AN EXTENSION OF THE VON KARMAN-POHLHAUSEN APPROXIMATION METHOD TO MAGNETOHYDRODYNAMIC FLOWS

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PREFACE

Magnetofluidmechanics is rapidly becoming a field of more importance than it has enjoyed in the past. Much effort has been made to obtain solutions to magnetofluidmechanical boundary layer problems. Exact solutions, however, have proven difficult and only a limited number of very simple cases can be solved by exact methods.

It is the purpose of this study to introduce an approximate technique which will allow magnetohydrodynamic boundary layer problems to be solved quickly and with reasonable accuracy. This method is also used to solve several problems for which no other known solutions exist at this time.

I am particularly indebted to Drs. M.K. Jovanovic, D.R. Haworth and J.D. Parker for their guidance and encouragement; also to Dr. O.H. Hamilton, Dr. H.P. Hotz and the Applied Mathematics Research Laboratory, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio.

TABLE OF CONTENTS

LIST OF FIGURES

NOMENCLATURE

x,y	=	coordinate distances.
u,v,w	=	velocity within the boundary layer.
U,V,W	=	velocity outside the boundary layer.
$\vec{q}$	=	velocity vector.
δ	=	boundary layer thickness.
δ^*	=	$\delta\int_0^1(1-f)d\eta$, displacement thickness.
θ	=	$\delta\int_0^1 f(1-f)d\eta$, momentum thickness.
$f(\eta)$	=	u/U, dimensionless velocity parameter.
η	=	y/δ, dimensionless distance.
U_∞	=	undisturbed free stream velocity.
τ_o	=	viscous shear stress.
μ	=	fluid viscosity.
P	=	pressure.
ρ	=	density.
σ	=	fluid electrical conductivity.
c_p	=	specific heat at constant pressure.
c_v	=	specific heat at constant volume.
k_c	=	thermal conductivity.
ρ_e	=	charge density.
μ_m	=	magnetic permeability.
C_f	=	surface viscous friction coefficient.
R_e	=	external circuit resistance.
m	=	m_1/U_∞.

m_1	=	$\sigma_0 B_0^2/\rho$.
$\vec{E}$	=	electric field strength.
$\vec{B}$	=	magnetic field flux density.
$\vec{B}_0$	=	magnetic field flux density at the surface.
$\vec{H}$	=	$\vec{B}/\mu_m$.
Λ	=	shape factor.
Λ_m	=	magnetic shape factor.
mx	=	$m_1 x/U_\infty$, the magnetic interaction parameter.
R_N	=	$U_\infty x/\nu$, Reynolds number.
ν	=	μ/ρ, kinematic viscosity.
α	=	constant of proportionality.
r_0	=	cylinder radius.
ϕ	=	angle in radians.
β	=	a measure of wedge angle.
k	=	$\dfrac{\beta}{2-\beta}$.
a,b,c,d,e	=	functional coefficients in x.
X,Y	=	body forces.
A_e	=	current flow area.
h_e	=	electrode separation distance.
I_e	=	external current.
$\vec{i}$	=	electric current.
$\hat{i},\hat{j},\hat{k}$	=	the orthogonal set of unit vectors.
β_e	=	$R_e A_e \sigma/h_e$, the conductivity ratio.
R_σ	=	$\sigma\mu_m U L$, magnetic Reynolds number.
f_1, f_2	=	functions of K.
K	=	functions of Z, U', m_1.

Z	=	θ^2/ν .
F	=	$2f_2 - 4K - 2Kf_1$.
$f(\eta)$	=	$a\eta + b\eta^2 + c\eta^3 + d\eta^4 + e\eta^5$, the dimensionless velocity.
$\Rightarrow$	=	"implying ..."
$(\ )'$	=	prime denotes d/dx.

CHAPTER I

THE PROBLEM

Magnetofluidynamics is discussed in this chapter in general terms to obtain familiarity with some of the associated difficulties.[1] Emphasis is placed upon those areas which are specially applicable to boundary layer fluid flow problems. Detailed analysis is deferred to later chapters.

Plasma Dynamics

Plasma dynamics is the study of the dynamics of ionized fluids and is not new, having long been important in the study of astrophysical problems such as the motion of interstellar gases. In recent years the engineering frontiers have been pushed into the realm of plasma dynamics and it is no longer the private property of the astrophysicist. Fluid flow problems today often involve ionized gases as the engineer considers the re-entry of orbiting vehicles as an example. Such vehicles may be winged and operate aerodynamically in a range of flight speeds leading to ionization of the fluid medium and for this

[1]When fluids are influenced by electromagnetic fields, the analysis of the results is by means of "electromagneto-fluidmechanics." If the fluid is incompressible, the term "hydro" will be found in place of the more general term "fluid" and when the fluid is in motion the word "dynamics" is usually found in place of "mechanics." By these conventions the study of an incompressible fluid in motion through a magnetic field would be called "magnetohydrodynamics." This convention will be followed and will be found in agreement with the literature.

reason considerable research is now being done in the area of ionized gas flows.

Just as the study of the boundary layer separation in fluid mechanics led to the important consideration of boundary layer control, so the study of the boundary layer in the magnetofluidynamical case may well lead to possible boundary layer control in these new flight regimes. Before application can be attempted, however, the magnetic field effects upon the plasma boundary layer must be first investigated. Two important boundary layer phenomena are separation and surface friction. Knowledge of both separation and surface friction are required if the total drag of the body is to be determined. With such an understanding of the fundamental forces at work in the boundary layer itself, the possibility of control of these forces can be considered; and with such control it may be possible to lower re-entry deceleration forces, reduce heat transfer to the vehicle and prevent instabilities during flight.

The engineer has learned to ply his trade with the three states of matter: solids, liquids, and gases. Now, he must include what is often termed the fourth state, the ionized gas. Engineers must then allow for the influence of electromagnetic fields in flow problems. A complete analysis would in fact have to include fluid dynamic, electromagnetic and thermal effects; but thermal effects on the fluid flow equations are often secondary and will be so considered in this dissertation. Further, the analysis will be made from the fluid dynamic standpoint; that is, fluid dynamics will be generalized to include electromagnetic effects.

Two methods suggest themselves, microscopic and macroscopic.

The first, the kinetic theory of plasma, assumes the gas is a collection of positive and negative ions, electrons and neutral particles. These charged bodies will be constantly influenced by the electromagnetic fields whereas in normal kinetic theories the particles are assumed to travel without influence between collisions in essentially straight lines. Collisions between the charged particles and neutral particles lead to interactions on the gas as a whole. Such an analysis necessarily requires a precise description of the collision phenomenon. In the non-ionized case one-particle distribution functions are used to describe the behavior leading to the well-known Boltzmann equation. But as already stated, the particles in a plasma are never free and in addition a satisfactory description of the plasma interaction and collision process is not now known. Even if the collision process were known the resulting generalized Boltzmann equation would have to be solved simultaneously with the Maxwell equations. Unfortunately, the complexity of such a system of equations is such that generally not even simple flow problems can be solved. However, they are still very important since the generalized Boltzmann equation, just as in the non-ionized case, should reduce to the macroscopic equations as a first approximation yielding valuable information on the transport coefficients. In the macroscopic postulate these coefficients are introduced as known or experimentally determined quantities. A thorough analysis of the microscopic approach may be found in <u>Magnetogasdynamics and Plasma Dynamics</u>, Chapter I.3(5).[2]

Engineers are seldom interested in the motion of individual particles but rather in the macroscopic quantities such as pressure,

[2]Numbers in parentheses indicate References in the Bibliography.

density, temperature, mean flow velocites, current density, etc. When considered macroscopically the dynamics of the plasma are postulated on the conservation laws of mass, momentum, energy, and charge. For most practical problems, therefore, the Maxwell-Boltzmann distribution function is simply too complex to be useful and for this reason subsequent analysis will be made from the macroscopic or continuum point of view. The resulting equations must, of course, be consistent with those derived from the microscopic consideration.

Although a plasma is in general a composite of particles of differing species, in many practical problems the variation in composition is small. Such a plasma will be considered in this dissertation. In this instance, the plasma becomes very much simplified, resulting in fluid dynamic relations very similar to the well-known Navier-Stokes equations but with the additional electromagnetic force terms appearing as body forces. These simplified equations will, of course, also have to be treated simultaneously with the Maxwell equations.

With each simplification, theoretical results will in general deviate further from the real case. Existing boundary layer concepts without magnetomotive effects are already complex after a simplification of the Navier-Stokes equations. While present boundary layer theory may yield results which compare favorably with experiment in the simpler cases, clearly it cannot be expected that such close agreement will be found in the magnetohydrodynamic case. Because of additional variables and equations, equivalent orders of difficulty will necessitate many more simplifying assumptions, especially regarding the character of the plasma itself. The result is that while the system of equations describing a given magnetohydrodynamic case may be of the same order of complexity the engineer must nevertheless expect to receive results which agree with the real case less closely than in equivalent gasdynamic cases; indeed, where before theory might deviate quantitatively only a few percent in a gasdynamic problem, a qualitative agreement may have to be accepted in a typical magnetohydrodynamic problem.

The Boundary Layer

As a first assumption, one might consider a fluid as a "perfect" one; i.e., homogeneous and inviscid. Many fluid flow cases may be solved with acceptable accuracy under these assumptions. It is known, however, that under some conditions the fluid will separate from the body; this is not always predicted by perfect fluid theory. In addition, surface friction is not accounted for by perfect fluid methods.

The obvious step is to generalize the fluid to include viscosity. When this is done the equations which result, the so-called Navier-Stokes equations, are too complex to be solved in general. Moreover, the viscosity of ordinary fluids is too small to be of significance in the main flow field. For these reasons the complete Navier-Stokes equations were little used until 1904 when L. Prandtl introduced the boundary layer concept to the Mathematical Congress in Heidelberg (11).

Prandtl showed that the flow about a body could be divided into two distinct regions, the main fluid flow and a relatively thin layer on the boundary of the solid body called the "boundary layer". In the main flow the viscous effects are too small to appreciably alter the perfect fluid results. In the thin boundary layer, however, the velocity gradients are large enough to result in the viscous effects dominating the fluid flow. With this concept Prandtl was able to predict for the first time the point of flow separation and the surface friction on a body in good agreement with experiment and in a case where perfect fluid theory implied no separation or surface friction.

Understanding Prandtl's boundary layer concept can be facilitated by considering the simple example of a semi-infinite flat plate at zero incidence in a uniform flow field as in Fig. 1. Because of viscosity the velocity, $u(x,o)$, will be identically zero for all positive x if there is no slip at the wall. However, as y increases without bound the velocity must equal U_∞ eventually; thus, u varies from zero at the wall

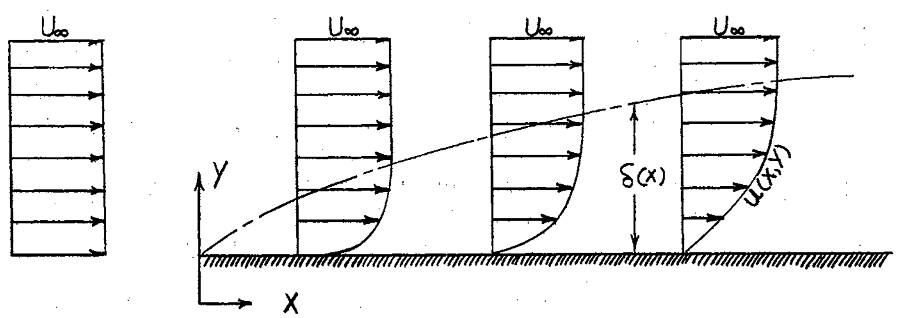

Fig. 1. Flat Plate Boundary Layer.

asymptotically to U_∞ at some distance from the wall. For convenience it is usually stated that when $u(x,y)$ is 99 percent of U_∞ the "edge" of the boundary layer has been reached. This velocity distribution is called the velocity profile and it should be noted in Fig. 1

that $\partial u/\partial y > 0$ for all y. It will be shown later that the slope of the velocity profile is the essential parameter when the separation point is considered. Note too that the boundary layer thickness, δ (x), increases with x. The surface friction is obtained by

$$\tau_0 = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} .$$

According to perfect fluid theory, the flow about a circular cylinder need not separate, although it is observed to do so in practice, except for very low Reynolds number. Thus, perfect fluid theory fails under these conditions. With the viscous boundary layer concept, however, separation is predicted. Prandtl's boundary layer theory states that separation will occur where the velocity gradient is zero at the wall and implies negative velocities near the wall at stations beyond this. Unfortunately, the velocity profiles and, therefore, the separation point depend upon the pressure distribution outside the boundary layer and the pressure distribution in turn upon the separation point. For this reason the pressure distribution must be known from experiment or guessed with sufficient accuracy. The predicted separation must then establish a flow pattern yielding the originally assumed pressure distribution. This procedure is very difficult since the Prandtl boundary layer theory is invalid beyond the separation point.

Experimental studies show strong curvature of the streamline patterns just downstream of a separation point implying the existence of a pressure gradient normal to the surface. The original boundary layer equations, however, made use of the assumption of

$$\partial P/\partial y \cong 0$$

and are, therefore, no longer valid in this area.[3] Further, the flow in the separated area is in general unsteady.

An additional difficulty also arises. The Prandtl assumption of a very thin boundary layer permitted the use of perfect fluid theory to establish the pressure gradient, but downstream of the separation point $\delta(x)$ increases at an enormous rate and is certainly no longer thin and greatly alters the pressure distribution - used to predict the separation in the first place (11).

Methods of Solution

Examples of exact solutions to the boundary layer equations show that except for the very simplest of cases the mathematical difficulties are considerable. An exact solution is assumed to be any solution, approximate or otherwise, to the original unchanged equations which govern the flow field. The most general case of fluid flow about a body of arbitrary shape cannot be solved by analytical methods known to date. Two methods are open to use. First is to simplify the problem to solvable level while still obtaining an exact solution. One then has confidence in the results but the problem may now be too trivial to be of value or application. A second approach is to retain as much as possible of the original problem but to simplify the solution by certain approximations to the equations themselves.

Boundary layer solutions are obtained only with difficulty and the trial and error requirements of matching the potential flow field, results in tedious solutions at best since each solution leads to a

[3] See Appendix A

resulting potential flow outside the boundary layer which may or may not agree with the assumed potential flow field. The problem must be solved again and again until the solution yields the assumed potential flow pattern.

The magnetofluidynamic boundary layer problem is an extension of the ordinary fluid dynamic situation. Boundary layer solutions are difficult in the gasdynamic case and are much more difficult in the magnetofluidynamic case. Much research has already been done in the field of magnetofluidynamics; however, so far as is known most of this work has been in connection with inviscid magnetofluidynamic flows (3) (8). Only recently have results appeared on viscous magnetofluidynamic flow problems.

Magnetohydrodynamic boundary layer equations appear to be rather simple in that for the cases to be considered only a single pondero-motive force term appears in the equations. However, even with this seemingly simple addition to the boundary layer equations it is very difficult indeed to solve such magnetohydrodynamic problems exactly. So far as can be determined to date, exact solutions to the magnetohy-drodynamic boundary layer problem have been published only for flow along flat surfaces with zero pressure gradient, the exception being Stokes' flow (10). Other efforts are being proposed, such as the solutions to the incompressible wedge flows (12), but as yet they are unfinished.

Mathematical complexity has resulted in even simple geometric shapes being difficult to handle by means of an exact mathematical approach. The magnetohydrodynamic boundary layer flow for a flat plate has been solved exactly by V.J. Rossow (6). An examination of his results will quickly disclose why exact solutions are few indeed and why they are proceeding with such difficulty. It also becomes readily apparent why so few papers have been published up to now and these only for plane surfaces.

It would be valuable to obtain approximate methods which would lead to rapid answers, even if this entailed some sacrifice in accuracy to exact methods. This was done by von Karman and Pohlhausen for ordinary boundary layer flow by assuming that it was sufficient to satisfy the differential equations on the average over the boundary layer thickness rather than to satisfy the equations at every point for every fluid particle. A mean value function is thus obtained from the momentum theorem which is developed as an integral of the equations of motion over the boundary layer thickness (7).

It is the purpose of this dissertation to introduce such an approximation method, based upon the ordinary fluid dynamic von Karman-Pohlhausen technique, suitably extended to magnetohydrodynamics. With such a technique magnetohydrodynamic boundary layer solutions could be obtained more quickly and for more varied geometrics than the flat plate.

This approximate method will first be applied to those cases where the exact solution is known in order that the results might be compared. The method will then be extended to the wedge and circular cylinder. No comparison is possible in these latter two cases as solutions for these are not known to have been published at this time.

CHAPTER II

VON KARMAN-POHLHAUSEN APPROXIMATION

This Chapter will introduce the von Karman-Pohlhausen approximation method for the solution of two-dimensional boundary layer problems. It is a momentum integral approximation and when applied in the normal way to the magnetohydrodynamic boundary layer, the method seems to fail; an examination discloses why and suggests an extension which could correct the difficulty.

The procedure used will be very much the same in subsequent chapters. Examples have been selected for which well documented exact solutions exist allowing a check on the accuracy and validity of the von Karman-Pohlhausen approximation when applied to a magnetohydrodynamic case. It will be seen that the procedures and fundamentals are very similar to those applied to the ordinary fluid dynamic case so that a detailed explanation will not be necessary.

The Method

Ordinarily the boundary layer equations, which represent the second law of motion and conservation of mass, are solved at every point within the boundary layer. It was von Karman and Pohlhausen who first thought that it might be sufficiently accurate to satisfy these equations on the average over the boundary layer thickness.

This was done by integrating the equation of motion over the boundary layer thickness. It is this integral equation which is then satisfied, rather than the equation of motion itself.

The von Karman-Pohlhausen approximation will be applied to three cases of magnetohydrodynamic boundary layer flow. All will have the magnetic field oriented perpendicularly to the surface of the body. These three cases are as follows:

1. Magnetic field fixed to the body
   a. Constant conductivity
   b. Variable conductivity
2. Magnetic field fixed to the flow
   a. Constant conductivity

Each case will be solved in turn and the flat plate results will be compared to the exact solution.

Body fixed Magnetic Field

Constant Conductivity

For this case and all subsequent cases covered in this dissertation, the following assumptions are made:

1. Steady two-dimensional laminar flow
2. ρ = constant
3. μ, c_p, c_v and k_c are constant
4. Magnetic field lines are perpendicular to the surface of the body and the free stream outside the boundary layer
5. μ_m, magnetic permeability, is constant throughout the flow field
6. Imposed electric field, $\vec{E}$, is zero
7. The excess charge density, ρ_e, is zero

8. Induced magnetic field, $\vec{B}_I$, is negligible and as a consequence the magnetic force term is linearized; i.e., $\vec{B} = \vec{B}_o + \vec{B}_I \cong \vec{B}_o$, where $\vec{B}_I \cong 0$ and $\vec{B}_o \equiv \vec{B}\,\big|_{y=0}$ is a constant

9. Conductivity, σ, is small

The boundary layer equations which result from these assumptions are very similar to the hydrodynamic equations, namely,[1]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

and

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} - \frac{\sigma B_o^2}{\rho}u + \nu\frac{\partial^2 u}{\partial y^2} \quad . \tag{2}$$

Equation (1) represents the conservation of mass and Eq. (2) is the equation of motion. The meaning of each term in Eq. (2) is indicated below:

$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$	$=$	$-\frac{1}{\rho}\frac{dP}{dx}$	$-\frac{\sigma B_o^2}{\rho}u$
(Time rate of change of momentum)		(Pressure gradient)	(Magnetic field body forces)

$$+\ \nu\frac{\partial^2 u}{\partial y^2} \quad . \tag{3}$$

(Frictional Shear Forces)

For convenience let $m_1 \equiv \frac{\sigma_o B_o^2}{\rho}$, where $\sigma_o \equiv \sigma\big|_{y=0} =$ a constant, and $m \equiv m_1 / U_\infty$.

With these definitions Eqs. (1) and (2) become:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} - m_1 u + \nu\frac{\partial^2 u}{\partial y^2}$$

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[1]See Appendix A and B for derivation.

The boundary conditions are as follows:

$$y = \delta \Rightarrow u = U$$

$$y = o \Rightarrow u = v = o$$

These conditions follow from a physical consideration of the boundary layer itself. The requirements at $y = \delta$ arise from the necessity of the boundary layer equations being asymptotic to the free stream solution. Thus, there would no longer be any change in u with y. Conditions at the wall, y = o, are imposed by the assumptions of no fluid slip or crossing of the boundary at the wall. The equation of motion may now be integrated as follows:

$$\int_0^\infty \left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{dP}{dx} + m_1 u \right] dy = \frac{1}{\rho}\int_0^\infty \mu \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) dy$$

Apply the first boundary condition to the equation of motion.

$$y = \delta \Rightarrow u = U(x)\ ,\ \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow U\frac{dU}{dx} = -\frac{1}{\rho}\frac{dP}{dx} - m_1 U$$

Also, from the continuity equation:

$$v = -\int_0^y \frac{\partial u}{\partial x} dy$$

Then,

$$\int_0^\infty \left[u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\int_0^y \frac{\partial u}{\partial x} dy - U\frac{dU}{dx} - m_1(U-u) \right] dy$$

$$= \frac{1}{\rho}\int_0^\infty \mu\, d\left(\frac{\partial u}{\partial y}\right) = \frac{1}{\rho}\mu\frac{\partial u}{\partial y}\Big|_0^\infty .$$

But, $\tau_0 = \mu\frac{\partial u}{\partial y}\Big|_{y=0}$ and $\frac{\partial u}{\partial y}\Big|_{y=\infty} = 0$.

Also, integration by parts will provide

$$-\int_0^\infty \frac{\partial u}{\partial y}\left[\int_0^y \frac{\partial u}{\partial x} dy\right] dy = \int_0^\infty u\frac{\partial u}{\partial x} dy - U\int_0^\infty \frac{\partial u}{\partial x} dy .$$

with these relations the integration becomes

$$\int_0^\infty \left[u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} - U\frac{\partial u}{\partial x} - U\frac{dU}{dx} - m_1(U-u) \right] dy = -\frac{\tau_0}{\rho} ,$$

$$\int_0^\infty \left[\frac{\partial}{\partial x}(u(U-u)) + \left(\frac{dU}{dx} + m_1\right)(U-u)\right] dy = \frac{\tau_0}{\rho} \quad .$$

Introducing the displacement thickness defined by

$$\delta^* = U^{-1}\int_0^\infty (U-u)\, dy$$

and the momentum thickness defined by

$$\Theta = U^{-2}\int_0^\infty u(U-u)\, dy$$

one obtains,

$$\frac{d}{dx}(U^2\Theta) + \left(\frac{dU}{dx} + m_1\right)U\delta^* = \frac{\tau_0}{\rho} \quad . \tag{4}$$

Equation (4) is unchanged from the hydrodynamic case except for the addition of m_ξ. The equation represents an integration of the momentum equation across the boundary layer thickness and is in general form, i.e., no simplification or assumptions have been made as to the configuration of the two-dimensional body and its associated pressure gradient outside the boundary layer. Although Eq. (4) has been derived for laminar flow, this result will be applicable to turbulent flow as well since the equation represents the averaged forces over the boundary layer thickness.

In general the coordinate system is arranged with y normal to the surface of the body and y = 0 on the wall. With x measured along the surface, Fig. 2, the integrated momentum equation takes the form of Eq. (4).

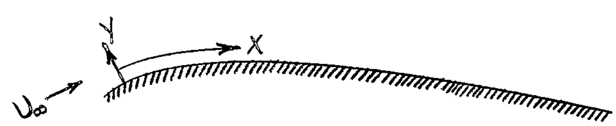

Fig. 2. Coordinate System

This equation is now an ordinary differential equation since a satisfactory velocity profile can be found which will allow the calculation of θ, δ^*, δ and τ_0. The boundary conditions must include no slip at the wall and continuity at the edge of the boundary layer where the inviscid flow solution is valid. Further, when adverse forces are impressed upon the flow the possibility of an inflexion point in the velocity profile must be accounted for to allow for separation if these forces persist. Of course to find the separation point at all it must be possible to calculate the point along the wall where

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0 \quad .$$

In this first case a body-fixed $\vec{B}$ perpendicular to the two-dimensional flow has been assumed. Additionally, it was assumed that m_1 and σ were constants.

With the use of Equations (1), (3), and (4) the problem may be solved in the usual way by the von Karman-Pohlhausen method. It is first assumed that the velocity profile may be approximated by a polynomial of order high enough to meet the boundary conditions and allow for the existence of an inflexion point. The assumption is often

$$\frac{u}{U} = f(\eta) = a\eta + b\eta^2 + c\eta^3 + d\eta^4$$

where $\eta = \frac{y}{\delta}$. It has been found in the gasdynamic case that this assumption yields acceptable accuracy for laminar flow. Remaining for determination are the coefficients a,b,c and d which, in general, are functions of x. These coefficients are determined by applying boundary conditions in the following way.

$$\eta = 1 \quad \Longrightarrow \quad u = U$$

$$\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = 0$$

which leads to the three equations

$$1 = a + b + c + d$$

$$o = a + 2b + 3c + 4d$$

$$o = 2b + 6c + 12d$$

$$\eta = 0 \Rightarrow u = v = 0$$

This condition has already been used to eliminate the η^{o} term of $f(\eta)$ and another boundary condition must be obtained. Using Eq. (2) at the wall the necessary condition is obtained, namely

$$0 = -\frac{1}{\rho}\frac{dP}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \quad .$$

Also, from Eq. (2) at $\eta = 1$

$$U\frac{dU}{dx} + \frac{1}{\rho}\frac{dP}{dx} + m_1 U = 0$$ is obtained.

The following definitions are now introduced:

$$\Lambda \equiv \frac{\delta^2}{\nu}\frac{dU}{dx}$$

$$\Lambda_m \equiv \frac{\delta^2}{\nu} m_1$$

With the above relations and definitions the final boundary condition becomes:

$$-\frac{\delta^2}{\nu}\left(\frac{dU}{dx} + m_1\right) = 2b$$

or,

$$-(\Lambda + \Lambda_m) = 2b \quad .$$

The four equations

$$1 = a + b + c + d$$

$$o = a + 2b + 3c + 4d$$

$$o = 2b + 6c + 12d$$

$$-(\Lambda + \Lambda_m) = 2b$$

may now be solved for a, b, c and d in terms of $(\Lambda+\Lambda_m)$. Since Λ_m and Λ contain $\delta(x)$ this quantity remains to be evaluated and is obtained through the solution of the integrated momentum equation.

Because

$$(\Lambda + \Lambda_m) = -\frac{1}{\rho U}\frac{dP}{dx}\frac{\delta^2}{\nu} ,$$

the flat plate case dP/dx = o implies $\Lambda + \Lambda_m = o$; thus, the velocity profile, u/U, becomes not only independent of x but also completely independent of the magnetic field yielding the ordinary gasdynamic velocity profiles.

Even for a flat plate one hardly expects that the velocity profiles will not be affected in any way by the magnetic field. This is corroborated by the exact flat plate solution which does indeed show a definite dependence upon the magnetic field as expected (6). Thus, in the flat plate case this integral method fails to introduce any magnetic effects into the velocity profile, resulting in an ordinary fluid dynamic profile which would, therefore, lead to gross errors in the surface friction coefficient. Case II results show the same difficulty. Case III, a flow-fixed magnetic field, will yield some results in that the stabilizing influence of the flow-fixed field is suggested by the analysis. In the conclusions which follow it will be seen why any quantitative results must be rejected.

Conclusions

The present method employed is based upon a straight forward integration of the equation of motion in the magnetohydrodynamic case and an assumption for the velocity profile in terms of arbitrary constants evaluated by means of the boundary conditions. This conventional straight forward application is inadequate because its use in the von Karman-Pohlhausen approximation depends upon a satisfactory velocity profile, obtained in terms of the shape factor $\Lambda(x)$. Additionally, the shape factor must represent those forces which alter the shape of the velocity profile and which directly determine the surface friction and separation point.

The von Karman-Pohlhausen approximation requires that the results of the boundary layer theory be consistent with the inviscid flow solution at the edge of the boundary layer and that the inviscid solution be unchanged by events inside the boundary layer, prior to separation. It is also assumed that the pressure force is independent of y and that it affects the entire profile at a given x with a constant value. In the fluid dynamic case when the velocity profile is obtained by the von Karman-Pohlhausen method, the boundary conditions used introduce just such a pressure gradient. This pressure gradient is the only force affecting separation and it is already a function of x only.

For the magnetohydrodynamic case, however, there is another force affecting the velocity profile besides the pressure gradient; i.e., the ponderomotive force. This new force is a function of y and x and its effects are not constant across the boundary layer.

Further, as can be seen from Eq. (2) this ponderomotive force can never be properly introduced into the von Karman-Pohlhausen approximation as it is normally applied because it vanishes at the wall, which is the only place where it can be introduced into $\Lambda(x)$ and the velocity profile. It was for this reason that the velocity profiles so obtained were identical to the fluid dynamic profiles and prevented a proper conclusion, even though the momentum integral was correct.

A non-zero result is obtained in the flow-fixed magnetic field case because the ponderomotive forces are not zero at the wall and can, therefore, be introduced into the velocity profile and $\Lambda(x)$. The result is not meaningful quantitatively, however, since the ponderomotive forces are not properly introduced into the velocity profile in this manner since it assumes an invariance in y and a value for all y equal to the boundary value at the wall. This result is clearly erroneous as the magnetohydrodynamic force at $y = \delta$ is not the value at the wall but zero. Further, results obtained in this manner do not agree with the exact solution (6).

A modification of the von Karman-Pohlhausen method is necessary. von Karman and Pohlhausen assumed that it was sufficiently accurate to satisfy the boundary layer equations in bulk or on the average; i.e., by satisyfing the momentum integral equation. This may now be extended in principle to the ponderomotive forces by considering the average ponderomotive force acting within the boundary layer. Applying the force in this manner results in a mean value which may still be a function of x but which is not independent of y, thus eliminating the difficulties previously noted with respect to the magnetohydrodynamic case. Mean ponderomotive force terms can then be obtained by integrating over the boundary layer from $y = o$ to $y = \delta$. With this modification the cases were solved and compared in detail with the exact solution (6).

The agreement was good only if a variable "weight-factor" was used with the mean ponderomotive force term. No rational way was found whereby this "weight-factor" could be predicted in advance for a given flow problem. A further difficulty was that the mean force term could be greater than zero even when the velocity profile had a zero slope at the wall. This implied separation without a pressure gradient even though an examination of Eq. (2) shows that when $u = 0$ the ponderomotive forces must be zero. At first examination the solutions seemed to be in agreement with the exact solutions (6); however, it was noted that the results of Reference (6) are valid only for $mx < .2$. For $mx > .2$ the series solution of Rossow (6) would have to include many more terms. This could only be done with great difficulty and to date only second order terms have been calculated. Another way had to be found if the von Karman-Pohlhausen method was to be used and the above difficulties avoided.

CHAPTER III

THE EXTENDED VON KARMAN-POHLHAUSEN METHOD

Constant Conductivity

From the previous chapter it was found that the main difficulty was the introduction of the magnetic field effect into the velocity profile relations. The assumption of a polynomial of higher order will also be of no avail unless a boundary condition which includes the magnetic field can be found. Such a boundary condition can be found in the following way.

Taking the $\partial/\partial y$ of Eq. (2)

$$\frac{\partial}{\partial y}\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + m_1 u + \frac{1}{\rho}\frac{dP}{dx} - \nu\frac{\partial^2 u}{\partial y^2}\right] = 0$$

leads directly to

$$u\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + m_1\frac{\partial u}{\partial y} - \nu\frac{\partial^3 u}{\partial y^3} = 0$$

From the continuity equation it can be seen that

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \quad .$$

Using this relation the above equation reduces to

$$u\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 u}{\partial y^2} + m_1\frac{\partial u}{\partial y} - \nu\frac{\partial^3 u}{\partial y^3} = 0 \quad .$$

The following equations are now available:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + m_1 u = -\frac{1}{\rho}\frac{dP}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (3)$$

$$\frac{d}{dx}(\theta U^2) + \left(\frac{dU}{dx} + m_1\right)U\delta^* = \frac{\tau_0}{\rho} \qquad (4)$$

$$u\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 u}{\partial y^2} + m_1\frac{\partial u}{\partial y} = \nu\frac{\partial^3 u}{\partial y^3} \qquad (5)$$

Previously, a, b, c and d were found from four boundary conditions obtained from

$$\eta = 1 \Rightarrow u = U(x)$$

and

$$\eta = 0 \Rightarrow u = v = 0.$$

The second boundary condition had already been used to show the coefficient of η^0 to be zero. The first boundary condition was $u = U(x)$ at $\eta = 1$. Two additional boundary conditions were also obtained from $u = U(x)$ at $\eta = 1$, namely

$$\left.\frac{\partial u}{\partial y}\right|_{\eta=1} = \left.\frac{\partial^2 u}{\partial y^2}\right|_{\eta=1} = 0 \ .$$

The fourth condition was obtained from Eq. (3) at $\eta = 0$; i.e.,

$$\nu\frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho}\frac{dP}{dx}$$

Additional boundary conditions could have been obtained from a higher derivative of u; i.e., $\frac{\partial^3 u}{\partial y^3}$. However, it would not contribute to the solution. This can be seen from Eq. (5) which shows that $\left.\frac{\partial^3 u}{\partial y^3}\right|_{\eta=0,1} = 0$ when $m_1 = 0$. It is for this reason that a fourth order polynomial is usually sufficient in the gasdynamic case and insufficient in the magnetohydrodynamic case where $m_1 \neq 0$. The use of Eq. (5) in determining the fifth boundary condition now allows the extended von Karman-Pohlhausen method to yield proper results. With these relations the following boundary conditions are obtained:

$$\eta = 1 \Rightarrow \quad u = U(x)$$

$$\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = 0$$

$$-\frac{1}{\rho}\frac{dP}{dx} = U\left(\frac{dU}{dx} + m_1\right)$$

$$\eta = 0 \Rightarrow \quad u = v = 0$$

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{1}{\rho}\frac{dP}{dx}$$

$$\nu \frac{\partial^3 u}{\partial y^3} = m_1 \frac{\partial u}{\partial y}$$

Assuming a fifth order order polynomial for u ,

$$\frac{u}{U} = f(\eta) = a\eta + b\eta^2 + c\,\eta^3 + d\eta^4 + e\,\eta^5, \tag{6}$$

one obtains the following set of equations for a, b, c, d and e from the above boundary conditions and the definitions of Λ and Λ_m:

$$\begin{aligned}
1 &= a + b + c + d + e \\
0 &= a + 2b + 3c + 4d + 5e \\
0 &= 2b + 6c + 12d + 20e \\
-(\Lambda + \Lambda_m) &= 2b \\
0 &= a\Lambda_m - 6C
\end{aligned}$$

These equations may then be solved for a, b, c, d and e yielding:

$$a = \frac{60 + 9\Lambda_m(1 + \Lambda/\Lambda_m)}{36 + \Lambda_m} \tag{7}$$

$$b = \frac{-\Lambda_m(1 + \Lambda/\Lambda_m)}{2} \tag{8}$$

$$c = \frac{\Lambda_m[20 + 3\Lambda_m(1 + \Lambda/\Lambda_m)]}{2(36 + \Lambda_m)} \tag{9}$$

$$d = \frac{30\Lambda_m(\Lambda/\Lambda_m) - 120 + 3\Lambda_m(2 - \Lambda_m)(1 + \Lambda/\Lambda_m)}{2(36 + \Lambda_m)} \tag{10}$$

$$e = \frac{72 - 12\Lambda_m(\Lambda/\Lambda_m) - \Lambda_m(6 - \Lambda_m)(1 + \Lambda/\Lambda_m)}{2(36 + \Lambda_m)} \tag{11}$$

From the definition of displacement thickness one obtains

$$\frac{\delta^*}{\delta} = \int_0^1 (1 - f)\, d\eta \; .$$

Using Eq. (6) , the displacement thickness is found as a function of a, b, c, d and e.

$$\frac{\delta^*}{\delta} = 1 - \frac{a}{2} - \frac{b}{3} - \frac{c}{4} - \frac{d}{5} - \frac{e}{6} \tag{12}$$

In a similar way $\frac{\theta}{\delta}$ may be found.

$$\frac{\theta}{\delta} = \frac{a}{2} + \frac{b}{3} + \frac{c}{4} + \frac{d}{5} + \frac{e}{6} - \frac{a^2}{3} - \frac{b^2}{5} - \frac{c^2}{7} - \frac{d^2}{9}$$
$$- \frac{e^2}{11} - \frac{ab}{2} - \frac{2}{5}ac - \frac{ad}{3} - \frac{2}{7}ae - \frac{bc}{3} - \frac{2}{7}bd - \frac{be}{4}$$
$$- \frac{dc}{4} - \frac{2}{9}ce - \frac{de}{5} \tag{13}$$

The momentum integral equation is now used to obtain $\Lambda_m = \Lambda_m(mx)$ in the following manner:

Multiply Eq. (4) by $\frac{\theta}{\nu U}$ and apply the definitions,

$$Z = \frac{\theta^2}{\nu} \quad ,$$

$$f_1 = \frac{\delta^*}{\theta} \quad ,$$

$$f_2 = \frac{\tau_0 \theta}{\mu U} = \frac{\theta}{\delta}\, a \quad ,$$

$$K = m_1 \left(\frac{U'}{m_1} + 1\right) Z \qquad \text{and} \qquad F = 2f_2 - 4K - 2Kf_1 \quad .$$

Equation (4) then becomes

$$U \frac{dz}{dx} = F + 4 m_1 Z \quad . \tag{14}$$

However, it can be shown from the definitions of Λ and Λ_m that

$$\frac{\Lambda}{\Lambda_m} = \frac{U'}{m_1} \tag{15}$$

and

$$\Lambda_m \frac{\theta^2}{\delta^2} = m_1 Z \tag{16}$$

Proceeding with Eq. (14) and noting that $m = \frac{m_1}{U_\infty}$ one obtains:

$$\frac{U}{U_\infty} \frac{d(m_1 Z)}{d(mx)} = F + 4(m_1 Z)$$

$$\frac{U}{U_\infty} \frac{d(m_1 Z)}{d\Lambda_m} \frac{d\Lambda_m}{d(mx)} = F + 4(m_1 Z) \quad ,$$

$$\frac{U}{U_\infty} \frac{d}{d\Lambda_m}\left[\Lambda_m \left(\frac{\theta}{\delta}\right)^2\right] \frac{d\Lambda_m}{d(mx)} = F + 4\Lambda_m \left(\frac{\theta}{\delta}\right)^2 \quad . \tag{17}$$

where m = a known constant for any particular case. The following quantities of Eq. (17) may now be evaluated in terms of Λ and Λ_m as indicated:

$$\frac{\theta}{\delta} = \text{Eq. (13) with Eqs. (7) - (11)} \quad ,$$

$\frac{\delta^*}{\delta}$ = Eq. (12) with Eqs. (7) - (11),

$$F = 2f_2 - 4K - 2Kf_1 \;,\; f_2 = \frac{\theta}{\delta} a \;,$$

$$K = \Lambda_m \left(\frac{\theta}{\delta}\right)^2 (\Lambda/\Lambda_m + 1) \quad \text{and} \quad f_1 = \frac{\delta^*}{\delta}\left(\frac{\theta}{\delta}\right)^{-1} .$$

The solution to Eq. (17) yields $\Lambda_m = \Lambda_m(mx)$ and, therefore,

$$\frac{u}{U_\infty} = \frac{U}{U_\infty}\left[a\eta + b\eta^2 + c\eta^3 + d\eta^4 + e\eta^5\right] \tag{18}$$

for each $\frac{U}{U_\infty}$ and $\frac{\Lambda}{\Lambda_m}$.[1] Also, $C_f \equiv \frac{2\tau_0}{\rho U_\infty^2}$ which yields:

$$\sqrt{R_N}\, C_f = 2a \frac{U}{U_\infty}\sqrt{mx/\Lambda_m} \quad . \tag{19}$$

Curves may now be drawn for $\sqrt{R_N}C_f$ vs mx and u/U_∞ vs $y\sqrt{U_\infty/\nu x}$ for values of mx. For each U/U_∞ there will be in general a family of such curves for each $\frac{\Lambda}{\Lambda_m}$. The velocity profile is given as u/U_∞ vs $y\sqrt{U_\infty/\nu x}$ rather than u/U_∞ vs η since $y\sqrt{U_\infty/\nu x}$ was used in reference (6) and is the usual one, although it can be clearly seen that from the standpoint of the von Karman-Pohlhausen method η is the more convenient.

For the special case of a plat plate, $\frac{dP}{dx} = 0$, $\frac{U}{U_\infty} = 1 - mx$ and the previous relations take a more simple form. From the boundary conditions at $\eta = 1, \frac{dU}{dx} = -m_1$. Then $\Lambda = -\Lambda_m$ and $\frac{\Lambda}{\Lambda_m} = -1$. Thus, the family of curves in $\frac{\Lambda}{\Lambda_m}$ reduces to the single curve for $\frac{\Lambda}{\Lambda_m} = -1$ in this case. The results have been plotted in Figs. 3 and 4 and it can be seen from Fig. 3 and Fig. 4 that agreement with reference (6) is quite good. Further, Rossow (6) points out that his solution is acceptable only for mx $<$ 0.2;

[1]Appendix C.

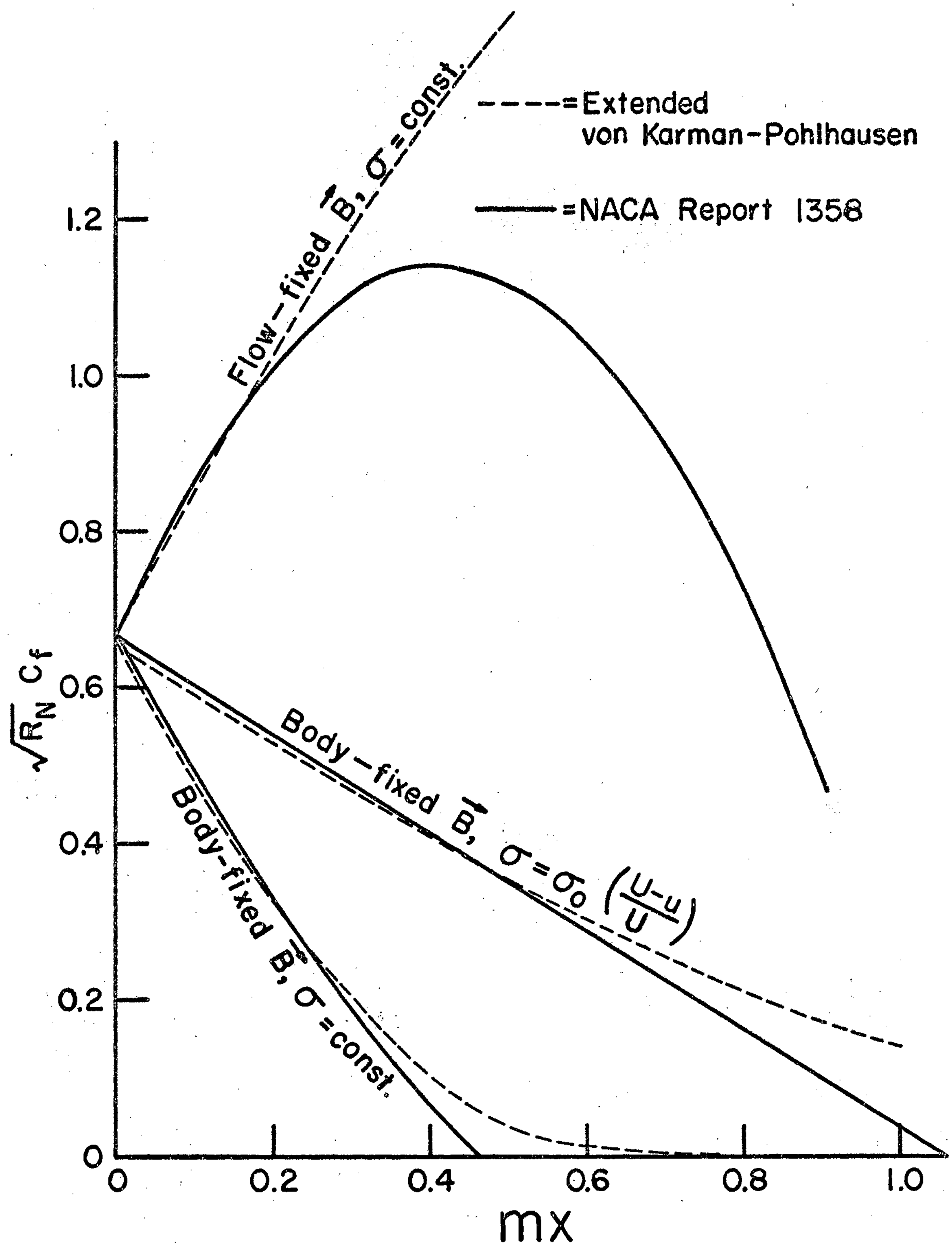

Fig. 3. $\sqrt{R_N}\ C_f$ vs mx, Flat Plate Flow

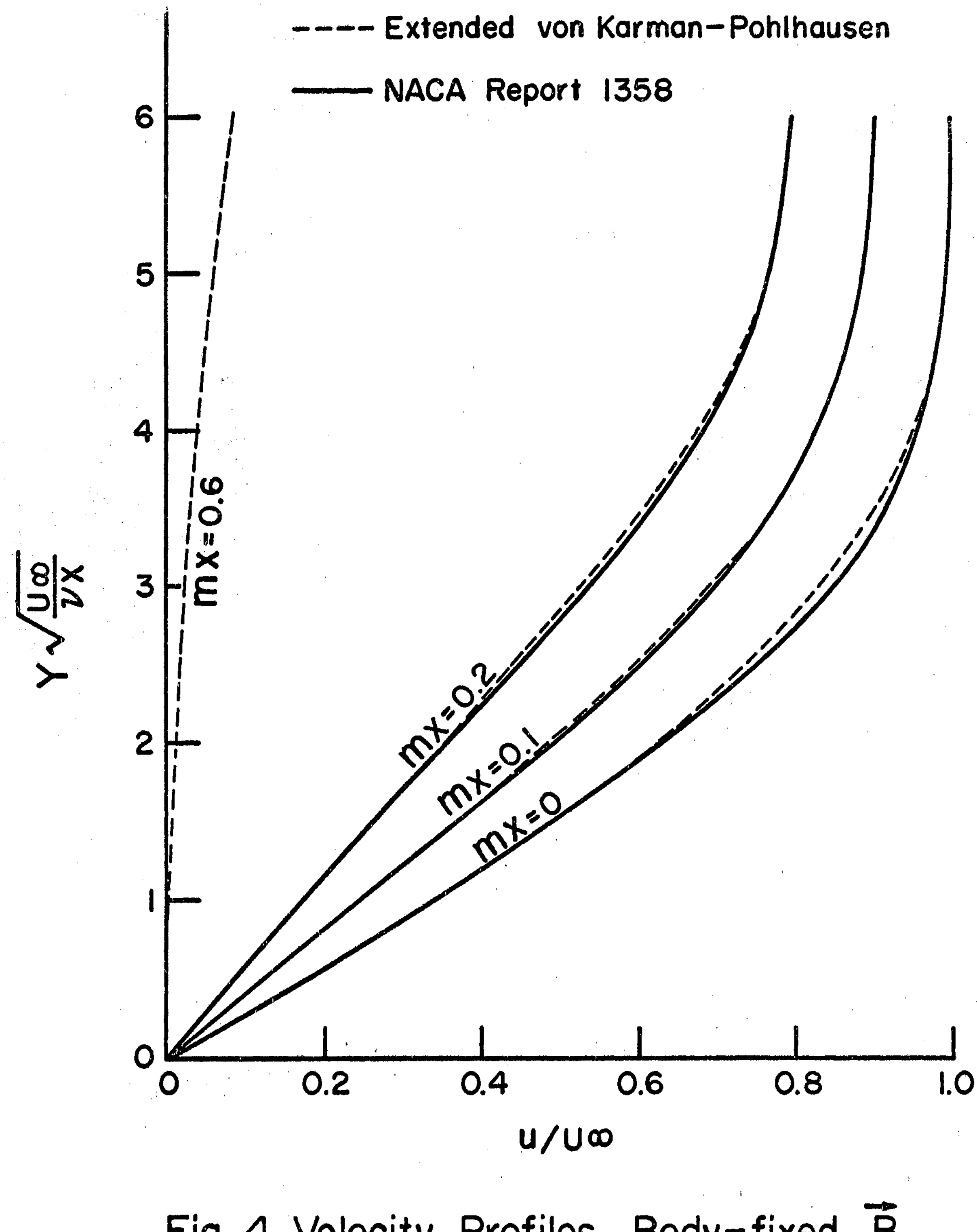

Fig. 4. Velocity Profiles, Body-fixed $\vec{B}$, σ = Constant, Flat Plate Flow

therefore, the indicated separation condition does not really exist for the flat plate case. The extended von Karman-Pohlhausen method, however, seems to be entirely in agreement with the expected physical results for much larger values of mx and indeed suggests that the flow cannot separate under these conditions because

$$a = \frac{60 + 9\Lambda_m}{36 + \Lambda_m} \quad .$$

Since $\Lambda_m = \frac{\delta^2}{\nu} m_1$ which is always greater than zero, it is clear that 'a' also is never zero. But 'a' must be zero for separation; hence, the flow cannot separate.

The case of a flow-fixed magnetic field may now be solved in a similar manner. Since the flow outside the boundary layer is not affected by the magnetic field in this case the pressure distribution is now independent of the magnetic field. The result is the same momentum integral equation developed for the previous body-fixed magnetic field case. The difference lies in the fact that $U'(x)$ is now independent of m_1 which was not the case previously. For the flat plate the parameter $\frac{\Lambda}{\Lambda_m}$ now reduces to $\frac{\Lambda}{\Lambda_m} = 0$.

These results are plotted in Figs. 3 and 5 and are again seen to be in excellent agreement with NACA Report 1358 (6). It should be noticed, however, that the results of NACA Report 1358 are limited to $mx < 0.2$. This difficulty does not present itself with the extended von Karman-Pohlhausen approximation which is a closed solution and applicable for large values of mx.

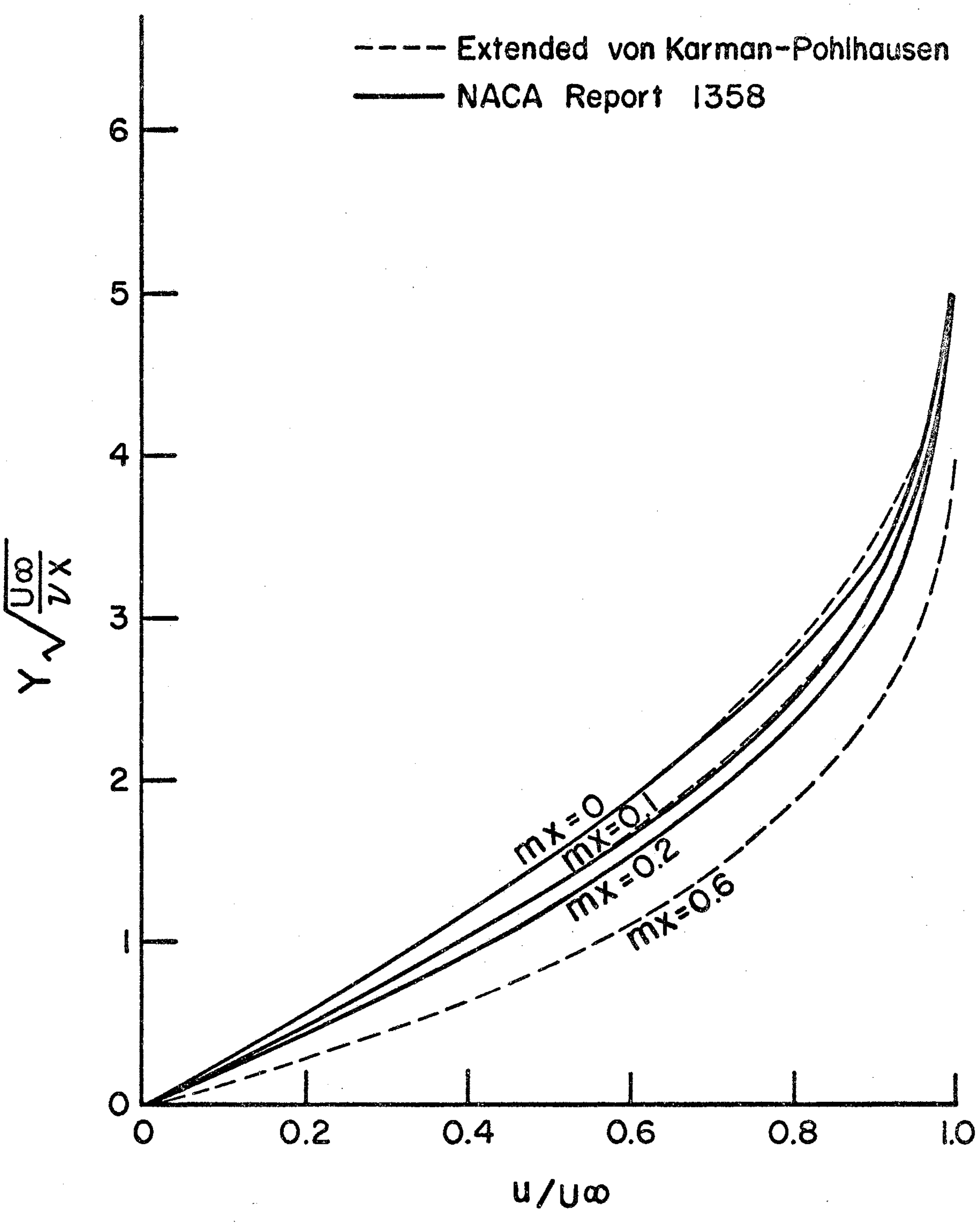

Fig. 5. Velocity Profiles, Flow-fixed $\vec{B}$, σ = Constant, Flat Plate Flow

Variable Conductivity, Body-fixed Magnetic Field

The assumption of a variable conductivity is in many cases more realistic. Of course the problem is to determine what constitutes a realistic conductivity distribution. Kantrowitz (2) found that for high Mach numbers of order 15, $\sigma = \sigma_o \left[\frac{U-u}{U}\right]$ where $\sigma_o = \sigma\Big|_{y=0}$. This assumption will, therefore, be used in this analysis. With this assumption the following equations are obtained:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + m_1\frac{u}{U}(U-u) = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} ,$$

$$u\frac{\partial^2 u}{\partial x\partial y} + v\frac{\partial^2 u}{\partial y^2} + m_1\frac{\partial u}{\partial y}\left(1-2\frac{u}{U}\right) = \nu\frac{\partial^3 u}{\partial y^3} ,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 ,$$

$$\frac{\tau_o}{\rho} = \frac{d}{dx}(\theta U^2) + \delta^* U\frac{dU}{dx} - \frac{m_1}{U}(\theta U^2) .$$

With the definitions,

$$Z = \frac{\theta^2}{\nu} , \quad K = \frac{\theta^2}{\nu}\frac{dU}{dx} , \quad f_1 = \frac{\delta^*}{\theta} , \quad f_2 = a\frac{\theta}{\delta}$$

and

$$F = 2f_2 - 4K - 2Kf_1$$

one may now reduce the momentum integral equation to:

$$U\frac{dZ}{dx} = F + 2m_1 Z .$$

Employing the definitions for Λ and Λ_m this equation may be further changed to:

$$\frac{U}{U_\infty} \frac{d}{d\Lambda_m}\left[\Lambda_m\left(\frac{\theta}{\delta}\right)^2\right] \frac{d\Lambda_m}{d(mx)} = F + 2\Lambda_m\left(\frac{\theta}{\delta}\right)^2 . \tag{20}$$

This equation is seen to be equivalent to Eq. (17) with the factor 4 replaced by 2.

Boundary conditions lead to the set of equations for a, b, c, d, and e:

$$\begin{aligned}
1 &= a + b + c + d + e \\
0 &= a + 2b + 3c + 4d + 5e \\
0 &= 2b + 6c + 12d + 20e \\
-\Lambda &= 2b \\
0 &= \Lambda_m a - 6c
\end{aligned}$$

These may now be solved for a, b, c, d, and e yielding:

$$a = \frac{60 + 9\Lambda_m(\Lambda/\Lambda_m)}{36 + \Lambda_m} \tag{21}$$

$$b = -\frac{\Lambda_m(\Lambda/\Lambda_m)}{2} \tag{22}$$

$$c = \frac{20\Lambda_m + 3\Lambda_m^2(\Lambda/\Lambda_m)}{2(36 + \Lambda_m)} \tag{23}$$

$$d = \frac{36\Lambda_m(\Lambda/\Lambda_m) - 120 - 30\Lambda_m - 3\Lambda_m^2(\Lambda/\Lambda_m)}{2(36 + \Lambda_m)} \tag{24}$$

$$e = \frac{72 - 18\Lambda_m(\Lambda/\Lambda_m) + 12\Lambda_m + \Lambda_m^2(\Lambda/\Lambda_m)}{2(36 + \Lambda_m)} \tag{25}$$

Equations (12) and (13) may now be evaluated in terms of $\frac{\Lambda}{\Lambda_m}$ and Λ . With these results Eq. (20) may be solved for Λ_m vs mx for various $\frac{\Lambda}{\Lambda_m}$ and $\frac{U}{U_\infty}$.[2] As in the previous results, for each $\frac{U}{U_\infty}$ families of curves in $\frac{\Lambda}{\Lambda_m}$ are obtained for Λ_m vs mx and, hence, $\sqrt{R_N}\ C_f$ vs mx and $\frac{u}{U_\infty}$ vs $y\sqrt{U_\infty/\nu x}$. For the special case of a flat plate $\sqrt{R_N}\ C_f$ vs mx becomes a single curve since $U = U_\infty$ and $\frac{\Lambda}{\Lambda_m} = 0$. Results are shown in Figures 3 and 6 which again show excellent agreement with reference (6).

Conclusions

The equations thus far derived by the extended von Karman-Pohlhausen approximation are for flat plate flow, $\frac{dP}{dx} = 0$, for which exact solutions exist. The exact solutions for the flat plate cases are obtained from Rossow (6). These results are summarized in Figs. 3-6.

Primarily, the advantage the extended von Karman-Pohlhausen method has over the exact solution is its range of convergence. The momentum-integral method is a closed solution, and therefore, applicable for all ranges of mx within the bounds of the original assumptions. On the other hand the exact solution's accuracy is dependent upon the maximum mx desired; as mx increases, the number of terms in the series must increase for a given accuracy. Unfortunately the number of additional terms required may be large for small increases in accuracy. Further, each additional term compounds the difficulty to such an extent that even with the use of electronic computers only a few terms of the series may be reasonably found. For this reason Rossow (6) confined himself to mx $<$ 0.2 and only second - order terms for constant conductivity

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[2]See Appendix C.

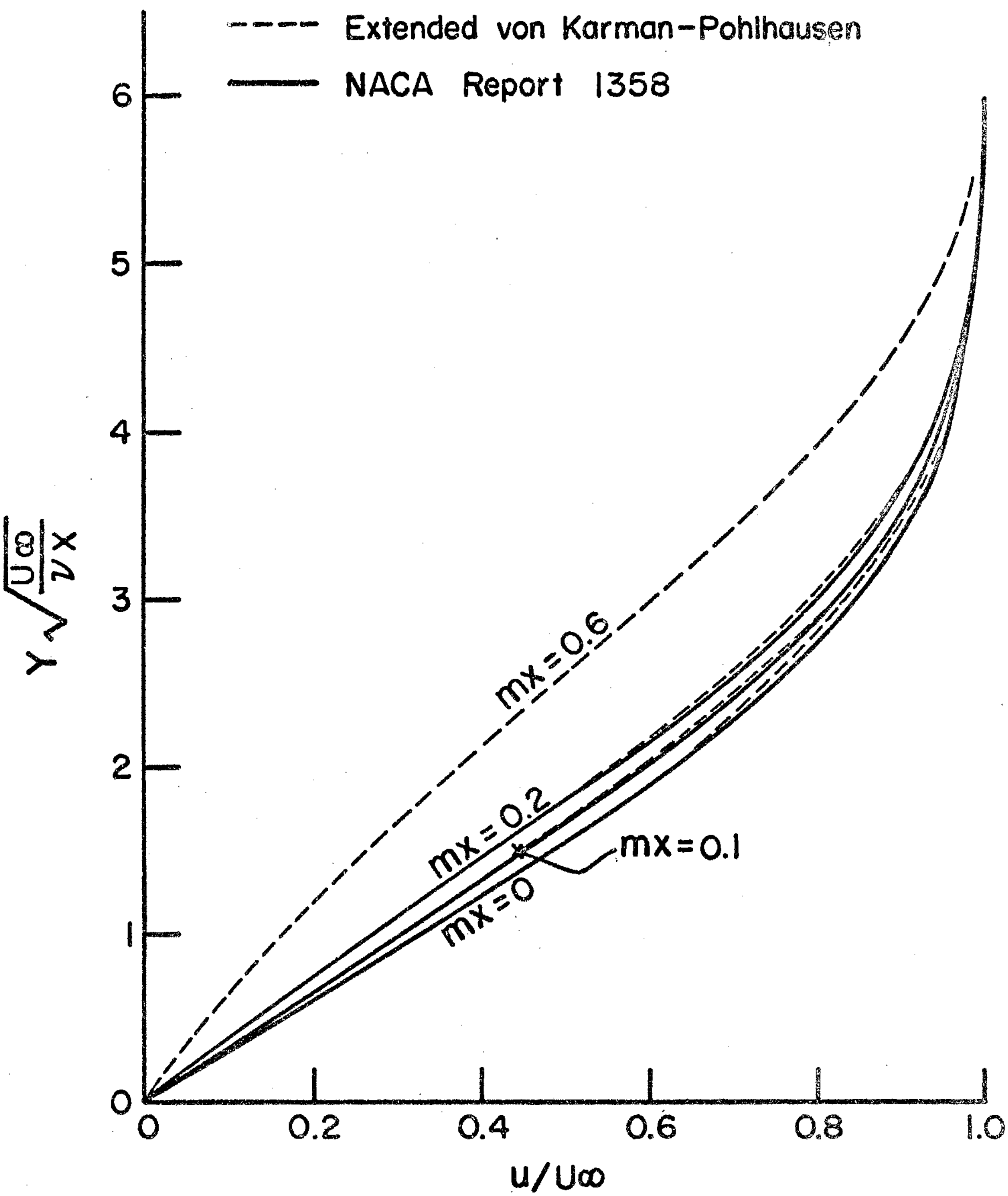

Fig. 6. Velocity Profiles, Body-fixed $\vec{B}$, $\sigma = \sigma_0 \left(\frac{U-u}{U}\right)$, Flat Plate Flow

cases and first - order terms for variable conductivity cases. In the cases of body-fixed magnetic fields the results of Rossow (6) are acceptable for mx $<$ 0.2. For mx $>$ 0.2 the exact solution deviates considerably from the expected result, yielding a separation point and back-flow velocity profile which clearly cannot occur since the magnetomotive force is zero for zero velocity; thus, the retarding force may asymptotically approach zero but never cause separation or back-flow. The extended von Karman-Pohlhausen method yields the proper results for all mx, implying no separation for the flat plate cases.

No separation occurs for the flow-fixed magnetic field case as well. This is also expected since the magnetic field is fixed to the flow and, therefore, tends to prevent any retardation of the flow; thus, the presence of a magnetic field tends to prevent the formation of a boundary layer. Ultimately, when $m_1 \rightarrow \infty$ the boundary layer will have a velocity distribution of $u/U = 1$ for $y > 0$ and there will be a discontinuity at the wall where the velocity will abruptly change from 0 to U . This implies an infinite velocity gradient at the wall, and, therefore, $C_f \rightarrow \infty$. The extended von Karman-Pohlhausen method yields this known physical result which is plotted in Fig. 3. The function $\sqrt{R_N}C_f$ is seen to increase as mx increases for all mx.

When the second order terms are included in the exact solution of Rossow (6) for the flow-fixed magnetic field, the function $\sqrt{R_N}C_f$ does not increase for mx $>$ 0.2; indeed, separation is erroneously predicted. It was pointed out by Rossow (6) that the solution may diverge for mx $>$ 0.2. Unfortunately, each term of the series can only be obtained with great difficulty and until additional terms are available only superficial conclusions regarding the possible convergence or divergence of the series can be drawn.

One of the principal advantages of the von Karman-Pohlhausen method is its ability to solve the boundary layer equations once and for all in terms of parameters dependent only upon the shape of the two-dimensional body. The shape factors, Λ , are then known functions of x only, evaluated from the potential flow solution. Similarly, in the magnetohydrodynamic case the solution may be found once and for all in terms of Λ and the magnetic shape factors, Λ_m . True, the solution is also dependent upon U/U_∞ ; however, a family of solutions can be obtained in U/U_∞ if desired. For the examples considered in this chapter $U/U_\infty = 1$ and $U/U_\infty = 1 - mx$.

Chapters IV and V will examine the method as applied to the wedge and circular cylinder, for which there are no other solutions at this time.

CHAPTER IV

WEDGE FLOW WITH BODY-FIXED MAGNETIC FIELD AND VARIABLE CONDUCTIVITY

Wedge flow of an incompressible, electrically conducting viscous fluid is to be analyzed in this chapter. The magnetic field, $\vec{B}$, is assumed to be oriented perpendicularly to the surface of the wedge and fixed to the body. Conductivity is assumed to be of the form $\sigma = \sigma_o \frac{1}{U}(U-u)$. All of the assumptions made in the foregoing chapters relating to the body-fixed $\vec{B}$, variable conductivity cases will also apply. It is further assumed that while the flow is to be studied in the neighborhood of the stagnation point, x is sufficiently greater than zero to ensure that the usual boundary layer assumptions are still valid.[1] Only the variable conductivity case will be solved since it represents the more realistic assumption and adequately serves to demonstrate the technique.

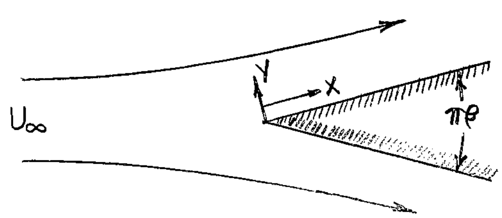

Fig. 7 Wedge Flow

[1]Appendix A

For wedge flows near the stagnation point as shown in Fig. 7 the velocity outside the boundary layer is well known from potential theory (7),

$$U(X,\xi) = \alpha X^{k} , \qquad \text{where} \qquad k = \frac{\beta}{2-\beta} .$$

The equation of motion and momentum integral equation are well established from previous chapters. Application of the extended von Karman-Pohlhausen method has also been covered in detail in the foregoing chapters and no further detail will be added here.

The wedge flow problem can also be solved by this method. Noting that

$$\frac{\Lambda}{\Lambda_m} = k \frac{U}{U_\infty} \frac{1}{mX}$$

one may solve Eqs. (18) - (20) in the manner outlined in the previous chapter. For a typical example a 90° wedge will be chosen and the flow conditions at U/U_∞ = 0.3, .5, .7 and .9 analyzed.

For a wedge angle of 90° it follows that

$$\beta = \frac{1}{2} \qquad \text{and} \qquad k = \frac{1}{3} .$$

Using these relations $\frac{\Lambda}{\Lambda_m}$ and Eqs. (18) - (20) simplify to the following equations for $U/U_\infty = .5$.

$$\frac{\Lambda}{\Lambda_m} = \frac{1}{6mX}$$

$$\frac{u}{U_\infty} = \frac{1}{2}\left[a\eta + b\eta^2 + c\eta^3 + d\eta^4 + e\eta^5\right] \tag{26}$$

$$\sqrt{R_N}\, C_f = a\sqrt{mx/\Lambda_m} \tag{27}$$

and

$$\frac{1}{2}\frac{d}{d\Lambda_m}\left[\Lambda_m\left(\frac{\theta}{\delta}\right)^2\right]\frac{d\Lambda_m}{d(mx)} = F + 2\Lambda_m\left(\frac{\theta}{\delta}\right)^2 \tag{28}$$

These equations may now be solved. It must be noted that for these cases the numerical integration may not be started at $mx = o$, $\Lambda_m = o$ as was done for the flat plate. The function Λ/Λ_m now appears in the equations since $\Lambda/\Lambda_m = \Lambda/\Lambda_m(mx)$ is no longer a constant.[2] The flat plate results show that mx versus Λ_m is linear for mx near zero. Equation (28) can then be solved iteratively for various values of Λ_m with a small fixed initial value of $mx \neq o$. The results are initially unstable and oscillatory as the computer integration scheme corrects for the eroneous first choices for Λ_m and approaches the solution. When the proper value of Λ_m is selected the results of mx vs Λ_m are stable and linear for mx near zero. It is this value of mx and Λ_m which is then used as the initial value in the computer program of Appendix C. More direct methods are available of course and the limit of $\frac{\Lambda_m}{mx}$ could be evaluated at $mx = o$ if desired. However, the method used is the easiest and does not require further programming.

Results are plotted in Figs. 8 and 9 and Figure 9 shows representative velocity profiles for $\frac{U}{U_\infty} = 0.5$. Again the

[2]Appendix C.

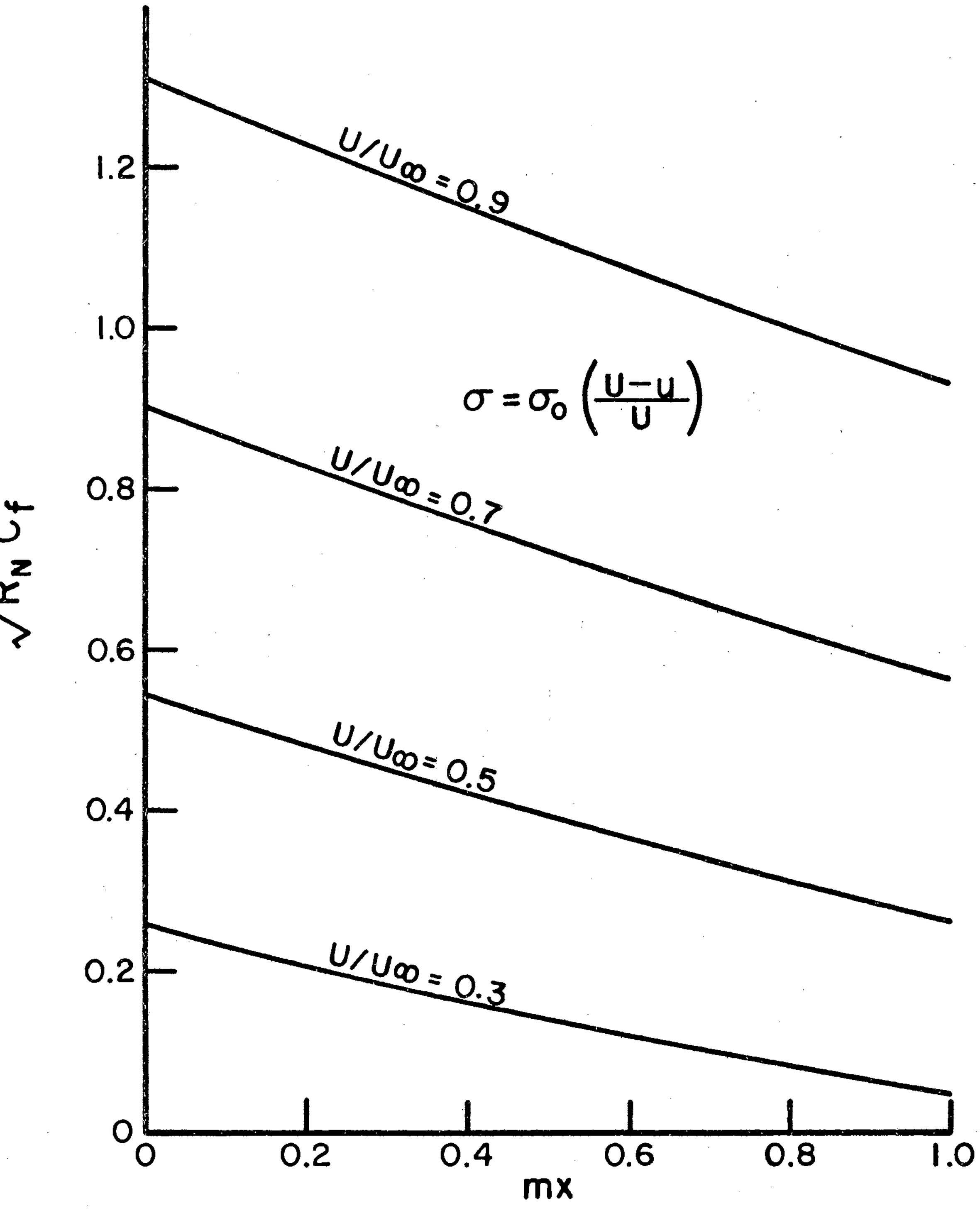

Fig. 8. $\sqrt{R_N}\ C_f$ vs mx, 90° Wedge Flow

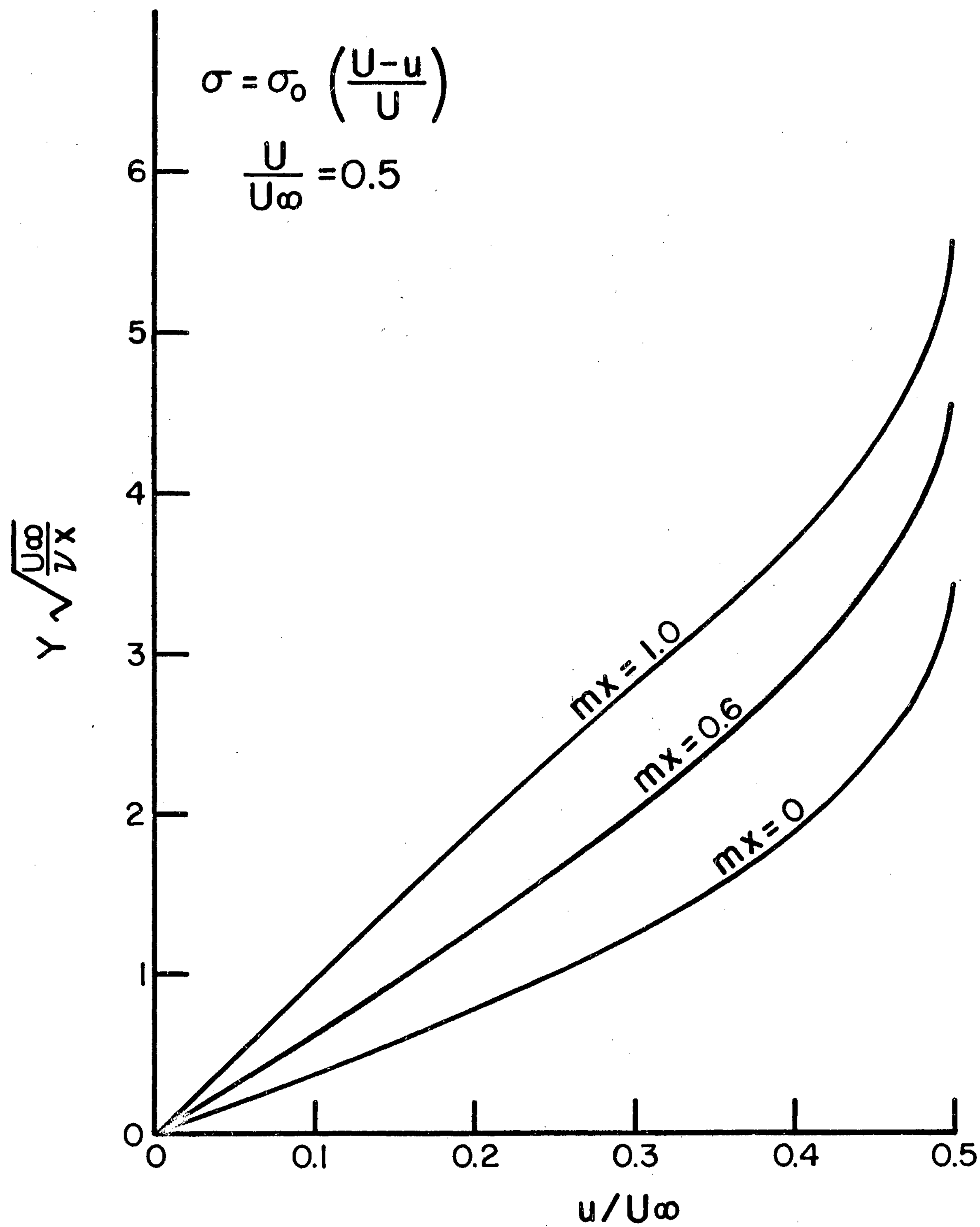

Fig. 9 Velocity Profiles, 90° Wedge Flow

retarding influence of the magnetic field can be seen. In a similar way the cases of $\frac{U}{U_\infty}$ = .3, .7, and .9 have been evaluated and are also included in Fig. 8.

CHAPTER V

CIRCULAR CYLINDER FLOW WITH BODY-FIXED MAGNETIC FIELD AND VARIABLE CONDUCTIVITY

This problem is an excellent example for the von Karman-Pohlausen method. Solutions are obtained which are difficult and tedious by exact methods. Magnetohydrodynamics compounds this difficulty so that exact solutions have not been found. This otherwise difficult magnetohydrodynamic case can be solved by the von Karman-Pohlhausen technique however. The result is a family of solutions in ϕ for $\sqrt{R_N}\, C_f$ vs mr_o and u/U_∞ vs $y\sqrt{\frac{U_\infty}{\nu x}}$ for each mr_o. This follows from the fact that unlike the ordinary case where one solution exists at each ϕ the solution at each ϕ in the magnetohydrodynamic case depends upon the magnetic field.

The incompressible laminar flow of an electrically conducting viscous fluid in the presence of a magnetic field perpendicular to its surface and fixed to the body may now be found for $\sigma = \sigma_o \frac{1}{U}(U-u)$. Procedures similar to those used for the preceding wedge flow case will be used.

From potential flow theory $U/U_\infty = 2\sin\phi$ (7). It is assumed that the boundary layer thickness is much less than the cylinder radius r_o; i.e., $\delta << r_o$. In such cases Schlichting (7) shows that the ordinary two-dimensional equations may be applied with x and y measured as shown in Figure 10.

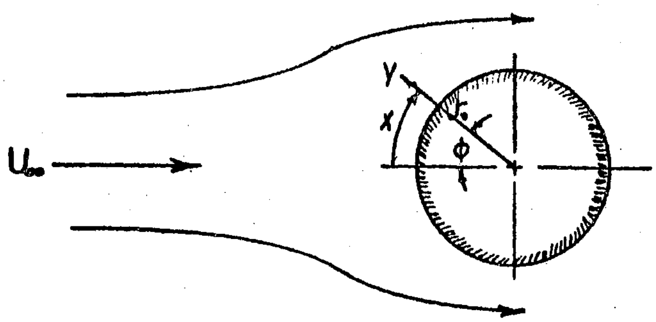

Fig. 10 Circular Cylinder Flow

For this particular case $U_\infty \neq 0$; therefore, $\frac{\Lambda}{\Lambda_m}$ and Eqs. (18)-(20) again can be simplified under these conditions to

$$\frac{\Lambda}{\Lambda_m} = \frac{2\phi \cos\phi}{mx} ,$$

$$\frac{u}{U_\infty} = 2\sin\phi\left[a\eta + b\eta^2 + c\eta^3 + d\eta^4 + e\eta^5\right] , \tag{29}$$

$$\sqrt{R_N}\, C_f = 4a\sin\phi\sqrt{mx/\Lambda_m} \tag{30}$$

and

$$2\sin\phi \frac{d}{d\Lambda_m}\left[\Lambda_m\left(\frac{\theta}{\delta}\right)^2\right]\frac{d\Lambda_m}{d(mx)} = F + 2\Lambda_m\left(\frac{\theta}{\delta}\right)^2 . \tag{31}$$

These equations may be solved in like manner to the previous wedge flow case ; the results are plotted in Figs. 11 and 12.[1]

[1]Appendix C.

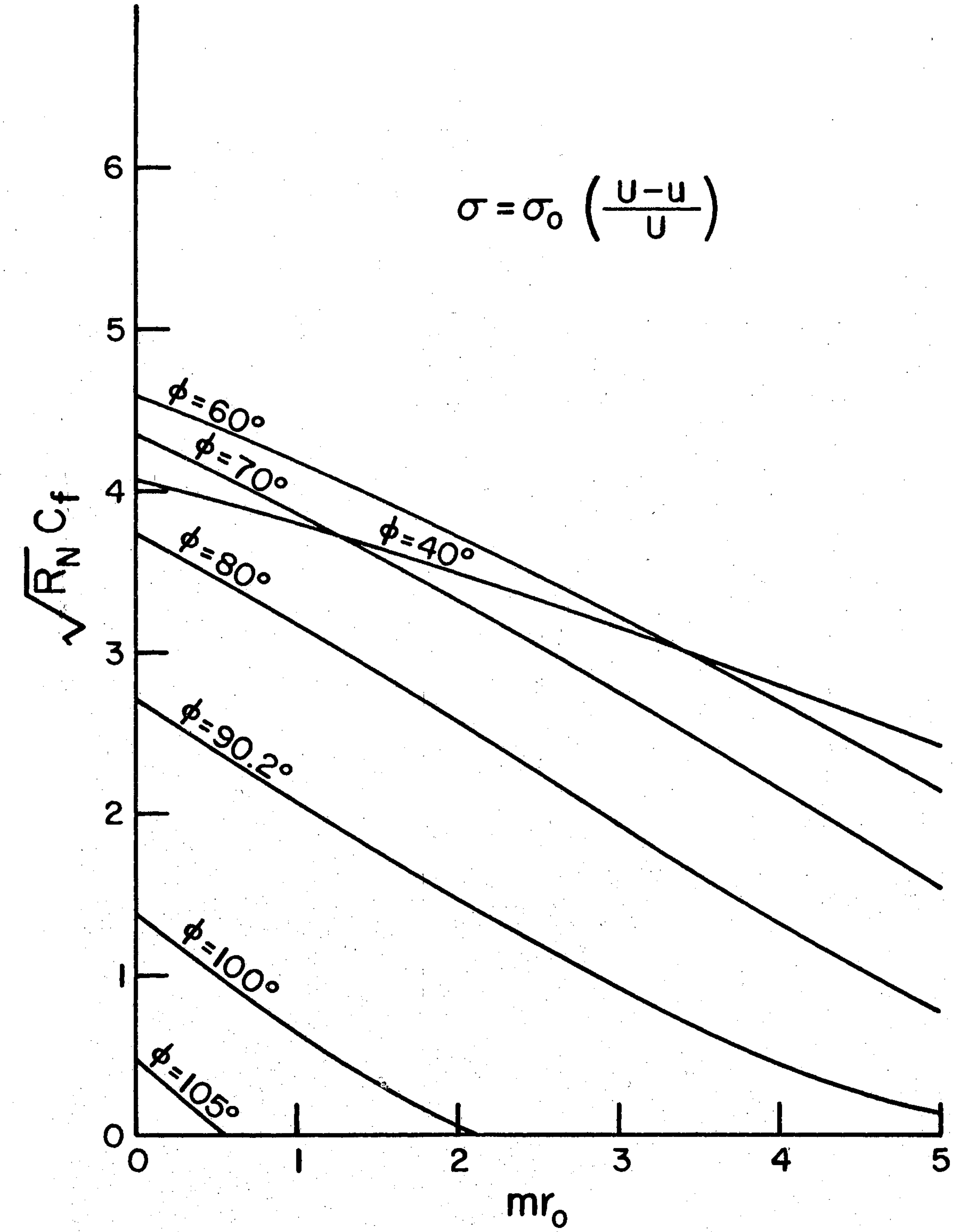

Fig. 11. $\sqrt{R_N}C_f$ vs mr_o, Circular Cylinder Flow

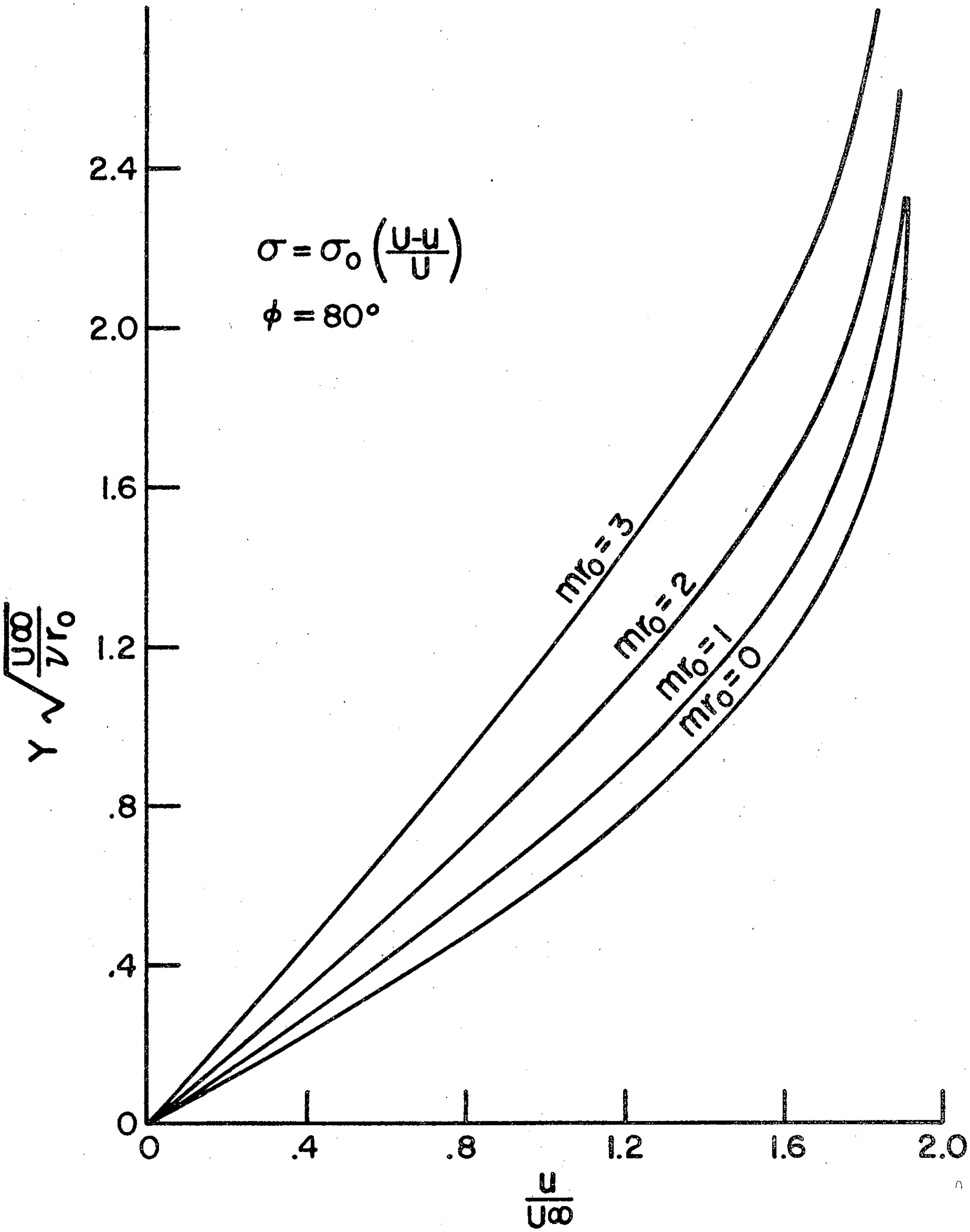

Fig. 12. Velocity Profiles, Circular Cylinder Flow

Figure 11 shows the $\sqrt{R_N}C_f$ vs mr_o for values of ϕ from 40° to 105°. The figure also shows the effect of the magnetic field upon the location of the separation point which is indicated by the intercepts on the mr_o axis. Representative velocity profiles for $\phi = 80°$ are given in Figure 12. The results again show the retarding effect of the magnetic field.

Separation angle vs mr_o is plotted in Figure 13, which shows that the separation angle moves toward $\phi = \frac{\pi}{2}$ as mr_o increases.

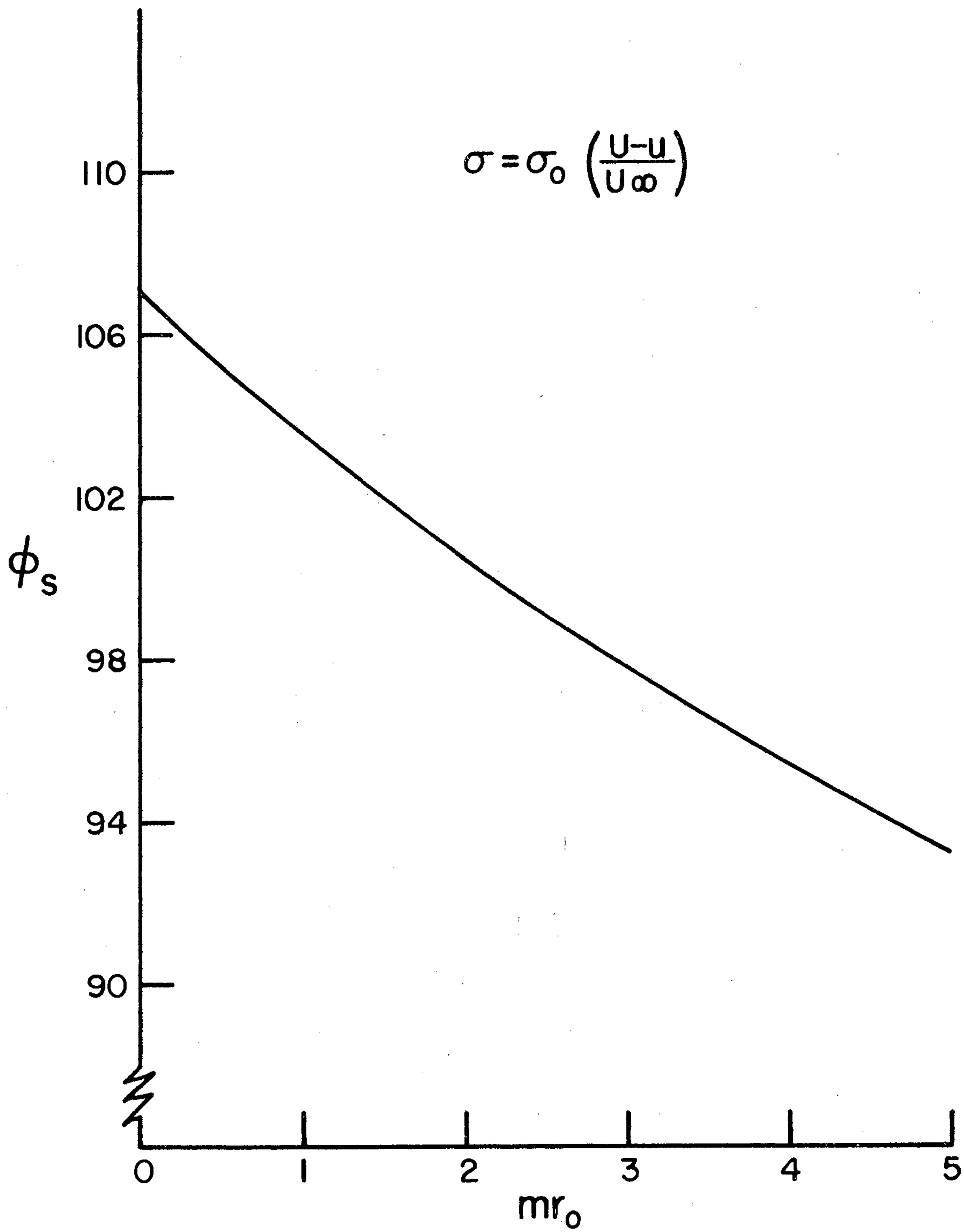

Fig. 13. Separation Point vs mr_o, Circular Cylinder Flow

CHAPTER VI

INTERPRETATION OF RESULTS

Summary and Conclusions

Chapter I disclosed some of the difficulties associated with magnetofluidmechanics and boundary layer solutions in particular. Theoretical solutions are very difficult and most of the known magnetohydrodynamic boundary layer flow problems for which exact solutions are available have been used for comparative purposes in this dissertation. Even though the problem is complex, theoretical solutions are still being sought not only for the physical insight they bring but also because very little experimental work has been published on magnetofluidmechanical boundary layer work. Clearly, even elementary results are superior to no results at all.

Approximate techniques have been available for many years to provide rapid calculation of hydrodynamic boundary layer problems. These techniques cannot be applied directly to magnetohydrodynamic boundary layers with success. Chapter I shows that some of the assumptions which lead to good agreement in the hydrodynamic case are invalid since the retarding body forces are no longer independent of y. Because of this fact the ponderomotive forces could not be introduced into the von Karman-Pohlhausen approximation since they vanished at the body surface. When this technique was applied directly to several cases the results were negative; there was no agreement with the exact solutions in body-fixed magnetic field cases and poor qualitative results in the flow-fixed magnetic field cases.

In Chapter II an extension of the von Karman-Pohlhausen approximation was presented. Another equation was introduced by differentiation of the equation of motion. This equation was no longer trivial in the magnetohydrodynamic case and a solution to the difficulty of the vanishing body force at the edge of the boundary layer was found which also allowed a fifth order polynomial assumption for u/U_∞ .

Comparing the results of this new method with the exact solution for a flat plate with $dP/dx = o$, Chapter III, showed that the new method was in close agreement with the exact solution in the range of mx in which the exact solution was reliable. For larger values of mx the exact solution implied erroneous results and, therefore, could not be used. On the other hand the extended von Karman-Pohlhausen solution did indicate the correct physical results for all mx. The facility of the method is well demonstrated by its ability to solve the wedge and circular cylinder problems.

Implications

Two primary advantages can be attributed to this new technique. First, it is possible to obtain solutions to magnetohydrodynamic boundary layer problems more easily than by the exact method of Rossow (6). The principal difficulties are encountered but once in the solution of problems by this technique since a large part of the analysis can be done in general terms. Even difficult examples involve only a few new algebraic relations which may be handled in a straight-forward way. In contrast, an exact solution to the magnetohydrodynamic boundary layer flow over even the flat plate results a system of linear ordinary differential equations with variable coefficients which must be solved to evaluate the series coefficients of the Blasius solutions. Since these coefficients are not known in closed form, the equations must be integrated numerically. Only the first one or two terms of the series are found because the work involved is sizable (6). More complex magnetogasdynamic problems such as the wedge and circular cylinder are even more difficult to solve by this exact method and attempts to obtain such solutions have not been successful at this time.

The second advantage which has resulted from this new method is that for the first time magnetohydrodynamic boundary layer flow problems for more complex shapes than a flat plate may be solved. This dissertation has included the flat plate, wedge and circular cylinder. It is the nature of this technique that any two-dimensional shape, for which the potential flow solution outside the boundary layer is known, may be solved, within the limits of the basic assumptions. Disadvantages to the method are those normally associated with the von Karman-Pohlhausen method itself, and these are usually confined to the accuracy of the results, expecially near the separation point. However, even the boundary layer equations themselves are in error near the separation point.

Suggestions

There are several areas of investigation which suggest themselves for additional study. They are enumerated below.

1. Other geometrics.

   It would be interesting to determine the exact limitations of the extended von Karman-Pohlhausen method in this respect. Configurations leading to flow separation should be considered in particular.

2. Other variations of σ .

   Only the two cases of σ = constant and $\sigma = \sigma_0\left(\frac{U-u}{U}\right)$ have been considered in this dissertation. Other variations in σ are possible and because of the nature of the extended von Karman-Pohlhausen technique complex functions of $\sigma = \sigma(u)$ should still be solvable.

3. Variable magnetic fields.

   The magnetic field has been assumed to be constant in this dissertation. Studies of magnetic fields which are functions of x and y would be important. This change would complicate the equations; however, as a first step one could assume a field variable in y but fixed in x. Thus, the momentum integral would be changed but the method of solution would remain the same since the magnetic field assumption of invariance in x would still be valid.

4. Improvements in the von Karman-Pohlhausen approximation.

   Several methods have been proposed which improve the accuracy of the von Karman-Pohlhausen method in the gasdynamic case (1). The most effective of these improvements involves the simultaneous solution of both the momentum integral equation and the energy integral equation. While this extension has provided definite improvements in accuracy, it necessarily complicates the solution. This extension should also be investigated for the magneto-hydrodynamic case.

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APPENDIXES

APPENDIX A

THE TWO-DIMENSIONAL INCOMPRESSIBLE BOUNDARY
LAYER EQUATIONS FOR STEADY FLOW

The incompressible, two-dimensional, steady flow equations of motion and continuity follow directly from the well known Navier-Strokes equations:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{X}{\rho} - \frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{A-1}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{Y}{\rho} - \frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{A-2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{A-3}$$

The boundary layer concept assumes that there is a very thin layer in the immediate neighborhood of the body in which the velocity gradient normal to the wall, $\frac{\partial u}{\partial y}$, assumes very large values. In this region $\tau_o = \mu \frac{\partial u}{\partial y}$ may be significant even though μ is very small since $\frac{\partial u}{\partial y}$ is itself very large.

In the rest of the flow field outside this boundary layer such velocity gradients do not appear and τ has negligible values. In this region the flow is frictionless and potential.

From these assumptions and known solutions to the Navier-Strokes equations it is well established, (7), that the boundary layer thickness

$\delta \sim \sqrt{\nu}$ where

$\delta << L$, the length of the body.

Also, in the boundary layer itself:

1. u is of the order of U, the flow velocity.
2. v is of the order of δ since v = 0 at the wall.
3. x is of the order L.
4. y is of the order δ .
5. U, L of order 1.

Define the dimensionless functions u*, v*, x*, y*, P*, R_N as follows:

$u^* = u/U$,

$v^* = v/U$,

$x^* = x/L$,

$y^* = y/L$,

$P^* = P/\rho U^2$

$R_N = UL/\nu$, of order $1/\delta^2 \Rightarrow$ large R_N.

Equations (A-1)-(A-3) may now be written in dimensionless form by multiplying through by L/U^2 , a constant, and using the relations above. Beneath each term appears its respective order of magnitude. It will be assumed for the time being that X and Y are significant but unknown. These relations will become the ponderomotive body forces in the derivation of the magnetohydrodynamic boundary layer equations in Appendix B. Proceeding as indicated above:

$$\frac{u}{U}\frac{\partial(\frac{u}{U})}{\partial(\frac{x}{L})} + \frac{v}{U}\frac{\partial(\frac{u}{U})}{\partial(\frac{y}{L})} = \frac{XL}{\rho U^2} - \frac{\partial(\frac{P}{\rho U^2})}{\partial(\frac{x}{L})} + \frac{\nu}{UL}\left[\frac{\partial^2(\frac{u}{U})}{\partial(\frac{x}{L})^2} + \frac{\partial^2(\frac{u}{U})}{\partial(\frac{y}{L})^2}\right]$$

$$\frac{u}{U}\frac{\partial(\frac{v}{U})}{\partial(\frac{x}{L})} + \frac{v}{U}\frac{\partial(\frac{v}{U})}{\partial(\frac{y}{L})} = \frac{YL}{\rho U^2} - \frac{\partial(\frac{P}{\rho U^2})}{\partial(\frac{y}{L})} + \frac{\nu}{UL}\left[\frac{\partial^2(\frac{v}{U})}{\partial(\frac{x}{L})^2} + \frac{\partial^2(\frac{v}{U})}{\partial(\frac{y}{L})^2}\right]$$

$$\frac{\partial(\frac{u}{U})}{\partial(\frac{x}{L})} + \frac{\partial(\frac{v}{U})}{\partial(\frac{y}{L})} = 0$$

Then
$$\underset{1}{u^*}\underset{\frac{1}{1}}{\frac{\partial u^*}{\partial x^*}} + \underset{\delta}{v^*}\underset{\frac{1}{\delta}}{\frac{\partial u^*}{\partial y^*}} = \left[\frac{XL}{\rho U}\right] - \frac{\partial P^*}{\partial x^*} + \underset{\delta^2}{\frac{1}{R_N}}\left[\underset{\frac{1}{1}}{\frac{\partial^2 u^*}{\partial x^{*2}}} + \underset{\frac{1}{\delta^2}}{\frac{\partial^2 u^*}{\partial y^{*2}}}\right]$$

$$\underset{1}{u^*}\underset{\frac{\delta}{1}}{\frac{\partial v^*}{\partial x^*}} + \underset{\delta}{v^*}\underset{\frac{\delta}{\delta}}{\frac{\partial v^*}{\partial y^*}} = \left[\frac{YL}{\rho U^2}\right] - \frac{\partial P^*}{\partial y^*} + \underset{\delta^2}{\frac{1}{R_N}}\left[\underset{\frac{\delta}{1}}{\frac{\partial^2 v^*}{\partial x^{*2}}} + \underset{\frac{\delta}{\delta^2}}{\frac{\partial^2 v^*}{\partial y^{*2}}}\right]$$

$$\underset{\frac{1}{1}}{\frac{\partial u^*}{\partial x^*}} + \underset{\frac{\delta}{\delta}}{\frac{\partial v^*}{\partial y^*}} = 0$$

Since $\delta \ll$ L and L is of order one, these equations may be reduced by neglecting the terms which contain δ or multiples thereof:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{X}{\rho} - \frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$0 = \frac{Y}{\rho} - \frac{1}{\rho}\frac{\partial P}{\partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

If Y = 0, as will be shown to be the case under the magnetohydrodynamic assumptions of Appendix B, $\frac{\partial P}{\partial y} = 0$ from the above equation and the so-called boundary layer equations for two-dimensional incompressible steady flow result:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{X}{\rho} - \frac{1}{\rho}\frac{dP}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

APPENDIX B

MAGNETOHYDRODYNAMIC, STEADY STATE, TWO-DIMENSIONAL AND INCOMPRESSIBLE BOUNDARY LAYER EQUATIONS

For many aero problems the magnetic Reynolds number R_σ is small. Under this condition the induced magnetic field due to flow may be neglected with respect to the applied field H_o.

Only two-dimensional flow will be considered and only the x and y velocity components will then appear. Similarly, for the magnetic field H_z will be zero. However, the electrical current $\vec{i}$ and field $\vec{E}$ have z - components only. For the case of small σ:[1]

$$H = H_o + \sigma H_1 + \sigma^2 H_2 + \cdots$$

$$E = E_o + \sigma E_1 + \sigma^2 E_2 + \cdots$$

where the subscript "o" refers to the externally applied fields. It is assumed that H_o has only the y-component H_{yo}. In the boundary layer the x-wise velocity component is much larger than the y-wise velocity component. So that:

$$\vec{q} \times \vec{H_o} = (u H_{yo} - v H_{xo})\hat{k} \cong u H_{yo} \hat{k} ,$$

$$\vec{q} \times \vec{H_1} = (u H_{y1} - v H_{x1})\hat{k} \cong u H_{y1} \hat{k}$$ and implies

that

$$\vec{i} = \sigma(\vec{E} + \mu_m \vec{q} \times \vec{H}) = \sigma(\vec{E_o} + \mu_m \vec{q} \times \vec{H_o}) + \sigma^2(\vec{E_1} + \mu_m \vec{q} \times \vec{H_1}) + \cdots .$$

---

[1]Reference (5), pp. 65-67.

In this problem there is only the z-component of $\vec{i}$, (neutral charge density), implying

$$i_z = \sigma(E_{z0} + \mu_m u H_{y0}) + \sigma^2(E_{z1} + \mu_m u H_{y1}) + \cdots .$$

The circuit of the electric current must be closed. It may be assumed that the current is taken up by suitable electrodes, connected externally by a circuit of resistance R_e. If h_e is the distance between electrodes (the depth of the field in the z-direction) and A_e their equivalent area, the external current is $I_e = A_e i_z$, $\Longrightarrow$ the boundary condition

$$h_e E = -R_e I_e = -R_e A_e i_z$$

or

$$E_{z0} + \sigma E_{z1} + \cdots = -\frac{R_e A_e \sigma}{h_e}\left[(E_{z0} + \mu_m u H_{y0}) + \sigma(E_{z1} + \mu_m u H_{y1}) + \cdots\right].$$

Let $\beta_e \equiv \dfrac{R_e A_e \sigma}{h_e}$, assumed small.

Then

$$E_{z0} = -\beta_e E_{z0} - \beta_e \mu_m u H_{y0} ,$$

$$E_{z1} = -\beta_e E_{z1} - \beta_e \mu_m u H_{y1} ,$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$E_{zn} = -\beta_e E_{zn} - \beta_e \mu_m u H_{yn} .$$

Using these relations the E_z's may be found:

$$E_{z0} = \frac{-\beta_e}{1+\beta_e} \mu_m u H_{y0} ,$$

$$E_{z1} = \frac{-\beta_e}{1+\beta_e} \mu_m u H_{y1} ,$$

$$\vdots \qquad \vdots$$

giving

$$i_z = \sigma\left[\frac{(-\beta_e+1-\beta_e)}{1+\beta_e}\mu_m u H_{y0}\right] + \sigma^2\left[\quad\right] + \cdots ,$$

$$i_z = \frac{1}{1+\beta_e}\left[\sigma\mu_m u H_{y0} + \sigma^2\mu_m u H_{y1} + \cdots\right] .$$

The ponderomotive force in the equations of motion is then

$$\vec{i} \times \vec{B} = \vec{F_e} ,$$

$$\vec{i} \times \vec{B} = \mu_m \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & i_z \\ H_x & H_y & 0 \end{vmatrix} = -(i_z\mu_m H_y)\hat{i} + (i_z\mu_m H_x)\hat{j} ,$$

$$F_{ex} = \frac{-1}{1+\beta_e}\left(\sigma\mu_m^2 u H_{y0}^2 + 2\sigma^2\mu_m^2 u H_{y0} H_{y1} + \cdots\right) ,$$

$$F_{ey} = \frac{+1}{1+\beta_e}\left(\sigma^2\mu_m^2 u H_{y0} H_{x1} + \cdots\right) .$$

For small β_e , σ and first approximation:

$$F_{ex} = -\sigma\mu_m^2 u H_{y0}^2 , \qquad F_{ey} = 0 .$$

For incompressible flow, ρ = constant, the fundamental equations of magnetohydrodynamics become

$$\nabla \cdot \vec{q} = 0 ,$$

$$\rho\frac{D\vec{q}}{Dt} - \mu_m(\vec{H}\cdot\nabla)\vec{H} = -\nabla\left(P + \mu_m\frac{H^2}{2}\right) + \mu\nabla^2\vec{q} ,$$

which reduces for this special case to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \quad \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0 ,$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} - \frac{\sigma\mu_m^2 H_{y0}^2}{\rho}u + \nu\frac{\partial^2 u}{\partial y^2} .$$

APPENDIX C

COMPUTER PROGRAMS AND THE SAMPLE RESULTS

The following nomenclature has been used:[1]

T(5)	=	$\frac{d\Lambda_m}{d(mx)}$, $\frac{d}{d\phi}\left[Sin(2\phi)\,EXP(-2\phi)\,\frac{\delta^2 U_\infty}{\nu r_o}\right]$
T(4)	=	Λ_m , $\left[Sin(2\phi)\,EXP(-2\phi)\,\frac{\delta^2 U_\infty}{\nu r_o}\right]$
T(3)	=	Δmx , $\Delta\phi$
T(2)	=	mx , ϕ
INT[2]	=	Computer Library Integration Subroutine
Y	=	$\frac{y}{\delta}$
F,F3	=	F,F/(2P/T(4))
P	=	$\frac{\theta}{\delta}$
P1	=	$\frac{\partial}{\partial(\Lambda_m)}\left(\frac{\theta}{\delta}\right)$
P2	=	$\frac{\partial}{\partial(mx)}\left(\frac{\theta}{\delta}\right)$, $\frac{\partial}{\partial\Lambda}\left(\frac{\theta}{\delta}\right)$
A1	=	Λ/Λ_m
A,B,C,...	=	a,b,c,...
A2,B2,C2,...	=	$\frac{\partial}{\partial(\Lambda_m)}(a,b,c,\cdots)$
A9,B9,C9,...	=	$\frac{\partial}{\partial(mx)}(a,b,c,\cdots)$, $\frac{\partial}{\partial\Lambda}(a,b,c,\cdots)$
Q	=	$36+\Lambda_m$, $2\cos\phi$
A3	=	$\sqrt{\frac{mx}{\Lambda_m}}$
A4	=	U/U_∞

[1] All programs are written in IBM FORTRAN II.

[2] The INT subroutine is of itself several times longer than the remainder of the entire program. For brevity, therefore, it is not included in detail.

A5	=	$Y\sqrt{U_\infty/\nu X}$, $Y\sqrt{U_\infty/\nu r_0}$
A6	=	$\sqrt{R_N}C_f$
A7	=	u/U_∞
B3	=	mr_0
B4	=	ϕ°

```
      FLAT PLATE,MHD BL,BODY-FIXED B,SIGMA CONST.

      DIMENSION T(15)
      COMMON T,A,B,C,D,E,A4
      T(2)=0.
      T(3)=.001
      T(4)=0.
      CALL INT (T,1,2,1.,1.,1.,1.,1.,1.,)
      DO 6 J=1,990
1     CALL INTM
      IF (T(4))8,8,11
11    A3=SQRTF(T(2)/T(4))
      A6=2.*A*A4*A3
8     ZJ=J
      ZJ=ZJ/100.
      JJ=ZJ
      IF (J-JJ*100)6,2,6
2     WRITE OUTPUT TAPE 3,3,T(2),A6
3     FORMAT(4HMX =F5.2,8X,19HFRICTION FUNCTION =E12.5,/13X,
     1 3HETA,15X,6HU/UINF)
      Y=.1
      DO 9 IB=1,10
      A5=Y/A3
      A7=A4*Y*(A+Y*(B+Y*(C+Y*(D+Y*E))))
      WRITE OUTPUT TAPE 3,10,A5,A7
10    FORMAT (8X,E12.5,8X,E12.5)
9     Y=Y+.1
6     CONTINUE
      CALL EXIT
      END (1,0,0,0,0,0,1,0,0,1,0,0,0,0,0)
```

```
 SUBROUTINE DAUX
 DIMENSION T(15)
 COMMON T,A,B,C,D,E,A4
 A1=1.
 A4=1.-T(2)
 Q=36.+T(4)
 A=(60.+9.*T(4)*(1.+A1))/Q
 B=-(T(4)*(1.+A1))/2.
 C=(-20.*T(4)*A1+(20.+3.*T(4))*(A1+1.)*T(4))/(2.*Q)
 D=(30.*(T(4)*A1-4.)+3.*T(4)*(2.-T(4))*(1.+A1))/(2.*Q)
 E=(12.*(6.-T(4)*A1)-(6.-T(4))*(1.+A1)*T(4))/(2.*Q)
 A2=(9.*(1.+A1)-A)/Q
 B2=(-1.-A1)/2.
 C2=(10.-C+3.*T(4)*(A1+1.))/Q
 D2=(18.*A1+3.-D-3.*T(4)*(1.+A1))/Q
 E2=(-9.*A1-3.-E+T(4)*(1.+A1))/Q
 P=A/2.+B/3.+C/4.+D/5.+E/6.-A*A/3.-B*B/5.-C*C/7.-D*D/9.-E*E/11.
1-D*E/5.-A*B/2.-2.*A*C/5.-A*D/3.-2.*A*E/7.-B*C/3.-2.*B*D/7.-B*E/4.
2-C*D/4.-2.*C*E/9.
 P1=A2/2.+B2/3.+C2/4.+D2/5.+E2/6.-A*(2.*A2/3.+B2/2.+2.*C2/5.+D2/3.
1+2.*E2/7.)-B*(2.*B2/5.+A2/2.+C2/3.+2.*D2/7.+E2/4.)-C*(2.*C2/7.
2+2.*A2/5.+B2/3.+D2/4.+2.*E2/9.)-D*(2.*D2/9.+A2/3.+2.*B2/7.+C2/4.
3+E2/5.)-E*(2.*E2/11.+2.*A2/7.+B2/4.+2.*C2/9.+D2/5.)
 F=2.*P*A-4.*T(4)*P*P*(1.+A1)-2.*T(4)*P*(1.+A1)*(1.-A/2.-B/3.
1-C/4.-D/5.-E/6.)
 T(5)=(F+4.*T(4)*P*P)/((P*P+2.*T(4)*P*P1)*(A4+A1*T(2)))
 RETURN
 END(1,0,0,0,0,0,1,0,0,1,0,0,0,0,0)
```

MX = 0.10 FRICTION FUNCTION = 0.45948E-00

ETA	U/UINF
0.59454E 00	0.13715E-00
0.11891E 01	0.27591E-00
0.17836E 01	0.41382E-00
0.23782E 01	0.54532E 00
0.29727E 01	0.66342E 00
0.35672E 01	0.76116E 00
0.41618E 01	0.83326E 00
0.47563E 01	0.87759E 00
0.53508E 01	0.89684E 00
0.59454E 01	0.90000E 00

MX = 0.20 FRICTION FUNCTION = 0.28683E-00

ETA	U/UINF
0.72126E 00	0.10488E-00
0.14425E 01	0.21581E-00
0.21638E 01	0.33270E-00
0.28850E 01	0.45057E-00
0.36063E 01	0.56162E 00
0.43276E 01	0.65727E 00
0.50488E 01	0.73014E 00
0.57701E 01	0.77617E 00
0.64913E 01	0.79669E 00
0.72126E 01	0.80000E 00

MX = 0.30 FRICTION FUNCTION = 0.13056E-00

ETA	U/UINF
0.98668E 00	0.67034E-01
0.19734E 01	0.14608E-00
0.29600E 01	0.24009E-00
0.39467E 01	0.34464E-00
0.49334E 01	0.45064E-00
0.59201E 01	0.54705E 00
0.69068E 01	0.62358E 00
0.78934E 01	0.67347E 00
0.88801E 01	0.69615E 00
0.98668E 01	0.70000E 00

MX = 0.40 FRICTION FUNCTION = 0.20885E-01

ETA	U/UINF
0.19042E 01	0.23981E-01
0.38085E 01	0.67331E-01
0.57127E 01	0.13672E-00
0.76170E 01	0.22826E-00
0.95212E 01	0.33107E-00
0.11425E 02	0.43092E-00
0.13330E 02	0.51380E 00
0.15234E 02	0.56958E 00
0.17138E 02	0.59553E 00
0.19042E 02	0.60000E 00

```
      FLAT PLATE,MHD BL,FLOW-FIXED B,SIGMA CONST.

      DIMENSION T(15)
      COMMON T,A,B,C,D,E,A4
      T(2)=0.
      T(3)=.001
      T(4)=0.
      CALL INT (T,1,2,1.,1.,1.,1.,1.,1.,)
      DO 6 J=1,3000
1     CALL INTM
      IF (T(4))8,8,11
11    A3=SQRTF(T(2)/T(4))
      A6=2.*A*A4*A3
8     ZJ=J
      ZJ=ZJ/100.
      JJ=ZJ
      IF (J-JJ*100)6,2,6
2     WRITE OUTPUT TAPE 3,3,T(2),A6
3     FORMAT (4HMX = F5.2,8X,19HFRICTION FUNCTION =E12.5,/13X,
     13HETA,15X,6HU/UINF)
      Y=.1
      DO 9 1B=1,10
      A5=Y/A3
      A7=A4*Y*(A+Y*(B+Y*(C+Y*(D+Y*E))))
      WRITE OUTPUT TAPE 3,10,A5,A7
10    FORMAT (8X,E12.5,8X,E12.5)
9     Y=Y+.1
6     CONTINUE
      CALL EXIT
      END(1,0,0,0,0,0,1,0,0,1,0,0,0,0,0)
```

```
      SUBROUTINE DAUX
      DIMENSION T(15)
      COMMON T,A,B,C,D,E,A4
      A1=0.
      A4=1.
      Q=36.+T(4)
      A=(60.+9.*T(4)*(1.+A1))/Q
      B=-(T(4)*(1.+A1))/2.
      C=(-20.*T(4)*A1+(20.+3.*T(4))*(A1+1.)*T(4))/(2.*Q)
      D=(30.*(T(4)*A1-4.)+3.*T(4)*(2.-T(4))*(1.+A1))/(2.*Q)
      E=(12.*(6.-T(4)*A1)-(6.-T(4)*(1.+A1)*T(4))/(2.*Q)
      A2=(9.*(1.+A1)-A)/Q
      B2=(-1.-A1)/2.
      C2=(10.-C+3.*T(4)*(A1+1.))/Q
      D2=(18.*A1+3.-D-3.*T(4)*(1.+A1))/Q
      E2=(-9.*A1-3.-E+T(4)*(1.+A1))/Q
      P=A/2.+B/3.+C/4.+D/5.+E/6.-A*A/3.-B*B/5.-C*C/7.-D*D/9.-E*E/11.
     1-D*E/5.-A*B/2.-2.*A*C/5.-A*D/3.-2.*A*E/7.-B*C/3.-2.*B*D/7.-B*E/4.
     2-C*D/4.-2.*C*E/9.
      P1=A2/2.+B2/3.+C2/4.+D2/5.+E2/6.-A*(2.*A2/3.+B2/2.+2.*C2/5.+D2/3.
     1+2.*E2/7.)-B*(2.*B2/5.+A2/2.+C2/3.+2.*D2/7.+E2/4.)-C*(2.*C2/7.
     2+2.*A2/5.+B2/3.+D2/4.+2.*E2/9.)-D*(2.*D2/9.+A2/3.+2.*B2/7.+C2/4.
     3+E2/5.)-E*(2.*E2/11.+2.*A2/7.+B2/4.+2.*C2/9.+D2/5.)
      F=2.*P*A-4.*T(4)*P*P*(1.+A1)-2.*T(4)*P*(1.+A1)*(1.-A/2.-B/3.
     1-C/4.-D/5.-E/6.)
      T(5)=(F+4.*T(4)*P*P)/((P*P+2.*T(4)*P*P1)*(A4+A1*T(2)))
      RETURN
      END(1,0,0,0,0,0,1,0,0,1,0,0,0,0,0)
```

MX = 0.10 FRICTION FUNCTION = 0.85561E 00

ETA	U/UINF
0.50153E 00	0.20273E-00
0.10031E 01	0.38370E-00
0.15046E 01	0.54374E 00
0.20061E 01	0.68181E 00
0.25077E 01	0.79595E 00
0.30092E 01	0.88428E 00
0.35107E 01	0.94599E 00
0.40123E 01	0.98233E 00
0.45138E 01	0.99757E 00
0.50153E 01	1.00000E 00

MX = 0.20 FRICTION FUNCTION = 0.10336E 01

ETA	U/UINF
0.48874E-00	0.23052E-00
0.97749E 00	0.42283E-00
0.14662E 01	0.58317E 00
0.19550E 01	0.71500E 00
0.24437E 01	0.82001E 00
0.29325E 01	0.89910E 00
0.34212E 01	0.95331E 00
0.39100E 01	0.98481E 00
0.43987E 01	0.99792E 00
0.48874E 01	1.00000E 00

MX = 0.30 FRICTION FUNCTION = 0.11895E 01

ETA	U/UINF
0.47620E-00	0.25217E-00
0.95240E 00	0.45229E-00
0.14286E 01	0.61180E 00
0.19048E 01	0.73823E 00
0.23810E 01	0.83624E 00
0.28572E 01	0.90871E 00
0.33334E 01	0.95786E 00
0.38096E 01	0.98629E 00
0.42858E 01	0.99812E 00
0.47620E 01	1.00000E 00

MX = 0.40 FRICTION FUNCTION = 0.13295E 01

ETA	U/UINF
0.46274E-00	0.26886E-00
0.92547E 00	0.47420E-00
0.13882E 01	0.63228E 00
0.18509E 01	0.75414E 00
0.23137E 01	0.84681E 00
0.27764E 01	0.91463E 00
0.32392E 01	0.96049E 00
0.37019E 01	0.98709E 00
0.41646E 01	0.99821E 00
0.46274E 01	1.00000E 00

```
      FLAT PLATE,MHD BL,BODY-FIXED B, SIGMA VAR.

      DIMENSION T (15)
      COMMON T,A,B,C,D,E,A4
      T(2)=0.
      T(3)=.001
      T(4)=0.
      CALL INT (T,1,2,1.,1.,1.,1.,1.,1.)
      DO 6 J=1,3000
1     CALL INTM
      IF (T (4)) 8,8,11
11    A3=SQRTF (T(2)/T(4))
      A6=2.*A*A4*A3
8     ZJ=J
      ZJ=ZJ/100.
      JJ=ZJ
      IF (J-JJ*100)6,2,6
2     WRITE OUTPUT TAPE 3,3,T(2),A6
3     FORMAT (4HMX =F5.2,8X,19HFRICTION FUNCTION= E12.5,/13X,
      13HETA,15X,6HU/UINF)
      Y=.1
      DO 9 1B=1,10
      A5=Y/A3
      A7=A4*Y*(A+Y*(B+Y*(C+Y*(D+Y*E))))
      WRITE OUTPUT TAPE 3,10,A5,A7
10    FORMAT (8X,E12.5,8X,E12.5)
9     Y=Y+.1
6     CONTINUE
      CALL EXIT
      END (1,0,0,0,0,0,1,0,0,1,0,0,0,0,0)
```

```
      SUBROUTINE DAUX
      DIMENSION T(15)
      COMMON T,A,B,C,D,E,A4
      A1=0.
      A4=1.
      Q=36.+T(4)
      A=(60.+9.*A1*T(4))/Q
      B=-(A1*T(4))/2.
      C=T(4)*(3.*A1*T(4)+20.)/(2.*Q)
      D=(36.*A1*T(4)-120.-T(4)*(3.*A1 *T(4)+30.))/(2.*Q)
      E=(72.-18.*A1*T(4)+T(4)*(12.+A1*T(4)))/(2.*Q)
      A2=(9.*A1-A)/Q
      B2=A1/2.
      C2=(3.*A1*T(4)+10.-C)/Q
      D2=(18.*A1-3.*A1*T(4)-15.-D)/Q
      E2=(-9.*A1+6.+A1*T(4)-E)/Q
      P=A/2.+B/3.+C/4.+D/5.+E/6.-A*A/3.-B*B/5.-C*C/7.-D*D/9.-E*E/11.
     1-D*E/5.-A*B/2.-2.*A*C/5.-A*D/3.-2.*A*E/7.-B*C/3.-2.*B*D/7.-B*E/4.
     2-C*D/4.-2.*C*E/9.
      P1=A2/2.+B2/3.+C2/4.+D2/5.+E2/6.-A*(2.*A2/3.+B2/2.+2.*C2/5.+D2/3.
     1+2.*E2/7.)-B*(2.*B2/5.+A2/2.+C2/3.+2.*D2/7.+E2/4.)-C*(2.*C2/7.
     2+2.*A2/5.+B2/3.+D2/4.+2.*E2/9.)-D*(2.*D2/9.+A2/3.+2.*B2/7.+C2/4.
     3+E2/5.)-E*(2.*E2/11.+2.*A2/7.+B2/4.+2.*C2/9.+D2/5.)
      F=2.*P*A-4.*T(4)*P*P*A1-2.*T(4)*P*A1*(1.-A/2.-B/3.
     1-C/4.-D/5.-E/6.)
      T(5)=(F+2.*T(4)*P*P)/((A4+A1*T(2))*(P*P+2.*T(4)*P*P1))
      RETURN
      END(1,0,0,0,0,0,1,0,0,1,0,0,0,0,0)
```

MX = 0.10 FRICTION FUNCTION = 0.58338E 00

ETA	U/UINF
0.53002E 00	0.15508E-00
0.10600E 01	0.31122E-00
0.15901E 01	0.46534E-00
0.21201E 01	0.61131E 00
0.26501E 01	0.74157E 00
0.31801E 01	0.84879E 00
0.37102E 01	0.92750E 00
0.42402E 01	0.97572E 00
0.47702E 01	0.99659E 00
0.53002E 01	1.00000E 00

MX = 0.20 FRICTION FUNCTION = 0.52428E 00

ETA	U/UINF
0.54557E 00	0.14410E-00
0.10911E 01	0.29224E-00
0.16367E 01	0.44269E-00
0.21823E 01	0.58929E 00
0.27279E 01	0.72346E 00
0.32734E 01	0.83633E 00
0.38190E 01	0.92071E 00
0.43646E 01	0.97320E 00
0.49101E 01	0.99620E 00
0.54557E 01	0.10000E 01

MX = 0.30 FRICTION FUNCTION = 0.46652E-00

ETA	U/UINF
0.56456E 00	0.13336E-00
0.11291E 01	0.27368E-00
0.16937E 01	0.42054E-00
0.22583E 01	0.56775E 00
0.28228E 01	0.70576E 00
0.33874E 01	0.82415E 00
0.39519E 01	0.91408E 00
0.45165E 01	0.97074E 00
0.50811E 01	0.99583E 00
0.56456E 01	0.10000E 01

MX = 0.40 FRICTION FUNCTION = 0.41042E-00

ETA	U/UINF
0.58720E 00	0.12276E-00
0.11744E 01	0.25535E-00
0.17616E 01	0.39867E-00
0.23488E 01	0.54649E 00
0.29360E 01	0.68828E 00
0.35232E 01	0.81212E 00
0.41104E 01	0.90752E 00
0.46976E 01	0.96830E 00
0.52848E 01	0.99545E 00
0.58720E 01	0.10000E 01

```
      90 WEDGE,MHD BL,X=0.50,BODY-FIXED B,SIGMA VAR.
      DIMENSION T(15)
      COMMON T,A,B,C,D,E,A4
      T(2)=.001
      T(3)=.0002
      T(4)=.024853
      Z4=1.
      CALL INT (T,1,2,1.,1.,1.,1.,1.,1.)
      DO 6 J=1,5000
1     CALL INTM
      IF(T(2)-Z4*.1)6,2,2
2     A3=SQRTF(T(2)/T(4))
      A6=2.*A*A4*A3
      WRITE OUTPUT TAPE 3,3,T(2),A6
3     FORMAT (4H MX=F5.2,8X,19HFRICTION FUNCTION=E12.5,/13X,
       13HETA,15X,6HU/UINF)
      Y=.1
      DO 9 IB=1,10
      A5=Y/A3
      A7=A4*Y*(A+Y*(B+Y*(D+Y*E)))
      WRITE OUTPUT TAPE 3,10,A5,A7
10    FORMAT (8X,E12.5,8X,E12.5)
9     Y=Y+.1
      Z4=Z4+1.
6     CONTINUE
      CALL EXIT
      END
```

```
      SUBROUTINE DAUX
      DIMENSION T(15)
      COMMON T,A,B,C,D,E,A4
      A4=.5
      A1=(1./(T(2)*3.))*A4
      Q=36.+T(4)
      A=(60+9.*A1*T(4))/Q
      B=-A1*T(4))/2.
      C=T(4)*(3.*A1*T(4)+20.)/(2.*Q)
      D=(36.*A1*T(4)-120.-T(4)*(3.*A1*T(4)+30.))/2.*Q)
      E=(72.-18.*A1*T(4)+T(4)*(12.+A1*T(4)))/(2.*Q)
      A2=(9.*A1-A)/Q
      B2=-A1/2.
      C2=(3.*A1*T(4)+10.-C)/Q
      D2=(18.*A1-3.*A1*T(4)-15.-D)/Q
      E2=(-9.*A1+6.+A1*T(4)-E)/Q
      A9=(9.*T(4)*(-A1/T(2)))/Q
      B9=(T(4)*A1/T(2))/2.
      C9=(-1.5*T(4)*T(4)*A1/T(2))/Q
      D9=(-18.*T(4)*A1/T(2)+1.5*T(4)*T(4)*A1/T(2))/Q
      E9=(9.*T(4)*A1/T(2)-.5*T(4)*T(4)*A1/T(2))/Q
      P=A/2.+B/3.+C/4.+D/5.+E/6.-A*A/3.-B*B/5.-C*C/7.-D*D/9.-E*E/11.
     1-D*E/5.-A*B/2.-2.*A*C/5.-A*D/3.-2.*A*E/7.-B*C/3.-2.*B*D/7.-B*E/4.
     2-C*D/4.-2.*C*E/9.
      P1=A2/2.+B2/3.+C2/4.+D2/5.+E2/6.-A*(2.*A2/3.+B2/2.+2.*C2/5.+D2/3.
     1+2.*E2/7.)-B*(2.*B2/5.+A2/2.+C2/3.+2.*D2/7.+E2/4.)-C*(2.*C2/7.
     2+2.*A2/5.+B2/3.+D2/4.+2.*E2/9.)-D*(2.*D2/9.+A2/3.+2.*B2/7.+C2/4.
     3+E2/5.)-E*(2.*E2/11.+2.*A2/7.+B2/4.+2.*C2/9.+D2/5.)
      P2=A9/2.=B9/3.+C9/4.+D9/5.+E9/6.-A*(2.*A9/3.+B9/2.+2.*C9/5.+D9/3.
     1+2.*E9/7.)-B*(2.*B9/5.+A9/2.+C9/3.+2.*D9/7.+E9/4.)-C*(2.*C9/7.
     2+2.*A9/5.+B9/3.+D9/4.+2.*E9/9.)-D*(2.*D9/9.+A9/3.+2.*B9/7.+C9/4.
     3+E9/5.)-E*(2.*E9/11.+2.*A9/7.+B9/4.+2.*C9/9.+D9/5.)
      F=2.*P*A-4.*T(4)*P*P*A1-2.*T(4)*P*A1*(1._A/2.-B/3.
     1-C/4.-D/5.-E/6.)
      T(5)=(F+2.*T(4)*P*P)/A4-2.*P*P2*T(4))/(P*P2*T(4))/(P*P+2.*P*P1*T(4))
      RETURN
      END
```

MX=0.10 FRICTION FUNCTION = 0.51209E 00

ETA	U/UINF
0.48463E-00	0.11474E-00
0.96927E 00	0.21222E-00
0.14539E 01	0.29393E-00
0.19385E 01	0.36076E-00
0.24232E 01	0.41331E-00
0.29078E 01	0.45215E-00
0.33924E 01	0.47822E-00
0.38771E 01	0.49303E-00
0.43617E 01	0.49906E-00
0.48463E 01	0.50000E-00

MX=0.20 FRICTION FUNCTION = 0.48332E 00

ETA	U/UINF
0.48308E-00	0.10783E-00
0.96616E 00	0.20031E-00
0.14492E 01	0.27976E-00
0.19323E 01	0.34702E-00
0.24154E 01	0.40202E-00
0.28985E 01	0.44440E-00
0.33816E 01	0.47399E-00
0.38646E 01	0.49146E-00
0.43477E 01	0.49882E-00
0.48308E 01	0.50000E-00

MX=0.30 FRICTION FUNCTION = 0.45488E 00

ETA	U/UINF
0.48709E-00	0.10206E-00
0.97419E 00	0.19022E-00
0.14613E 01	0.26762E-00
0.19484E 01	0.33514E-00
0.24355E 01	0.39221E-00
0.29226E 01	0.43762E-00
0.34097E 01	0.47029E-00
0.38967E 01	0.49009E-00
0.43838E 01	0.49861E-00
0.48709E 01	0.50000E 00

MX=0.40 FRICTION FUNCTION = 0.42663E-00

ETA	U/UINF
0.49483E-00	0.96878E-01
0.98966E 00	0.18101E-00
0.14845E 01	0.25642E-00
0.19793E 01	0.32410E-00
0.24742E 01	0.38305E-00
0.29690E 01	0.43127E-00
0.34638E 01	0.46681E-00
0.39586E 01	0.48879E-00
0.44535E 01	0.49841E-00
0.49483E 01	0.50000E-00

```
      CIRCULAR CYL MHD BL,BODY-FIXED B,SIGMA VARIABLE
      DIMENSION T(15)
      COMMON T,A,B,C,D,E,F,G,H,P,A4,B3,Q3
      B3=1.
      T(2)=.005
      T(3)=.0005
      T(4)=.0526
      Z4=1.
      CALL INT (T,1,0,1.E-8,100.,1.,.01,5.8E-5,.5)
      DO 6 J=1,32000
1     CALL INIM
      B4=T(2)/.017453293
      IF(B4-44.)12,16,16
16    IF(B4-45.5)17,18,18
17    T(2)=T(2)+.034906586
      T(4)=T(4)+T(5)*.034906586
18    IF(B4-89.2)12,14,14
14    IF(B4-90.1)15,13,13
15    T(2)=T(2)+.017453293
      T(4)=T(4)+T(5)*.017453293
13    IF(B4-111.)12,12,8
12    IF(A)2,2,19
19    IF(B4-Z4*5.)6,2,2
2     Q10=T(4)*EXPF(2.*T(2))/SINF(2.*T(2))
      A3=SQRTF(Q10)
      A6=2.*A*A4/A3
      WRITE OUTPUT TAPE 3,3,B4,A6
3     FORMAT (8H THETA =F6.1,8X,18HFRICTIIN FUNCTION=E12.5,
     1/13X,5HETA R,15X,6HU/UINF)
      Y=.1
      DO 9 IB=1,10
      A5=Y*A3
      A7=A4*Y*(A+Y*(B+Y*(C+Y*(D+Y*(E+Y*(F+Y*(G+Y*H)))))))
      WRITE OUTPUT TAPE 3,10,A5,A7
10    FORMAT (8X,E12.5,8X,E12.5)
9     Y=Y+.1
      Z4=Z4+1.
      IF(A)8,8,6
6     CONTINUE
      CALL EXIT
      END
```

```
      SUBROUTINE DAUX
      DIMENSION T(15)
      COMMON T,A,B,C,D,E,F,G,H,P,A4,B3,Q3
      Q4=T(2)
      Q5=T(4)
      A4=2.*SINF(Q4)
      Q=2.*COSF(Q4)
      Q7=EXPF(2.*Q4)
      Q8=COSF(2.*Q4)
      Q9=SINF(2.*Q4)
      Q1=B3*Q5*Q7/Q9
      Q2=2.*Q5*Q7/A4
      W=.98
      A=(336.*W+18.*Q2)/(126.+Q1)
      B=-Q2*.5
      C=(56.*61*W+3.*Q1*Q2)/(126.+Q1)
      D=-8.75*A-7.5*B-3.75*C
      E=21.*A+16.*B+6.*C
      F=-21.*A-15.*B-5.*C
      G=10.*A+48.*B/7.+15.*C/7.
      H=-15.*A/8.-5.*B/4.-3.*C/8.
      A2=-A/(126.+Q1)
      C2=(56.*W+3.*Q2-C)/(126.+Q1)
      D2=-8.75*A1-3.75*C2
      E2=21.*A2+6.*C2
      F2=-21.*A2-5.*C2
      G2=10.*A2+15.*C2/7.
      H2=-15.*A2/8.-3.*C2/8.
      A9=18./(126.+Q1)
      B9=-.5
      C9=3.*Q1/(126.+Q1)
      D9=-8.75*A9-7.5*B2-3.75*C9
      E9=21.*A9+16.*B9+6.*C9
      F9=-21.*A9-15.*B9-5.*C0
      G9=10.*A9+48.*B9/7.+15.*C9/7.
      H9=-15.*A9/8.-5.*B9/4.-3.*C9/8.
      P=2.*A*(.25-A/6.-B/4.-C/5.-D/6.-E/7.-F/8.-G/9.-H/10.)+2.*B*(1./6.
     1-B/10.-C/6.-D/7.-E/8.-F/9.-G/10.-H/11.)+2.*C*(.125-C/14.-D/8.-E/9.
     2-F/10.-G/11.-H/12.)+2.*D*(.1-D/18.-E/10.-F/11.-G/12.-H/13.)+2.*E*
     3(1./12.-E/22.-F/12.-G/13.-H/14.)+2.*F*(1./14.-F/26.-G/14.-H/15.)
     4+2.*G*(1./16.-G/30.-H/16.)+2.*H*(1./18.-H/34.)
      P1=2.*A2*(.25-A/3.-B/4.-C/5.-D/6.-E/7.-F/8.-G/9.-H/10.)+2.*C2*(.12
     15-A/5.-B/6.-C/7.-D/8.-E/9.-F/10.-G/11.-H/12.)+2.*D2*(.1-A/6.-B/7.
     2-C/8.-D/9.-E/10.-F/11.-G/12.-H/13.)+2.*E2*(1./12.-A/7.-B/8.-C/9.-
     3D/10.-E/11.-F/12.-G/13.-H/14.)+2.*F2*(1./14.-A/8.-B/9.-C/10.-D/11.
     4-E/12.-F/13.-G/14.-H/15.)+2.*G2*(1./16.-A/9.-B/10.-C/11.-D/12.-E/
     513.-F/14.-G/15.-H/16.)+2.*H2*(1./18.-A/10.-B/11.-C/12.-D/13.-E/14.
     6-F/15.-G/16.-H.17.)
      P2=2.*A9*(.25-A/3.-B/4.-C/5.-D/6.-E/7.-F/8.-G/9.-H/10.)+2.*C9*(.12
     15-A/5.-B/6.-C/7.-D/8.-E/9.-F/10.-G/11.-H/12.)+2.*D9*(.1-A/6.-B/7.
     2-C/8.-D/9.-E/10.-F/11.-C/12.-H/13.)+2.*E9*(1./12.-A/7.-B/8.-C/9.-
     3D/10.-E/11.-F/12.-G/13.-H/14.)+2.*F9*(1./14.-A/8.-B/9.-C/10.-D/11.
     4-E/12.-F/13.-G/14.-H/15.)+2.*G9*(1./16.-A/9.-B/10.-C/11.-D/12.-E/
     513.-F/14.-G/15.-H/16.)+2.*H9*(1./18.-A/10.-B/11.-C/12.-D/13.-E/14.
     6-F/15.-G/16.-H/17.)+2.*B9*(1./6.-A/4.-B/5.-C/6.-D/7.-E/8.-
     7F/9.-G/10.-H/11.)
```

```
 Q3=1.-A/2.-B/3.-C/4.-D/5.-E/6.-F/7.-G/8.-H/9.
 F3=A/Q5-2.*Q7*(2.*P+Q3)/A4
 Q6=((.5*F3*Q/Q7+P*B3/A4)/Q7+P*(Q8/Q9-1.)/Q7-P1*2.*B3*Q5/Q9+
12.*P1*B3*Q5*Q8/(Q9*Q9)-4.*P2*Q5/A4+2.*P2*Q5*Q/(A4*A4))/
2(.5*P/(Q5*Q7)+P1*B3/Q9+P2*2./A4)
 T(5)=Q6
 RETURN
 END
```

```
THETA =    5.0               FRICTION FINCTION= 0.58377E 00
               ETA R                   U/UINF
            0.25076E-00            0.62941E-01
            0.50152E 00            0.10769E-00
            0.75228E 00            0.13742E-00
            0.10030E 01            0.15537E-00
            0.12538E 01            0.16492E-00
            0.15046E 01            0.16918E-00
            0.17553E 01            0.17066E-00
            0.20061E 01            0.17100E-00
            0.22568E 01            0.17104E-00
            0.25076E 01            0.17104E-00
THETA =   10.0               FRICTION FUNCTION= 0.11556E 01
               ETA R                   U/UINF
            0.25170E-00            0.12511E-00
            0.50341E 00            0.21417E-00
            0.75511E 00            0.27340E-00
            0.10068E 01            0.30922E-00
            0.12585E 01            0.32829E-00
            0.15102E 01            0.33681E-00
            0.17619E 01            0.33977E-00
            0.20136E 01            0.34045E-00
            0.22653E 01            0.34052E-00
            0.25170E 01            0.34053E-00
THETA =   15.0               FRICTION FUNCTION= 0.17062E 01
               ETA R                   U/UINF
            0.25330E-00            0.18602E-00
            0.50659E 00            0.31868E-00
            0.75989E 00            0.40711E-00
            0.10132E 01            0.46070E-00
            0.12665E 01            0.48928E-00
            0.15198E 01            0.50207E 00
            0.17731E 01            0.50652E 00
            0.20264E 01            0.50755E 00
            0.22797E 01            0.50766E 00
            0.25330E 01            0.50766E 00
```

VITA

Rodney Dale Hugelman

Candidate for the Degree of

Doctor of Philosophy

Thesis: MAGNETOHYDRODYNAMIC BOUNDARY LAYER SEPARATION AND SURFACE FRICTION COEFFICIENTS

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