \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right] +$$

$$\sum P_i S'_{c_i}(x) + \sum M_i S''_{d_i}(x)$$

Letting $\beta = \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right]$

$$F(x) = R_1 S'_o(x) + R_2 S'_{c_2}(x) + \beta + \sum P_i S'_{c_i}(x) +$$

$$\sum M_i S''_{d_i}(x)$$

C. Laplace Transform of Load Function

$$f(s) = R_1 + R_2 e^{-c_2 s} + L\beta + \sum P_i e^{-c_i s} + \sum M_i s e^{-d_i s}$$

D. Inverse Transform of $\frac{1}{EI}\left[\frac{f(s)}{s^4}\right]$

$$\frac{1}{EI} L^{-1}\left[\frac{f(s)}{s^4}\right] = \frac{1}{EI}\left\{\frac{R_1 x^3}{3!} + R_2 \frac{(x - c_2)^3}{3!} S_{c_2}(x) + \right.$$

$$L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i \frac{(x - c_i)^3}{3!} S_{c_i}(x) +$$

$$\left.\sum M_i \frac{(x - d_i)^2}{2!} S_{d_i}(x)\right\}$$

E. Substituting (A) and (D) Into Eq., (4-2b)

$$Y(x) = Y'(0)\, x + \frac{1}{EI}\left\{\frac{R_1 x^3}{3!} + R_2 \frac{(x - c_2)^3}{3!} S_{c_2}(x) +\right.$$

$$L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i \frac{(x - c_i)^3}{3!} S_{c_i}(x) +$$

$$\left.\sum M_i \frac{(x - d_i)^2}{2!} S_{d_i}(x)\right\}$$

F. Evaluation of Unknown Boundary Conditions and Reactions

$$Y(c_2) = 0 = Y'(0)\, c_2 + \frac{1}{EI}\left\{\frac{R_1 c_2^3}{3!} + L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_2} +\right.$$

$$\left.\sum P_i \frac{(c_2 - c_i)^3}{3!} S_{c_i}(c_2) + \sum M_i \frac{(c_2 - d_i)^2}{2!} S_{d_i}(c_2)\right\}$$

$$Y(\ell) = 0 = Y'(0)\ell + \frac{1}{EI}\left\{\frac{R_1 \ell^3}{3!} + R_2 \frac{(\ell - c_2)^3}{3!} + L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} +\right.$$

$$\left.\sum P_i \frac{(\ell - c_i)^3}{3!} + \sum M_i \frac{(\ell - d_i)^2}{2!}\right\}$$

$$\sum M_{x=\ell} = 0 = R_1 \ell + R_2 (\ell - c_2) + \sum Q_i (\ell - \bar{x}_i) + \sum P_i (\ell - c_i) + \sum M_i$$

Letting

$$K_1 = \left\{ L^{-1} \left[\frac{L\beta}{s^4}\right]_{x=c_2} + \sum P_i \frac{(c_2 - c_i)^3}{3!} S_{c_i}(x) + \sum M_i \frac{(c_2 - d_i)^2}{2!} S_{d_i}(c_2) \right\}$$

$$K_2 = \left\{ L^{-1} \left[\frac{L\beta}{s^4}\right]_{x=\ell} + \sum P_i \frac{(\ell - c_i)^3}{3!} + \sum M_i \frac{(\ell - d_i)^2}{2!} \right\}$$

$$K_3 = \left\{ \sum Q_i (\ell - \bar{x}_i) + \sum P_i (\ell - c_i) + \sum M_i \right\}$$

And solving these equations by determinates; R_1, R_2, and $Y'(0)$ are:

$$R_1 = \frac{3}{\ell (c_2)(\ell - c_2)} \left\{ \frac{K_1 \ell}{c_2} - K_2 + K_3 \frac{(\ell - c_2)^2}{6} \right\}$$

$$R_2 = \frac{3}{c_2 (\ell - c_2)^2} \left\{ -\frac{K_1 \ell}{c_2} + K_2 + \frac{K_3}{6} \left[(c_2)^2 - \ell^2 \right] \right\}$$

$$Y'(0) = \frac{1}{2EI\,(c_2)^2\,\ell\,(\ell - c_2)} \left\{ K_1 \ell \left[(\ell - c_2)^2 - \ell^2 \right] + \right.$$

$$\left. K_2\,(c_2)^3 - K_3\,\frac{(c_2)^3\,(\ell - c_2)^2}{6} \right\}$$

G. General Elastic/Deflection Curve Equation

The same as (E) where R_1, R_2, and $Y'(0)$ are the values found in (F).

H. Slope, Bending Moment, and Shearing Force Equations

a) Slope

$$\phi(x) = Y'(0) + \frac{1}{EI} \left\{ \frac{R_1 x^2}{2} + R_2\,\frac{(x - c_2)^2}{2}\,S_{c_2}(x) + \right.$$

$$\frac{d}{dx}\,L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i\,\frac{(x - c_i)^2}{2}\,S_{c_i}(x) +$$

$$\left. \sum M_i\,(x - d_i)\,S_{d_i}(x) \right\}$$

b) Bending Moment

$$M_b(x) = R_1 x + R_2\,(x - c_2)\,S_{c_2}(x) + \frac{d^2}{dx^2}\,L^{-1}\left[\frac{L\beta}{s^4}\right] +$$

$$\sum P_i (x - c_i) S_{c_i}(x) + \sum M_i S_{d_i}(x)$$

c) Shearing Force

$$V(x) = R_1 + R_2 S_{c_2}(x) + \frac{d^3}{dx^3} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i S_{c_i}(x)$$

5-3 General Solution of a Two Span Beam Fixed at One End

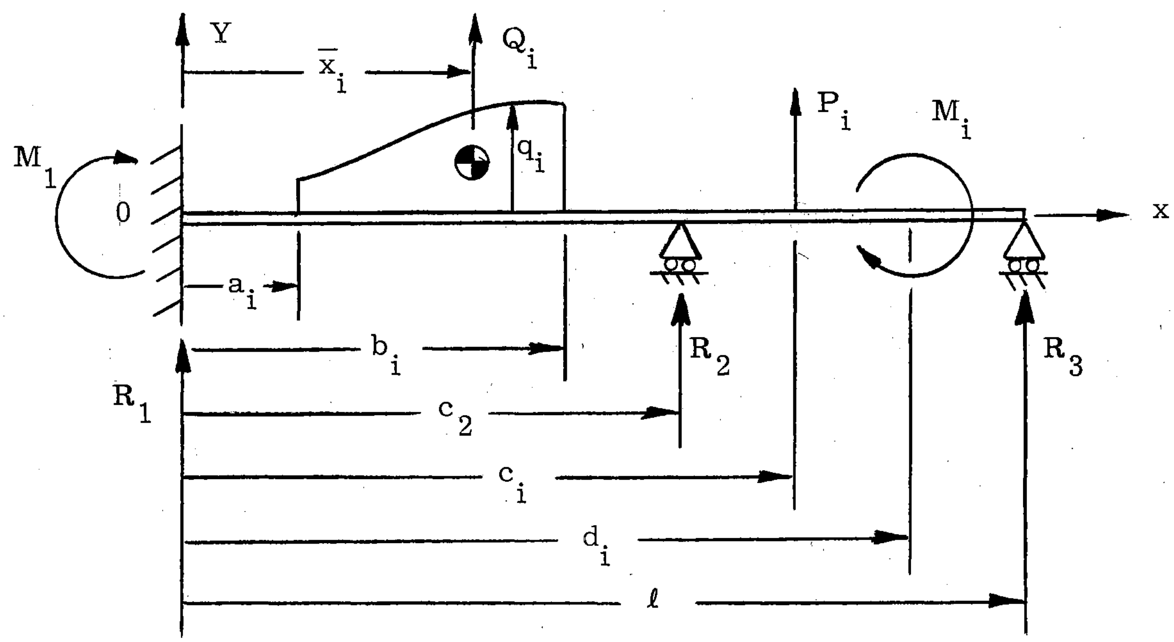


Fig. 5-3. Two Span Beam Fixed at One End

A. Boundary Conditions

$Y(0) = Y'(0) = Y''(0) = Y'''(0) = 0$
$Y(c_2) = Y(\ell) = 0$

B. Load Function

$$F(x) = R_1 S'_o(x) + R_2 S'_{c_2}(x) + M_i S''_o(x) +$$

$$\sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right] + \sum P_i S'_{c_i}(x) +$$

$$\sum M_i S''_{d_i}(x)$$

Letting $\beta = \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right]$

$$F(x) = R_1 S'_o(x) + M_1 S''_o(x) + R_2 S'_{c_2}(x) + \beta +$$

$$\sum P_i S'_{c_i}(x) + \sum M_i S_{d_i}(x)$$

C. Laplace Transform of Load Function

$$f(s) = R_1 + M_1 s + R_2 e^{-c_2 s} + L\beta + \sum P_i e^{-c_i s} + \sum M_i s e^{-d_i s}$$

D. Inverse Transform of $\frac{1}{EI}\left[\frac{f(s)}{s^4}\right]$

$$\frac{1}{EI} L^{-1} \left[\frac{f(s)}{s^4}\right] = \frac{1}{EI} \left\{ \frac{R_1 x^3}{3!} + \frac{M_1 x^2}{2!} + R_2 \frac{(x - c_2)^3}{3!} S_{c_2}(x) + \right.$$

$$L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i \frac{(x - c_i)^3}{3!} S_{c_i}(x) +$$

$$\left.\sum M_i \frac{(x - d_i)^2}{2!} S_{d_i}(x)\right\}$$

E. Substituting (A) and (D) into Eq. (4-2b)

$$Y(x) = \frac{1}{EI}\left\{\frac{R_1 x^3}{3!} + \frac{M_1 x^2}{2!} + R_2 \frac{(x - c_2)^3}{3!} S_{c_2}(x) +\right.$$

$$\left. L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i \frac{(x - c_i)^3}{3!} S_{c_i}(x) + \sum M_i \frac{(x - d_i)^2}{2!} S_{d_i}(x)\right\}$$

F. Evaluation of Unknown Boundary Conditions and Reactions

$$Y(c_2) = 0 = \frac{1}{EI}\left\{\frac{R_1 c_2^3}{3!} + \frac{M_1 c_2^2}{2!} + L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_2} +\right.$$

$$\left.\sum P_i \frac{(c_2 - c_i)^3}{3!} S_{c_i}(c_2) + \sum M_i \frac{(c_2 - d_i)^2}{2!} S_{d_i}(c_2)\right\}$$

$$Y(\ell) = 0 = \frac{1}{EI}\left\{\frac{R_1 \ell^3}{3!} + \frac{M_1 \ell^2}{2!} + R_2 \frac{(\ell - c_2)^3}{3!} +\right.$$

$$\left. L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \sum P_i \frac{(\ell - c_i)^3}{3!} + \sum M_i \frac{(\ell - d_i)^2}{2!}\right\}$$

$$\sum M_{x=\ell} = 0 = R_1\ell + M_1 + R_2(\ell - c_2) + \sum Q_i(\ell - \bar{x}_i) +$$

$$\sum P_i(\ell - c_i) + \sum M_i$$

R_1, R_2 and M_1 can now be solved from these equations by determinates similar to section 5-2 (F). Likewise, the general elastic/deflection curve slope, bending moment, and shearing force equations can be obtained and, therefore, will not be carried out in detail in this section.

5-4 General Solution of a Three Span Simple Beam

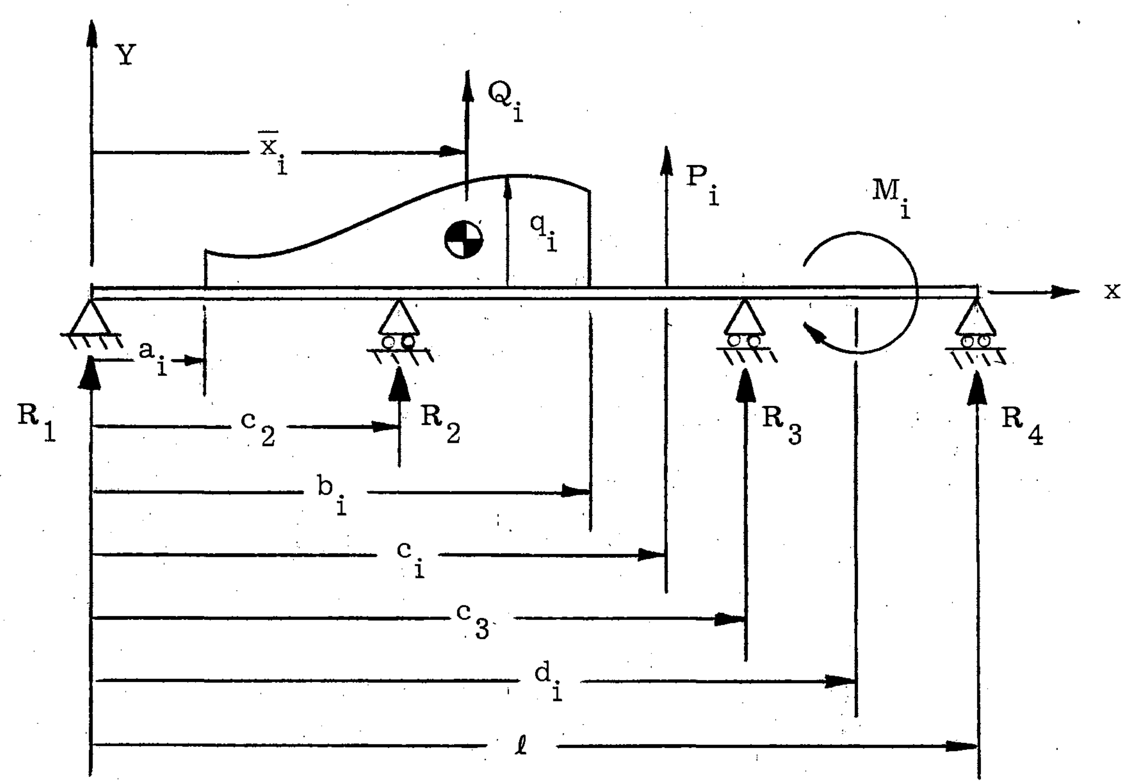


Fig. 5-4. Three Span Simple Beam

A. Boundary Conditions

$$Y(0) = Y''(0) = Y'''(0) = 0$$
$$Y(c_2) = Y(c_3) = Y(\ell) = 0$$

B. Load Function

$$F(x) = R_1 S'_o(x) + R_2 S'_{c_2}(x) + R_3 S'_{c_3}(x) + \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right] + \sum P_i S'_{c_i}(x) + \sum M_i S''_{d_i}(x)$$

Letting $\beta = \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right]$

$$F(x) = R_1 S'_o(x) + R_2 S'_{c_2}(x) + R_3 S'_{c_3}(x) + \beta + \sum P_i S'_{c_i}(x) + \sum M_i S''_{d_i}(x)$$

C. Laplace Transform of Load Function

$$f(s) = R_1 + R_2 e^{-c_2 s} + R_3 e^{-c_3 s} + L\beta + \sum P_i e^{-c_i s} + \sum M_i s e^{-d_i s}$$

D. Inverse Transform of $\frac{1}{EI}\left[\frac{f(s)}{s^4}\right]$

$$\frac{1}{EI}\, L^{-1}\left[\frac{f(s)}{s^4}\right] = \frac{1}{EI}\left\{\frac{R_1 x^3}{3!} + R_2\,\frac{(x - c_2)^3}{3!}\, S_{c_2}(x) + \right.$$

$$R_3\,\frac{(x - c_3)^3}{3!}\, S_{c_3}(x) + L^{-1}\left[\frac{L\beta}{s^4}\right] +$$

$$\left.\sum P_i\,\frac{(x - c_i)^3}{3!}\, S_{c_i}(x) + \sum M_i\,\frac{(x - d_i)^2}{2!}\, S_{d_i}(x)\right\}$$

E. Substituting (A) and (D) into Eq. (4-2b)

$$Y(x) = Y'(0)\, x + \frac{1}{EI}\left\{\frac{R_1 x^3}{3!} + R_2\,\frac{(x - c_2)^3}{3!}\, S_{c_2}(x) + \right.$$

$$R_3\,\frac{(x - c_3)^3}{3!}\, S_{c_3}(x) + L^{-1}\left[\frac{L\beta}{s^4}\right] +$$

$$\left.\sum P_i\,\frac{(x - c_i)^3}{3!}\, S_{c_i}(x) + \sum M_i\,\frac{(x - d_i)^2}{2!}\, S_{d_i}(x)\right\}$$

F. Evaluation of Unknown Boundary Conditions and Reactions

$$Y(c_2) = 0 = Y'(0)\, c_2 + \frac{1}{EI}\left\{\frac{R_1 c_2^3}{3!} + L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_2} + \right.$$

$$\sum P_i \frac{(c_2 - c_i)^3}{3!} S_{c_i}(c_2) + \sum M_i \frac{(c_2 - d_i)^2}{2!} S_{d_i}(c_2) \Bigg\}$$

$$Y(c_3) = 0 = Y'(0)\, c_3 + \frac{1}{EI} \Bigg\{ \frac{R_1 c_3^{\,3}}{3!} + R_2 \frac{(c_3 - c_2)^3}{3!} +$$

$$L^{-1} \left[\frac{L\beta}{s^4}\right]_{x=c_3} + \sum P_i \frac{(c_3 - c_i)^3}{3!} S_{c_i}(c_3) +$$

$$\sum M_i \frac{(c_3 - d_i)^2}{2!} S_{d_i}(c_3) \Bigg\}$$

$$Y(\ell) = 0 = Y'(0)\, \ell + \frac{1}{EI} \Bigg\{ \frac{R_1 \ell^3}{3!} + R_2 \frac{(\ell - c_2)^3}{3!} +$$

$$R_3 \frac{(\ell - c_3)^3}{3!} + L^{-1} \left[\frac{L\beta}{s^4}\right]_{x=\ell} + \sum P_i \frac{(\ell - c_i)^3}{3!} +$$

$$\sum M_i \frac{(\ell - d_i)^2}{2!} \Bigg\}$$

$$\sum M_{x=\ell} = 0 = R_1 \ell + R_2 (\ell - c_2) + R_3 (\ell - c_3) +$$

$$\sum Q_i (\ell - \overline{x}_i) + \sum P_i (\ell - c_i) + \sum M_i$$

From these equations, R_1, R_2, R_3 and $Y'(0)$ can now be solved by determinates similar to section 5-2 (F). Likewise, the general elastic/deflection curve, slope, bending moment, and shearing force equations can be obtained and, therefore, will not be carried out in detail in this section.

These solutions for two and three span beams prove the applicability of the method. They also indicate a practical limitation concerning continuous beams. That is, for every n spans there are $n + 1$ simultaneous linear equations with numerical coefficients to be solved. When the number of spans are few this is a reasonable task. When the number increases, the method ceases to be an efficient method of analysis for continuous beams. Further treatment of continuous beams will not be considered here.

CHAPTER VI

PART II - BUCKLING OF STRIPS BY END MOMENTS

When a strip of constant rectangular cross section is subjected to end moments about the short principal axis of the cross section, lateral buckling may occur. The value of this moment may be much lower than that found by ordinary flexure theory.

The differential equation expressing this type of failure (8) is

$$\frac{d^2\Phi}{dx^2} = -\frac{M^2\Phi}{EI_u J_e G} \qquad (6-1)$$

where

Φ	=	Angle of rotation of cross section from initial position
x	=	Longitudinal axis of strip
M	=	End moment
I_u	=	Moment of inertia about the long principal axis
J_e	=	$1/3 \sum c^3 d$, equivalent polar moment of inertia
G	=	Modulus of rigidity.

The solution of Eq. (6-1) for the critical moment, M_{cr}, is obtained by the Laplace transform method as follows.

Letting $k^2 = \frac{M^2}{EI_u J_e G}$ and taking the Laplace transform of Eq. (6-1) the following expressions are obtained.

$$s^2\phi(s) - s\Phi(0) - \Phi'(0) - k^2\phi(s) = 0 \tag{6-2a}$$

$$\phi(s) = \frac{s\Phi(0) + \Phi'(0)}{s^2 + k^2} \quad . \tag{6-2b}$$

Since $\Phi(0) = 0$, (6-2b) becomes

$$\phi(s) = \frac{\Phi'(0)}{s^2 + k^2} \quad . \tag{6-2c}$$

Taking the inverse transform of (6-2c)

$$\Phi(x) = \Phi'(0) \frac{1}{k} \sin kx \quad . \tag{6-3}$$

From the initial boundary condition $\Phi(\ell) = 0$ the solution to (6-3) is found.

$$\Phi(\ell) = 0 = \Phi'(0) \frac{1}{k} \sin k\ell \quad .$$

$\Phi'(0) = 0$ is discarded as a trivial solution, therefore

$$k\ell = n\pi \, , \, (n = 0, 1, 2, \ldots) \tag{6-4}$$

is the solution.

Substituting $k = \sqrt{\dfrac{M^2}{EI_u J_e G}}$ into (6-4) the least or critical value M_{cr} is found to be

$$M_{cr} = \frac{\pi\sqrt{EI_u J_e G}}{\ell} \quad . \tag{6-5}$$

CHAPTER VII

PART II - MISCELLANEOUS INVESTIGATIONS USING THE LAPLACE TRANSFORM METHOD

7-1 General

In addition to the types of structures considered in the previous chapters, the following items have been investigated in a general manner to determine the applicability of the Laplace transform method to their solutions. The findings have been briefly summarized, expressing the relative merits of additional detailed study and development.

7-2 Columns

Thomson (13) and Wagner (16) have made extensive use of the Laplace transform in the analysis of columns. They have considered the centrally loaded, constant and multiple cross section column with the usual types of end conditions.

The investigation proved the Laplace transform could be extended to eccentrically loaded, constant cross section columns, the results of which yielded the well-known "secant formula". With the exception of multiple cross section columns, it was concluded that the Laplace transform method has no outstanding advantage over other methods of analysis.

However, in the related area of "beam column" analysis, the Laplace transform can be used to develop an efficient and powerful procedure of analysis. Strandhagen (12) has laid the basis for this method. He applied the transform in determining the deflections of single span, constant cross section members that were subjected to various kinds of distributed transverse loads and to axial loads. The development of a method similar to that for elementary beams should be possible, i.e., a set of generalized solutions for the various types of beam columns loaded by any system of distributed loads, concentrated loads, and applied moments over any portion of the span.

It was concluded from the investigation that an extension might also include continuous beam columns, single and multiple span beam columns on elastic foundations, and beam columns with multiple cross sections.

7-3 Dead Load Deflections

Bridge beams and similar structures are usually cambered to compensate for dead load deflections. To determine the amount of camber, it is necessary to know the deflection values at many points along the span or spans.

In the case of a constant cross section beam, regardless of the number of spans, it was found that the method developed for elementary beams is ideally suited, for this task, over the classical methods. Besides knowing the kind and amount of dead load, the only additional information required is either the moment or reaction values at one support as determined by other methods. Knowing either of these values, the deflection can be easily computed at any point in any span of the beam.

7-4 Frames, Grid Structures, and End Fixity

The general nature of the investigation revealed little of importance in the application of the Laplace transform to the analysis of these items. However, the existence of important applications should not be excluded on the basis of this preliminary study.

In general, these topics are quite similar to section 7-3; that is, once an end condition (reaction or moment) is determined by other methods, the procedures in Chapter IV can be used to determine the values of deflection, slope, moment, or shear.

7-5 Flat Plates

The literature survey indicated the feasibility of using the Laplace transform method in the analysis of flat plates. References (6), (10), (11), and (17) were the only ones found pertaining to the solution of plates by the Laplace transform. Since three of these have been written in approximately the last year, it is reasonable to assume that this is an active area of development for the transformation.

An extension that is readily apparent would be the development of an analogy or direct relationship between the analysis of a flat plate, by the Laplace transform, and that of a grid structure. Additional extensions may be possible to slightly curved plates, thereby allowing the analysis of shell structures.

CHAPTER VIII

PART III - IMPACT ANALYSIS USING THE LAPLACE TRANSFORM METHOD

8-1 General

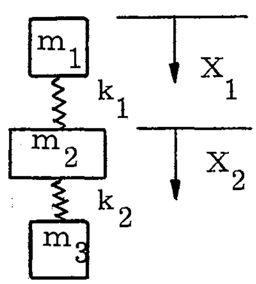
When an item, be it a machine, household applicance, or guided missile is to be shipped via a commercial or military mode of transportation, one of the primary factors in its design criteria is the "G" factor.

This "G" factor is the maximum or peak acceleration to which the item may be subjected to while in transit or during handling and loading operations. It is used in the design procedure either as a "factor of safety" or as a "limiting factor". As a factor of safety it is used to determine the design load or working stress for the structural elements of the item. In those cases where the strength of the elements is limited by space, size, materials, and other criteria, the "G" factor is used in the design of a shock mitigating system which will limit the maximum acceleration to the required safe level.

Although easy to define and use, the "G" factor is difficult to compute due to the many variables and parameters on which it depends. In some designs the "G" factor can be assumed. The assumption is based on past experience resulting from trial and error methods. Otherwise, it must be determined analytically and verified by a minimum number of simulated tests. The two most common methods of simulating the impacts and shocks which an item is likely to encounter are the drop and inclined-plane tests. The size and weight of the item usually dictates which method will be used. - the smaller and lighter items being dropped while the larger and heavier ones are tested on the inclined-plane. The drop test is the more common of the two due to its simplicity and lack of requirements for special equipment and facilites as in the inclined-plane tests.

It will now be shown how the Laplace transform method can be used in analytically determining a "G" factor which can be verified by instrumented drop tests.

When an item is to be shipped it is usually packed in a shock mitigating system and placed inside of a shipping container. For the purpose of analysis this mechanical system can be idealized by a system of spring-mass components as follows:



where

m_1 = the mass of a structural element of the item

k_1 = the inherent elastic property of m_1

m_2 = the total mass of the item

k_2 = the spring rate of the mitigating system

m_3 = the mass of the shipping container.

The mass m_1 is usually very small in comparison to m_2, therefore it will be neglected in the analysis. The shipping container will be assumed rigid and to have little or no deformation between it and the floor upon impact. Also, no rebound is assumed for the container. The static deflection $m_2 g/k_2$ is usually very small with respect to the dynamic deflection and therefore it will be neglected.

The restoring force $k_2 x_2$ can be either linear or nonlinear. However, nonlinear is by far the most common in actual design practice. For the analysis of a linear system with and without viscous damping, the reader is referred to Thomson. (13).

8-2 Impact Analysis for Cushioning with Cubic Elasticity

Cushioning which has a small amount of nonlinearity can easily be represented by a load-displacement function of the type $F(x_2) = k_o x_2 \pm r x_2^3$ where k_o is the spring rate that would exist if the elasticity was linear and r is a parameter associated with the degree of nonlinearity. Although the Laplace transform method is not applicable to the solution of nonlinear differential equations, it can be used to good advantage with the perturbation method to obtain an approximate solution when the value of r is small. This procedure will now be shown.

The equation of motion for this type of cushioning system with r positive and damping neglected is

$$m_2\ddot{X}_2 + k_0 X_2 + rX_2^3 = 0 \tag{8-1}$$

with the initial conditions

$$X_{20}\,(0) = 0 \text{ and } X'_{20}\,(0) = A_0 = \sqrt{2gh}\ .$$

Equation (8-1) can be rewritten as

$$\ddot{X}_2 + \omega_o^2\, X_2 + \alpha X_2^3 = 0 \tag{8-2}$$

where $\omega_o = \sqrt{\frac{k_o}{m_2}}$, the natural angular frequency if nonlinear terms are missing and $\alpha = \frac{r}{m_2}$ is a positive constant. A solution for Eq. (8-2) is of the form $X_2(t) = X_{20}\,(t) + \alpha\, X_{21}\,(t) + \alpha^2\, X_{22}\,(t)$ (8-3)

where the subscript 20 is the generating solution and 21 is the first-order correction term and 22 is the second-order correction term. No powers of α greater than 2 will be retained. Also, since the solution will be oscillatory it will be necessary to assume

$$\omega^2 = \omega_o^2 + \alpha\, b_1\,(A) + \alpha^2 b_2\,(A) \tag{8-4}$$

where A is the amplitude and ω is the actual fundamental frequency of oscillation, and b_1 (A) and b_2 (A) are functions of the amplitude A. This assumption is necessary in order to be able to remove secular terms (oscillatory terms having an amplitude increasing indefinitely with time) as they arise. Equation (8-4) can be rewritten as

$$\omega_o^2 = \omega^2 - \alpha\, b_1\,(A) - \alpha^2 b_2\,(A) \quad .$$

Equations (8-3) and (8-4) are then substituted into Eq. (8-2) resulting in

$$\ddot{X}_{20} + \alpha\ddot{X}_{21} + \alpha^2\ddot{X}_{22} + \omega^2 X_{20} + \alpha\omega^2 X_{21} + \alpha^2\omega^2 X_{22} -$$

$$\alpha b_1 X_{20} - \alpha^2 b_1 X_{21} - \alpha^2 b_2 X_{20} + \alpha X_{20}^3 + 3\alpha^2 X_{20}^2\, X_{21} = 0$$

where no powers of α greater than 2 have been retained.

Equating like powers of α the following equations are obtained:

$$\alpha^0 \quad \ddot{X}_{20} + \omega^2 X_{20} = 0$$

$$\alpha^1 \quad \ddot{X}_{21} + \omega^2 X_{21} = b_1 X_{20} - X_{20}^3$$

$$\alpha^2 \quad \ddot{X}_{22} + \omega^2 X_{22} = b_1 X_{21} + b_2 X_{20} - 3X_{20}^2\, X_{21} \quad .$$

Using the Laplace transformation these equations can be solved in the following manner to obtain the approximate solution to Eq. (8-2).

The generating solution is found from

$$\ddot{X}_{20} + \omega^2 X_{20} = 0 \quad .$$

Taking the Laplace transform

$$s^2 X_{20}(s) - sX_{20}(0) - X'_{20}(0) + \omega^2 X_{20}(s) = 0 \quad .$$

By substituting in the boundary conditions

$$X_{20}(0) = 0 \qquad \text{and} \qquad X'_{20}(0) = A_o = \sqrt{2gh}$$

the subsidiary equation becomes

$$X_{20}(s) = \frac{A_o}{s^2 + \omega^2} = \frac{\sqrt{2gh}}{s^2 + \omega^2} \quad .$$

Performing the inverse transformation the generating solution is found to be

$$X_{20}(t) = \frac{A_o}{\omega} \sin \omega t = \frac{\sqrt{2gh}}{\omega} \sin \omega t \quad . \tag{8-5}$$

The first order correction terms can be found from

$$\ddot{X}_{21} + \omega^2 X_{21} = b_1 X_{20} - X_{20}^3 \quad .$$

Substituting in the value for X_{20} and using the identity $\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$, the following form is obtained

$$\ddot{X}_{21} + \omega^2 X_{21} = \left(\frac{b_1 A_o}{\omega} - \frac{3 A_o^3}{4\omega^3}\right) \sin \omega t + \frac{A_o^3}{4\omega^3} \sin 3 \omega t \quad .$$

Taking the Laplace transform and using the initial conditions $X_{21}(0) = X'_{21}(0) = 0$, the subsidiary equation becomes

$$X_{21}(s) = \frac{\left(b_1 A_o - \frac{3A_o^3}{4\omega^2}\right)}{(s^2 + \omega^2)^2} + \frac{\frac{3A_o^3}{4\omega^2}}{(s^2 + \omega^2)(s^2 + 9\omega^2)} \quad .$$

Performing the inverse transformation the solution is

$$X_{21}(t) = \left(\frac{b_1 A_o}{2\omega^3} - \frac{3 A_o^3}{8\omega^5}\right) \sin \omega t -$$

$$\omega t \left(\frac{b_1 A_o}{2\omega^3} - \frac{3 A_o^3}{8\omega^5}\right) \cos \omega t +$$

$$\frac{A_o^3}{32 \omega^5} (3 \sin \omega t - \sin 3 \omega t) \quad .$$

The second term is the secular term. For it to vanish, its coefficient must be equal to zero. Therefore the value of b_1 must be $\frac{3A_o^2}{4\omega^2}$.

The first order correction terms are then

$$X_{21}(t) = \frac{A_o^3}{32\,\omega^5}\,(3 \sin \omega t - \sin 3\,\omega t) \quad . \tag{8-6}$$

The second order correction terms can be found from

$$\ddot{X}_{22} + \omega^2 X_{22} = b_1 X_{21} + b_2 X_{20} - 3X_{20}^2\, X_{21} \quad .$$

Substituting in the values for b_1, X_{20}, X_{21} and the identities for $\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$ and $\sin^2 x \sin 3x = \frac{1}{2} \sin 3x - \frac{1}{4} \sin 5x - \frac{1}{4} \sin x$ the following form is obtained

$$\ddot{X}_{22} + \omega^2 X_{22} = \left(\frac{b_2 A_o}{\omega} - \frac{21\,A_o^5}{128\,\omega^7}\right) \sin \omega t + \frac{12\,A_o^5}{128\,\omega^7} \sin 3\,\omega t - \frac{3\,A_o^5}{128\,\omega^7} \sin 5\,\omega t \quad .$$

Taking the Laplace transform and using the initial conditions $X_{22}(0) = X'(0)_{22} = 0$ the subsidiary equation becomes

$$X_{22}(s) = \frac{\left(\frac{b_2 A_o}{\omega} - \frac{21\,A_o^5}{128\,\omega^7}\right)\omega}{(s^2 + \omega^2)^2} + \frac{\left(\frac{12\,A_o^5}{128\,\omega^7}\right) 3\omega}{(s^2 + \omega^2)(s^2 + 9\,\omega^2)} + \frac{\left(-\frac{3\,A_o^5}{128\,\omega^7}\right) 5\omega}{(s^2 + \omega^2)(s^2 + 25\,\omega^2)} \quad .$$

Performing the inverse transformation the solution is

$$X_{22}(t) = \left(\frac{b_2 A_o}{2\omega^3} - \frac{21 A_o^5}{256\,\omega^9}\right) \sin \omega t - \left(\frac{b_2 A_o}{2\omega^2} - \frac{21 A_o^5}{256\,\omega^8}\right) t \cos \omega t +$$

$$\frac{36 A_o^5}{1024\,\omega^9} \sin \omega t - \frac{12 A_o^5}{1024\,\omega^9} \sin 3\omega t -$$

$$\frac{5 A_o^5}{1024\,\omega^9} \sin \omega t + \frac{A_o^5}{1024\,\omega^9} \sin 5\omega t \quad .$$

The second term is the secular term. Therefore b_2 must be $\frac{21 A_o^4}{128\omega^6}$.

The second order correction terms are then

$$X_{22}(t) = \frac{31 A_o^5}{1024\,\omega^9} \sin \omega t - \frac{12 A_o^5}{1024\,\omega^9} \sin 3\omega t + \frac{A_o^5}{1024\,\omega^9} \sin 5\omega t \quad . \tag{8-7}$$

The solution to Eq. (8-2) to the second-order correction is therefore

$$X_2(t) = \frac{A_o}{\omega} \sin \omega t + \frac{\alpha A_o^3}{32\,\omega^5} (3 \sin \omega t - \sin 3\omega t) +$$

$$\frac{\alpha^2 A_o^5}{1024\,\omega^9} (31 \sin \omega t - 12 \sin 3\omega t + \sin 5\omega t) \tag{8-8}$$

$$\omega^2 = \omega_o^2 + \frac{\alpha\, 3 A_o^2}{4\omega^2} + \frac{\alpha^2\, 21 A_o^4}{128\,\omega^6}$$

where $A_o = \sqrt{2gh}$.

Differentiating $X_2(t)$ twice with respect to time, the acceleration is

$$\ddot{X}_2(t) = -A_o\,\omega \sin \omega t + \frac{\alpha A_o^3}{32\,\omega^3}(-3 \sin \omega t + 9 \sin 3\,\omega t)$$

$$+ \frac{\alpha^2 A_o^5}{1024\,\omega^7}(-31 \sin \omega t + 108 \sin 3\,\omega t - 25 \sin 5\,\omega t). \quad (8\text{-}9)$$

The "G" factor or maximum acceleration occurs when $\omega t = \pi/2$, therefore

$$\text{"G"} = \ddot{X}_2(t)_{max} = -A_o\,\omega - \frac{12\,\alpha A_o^3}{32\,\omega^3} - \frac{164\,\alpha^2 A_o^5}{1024\,\omega^7}. \quad (8\text{-}10)$$

In his treatment on impact analysis, Mindlin (7) has solved this same type of cushioning problem using the elliptical integral, the use of which results in an exact solution.

The purpose in using the Laplace transform to obtain an approximate general solution is to illustrate how it may be used to develop a method of impact analysis when the system is nonlinear. Additional extensions can be made that include other types of cushioning for which the Laplace transforms exist, systems in which damping is present, etc.

CHAPTER IX

SUMMARY AND CONCLUSIONS

Utilizing the Laplace transform method, the formulation of a generalized procedure of analysis for elementary static beam systems has been the primary objective of this thesis.

This objective has been achieved. The procedure developed is applicable to any single span, constant cross section beam that may be subjected to a static transverse system of distributed loads, concentrated loads, and/or applied moments. The only restriction imposed is that the Laplace transform of the load function must exist. Since most practical load systems do have Laplace transforms, this restriction detracts little from the generality of the procedure.

The principal advantages of the transform method are found in the ease with which complex load functions can be dispatched, the capability of being able to write the solution as one equation for the entire span, and the reduction in solution time over classical methods.

One's attention is called to the important fact that the seven generalized solutions which are summarized in Appendix A constitute virtually an unlimited number of elementary beam systems. To assemble an equivalent number of systems from existing handbooks would be very difficult, if not impossible, and certainly would require a multitude of handbooks.

An additional extension of the Laplace transform method is possible in the development of a similar procedure for single span beams with variable or multiple cross sections.

The procedures for the elementary beams were extended directly to the analysis of continuous beams with constant cross sections. It was found that as the number of spans increased, the amount of computation required to obtain the solution increased. That is, for every n spans there were $n + 1$ simultaneous equations to solve. When the analysis is being performed by hand, this task becomes unreasonable when n exceeds three.

This suggests an extension could be made of the Laplace transform method, in conjunction with computer techniques, in the development of a procedure for continuous beams. A further extension might also be made to continuous beams with variable or multiple cross sections.

A general investigation was conducted to determine additional areas, in the static structures field, for which the Laplace transformation would have important applications.

The areas investigated were columns, frames, grid structures, flat plates, and beams with varying degrees of end fixity. From this investigation, it was concluded that beam columns and plates have the best potential for further development. The formulation of a procedure for beam column analysis, similar to elementary beams, is possible and would be both important and desirable.

A similar procedure developed for flat plates can perhaps be extended to slightly curved plates, thereby including the analysis of shell structures. It is also possible that an analogy could be developed to extend the analysis of flat plates to grid structures. In general, the remaining topics were found to have no apparent characteristics which would enable the Laplace transform method to have an advantage over established methods of analysis.

In the field of impact analysis the Laplace transformation is an important tool. In this thesis it was used to determine the maximum acceleration for a slightly nonlinear system. Although not applicable to nonlinear analysis, the Laplace transform can be used in conjunction with the perturbation method when the degree of nonlinearity is small. The system represented an item that was cushioned in a material which had the characteristics of an undamped, massless, hard spring. The solution was obtained for a drop of any height, h.

Further applications in this field were not attempted at this time. However, there are many types of impact systems which deserve detailed study and development through the use of the Laplace transformation. For instance, systems in which damping is present, systems where the load-displacement characteristics can be represented by transformable functions, systems where the mass of cushioning is accounted for in the analysis, etc.

In conclusion, the Laplace transformation is considered to be an efficient and powerful method of analysis in the static and dynamic structures field where it's development and applications are far from having been exhausted.

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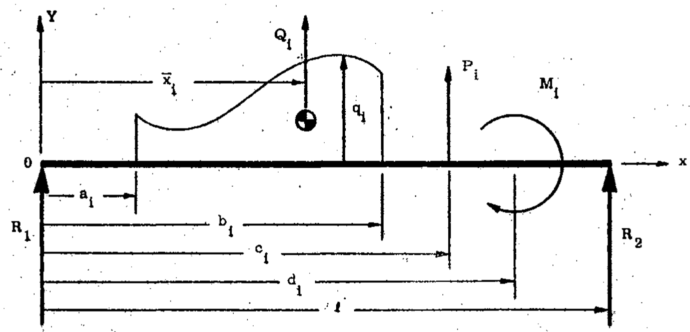
APPENDIX A

ELEMENTARY BEAM FORMULAS FOR ARBITRARY STATIC LOADING CONDITIONS

SIMPLE BEAM



END REACTIONS

$$R_1 = -\frac{1}{\ell}\left[\sum Q_i(\ell - \bar{x}_i) + \sum P_i(\ell - c_i) + \sum M_i\right]$$

$$R_2 = -\frac{1}{\ell}\left[\sum Q_i\bar{x}_i + \sum P_i c_i - \sum M_i\right]$$

DEFLECTION AT ANY POINT

$$Y(x) = Y'(0)x + \frac{1}{EI}\left\{R_1\frac{x^3}{3!} + L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i\frac{(x - c_i)^3}{3!}S_{c_i}(x) + \sum M_i\frac{(x - d_i)^2}{2!}S_{d_i}(x)\right\}$$

SLOPE AT ANY POINT

$$\phi(x) = Y'(0) + \frac{1}{EI}\left\{R_1\frac{x^2}{2} + \frac{d}{dx}L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i\frac{(x - c_i)^2}{2}S_{c_i}(x) + \sum M_i(x - d_i)S_{d_i}(x)\right\}$$

BENDING MOMENT AT ANY SECTION

$$M_b(x) = R_1 x + \frac{d^2}{dx^2}L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i(x - c_i)S_{c_i}(x) + \sum M_i S_{d_i}(x)$$

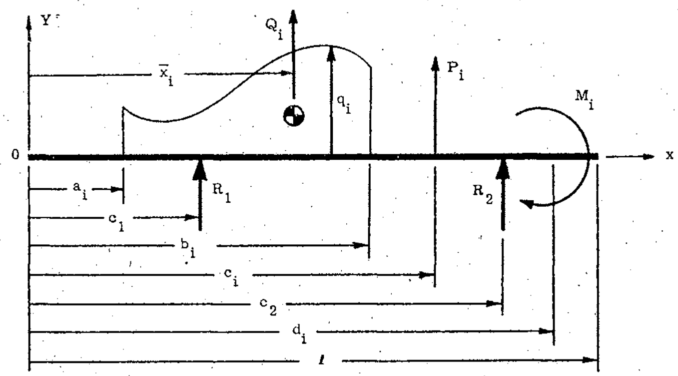
SHEARING FORCE AT ANY SECTION

$$V(x) = R_1 + \frac{d^3}{dx^3}L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i S_{c_i}(x)$$

WHERE

$$Y'(0) = -\frac{1}{EI}\left\{R_1\frac{\ell^2}{3!} + \frac{1}{\ell}L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \sum P_i\frac{(\ell - c_i)^3}{3!\ell} + \sum M_i\frac{(\ell - d_i)^2}{2!\ell}\right\}$$

SIMPLE OVERHANGING BEAM



END REACTIONS

$$R_1 = \frac{1}{(c_2 - c_1)}\left\{\sum Q_i\bar{x}_i + \sum P_i c_i - \sum M_i - c_2\left(\sum Q_i + \sum P_i\right)\right\}$$

$$R_2 = \frac{1}{(c_2 - c_1)}\left\{\sum M_i - \sum Q_i\bar{x}_i - \sum P_i c_i + c_1\left(\sum Q_i + \sum P_i\right)\right\}$$

DEFLECTION AT ANY POINT

$$Y(x) = Y(0) + Y'(0)x + \frac{1}{EI}\left\{R_1\frac{(x-c_1)^3}{3!}S_{c_1}(x) + R_2\frac{(x-c_2)^3}{3!}S_{c_2}(x) + L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i\frac{(x-c_i)^3}{3!}S_{c_i}(x) + \sum M_i\frac{(x-d_i)^2}{2!}S_{d_i}(x)\right\}$$

SLOPE AT ANY POINT

$$\phi(x) = Y'(0) + \frac{1}{EI}\left\{R_1\frac{(x-c_1)^2}{2}S_{c_1}(x) + R_2\frac{(x-c_2)^2}{2}S_{c_2}(x) + \frac{d}{dx}L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i\frac{(x-c_i)^2}{2}S_{c_i}(x) + \sum M_i(x-d_i)S_{d_i}(x)\right\}$$

BENDING MOMENT AT ANY SECTION

$$M_b(x) = R_1(x-c_1)S_{c_1}(x) + R_2(x-c_2)S_{c_2}(x) + \frac{d^2}{dx^2}L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i(x-c_i)S_{c_i}(x) + \sum M_i S_{d_i}(x)$$

SHEARING FORCE AT ANY SECTION

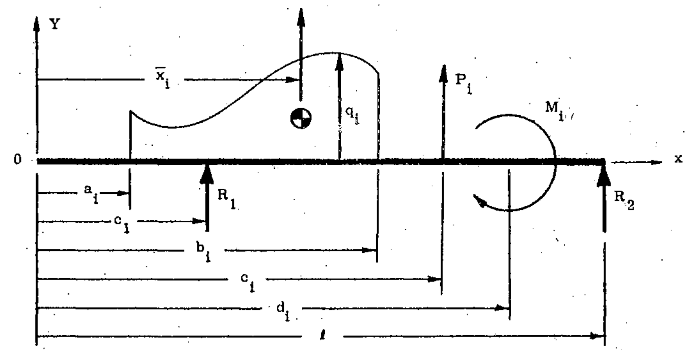
$$V(x) = R_1 S_{c_1}(x) + R_2 S_{c_2}(x) + \frac{d^3}{dx^3}L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i S_{c_i}(x)$$

WHERE

$$Y(0) = \frac{1}{(c_2 - c_1)EI}\left\{c_1 R_1\frac{(c_2-c_1)^3}{3!} + c_1 L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_2} - c_2 L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_1} + c_1\sum P_i\frac{(c_2-c_i)^3}{3!}S_{c_i}(c_2) - c_2\sum P_i\frac{(c_1-c_i)^3}{3!}S_{c_i}(c_1) + c_1\sum M_i\frac{(c_2-d_i)^2}{2!}S_{d_i}(c_2) - c_2\sum M_i\frac{(c_1-d_i)^2}{2!}S_{d_i}(c_1)\right\}$$

$$Y'(0) = \frac{1}{(c_1 - c_2)EI}\left\{R_1\frac{(c_2-c_1)^3}{3!} + L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_2} - L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_1} + \sum P_i\frac{(c_2-c_i)^3}{3!}S_{c_i}(c_2) - \sum P_i\frac{(c_1-c_i)^3}{3!}S_{c_i}(c_1) + \sum M_i\frac{(c_2-d_i)^2}{2!}S_{d_i}(c_2) - \sum M_i\frac{(c_1-d_i)^2}{2!}S_{d_i}(c_1)\right\}$$

SIMPLE BEAM WITH ONE OVERHANG



END REACTIONS

$$R_1 = -\frac{1}{(\ell - c_1)}\left[\sum Q_i(\ell - \bar{x}_i) + \sum P_i(\ell - c_i) + \sum M_i\right]$$

$$R_2 = -\frac{1}{(\ell - c_1)}\left[\sum Q_i(\bar{x}_i - c_1) + \sum P_i(c_i - c_1) - \sum M_i\right]$$

DEFLECTION AT ANY POINT

$$Y(x) = Y(0) + Y'(0)x + \frac{1}{EI}\left\{R_1\frac{(x - c_1)^3}{3!}S_{c_1}(x) + L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i\frac{(x - c_i)^3}{3!}S_{c_i}(x) + \sum M_i\frac{(x - d_i)^2}{2!}S_{d_i}(x)\right\}$$

SLOPE AT ANY POINT

$$\phi(x) = Y'(0) + \frac{1}{EI}\left\{R_1\frac{(x - c_1)^2}{2}S_{c_1}(x) + \frac{d}{dx}L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i\frac{(x - c_i)^2}{2}S_{c_i}(x) + \sum M_i(x - d_i)S_{d_i}(x)\right\}$$

BENDING MOMENT AT ANY SECTION

$$M_b(x) = R_1(x - c_1)S_{c_1}(x) + \frac{d^2}{dx^2}L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i(x - c_i)S_{c_i}(x) + \sum M_i S_{d_i}(x)$$

SHEARING FORCE AT ANY SECTION

$$V(x) = R_1 S_{c_1}(x) + \frac{d^3}{dx^3}L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i S_{c_i}(x)$$

WHERE

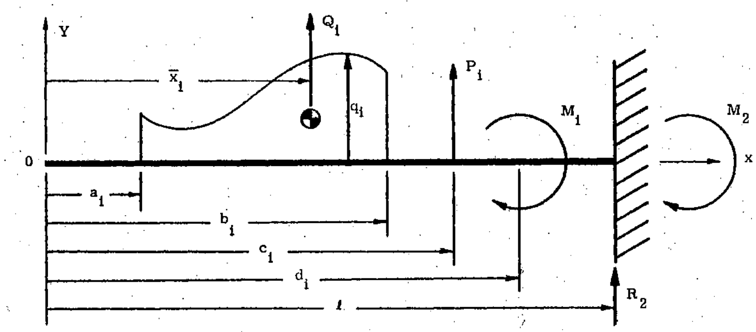
$$Y(0) = \frac{1}{(\ell - c_1)EI}\left\{c_1R_1\frac{(\ell - c_1)^3}{3!} - \ell L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_1} + c_1L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} - \ell\sum P_i\frac{(c_1 - c_i)^3}{3!}S_{c_i}(c_1) + \right.$$

$$\left. c_1\sum P_i\frac{(\ell - c_i)^3}{3!} - \ell\sum M_i\frac{(c_1 - d_i)^2}{2!}S_{d_i}(c_1) + c_1\sum M_i\frac{(\ell - d_i)^2}{2!}\right\}$$

$$Y'(0) = \frac{1}{(c_1 - \ell)EI}\left\{R_1\frac{(\ell - c_1)^3}{3!} - L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_1} + L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} - \sum P_i\frac{(c_1 - c_i)^3}{3!}S_{c_i}(c_1) + \sum P_i\frac{(\ell - c_i)^3}{3!} - \right.$$

$$\left. \sum M_i\frac{(c_1 - d_i)^2}{2!}S_{d_i}(c_1) + \sum M_i\frac{(\ell - d_i)^2}{2!}\right\}$$

CANTILEVER BEAM



END REACTIONS

$$R_2 = -\left[\sum Q_i + \sum P_i\right]$$

$$M_2 = -\left[\sum Q_i(\ell - \bar{x}_i) + \sum P_i(\ell - c_i) + \sum M_i\right]$$

DEFLECTION AT ANY POINT

$$Y(x) = Y(0) + Y'(0)x + \frac{1}{EI}\left\{L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i \frac{(x - c_i)^3}{3!} S_{c_i}(x) + \sum M_i \frac{(x - d_i)^2}{2!} S_{d_i}(x)\right\}$$

SLOPE AT ANY POINT

$$\phi(x) = Y'(0) + \frac{1}{EI}\left\{\frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i \frac{(x - c_i)^2}{2} S_{c_i}(x) + \sum M_i (x - d_i) S_{d_i}(x)\right\}$$

BENDING MOMENT AT ANY SECTION

$$M_b(x) = \frac{d^2}{dx^2} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i (x - c_i) S_{c_i}(x) + \sum M_i S_{d_i}(x)$$

SHEARING FORCE AT ANY SECTION

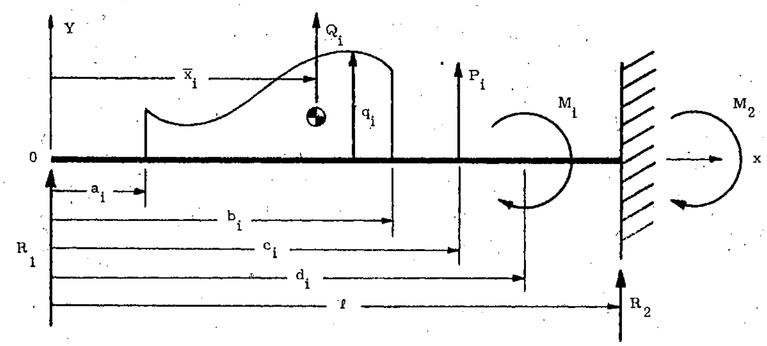
$$V(x) = \frac{d^3}{dx^3} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i S_{c_i}(x)$$

WHERE

$$Y(0) = \frac{1}{EI}\left\{\left(\frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} + \sum P_i \frac{(\ell - c_i)^2}{2} + \sum M_i(\ell - d_i)\right\}\ell - \frac{1}{EI}\left\{L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \sum P_i \frac{(\ell - c_i)^3}{3!} + \sum M_i \frac{(\ell - d_i)^2}{2!}\right\}$$

$$Y'(0) = -\frac{1}{EI}\left\{\left(\frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} + \sum P_i \frac{(\ell - c_i)^2}{2} + \sum M_i(\ell - d_i)\right\}$$

PROPPED CANTILEVER BEAM



END REACTIONS

$$R_1 = -\frac{3}{\ell^3}\left\{\ell\left(\frac{d}{dx}L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} - L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \ell\sum P_i\frac{(\ell-c_i)^2}{2} - \sum P_i\frac{(\ell-c_i)^3}{3!} + \ell\sum M_i(\ell-d_i) - \sum M_i\frac{(\ell-d_i)^2}{2!}\right\}$$

$$R_2 = -\left[R_1 + \sum Q_i + \sum P_i\right]$$

$$M_2 = -\left[R_1\ell + \sum Q_i(\ell-\bar{x}_i) + \sum P_i(\ell-c_i) + \sum M_i\right]$$

DEFLECTION AT ANY POINT

$$Y(x) = Y'(0)x + \frac{1}{EI}\left\{R_1\frac{x^3}{3!} + L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i\frac{(x-c_i)^3}{3!}S_{c_i}(x) + \sum M_i\frac{(x-d_i)^2}{2!}S_{d_i}(x)\right\}$$

SLOPE AT ANY POINT

$$\phi(x) = Y'(0) + \frac{1}{EI}\left\{R_1\frac{x^2}{2} + \frac{d}{dx}L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i\frac{(x-c_i)^2}{2}S_{c_i}(x) + \sum M_i(x-d_i)S_{d_i}(x)\right\}$$

BENDING MOMENT AT ANY SECTION

$$M_b(x) = R_1 x + \frac{d^2}{dx^2}L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i(x-c_i)S_{c_i}(x) + \sum M_i S_{d_i}(x)$$

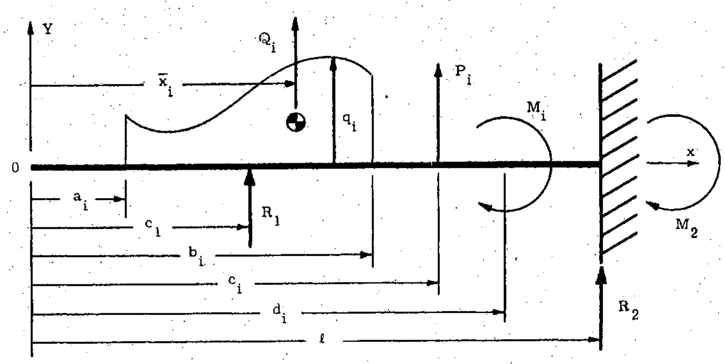
SHEARING FORCE AT ANY SECTION

$$V(x) = R_1 + \frac{d^3}{dx^3}L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i S_{c_i}(x)$$

WHERE

$$Y'(0) = \frac{3}{2EI}\left\{\frac{1}{3}\left(\frac{d}{dx}L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} - \frac{1}{\ell}L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \sum P_i\frac{(\ell-c_i)^2}{3!} - \sum P_i\frac{(\ell-c_i)^3}{3!\ell} + \sum M_i\frac{(\ell-d_i)}{3} - \sum M_i\frac{(\ell-d_i)^2}{2!\ell}\right\}$$

PROPPED CANTILEVER BEAM WITH OVERHANG



END REACTIONS

$$R_1 = \frac{3}{(\ell - c_1)^3}\left[-\left\{ L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_1} + \sum P_i \frac{(c_1 - c_i)^3}{3!} S_{c_i}(c_1) + \sum M_i \frac{(c_1 - d_i)^2}{2!} S_{d_i}(c_1) \right\} + \left\{ L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \sum P_i \frac{(\ell - c_i)^3}{3!} + \sum M_i \frac{(\ell - d_i)^2}{2!} \right\} + (c_1 - \ell)\left\{ \left(\frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} + \sum P_i \frac{(\ell - c_i)^2}{2} + \sum M_i(\ell - d_i) \right\} \right]$$

$$R_2 = -\left[\sum Q_i + \sum P_i + R_1\right]$$

$$M_2 = -\left[R_1(\ell - c_1) + \sum Q_i(\ell - \bar{x}_i) + \sum P_i(\ell - c_i) + \sum M_i\right]$$

DEFLECTION AT ANY POINT

$$Y(x) = Y(0) + Y'(0)x + \frac{1}{EI}\left\{ R_1 \frac{(x - c_1)^3}{3!} S_{c_1}(x) + L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i \frac{(x - c_i)^3}{3!} S_{c_i}(x) + \sum M_i \frac{(x - d_i)^2}{2!} S_{d_i}(x) \right\}$$

SLOPE AT ANY POINT

$$\phi(x) = Y'(0) + \frac{1}{EI}\left\{ R_1 \frac{(x - c_1)^2}{2} S_{c_1}(x) + \frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i \frac{(x - c_i)^2}{2} S_{c_i}(x) + \sum M_i(x - d_i) S_{d_i}(x) \right\}$$

BENDING MOMENT AT ANY SECTION

$$M_b(x) = R_1(x - c_1) S_{c_1}(x) + \frac{d^2}{dx^2} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i(x - c_i) S_{c_i}(x) + \sum M_i S_{d_i}(x)$$

SHEARING FORCE AT ANY SECTION

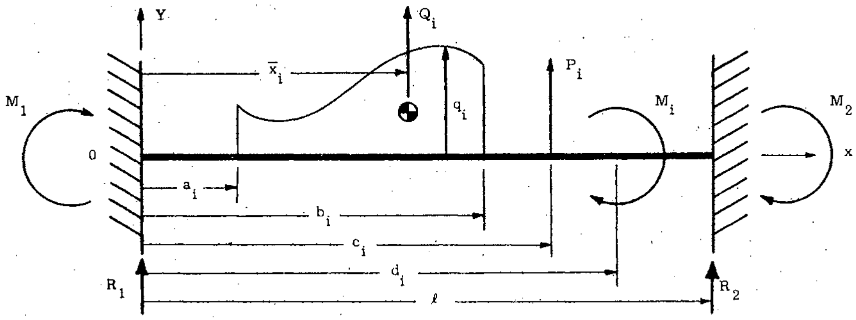
$$V(x) = R_1 S_{c_1}(x) + \frac{d^3}{dx^3} L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i S_{c_i}(x)$$

WHERE

$$Y(0) = \frac{1}{2(\ell - c_1)EI}\left[3c_1\left\{ L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \sum P_i \frac{(\ell - c_i)^3}{3!} + \sum M_i \frac{(\ell - d_i)^2}{2!} \right\} + c_1(c_1 - \ell)\left\{ \left(\frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} + \sum P_i \frac{(\ell - c_i)^2}{2} + \sum M_i(\ell - d_i) \right\} - (2\ell + c_1)\left\{ L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_1} + \sum P_i \frac{(c_1 - c_i)^3}{3!} S_{c_i}(c_1) + \sum M_i \frac{(c_1 - d_i)^2}{2!} S_{d_i}(c_1) \right\} \right]$$

$$Y'(0) = \frac{1}{2(\ell - c_1)EI}\left[3\left\{ L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=c_1} + \sum P_i \frac{(c_1 - c_i)^3}{3!} S_{c_i}(c_1) + \sum M_i \frac{(c_1 - d_i)^2}{2!} S_{d_i}(c_1) \right\} - 3\left\{ L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \sum P_i \frac{(\ell - c_i)^3}{3!} + \sum M_i \frac{(\ell - d_i)^2}{2!} \right\} + (\ell - c_1)\left\{ \left(\frac{d}{dx} L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} + \sum P_i \frac{(\ell - c_i)^2}{2} + \sum M_i(\ell - d_i) \right\} \right]$$

FIXED BEAM



END REACTIONS

$$R_1 = -\frac{12}{\ell^3}\left\{\frac{\ell}{2}\left(\frac{d}{dx}\,L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} - L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \frac{\ell}{2}\sum P_i\frac{(\ell-c_i)^2}{2!} - \sum P_i\frac{(\ell-c_i)^3}{3!} + \frac{\ell}{2}\sum M_i(\ell-d_i) - \sum M_i\frac{(\ell-d_i)^2}{2!}\right\}$$

$$R_2 = -\left[R_1 + \sum Q_i + \sum P_i\right]$$

$$M_1 = \frac{6}{\ell^2}\left\{\frac{\ell}{3}\left(\frac{d}{dx}\,L^{-1}\left[\frac{L\beta}{s^4}\right]\right)_{x=\ell} - L^{-1}\left[\frac{L\beta}{s^4}\right]_{x=\ell} + \frac{\ell}{3}\sum P_i\frac{(\ell-c_i)^2}{2} - \sum P_i\frac{(\ell-c_i)^3}{3!} + \frac{\ell}{3}\sum M_i(\ell-d_i) - \sum M_i\frac{(\ell-d_i)^2}{2!}\right\}$$

$$M_2 = -\left[R_1\ell + \sum Q_i(\ell-\bar{x}_i) + \sum P_i(\ell-c_i) + \sum M_i + M_1\right]$$

DEFLECTION AT ANY POINT

$$Y(x) = \frac{1}{EI}\left\{R_1\frac{x^3}{3!} + M_1\frac{x^2}{2!} + L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i\frac{(x-c_i)^3}{3!}S_{c_i}(x) + \sum M_i\frac{(x-d_i)^2}{2!}S_{d_i}(x)\right\}$$

SLOPE AT ANY POINT

$$\phi(x) = \frac{1}{EI}\left\{R_1\frac{x^2}{2} + M_1x + \frac{d}{dx}\,L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i\frac{(x-c_i)^2}{2}S_{c_i}(x) + \sum M_i(x-d_i)S_{d_i}(x)\right\}$$

BENDING MOMENT AT ANY SECTION

$$M_b(x) = R_1x + M_1 + \frac{d^2}{dx^2}\,L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_i(x-c_i)S_{c_i}(x) + \sum M_iS_{d_i}(x)$$

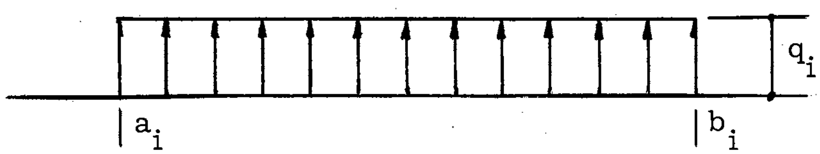
SHEARING FORCE AT ANY SECTION

$$V(x) = R_1 + \frac{d^3}{dx^3}\,L^{-1}\left[\frac{L\beta}{s^4}\right] + \sum P_iS_{c_i}(x)$$

APPENDIX B

THE LAPLACE TRANSFORMS OF USEFUL $\beta = \sum \left[q_i S_{a_i}(x) - q_i S_{b_i}(x) \right]$ FACTORS AND THEIR INVERSE TRANSFORMS

a. Uniformly Distributed Load



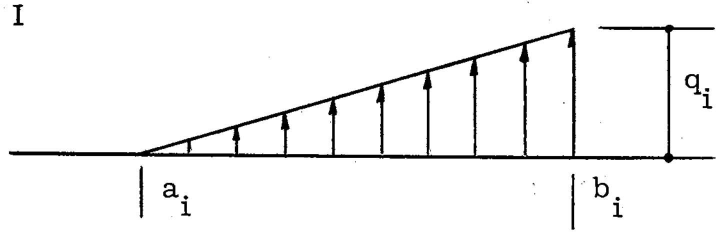
$$\beta = \sum \left[q_i \, S_{a_i}(x) - q_i \, S_{b_i}(x) \right]$$

$$L\beta = \sum \left\{ q_i \frac{e^{-a_i s}}{s} - q_i \frac{e^{-b_i s}}{s} \right\}$$

$$L^{-1}\left[\frac{L\beta}{s^4}\right] = \sum \left\{ q_i \frac{(x - a_i)^4}{4!} S_{a_i}(x) - q_i \frac{(x - b_i)^4}{4!} S_{b_i}(x) \right\}$$

b. Triangularly Distributed Load

Case I



$$\beta = \sum \left\{ \frac{q_i}{b_i - a_i} (x - a_i) S_{a_i}(x) - q_i S_{b_i}(x) - \right.$$

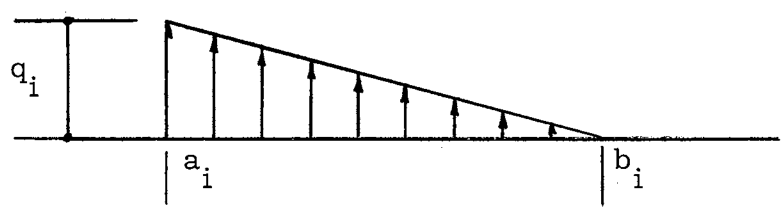
$$\left. \frac{q_i}{b_i - a_i} (x - b_i)\, S_{b_i}\, (x) \right\}$$

$$L\beta = \sum \left\{ \frac{q_i}{b_i - a_i} \; \frac{e^{-a_i s}}{s^2} - q_i \frac{e^{-b_i s}}{s} - \frac{q_i}{b_i - a_i} \; \frac{e^{-b_i s}}{s^2} \right\}$$

$$L^{-1}\left[\frac{L\beta}{s^4}\right] = \sum \left\{ \frac{q_i}{b_i - a_i} \; \frac{(x - a_i)^5}{5!} S_{a_i}(x) - q_i \frac{(x - b_i)^4}{4!} S_{b_i}(x) - \right.$$

$$\left. \frac{q_i}{b_i - a_i} \; \frac{(x - b_i)^5}{5!} S_{b_i}\, (x) \right\}$$

Case II



$$\beta = \sum \left\{ q_i\, S_{a_i}\, (x) - \frac{q_i}{b_i - a_i} (x - a_i)\, S_{a_i}(x) + \right.$$

$$\left. \frac{q_i}{b_i - a_i} (x - b_i)\, S_{b_i}\, (x) \right\}$$

$$L\beta = \sum \left\{ q_i \frac{e^{-a_i s}}{s} - \frac{q_i}{b_i - a_i} \; \frac{e^{-a_i s}}{s^2} + \right.$$

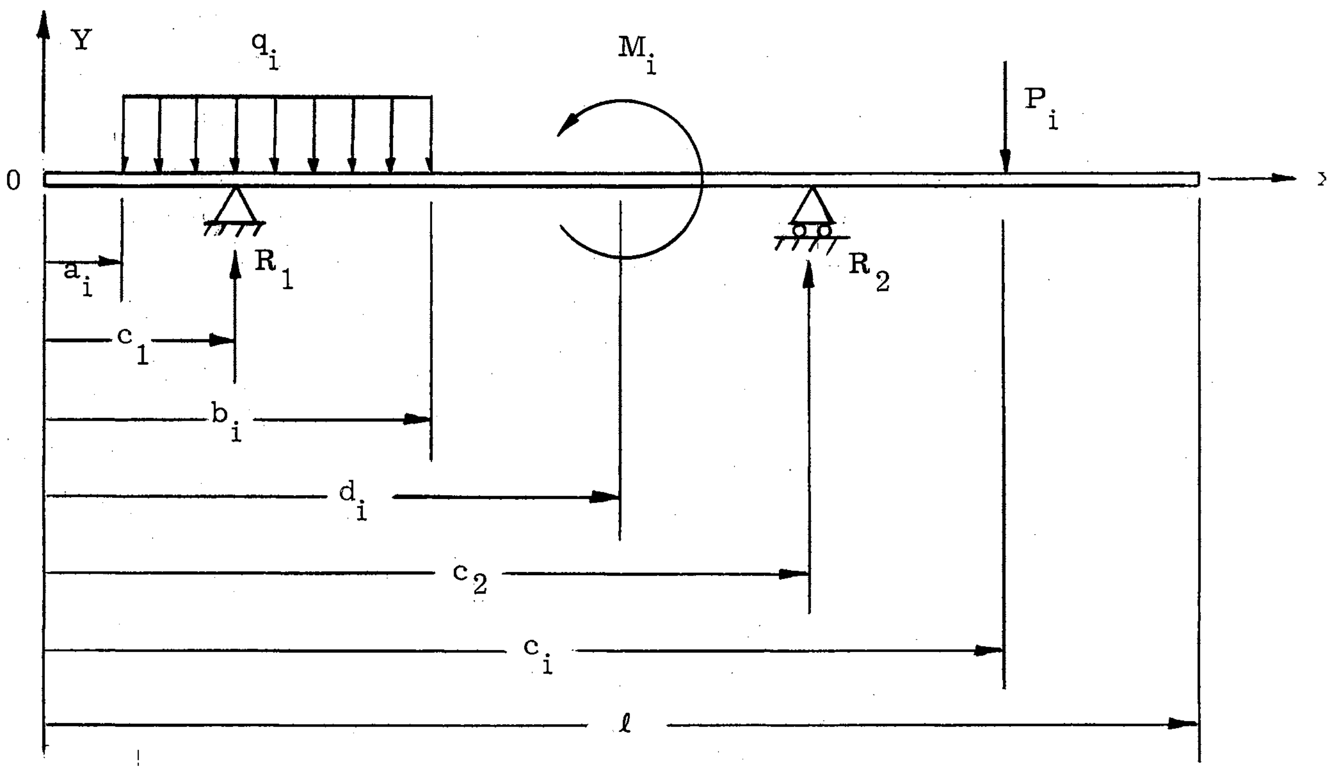
$$\left.\frac{q_i}{b_i - a_i} \frac{e^{-b_i s}}{s^2}\right\}$$

$$L^{-1}\left[\frac{L\beta}{s^4}\right] = \sum \left\{ q_i \frac{(x - a_i)^4}{4!} S_{a_i}(x) - \frac{q_i}{b_i - a_i} \right.$$

$$\left. \frac{(x - a_i)^5}{5!} S_{a_i}(x) + \frac{q_i}{b_i - a_i} \frac{(x - b_i)^5}{5!} S_{b_i}(x) \right\}$$

APPENDIX C

A NUMERICAL EXAMPLE



Find the deflection at $x = 11$ ft. by the method of

I. Area Moment

II. Section 4-7-2

III. Appendices A and B.

Given:

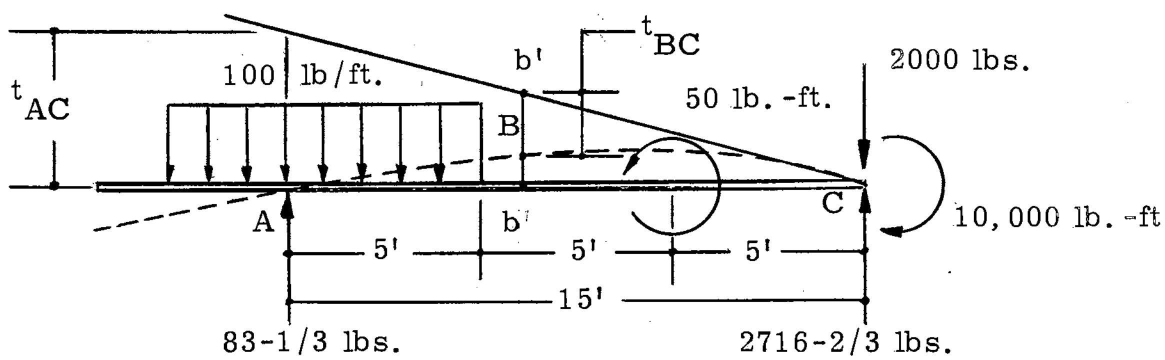
a_i	=	2'	c_1	=	5'
b_i	=	10'	c_2	=	20'
c_i	=	25'	P_i	=	-2,000 lbs.
d_i	=	15'	M_i	=	-50 lb.-ft.
ℓ	=	30'	q_i	=	-100 lb./ft.

I. Solution by Area Moment

From statics, R_1 and R_2 are found to be:

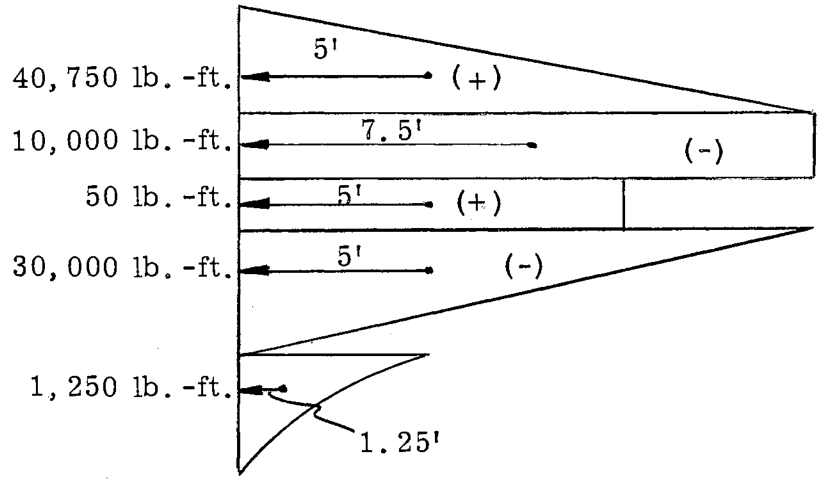
$$R_1 = -\left[\frac{(5)\,(2000) - 50 - (100)\,(8)\,(14)}{15}\right] = 83\text{-}1/3 \text{ lbs.}$$

$$R_2 = 2800 - 83\text{-}1/3 = 2716\text{-}2/3 \text{ lbs.}$$



Using the tangential deviations and geometrical relationships from the above sketch, the deflection at $x = 11$ ft. is found as follows.

The bending moment by parts, as drawn from C to A is



with the moment arms as shown.

The area moments are computed and algebraically summed to give the tangential deviation, t_{AC},

$$+ (40,750)(7.5)(5) = +1,528,125.00/EI$$

$$-(10,000)(15)(7.5) = -1,125,000.00/EI$$

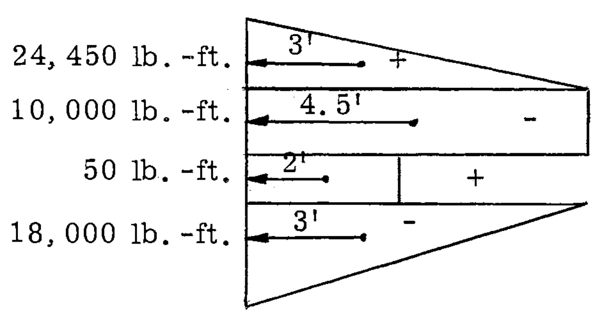
$$+(50)(10)(5) = +\quad 2,500.00/EI$$

$$-(30,000)(7.5)(5) = -1,125,000.00/EI$$

$$-(.1250)(1.667)(1.25) = -\quad 2,604.17/EI$$

$$t_{AC} = \sum \frac{A_{m_i} \overline{x_i}}{EI} = -721,979.17/EI$$

The bending moment by parts, as drawn from C to B is,



with the moment arms as shown.

The area moments are computed and algebraically summed to give the tangential deviation, t_{BC},

$$+ (24,450)(4.5)(3) = + 330,075.00/EI$$

$$- (10,000)(9)(4.5) = - 405,000.00/EI$$

$$+ (50)(4)(2) = + 400.00/EI$$

$$- (18,000)(4.5)(3) = - 243,000.00/EI$$

$$t_{BC} = \sum \frac{A_{m_i} \bar{x}_i}{EI} = - 317,525.00/EI$$

The deflection Y(11) is found by the following geometrical relationships.

$$bb' = 0.6\, t_{AC} = (0.6)\left(\frac{-721,979.17}{EI}\right) = \frac{-433,187.50}{EI}$$

$$Y(11) = bb' - t_{BC} = \frac{-433,187.50 + 317,525.00}{EI} = \frac{-115,662.50}{EI}$$

II. Solution by Section 4-7-2

A. Boundary Conditions

$$Y''(0) = Y'''(0) = Y(5') = Y(20') = 0$$

B. Load Function

$$F(x) = R_1\, S'_5(x) + R_2\, S'_{20}(x) - 100\, S_2(x) + 100\, S_{10}(x) -$$

$$2000\, S'_{25}(x) - 50\, S''_{15}(x)$$

C. Laplace Transform of Load Function

$$f(s) = R_1 e^{-5s} + R_2 e^{-20s} - 100 \frac{e^{-2s}}{s} + 100 \frac{e^{-10s}}{s} - 2000\, e^{-25s} -$$

$$50s\, e^{-15s}$$

D. Inverse Transform of $\frac{1}{EI}\left[\frac{f(s)}{s^4}\right]$

$$\frac{1}{EI}\, L^{-1}\left[\frac{f(s)}{s^4}\right] = \frac{1}{EI}\left\{R_1 \frac{(x - 5)^3}{3!} S_5 (x) + R_2 \frac{(x - 20)^3}{3!} S_{20} (x) -\right.$$

$$100 \frac{(x - 2)^4}{4!} S_2 (x) + 100 \frac{(x - 10)^4}{4!} S_{10} (x) -$$

$$\left. 2000 \frac{(x - 25)^3}{3!} S_{25} (x) - 50 \frac{(x - 15)^2}{2!} S_{15} (x)\right\}$$

E. Substituting (A) and (D) into Eq. (4-2b)

$$Y(x) = Y(0) + Y'(0)\, x + \frac{1}{EI}\left\{R_1 \frac{(x - 5)^3}{3!} S_5 (x) +\right.$$

$$R_2 \frac{(x - 20)^3}{3!} S_5 (x) - 100 \frac{(x - 2)^4}{4!} S_2 (x) +$$

$$100 \frac{(x - 10)^4}{4!} S_{10} (x) - 2000 \frac{(x - 25)^3}{3!} S_{25} (x) -$$

$$\left. 50 \frac{(x - 15)^2}{2!} S_{15} (x)\right\}$$

F. Evaluation of Unknown Boundary Conditions and Reactions

By statics,

$$R_1 = \frac{11,200 + 50 - 10,000}{15} = 83\text{-}1/3 \text{ lbs.}$$

$$R_2 = 2,800 - 83\text{-}1/3 = 2,716\text{-}2/3 \text{ lbs.}$$

$$Y(5') = 0 = Y(0) + 5Y'(0) + \frac{1}{EI}\left\{R_1 \cancelto{0}{\frac{(5-5)^3}{3!}} S_5(x) + \right.$$

$$R_2 \frac{(5-20)^3}{3!} \cancelto{0}{S_{20}}(5) - 100 \frac{(5-2)^4}{4!} \cancelto{1}{S_2}(5) +$$

$$100 \frac{(5-10)^4}{4!} \cancelto{0}{S_{10}}(5) - 2000 \frac{(5-25)^3}{3!} \cancelto{0}{S_{25}}(5) -$$

$$\left. 50 \frac{(5-15)^2}{2!} \cancelto{0}{S_{15}}(5) \right\}$$

$$Y(20') = 0 = Y(0) + 20\,Y'(0) + \frac{1}{EI}\left\{R_1 \frac{(20-5)^3}{3!} \cancelto{1}{S_5(20)} + \right.$$

$$R_2 \cancelto{0}{\frac{(20-20)^3}{3!}} S_{20}(20) - 100 \frac{(20-2)^4}{4!} \cancelto{1}{S_2}(20) +$$

$$100 \frac{(20-10)^4}{4!} \cancelto{1}{S_{10}}(20) - 2000 \frac{(20-25)^3}{3!} \cancelto{0}{S_{25}}(20)$$

$$\left. - 50 \frac{(20-15)^2}{2!} \cancelto{1}{S_{15}}(20) \right\}$$

Substituting in the values for R_1 and R_2, these equations reduce to

$$Y(0) + 5Y'(0) = \frac{337.5}{EI}$$

$$Y(0) + 20Y'(0) = \frac{349,483.34}{EI}$$

Solving simultaneously,

$$Y(0) = -\frac{116,044.45}{EI}$$

$$Y'(0) = \frac{23,276.39}{EI}$$

G. General Elastic/Deflection Curve Equation

$$Y(x) = -\frac{116,044.45}{EI} + \frac{23,276.39}{EI}\,x + \frac{1}{EI}\Bigg\{$$

$$83\text{-}1/3\ \frac{(x-5)^3}{3!}\,S_5(x) + 2716\text{-}2/3\ \frac{(x-20)^3}{3!}\,S_{20}(x) -$$

$$100\ \frac{(x-2)^4}{4!}\,S_2(x) + 100\ \frac{(x-10)^4}{4!}\,S_{10}(x) -$$

$$2000\ \frac{(x-25)^3}{3!}\,S_{25}(x) - 50\ \frac{(x-15)^2}{2!}\,S_{15}(x)\Bigg\}$$

$$Y(11) = -\frac{116,044.45}{EI} + \frac{23,276.39}{EI}(11) + \frac{1}{EI}\Bigg\{$$

$$83\text{-}1/3\ \frac{(11-5)^3}{3!} - 100\ \frac{(11-2)^4}{4!} + 100\ \frac{(11-10)^4}{4!}\Bigg\}$$

$$= +\frac{115,662.50}{EI}$$ (+ sign indicates deflection is upward.)

H. Slope, Bending Moment and Shearing Force Equation

a. Slope

$$\phi(x) = \frac{23,276.39}{EI} + \frac{1}{EI}\Bigg\{83\text{-}1/3\ \frac{(x-5)^2}{2}\ S_5(x) +$$

$$2716\text{-}2/3\ \frac{(x-20)^2}{2}\ S_{20}(x) - 100\ \frac{(x-2)^3}{6}\ S_2(x) +$$

$$100\ \frac{(x-10)^3}{6}\ S_{10}(x) - 2000\ \frac{(x-25)^2}{2}\ S_{25}(x) -$$

$$50\ (x-15)\ S_{15}(x)\Bigg\}$$

b. Bending Moment

$$M_b(x) = 83\text{-}1/3\ (x-5)\ S_5(x) + 2716\text{-}2/3\ (x-20)\ S_{20}(x) -$$

$$100\ \frac{(x-2)^2}{2}\ S_2(x) + 100\ \frac{(x-10)^2}{2}\ S_{10}(x) -$$

$$2000\ (x-25)\ S_{25}(x) - 50\ S_{15}(x)$$

c. Shearing Force

$$V(x) = 83\text{-}1/3\ S_5\ (x) + 2716\text{-}2/3\ S_{20}\ (x) - 100\ (x - 2)\ S_2\ (x) +$$

$$100\ (x - 10)\ S_{10}\ (x) - 2000\ S_{25}\ (x)$$

III. Solution by Appendices A and B

$$R_1 = \frac{1}{15}\left[-4800 - 50,000 + 50 + 56,000\right] = 83\text{-}1/3 \text{ lbs.}$$

$$R_2 = \frac{1}{15}\left[-50 + 4,800 + 50,000 - 14,000\right] = 2716\text{-}2/3 \text{ lbs.}$$

$$Y(0) = \frac{1}{15EI}\left\{5\left(\frac{250}{3}\right)\frac{(15)^3}{6} + 5\left[-100\frac{(18)^4}{24} + 100\frac{(10)^4}{24}\right] -\right.$$

$$\left. 20\left[-100\frac{(3)^4}{24}\right] + 5\left[-50\frac{(5)^2}{2}\right]\right\} = -\frac{116,044.45}{EI}$$

$$Y'(0) = \frac{1}{-15EI}\left\{\left(\frac{250}{3}\right)\frac{(15)^3}{6} + \left[-100\frac{(18)^4}{24} + 100\frac{(10)^4}{24}\right] -\right.$$

$$\left.\left[-100\frac{(3)^4}{24}\right] + \left[-50\frac{(5)^2}{2}\right]\right\} = \frac{23,276.39}{EI}$$

$$Y(x) = -\frac{116,044.45}{EI} + \frac{23,276.39}{EI}x + \frac{1}{EI}\left\{83\text{-}1/3\frac{(x-5)^3}{3!}S_5\ (x) +\right.$$

$$2716\text{-}2/3\frac{(x-20)^3}{3!}S_{20}\ (x) - 100\frac{(x-2)^4}{4!}S_2\ (x) +$$

$$100 \frac{(x - 10)^4}{4!} S_{10}(x) - 2000 \frac{(x - 25)^3}{3!} S_{25}(x) -$$

$$\left. 50 \frac{(x - 15)^2}{2!} S_{15}(x) \right\}$$

$$Y(11) = - \frac{116,044.45}{EI} + \frac{23,276.39}{EI}(11) + \frac{1}{EI}\left\{ 83\text{-}1/3 \frac{(11 - 5)^3}{3!} - \right.$$

$$\left. 100 \frac{(11 - 2)^4}{4!} + 100 \frac{(11 - 10)^4}{4!} \right\} = + \frac{115,662.50}{EI}$$

(+ sign indicates deflection is upward.)

$$\phi(x) = \frac{23,276.39}{EI} + \frac{1}{EI}\left\{ 83\text{-}1/3 \frac{(x - 5)^2}{2} S_5(x) + \right.$$

$$2716\text{-}2/3 \frac{(x - 20)^2}{2} S_{20}(x) - 100 \frac{(x - 2)^3}{6} S_2(x) +$$

$$100 \frac{(x - 10)^3}{6} S_{10}(x) - 2000 \frac{(x - 25)^2}{2} S_{25}(x) -$$

$$\left. 50 (x - 15) S_{15}(x) \right\}$$

$$M_b(x) = 83\text{-}1/3 (x - 5) S_5(x) + 2716\text{-}2/3 (x - 20) S_{20}(x) -$$

$$100 \frac{(x - 2)^2}{2} S_2(x) + 100 \frac{(x - 10)^2}{2} S_{10}(x) -$$

$$2000 (x - 25) S_{25}(x) - 50 S_{15}(x)$$

$$V(x) = 83\text{-}1/3\, S_5(x) + 2716\text{-}2/3\, S_{20}(x) - 100\,(x-2)\, S_2(x) +$$

$$100\,(x-10)\, S_{10}(x) - 2000\, S_{25}(x)$$

VITA

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