RESPONSE OF PIPING SYSTEMS TO RANDOM INPUTS

By

LARRY HOVE

Bachelor of Science

Oklahoma State University

Stillwater, Oklahoma

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L. E. H.

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NOMENCLATURE

a	Velocity of Wave Propagation.
b	Pipe Wall Thickness.
c_n	Fourier Coefficient.
j	$\sqrt{-1}$.
ℓ	Length.
n	Exponent of $\overline{q}$ for Mean Flow.
p	Variation from a Mean Pressure.
$\overline{p}$	Mean Pressure.
$\overline{p}_f$	Pressure Required to Overcome the Frictional Resistance of the Pipe.
p_t	Instantaneous Pressure $= \overline{p} + p$.
q	Variation from a Mean Volume Flow Rate.
$\overline{q}$	Mean Volume Flow Rate.
q_t	Instantaneous Volume Flow Rate $= \overline{q} + q$.
r	Refers to the Receiving End of the Piping System.
s	Laplace Transform with Respect to Time Variable.
t	Time.
u	Velocity in the x-direction.
x	Distance from Receiving End of the Pipe.
A	Cross Sectional Area.
B	System Function.
C	Coefficient of Capacitance.
D	Inside Diameter of the Pipe.

E	. . . .	Modulus of Elasticity of the Pipe.
G	. . . .	Transfer Function.
H	. . . .	Transfer Function.
K	. . . .	Bulk Modulus of the Fluid.
K'	. . . .	System Modulus of Elasticity.
L	. . . .	Coefficient of Inertia.
M	. . . .	System Function.
N	. . . .	Denotes Nonlinear Operation.
$P^{\cdot}$	. . . .	Pressure Variation Transformed with Respect to Time.
$\hat{P}$	. . . .	Pressure Variation Transformed with Respect to Time and Distance.
Q	. . . .	Flow Rate Variation Transformed with Respect to Time.
$\hat{Q}$	. . . .	Flow Rate Variation Transformed with Respect to Time and Distance.
R	. . . .	Coefficient of Resistance.
T	. . . .	System Function.
U	. . . .	Unit Step Function.
V	. . . .	Unit Step Function Series.
Z_c	. . . .	Characteristic Impedance.
α	. . . .	Attenuation Constant.
β	. . . .	Phase Constant.
γ	. . . .	Propagation Constant.
δ	. . . .	Dirac Delta Function.
η	. . . .	Number of Impeller Blades in a Pump.
λ	. . . .	Laplace Transform with Respect to x Variable.
ν	. . . .	Kinematic Viscosity.
ρ	. . . .	Mass Density of the Fluid.

τ . . . . Time.

ϕ . . . . Correlation Function.

ω . . . . Circular Frequency.

Φ . . . . Spectral Density.

INTRODUCTION

Variations from the mean flow in a piping system lead to pressure fluctuations. If of sufficient magnitude, these pressure fluctuations can cause failure or can affect the performance of the system. Thus, pressure fluctuations of large magnitude have been of interest to the engineer for many years.

The mechanisms, or "inputs", causing large pressure variations are usually mathematically expressible as functions of time, or "deterministic". Examples of such inputs are: a positive displacement pump (periodic function); an instantaneous valve closure (step function); a gradual valve closure (ramp function); and a power failure in a centrifugal pumping system. Because the engineer has been chiefly concerned with large amplitude pressure fluctuations, most of the previous work in predicting pressure variations in a piping system has been done with deterministic inputs.

Recently, the noise associated with pressure fluctuations, both large and small, in piping systems has gained the interest of engineers associated with naval defense. This interest has been stimulated by major improvements in underwater sound detection systems. Thus, attention must be given to all sources causing pressure variations. Other than the inputs described in the preceding paragraph, two other sources of pressure fluctuations might be found. One is the flow noise generated by pipe discontinuities such as wall roughness, valves, orfices, and changes in cross section. The other is a rotating pump. Both of these inputs are "random" processes, that is, processes not mathematically expressible as functions of time.

Previous work in the analysis of systems with random inputs has been chiefly concentrated in the fields of electronics, vibrations, and acoustics. Very little, if any, work has been published dealing with the analysis of complex hydraulic systems with random inputs. This thesis uses the basic statistical theory of systems analysis and applies it to piping systems with random inputs. Specifically, it determines the spectral density of the pressure fluctuations at the source of flow in terms of the system parameters and the spectral density of the flow rate fluctuations at the source or the termination. Methods for determining spectral densities of flow rate variations are presented. The analysis of an example piping system is demonstrated, and the results are manipulated to allow digital computation. Appedices A and B are to familiarize the reader with the physical system and the statistical theory.

CHAPTER I

REVIEW OF THE LITERATURE

1.1. Analysis of Piping Systems with Deterministic Inputs.

Pressure fluctuations in closed conduits first came to the attention of Joukowsky[1]* and Allievi[2] at the turn of the century in their studies of "water-hammer" caused by instantaneous valve closure in penstocks of dams. The theory of pipeline pressure surge prediction was advanced considerably by the introduction of transform calculus to the problem by Wood[3] and Rich[4]. Textbooks by Thompson[5], Rouse[6], and Rich[7] illustrate modern methods in predicting pressure fluctuations in piping systems.

Waller[8] presented experimental evidence that the nonlinear frictional resistance to flow could be approximated resulting in a set of linear partial differential equations applicable to specified flow ranges. Waller[9] also presented a frequency response solution to the problem of transient flow in pipelines.

Many books and papers [10, 11, 12, 13, 14] in the acoustics field deal with the creation and transmission of noise in liquid-filled pipes. Although not directly connected to this thesis, they do provide an insight into the physical aspects of the problem.

The results of previous investigations applicable to this thesis are

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*Numbers in parentheses refer to references in Bibliography.

summarized in Appendix A.

1.2. Analysis of Systems with Random Inputs.

The basic theory of statistical methods used in the analysis of linear systems can be found in many recent books. Notable examples are by Azeltine[15], Harris and Crede[16], Chang[17], and Newton, Gould, and Kaiser[18]. Appendix B contains a summary of basic statistical theory and its application to linear systems.

CHAPTER II

THE SYSTEM: OUTPUT SPECTRAL DENSITY DETERMINATION

The spectral density of pressure fluctuations at the source of flow is called the "output". The engineer dealing with noise problems in a piping system will probably be interested in one of two situations. One is the prediction of pressure fluctuations at any point in a system. The other is, if a maximum allowable level of pressure variations in the terminating system is given, what is the allowable level of pressure fluctuations at the source. In both cases, the spectral density of pressure fluctuations at the source can be considered the desired end result, or output. In this chapter the method of analysis for both situations is presented.

2.1. The General System

In any physical piping system, the receiving end, or termination, must be either open or closed. One of two boundary conditions is known:

$$p(r,t) = 0 \quad \text{(open end);} \tag{1}$$

or

$$q(r,t) = 0 \quad \text{(closed end)} . \tag{2}$$

Since open end termination is more likely to occur, all terminations in this thesis are considered open.

There are two possible sources of pressure fluctuations in a piping system. One is pipe discontinuities such as wall roughness, valves, orfices, and changes in cross section. The flow noise generated by these sources is assummed negligible in this thesis. This assumption should be valid in a well designed system. The other source is the prime mover, and it will be considered the only source.

The functions p(x, t) and q(x, t) are random processes in time, or "stochastic" processes. They are stationary, ergodic processes. The functions are correlated with each other, that is, the associated cross-correlation function is not zero.

The frequency response of the simple system shown in Figure 1 at any point x is as derived in Appendix A:

$$P(x_1, j\omega) = P(r, j\omega) \cosh \gamma_1 x_1 + Z_{c1} Q(r, j\omega) \sinh \gamma_1 x_1 ; \tag{3}$$

and

$$Q(x_1, j\omega) = Q(r, j\omega) \cosh \gamma_1 x_1 + \frac{P(r, j\omega)}{Z_{c1}} \sinh \gamma_1 x_1 \tag{4}$$

where

$$\gamma^2 = j\omega \, C \, (R + j\omega L) ; \tag{5}$$

$$Z_c^2 = \frac{R + j\omega L}{j\omega C} ; \tag{6}$$

$$C = \frac{A}{K'} ; \tag{7}$$

$$L = \frac{\rho}{A} ; \tag{8}$$

and

$$R = \frac{n\bar{p}_f}{\ell q} . \tag{9}$$

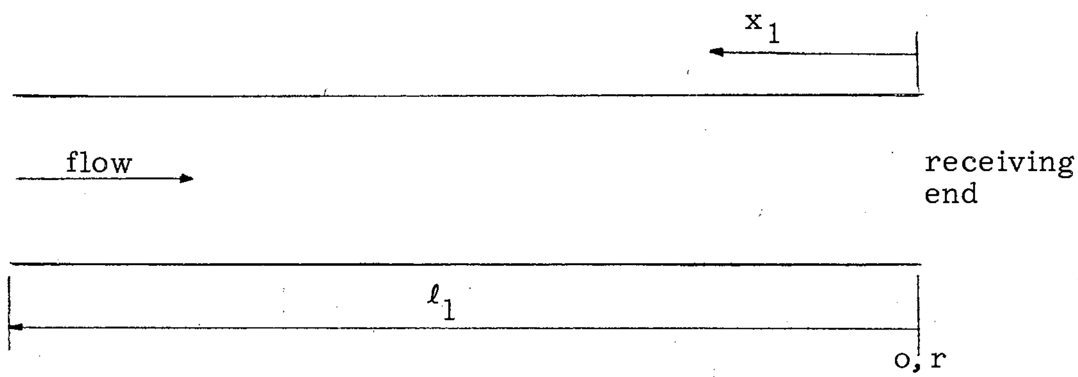


Fig. 1. A simple piping system.

For open termination,

$$P(x_1, j\omega) = Z_{c_1} Q(r, j\omega) \sinh \gamma_1 x_1 , \tag{10}$$

and

$$Q(x_1, j\omega) = Q(r, j\omega) \cosh \gamma_1 x_1 . \tag{11}$$

At $x_1 = \ell_1$, the source end,

$$P(\ell_1, j\omega) = Q(r, j\omega) Z_{c1} \sinh \gamma_1 \ell_1$$

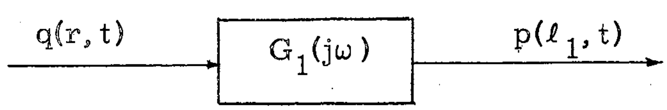
$$= G_1(j\omega) Q(r, j\omega) , \tag{12}$$

and

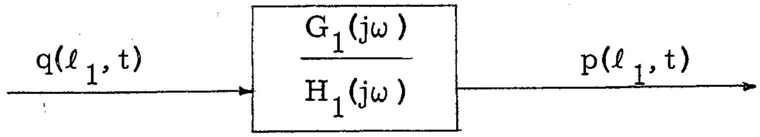
$$Q(\ell_1, j\omega) = Q(r, j\omega) \cosh \gamma_1 \ell_1$$

$$= H_1(j\omega) Q(r, j\omega) \tag{13}$$

where $G_1(j\omega)$ and $H_1(j\omega)$ are system transfer functions. The block diagrams



and



are evident from Equations (12) and (13). Then (Appendix B, Section B. 5),

$$\Phi_{pp}(\ell_1, \omega) = |G_1(j\omega)|^2 \; \Phi_{qq}(r, \omega) \; , \qquad (14)$$

or

$$\Phi_{pp}(\ell_1, \omega) = \frac{|G_1(j\omega)|^2}{|H_1(j\omega)|^2} \; \Phi_{qq}(\ell_1, \omega) \; . \qquad (15)$$

In the following sections, the preceding theory is extended to more complex systems.

2. 2. Series Piping Systems.

A series piping system is shown in Figure 2.

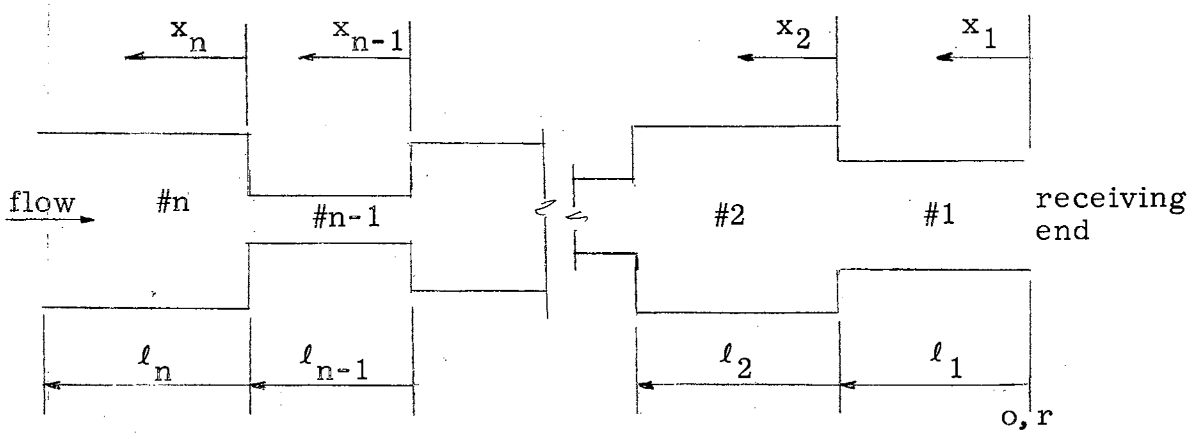


Fig. 2. A series piping system.

The following notation is employed:

$$B_i(j\omega) = \cosh \gamma_i \ell_i \quad ; \tag{16}$$

$$T_i(j\omega) = Z_{ci} \sinh \gamma_i \ell_i \quad ; \tag{17}$$

and

$$M_i(j\omega) = \frac{1}{Z_{ci}} \sinh \gamma_i \ell_i \tag{18}$$

where i refers to the pipe number, $i = 1, 2, 3, \cdots, n$. Then

$$P(\ell_i, j\omega) = B_i(j\omega)\, P(\ell_{i-1}, j\omega) + T_i(j\omega)\, Q(\ell_{i-1}, j\omega) , \tag{19}$$

and

$$Q(\ell_i, j\omega) = B_i(j\omega)\, Q(\ell_{i-1}, j\omega) + M_i(j\omega)\, P(\ell_{i-1}, j\omega). \tag{20}$$

For open end termination, substitution for $P(\ell_{i-1}, j\omega)$ and $Q(\ell_{i-1}, j\omega)$ in the above equations will yield $P(\ell_i, j\omega)$ and $Q(\ell_i, j\omega)$ in the form

$$P(\ell_i, j\omega) = G_i(j\omega)\, Q(r, j\omega) \quad , \tag{21}$$

and

$$Q(\ell_i, j\omega) = H_i(j\omega)\, Q(r, j\omega) \tag{22}$$

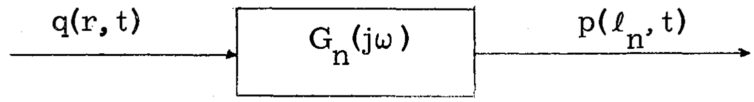
where

$$G_i = G_{i-1} B_i + H_{i-1} T_i \quad , \tag{23}$$

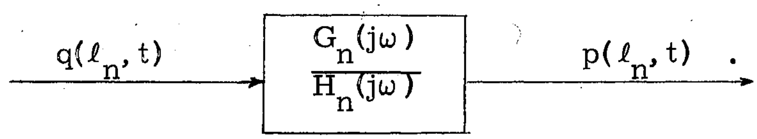
and

$$H_i = H_{i-1} B_i + G_{i-1} M_i \quad . \tag{24}$$

For $i = n$, Equations (21) and (22) give the block diagrams



and



The desired results are:

$$\Phi_{pp}(\ell_n, \omega) = |G_n(j\omega)|^2 \, \Phi_{qq}(r, \omega) \quad ; \tag{25}$$

or

$$\Phi_{pp}(\ell_n, \omega) = \left|\frac{G_n(j\omega)}{H_n(j\omega)}\right|^2 \Phi_{qq}(\ell_n, \omega) \quad . \tag{26}$$

The transfer functions $G_n(j\omega)$ and $H_n(j\omega)$ are combinations of the system functions $B_i(j\omega)$, $T_i(j\omega)$, and $M_i(j\omega)$, $i = 1, 2, 3, \cdots, n$. $G_1(j\omega)$ and $H_1(j\omega)$ are given in the preceding section. The transfer functions for $n = 2$, 3, and 4 follow.

$n = 2$:

$$G_2(j\omega) = B_2 T_1 + T_2 B_1 \quad ; \tag{27}$$

and

$$H_2(j\omega) = B_2 B_1 + M_2 T_1 \quad . \tag{28}$$

$n = 3$:

$$G_3(j\omega) = B_3 G_2 + T_3 H_2 \quad ; \tag{29}$$

and

$$H_3(j\omega) = B_3 H_2 + M_3 G_2 \quad . \tag{30}$$

n = 4:

$$G_4(j\omega) = B_4 G_3 + T_4 H_3 \;; \tag{31}$$

and

$$H_4(j\omega) = B_4 H_3 + M_4 G_3 \;. \tag{32}$$

This theory can be extended to any number of series components, but it is noted that the transfer functions become increasingly complicated as the number of components increases.

2. 3. Parallel Piping Components.

In Figure 3 a piping system is shown containing pipes in parallel. The parallel components are not necessarily dimensionally or materially identical. The system to the right and left of the parallel pipes consists of either a single pipe or a series of pipes.

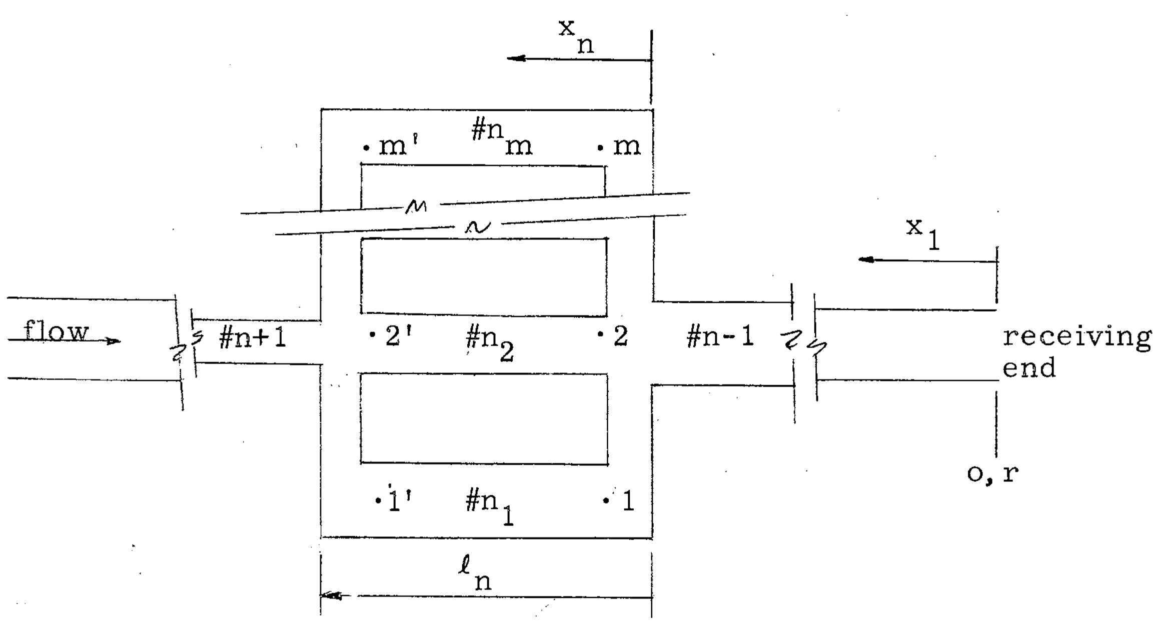


Fig. 3. Piping system with parallel components.

From the preceding section,

$$P(\ell_{n-1}, j\omega) = G_{n-1}(j\omega)\, Q(r, j\omega), \tag{33}$$

and

$$Q(\ell_{n-1}, j\omega) = H_{n-1}(j\omega)\, Q(r, j\omega) \;. \tag{34}$$

To analyze the complete system, $P(\ell_n, j\omega)$ and $Q(\ell_n, j\omega)$ must be found in terms of a transfer function and $Q(r, j\omega)$.

The following notation is employed:

$$B_k(j\omega) = \cosh \gamma_k \ell_k \;; \tag{35}$$

$$T_k(j\omega) = Z_{ck} \sinh \gamma_k \ell_k \;; \tag{36}$$

and

$$M_k(j\omega) = \frac{1}{Z_{ck}} \sinh \gamma_k \ell_k \tag{37}$$

where k refers to the parallel pipe number, $k = 1, 2, \cdots, m$. The relationships

$$P(\ell_{n-1}, j\omega) = P(1, j\omega) = P(2, j\omega) = \cdots = P(m, j\omega), \tag{38}$$

$$Q(\ell_{n-1}, j\omega) = Q(1, j\omega) + Q(2, j\omega) + \cdots + Q(m, j\omega), \tag{39}$$

$$P(\ell_n, j\omega) = P(1', j\omega) = P(2', j\omega) = \cdots = P(m', j\omega), \tag{40}$$

and

$$Q(\ell_n, j\omega) = Q(1', j\omega) + Q(2', j\omega) + \cdots + Q(m', j\omega) \tag{41}$$

are valid. Using Equations (38) and (40),

$$
\left.\begin{aligned}
P(\ell_n, j\omega) &= B_1(j\omega)\, P(\ell_{n-1}, j\omega) + T_1(j\omega)\, Q(1, j\omega) \;; \\
P(\ell_n, j\omega) &= B_2(j\omega)\, P(\ell_{n-1}, j\omega) + T_2(j\omega)\, Q(2, j\omega) \;; \\
&\cdots \\
P(\ell_n, j\omega) &= B_m(j\omega)\, P(\ell_{n-1}, j\omega) + T_m(j\omega)\, Q(m, j\omega) \;.
\end{aligned}\right\} \quad (42)
$$

Also,

$$
\left.\begin{aligned}
Q(1', j\omega) &= B_1(j\omega)\, Q(1, j\omega) + M_1(j\omega)\, P(\ell_{n-1}, j\omega) \;; \\
Q(2', j\omega) &= B_2(j\omega)\, Q(2, j\omega) + M_2(j\omega)\, P(\ell_{n-1}, j\omega) \;; \\
&\cdots \\
Q(m', j\omega) &= B_m(j\omega)\, Q(m, j\omega) + M_m(j\omega)\, P(\ell_{n-1}, j\omega) \;.
\end{aligned}\right\} \quad (43)
$$

Equations (39), (41), (42), and (43) can be manipulated as follows to form $m + 1$ independent equations. Substitute Equations (33) and (34) for all $P(\ell_{n-1}, j\omega)$ s and $Q(\ell_{n-1}, j\omega)$ s . In Equations (42) subtract each equation from the first equation. Add Equations (43) and substitute Equation (41). The results are:

$$
\left.\begin{aligned}
T_1 Q(1, j\omega) - T_2 Q(2, j\omega) &= G_{n-1}(B_2 - B_1)\, Q(r, j\omega) \;; \\
T_1 Q(1, j\omega) - T_3 Q(3, j\omega) &= G_{n-1}(B_3 - B_1)\, Q(r, j\omega) \;; \\
&\cdots \\
T_1 Q(1, j\omega) - T_k Q(k, j\omega) &= G_{n-1}(B_k - B_1)\, Q(r, j\omega) \;; \\
&\cdots \\
T_1 Q(1, j\omega) - T_m Q(m, j\omega) &= G_{n-1}(B_m - B_1)\, Q(r, j\omega) \;;
\end{aligned}\right\} \quad (44)
$$

$$\left.\begin{aligned} &Q(\ell_n, j\omega) - B_1 Q(1, j\omega) - B_2 Q(2, j\omega) - B_3 Q(3, j\omega) - \cdots - \\ &\qquad B_k\, Q(k, j\omega) - \cdots - B_m\, Q(m, j\omega) = G_{n-1}(M_1 + \\ &\qquad M_2 + M_3 + \cdots + M_k + \cdots + M_m)\, Q(r, j\omega) \;; \\ &Q(1, j\omega) + Q(2, j\omega) + Q(3, j\omega) + \cdots + Q(k, j\omega) + \cdots + \\ &\qquad Q(m, j\omega) = H_{n-1}\, Q(r, j\omega). \end{aligned}\right\}$$

In matrix form,

$$\begin{bmatrix} 0 & T_1 & -T_2 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & T_1 & 0 & -T_3 & \cdots & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ 0 & T_1 & 0 & 0 & \cdots & -T_k & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ 0 & T_1 & 0 & 0 & \cdots & 0 & \cdots & -T_m \\ 1 & -B_1 & -B_2 & -B_3 & \cdots & -B_k & \cdots & -B_m \\ 0 & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} Q(\ell_n, j\omega) \\ Q(1, j\omega) \\ Q(2, j\omega) \\ Q(3, j\omega) \\ \cdot \\ \cdot \\ \cdot \\ Q(k, j\omega) \\ \cdot \\ \cdot \\ \cdot \\ Q(m, j\omega) \end{bmatrix} = Q(r, j\omega) \begin{bmatrix} G_{n-1}(B_2 - B_1) \\ G_{n-1}(B_3 - B_1) \\ \cdot \\ G_{n-1}(B_k - B_1) \\ \cdot \\ G_{n-1}(B_m - B_1) \\ G_{n-1} \sum_{k=1}^{m} M_k \\ H_{n-1} \end{bmatrix} \tag{45}$$

Equation (45) is solved for $Q(\ell_n, j\omega)$ and any one of the $Q(k, j\omega)$s. Substitution of $Q(k, j\omega)$ and Equation (33) into the kth equation of Equations (42) will give $P(\ell_n, j\omega)$. The results in the desired form are:

$$P(\ell_n, j\omega) = G_n(j\omega)\, Q(r, j\omega) \;; \tag{46}$$

and

$$Q(\ell_n, j\omega) = H_n(j\omega)\, Q(r, j\omega). \tag{47}$$

For m = 2,

$$H_n(j\omega) = \frac{H_{n-1}(T_1B_2 + T_2B_1) + G_{n-1}[B_1^2 - B_2^2 + (M_1 + M_2)(T_1 + T_2)]}{T_1 + T_2}, \tag{48}$$

and

$$G_n(j\omega) = B_1\, G_{n-1} + T_1 \left[\frac{G_{n-1}\,(B_1 - B_2) - T_2 H_{n-1}}{T_1 + T_2}\right]. \tag{49}$$

If the parallel components are dimensionally and materially identical,

$$G_n(j\omega) = B_1\, G_{n-1} + \frac{1}{m}\, T_1\, H_{n-1}, \tag{50}$$

and

$$H_n(j\omega) = B_1\, H_{n-1} + m\, M_1\, G_{n-1} \tag{51}$$

for any number of pipes.

CHAPTER III

THE SYSTEM: PARAMETERS AND INPUT SPECTRAL DENSITY DETERMINATION

Chapter II demonstrated the spectral density determination of the pressure fluctuations at the source. It was shown to be the product of a transfer function and the spectral density of the flow variations at either end. In this chapter, the system parameters are divided into three dependent groups for discussion. They are: physical parameters; frequency parameters; and system functions. In the introduction to Chapter II, two applications of the theory are stated; one of prediction, the other of specification. For the former, an analysis of the prime mover, or source, yields a flow rate spectral density. In the latter, a specified level of pressure fluctuations at some point in the system is converted into a spectral density. This chapter discusses the determination of both spectrums and their use.

3. 1. Physical Parameters

The three parameters, C, L, and R, defined by Equations (7), (8), and (9) are called the physical parameters. Each has a definite physical significance as discussed in the following paragraphs.

C, the coefficient of capacity, is a measure of the elasticity of the fluid and pipe.

L, the coefficient of inertia, is indicative of the inertial effect of the fluid mass.

R, the coefficient of resistance, determines the friction encountered by the fluid at the pipe walls and in the fluid itself. This friction is the mechanism of energy dissipation in the system.

3.2. Frequency Parameters.

The parameters γ and Z_c are functions of the physical parameters and the frequency. They are called the "frequency" parameters, and both are complex quantities.

γ, the propagation coefficient, shows the effects of the system on a propagated wave. It is given by Equation (5) as

$$\gamma(j\omega) = \sqrt{j\omega C(R + j\omega L)}\ . \tag{52}$$

The above equation can be put in the form

$$\gamma(j\omega) = \alpha + j\beta \tag{53}$$

where $\alpha + j\beta$ is one of the two roots of Equation (52). The real part of the other root is negative and, as pointed out in the following discussion, is physically unrealizable.

α, the attenuation constant, is given by

$$\alpha = \left[\frac{\omega C}{2}\left(\sqrt{R^2 + \omega^2 L^2} - \omega L\right)\right]^{\frac{1}{2}}. \tag{54}$$

It determines the damping in the vibrating system. It is evident α cannot be negative in a passive system. As would be expected, the amount of damping is dependent on the frequency ω and the physical parameters.

β, the phase constant, is given by

$$\beta = \left[\frac{\omega C}{2}\left(\sqrt{R^2 + \omega^2 L^2} + \omega L\right)\right]^{\frac{1}{2}}. \tag{55}$$

It affects the period of oscillation of a propagated wave. β is also dependent on the frequency and physical parameters.

Z_c, the characteristic impedance, is given by Equation (6) as

$$Z_c(j\omega) = \sqrt{\frac{R + j\omega L}{j\omega C}} \quad . \tag{56}$$

By determining the two roots of Equation (56), it can be put in the form

$$Z_c(j\omega) = X_c + jY_c \tag{57}$$

where $X_c + jY_c$ is one of the roots. The real part of the other root is negative and, as will be pointed out, is physically unrealizable.

Then,

$$X_c = \left[\frac{1}{2C}\left(\sqrt{L^2 + \frac{R^2}{\omega^2}} + L\right)\right]^{\frac{1}{2}} \quad , \tag{58}$$

and

$$Y_c = -\left[\frac{1}{2C}\left(\sqrt{L^2 + \frac{R^2}{\omega^2}} - L\right)\right]^{\frac{1}{2}} \quad . \tag{59}$$

For discussion purposes, let $\frac{R^2}{\omega^2} \rightarrow 0$. Then,

$$Z_c = \sqrt{\frac{L}{C}} \quad . \tag{60}$$

Introducing the velocity of wave propagation in the fluid as

$$a = \sqrt{\frac{K'}{\rho}} \tag{61}$$

Equation (60) becomes

$$Z_c = \frac{(K'/A)}{a} \quad . \tag{62}$$

It is concluded that Z_c is not especially dependent on the frequency, but it is heavily dependent on the physical characteristics of the system. Also, Z_c is in the form of an impedance, that is, the complex ratio of pressure to flow rate. Thus, Z_c is called the characteristic impedance.

Speaking in terms of impedances, X_c is the "resistance", and Y_c is the "reactance". In a passive system, the resistance cannot be negative; therefore, the qualification of the roots of Z_c is valid.

To simplify computations, Z_c can be written in terms of α and β as

$$Z_c(j\omega) = \frac{1}{\omega C}(\beta - j\alpha) \quad . \tag{63}$$

3.3. System Functions.

The system functions B, T, and M are defined by either Equations (16), (17), and (18) or Equations (35), (36), and (37). They are functions of the frequency parameters and take their physical significance from these parameters. They are called system functions, because they determine the response of the system.

In Chapter II the output was expressed in terms of the input and a transfer function. The transfer function was determined from combinations of the system functions. Thus, it is desirable to have B, T, and M in easily computed forms.

Using hyperbolic identities and complex algebra,

$$\begin{aligned} B(j\omega) &= \cosh \gamma\ell \\ &= \cosh \alpha\ell \cos \beta\ell + j \sinh \alpha\ell \sin \beta\ell \; ; \end{aligned} \tag{64}$$

$$\begin{aligned} T(j\omega) &= Z_c \sinh \gamma\ell \\ &= \frac{1}{\omega C}[\alpha \cosh \alpha\ell \sin \beta\ell + \beta \sinh \alpha\ell \cos \beta\ell] \\ &\quad + j\frac{1}{\omega C}[\beta \cosh \alpha\ell \sin \beta\ell - \alpha \sinh \alpha\ell \cos \beta\ell] \; ; \end{aligned} \tag{65}$$

and

$$M(j\omega) = \frac{1}{Z_c} \sinh \gamma \ell$$

$$= \frac{\omega C}{\beta^2+\alpha^2} [\beta \sinh \alpha \ell \cos \beta \ell -$$

$$\alpha \cosh \alpha\ell \sin \beta\ell] + j \frac{\omega C}{\beta^2+\alpha^2} \tag{66}$$

$$[\beta \cosh \alpha\ell \sin \beta\ell + \alpha \sinh \alpha\ell \cos \beta\ell] .$$

3.4. Input Spectral Densities.

From Chapter II, the spectral density of the pressure or the flow rate fluctuations at any point in a piping system can be obtained from one of the following equations:

$$\Phi_{pp}(\ell_n, \omega) = |G_n(j\omega)|^2 \Phi_{qq}(r, \omega) ; \tag{67}$$

$$\Phi_{qq}(\ell_n, \omega) = |H_n(j\omega)|^2 \Phi_{qq}(r, \omega) ; \tag{68}$$

or

$$\Phi_{pp}(\ell_n, \omega) = \frac{|G_n(j\omega)|^2}{|H_n(j\omega)|^2} \Phi_{qq}(\ell_n, \omega). \tag{69}$$

The transfer functions G_n and H_n for complex piping systems are given in Chapter II. The preceding sections dealt with the system parameters that are included in the transfer functions. It remains now to determine an "input" spectral density. Just what is to be termed input depends on what is known.

If a pressure-frequency plot for some point in the system is available, the spectral density of the pressure can be obtained using the theory in Appendix B, Section B. 6. If the pressure is an actual measurement, the spectral density of the pressure or flow rate can be predicted at any point in the system. If it is a specification, a limiting spectral density of the pressure or flow rate at any point can be obtained.

In the case where the problem is to predict the pressure fluctuations before the system is built, complications arise. It is apparent that the spectral density of pressure or flow rate at some point must be known. An analysis of the prime mover to obtain a flow rate spectrum is the most logical approach. An example of this type of analysis is in the following section.

In all cases, once any one of the spectral quantities in Equations (67), (68), or (69) is known, simple manipulations of the equations will give expressions for both spectral densities at any point in the system. With the theory in Appendix B, Section B. 6, these can be changed to amplitude-frequency plots.

3. 5. Centrifugal Pump Analysis.

In systems with random inputs, the prime mover, or source, is usually a rotating pump. The most common of these is a centrifugal pump. This section is devoted to the determination of flow rate spectral density of a centrifugal pump.

Due to the lack of published material dealing with the minute details of centrifugal pump operation, most of the theory in this section is based on assumptions. It is felt that none of the conditions imposed are without basis, but they do come mostly from intuition. In any event, the methods proposed seem useful in the analysis of centrifugal pumps with adaptations to other pumps possible.

The volute-type centrifugal pump shown in Figure 4 consists of an impeller rotating within a case. Fluid enters the impeller in the eye, flows radially outward, and is discharged around the circumference into the casing.

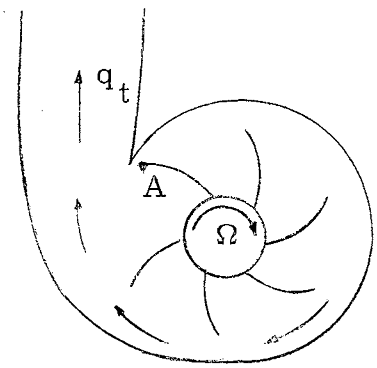


Fig. 4. A volute-type centrifugal pump.

The average rotational speed of the impeller is Ω revolutions per second. The number of impeller blades is η.

The total flow q_t from the pump is the sum of the total flow from each blade q_{tk}. Then,

$$q_t(t) = \sum_{k=1}^{\eta} q_{tk}(t) \tag{70}$$

Graphically, the flow from one blade through one revolution starting at point A in Figure 4 should approximate Figure 5.

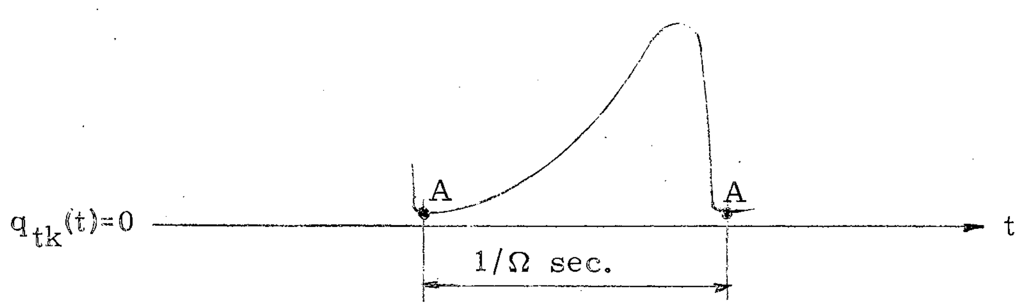


Fig. 5. Flow from one impeller blade through one revolution.

By further approximation of the flow rate pulse shown in Figure 5, the flow rate from one impeller is assumed to be as shown in Figure 6.

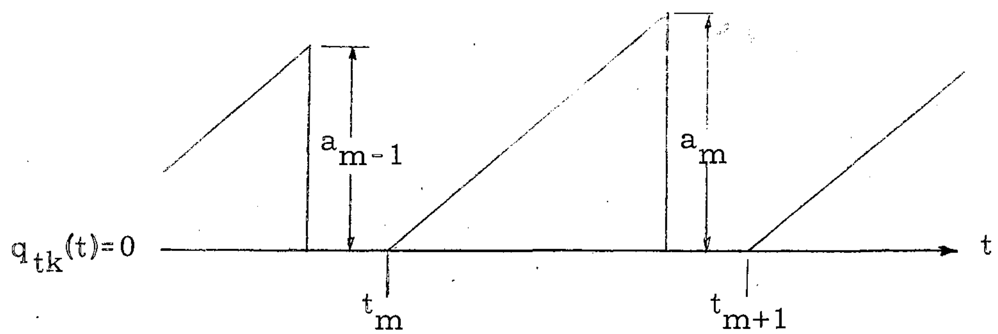


Fig. 6. Approximated flow from one impeller blade.

The total flow is the sum of η series of pulses which will be superimposed on one another but started at different instants of time. The flow is given by the mathematical expression

$$q_t(t) = \sum_{i=-\infty}^{\infty} a_i f(t-t_i) \tag{71}$$

where the pulse amplitudes a_i and times of occurrence t_i are random values. $f(t-t_i)$ is the general waveform of unit amplitude displaced t_i seconds.

The wave form $f(t)$ is a sawtooth pulse. To express $f(t)$ mathematically the duration of the pulse must be assumed. It is known to be less than $\frac{1}{\Omega}$ seconds. Denoting the pulse duration by $\frac{\epsilon}{\Omega}$ where ϵ is some number between 0 and 1 ,

$$f(t) = \frac{\Omega}{\epsilon} t\, U(t) - \frac{\Omega}{\epsilon}\left(t - \frac{\epsilon}{\Omega}\right) U\left(t - \frac{\epsilon}{\Omega}\right) - U\left(t - \frac{\epsilon}{\Omega}\right) . \tag{72}$$

All that is known about a_i and t_i is that the average number of occurrences per second is $\eta\Omega$. From this an average a_i can be obtained. The area under $\eta\Omega$ of the impulses must equal the mean flow $\overline{q}$. Therefore,

$$(a_i)_{av} = \frac{2\overline{q}}{\epsilon\eta} . \tag{73}$$

As would be expected, the choice of ϵ plays an important role in determining $(a_i)_{av}$. ϵ is probably best determined by a close study of the geometry of the pump.

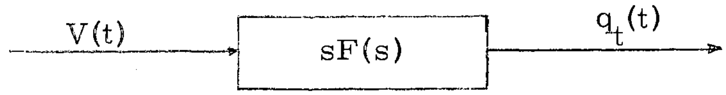
It is reasonable to assume that the flow rate q_t is a stationary, ergodic process. Using the previous results and discussions, the derivation for the spectral density of the flow rate follows.

The statistical theory is from Appendix B with one exception. In Section B. 4 an expression is derived for spectral densities using Fourier transforms. This can be done using Laplace transforms with the end result being

$$\Phi_{xx}(s) = \lim_{T \to \infty} \frac{1}{T} \widetilde{X_T(s) X_T(-s)} , \tag{74}$$

where the function x(t) is truncated at 0 and T. The spectral density is then found by letting $s = j\omega$. Equation (74) avoids difficulties with functions not having Fourier transforms.

If the Laplace transform of the wave form given by Equation (72) is considered a transfer function, Equation (71) can be represented by the block diagram



where V(t) is a step series defined by

$$V(t) = \sum_{i=-\infty}^{\infty} a_i \, U(t - t_i) \ . \tag{75}$$

Then,

$$\Phi_{q_t q_t}(\omega) = \omega^2 \, F(j\omega) \, F(-j\omega) \, \Phi_{VV}(\omega) \tag{76}$$

where

$$F(j\omega) \, F(-j\omega) = \frac{\Omega^2}{\omega^4 \epsilon^2} \left(2 + \frac{\omega^2 \epsilon^2}{\Omega^2} - 2 \cos \frac{\omega \epsilon}{\Omega} - \frac{2\omega\epsilon}{\Omega} \sin \frac{\omega\epsilon}{\Omega}\right) . \tag{77}$$

The step function series given by Equation (75) is shown in Figure 7.

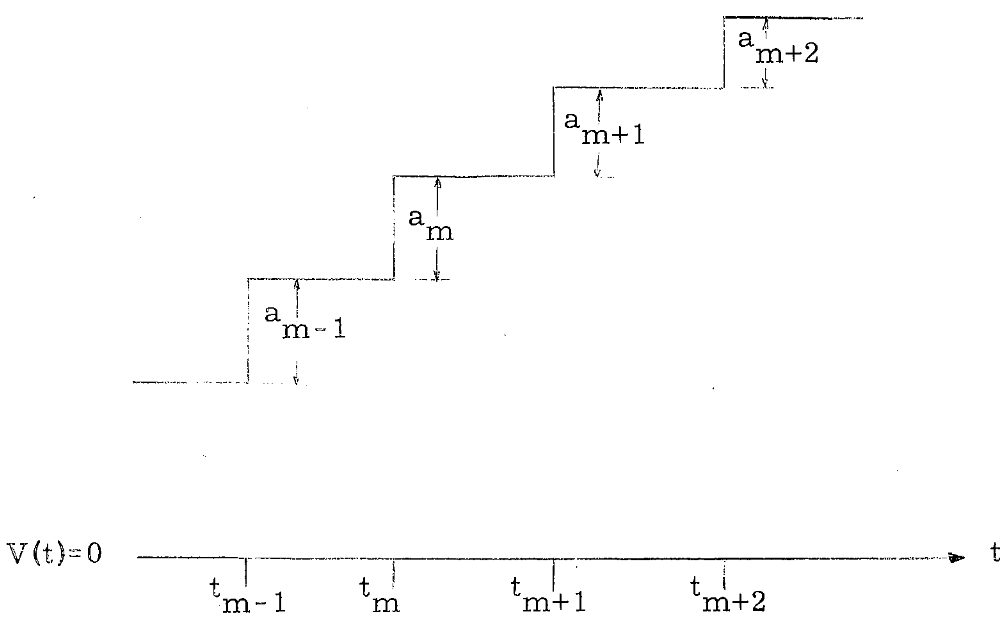


Fig. 7. A step function series.

The expected value of $\overline{V(t)^2}$ is the mean-square value of the amplitude times the number of inputs. Therefore, the correlation functions $\phi_{VV}(0)$ and $\phi_{VV}(\tau)$ are infinite. However, the spectral density must be finite due to physical limitations of the pump. Thus, the spectral density is calculated in a somewhat different manner.

The function V(t) is truncated at 0 and T. Then the number of input amplitudes is $\eta\Omega T$. The contribution of one step a_i at $t = t_i$ is

$$\Phi_{V_i V_i}(s) = \lim_{T\to\infty} \frac{1}{T} \widetilde{V_{T_i}(s)\, V_{T_i}(-s)} \tag{78}$$

where

$$V_{T_i}(t_i) = a_i\, U(t_i) \quad , \tag{79}$$

and

$$V_{T_i}(s) = \frac{a_i}{s} \; . \tag{80}$$

Then,

$$\Phi_{V_i V_i}(\omega) = \frac{1}{\omega^2} \lim_{T\to\infty} \frac{1}{T} \widetilde{a_i^2} \; . \tag{81}$$

At this point it would be desirable to know more about the amplitudes a_i. A decision is required as to whether each amplitude is independent of or dependent upon any amplitude occurring at a previous or later time. It appears that the value of one amplitude should have little affect on the value of the next amplitude and less affect on succeeding amplitudes. Therefore, the amplitudes are considered completely independent of one another.

The spectral density of the step series is then the sum of spectrums of each step, or

$$\Phi_{VV}(\omega) = \frac{1}{\omega^2} \lim_{T\to\infty} \frac{1}{T} \widetilde{\eta \Omega T a^2}$$

$$= \frac{\eta \Omega \widetilde{a^2}}{\omega^2} \quad . \tag{82}$$

Determining the ensemble averaged value of the input amplitude squared is a difficult problem. Nothing is known, nor is there likely to be any thing known, about either its probability distribution or its variance with respect to time. All things taken into consideration, it appears logical that the square of the average value of a_i given by Equation (73) will approximate $\widetilde{a_i^2}$. Then,

$$\Phi_{VV}(\omega) = \frac{4\Omega \bar{q}^2}{\eta \epsilon^2 \omega^2} \quad . \tag{83}$$

Substituting Equations (77) and (83) into Equation (76),

$$\Phi_{q_t q_t}(\omega) = \frac{8\Omega^3 \bar{q}^2}{\omega^4 \epsilon^4 \eta} \left(1 + \frac{\omega^2 \epsilon^2}{2\Omega^2} - \cos\frac{\omega\epsilon}{\Omega} - \frac{\omega\epsilon}{\Omega} \sin\frac{\omega\epsilon}{\Omega}\right) . \tag{84}$$

By definition,

$$\bar{q}(t) = \bar{q}(t+\tau) = \bar{q} \; ; \tag{85}$$

$$\bar{q}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} q_t(t)\, dt \quad ; \tag{86}$$

and

$$\bar{q}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} q_t(t+\tau)\, dt. \tag{87}$$

Then,

$$\phi_{\bar{q}q_t}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \bar{q}(t)\, q_t(t+\tau)\, dt = \bar{q}^2 \quad ; \tag{88}$$

$$\phi_{q_t \bar{q}}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} q_t(t)\, \bar{q}(t+\tau)\, dt = \bar{q}^2 ; \tag{89}$$

and

$$\phi_{\bar{q}\bar{q}}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \bar{q}(t)\, \bar{q}(t)\, dt = \bar{q}^2 . \tag{90}$$

From the definition of spectral density,

$$\Phi_{\bar{q} q_t}(\omega) = \Phi_{q_t \bar{q}}(\omega) = \Phi_{\bar{q}\bar{q}}(\omega) = \mathcal{F}[\bar{q}^2] = \bar{q}^2\, \delta(\omega) \tag{91}$$

where the Dirac delta function $\delta(\omega)$ is undefined at $\omega = 0$ and zero elsewhere.

Since $q(t) = q_t(t) - \bar{q}(t)$,

$$\Phi_{qq}(\omega) = \Phi_{q_t q_t}(\omega) - \Phi_{q_t \bar{q}}(\omega) - \Phi_{\bar{q} q_t}(\omega) + \Phi_{\bar{q}\bar{q}}(\omega) . \tag{92}$$

Substituting Equations (84) and (91) into the above equation,

$$\Phi_{qq}(\omega) = \frac{8\Omega^3 \bar{q}^2}{\omega^4 \epsilon^4 \eta} \left(1 + \frac{\omega^2 \epsilon^2}{2\Omega^2} - \cos\frac{\omega\epsilon}{\Omega} - \frac{\omega\epsilon}{\Omega} \sin\frac{\omega\epsilon}{\Omega}\right) - \bar{q}^2\, \delta(\omega) . \tag{93}$$

Equation (93) is shown graphically in the next chapter.

CHAPTER IV

ILLUSTRATIVE EXAMPLE

The purpose of this chapter is to demonstrate by calculation the use of the theory developed in previous chapters as applied to a typical piping system. Two different types of problems are solved using one discrete frequency. A complete solution would involve computations for all frequencies of interest. It appears that the only practical method of numerical solution is by digital computation.

4.1. The Piping System.

The piping system shown in Figure 8 is used for illustration. Tables I and II summarize the necessary calculations. The functions dependent on the frequency are computed for $\omega = 12$ radians per second; however, any convenient value of ω could have been used.

From Equation (61),

$$C = \frac{1}{La^2} . \tag{94}$$

Nomographs are available that give the velocity of propagation, a, for most sizes and kinds of pipes. Using Equation (94), C can be calculated without computing K'. Column 2 of Table I contains the velocity of propagation for the different pipes.[19]

Columns 3 and 4 were computed from basic fluid mechanics theory. The Darcy-Weisbach formula was used to compute head losses.

System Data

Pump	Pipe	Fluid
Type: Centrifugal	Type: Commercial Steel	Water @ 70° F
Speed: 3600 rpm	Thickness: 0.25 in.	ρ = 1.936 slugs/ft^3
No. of Blades: 8	n: 1.75	ν = 1.059 (10^{-5})ft^2/sec
Head: 1600 psi		
Flow Rate: 4 cfs		

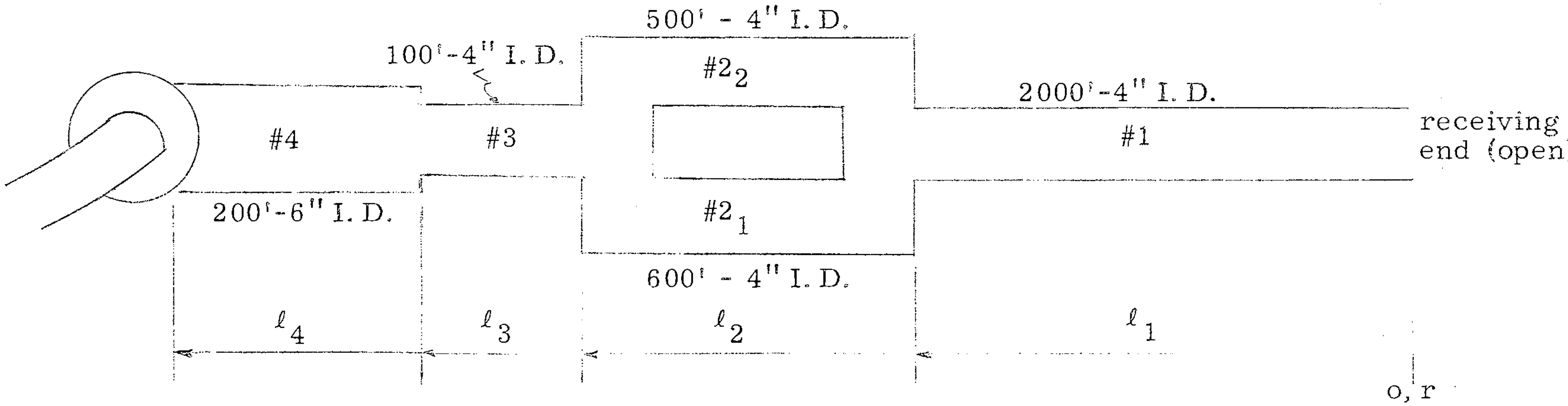


Fig. 8. The piping system.

TABLE I

EXAMPLE PIPING SYSTEM DATA, PHYSICAL PARAMETERS, AND FREQUENCY PARAMETERS

$\omega = 12$ rad/sec

1 Pipe Number	2 a ft/sec	3 $\overline{q}$ cfs	4 $\overline{p}_f \times 10^{-4}$ psf	5 $C \times 10^9$ ft^4/lb (94)	6 L $lb\text{-}sec^2/ft^6$ (8)	7 R $lb\text{-}sec/ft^6$ (9)	8 $\alpha \times 10^4$ 1/ft (54)	9 $\beta \times 10^3$ rad/ft (55)
1	4350	4.0	20.2	2.23	22.19	44.24	2.17	2.67
2_1	4350	1.91	1.42	2.23	22.19	21.80	1.16	2.67
2_2	4350	2.09	1.42	2.23	22.19	23.86	1.16	2.67
3	4350	4.0	1.01	2.23	22.19	44.24	2.17	2.67
4	4200	4.0	0.249	5.75	9.86	5.45	1.18	2.85

TABLE II

SYSTEM FUNCTIONS AND TRANSFER FUNCTIONS

FOR EXAMPLE PIPING SYSTEM

$\omega = 12$ rad/sec

1	2	3	4	5	6	7	8	9
Pipe Number	$B(j\omega)$ (16)	$T(j\omega) \times 10^{-4}$ (17)	$M(j\omega) \times 10^{6}$ (18)	$G(j\omega) \times 10^{-4}$ (23)*	$H(j\omega)$ (24)*	$\lvert G(j\omega)\rvert^2 \times 10^{-8}$	$\lvert H(j\omega)\rvert^2$	$\frac{\lvert G(j\omega)\rvert^2}{\lvert H(j\omega)\rvert^2} \times 10^{-9}$
1	0.65 +0.36j	3.36 +8.62j	1.92 +9.0j	3.36 +8.62j	0.65 +0.36j	85.7	0.549	15.65
2_1	-0.003 +0.07j	0.44 +10.01j	-0.44 +10.05j	2.30 +3.53j	-0.23 +0.64j	1.776	0.463	0.384
2_2	0.24 +0.056j	0.28 +8.74j	-0.55 +9.75j	2.30 +3.53j	-0.23 +0.64j	1.776	0.463	0.384
3	0.97 +0.006j	0.42 +2.62j	-0.006 +2.64j	0.42 +3.08j	-0.32 +0.67j	9.674	0.545	1.757
4	0.84 +0.013j	0.17 +2.23j	-0.06 +13.1j	-1.23 +2.02j	-0.68 +0.61j	5.62	0.835	0.674

*For parallel pipes use Equations (48) and (49).

The physical and frequency parameters are in columns 5 through 9 of Table I. Numbers in parentheses refer to the equation in the thesis used to obtain the parameter.

Table II contains the system and transfer functions. Numbers in parentheses refer to the equation used for calculation. Columns 7, 8, and 9 are the transfer functions needed to predict pressure fluctuations.

4.2. Illustrative Problem 1.

An amplitude-frequency plot of pressure pulsations at the pump shows a value of 5 pounds per square inch at center frequency $\omega = 12$ radians per second. The associated bandwidth is 8 radians per second. The pressure pulsations at the pipe junctions are to be found.

The following equations are used in the solution:

$$\Phi_{pp}(\ell_n, 12) = \frac{\pi \left| p(\ell_n, t)_{\omega=12} \right|^2}{\Delta\omega} ; \quad (95)$$

$$\Phi_{qq}(r, 12) = \frac{1}{\left| G_4(j\omega)_{\omega=12} \right|^2} \Phi_{pp}(\ell_4, 12) ; \quad (96)$$

and

$$\Phi_{pp}(\ell_n, 12) = \left| G_n(j\omega)_{\omega=12} \right|^2 \Phi_{qq}(r, 12) . \quad (97)$$

From Equation (95),

$$\Phi_{pp}(\ell_4, 12) = 17.2\ (10^4) . \quad (98)$$

Then,

$$\Phi_{qq}(r, 12) = 3.08\ (10^{-4}) . \quad (99)$$

Using Equation (99) in Equation (97),

$$\Phi_{pp}(\ell_1, 12) = 264\,(10^4) \;; \tag{100}$$

$$\Phi_{pp}(\ell_2, 12) = 5.46\,(10^4) \;; \tag{101}$$

$$\Phi_{pp}(\ell_3, 12) = 29.8\,(10^4) \;. \tag{102}$$

The desired results are obtained using Equation (95). They are:

$$p(\ell_1, t)\Big|_{\omega=12} = 18.6 \text{ psi} \;; \tag{103}$$

$$p(\ell_2, t)\Big|_{\omega=12} = 2.6 \text{ psi} \;; \tag{104}$$

$$p(\ell_3, t)\Big|_{\omega=12} = 6.05 \text{ psi} \;. \tag{105}$$

The values of the pressure pulsations at the various points are average values over the bandwidth $\Delta\omega = 8$. The magnitudes are not unreasonable with the possibility of even larger ones occurring at other points in the system.

4.3. Illustrative Problem 2.

The flow rate spectral density of the pump is given by Equation (93) and shown graphically in Figure 9. The pulse duration factor ϵ is assumed to be 0.75. At $\omega = 12$ the flow rate spectral density has a value of 0.0523. The pressure pulsations at the source and the pipe junctions are to be found.

Equations (95) and (97) are used in the solution. Also,

$$\Phi_{qq}(r, 12) = \frac{1}{\left| H_4(j\omega)\Big|_{\omega=12} \right|^2} \, \Phi_{qq}(\ell_4, 12) \;. \tag{106}$$

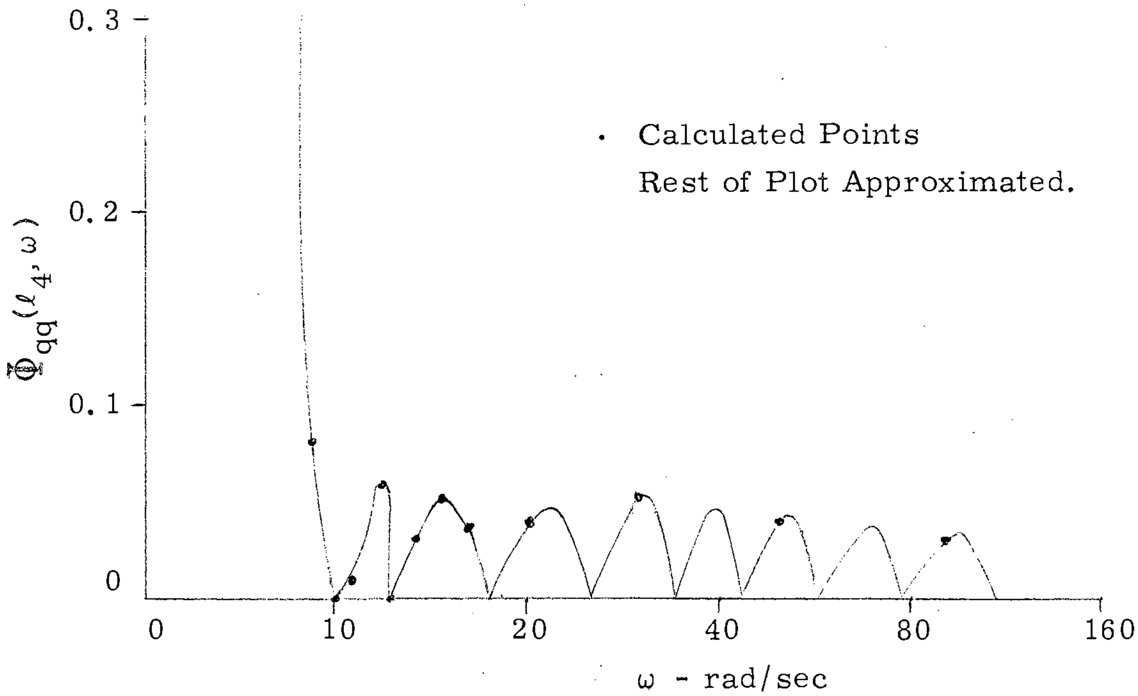


Fig. 9. Flow rate spectrum for the centrifugal pump.

From Equation (106),

$$\Phi_{qq}(r, 12) = 0.0633. \quad (107)$$

Then, from Equation (97),

$$\Phi_{pp}(\ell_1, 12) = 543\,(10^6) \; ; \quad (108)$$

$$\Phi_{pp}(\ell_2, 12) = 1.123\,(10^6) \; ; \quad (109)$$

$$\Phi_{pp}(\ell_3, 12) = 61.2\,(10^6) \; ; \quad (110)$$

$$\Phi_{pp}(\ell_4, 12) = 35.6\,(10^6) \; . \quad (111)$$

The average values for $\Delta\omega = 1$ are found from Equation (95). They are:

$$p(\ell_1, t)_{\omega=12} = 91.5 \text{ psi} \; ; \quad (112)$$

$$p(\ell_2, t)\Big|_{\omega=12} = 4.2 \text{ psi}; \tag{113}$$

$$p(\ell_3, t)\Big|_{\omega=12} = 30.7 \text{ psi}; \tag{114}$$

$$p(\ell_4, t)\Big|_{\omega=12} = 23.4 \text{ psi} . \tag{115}$$

4.4. Digital Computer Solution.

Digital computation is apparently the only practical method of calculating the necessary values to analyze a complex piping system. Except for the complex variable aspect, a computer program to accomplish this is easily formulated. Work has been done[20] and is presently in progress at this University toward this type of solution.

SUMMARY AND CONCLUSIONS

This thesis presents a method for determining the response of complex piping systems to random inputs. The partial differential equations which describe the pressure and flow in a cylinder are derived in Appendix A. The frequency response solutions of the describing equations are given by Equation (A-34) and (A-35) in Appendix A. These equations are used in developing the theory in the thesis. Appendix B contains the necessary basic statistical concepts.

Pressure fluctuations in piping systems have been of interest to the engineer for many years, but the methods of prediction have been confined to deterministic inputs. In many piping systems the input flow rate is a random process, for example, a system containing a rotating pump. Thus, the pressure variation is a random process. Generally, the fluctuations are small in magnitude, but from an acoustic standpoint they are important.

In this thesis, linear system statistical theory is applied to the analysis of piping systems. The spectral density of the pressure fluctuations at some point in the system is found to be the product of a transfer function and the spectral density of the flow rate at some point. This is first shown by Equations (14) and (15).

The transfer functions for series and parallel pipes are given by Equations (23), (24), and (45) . They are complicated functions of the frequency, and manual numerical calculations for all frequencies of interest is not practical.

The parameters that comprise the transfer functions are discussed in Sections 3. 1, 3. 2, and 3. 3 of Chapter III.

The spectral density of either pressure or flow at some point in the system must be known before prediction of pressure fluctuations at other points is possible. In Chapter III, pressure measurements are shown to give the necessary information. Also, a method is proposed for obtaining the flow rate spectral density of a centrifugal pump. This is given by Equation (93).

The methods of analysis developed in the thesis are demonstrated by an illustrative example in Chapter IV. The numerical calculations are made for only one frequency. It is evident that the only practical means of complete solution to such a problem is a digital computer. A research group at this University is presently engaged in this type of solution.

It is concluded that the theory in this thesis, with the possible exception of the centrifugal pump analysis, provides a reasonable method for determining the response of piping systems to random inputs.

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APPENDIX A

DERIVATION AND SOLUTION OF THE SYSTEM DESCRIBING EQUATIONS

The derivation and solution of the differential equations describing one-dimensional, compressible, turbulent flow in a non-rigid cylinder including the effects of friction follows. Selected reading references are References 3, 5, 8, 9, and 21 of the Bibliography.

A. 1. <u>Equation of Motion.</u>

Figure A-1 shows the system to be studied.

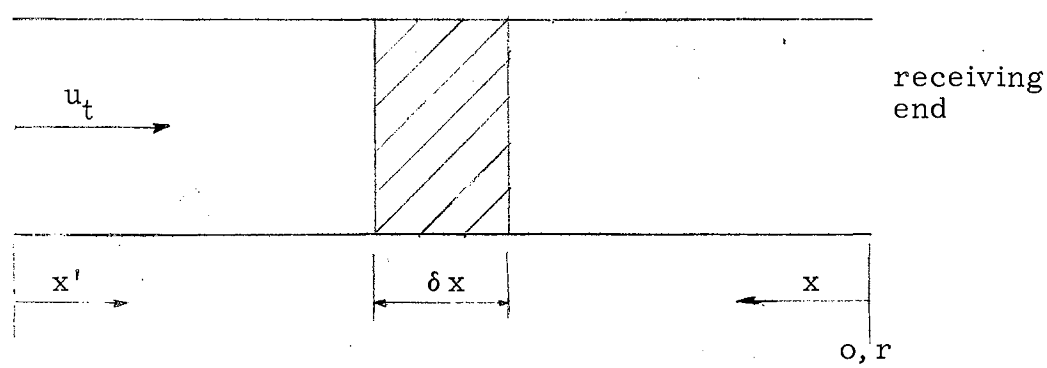

Fig. A-1. The System

The equation of motion of a fluid element for one-dimensional flow in the x' direction is

$$\frac{\partial u_t}{\partial t} + u\frac{\partial u_t}{\partial x'} = -\frac{1}{\rho}\frac{\partial p_t}{\partial x'} + \frac{1}{\rho}\left(\frac{\partial \sigma'_{x'}}{\partial x'} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right). \qquad \text{(A-1)}$$

The terms $\sigma'_{x'}$, τ_{yx}, and τ_{zx} represent normal and shearing stresses on the element of fluid. They are functions of the dynamic and eddy viscosities of the fluid, the roughness and size of the pipe, and the velocity of the fluid. In general, the last term of Equation (A-1) is a function of system parameters and some nonlinear operation on the fluid velocity. Equation (A-1) is rewritten in terms of x as

$$\frac{\partial u_t}{\partial t} - u_t\frac{\partial u_t}{\partial x} = \frac{1}{\rho}\frac{\partial p_t}{\partial x} + N(u_t)\,. \qquad \text{(A-2)}$$

The last term on the left side of Equation (A-2) will be neglected, because the change in velocity with respect to x can be considered negligible as compared to its change with respect to time. Integrating Equation (A-2) over a control volume $A\delta x$,

$$\delta x\frac{\partial}{\partial t}\int_A u_t\,dA = \frac{\delta x}{\rho}\frac{\partial}{\partial x}\int_A p_t\,dA + \delta x\int_A N(u_t)\,dA\ , \qquad \text{(A-3)}$$

or

$$-\frac{\partial p_t}{\partial x} + \frac{\rho}{A}\frac{\partial q_t}{\partial t} + N(q_t) = 0\ . \qquad \text{(A-4)}$$

When the volume flow rate is constant at a cross section, $q_t = \overline{q}$, Equation (A-4) reduces to the slope of the pressure grade line, or

$$\frac{\partial \overline{p}}{\partial x} = N(\overline{q}) = \frac{f\rho\overline{q}^2}{2DA^2}\ . \qquad \text{(A-5)}$$

A. 2. Linearized Equation of Motion.

Equation (A-4) is one of the two describing differential equations needed, but the nonlinear term $N(q_t)$ makes it unsuitable for an exact solution. This difficulty can be circumvented by "linearizing" $N(q_t)$ by a perturbation process first proposed and subsequently verified experimentally by Waller[8].

The nonlinear term is expressed as

$$N(q_t) = B(\overline{q} + q)^n , \tag{A-6}$$

where B and n are system parameters independent of the volume flow rate. Equation (A-6) is rewritten as

$$N(q_t) = B\,\overline{q}^n\,(1 + \frac{q}{\overline{q}})^n , \tag{A-7}$$

and a binomial expansion yields

$$N(q_t) = B\,\overline{q}^n\,[1 + n(\frac{q}{\overline{q}}) + \frac{n(n-1)}{2!}(\frac{q}{\overline{q}})^2 + \cdots + \frac{n!}{(n-m)!\,m!}(\frac{q}{\overline{q}})^m + \cdots] . \tag{A-8}$$

The series of Equation (A-8) is convergent for $(q/\overline{q}) < 1$ and is sufficiently approximated by its first two terms. Substituting Equation (A-8) into Equation (A-4),

$$-\frac{\partial p_t}{\partial x} + \frac{\rho}{A}\frac{\partial q_t}{\partial t} + B\,\overline{q}^n + n\,B\,\overline{q}^{n-1}\,q = 0 . \tag{A-9}$$

For $q = 0$ and $p = 0$, Equation (A-9) reduces to

$$- \frac{\partial \bar{p}}{\partial x} + B \bar{q}^{n} = 0. \tag{A-10}$$

Comparison of Equation (A-10) and Equation (A-5) shows that

$$B = \frac{f \rho \bar{q}^{2-n}}{2 DA^{2}} = \frac{\bar{p}_f}{\ell \bar{q}^{n}} . \tag{A-11}$$

Equation (A-9) then becomes in terms of the variations

$$- \frac{\partial p}{\partial x} + \frac{\rho}{A} \frac{\partial q}{\partial t} + \frac{n \bar{p}_f}{\ell \bar{q}} q = 0 , \tag{A-12}$$

where the pressure needed to overcome the frictional resistance of the pipe $\bar{p}_f$ and the mean volume flow rate $\bar{q}$ are determined by elementary fluid mechanics. The value of n is estimated from experience or experiment and should range from 1.65 to 2.05.

A. 3. Continuity Equation .

The equation of continuity for one-dimensional flow in the x' direction is

$$\frac{\partial(\rho u_t)}{\partial x'} = - \frac{\partial \rho}{\partial t} . \tag{A-13}$$

Considering the change in mass density with respect to x' negligible as compared to its change with respect to time, Equation (A-13) becomes in terms of x

$$- \rho \frac{\partial u_t}{\partial x} = - \frac{\partial \rho}{\partial t} . \tag{A-14}$$

Also, ρV = constant; therefore,

$$\frac{\partial \rho}{\rho} = - \frac{\partial V}{V} = \frac{\partial p_t}{K'} , \quad \text{(A-15)}$$

where the bulk modulus K' of the fluid and the pipe wall is

$$K' = \frac{KbE}{KD + bE} . \quad \text{(A-16)}$$

Substituting Equation (A-15) into Equation (A-14),

$$- \frac{\partial u_t}{\partial x} + \frac{1}{K'} \frac{\partial p_t}{\partial t} = 0 . \quad \text{(A-17)}$$

Integrating Equation (A-17) over the control volume $A\delta x$,

$$- \delta x \frac{\partial}{\partial x} \int_A u_t \, dA + \frac{\delta x}{K'} \frac{\partial}{\partial t} \int_A p_t \, dA = 0 , \quad \text{(A-18)}$$

or

$$- \frac{\partial q_t}{\partial x} + \frac{A}{K'} \frac{\partial p_t}{\partial t} = 0 . \quad \text{(A-19)}$$

Equation (A-19) becomes in terms of the variations

$$- \frac{\partial q}{\partial x} + \frac{A}{K'} \frac{\partial p}{\partial t} = 0 . \quad \text{(A-20)}$$

A. 4. Solution of the Differential Equations.

The describing differential Equations (A-12) and (A-20) for the variations in pressure and volume flow rate are:

$$- \frac{\partial p(x,t)}{\partial x} + L \frac{\partial q(x,t)}{\partial t} + R q(x,t) = 0 ; \quad \text{(A-21)}$$

and

$$- \frac{\partial q(x,t)}{\partial x} + C \frac{\partial p(x,t)}{\partial t} = 0 \quad \text{(A-22)}$$

where

$$L = \frac{\rho}{A} \quad ; \tag{A-23}$$

$$R = \frac{n \overline{p}_f}{\ell \overline{q}} \quad ; \tag{A-24}$$

and

$$C = \frac{A}{K'} \quad . \tag{A-25}$$

Equations (A-21) and (A-22) can be solved by Laplace transform methods for solutions in the frequency domain.

Transforming Equations (A-21) and (A-22) with the Laplace integral

$$F(s) = \int_o^\infty f(t)\, e^{-st}\, dt \quad ,$$

$$- \frac{\partial P(x,s)}{\partial x} + (R + sL)\, Q(x,s) = 0 \, , \tag{A-26}$$

and

$$- \frac{\partial Q(x,s)}{\partial x} + s\, C P(x,s) = 0 \tag{A-27}$$

where p (x, o) and q(x, o) are zero. Transforming the above equations with the Laplace integral

$$\hat{F}(\lambda) = \int_o^\infty f(x)\, e^{-\lambda x}\, dx \quad ,$$

$$- \lambda \hat{P}(\lambda, s) + (R + sL)\, \hat{Q}(\lambda, s) = -P(r, s) \quad , \tag{A-28}$$

and

$$- \lambda \hat{Q}(\lambda, s) + sC\, \hat{P}(\lambda, s) = -Q(r, s) \quad . \tag{A-29}$$

Solving Equations (A-28) and (A-29) simultaneously for $\hat{P}(\lambda, s)$ and $\hat{Q}(\lambda, s)$,

$$\hat{P}(\lambda, s) = Z_c\, Q(r, s) \left(\frac{\gamma}{\lambda^2 - \gamma^2}\right) + P(r, s) \left(\frac{\lambda}{\lambda^2 - \gamma^2}\right) , \qquad \text{(A-30)}$$

and

$$\hat{Q}(\lambda, s) = \frac{P(r, s)}{Z_c} \left(\frac{\gamma}{\lambda^2 - \gamma^2}\right) + Q(r, s) \left(\frac{\lambda}{\lambda^2 - \gamma^2}\right) \qquad \text{(A-31)}$$

where

$$\gamma^2 = sC\,(R + sL) , \qquad \text{(A-32)}$$

and

$$Z_c^2 = \frac{R + sL}{sC} . \qquad \text{(A-33)}$$

Evaluating Equations (A-30) and (A-31) with the inversion integral

$$f(x) = \frac{1}{2\pi j} \int_{\epsilon - j\infty}^{\epsilon + j\infty} \hat{F}(\lambda)\, e^{\lambda t}\, d\lambda ,$$

$$P(x, s) = P(r, s) \cosh \gamma x + Z_c\, Q(r, s) \sinh \gamma x , \qquad \text{(A-34)}$$

and

$$Q(x, s) = Q(r, s) \cosh \gamma x + \frac{P(r, s)}{Z_c} \sinh \gamma x . \qquad \text{(A-35)}$$

The frequency response of the system is obtained by replacing s by $j\omega$ in Equations (A-32) through (A-35).

APPENDIX B

BASIC STATISTICAL THEORY

The following material is the basic statistical theory used in this thesis. Suggested reading references are References 15, 16, 17, and 18 of the Bibliography.

B. 1. Probability and Probability Density.

The chance that an event in a random process will occur can be expressed by a number between 0 and 1. If the event is certain, the number is 1, and if impossible, the number is 0. This number is called the "probability" of occurence. The calculation of the probabilities of one event or any number of events is a logical and elementary process.

Probability density has more physical significance than probability. Since there are infinitely many values that a random function can have, the probability of finding the function at a particular value at a given time is zero. The probability of finding the function at a particular value in an interval of the function is not necessarily zero and is the "probability density". If the random function is $x(t)$, the probability density $p(x)$ for x equal to some value x_o in an interval dx is defined mathematically as

$$p(x_o)\,dx = Pr(x_o \leq x \leq x_o + dx)\,, \quad dx \rightarrow 0 \qquad \text{(B-1)}$$

where Pr( ) denotes the probability of the event in parentheses.

Since probability and probability density are not instrumental in the frequency description of a system, the preceding discussion is considered sufficient to allow continuation into more appropriate statistical theory.

B. 2. Averages.

The "time" average of a function x(t), whether it is random or not, is given by

$$\overline{x(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt \quad . \tag{B-2}$$

This can be extended to a function of x(t) as

$$\overline{f[x(t)]} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f[x(t)]\, dt \quad . \tag{B-3}$$

Another type of average is the "ensemble" average. Whereas the time average denotes an average value of a random function over a period of time, the ensemble average denotes an average value of a random function at a particular time for any number of generations. The mathematical statement for the ensemble average is easily obtained and is stated as

$$\widetilde{x(t_o)} = \int_{-\infty}^{\infty} x\, p(x)\, dx \quad . \tag{B-4}$$

This can be generalized to a function of x by

$$\widetilde{f[x(t_o)]} = \int_{-\infty}^{\infty} f(x)\, p(x)\, dx \, . \tag{B-5}$$

Two terms which are useful in describing a random process are:

1. A "stationary" process is one having statistical properties that do not change with time.

2. An "ergodic" process is one having equal time and ensemble averages.

B. 3. Correlation Functions.

The ensemble averaged product of two functions of time, one displaced τ seconds, is defined as a "correlation" function ϕ. If the two functions are the same, the correlation function is called the "autocorrelation" function and is given by

$$\phi_{xx}(t_o, \tau) \triangleq \widetilde{x(t_o)\, x(t_o + \tau)}. \qquad \text{(B-6)}$$

If the functions are different, the correlation function is designated the "cross-correlation" function and is expressed as

$$\phi_{xy}(t_o, \tau) \triangleq \widetilde{x(t_o)\, y(t_o + \tau)}. \qquad \text{(B-7)}$$

For ergodic processes, the correlation functions can also be found by

$$\phi_{xx}(\tau) = \overline{x(t)\, x(t + \tau)} \qquad \text{(B-8)}$$

and

$$\phi_{xy}(\tau) = \overline{x(t)\, y(t + \tau)}. \qquad \text{(B-9)}$$

Correlation functions can be used to describe certain systems, but the concept of spectral densities discussed in the following section is usually better.

B. 4. Spectral Densities.

The "spectral density" $\Phi_{xx}(\omega)$ of an ergodic random process x(t) is defined as the Fourier transform of the autocorrelation function $\phi_{xx}(\tau)$. In symbols

$$\Phi_{xx}(\omega) \triangleq \int_{-\infty}^{\infty} \phi_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau\,. \qquad \text{(B-10)}$$

It follows that

$$\Phi_{xy}(\omega) = \int_{-\infty}^{\infty} \phi_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau \qquad \text{(B-11)}$$

where $\Phi_{xy}(\omega)$* is the "cross-spectral density" of the ergodic random processes x(t) and y(t) .

Several properties of $\Phi_{xx}(\omega)$ can be deduced from its definition.[15]. Without proof, they are:

1. $\Phi_{xx}(\omega)$ is an even function of ω.
2. $\Phi_{xx}(\omega)$ is real.
3. $\Phi_{xx}(\omega)$ is positive.

A rigorous mathematical investigation of the operations on a random variable leading to the expression for its spectral density will show that very badly behaved functions are created. Any seemingly obvious operation should be approached with caution and a thorough understanding of the mathematics involved. The remainder of this section will be devoted to the derivation of an expression for the spectral density useful in system analysis. The mathematics involved can be shown to be correct.[15].

Let the truncated function $x_T(t)$ be defined as the same random function x(t) in the interval $-T \leq t \leq T$ but zero outside the interval. Then the equality

$$\frac{1}{2T}\int_{-T}^{T-\tau} x(t)\, x(t+\tau)\, dt = \frac{1}{2T}\int_{-\infty}^{\infty} x_T(t)\, x_T(t+\tau)\, dt \qquad \text{(B-12)}$$

can be written. Introducing the Fourier inversion integral

---

* A more correct notation would be $\Phi_{xy}(j\omega)$, because the cross-spectral density is complex.

$$x_T(t+\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_T(j\omega)e^{j\omega(t+\tau)}\,d\omega\ , \tag{B-13}$$

Equation (B-12) becomes

$$\frac{1}{2T}\int_{-T}^{T-\tau} x(t)\,x\,(t+\tau)\,dt = \left(\frac{1}{2T}\right)\left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} x_T(t)\int_{-\infty}^{\infty} X_T(j\omega)e^{j\omega(t+\tau)}\,d\omega\,dt$$

$$= \left(\frac{1}{2T}\right)\left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} X_T(j\omega)e^{j\omega\tau}\,d\omega\int_{-\infty}^{\infty} x_T(t)e^{j\omega t}\,dt$$

$$= \left(\frac{1}{2T}\right)\left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} X_T(j\omega)\,X_T(-j\omega)e^{j\omega\tau}\,d\omega\ . \tag{B-14}$$

Performing the ensemble average over the above equation,

$$\frac{1}{2T}\int_{-T}^{T-\tau} \widetilde{x(t)x(t+\tau)}dt = \left(\frac{1}{2T}\right)\left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} \widetilde{X_T(j\omega)X_T(-j\omega)}e^{j\omega\tau}\,d\omega\ , \tag{B-15}$$

or

$$\frac{1}{2T}\,\phi_{xx}(\tau)\int_{-T}^{T-\tau} dt = \left(\frac{1}{2T}\right)\left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} \widetilde{X_T(j\omega)X_T(-j\omega)}e^{j\omega\tau}\,d\omega\ . \tag{B-16}$$

Integrating the left side and taking the limit as $T\to\infty$ of both sides of Equation (B-16),

$$\lim_{T\to\infty}\frac{2T-\tau}{2T}\,\phi_{xx}(\tau) = \lim_{T\to\infty}\left(\frac{1}{2T}\right)\left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} \widetilde{X_T(j\omega)X_T(-j\omega)}e^{j\omega\tau}\,d\omega\ , \tag{B-17}$$

or

$$\phi_{xx}(\tau) = \lim_{T\to\infty} \left(\frac{1}{2T}\right)\left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} \widetilde{X_T(j\omega)X_T(-j\omega)}e^{j\omega\tau}\, d\omega\ , \tag{B-18}$$

or

$$\mathcal{F}\,[\phi_{xx}(\tau)] = \lim_{T\to\infty} \frac{1}{2T}\, \widetilde{X_T(j\omega)X_T(-j\omega)}\ . \tag{B-19}$$

From the definition of spectral density, Equation (B-19) can be written as

$$\Phi_{xx}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\, \widetilde{X_T(j\omega)X_T(-j\omega)}\ . \tag{B-20}$$

By a similiar derivation it can also be shown that the cross-spectral density $\Phi_{xy}(\omega)$ is given as

$$\Phi_{xy}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\, \widetilde{X_T(-j\omega)\ Y_T(j\omega)}. \tag{B-21}$$

Equations (B-20) and (B-21) are useful tools in the analysis of a system with random inputs as will be shown in the next section.

B. 5. System Application

A system with multiple random inputs is shown in Figure B-1. The inputs are denoted by $x_i(t)$ and the respective system transfer functions by $H_i(t)$, $i = 1, 2, 3, \cdots, n$. After truncating the inputs as in the previous section, the following equations can be written:

$$X_T(j\omega) = \sum_{i=1}^{n} H_i\ (j\omega)\ X_{Ti}(j\omega); \tag{B-22}$$

and

$$X_T(-j\omega) = \sum_{k=1}^{n} H_k(-j\omega)\ X_{Tk}(-j\omega)\ . \tag{B-23}$$

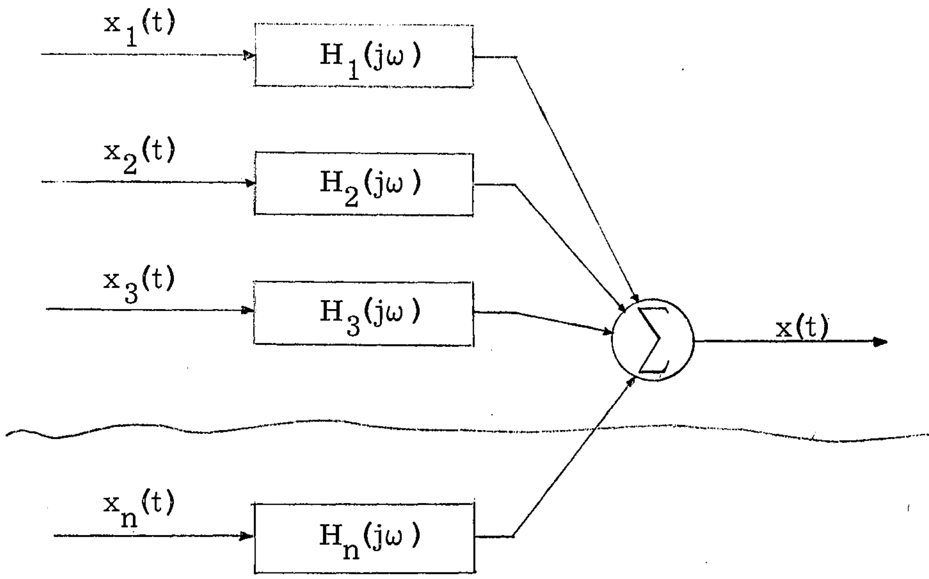

Fig. B-1. System with multiple random inputs

Multiplying Equations (B-22) and (B-23) and dividing the results by 2T,

$$\frac{1}{2T} X_T(j\omega)X_T(-j\omega) = \frac{1}{2T}\sum_{i=1}^{n}\sum_{k=1}^{n} H_i(j\omega)H_k(-j\omega)X_{Ti}(j\omega)X_{Tk}(-j\omega) \ . \tag{B-24}$$

If the ensemble average is taken over the above equation and T is allowed to approach infinity,

$$\Phi_{xx}(\omega) = \sum_{i=1}^{n}\sum_{k=1}^{n} H_i(j\omega)H_k(-j\omega)\,\Phi_{x_k x_i}(\omega) \ . \tag{B-25}$$

Equation (B-25) can be used to obtain the spectral density of the output in terms of the spectral and cross-spectral densities of the inputs and the system functions. As illustrative examples, single input and double input systems are analyzed.

Single input:

$$\Phi_{xx}(\omega) = \sum_{i=1}^{1} \sum_{k=1}^{1} H_i(j\omega) H_k(-j\omega)\, \Phi_{x_k x_i}(\omega)$$

$$= \sum_{i=1}^{1} H_i(j\omega)\, H_1(-j\omega)\, \Phi_{x_1 x_i}(\omega)$$

$$= H_1(j\omega)\, H_1(-j\omega)\, \Phi_{x_1 x_1}(\omega)$$

$$= |H_1(j\omega)|^2 \;\Phi_{x_1 x_1}(\omega) \;. \qquad \text{(B-26)}$$

Double input:

$$\Phi_{xx}(\omega) = \sum_{i=1}^{2} \sum_{k=1}^{2} H_i(j\omega)\, H_k(-j\omega)\, \Phi_{x_k x_i}(\omega)$$

$$= \sum_{i=1}^{2} H_i(j\omega)\, H_1(-j\omega)\, \Phi_{x_1 x_i}(\omega) + H_i(j\omega) H_2(-j\omega)\, \Phi_{x_2 x_i}(\omega)$$

$$= H_1(j\omega)\, H_1(-j\omega)\, \Phi_{x_1 x_1}(\omega) + H_1(j\omega)\, H_2(-j\omega)\, \Phi_{x_2 x_1}(\omega) +$$

$$H_2(j\omega)\, H_1(-j\omega)\, \Phi_{x_1 x_2}(\omega) + H_2(j\omega)\, H_2(-j\omega)\, \Phi_{x_2 x_2}(\omega) \;. \qquad \text{(B-27)}$$

Thus, the spectral density of a system output is determined once the spectral densities of the input(s) are found. The determination of the spectral density of a random variable is discussed in the next section.

B. 6. Determination of Spectral Densities.

In Section B. 4 an expression was derived for the spectral density of a function in terms of the function's Fourier transform. The relationship so derived is useful in mathematical manipulations as shown in Section B. 5, but it is of little use in the actual calculation of the spectral density. Because a random variable cannot usually be expressed as a function of time, its Fourier transform cannot be found.

There are methods for determining spectral densities analytically, but all involve prior knowledge of the variable such as its amplitude distribution, frequency of occurrence, etc. Perhaps the simplest way of determining the spectral density of a random variable is a mechanical one. A short derivation follows to fully explain this method.

From the definition of spectral density, if x(t) is ergodic and $\tau = 0$, then

$$\phi_{xx}(o) = \overline{x(t)^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xx}(\omega)\, d\omega \quad , \qquad \text{(B-28)}$$

Replacing x(t) in Equation (B-28) by the real part of its Fourier series representation

$$x(t) = R \sum_{n=-\infty}^{\infty} c_n e^{j\omega_n t} = \sum_{n=-\infty}^{\infty} |c_n| \cos(\omega_n t - \theta_n) \qquad \text{(B-29)}$$

where

$$c_n = \frac{1}{2T} \int_{-T}^{T} x(t)\, e^{-j\omega_n t}\, dt \qquad \text{(B-30)}$$

and considering only the frequency interval $\Delta\omega$ with center frequency ω_n,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |c_n|^2 \cos^2(\omega_n t - \theta_n) dt = \frac{1}{2\pi} \int_{\omega_n - \frac{\Delta\omega}{2}}^{\omega_n + \frac{\Delta\omega}{2}} \Phi_{xx}(\omega_n)\, d\omega . \tag{B-31}$$

Rewriting the above equation,

$$\frac{|c_n|^2}{\omega_n} \lim_{T\to\infty} \frac{1}{2T} \int_{-\omega_n T - \theta_n}^{\omega_n T - \theta_n} \cos^2(\omega_n t - \theta_n)\, d(\omega_n t - \theta_n) =$$

$$\frac{1}{2\pi} \Phi_{xx}(\omega_n) \int_{\omega_n - \frac{\Delta\omega}{2}}^{\omega_n + \frac{\Delta\omega}{2}} d\omega , \tag{B-32}$$

and integrating gives

$$\frac{|c_n|^2}{\omega_n} \lim_{T\to\infty} \frac{1}{2T} \left[\frac{\omega_n T - \theta_n}{2} - \frac{-\omega_n T - \theta_n}{2} + \frac{\sin 2(\omega_n T - \theta_n)}{4} - \right.$$

$$\left. \frac{\sin 2(-\omega_n T - \theta_n)}{4} \right] = \frac{1}{2\pi} \Phi_{xx}(\omega_n)\, \Delta\omega , \tag{B-33}$$

or

$$\Phi_{xx}(\omega_n) = \frac{\pi |c_n|^2}{\Delta\omega} . \tag{B-34}$$

An alternate expression is

$$\Phi_{xx}(f_n) = \frac{|c_n|^2}{2\Delta f} . \tag{B-35}$$

Equations (B-34) or (B-35) provide an easy method for determining the spectral density of a random variable if its amplitude-frequency plot made by a wave analyzer is available. The wave analyzer gives the absolute value of the Fourier coefficient c_n for some frequency band width $\Delta\omega$. Thus, the spectral density may be computed either by hand or electronic computer using Equation (B-34) or Equation (B-35).

VITA

Larry Edward Hove

Candidate for the Degree of

Master of Science

Thesis: RESPONSE OF PIPING SYSTEMS TO RANDOM INPUTS

Major Field: Civil Engineering

Biographical:

Personal Data: Born in Altus, Oklahoma, July 5, 1939, the son of Segurd G. and Edna Lee Hove.

Education: Attended elementary and secondary school in Altus, Oklahoma; graduated from Altus High School in 1957; received the Bachelor of Science Degree from the Oklahoma State University, with a major in Civil Engineering, in May, 1961; completed requirements for the Master of Science Degree, with a major in Civil Engineering, in August, 1962.

Professional Experience: Employed by the U. S. Army Corps of Engineers, Tulsa District, as structural engineer from June to September, 1961. Employed by the School of Civil Engineering, Oklahoma State University, as Graduate Assistant from September, 1961 to August, 1962. Associate Member of the American Society of Civil Engineers. Member of Chi Epsilon. Engineer-in-Training (Oklahoma).