PROPERTIES OF PERIPHERALLY CONTINUOUS FUNCTIONS AND CONNECTIVITY MAPS

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PREFACE

This paper will be concerned with some special, not necessarily continuous, transformations, namely connectivity maps, local connectivity maps, and peripherally continuous functions. Chapter I is an introductory chapter giving the definitions of the above mentioned transformations. The material in Chapter II is somewhat of a general nature with attention being directed to the graph of a function. Chapter III deals with the specific problem of showing the equivalence of the above mentioned transformations on a suitably restricted space. In Chapter IV certain results for continuous functions [9] (numbers in brackets refer to the bibliography at the end of the paper) are extended to peripherally continuous functions. A summary of results is contained in Chapter V.

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CHAPTER I

INTRODUCTION

This paper will be devoted to the development of certain properties of connectivity maps, local connectivity maps, peripherally continuous transformations, and relationships among these non-continuous mappings. John Nash and O. H. Hamilton have defined in [6] and [3], respectively, the connectivity map and the peripherally continuous transformation. J. Stallings has defined in [8] the local connectivity map. In his doctoral thesis and in [4] Paul E. Long developed certain properties and relationships among these and other discontinuous mappings. The present paper is an outgrowth of papers [3], [4], and [8]. The results in this paper rely heavily upon material in Whyburn [9] and in Moore [5].

In Chapter II some properties of connectivity maps and peripherally continuous transformations are given. Among these are some sufficient conditions for continuity. Attention is focused on the graph map of a function, and it is found that if the graph of a connectivity map is semi-locally connected the function is continuous. Two conditions are also given such that if either is imposed on the graph of a peripherally continuous function, the function is a connectivity map. The latter part of the chapter is concerned with sequences of functions, and one result is that the limit of a uniformly convergent sequence of peripherally continuous functions is peripherally continuous.

In Chapter III a proof is given that on a locally peripherally connected space having the Brouwer Property II every peripherally continuous function is a connectivity map. In [8] Stallings has shown that on a locally peripherally connected polyhedron every local connectivity map, and hence every connectivity map, is peripherally continuous. Combining these results, we have that on a locally peripherally connected polyhedron having the Brouwer Property II local connectivity maps, connectivity maps, and peripherally continuous functions are equivalent.

Chapter IV is concerned with upper semi-continuous decompositions of the domain space, and factorization of connectivity maps and of peripherally continuous functions. It is well known that a continuous function f can be factored into a composite $f = f_2 f_1$ where both factors are continuous, f_1 is monotone, and f_2 is light [9]. The main result of Chapter IV is that this factorization also holds for peripherally continuous functions where the factor f_2 is peripherally continuous. A similar result is obtained for connectivity maps if an added assumption is made on the function.

The definitions of the above mentioned transformations will now be given. These and other functions defined throughout this paper are not necessarily continuous although all results apply to continuous functions.

<u>Definition 1.1</u>. A connectivity map from a space S into a space T is a mapping f such that the induced map g of S into $S \times T$, defined by $g(p) = (p \times f(p))$, transforms connected subsets of S onto connected subsets of $S \times T$ [6].

<u>Definition 1.2</u>. A mapping f of a space S into a space T is called peripherally continuous if and only if for each point p in S and each pair of open sets U and V containing p and f(p), respectively, there

is an open set $D \subset U$ containing p such that f transforms the boundary F of D into V [3].

Definition 1.3. A mapping f from a space S into a space T is called a local connectivity map if there is a covering $\{U_\alpha\}$ of S by open sets such that f restricted to U_α is a connectivity map for every α [8].

If U is a subset of a space S the boundary of U will be denoted by F(U); and if f is a transformation of the space S into the space T and M is a subset of S the graph map of M under f, which consists of points $(p \times f(p))$, p in M, will be denoted by g(M). Other definitions will be given as needed throughout the paper. All topological spaces considered are at least regular and T_1 unless otherwise stated, and all metric spaces are separable.

CHAPTER II

SOME PROPERTIES OF DISCONTINUOUS FUNCTIONS

Let f be a mapping of the space S into the space T and g the induced graph map of S into $S \times T$. This chapter is concerned with the relationship between f and g, and conditions on g(S), the graph of f, which will impose certain conditions on f. A well known result is that f is continuous if and only if g is a homeomorphism [2, p. 76]. If S and T are compact Hausdorff spaces and S is locally connected, then g being continuous implies g(S) is locally connected [9, p. 26]. If S and T are T_1 spaces and S is locally compact, then g being continuous implies g(S) is locally compact [9, p. 279]. If one requires that g(S) be compact, then it is easy to show that g, and hence f, is continuous.

Unless explicitly stated the terms mapping, function, and transformation will not imply continuity. The following propositions are concerned with properties of f and the graph map g.

Proposition 2.1. If f is an open mapping of the space S into the space T, then the graph map g is an open mapping of S into $S \times T$.

Proof. Let U be an open set in S and $p \times f(p)$ a point of g(U). Then p is in U, f(p) is in f(U), and f(U) is open by hypothesis. Therefore $U \times f(U)$ is open in $S \times T$ and $(U \times f(U)) \cap g(S)$ is an open set in g(S) containing $p \times f(p)$ and contained in g(U). Therefore $p \times f(p)$ is an interior point of g(U) and g(U) is open. Thus g is an open mapping.

Proposition 2.2. If f is a mapping of the compact T_1 space S into the T_1 space T and if g is closed, then f is a closed mapping.

Proof. Let A be a closed subset of S and let q be a limit point of f(A). Then there is a sequence $\{f(p_n)\}$ of distinct points of f(A) converging to q such that p_n is in A. Since A is a closed subset of a compact set, A is compact and some subsequence $\{p_{ni}\}$ of $\{p_n\}$ has a sequential limit point p in A. Then $p_{ni} \times f(p_{ni}) \to p \times q$, and $p_{ni} \times f(p_{ni})$ is in g(A) and g(A) closed implies that $p \times q$ is in g(A). Thus f(p) = q and f(p) is in f(A). Therefore f(A) is closed and f is a closed mapping.

Example 1. This example is to show that if f is a closed mapping the graph map g is not necessarily closed. Let $S = (0,1] \cup \{2\}$, and $T = [-1,1]$, and define a mapping f of S into T by $f(x) = \sin(\frac{1}{x})$, $x \neq 2$, and f(2) = 0. Then f is a closed mapping, but the set (0,1] is closed in S and g((0,1]) is not closed in g(S) since (2,f(2)) is a limit point of g((0,1]). Thus g is not a closed mapping.

Proposition 2.3. If f is a mapping of the T_1 space S into the T_1 space T and K is a connected subset of g(S), then $g^{-1}(K)$ is connected in S.

Proof. Suppose $g^{-1}(K) = M \cup N$, where M and N are mutually separated. Then $K = g(M) \cup g(N)$ and $g(M) \cap g(N) = \emptyset$ since $M \cap N = \emptyset$. Therefore one of g(M) and g(N) must contain a limit point of the other. Let $p \times f(p)$ be a limit point of g(N) belonging to g(M). Then there is a sequence $\{q_n \times f(q_n)\}$ of points in g(N) converging to $p \times f(p)$. Now q_n is in N, point p is in M and $q_n \to p$. This implies that p is a limit point of N belonging to M contradicting M and N being mutually separated. Therefore $g^{-1}(K)$ is connected.

Proposition 2.4. If f is a mapping of the space S into the space T, then f is peripherally continuous if and only if g is peripherally continuous.

Proof. Suppose f is peripherally continuous. Let p be a point of S and let U and V be open sets containing p and $p \times f(p)$, respectively, where V is of the form $H \times K$ with H open in S and K open in T. Then $H \cap U$ is an open set containing p and K is an open set containing f(p). Since f is peripherally continuous there is an open set $D \subset (H \cap U)$ containing p such that $f(F(D)) \subset K$. Thus $g(F(D)) \subset V$ and g is peripherally continuous.

Conversely, suppose g is peripherally continuous. Let p be a point of S and U and V open sets containing p and f(p), respectively. Then $U \times V$ is an open set containing $p \times f(p)$ and hence there exists an open set $D \subset U$ and containing p such that $g(F(D)) \subset (U \times V)$. Therefore $f(F(D)) \subset V$ and f is peripherally continuous.

Definition 2.1. A space S is said to be semi-locally connected at a point p if for every open set U containing p there is an open set V containing p and contained in U such that S - V has a finite number of components. A space S is semi-locally connected if S is semi-locally connected at each of its points [9, p. 19].

The next two theorems give conditions which imply continuity of a function.

Theorem 2.1. If f is a connectivity mapping of the T_1 space S into the T_1 space T and g(S) is semi-locally connected, then f is continuous.

Proof. Suppose f is not continuous at a point p in S. Then g is not continuous at p and hence there is a sequence $\{p_n\}$ of points of S converging to p but $\{p_n \times f(p_n)\}$ does not converge to $p \times f(p)$. Since g(S) is semi-locally connected there is an open set U containing $p \times f(p)$ such that $p_n \times f(p_n)$ is not in U for infinitely many n and g(S) - U has only a finite number of components. Then infinitely many $p_n \times f(p_n)$ lie in a single component K of g(S) - U. Now $K \cup \{p \times f(p)\}$ is not connected but $g^{-1}(K \cup \{p \times f(p)\}) = g^{-1}(K) \cup \{p\}$ is connected since by Proposition 2.3 $g^{-1}(K)$ is connected and p is a limit point of $g^{-1}(K)$. The reason that p is a limit point of $g^{-1}(K)$ is that infinitely many $p_n \times f(p_n)$ lie in K and hence infinitely many p_n lie in $g^{-1}(K)$ and $p_n \rightarrow p$. Since f is a connectivity map and $g^{-1}(K) \cup \{p\}$ is connected, the set $g(g^{-1}(K) \cup \{p\}) = K \cup \{p \times f(p)\}$ is connected. This is a contradiction and therefore f is continuous.

Corollary 2.1. If f is a connectivity mapping of the T_1 space S into the T_1 space T and g(S) is locally compact and locally connected, then f is continuous.

Proof. Every locally compact and locally connected space is semi-locally connected [9, p. 20]. Hence Theorem 2.1 implies f is continuous.

Example 2. This example is to show that one cannot expect the graph of a locally connected, locally compact set to be locally connected and locally compact for connectivity maps or peripherally continuous functions even if the mapping is open. Let $f(x) = \sin(\frac{1}{x})$, $0 < x < 1$, and $f(0) = 0$. This is a connectivity map since the graph of any connected subset of $[0,1)$ is connected. The mapping is also peripherally continuous since it is continuous everywhere except at zero and is peripherally continuous at that point. The mapping is also open, since if I is any open interval contained in $(0,1)$, then $f(I)$ is an open interval, and if I is an open interval containing zero, then $f(I) = [-1,1]$. The graph of this function is neither locally connected nor locally compact. This example also shows that even though a connectivity map or peripherally continuous function is open it need not be continuous.

Theorem 2.2. Let f be a mapping of the T_1 space S onto the semi-locally connected T_1 space T with the following properties:

(a) f is finite - to - 1 onto T,

(b) the inverse image $f^{-1}(H)$ of a closed set H in T has closed components, and

(c) if H is a connected subset of T, then every component of $f^{-1}(H)$ maps onto all of H.

Then f is a continuous function.

Proof. If f is not continuous at a point p in S, then there is an open set V containing f(p) such that if U is any open set containing p, f(U) is not a subset of V. Since T is semi-locally connected, there is an open set $W \subset V$ and containing f(p) such that T - W has a finite number of components, $C_1, \ldots, C_n$. Now T - W closed implies that the C_i are closed, i = 1, ... ,n, and f finite - to - 1 implies that $f^{-1}(C_i)$, i = 1, ... ,n, has a finite number of components K_{ij}, j = 1, ... ,n_i, since each component maps onto all of C_i.

The point p is a limit point of at least one component of $f^{-1}(C_i)$ for some i. For suppose that for every i, i = 1, ... ,n, p is not a limit point of any component of $f^{-1}(C_i)$. Then there is an open set U_{ij} such that p is in U_{ij} and $(U_{ij}) \cap (K_{ij}) = \emptyset$, j = 1, ... ,$n_i$, and i = 1, ... ,n. If U denotes the intersection of all the sets U_{ij}, then U is an open set containing p and $f(U) \cap (T - W) = \emptyset$. Thus f(U) is a subset of W and therefore is contained in V. This contradicts the hypothesis that f(U) is not contained in V for any open set U. Thus p is a limit point of some component of some $f^{-1}(C_i)$. But p is not in $f^{-1}(C_i)$ contradicting the hypothesis that $f^{-1}(C_i)$ has closed components. Therefore f is continuous.

Definition 2.2. A mapping f of the space S into the space T is called a C - mapping if for every point p in S, every pair of open sets U and V containing p and f(p), respectively, and every non-degenerate subset M of S having the property that $M \cup \{p\}$ is connected, then there is a point q in $U \cap M$ distinct from p such that f(q) is in V(Long).

In [4] Long has shown that if f is peripherally continuous and N is closed, then $f^{-1}(N)$ has closed components. He has also shown this for the C - mapping, and in [3] Hamilton has shown this for the connectivity map. Therefore, Theorem 2.2 remains valid if part (b) of the hypothesis is replaced by the requirement that the mapping be one of the above. Also, Theorem 2.2 implies that there does not exist a 2 - to - 1 C - map, connectivity map, or peripherally continuous function f on an n - cell, n = 1, 2, 3, onto itself such that if H is a connected subset of the n - cell, then each component of $f^{-1}(H)$ maps onto all of H. For if such a function did exist it would be continuous by Theorem 2.2 contradicting the fact that no such 2 - to - 1 mapping exists [1], [7].

<u>Example 3</u>. This example is to show that if f is a one - to - one connectivity map or peripherally continuous function, f^{-1} need not be a connectivity map or peripherally continuous function. Let A be the set $[-1,0) \cup (0,1)$, B = (-1,1), and let f(x) = x - 1 if $x \in (0,1)$ and f(x) = x + 1 if $x \in [-1,0)$. The mapping thus defined is continuous and therefore a connectivity map and a peripherally continuous function. It is also one - to - one. However, $f^{-1}(x) = x - 1$ if $x \in [0,1)$ and $f^{-1}(x) = x + 1$ if $x \in (-1,0)$. Now no connected set containing zero has a connected graph and hence f^{-1} is not a connectivity map nor is it peripherally continuous at zero.

The next two theorems give conditions on the graph map g which will imply that a peripherally continuous function is a connectivity map.

<u>Theorem 2.3</u>. Let f be a peripherally continuous mapping from the T_1 space S into the T_1 space T. If for every connected set K in S, g(K) has

a finite number of components, then f is a connectivity map.

Proof. Since f is peripherally continuous, g is peripherally continuous by Proposition 2.4.

Let K be a connected subset of S and suppose g(K) is not connected. By hypothesis g(K) has a finite number of components, $C_1, \ldots, C_n$. Then $g(K) = \bigcup_{i=1}^{n} C_i$, $K = \bigcup_{i=1}^{n} g^{-1}(C_i)$, and $g^{-1}(C_i) \cap g^{-1}(C_j) = \emptyset$ since $C_i \cap C_j = \emptyset$, $i \neq j$. Since K is connected not all the $g^{-1}(C_i)$ are mutually separated. Let p be a point of $g^{-1}(C_i)$ which is a limit point of $\bigcup_{j=1}^{n} g^{-1}(C_j)$, $j \neq i$. Then p must be a limit point of some $g^{-1}(C_j)$. Now $p \times f(p)$ is in C_i and there is an open set V containing $p \times f(p)$ such that $V \cap C_k = \emptyset$, $k \neq i$, since the C_k are mutually separated. Let U be any open set containing p. Then g peripherally continuous implies there is an open set $D \subset U$ and containing p such that $g(F(D)) \subset V$. By Proposition 2.3, $g^{-1}(C_j)$ is connected since C_j is connected. Since p is a limit point of $g^{-1}(C_j)$, then $g^{-1}(C_j)$ is non-degenerate and the open set D can be chosen such that $g^{-1}(C_j)$ has points interior to D and exterior to D. Therefore $g^{-1}(C_j)$ has points in common with F(D) since $g^{-1}(C_j)$ is connected. Thus g(F(D)) is not a subset of V. This involves a contradiction and hence g(K) must be connected. Therefore f is a connectivity map.

Theorem 2.4. Let f be a peripherally continuous mapping of the T_1 space S into the T_1 space T. If for every non-degenerate connected set K in S, g(K) has no degenerate components, then f is a connectivity map.

Proof. Suppose f is not a connectivity map. Then there is a non-degenerate connected set K in S such that $g(K) = M \cup N$, where M and N are mutually separated. By hypothesis the components of M and N are non-degenerate. Hence, $g^{-1}(M)$ and $g^{-1}(N)$ have non-degenerate components. For suppose the point p is a component of $g^{-1}(M)$. Then $g(p) = p \times f(p)$ lies in some non-degenerate component C of M and $g^{-1}(C)$ is connected.

Therefore $g^{-1}(C) = p$ and this contradicts g being a one - to - one mapping.

Now $M \cap N = \emptyset$ implies $g^{-1}(M) \cap g^{-1}(N) = \emptyset$, and $K = g^{-1}(M) \cup g^{-1}(N)$ being connected implies $g^{-1}(M)$ and $g^{-1}(N)$ are not mutually separated. Let p be a point of $g^{-1}(M)$ which is a limit point of $g^{-1}(N)$. Then $p \times f(p)$ is in M and there is an open set V containing $p \times f(p)$ such that $V \cap N = \emptyset$ since M and N are mutually separated. Let U be an open set containing p. Then $U \cap g^{-1}(N) \neq \emptyset$ since p is a limit point of $g^{-1}(N)$. Hence U intersects some non-degenerate component C of $g^{-1}(N)$. Since g is peripherally continuous there is an open set W containing p and contained in U such that C does not lie wholly in W and $g(F(W)) \subset V$. Then $C \not\subset W$, but $C \cap W \neq \emptyset$ implies the connected set C has points interior to W and points exterior to W which means that $F(W) \cap C \neq \emptyset$. This is a contradiction since $g(F(W)) \subset V$, $g(C) \subset N$ and $V \cap N = \emptyset$. Thus f is a connectivity map.

<u>Theorem 2.5</u>. Let f be a connectivity map from the T_1 space S into the T_1 space T. If V is an open subset of T and K is a non-degenerate component of $f^{-1}(V)$, then any point p in the closure of K such that p is not in K has the property that $f(p)$ is in $F(V)$.

<u>Proof</u>. Let p be a limit point of K which is not in K. Since $K \cup \{p\}$ is connected and connectivity functions map connected sets onto connected sets, $f(K \cup \{p\}) = f(K) \cup \{f(p)\}$ is connected. Now $f(K)$ is contained in V and $f(p)$ is not in V, and $f(K) \cup \{f(p)\}$ connected implies $f(p)$ is a limit point of $f(K)$. Hence $f(p)$ is a limit point of V not in V. Therefore $f(p)$ is in $F(V)$.

<u>Theorem 2.6</u>. Let f be a connectivity mapping of the locally connected and connected T_1 space S into the T_1 space T. If V is an open subset of T, then $f^{-1}(V)$ is dense-in-itself.

Proof. Suppose $f^{-1}(V)$ is not dense-in-itself. Then there is a point p in $f^{-1}(V)$ and an open set U containing p such that $U - \{p\}$ contains no point of $f^{-1}(V)$. Since S is locally connected the component C of U containing p is open. Then $C \times V$ is an open set in $S \times T$ containing only the point $p \times f(p)$ of g(C). This implies g(C) is not connected contradicting the hypothesis that f is a connectivity map. Therefore every point of $f^{-1}(V)$ is a limit point of $f^{-1}(V)$ and hence $f^{-1}(V)$ is dense-in-itself.

The following two theorems are concerned with sequences of functions and properties of the limit function.

Theorem 2.7. Let f_n be a mapping of the space S into the metric space T, n = 1, 2, ... If $f_n \to f$ uniformly on S and M is a subset of S, then $\overline{f(M)}$ is the limiting set of $\{f_n(M)\}$.

Proof. Denote by L the limiting set of $\{f_n(M)\}$. Then $\overline{f(M)} \subset L$. Denote by d the metric for T and suppose there is a point z in L which does not belong to $\overline{f(M)}$. Then $b = d(z,\overline{f(M)}) > 0$ since z is not a limit point of $\overline{f(M)}$. If $\epsilon = \frac{b}{2}$ and U is the spherical neighborhood of z with radius ϵ, then $U \cap \overline{f(M)} = \emptyset$. Since the convergence is uniform a positive integer N exists such that for every pair of positive integers n and m such that $n > N$ and $m > N$, $d(f_n(x),f_m(x)) < \frac{\epsilon}{2}$ for every x in S. Since L is the limiting set of $\{f_n(M)\}$ there is an integer $n > N$ and a point y in M such that $f_n(y)$ is in U. Then $d(f_n(y),f_k(y)) < \frac{\epsilon}{2}$ for every $k > N$. Since $f_k(y)$ is in U for every $k > N$, then f(y) is in U. This implies z is a limit point of f(M). This involves a contradiction and hence $L = \overline{f(M)}$.

Corollary 2.2. If, in Theorem 2.7, each f_n maps connected sets onto connected sets, M is connected, and $\overline{\bigcup_{n=1}^{\infty} f_n(M)}$ is compact, then $\overline{f(M)}$ is a continuum.

<u>Proof</u>. This follows directly from Theorem 58 page 23 of [5].

<u>Theorem 2.8</u>. Let f_n be a peripherally continuous mapping of the space S into the metric space T, n = 1, 2, ... If $f_n \to f$ uniformly on S, then f is peripherally continuous.

<u>Proof</u>. Let p be a point of S and U and V open sets containing p and f(p), respectively. There is an $\epsilon > 0$ such that the spherical neighborhood R of f(p) with radius ϵ is contained in V. Let R' be the spherical neighborhood of f(p) of radius $\frac{\epsilon}{4}$. Since the convergence is uniform there is a positive integer N such that for every $n > N$, $d(f_n(x),f(x)) < \frac{\epsilon}{4}$ for every x in S. Choose $n_o > N$. Then $f_{n_o}(p)$ is in R' and since f_{n_o} is peripherally continuous at p there is an open set D containing p and contained in U such that $f_{n_o}(F(D)) \subset R'$.

If y is a point of F(D), then

$$d(f(y),f(p)) \leqslant d(f_{n_o}(y),f(y)) + d(f_{n_o}(y),f(p)).$$

Now $d(f_{n_o}(y),f(y)) < \frac{\epsilon}{4}$ by the uniform convergence and $d(f_{n_o}(y),f(p))$ is less than $\frac{\epsilon}{4}$ since $f_{n_o}(y)$ is in R'. Hence $d(f(y),f(p)) < \frac{\epsilon}{2}$, and f(y) is in R. Thus $f(F(D)) \subset R \subset V$ and D is the required neighborhood. Hence f is peripherally continuous.

<u>Example 4</u>. This example is to show that the limit of a sequence of connectivity maps or peripherally continuous functions need not be of the same type. The sequence of continuous functions $f_n(x) = x^n$ on $[0,1]$ converges to $f(x) = 0$, $x \neq 1$ and $f(1) = 1$, which is neither a connectivity map nor peripherally continuous.

CHAPTER III

THE EQUIVALENCE OF CONNECTIVITY MAPS AND PERIPHERALLY CONTINUOUS FUNCTIONS ON A CERTAIN SPACE

In this chapter equivalence, under certain conditions on the domain space, of the peripherally continuous function, connectivity map, and local connectivity map is established.

O. H. Hamilton [3] stated a theorem, which was somewhat generalized and completely proved by Stallings [8], to the effect that on a locally peripherally connected polyhedron into a regular Hausdorff space, every local connectivity map, and hence every connectivity map, is peripherally continuous. One of the main results of this paper is the converse of this theorem.

Definition 3.1. A space S is said to be locally peripherally connected at the point p if for every open set U containing p there is an open set V containing p and contained in U such that F(V) is connected. A space is locally peripherally connected if it is locally peripherally connected at every point [8].

Stallings in [8] has shown that if f is a peripherally continuous mapping of the locally peripherally connected space S into the space T and p is a point of S, then for every pair of open sets U and V containing p and f(p), respectively, there is a connected neighborhood N of p contained in U such that F(N) is connected and $f(F(N)) \subset V$. This property is used to a great extent throughout the rest of this paper.

Another property that will be imposed on the domain space of a function is the following.

Property II(Brouwer Property). If M is a closed connected subset of S and C is a component of S - M, then the boundary of C is a closed and connected set [10].

The following lemma plays an important role in the proof of Theorem 3.1.

Lemma 3.1. Let W be an open connected subset of the locally peripherally connected Moore space S such that F(W) is connected. Let W_1 and W_2 be open connected sets such that $W_1 \cap W_2 \neq \emptyset$, $F(W_1)$ and $F(W_2)$ are connected, and $\overline{W}_1 \cup \overline{W}_2 \subset W$. If $W_3 = (W_1 \cup W_2) \cup (\bigcup C_\alpha)$, where $\{C_\alpha\}$ is the collection of all components of $\overline{W} - (W_1 \cup W_2)$ such that $F(C_\alpha) \subset F(W_1) \cup F(W_2)$, and if C is the component of $\overline{W} - (W_1 \cup W_2)$ containing the connected set F(W), then

(1) $F(W_3) \subset F(W_1) \cup F(W_2)$,

(2) $\overline{W} = C \cup W_3$,

(3) W_3 is open and connected, and

(4) if the space S has Property II, $F(W_3)$ is connected.

Proof of (1). Suppose there is an $x \in F(W_3) - (F(W_1) \cup F(W_2))$. Then there is an open connected set G such that F(G) is connected, $x \in G$, and $\overline{G} \cap (\overline{W}_1 \cup \overline{W}_2) = \emptyset$. Since $F(C_\alpha) \subset (F(W_1) \cup F(W_2))$, then $x \notin C_\alpha$ for any α. Therefore x is a limit point of $\bigcup C_\alpha$ such that $x \notin \bigcup C_\alpha$. This implies G must intersect infinitely many C_α. If $C_\alpha \subset G$ for some α, then $F(C_\alpha) \subset G$ since C_α is closed. This is a contradiction since $F(C_\alpha)$ is contained in $F(W_1) \cup F(W_2)$. Therefore, if $C_\alpha \cap G \neq \emptyset$, then C_α has points interior to G and points exterior to G. This implies $F(G) \cap C_\alpha \neq \emptyset$ since C_α is connected. Now $F(G) \cap F(C_\alpha) = \emptyset$ since $\overline{G} \cap (\overline{W}_1 \cup \overline{W}_2) = \emptyset$. Hence $F(G) = (F(G) - C_\alpha) \cup (F(G) \cap C_\alpha)$, where $F(G) - C_\alpha$ and $F(G) \cap C_\alpha$ are non-empty and mutually separated. This contradicts F(G) being connected.

Thus $F(W_3) \subset (F(W_1) \cup F(W_2))$.

Proof of (2). If K is a component of $W - (W_1 \cup W_2)$ such that $K \cap C = \emptyset$, then $F(K) \subset (F(W_1) \cup F(W_2))$. For suppose there is a point x in $F(K) - (F(W_1) \cup F(W_2))$. Since K is closed, $x \in K$. Now $K - (F(W_1) \cup F(W_2))$ is equal to $\bigcup K_\alpha$, where $\{K_\alpha\}$ is the collection of components of the set $K - (F(W_1) \cup F(W_2))$. Then each K_α is also a component of $W - (C \cup \overline{W}_3)$ since $K \cap C_\alpha = \emptyset$, $K \cap C = \emptyset$ and hence $(K - (F(W_1) \cup F(W_2))) \cap (C \cup \overline{W}_3)$ is empty. Since $W - (C \cup \overline{W}_3)$ is open K_α is open and $F(K_\alpha) \subset C \cup \overline{W}_3$. But $\overline{K}_\alpha \cap C = \emptyset$ implies $F(K_\alpha) \subset \overline{W}_3$. Now $\overline{W}_3 = (\overline{W}_3 - F(W_3) \cup F(W_3)$ and $\overline{W}_3 - F(W_3)$ is an open set disjoint from K_α. Therefore $F(K_\alpha) \subset F(W_3)$. Now $K = (K - F(W_3)) \cup (K \cap F(W_3))$. Therefore, since $x \in (F(K) - F(W_3))$, then $x \in K_\alpha$ for some α. But $x \notin$ interior K_α since interior K_α is contained in interior K. Therefore $x \in F(K_\alpha)$. This is a contradiction since $F(K_\alpha) \subset F(W_3) \subset (F(W_1) \cup F(W_2))$. Hence $F(K) \subset (F(W_1) \cup F(W_2))$. Now suppose there is a point $x \in (W - (C \cup W_3))$. Then x is in some component K of $W - (W_1 \cup W_2)$. By the above argument $F(K) \subset (F(W_1) \cup F(W_2))$ and hence $K = C_\alpha$ for some α. But $C_\alpha \subset W_3$. This contradiction implies that $\overline{W} = C \cup W_3$.

Proof of (3). Since $C \cap W_3 = \emptyset$ and C is closed $\overline{W} - C = W - C = W_3$ is open. Also, W_3 is connected since $W_1 \cup W_2$ is connected and each C_α is connected and $C_\alpha \cap \overline{W_1 \cup W_2} \neq \emptyset$.

Proof of (4). Since W_3 is open, $F(W_3) \cap W_3 = \emptyset$ and hence $F(W_3) \subset C$. Therefore $\overline{W}_3 \cap C = F(W_3)$. Since $\overline{W}_3$ is closed and S has Property II, every component of $S - \overline{W}_3$ has connected boundary. The continuum C contains $F(W_3)$, and W_3 is connected. Hence, by Theorem 34 page 103 of [5], $\overline{W}_3 \cap C = F(W_3)$ is connected.

The following theorem is the converse of Hamilton's and Stallings' theorem [3], [8].

Theorem 3.1. If f is a peripherally continuous mapping of the locally peripherally connected Moore space S having Property II into the space T and if $S \times T$ is completely normal, then f is a connectivity map.

Proof. Suppose that f is not a connectivity map and let A be a connected subset of S such that $g(A) = M \cup N$, where M and N are mutually separated. Let $g^{-1}(M) = H$ and $g^{-1}(N) = K$. Then $A = H \cup K$, where $H \cap K$ is empty. Since A is connected H and K are not separated and hence one must contain a limit point of the other. Let p be a point of H which is a limit point of K. Since $S \times T$ is completely normal there exist open disjoint sets U and V in $S \times T$ containing M and N, respectively.

Let R be an open set containing p such that A is not contained entirely in R. Then f peripherally continuous and S locally peripherally connected implies there is an open connected set W containing p and contained in R such that W and F(W) are both connected and $g(F(W)) \subset U$ [8]. Since p is a limit point of K there is a point q of K in W.

Let Q be the collection of all open connected sets D such that q is in D, $\overline{D} \subset W$, D and F(D) are connected, and $g(F(D)) \subset V$. The collection Q is non-empty since f is peripherally continuous at the point q. Denote by Q^+ the point-set union of all sets in Q. Then Q^+ is an open subset of W. Consider the boundary $F(Q^+)$ of Q^+. If $F(Q^+) \cap A = \emptyset$, then $A = (A - Q^+) \cup (A \cap Q^+)$ and $A - Q^+$ and $A \cap Q^+$ are mutually separated. For $A - Q^+ \neq \emptyset$ since A does not lie wholly in Q^+ and $A \cap Q^+ \neq$ since q is in $A \cap Q^+$. Further, $A \cap Q^+$ is open in A and hence cannot contain a limit point of $A - Q^+$, and any limit point of $A \cap Q^+$ which is in $A - Q^+$ is in $F(Q^+)$ which is disjoint from A. Thus, since $A - Q^+$ and $A \cap Q^+$ are disjoint, they are mutually separated and this contradicts A being connected. Therefore $F(Q^+) \cap A \neq \emptyset$.

Since $F(Q^+) \cap A \neq \emptyset$, either $F(Q^+)$ contains a point of H or a point of K. Suppose there is a point h in $F(Q^+) \cap H$. Then there is an open set E containing h but not q such that F(E) is connected and $g(F(E)) \subset U$. Since h is a limit point of Q^+, E must intersect some set D belonging to the collection Q. Now $E \not\subset D$ since h is in E - D and $D \not\subset E$ since q is in D - E. Thus E and D both have points interior and exterior to one another and F(D) and F(E) being connected implies $F(D) \cap F(E) \neq \emptyset$. But this contradicts the fact that $g(F(D)) \subset V$, $g(F(E)) \subset U$ and $U \cap V = \emptyset$. Hence $F(Q^+) \cap H = \emptyset$ and therefore $F(Q^+) \cap K \neq \emptyset$.

Let k be a point of $F(Q^+) \cap K$. Now k is not a point of F(W) since g(F(W)) is contained in U and g(k) is in V. Thus k is in W and there is an open connected set W_1 containing k and contained in W such that $F(W_1)$ is connected, $\bar{W}_1 \subset W$ and $g(F(W_1)) \subset V$. Since k is a limit point of Q^+ there is a set W_2 in the collection Q such that $W_1 \cap W_2 \neq \emptyset$.

Now form the set W_3 referred to in Lemma 3.1. By this lemma the set W_3 is open, connected, $F(W_3)$ is connected, $\bar{W}_3 \subset W$, and $q \in W_3$. Further, $g(F(W_3)) \subset V$ since $F(W_3) \subset F(W_1) \cup F(W_2)$. Therefore W_3 possesses all the requirements to belong to Q, but W_3 is not in Q since k is in $(W_3 \cap F(Q^+))$. Therefore the assumption that g(A) is not connected leads to a contradiction. Hence f is a connectivity map.

The explicit statement of Stallings' theorem [8] is as follows: If f is a local connectivity mapping of the locally peripherally connected polyhedron P into a regular Hausdorff space Y, then f is peripherally continuous. Since every connectivity map is a local connectivity map this theorem implies that under these conditions a connectivity map is also peripherally continuous. In view of this and Theorem 3.1, if S is a locally peripherally connected polyhedron having Property II, and T is a regular Hausdorff space such that $S \times T$ is completely normal, then local connectivity maps, connectivity maps, and peripherally continuous functions from S to T are equivalent. In particular, if f is a mapping of the n-cell I, n = 2, 3, ... , into itself, then there is no distinction among these functions, since for $n > 1$, the n-cell has all the required properties. In the case of a 1-cell these functions are no longer equivalent. In [4] Long has shown that a connectivity map of a 1-cell into itself is peripherally continuous and gives an example to show that the converse is not true. The following theorem will complete the theory of equivalence of the local connectivity map and the connectivity map of a n-cell, n = 1, 2, ... , into itself.

Theorem 3.2. If f is a local connectivity map of the closed unit interval I into itself, then f is a connectivity map.

Proof. Since f is a local connectivity map there is an open covering $\{U_\alpha\}$ of I such that f restricted to U_α is a connectivity map for each α. Since I is compact the covering $\{U_\alpha\}$ can be reduced to an irreducible number of intervals, $[a_1,b_1)$, (a_2,b_2), ... , $(a_n,b_n]$, where $a_i < a_{i+1} < b_i < b_{i+1}$, $a_1 = 0$, $b_n = 1$, and f is a connectivity map on each interval.

Let K be any connected subset of I. Then K has one the forms $[a,b]$, (a,b), $[a,b)$, or $(a,b]$. Without loss of generality one can assume that $a_1 \leqslant a < b_1$ and $a_n < b \leqslant b_n$. Only the case where $K = [a,b]$ will be considered since the other three cases are similar.

The set K can be written $K = K \cap [a_1,b_1) \cup K \cap (a_2,b_2) \ldots \cup K \cap (a_n,b_n]$ where $K \cap [a_1,b_1)$, $K \cap (a_2,b_2)$, ... , $K \cap (a_n,b_n]$ are all connected, and each set intersects the preceeding one and the succeeding one. Since f is a connectivity map on each interval the graph of each of the above sets is connected and each has a point in common with the succeeding one. Hence g(K) is connected and f is a connectivity map.

Example 5. This example is to show that a local connectivity map need not be either a connectivity map or a peripherally continuous function. Let A be the set of points (x,y) such that $0 < x < 1$ and $y = \sin(\frac{1}{x})$, $p = (0,1)$ and $q = (0,-1)$. Let $S = A \cup \{p \cup q\}$ and $T = A \cup \{p' \cup q'\}$ where $p' = (1,0)$ and $q' = (-1,0)$. Define the mapping f of S into T as follows. Let f be the identity mapping on A and let $f(p) = p'$ and $f(q) = q'$. Let H be the open rectangle given by $-1 < x < 1$, $-1 < y < 2$, and let K be the open rectangle given by $-1 < x < 1$, $-2 < y < 1$. Then the two sets $H' = H \cap S$ and $K' = K \cap S$ form an open covering of S. Also $q \notin H'$ and $p \notin K'$. Since f is continuous when restricted to any connected subset of either H' or K', then f is a connectivity map on H' and on K'. However, f is not a connectivity map on S since f(S) is not connected. Further, f is not peripherally continuous at either p or q since there is no neighborhood of p or of q whose boundary maps into a neighborhood of p' or of q', respectively.

Theorem 2.8, page 13, of this paper states that the limit function of a uniformly convergent sequence of peripherally continuous functions is peripherally continuous. If the space S satisfies the hypothesis of Theorem 3.1 and Stallings' theorem, then the limit of a uniformly convergent sequence of connectivity maps is a connectivity map.

CHAPTER IV

UPPER SEMI-CONTINUOUS DECOMPOSITIONS AND THE RELATED FACTORIZATION OF CONNECTIVITY MAPS AND PERIPHERALLY CONTINUOUS FUNCTIONS

This chapter is devoted to the study of the relationship between discontinuous functions and the decomposition of the domain space into upper semi-continuous collections. Related to this is the factorization of functions and the properties possessed by the factors. For continuous transformations Whyburn, in [9], Chapters VII and VIII, has a thorough exposition. The theorems in the present chapter will show that many of the classical results for continuous functions have their counterpart for some special discontinuous functions, namely the peripherally continuous functions, and the connectivity maps.

In order that the work have a semblance of completeness some definitions and theorems will now be quoted from Whyburn [9] and Moore [5]. In what follows continuity is not assumed for any function unless explicitly stated. The work in [9] assumes that all spaces are separable and metric. For this reason many of the theorems in this chapter assume this also.

Definition 4.1. A collection G of mutually exclusive closed point sets is said to be upper semi-continuous if and only if it is true that if g is a point set of the collection G and $\{g_n\}$ is a sequence of point sets of this collection and, for each n, a_n and b_n are points of g_n and the sequence $\{a_n\}$ has a sequential limit point lying in g, then every infinite subsequence of $\{b_n\}$ has a subsequence having a sequential limit point that lies in g [5, p. 273].

A useful characterization of upper semi-continuity is

Theorem 4.1. In a compact metric space S a necessary and sufficient condition that a collection G of closed sets be upper semi-continuous is that for any sequence $\{g_n\}$ of elements of G with $g \cap (\lim\inf g_n) \neq \emptyset$, where g is in G, then $\lim\sup g_n \subset g$ [9, p. 122].

Definition 4.2. A collection G of disjoint closed sets in a space S is said to be semi-closed if any convergent sequence of sets of G whose limit set intersects $S - G^+$ converges to a single point of $S - G^+$, where G^+ denotes the point-set union of all sets in G [9, p. 131].

Definition 4.3. In a metric space S a collection of sets is called a null collection if and only if for any $\epsilon > 0$, at most a finite number of its elements are of diameter $> \epsilon$ [9, p. 67].

Definition 4.4. A collection G of disjoint subsets of the space S will be called non-separated provided that no element of G separates in S two points belonging to any other single element of G [9, p. 42].

Definition 4.5. A non-separated collection G of subsets of the space S will be called saturated provided that if g is in G and p is any point of S - g, there exists at least one element g' of G which separates p and g in S [9, p. 45].

If f is a continuous function from a space S into a space T and C is a closed subset of T, the set $f^{-1}(C)$ is closed. For a connectivity map or a peripherally continuous function the components of $f^{-1}(C)$ are closed [3], [4]. The following two theorems and the resulting corollaries give some more information concerning $f^{-1}(C)$.

Theorem 4.2. Let f be a connectivity mapping of the compact, semi-locally connected space S into the T_1 space T. If C is a closed subset of T, then the components of $f^{-1}(C)$ form a semi-closed collection.

Proof. Since f is a connectivity map and C is closed, the components of $f^{-1}(C)$ are closed [3]. Let $\{M_n\}$ be a convergent sequence of components of $f^{-1}(C)$ with a non-empty limiting set M. Since S is compact $\overline{\bigcup M_n}$ is compact and hence M is connected [9, p. 15].

Suppose M is non-degenerate and that $M \cap (S - f^{-1}(C)) \neq \emptyset$. Let x be a point of M such that x is not in $f^{-1}(C)$ and let y be a point of M distinct from x. Then there is a sequence of points $\{y_n\}$ of M converging to y. Since S is semi-locally connected there is an open set U containing x such that y_n is not in U, n = 1, 2, ... , and S - U has a finite number of components, $K_1, \ldots, K_j$. Since there are only a finite number of the K_i, some K_i must intersect infinitely many M_n since y_n is in S - U for all n. Denote these by $M_{n'}$. Then $D = K_i \cup (\bigcup M_{n'}) \cup \{x\}$ is a connected subset of S and f a connectivity map implies that the graph g(D) is connected.

Now $U \cap K_i = \emptyset$ and $f(\bigcup M_{n'}) \cap (T - C) = \emptyset$ since $f(\bigcup M_{n'}) \subset C$. But x is in U and f(x) is in T - C since x is not in $f^{-1}(C)$. Thus $U \times (T - C)$ is an open set in $S \times T$ containing only the point g(x) of g(D). This contradicts g(D) being connected. Therefore either M is contained in $f^{-1}(C)$ or M is a single point. Thus the components of $f^{-1}(C)$ form a semi-closed collection.

Theorem 4.3. Let f be a peripherally continuous mapping of the compact, locally peripherally connected space S into the regular T_1 space T. If C is a closed subset of T, then the components of $f^{-1}(C)$ form a semi-closed collection.

Proof. The components of $f^{-1}(C)$ are closed by [4, p. 639]. Let $\{g_n\}$ be a convergent sequence of components of $f^{-1}(C)$ and let $\lim g_n = L$. Suppose $L \cap (S - f^{-1}(C)) \neq \emptyset$ and let $a \in L \cap (S - f^{-1}(C))$. Let b be any other point of L. If no such point exists, then $L = \{a\}$ and $f^{-1}(C)$ is semi-closed. Since $\{g_n\}$ is a sequence of connected sets and $\overline{\bigcup g_n}$ is compact, L is connected [9, p. 15]. Since L contains the two distinct points a and b, L is non-degenerate and hence $\{g_n\}$ has no null subsequence. Without loss of generality we can assume there is an $\epsilon > 0$ such that diameter $g_n \geq \epsilon$ for every n.

Let $\{U_n\}$ and $\{V_n\}$ be sequences of open sets closing down on a and f(a), respectively, such that diameter $U_n < \epsilon$, $F(U_n)$ is connected, and $f(F(U_n)) \subset V_n$, for every n. Since diameter $g_n \geq \epsilon$, diameter $U_n < \epsilon$, and $F(U_n)$ and g_n are connected, it follows that $F(U_n) \cap g_n \neq \emptyset$. Let a_n be a point of $F(U_n) \cap g_n$. Since the sequences $\{U_n\}$ and $\{V_n\}$ are closing down on a and f(a), respectively, $a_n \to a$ and $f(a_n) \to f(a)$. But $a_n \in g_n$ implies that $f(a_n) \in C$ and $a \in L \cap (S - f^{-1}(C))$ implies $f(a) \notin C$. Thus, f(a) is a limit point of C not in C contradicting that C is closed. Therefore either $L \subset f^{-1}(C)$ or L is a single point, and the components of $f^{-1}(C)$ form a semi-closed collection.

<u>Corollary 4.1</u>. Under the hypothesis of Theorem 4.2 or Theorem 4.3, the collection of components of $f^{-1}(C)$ form an upper semi-continuous collection.

<u>Proof</u>. The set $f^{-1}(C)$ is a semi-closed set by the two previous theorems. Theorem (5.2) of [9, p. 132] states that the collection of all components of any semi-closed set forms an upper semi-continuous collection in a compact space.

<u>Corollary 4.2</u>. Under the hypothesis of Theorem 4.2 or Theorem 4.3, the collection of non-degenerate components of $f^{-1}(C)$ forms an F_σ set.

<u>Proof</u>. The set $f^{-1}(C)$ is a semi-closed set by the two previous theorems. By Theorem (5.41) of [9, p. 132] the union of all non-degenerate components of a semi-closed set is an F_σ set, if S is compact.

Corollary 4.3. Under the hypothesis of Theorem 4.2 or Theorem 4.3, for every $\epsilon > 0$, the union of all components of $f^{-1}(C)$ which have diameter $\geqslant \epsilon$ is closed.

Proof. The set $f^{-1}(C)$ is semi-closed by the two previous theorems. By Theorem (5.4) of [9, p. 132], in a compact space a necessary and sufficient condition that a set be semi-closed is that for every $\epsilon > 0$ the union of all components of diameter $\geqslant \epsilon$ is a closed set.

For a continuous mapping f of a compact space S into a space T it is known [9, p. 142] that the collection of components of the sets $f^{-1}(y)$, where y is a point of T, forms an upper semi-continuous decomposition of S. The next three theorems give some information, in this respect, for connectivity maps and peripherally continuous functions.

Theorem 4.4. Let f be a connectivity mapping of the metric space S into the T_1 space T. If S is decomposed into the collection S' of all components of the sets $f^{-1}(y)$, where y is a point of T, and if S' is a null collection, then S' is upper semi-continuous.

Proof. Since T is a T_1 space, y a point of T implies y is a closed set, and f a connectivity map implies the components of $f^{-1}(y)$ are closed [3]. Therefore S' is a null collection of disjoint closed sets and is therefore upper semi-continuous [9, p. 134].

Theorem 4.5. Let f be a connectivity mapping of the compact metric space S into the T_1 space T. If S', the decomposition of S into the collection of all components of the sets $f^{-1}(y)$, where y is a point of T, is a saturated collection, then S' is upper semi-continuous.

Proof. The collection S' is non-separated since S' is composed of disjoint continua. Thus S' is a non-separated, saturated collection and is therefore upper semi-continuous [9, p. 128].

Theorems 4.4 and 4.5 remain valid if the hypothesis that f be a connectivity mapping be replaced by f being a peripherally continuous function since if C is a closed subset of T, then $f^{-1}(C)$ has closed components [4].

<u>Theorem 4.6</u>. Let f be a peripherally continuous mapping of the locally peripherally connected, compact metric space S into the regular T_1 space T. Let S' be the decomposition of S into the components of the sets $f^{-1}(y)$, where y is a point of T. Then S' is an upper semi-continuous decomposition of S.

<u>Proof</u>. The elements of S' are closed since y is closed in T and f is peripherally continuous [4, p. 639]. Since S is compact S' is a collection of disjoint compact continua filling up S. By Theorem 4.1, S' is upper semi-continuous if and only if $\{g_n\}$ is any sequence of sets from S' and $g \cap (\lim\inf g_n) \neq \emptyset$, then $\lim\sup g_n \subset g$, where g is in S'.

To this end let a be a point of $g \cap (\lim\inf g_n)$, let $L = \lim\sup g_n$ and let p be a point of L distinct from a. If no such point exists, then $L = \{a\}$ and hence $L \subset g$. Now there exists a convergent subsequence $\{g_{ni}\}$ of $\{g_n\}$ such that $\lim g_{ni} = K \subset L$ and p is in K [5, p. 24]. Then we have $\lim\inf g_n \subset \lim\inf g_{ni}$ and hence a is a point of K. Since the g_{ni} are connected and $\overline{\bigcup g_{ni}}$ is compact the set K is connected [9, p. 15]. Now $\{g_{ni}\}$ is not a null sequence since K is non-degenerate. Hence there is an $\epsilon > 0$ such that infinitely many g_{ni} have diameter $\geq \epsilon$. Without loss of generality assume all the g_{ni} have diameter $\geq \epsilon$. Let q be an arbitrary point of K. Let $\{U_n\}$ and $\{V_n\}$ be sequences of open sets closing down on q and f(q), respectively, such that diameter $U_n < \epsilon$, $F(U_n)$ is connected, and $f(F(U_n)) \subset V_n$, for every n. Let $\{W_n\}$ and $\{R_n\}$ be sequences of open sets closing down on a and f(a), respectively, such that diameter $W_n < \epsilon$, $F(W_n)$ is connected, and $f(F(W_n)) \subset R_n$, for every n.

Every open set containing a intersects all but a finite number of g_{ni}, and every open set containing q intersects all but a finite number of g_{ni}. Hence there is an integer k such that for every $i > k$, $F(U_{ni}) \cap g_{ni} \neq \emptyset$ and $F(W_{ni}) \cap g_{ni} \neq \emptyset$ since diameter $g_{ni} \geq \epsilon$, diameter U_{ni} is less than ϵ, diameter $W_{ni} < \epsilon$, and all the g_{ni} are connected. Let a_i be a point of $F(W_{ni}) \cap g_{ni}$ and let q_i be a point of $F(U_{ni}) \cap g_{ni}$. Then $a_i \to a$, $q_i \to q$, $f(a_i) \to f(a)$, and $f(q_i) \to f(q)$. But a_i and q_i belonging to g_{ni} imply $f(a_i) = f(q_i)$. Therefore $f(a) = f(q) = f(g)$. Since q was an arbitrary point of K we have $f(K) \subset f(g)$. Therefore $K \subset f^{-1}(f(g))$ and since K is connected and $K \cap g \neq \emptyset$, $K \subset g$. Now p was an arbitrary point of L which is in K. Thus p is in g and $L \subset g$. This shows that $\lim\sup g_n \subset g$ and hence S' is an upper semi-continuous decomposition of S.

Using the upper semi-continuous decomposition of the domain space a continuous function f can be factored into a composite $f = f_2 f_1$, where f_1 is a monotone continuous function, and f_2 is a light continuous function [9, p. 141]. The definition of monotone and light will now be given, and some analogous results for peripherally continuous functions and connectivity maps will be proved.

<u>Definition 4.6</u>. A mapping f of a space S into a space T is called monotone if $f^{-1}(y)$ is connected for every point y in T. The mapping is called light if $f^{-1}(y)$ is totally disconnected [9, p. 130].

In [9] this definition is given in terms of continuous mappings. Here continuity is not assumed.

<u>Theorem 4.7</u>. Let f be a peripherally continuous mapping of the locally peripherally connected, compact metric space S into the regular

T_1 space T. Then f can be factored into the composite $f = f_2 f_1$, where f_1 is a continuous monotone mapping of S onto the upper semi-continuous decomposition S' of S into the components of $f^{-1}(y)$, y a point of T, and f_2 is a light, peripherally continuous mapping from S' into T.

Proof. The existence of the upper semi-continuous decomposition S' of S is guaranteed by Theorem 4.6. As in the analogous theorem concerning continuous functions, the mapping f_1 of S onto S', defined by $f(x) = g$, where g is the element in S' containing x, is monotone and continuous [9, p. 127].

Define the mapping f_2 of S' into T by $f_2(g) = y$, where g is a component of $f^{-1}(y)$. Then f_2 is light since the elements of $f_2^{-1}(y)$ are the components of $f^{-1}(y)$ and these form a totally disconnected set in S'. For if H is any non-degenerate subcollection of $f_2^{-1}(y)$, then H is connected in S' if and only if H^+, the point-set union in S of sets in H, is connected [5, p. 275]. Now H being non-degenerate implies H^+ contains more than one component of $f^{-1}(y)$ and hence is not connected. Thus H is not connected and f_2 is a light mapping.

By definition of f_1 and f_2, $f = f_2 f_1$. Therefore it remains to show that f_2 is peripherally continuous.

With this in mind, let g be an element in S' and let U be a connected region in S' containing g, and V an open set in T containing $f_2(g)$. Then U^+, the point-set union in S of elements in U, is open in S [9, p. 123], and since g is compact there is an open set $W \subset U^+$ containing g such that if h is in S' and $h \cap W \neq \emptyset$, then $h \subset U^+$ [5, p. 277]. Since f is peripherally continuous, for every x in g there is an open set R_x containing x such that $\overline{R}_x \subset W$, $F(R_x)$ is connected and $f(F(R_x)) \subset V$. Now $\{R_x\}$, x in g, is an open covering of g and g compact implies there is a finite subcollection $R_1, \ldots, R_n$ covering g. Hence $R = \bigcup_{i=1}^{n} R_i$ is an open set containing g and contained in W, and $F(R) \subset \bigcup_{i=1}^{n} F(R_i)$ implies $f(F(R)) \subset V$.

Let H be the collection of all elements h in S' such that $h \subset R$. The collection $H \neq \emptyset$ since $g \subset R$. Also, $H \subset U$ and H is open in S' [5, p. 277]. Let B be the collection of all elements h in S' such that $h \cap F(R) \neq \emptyset$. Now if $B = \emptyset$, then $U^+ = A^+ \cup K^+$, where A is the collection of elements in U lying wholly in $U^+ - \overline{R}$ and K is the collection of elements in U lying wholly in R. But A^+ and K^+ are mutually separated. This means U^+ is not connected and hence U is not connected in S' [5, p. 275]. This contradiction implies $B \neq \emptyset$.

Now $f(F(R)) \subset V$ and $f(h) = f_2(h)$ is a single point in V for every h in B. Hence $f(B^+) = f_2(B)$ is contained in V. We need only show that $F(H) \subset B$ and then H will be the desired open set contained in U for the peripheral continuity of f_2. To this end let h be an element of F(H) and suppose $h \cap F(R) = \emptyset$. Then since h is not in H, we have $h \subset (S - \overline{R})$ which is open in S. Let M be the collection of all g in S' such that $g \subset (S - \overline{R})$. Then M is an open set in S' containing h, but containing no element of H. This contradicts h being a boundary element of H. Thus $F(H) \subset B$ and the proof is completed.

If it is assumed that the collection S' of components of $f^{-1}(y)$, y a point of T, gives an upper semi-continuous decomposition of S, then the next theorem gives an analogous factorization of a connectivity mapping from the space S into the space T. The following lemma is needed in the proof of this theorem.

<u>Lemma 4.1.</u> If G is an upper semi-continuous decomposition of S, then $G \times T$ is an upper semi-continuous decomposition of $S \times T$.

Proof. Since $G^+ = S$, $(G \times T)^+ = S \times T$. Let $\{g_n \times t_n\}$ be a sequence of elements of $G \times T$ and let $s_n \times t_n$ be a point in $g_n \times t_n$ such that $s_n \times t_n \to s \times t$, where $s \times t$ is in $g \times t$ and $g \times t$ is an element of $G \times T$. Then $s_n \to s$ where s is in g, and $t_n \to t$. Let $\{a_n \times t_n\}$ be any other sequence such that $a_n \times t_n$ is in $g_n \times t_n$ and let $\{a_{ni} \times t_{ni}\}$ be any subsequence of $\{a_n \times t_n\}$. Since G is upper semi-continuous there is a subsequence $\{a_{ni'}\}$ of $\{a_{ni}\}$ such that $a_{ni'} \to a$, where a is in g. Hence $a_{ni'} \times t_{ni'} \to a \times t$, where $a \times t$ is in $g \times t$, and $G \times T$ is upper semi-continuous.

Theorem 4.8. Let f be a connectivity map from the compact metric space S into the T_1 space T. If the decomposition S' of S into the collection of all components of the sets $f^{-1}(y)$, where y is a point of T, is upper semi-continuous, then f can be factored into the composite $f = f_2 f_1$, where f_1 is a monotone continuous mapping of S onto S', and f_2 is a light, connectivity mapping from S' into T.

Proof. Just as in Theorem 4.7, define a mapping f_1 of S onto S' by $f_1(x) = h$, where h is the element of S' containing x. By hypothesis S' is upper semi-continuous, and hence f_1 is monotone and continuous [9, p. 127].

Again, as in Theorem 4.7, define the mapping f_2 of S' into T by $f_2(h) = y$, where h is a component of $f^{-1}(y)$. Then f_2 is light since $f_2^{-1}(y)$ is totally disconnected in S', and by definition of f_1 and f_2, $f = f_2 f_1$. It remains to show that f_2 is a connectivity map.

To prove this let g and g_2 be the graph maps of f and f_2, respectively, and let H be a connected set in S'. Then $g(H^+) = [g_2(H)]^+$. For if x is a point in $g(H^+)$, then $x = p \times f(p)$, where p is in H^+. Let h be the element in H containing p. Then $h \times f_2(h)$ is a point in $g_2(H)$.

Hence $[h \times f_2(h)]^+$ is contained in $[g_2(H)]^+$, and since p is in h and $f(p) = f_2(h)$, then $p \times f(p)$ is in $[h \times f_2(h)]^+$. Therefore $p \times f(p)$ is in $[g_2(H)]^+$ and $g(H^+)$ is contained in $[g_2(H)]^+$.

Conversely, let x be a point in $[g_2(H)]^+$. Then there is an $h \times f_2(h)$ such that x is in $[h \times f_2(h)]^+$, and therefore there is a point p in h such that $x = p \times f(p)$ since $f_2(h) = f(p)$. Thus x is in $g(H^+)$ and hence $[g_2(H)]^+$ is contained in $g(H^+)$. Therefore $g(H^+) = [g_2(H)]^+$.

Now S' being upper semi-continuous implies that H is connected if and only if H^+ is connected [5, p. 275]. Since f is a connectivity map, $g(H^+)$ is connected and therefore $[g_2(H)]^+$ is connected. Then $S' \times T$ being upper semi-continuous by Lemma 4.1 implies $g_2(H)$ is connected. Thus f_2 is a connectivity map and the theorem is proved.

<u>Theorem 4.9</u>. Let f be a connectivity mapping of the compact, semi-locally connected metric space S into the T_1 space T. Let y be a point of T and S' the decomposition of S into the components of $f^{-1}(y)$ and the individual points of $S - f^{-1}(y)$. Then f can be factored into the composite $f = f_2 f_1$, where f_1 is a continuous mapping of S onto S' which is monotone on S and one-to-one on $S - f^{-1}(y)$, and f_2 is a connectivity map from S' into T which is light on the set of components of $f^{-1}(y)$.

<u>Proof</u>. Since T is a T_1 space, the point y is a closed subset of T and Theorem 4.2 and Corollary 4.1 imply that the components of $f^{-1}(y)$ form a semi-closed, upper semi-continuous collection. By Theorem (5.1) page 131 of [9], the collection S', consisting of the components of $f^{-1}(y)$ and the individual points of $S - f^{-1}(y)$, is an upper semi-continuous decomposition of S into disjoint continua.

As in Theorem 4.7, define the mapping f_1 of S onto S' by $f(x) = g$, where g is the element in S' containing x. By [9, p. 127] f_1 is monotone and continuous. Define the mapping f_2 of S' into T by $f_2(g) = z$, where $z = f(g)$. The mapping f_1 is one-to-one on $S - f^{-1}(y)$ by definition, and f_2 is light on the set of components of $f^{-1}(y)$ since $f_2^{-1}(y) = f^{-1}(y)$ which is totally disconnected in S'. By definition of f_1 and f_2, $f = f_2 f_1$. It remains to show that f_2 is a connectivity map. The proof of this is identical to the proof of the corresponding fact in Theorem 4.8.

Theorem 4.10. Let f be a peripherally continuous mapping of the locally peripherally connected, compact metric space S into the regular T_1 space T. Let y be a point of T and S' the decomposition of S into the components of $f^{-1}(y)$ and the individual points of $S - f^{-1}(y)$. Then f can be factored into the composite $f = f_2 f_1$, where f_1 is a continuous mapping of S into S' which is monotone on S and one-to-one on $S - f^{-1}(y)$, and f_2 is a peripherally continuous mapping of S' into T which is light on the set of components of $f^{-1}(y)$.

Proof. Since T is a T_1 space, the point y is a closed subset of T and Theorem 4.3 and Corollary 4.1 implies that the components of $f^{-1}(y)$ form a semi-closed, upper semi-continuous collection. By Theorem (5.1) page 131 of [9], the collection S', consisting of the components of $f^{-1}(y)$ and the individual points of $S - f^{-1}(y)$, is an upper semi-continuous decomposition of S into disjoint continua.

As in Theorem 4.7, define the mapping f_1 of S onto S' by $f(x) = g$, where g is the element in S' containing x. By [9, p. 127] f_1 is monotone and continuous. Define the mapping f_2 of S' into T by $f_2(g) = z$, where $z = f(g)$. The mapping f_1 is one-to-one on $S - f^{-1}(y)$ by definition, and f_2 is light on the sets of components of $f^{-1}(y)$ since

$f_2^{-1}(y) = f^{-1}(y)$ which is totally disconnected in S'. By definition of f_1 and f_2, $f = f_2 f_1$. It remains to show that f_2 is peripherally continuous. The proof of this is identical to the proof of the corresponding fact in Theorem 4.7.

For continuous mappings there is an "ϵ - δ" characterization of the property of being light [9, p. 131]. That this is also true for peripherally continuous transformations is the content of the next theorem.

Theorem 4.11. Let f be a peripherally continuous mapping of the locally peripherally connected, compact metric space S onto the compact metric space T. A necessary and sufficient condition that f be light is that for every $\epsilon > 0$ there is a $\delta > 0$ such that if A is any continuum in T such that diameter $A < \delta$, then any component of $f^{-1}(A)$ has diameter less than ϵ.

Proof. Assume that f is a light mapping and suppose there is an $\epsilon > 0$ such that for every $\delta > 0$ there is a continuum A_δ in T such that diameter $A_\delta < \delta$ but there is a component C_δ of $f^{-1}(A_\delta)$ such that diameter $C_\delta \geq \epsilon$. Choose a sequence of positive real numbers $\{\delta_n\}$ converging to zero. Then for each δ_n there is a continuum $A_n \subset T$, and a component C_n of $f^{-1}(A_n)$ such that diameter $A_n < \delta_n$ and diameter $C_n \geq \epsilon$. Since S and T are compact the sequence $\{A_n\}$ and $\{C_n\}$ can be chosen to be convergent.

Since diameter $A_n < \delta_n$ and $\delta_n \to 0$, diameter $A_n \to 0$ and hence $\lim A_n = \{p\}$, a single point in T. Now diameter $C_n \geq \epsilon$ for every n implies that $\{C_n\}$ is a sequence of non-degenerate connected sets and S compact implies $\lim C_n = C$ is a non-degenerate continuum [5, p. 23].

Let x be a point of C and let $\{U_n\}$ be a sequence of open sets closing down on x such that diameter $U_n < \epsilon$, and $\{V_n\}$ a sequence of open sets closing down on f(x) such that $F(U_n)$ is connected and $f(F(U_n)) \subset V_n$. Then since U_n intersects infinitely many C_k, $F(U_n)$ intersects infinitely many C_k. Let x_n be a point in $F(U_n) \cap C_k$. Then $f(x_n)$ is in V_n and we have $x_n \to x$ and $f(x_n) \to f(x)$. But x_n a point of C_k implies that $f(x_n)$ is in $f(A_k)$ and $\lim A_k = \{p\}$. Therefore $f(x_n) \to p$. Hence $f(x) = p$ and x is a point of $f^{-1}(p)$. Since x was an arbitrary point of C we have $C \subset f^{-1}(p)$. But $f^{-1}(p)$ is totally disconnected since f is light. This involves a contradiction since C is non-degenerate. Therefore the condition holds.

If the condition is assumed to hold, then f is light regardless of whether or not f is peripherally continuous. For let p be a point of T and suppose there is a non-degenerate component C of $f^{-1}(p)$. If $0 < \epsilon <$ diameter C, then there is a $\delta > 0$ such that if A is any continuum containing p with diameter $< \delta$, then every component of $f^{-1}(A)$ has diameter $< \epsilon$. But C is contained in $f^{-1}(A)$ and hence is contained in some component C'. Then diameter $C' < \epsilon$ on the one hand and $\epsilon <$ diameter $C <$ diameter C' on the other, which is a contradiction and thus $f^{-1}(p)$ is totally disconnected [9, p. 131].

The following theorem gives a necessary and sufficient condition for a monotone peripherally continuous function to be open. This result is then used to obtain a sufficient condition for continuity of a peripherally continuous function.

Theorem 4.12. Let f be a monotone peripherally continuous mapping of the locally peripherally connected, compact metric space S onto the regular T_1 space T. Then f is an open mapping if and only if for every sequence $\{y_n\}$ of points of T with sequential limit point y ,

$\lim f^{-1}(y_n) = f^{-1}(y)$.

Proof. Suppose f is an open mapping and let $\{y_n\}$ be a sequence of points of T with sequential limit point y. Let $g_n = f^{-1}(y_n)$ and $g = f^{-1}(y)$. Since f is monotone and peripherally continuous and S is compact g_n and g are compact continua. Let x be a point of g and suppose there is an open set U containing x such that $U \cap g_n = \emptyset$ for infinitely many n. Then, since f is open, f(U) is an open set in T containing y such that $f(U) \cap y_n = \emptyset$ for infinitely many n. This contradicts the hypothesis that the sequence $\{y_n\}$ converges to y. Therefore every open set containing x intersects all but a finite number of g_n, and thus x is in $\lim\inf g_n$. Hence $g \subset \lim\inf g_n$. Since S', the collection $f^{-1}(y)$, y a point of T, is upper semi-continuous, $\lim\sup g_n \subset g$. Therefore we have $g \subset \lim\inf g_n \subset \lim\sup g_n \subset g$, which implies that $\lim g_n = g$. In other words, $\lim f^{-1}(y_n) = f^{-1}(y)$.

Conversely, suppose U is an open set in S such that f(U) is not open. Then there is a point y in f(U) and a sequence $\{y_n\}$ of points of T - f(U) such that $y_n \to y$. By hypothesis $\lim f^{-1}(y_n) = f^{-1}(y)$. Now $U \cap f^{-1}(y_n) = \emptyset$ for every n since $y_n \notin f(U)$. But $U \cap g \neq \emptyset$ since $y = f(g)$ and $y \in f(U)$, where $g = f^{-1}(y)$. Hence U intersects all but a finite number of $f^{-1}(y_n)$. This contradiction implies f(U) is open and therefore f is an open mapping.

Theorem 4.13. Let f be a peripherally continuous mapping of the locally peripherally connected, compact metric space S onto the compact, regular T_1 space T. If f is an open monotone mapping, then f is continuous.

Proof. Let $\{x_n\}$ be a sequence of points in S with sequential limit point x. Let $y_n = f(x_n)$. Since T is compact, some subsequence

$\{y_{ni}\}$ of $\{y_n\}$ has a sequential limit point y in T. By Theorem 4.12, $\lim f^{-1}(y_{ni}) = f^{-1}(y)$. Since $x_n \to x$, $x_{ni} \to x$ and $x_{ni} \in f^{-1}(y_{ni})$. Therefore $x \in f^{-1}(y)$ and $y = f(x)$. Since every sequence $\{x_n\}$ converging to x has a subsequence $\{x_{ni}\}$ converging to x such that $f(x_{ni}) \to f(x)$, f is continuous at x.

Theorem 4.13 implies, in particular, that if f is a one-to-one, open, peripherally continuous transformation of an n-cell onto itself, $n \geqslant 2$, then f is continuous. This is also true for a connectivity map since the two mappings are equivalent on n-cells, $n \geqslant 2$. In [4] Long has shown that a one-to-one connectivity map of a 1-cell into itself is continuous and gives an example to show that this is not necessarily true for peripherally continuous functions.

The next theorem gives a sufficient condition on the upper semi-continuous decomposition space S' to imply continuity of a peripherally continuous function.

Theorem 4.14. Let f be a peripherally continuous mapping of the locally peripherally connected, compact metric space S into the regular T_1 space T. If S', the upper semi-continuous decomposition of S into the components of sets $f^{-1}(y)$, y a point of T, has no null subcollection, then f is continuous.

Proof. Let $\{x_n\}$ be a sequence of points of S with the sequential limit point x. Suppose infinitely many x_n are in the same element h in S'. Since h is compact, some subsequence $\{x_{ni}\}$ of $\{x_n\}$ has a sequential limit point in h which must be x since $x_n \to x$. Hence $x_{ni} \to x$ and $f(x_{ni}) = f(x)$ implies that $f(x_{ni}) \to f(x)$.

Now suppose only a finite number of x_n are in any one element in S'. Without loss of generality we can assume that each x_n is in a distinct h_n.

Let h be the element in S' containing x. Then $h \cap (\lim\inf h_n) \neq \emptyset$ and S' upper semi-continuous implies $\lim\sup h_n \subset h$. Let $L = \lim\sup h_n$. Then $x \in L$, and by Theorem 58 page 23 of [5], L is a continuum. Since S is compact, L is compact, and by [5, p. 24] some subsequence $\{h_{ni}\}$ of $\{h_n\}$ is convergent. Let $K = \lim h_{ni}$. Then $\lim\inf h_n \subset \lim\inf h_{ni}$ implies $x \in K$. The set K is non-degenerate since S' has no null subcollection. Let ϵ = diameter K. Then there is an ϵ_1 such that $0 < \epsilon_1 < \epsilon$ such that infinitely many h_{ni} have diameter $\geqslant \epsilon_1$. Let $\{U_n\}$ and $\{V_n\}$ be sequences of open sets closing down on x and f(x), respectively, such that diameter $U_n < \epsilon_1$, $F(U_n)$ is connected, and $f(F(U_n)) \subset V_n$. Then since diameter $U_n < \epsilon_1$, $F(U_n)$ intersects infinitely many h_{ni}. Let $a_n \in F(U_n) \cap h_{ni}$. Then $f(a_n) \in V_n$ and $f(a_n) \to f(x)$ since $\{V_n\}$ closes down on f(x). But $f(a_n) = f(x_{ni})$ and thus $f(x_{ni}) \to f(x)$.

The above argument shows that for every sequence $\{x_n\}$ such that $x_n \to x$ there is a subsequence $\{x_{ni}\}$ of $\{x_n\}$ such that $f(x_{ni}) \to f(x)$. This implies f is continuous at x.

As Example 3 of Chapter II indicates, one cannot expect the inverse image of a connected set to be connected under a connectivity map or peripherally continuous function. However, the next theorem gives a condition which will force a peripherally continuous function to have this property. This result is then used to obtain a sufficient condition for continuity of a peripherally continuous function.

<u>Theorem 4.15</u>. Let f be an open, monotone, peripherally continuous mapping of the locally peripherally connected, compact metric space S onto the regular T_1 space T. If K is a connected subset of T, then $f^{-1}(K)$ is a connected subset of S.

<u>Proof</u>. Suppose $f^{-1}(K) = M \cup N$, where M and N are mutually

separated. Then $K = f(M) \cup f(N)$. Suppose there is a point y in $f(M) \cap f(N)$. Then there is a point x_1 in M such that $f(x_1) = y$ and a point x_2 in N such that $f(x_2) = y$. Hence $f^{-1}(y) \cap M \neq \emptyset$ and $f^{-1}(y) \cap N \neq \emptyset$. This is a contradiction since $f^{-1}(y)$ is connected and M and N are mutually separated. Therefore $f(M) \cap f(N) = \emptyset$. Since K is connected one of f(M) and f(N) must contain a limit point of the other. Let $y \in f(M)$ such that y is a limit point of f(N). Then there is a sequence $\{y_n\}$ of points of f(N) such that $y_n \to y$. By Theorem 4.12, $\lim f^{-1}(y_n) = f^{-1}(y)$. This is a contradiction since $f^{-1}(y_n) \subset N$ for every n, $f^{-1}(y) \subset M$, and M and N are mutually separated. The assumption that $f^{-1}(K)$ is not connected leads to a contradiction and hence $f^{-1}(K)$ is connected.

Theorem 4.16. Let f be an open, monotone, peripherally continuous mapping of the locally peripherally connected, compact metric space S onto the semi-locally connected, regular T_1 space T. Then f is continuous.

Proof. Suppose f is not continuous. Then there is a point x in S and a sequence $\{x_n\}$ of points of S such that $x_n \to x$ but $f(x_n) \not\to f(x)$. Since T is semi-locally connected there is an open set U containing f(x) such that T - U has a finite number of components, $K_1, \ldots, K_n$, and infinitely many $f(x_n)$ are in T - U. Hence infinitely many $f(x_n)$ are in some K_i. By Theorem 4.15, $f^{-1}(K_i)$ is connected, and x is a limit point of $f^{-1}(K_i)$ since infinitely many x_n are in $f^{-1}(K_i)$ and $x_n \to x$.

Since $H = f^{-1}(K_i) \cup \{x\}$ is non-degenerate, ϵ = diameter $H > 0$. Let $\{U_n\}$ and $\{V_n\}$ be sequences of open sets closing down on x and f(x), respectively, such that diameter $U_n < \epsilon$, $V_n \subset U$, $F(U_n)$ is connected, and $f(F(U_n)) \subset V_n$. Since diameter $U_n < \epsilon$ and $F(U_n)$ and

$f^{-1}(K_i)$ are connected, $F(U_n) \cap f^{-1}(K_i) \neq \emptyset$. Let a_n be a point in $F(U_n) \cap f^{-1}(K_i)$. Then $a_n \to x$ since $\{U_n\}$ closes down on x, and $f(a_n) \to f(x)$ since $f(a_n)$ is in V_n and $\{V_n\}$ closes down on f(x). This implies that f(x) is a limit point of K_i. This is a contradiction since $f(x) \in U$ and $f(a_n) \notin U$ for every n. Therefore f must be continuous.

Corollary 4.4. If in Theorem 4.16 the hypothesis that T be semi-locally connected is replaced by T being locally connected and locally compact, then f is continuous.

Proof. Every locally connected, locally compact space is semi-locally connected [9, p. 20]. Hence Theorem 4.16 implies f is continuous.

Example 6. Theorems 4.12, 4.15, and 4.16 are concerned with open, monotone, peripherally continuous transformations. The following example, due to Hamilton [3], shows that an open, monotone, peripherally continuous mapping of a closed 2-cell into itself need not be continuous. The mapping is defined as follows. Let I be the closed circular 2-cell of radius 1 and center at the origin. Let $(r,\emptyset)$ be the polar coordinates of points in I, and let C be a topological ray with endpoint at (0,0) which has each point of F(I), the boundary of I, as a limit point, but whose intersection with a circle of radius r, $0 \leqslant r < 1$, consists of one and only one point. For each point $(r,\emptyset)$ of I, $0 < r \leqslant 1$, let $f(r,\emptyset)$ be the point of C whose distance from the origin is 1 - r. Let f(0,0) be a particular point O' of F(I). Hamilton has shown that f is a connectivity map and hence is peripherally continuous. The mapping is monotone since the inverse image of any point on the topological ray C is a circle and $f^{-1}(O') = (0,0)$. The mapping is open since the image of any open disc is an open segment or an open subray of the ray C if it does not contain (0,0). If it contains (0,0), then its image is a subray of C plus the point O' which is open in $C \cup \{O'\}$. This mapping has one discontinuity, namely the origin. Note that the image $C \cup \{O'\}$ is neither compact nor semi-locally connected. If it were compact, then Theorem 4.13 would imply continuity, and if it were semi-locally connected Theorem 4.16 would imply continuity.

Example 7. The following is an example of a peripherally continuous transformation of a 2-cell onto itself, and hence is also a connectivity map. The mapping is described as follows. Let I be the closed circular 2-cell with center at (0,0) and radius 1. Let (r,ϕ) be the polar coordinates of points of I. Let $f(r,\phi) = (r',\phi')$ where $r' = -2^{2n+2}r + 3$ if $\frac{1}{2^{2n+1}} \leqslant r \leqslant \frac{1}{2^{2n}}$, $n = 0, 1, \ldots$, $r' = 2^{2n+1}r - 3$ if $\frac{1}{2^{2n}} \leqslant r \leqslant \frac{1}{2^{2n-1}}$, $n = 1, 2, \ldots$, and $r' = 0$ if $r = 0$. Let $\phi' = \phi + r'2\pi$. The function r' maps the interval $0 \leqslant r \leqslant 1$ onto the interval $-1 \leqslant r' \leqslant 1$ infinitely many times. The graph of r' would be as follows.

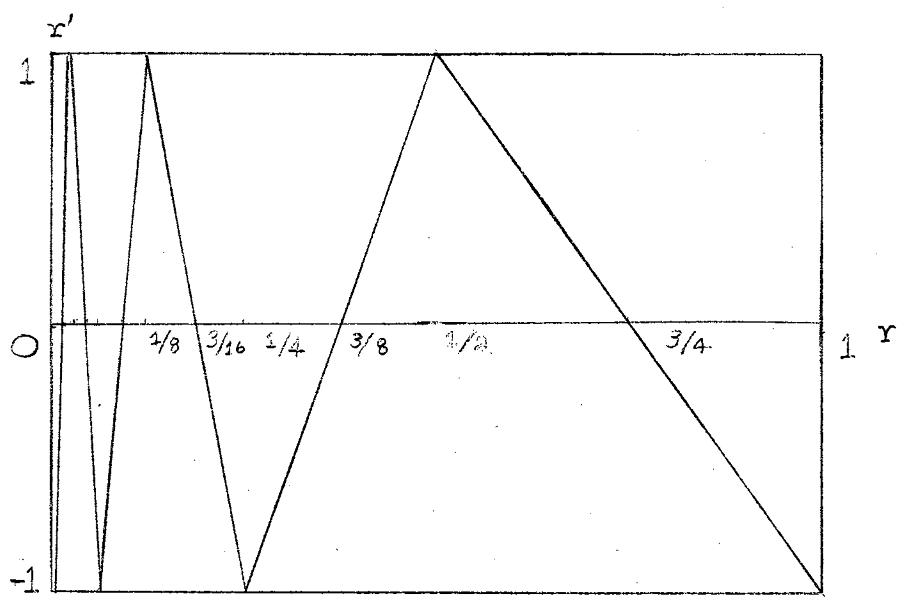

Figure 1

The function $\phi' = \phi + r'2\pi$ gives a twisting effect. The image of $0 \leqslant r \leqslant 1$ and $\phi = 0$ would be as follows.

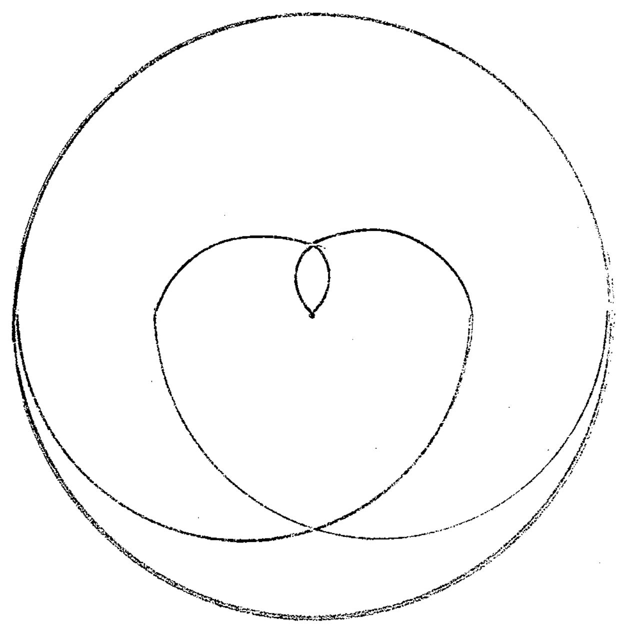

Figure 2

The image of $0 \leqslant r \leqslant 1$, $\phi = \phi_o$ would be the same as that in Figure 2 rotated through the angle ϕ_o.

From the nature of the function r' the set given by $\frac{1}{2^{2n+1}} \leqslant r \leqslant \frac{1}{2^{2n}}$, ϕ fixed, n = 0, 1, ..., and $\frac{1}{2^{2n}} \leqslant r \leqslant \frac{1}{2^{2n-1}}$, ϕ fixed, n = 1, 2, ..., have the same image for the same value of ϕ. For $\phi = 0$ the image would be as in Figure 2. The image for any other value of ϕ would be that of Figure 2 rotated through the angle ϕ.

The mapping f is continuous everywhere except at (0,0). Considering the function r' and Figure 1, it can be seen that all circles with center at (0,0) with radius $r = 3/2^n$, n = 1, 2, ..., map onto (0,0). Thus if U and V are open sets containing (0,0) and f(0,0), respectively, there exists a positive integer n such that the circle C_n of radius

$r = 3/2^n$ with center at (0,0) is contained in U and $f(C_n) = (0,0) \in V$. Therefore f is peripherally continuous at (0,0).

The function f is not monotone due to the fact that r' is not a monotone function of r. Further, f is not an open mapping. For, referring to Figure 3, the image f(A) of the region A bounded by $r = 11/16$ and $r = 13/16$, $\phi = 0$ and $\phi = \phi_0$, is the shaded shell, and the image (0,0) of the point $p = (3/4, \phi_0/2)$ is not an interior point of f(A) while p is an interior point of A. Thus f does not necessarily map open sets onto open sets. The interior of circles with center at (0,0) map onto all of I. All circles with center at (0,0) with radius $r = 1/2^n$, n = 0, 1, ..., are mapped onto F(I).

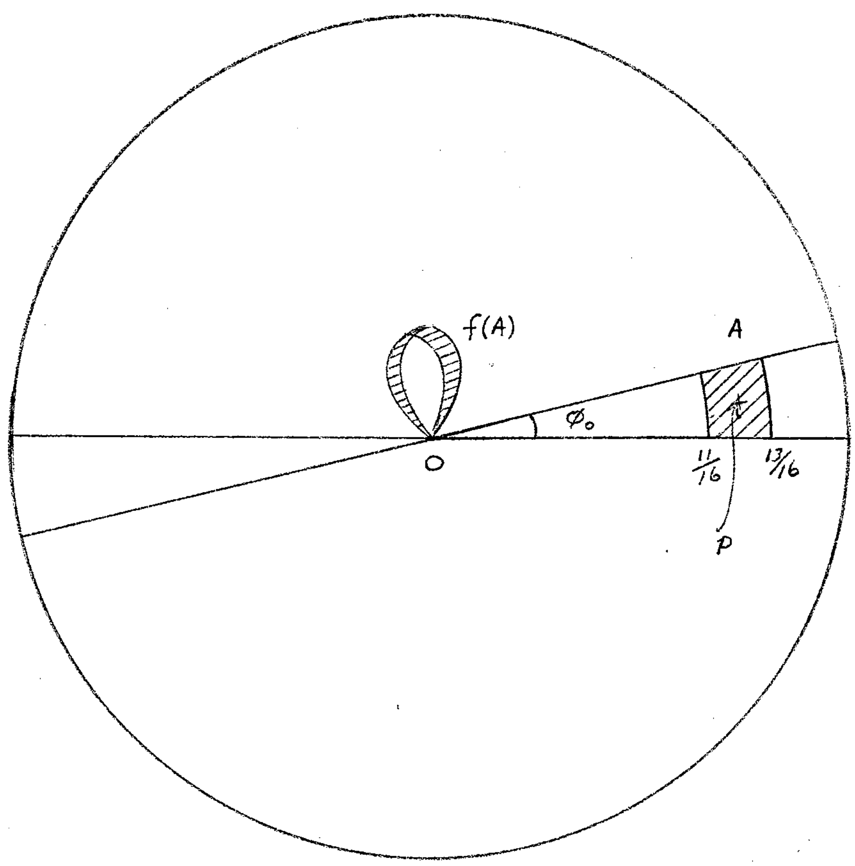

Figure 3

CHAPTER V

SUMMARY

This paper is primarily concerned with two types of non-continuous functions, namely connectivity mappings, and peripherally continuous transformations.

Focusing attention on the graph of a function, it is found that a connectivity map will be continuous if its graph is semi-locally connected. As a corollary to this, a connectivity map will be continuous if its graph is locally compact and locally connected. Sufficient conditions on a finite-to-one, onto mapping f, in order that it be continuous, is that the inverse image $f^{-1}(H)$ of a closed set H has closed components, and f be quasi-monotone in the sense that if H is connected, every component of $f^{-1}(H)$ maps onto all of H.

Sufficient conditions that a peripherally continuous function be a connectivity map is that the graph of a connected set either has a finite number of components, or has no degenerate components. A connectivity mapping f of S into T preserves boundary points in the sense that if V is open in T and K is a non-degenerate component of $f^{-1}(V)$, then any point p in $\overline{K}$, such that p is not in K, has the property that f(p) is in F(V). In this same setting it is found that $f^{-1}(V)$ is dense-in-itself. As a final result in Chapter II, the limit of a uniformly convergent sequence of peripherally continuous functions is peripherally continuous.

One of the main results of the paper is contained in Chapter III.

On a locally peripherally connected polyhedron having Brouwer Property II such that $S \times T$ is completely normal and T is a regular Hausdorff space, connectivity maps, local connectivity maps, and peripherally continuous functions from S into T are indistinguishable. This implies, in particular, that there is no distinction among these functions on an n-cell into itself, n = 2, 3, ... An example is given to show that in general a local connectivity map is not necessarily a connectivity map.

In Chapter IV some classical results for continuous functions are extended to peripherally continuous functions, and connectivity maps. One of the more interesting results being that a peripherally continuous function f on a locally peripherally connected, compact metric space induces an upper semi-continuous decomposition of the domain space, which in turn allows a factorization $f = f_2 f_1$, where f_1 is a monotone, continuous function and f_2 is a light, peripherally continuous function. This is in complete analogy with the continuous case. With the hypothesis that a connectivity map f induces an upper semi-continuous decomposition of the domain space, f can also be factored $f = f_2 f_1$, where in this case f_2 is a light, connectivity map.

An "ϵ - δ" characterization of a peripherally continuous function f, on a locally peripherally connected, compact metric space S into a compact metric space T, being light is that for every $\epsilon > 0$ there is a $\delta > 0$ such that if A is any subcontinuum of T such that diameter $A < \delta$, then any component of $f^{-1}(A)$ has diameter less than ϵ.

A necessary and sufficient condition that a monotone, peripherally continuous transformation f of a locally peripherally connected, compact metric space S onto a regular T_1 space T be open is that if $y_n \to y$ in T, then $\lim f^{-1}(y_n) = f^{-1}(y)$. This condition is then used to show that if T is compact and f is open and monotone, then f is continuous. An example is given to show that an open, monotone, peripherally continuous function is not necessarily continuous.

Another sufficient condition for continuity of a peripherally continuous function f is that the upper semi-continuous decomposition S' of the domain S into the components of $f^{-1}(y)$, where y varies over the range T, is that S' contain no null subcollection.

An example is given to show that in general the inverse image of a connected set is not necessarily connected under a peripherally continuous function. However, if the space is a locally peripherally connected, compact metric space and the function is open and monotone, then it possesses this property. Under these conditions a sufficient condition that the function be continuous is that the range be semi-locally connected. As a particular application, every monotone, open, peripherally continuous transformation of an n-cell, $n \geqslant 2$, onto itself must be continuous.

Some questions for further study might include the following. Is it possible for a peripherally continuous transformation of an n-cell I, $n \geqslant 2$, into I to be discontinuous at every point of a dense subset? Can it be discontinuous at every point of an open set? Related to this is a question posed by Long [4]. Is the set of points of discontinuity of a peripherally continuous function or connectivity map a set of the first category? It would be both interesting and useful to have an example of a peripherally continuous transformation of an n-cell, $n \geqslant 2$, into itself which has an infinite number of points of discontinuity. To the writer's knowledge no such example has been given.

Further, under what conditions will f^{-1} be a connectivity map if f is a one-to-one connectivity map? Also, can any of the results for peripherally continuous transformations be extended to the more general C - mapping? Finally, the proofs of theorems in Chapter IV rely on the fact that the domain space is locally peripherally connected. Can a method be devised so that this hypothesis can be dropped or modified?

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