SOLUTIONS TO THE TRANSIENT HEAT CONDUCTION EQUATION WITH VARIABLE THERMAL CONDUCTIVITY

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PREFACE

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CHAPTER I

INTRODUCTION

For the past twenty years, a large portion of engineering emphasis has been placed on the problems associated with the development of suitable vehicles for the exploration of space. Since immediate solutions to these problems are desired, the devices conceived must be constructed of existing materials whose thermal properties are generally dependent upon temperature. In such applications, these materials are subjected to extreme temperature conditions. For example, consider the widely varying boundary temperatures imposed upon insulating materials around rocket nozzles and upon heat shields of re-entry vehicles. To obtain the optimum engineering design, an accurate prediction of the temperature distribution and energy transfer rates within these materials must be made.

With few exceptions, the present analysis of such problems makes use of previously derived solutions based upon constant thermal properties. The constant conductivity used in these solutions represents the average of the thermal conductivity of the material in question over the range of temperatures encountered. Although this technique has been found to be sufficiently accurate for small temperature differences, the results obtained for large differences leave much to be desired. The few specific solutions now available which do account to some degree for the variation of material properties with temperature are so complex in form that their contribution is often slight when compared to the energy expended in attempting to make them applicable. Hence, there does exist a real need for a simple, yet accurate, method of analysis for these problems.

The objective of this study was to develop a general, straightforward, analytical procedure for accurately describing the transient temperature distribution within materials whose boundaries are subjected to known temperatures, either varying or constant. These materials have been assumed to have thermal conductivities which vary linearly with temperature while their volumetric specific heats are taken to be constant. Such property variations are characteristic of most engineering materials which are presently available.

Although the major part of this study was directed toward problems involving one spatial, cartesian dimension, a two-dimensional analysis is also included. The method used can quite easily be extended to problems of three dimensions. Both the one- and two-dimensional solutions have been programmed for use on an IBM 1620 digital computer.

For the specific class of one-dimensional problems described by the conditions of uniform initial temperatures with constant and equal step function boundary temperatures, charts are presented in dimensionless form which enable an immediate evaluation to be made of the transient temperature at any spatial position.

CHAPTER II

REVIEW OF PREVIOUS INVESTIGATIONS

The engineer concerned with the problems of energy transfer by conduction is well aware of the complications involved in attempting to apply even the solutions which are based on constant thermal properties to a specific problem. Such solutions normally have the form of an infinite series and are, therefore, quite difficult to employ for accurate results. The infinite series solution presented by Chen (4)* for a one-dimensional problem with an arbitrary heating rate is a typical example. In addition, the present interest in space exploration has required that problems of variable thermal properties be treated. For such problems, the governing equation is non-linear and the familiar mathematical methods fail to yield exact solutions of even the series type. For these reasons, two general paths for investigating heat conduction problems have been followed.

The first is well represented by the analysis of certain constant thermal property solutions by Erdogan (8). The objective of his study was to develop sufficiently accurate solutions which were easier to apply than the exact solution. Although the solutions obtained were indeed simplified, they were in the form of finite series which are still difficult to employ. This simply illustrates that a continuation of such an

* Numbers in parentheses refer to references in the Bibliography.

investigation would be of engineering value. The other research path was directed toward the development of simple accurate solutions of the problems in which the variations of the thermal properties were important and had to be considered. Since this study is concerned with such a development, no further mention will be made of the available solutions for the problems of constant thermal properties.

A general discussion of the various techniques which have been used to approach the variable property problems will be presented accompanied by a brief outline of the specific problems which have been treated. As very little work was done along this line until 1957, primary concern will be given to the techniques which have been employed since that time. The research accomplished previous to 1957 is described quite well in Friedmann's dissertation (9) and paper (10). Also, emphasis is placed on the techniques which have been applied to problems of finite dimensions.

Integral Transformation Techniques

Perhaps the first attempt to provide a general procedure for generating solutions to the problems with variable thermal properties was made by Friedmann (9)(10). Using the integral transformation (see Ref. 7),

$$u = \int_{t_o}^{t} \frac{k(\lambda)}{k_M} \, d\lambda \tag{II-1}$$

where t = temperature

t_o = constant reference temperature

k = thermal conductivity

k_M = constant value of thermal conductivity which is either evaluated at a specific temperature or assumed to be the arithmetic average for the temperature range considered and λ = integration variable

Friedmann showed that the one-dimensional Fourier heat conduction equation [Appendix A, Eq. (A-10)] could be written as

$$\frac{\overline{\rho c}(t)}{k(t)} \frac{\partial u}{\partial \theta} = \frac{\partial^2 u}{\partial x^2} \tag{II-2}$$

where $\overline{\rho c}(t)$ denotes the variable volumetric specific heat. From this form, approximate and perhaps some exact solutions could be generated by choosing particular functions for the volumetric specific heat and the thermal conductivity. Regardless of the specific functions assigned to represent these quantities, the form does lend itself to numerical and analog approaches.

Friedmann employed an analog computer to generate solutions for two problems which were characterized by energy transfer in only one dimension. Both problems were considered to involve materials whose thermal properties varied as linear functions of temperature,

$$\begin{aligned} k &= k_o(1 - at) \\ \overline{\rho c} &= (\rho c)_o(1+bt) \quad . \end{aligned} \tag{II-3}$$

The initial and boundary conditions which defined the problems were chosen to be

$$\begin{aligned} &(1) \quad t(0,\theta) = t_1 && (2) \quad k \frac{\partial t}{\partial x}(0,\theta) = -q \\ &\quad\;\;\; t(\ell,\theta) = t_o && \quad\;\;\; t(\ell,\theta) = t_o \\ &\quad\;\;\; t(x,0) = t_o && \quad\;\;\; t(x,0) = t_o \quad . \end{aligned} \tag{II-4}$$

The solution was obtained by approximating the spatial derivative of Eq. (II-2) by a finite difference quantity and by using function generators to approximate the desired property variations. Hence, the analog was continuous in time. At that time, according to his literature survey, these were the only available solutions which assumed reasonable functions for the thermal property variations.

After Friedmann's analysis, Roy and Thompson (19) made use of a similar transformation to solve two more problems of finite dimensions. These problems were characterized by periodically varying boundary conditions. Although the investigators assumed linear property variations, they also assumed that the variations were small. Moreover, they defined the system as being in a quasi-steady state which permitted them to define the time dependence of the solution as equal to that of the boundary conditions.

Another solution now available which employs such a transformation is discussed by Chu and Abramson (5). They used numerical techniques to solve the transformed equation for the specific case of constant but unequal energy transfer rates at the ends of a rod of finite length. The thermal diffusivity and conductivity of the rod were both assumed to vary linearly with temperature. Even though they did not present a complete set of results, the accuracy and simplicity of their method was clearly indicated.

Integral Equation Techniques

A second procedure for solving problems of the variable property class was mentioned by Goodman (11)(12)(13)(14) shortly after the integral transform technique was presented. As summarized by Costello (6), Goodman integrated the one-dimensional Fourier heat conduction equation to obtain the heat balance integral

$$\int_0^{\ell} \overline{\rho c}\, \frac{\partial t}{\partial \theta}\, dx = \left(k\, \frac{\partial t}{\partial x}\right)_{x=\ell} - \left(k\, \frac{\partial t}{\partial x}\right)_{x=o} \quad . \tag{II-5}$$

He then assumed for the temperature distribution a polynomial expression in the spatial variable x which involved arbitrary time dependent coefficients. To arrive at the final solution, this assumed distribution was forced to satisfy the imposed boundary conditions and the heat balance integral. Therefore, the number of time dependent coefficients which were employed was limited to the sum of the boundary conditions plus the single condition of the integral equation.

To increase the number of coefficients and hence the accuracy of the solution requires that the solution be forced to satisfy the Fourier equation either at specific points within the region considered or averaged over subdivisions of that region. Koh (15) suggests another possible means for improving the accuracy of the technique. He argued that a refinement can be made if an exponential function rather than a polynomial is first assumed to represent the general distribution. Again, the number of coefficients considered must be determined by the number of constraints available.

Although most of the problems which have been approached by employing this technique are of the semi-infinite type or have incorporated a constant property assumption (11)(12)(13)(14)(18), the method is applicable to problems of finite dimensions which involve variable properties. For example, Koh considered the problem of one-dimensional conduction with variable properties and arbitrary heating rates. The wall in question was assumed to be insulated on one surface and forced to accept a certain energy flow at the other surface. Unfortunately, the solution was only carried to the point where the property variations had to be specified before completing the analysis. Hence, the actual value of the technique is somewhat uncertain.

Other Approaches

Although primarily concerned with the phenomena of diffusion, Philip (17) has developed a method by which exact solutions of the one-dimensional Fourier equation can be generated for general property variations. This particular method employs the Boltzmann transformation of the independent variables, $\eta = (x\theta^{-\frac{1}{2}})$, to reduce the partial differential equation to one of ordinary derivatives. The problems treated are limited to those with initial and boundary conditions described by the relations,

$$\begin{aligned} t(x,0) &= 0 \\ t(0,\theta) &= 1 \quad . \end{aligned} \tag{II-6}$$

Using the same transformation, Yang (24) obtained an integral solution for the semi-infinite solid with a constant surface temperature. As the integral involved in the solution could not be directly solved, results could only be obtained by using some approximate technique to evaluate the integral. A more convenient analysis of a similar problem is the numerical solution by Luke (16). Luke directly approximated the solution by applying finite difference techniques to the governing differential equation which he had transformed in a way similar to that used by Friedmann.

Tsang (23) assumed a general trigometric series solution and approximated a solution to the finite solid problem characterized by the conditions

$$
\begin{aligned}
t(x,0) &= 1 \\
t(0,\theta) &= 0 \qquad \text{(II-7)} \\
t(\ell,\theta) &= 0 \; .
\end{aligned}
$$

The method Tsang used appeared to be quite similar to a small perturbation technique.

Methods incorporating the use of variational calculus, successive approximations and least squares have been applied to give results similar to those obtained by the heat balance integral method (6). The accuracy of these solutions depends not only on the assumed temperature distributions used as a first approximation but on the degree to which the solutions are made to satisfy the governing equation.

Other techniques have been considered by different investigators (7) (9). For example, a separation of variables approach was mentioned by Friedmann (9) as having been employed for certain problems. However, due to the development of the procedures discussed here, these remaining techniques are not currently considered to be of engineering interest.

In regard to the problems which have been discussed, the solutions generated from the heat balance integral are of closed form but are only as accurate as the distribution assumed in the analysis. The remaining solutions, because of their complexity, must rely on digital and analog computers to complete the analysis. Hence, no closed form solutions of guaranteed accuracy have been found for the problems of general engineering interest.

CHAPTER III

ANALYTICAL APPROACH

The following discussion pertains to the development of numerical solutions for transient heat conduction problems of one and two rectangular dimensions with variable thermal conductivity. The solutions are general inasmuch as they enable problems of any known realistic initial and boundary temperatures to be solved. They are, however, limited in application to problems which involve materials which closely approximate the assumed conditions of constant volumetric specific heat and linear variation of thermal conductivity with temperature.

Arguments for a Numerical Approach

Certain conclusions concerning a possible analytical program for solving transient heat conduction problems with variable thermal conductivity were evident after the review of the previous methods of investigations had been completed. Considering the objectives of this present work, a continuation of the various transformation techniques used by several investigators did not seem worthwhile. Although these techniques allowed the non-linear partial differential equation which governed the problem to be written as an ordinary non-linear equation or in a more convenient non-linear partial differential equation form, the solutions obtained were generally complicated and unhandy to employ.

Moreover, the assumptions made in grouping the variables to be transformed might conceivably limit the problems which could be solved through such an analysis. For example, application of the transformation ($x\theta^{-\frac{1}{2}}$) to a problem of finite dimensions with boundary conditions different from each other and different from the initial condition would be quite difficult since only two arbitrary constants are present in the integral solution.

Similar difficulties were present with the heat balance integral technique developed by Goodman (11)(12)(13)(14). Although this technique appears to be applicable to many problems, the solutions which were relatively simple and convenient were also undesirable because their accuracy seemed questionable.

The separation of variables method which was briefly mentioned in Chapter I was extensively considered in the course of this study because of the simplicity of its application. However, the study only assured this investigator that the solutions which might be obtained would indeed be complicated. Also, the method would only yield particular results and could not be used on all problems. This is due to the non-linearity of the governing equation.

A method involving the use of an analog computer would certainly yield valuable results but would require a separate analysis of each specific problem. Such an analysis on an analog device would consume a considerable amount of time and still would not directly yield the exact information desired. The elimination of these techniques reduced the list of the present methods of approach to that of the numerical class.

Numerical solutions written for one and two dimensions would enable problems involving the classical boundary conditions to be immediately analyzed. Such problems are characterized by constant boundary temperatures or by boundary temperatures which vary according to some known function of space, time, or both. For these problems, the engineer would need only to define the specific problem in question and supply this information with the programmed solution to a digital computer. The results received from the analysis would directly represent the problem considered. Also, with reference to the classical problems, numerical solutions would conceivably enable graphical solutions of certain classes of problems to be developed without great difficulty. These solution charts could then be used to provide the engineer with an immediate indication of the temperature distribution histories which should be expected. As numerical solutions represent the general governing equations, the engineer could easily modify the existing solution so that problems with unusual boundary conditions could be treated.

The above advantages offered by a numerical approach provided a more than sufficient reason for developing an analytical program based upon that technique.

General Considerations

As derived in Appendix A, the general heat conduction equation for systems which do not generate energy internally can be written as

$$\overline{\rho c}\,\frac{\partial t}{\partial \theta} = \frac{\partial}{\partial x}\left(k\,\frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\,\frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\,\frac{\partial t}{\partial z}\right) \qquad \text{(III-1)}$$

where $\overline{\rho c}$ = volumetric specific heat (Btu/ft^3 °F)

t = temperature (°F)

θ = time (hrs)

k = thermal conductivity (Btu/hr ft °F)

x, y, z = spatial variables (ft) .

Assuming the volumetric specific heat is constant and the thermal conductivity variation with temperature can be approximated by the relation

$$k = k_o(1 + \beta t) \tag{III-2}$$

where k_o = thermal conductivity at 0°F

β = thermal conductivity temperature coefficient $(°F)^{-1}$

allows Eq. (III-1) to be transformed into a relation involving a new variable. This variable, g, is defined by the equation,

$$g = \frac{k}{\rho c} \quad . \tag{III-3}$$

In this form, Eq. (III-1) becomes

$$\frac{\partial g^2}{\partial \theta} = g\left\{\frac{\partial^2 g^2}{\partial x^2} + \frac{\partial^2 g^2}{\partial y^2} + \frac{\partial^2 g^2}{\partial z^2}\right\} \tag{III-4}$$

which appears to be considerably more convenient to handle.

Although both Eqs. (III-1) and (III-4) govern problems which involve materials with thermal conductivities that vary linearly with temperature, only Eq. (III-1) can be used when the materials considered have constant conductivities. This is true because the variable g of Eq. (III-4) assumes a constant value in such cases and a trivial solution results. Therefore, numerical programs developed from the non-transformed equation would enable a larger class of problems to be treated. Also, these programs could be checked by comparison to the available solutions for the constant conductivity case. For these reasons, the convenience of form offered by Eq. (III-4) was not used and the solutions were generated from Eq. (III-1).

One Dimensional Solution

By considering only one spatial dimension and by applying the assumptions stated above, Eq.(III-1) can be reduced to

$$\frac{1}{\alpha_o} \frac{\partial t}{\partial \theta} = (1 + \beta t) \frac{\partial^2 t}{\partial x^2} + \beta\left(\frac{\partial t}{\partial x}\right)^2 \qquad \text{(III-5)}$$

where the symbol α_o represents the ratio of constants k_o and ρc. To solve this equation numerically requires that each derivative be approximated by a finite difference quantity. Employing the technique discussed in Ref. (20), the time derivative can be approximated to the order of the time increment $\Delta\theta$ by a forward difference. The space derivatives can be approximated to the order of the square of the spatial increment, Δx, by central differences. These finite differences are defined by the relations,

$$\begin{aligned} \frac{\partial t}{\partial \theta} &\rightarrow \frac{t_{n,\theta+1} - t_{n,\theta}}{\Delta\theta} \quad ; \quad e = 0(\Delta\theta) \\ \frac{\partial t}{\partial x} &\rightarrow \frac{t_{n+1,\theta} - t_{n-1,\theta}}{2\Delta x} \quad ; \quad e = 0(\Delta x^2) \\ \frac{\partial^2 t}{\partial x^2} &\rightarrow \frac{t_{n+1,\theta} + t_{n-1,\theta} - 2t_{n,\theta}}{(\Delta x)^2} \quad ; \quad e = 0(\Delta x^2) \quad . \end{aligned} \qquad \text{(III-6)}$$

In the above relations, n denotes the spatial location of the point whose temperature is being considered and θ specifies the time at which the evaluation is made. The order of magnitude of the errors involved in these approximations is indicated to the right of each definition.

Substituting these quantities into Eq. (III-5) yields the finite difference equation,

$$\begin{aligned} \frac{1}{\alpha_o}\left(\frac{t_{n,\theta+1} - t_{n,\theta}}{\Delta\theta}\right) = (1 + \beta t_{n,\theta})\left(\frac{t_{n+1,\theta} + t_{n-1,\theta} - 2t_{n,\theta}}{\Delta x^2}\right) \\ + \beta\left(\frac{t_{n+1,\theta} - t_{n-1,\theta}}{2\Delta x}\right)^2 \quad . \end{aligned} \qquad \text{(III-7)}$$

Since the second term on the right hand side of this equation contains a square of the finite difference derivative approximation, the error associated with that term is doubled. However, the order of error for this term is not changed. Hence, Eq. (III-7) is in error to $O(\Delta\theta)$ or $O(\Delta x^2)$ depending upon which term is the larger.

Solving Eq. (III-7) for the temperature $t_{n,\theta+1}$ indicates that

$$t_{n,\theta+1} = \left\{1 - 2\frac{\alpha_o \Delta\theta}{\Delta x^2}(1 + \beta t_{n,\theta})\right\} t_{n,\theta} + \frac{\alpha_o \Delta\theta}{\Delta x^2}(1 + \beta t_{n,\theta})(t_{n+1,\theta} + t_{n-1,\theta}) + \frac{\alpha_o \Delta\theta}{\Delta x^2}\left(\frac{\beta}{4}\right)(t^2_{n+1,\theta} - 2t_{n+1,\theta}t_{n-1,\theta} + t^2_{n-1,\theta}) . \qquad \text{(III-8)}$$

Equation (III-8) defines the temperature at node n for a time $(\theta + \Delta\theta)$ as a function of the temperatures of nodes n - 1, n, and n + 1 at the time θ. Hence, if the initial temperature distribution and the imposed boundary temperatures are known, the temperature distribution can be obtained for each time after an elapsed period of $\Delta\theta$. This equation is, therefore, the numerical solution of the one-dimensional heat conduction equation with constant volumetric specific heat and linear variation in thermal conductivity with temperature.

Stability of the One-Dimensional Solution

Although Eq. (III-8) does represent the one-dimensional numerical solution, some consideration must be given to its stability and accuracy before attempting to apply it to a problem.

The errors which are inherent in any numerical analysis due to the finite difference approximations have already been mentioned. Equation (III-8) was said to be in error to $O(\Delta\theta)$ or $O(\Delta x^2)$ depending upon which one is the larger. Regardless of the particular governing order, the accuracy of the solution is always improved when smaller values of both $\Delta\theta$ and Δx are taken. Hence, care should be taken in specifying these quantities.

Actually, the increments are not entirely independent of each other but are related through the stability criteria for the solution. Since the solution is admittedly in error to some degree, a criteria must be found which will guarantee that for certain conditions this error will not increase as a function of time.

Through an analysis similar to that presented on page 298 of Ref. (21), the necessary stability criteria for Eq. (III-8) is found to be represented by the inequality

$$\frac{(\Delta x)^2}{\alpha_o \Delta\theta (1+\beta t_M)} \geq 2 \sin^2\left(\frac{\mu\Delta x}{2}\right) \left\{1 + \cos^2\left(\frac{\mu\Delta x}{2}\right) - \left(\frac{1}{1+\beta t_M}\right)\cos^2\left(\frac{\mu\Delta x}{2}\right)\right\} \quad . \text{(III-9)}$$

In the above, μ denotes a constant which depends upon the increment Δx and the number of nodes considered. The term $(1+\beta t_M)$ must always be greater than or equal to unity. Hence, for problems which involve a negative β, the temperature t_M should be taken equal to zero. For problems of positive β, t_M should be set equal to the highest temperature encountered. The remaining quantities have been previously defined.

A sufficient condition for stability which is of simpler form can be determined by maximizing the right hand side of Eq. (III-9). This condition is represented by the inequality,

$$\frac{(\Delta x)^2}{\alpha_o \Delta\theta (1+\beta t_M)} \geq 2.0$$

which indicates that to insure stability, all terms of Eq. (III-8) must be positive when all temperatures involved are positive. In this study, Eq. (III-10) was taken as the required stability condition.

To summarize, the accuracy of the numerical results is directly dependent on the size of the Δx and $\Delta\theta$ increments which are used. The stability of the solution is guaranteed when Eq. (III-10) is satisfied.

Applications of the One-Dimensional Solution

With the above definition of the required conditions for stability, the one-dimensional solution can now be applied to a specific problem. However, since small increments of time and space must be assumed if accurate results are to be obtained, the great effort involved in carrying out the analysis does not allow the solution as presented to represent a useful tool. In fact, before a problem can be conveniently solved, the solution must be programmed for analysis by a high speed digital computer. Hence, a program of the one-dimensional solution was developed for use on an IBM 1620 computer. As such a program represents only a convenient way of providing information to the computer, no discussion will be made about its development. For the reader's convenience, the complete program is outlined in Appendix B.

With reference to this program, only problems of known initial and boundary temperatures can be treated. These temperatures can be constant or varying with space and time. The thermal properties of the materials considered should closely approximate the assumed material properties used in the development of the solution. The density, ρ, and the specific heat, c, when averaged for the temperature range of the problem should indicate for the volumetric specific heat a value which is characteristic of that property at any temperature within that range. In addition, the thermal conductivity of the material should be closely approximated by a linear function of temperature. Although the solution as represented by Eq. (III-8) can be applied to walls of infinite thickness, the program described in Appendix B will only allow problems with finite thick walls to be treated.

The actual application of the program to a particular problem was quite simple when certain considerations were made before attempting to obtain actual results. These considerations and the resulting application of the solution can be best explained by means of an example problem. In Appendix C, such an example is described in detail.

When properly employed, this specific program of the one-dimensional solution enables the transient temperature distribution within a wall characterized by the conditions previously defined to be obtained for virtually any desired time. The results indicate the total time elapsed since the wall was in its initial state, the temperature and thermal conductivity at each node for that time, and the exact location within the wall of each node. By plotting this information, a complete history of the temperatures within the wall can be determined. This history would enable energy transfer rates to be specified. Hence, a general analysis of the problem can be obtained.

The solution can be conveniently used to generate solution charts for problems of a particular type which may frequently occur in one specific area of research. Such a class of problems was considered in this study. This problem and the solution charts which were obtained are discussed in Chapter V.

Two-Dimensional Solution

By employing the same assumptions as made in the one-dimensional analysis but considering two spatial dimensions, Eq. (III-1) reduces to

$$\frac{1}{\alpha_o}\frac{\partial t}{\partial \theta} = (1+\beta t)\left\{\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2}\right\} + \beta\left\{\left(\frac{\partial t}{\partial x}\right)^2 + \left(\frac{\partial t}{\partial y}\right)^2\right\}. \qquad \text{(III-11)}$$

This equation can be approximated by finite difference quantities which are exactly equivalent to those defined by Eq. (III-6) and written as

$$\begin{aligned} t_{n,m,\theta+1} = & \left\{1-2\left(\frac{\alpha_o\Delta\theta}{\Delta x^2} + \frac{\alpha_o\Delta\theta}{\Delta y^2}\right)(1+\beta t_{n,m,\theta})\right\}t_{n,m,\theta} \\ & + (1+\beta t_{n,m,\theta})\left\{\frac{\alpha_o\Delta\theta}{\Delta x^2}(t_{n+1,m,\theta} + t_{n-1,m,\theta}) \right. \\ & \left. + \frac{\alpha_o\Delta\theta}{\Delta y^2}(t_{n,m+1,\theta} + t_{n,m-1,\theta})\right\} \\ & + \frac{\beta}{4}\left(\frac{\alpha_o\Delta\theta}{\Delta x^2}\right)\left\{t_{n+1,m,\theta} - t_{n-1,m,\theta}\right\}^2 \\ & + \frac{\beta}{4}\left(\frac{\alpha_o\Delta\theta}{\Delta y^2}\right)\left\{t_{n,m+1,\theta} - t_{n,m-1,\theta}\right\}^2 . \end{aligned} \qquad \text{(III-12)}$$

In Eq. (III-12), n denotes the x spatial variable, m represents the y variable and θ corresponds to the time variable. Therefore, Eq. (III-12) defines the temperature at node (n,m) for a time $(\theta + \Delta\theta)$ as a function of the temperatures at the nodes (n-1,m), (n,m-1), (n,m), (n+1,m) and (n,m+1) for the time θ . This enables a new distribution to be calculated for times after integral periods of $\Delta\theta$ have elapsed if the initial and boundary temperatures are known. Hence, just as Eq. (III-8) represented the one-dimensional solution, Eq. (III-12) represents the two-dimensional solution of the heat conduction equation for the same property assumptions previously defined.

Stability of the Two-Dimensional Solution

With reference to the accuracy of Eq. (III-12), the finite difference approximations have made that equation in error to the order of the largest of the terms $O(\Delta\theta)$, $O(\Delta x^2)$ and $O(\Delta y^2)$. Since uniform spatial accuracy is desired, the magnitude of the product of Δx and the over-all temperature gradient in the x direction should be comparable to that product for the y direction. Moreover, both Δx and Δy should be small compared to their corresponding total dimension. Both these conditions may be relaxed somewhat when the imposed conditions on the specific problem being analyzed are considered.

The time increment $\Delta\theta$ which will allow this accuracy to be maintained can then be determined from the stability criteria for the solution. Without deriving the exact stability relations, a sufficient condition can be defined in the same manner as was applied in the one-dimensional analysis. Hence, by forcing all the terms of Eq. (III-12) to be positive for positive temperatures, the inequality

$$\frac{(\Delta x^2)(\Delta y^2)}{\alpha_o(\Delta\theta)(\Delta x^2 + \Delta y^2)(1+\beta t_M)} \geq 2.0 \qquad \text{(III-13)}$$

must be satisfied. This inequality is then a sufficient condition for stability of Eq. (III-12).

Applications of the Two-Dimensional Solution

Since the same assumptions were made in developing Eq. (III-12) as were made in obtaining Eq. (III-8), the two-dimensional solution is limited in application to the general problems described for the one-dimensional solution. The same is true of the program for the two-dimensional solution which is presented in Appendix B. An example problem illustrating the application of the two-dimensional program can be found in Appendix C. The information received from the two-dimensional program corresponds exactly to that received from the one-dimensional solution.

CHAPTER IV

EXPERIMENTAL PROGRAM

The use of numerical techniques in generating the general analytical solutions described in Chapter III suggested that some means for verifying the results from these solutions should be found. Since the available solutions obtained by previous investigations were either approximate or dependent upon computers to complete the analysis, they did not present a desirable comparison datum. For this reason, an experimental program was developed.

The results obtained from the experimental program were compared directly to the corresponding results obtained from the one-dimensional numerical analysis. The two-dimensional numerical solution was not considered. This was believed to be sufficient because both numerical solutions were based on the same assumptions. Hence, a verification of either solution would imply a verification of the other.

The general development of the experimental program and the comparison made are the subject of this chapter.

General Considerations

The numerical solutions were developed for materials with a relatively constant volumetric specific heat and a thermal conductivity which varied linearly with temperature. Hence, in order to allow the experimental results to be compared to the numerical results, the material chosen for the experimental tests had to have thermal properties which resembled those assumed in the numerical solutions. Using Ref. (22) as a guide, a search was made for a readily available material of engineering importance whose thermal properties were extremely representative of the assumed. The choice of k-monel for the test material was the result of that consideration.

The thermal conductivity of k-monel at temperatures between 0 and 1200°F can be expressed quite exactly by the relation

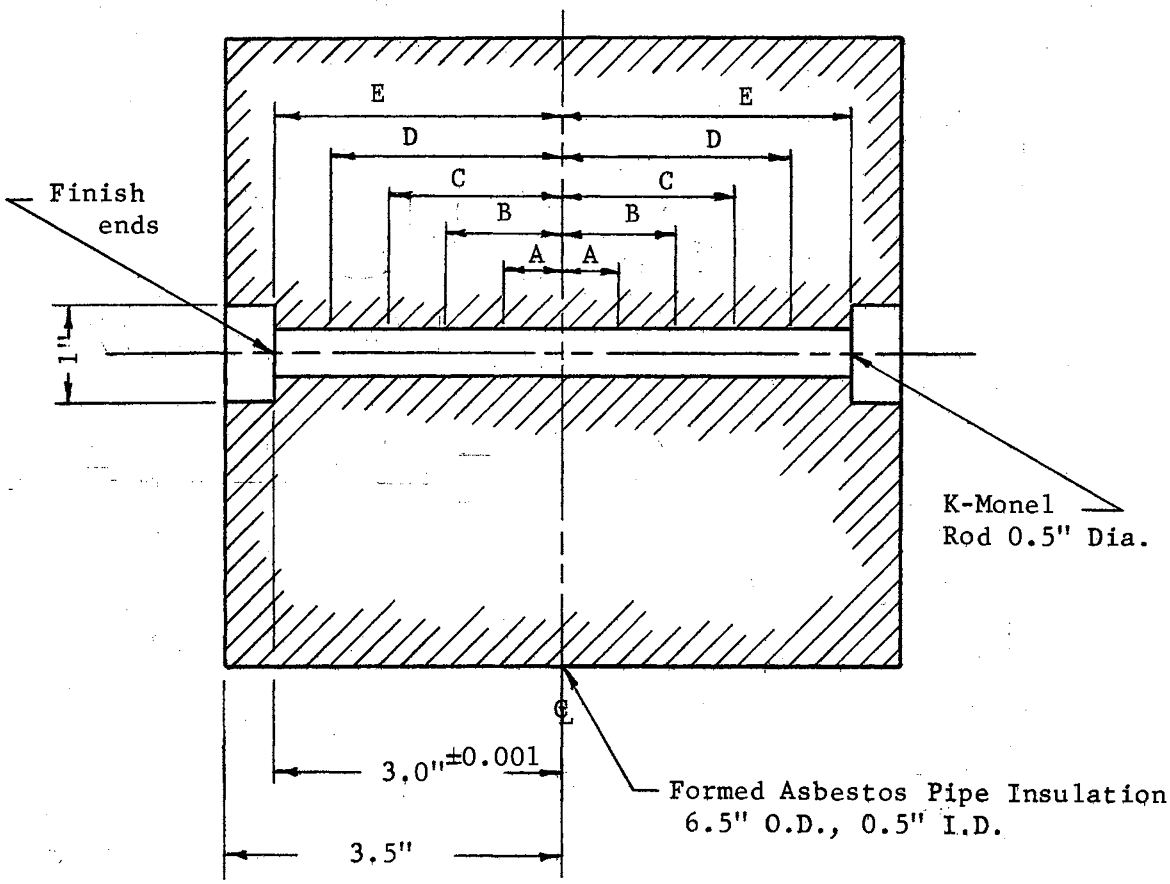
$$k = 9.5(1+0.00093\ t)\ \text{Btu/hr ft }^\circ\text{F}$$

where t represents the temperature in °F. The relatively high value of the thermal conductivity temperature coefficient ($\beta = 0.00093/^\circ F$) insures that the effect of variable conductivity is appreciable for most boundary temperature differences. This coupled with the fact that the representative thermal diffusivity of k-monel for that range of temperatures ($\alpha_o = 0.16\ ft^2/hr$) characterizes many other engineering materials, was sufficient reason for justifying the use of k-monel.

Having found a suitable material, the next step was to define more exactly the problem which was to be treated. Some degree of experimental simplicity could be achieved if the initial and boundary temperatures which were to be used might be obtained experimentally without difficulty. Of course, these conditions also had to be of such a nature that the objective of the program could be correctly achieved. That is to say that the boundary conditions should not offer any complications to the general analysis. With these thoughts in mind, the problem was defined to be that of uniform initial temperature with constant step function boundary temperatures. The problems involved in realizing such boundary temperatures experimentally were thought to be slight when compared to the theoretical simplicity which they provided in the analysis.

To obtain an indication of what boundary temperature difference should be imposed, the one-dimensional numerical solution was employed. Problems involving k-monel walls were considered with boundary temperature differences of 200, 400, 600 and 800 °F. The steady state distributions obtained were then compared to the distributions predicted by assuming constant thermal properties. This comparison indicated that a boundary temperature difference of 600°F was sufficient to cause a substantial difference between the results. For such boundary temperatures, the assumption of constant conductivity would yield temperatures which were different from those predicted by the variable conductivity numerical solution by up to 26°F. Thus, for k-monel, a boundary temperature difference of 600°F would clearly illustrate the effect of variable conductivity.

Consideration of the energy input required to maintain one surface at 600°F indicated that a circular rod of 0.5-inch diameter would be the most convenient size to use for a specimen. As eleven thermocouple positions were thought to be adequate for defining the temperature distribution within the specimen considered, a rod of 0.5 ft. in length was chosen. This allowed a thermocouple to be mounted flush at both ends of the rod and nine thermocouples, equally spaced 0.05 feet apart, to be attached internally along the center axis of the rod, Figure 1. Hence



NOTE: Drill eleven No. 56 holes along radial line to depth of 0.25" $^{+0.001}_{-0.0}$ at following locations:

With first hole on centerline,

A = 0.6" ±0.001

B = 1.2" ±0.001

C = 1.8" ±0.001

D = 2.4" ±0.001

E = 2.968" ±0.001

Figure 1. First Test Section Design

the problem was reduced to that of designing a device which would enable the desired boundary temperatures to be imposed on the test specimen.

First Apparatus

The apparatus thus conceived was quite simple in principle and application, Figure 2. Two large copper rods, one maintained at approximately 600°F by a controlled calrod heating element and the other maintained at approximately 32°F by an ice bath, were simultaneously brought into contact with the opposite ends of the k-monel test specimen illustrated on Figure 1. The specimen was initially at the uniform temperature of the ambient air, approximately 76°F. To insure that both sources simultaneously made contact with the ends of the test rod, a positive displacement mechanical linkage of the Scotch Yoke type was employed. Also, in order to approximate one-dimensional axial heat flow, the k-monel rod was insulated against radial heat flow. A standard asbestos pipe insulation, three inches thick, was used. Iron-constantan thermocouples (30 gage), mounted along the center axis of the test specimen at the positions previously described and attached to strip chart recording devices, provided a history of the temperature at each of the stated positions. These thermocouples were mounted in the rod by applying the technique described on pages 142-166 of Ref. (1).

The first runs made while calibrating the control and recording instruments indicated that the step function boundary conditions which had been desired were not going to be obtained. The large contact resistance between the copper rods and the k-monel rod would not allow such an instantaneous change of temperature to be achieved. In an effort to

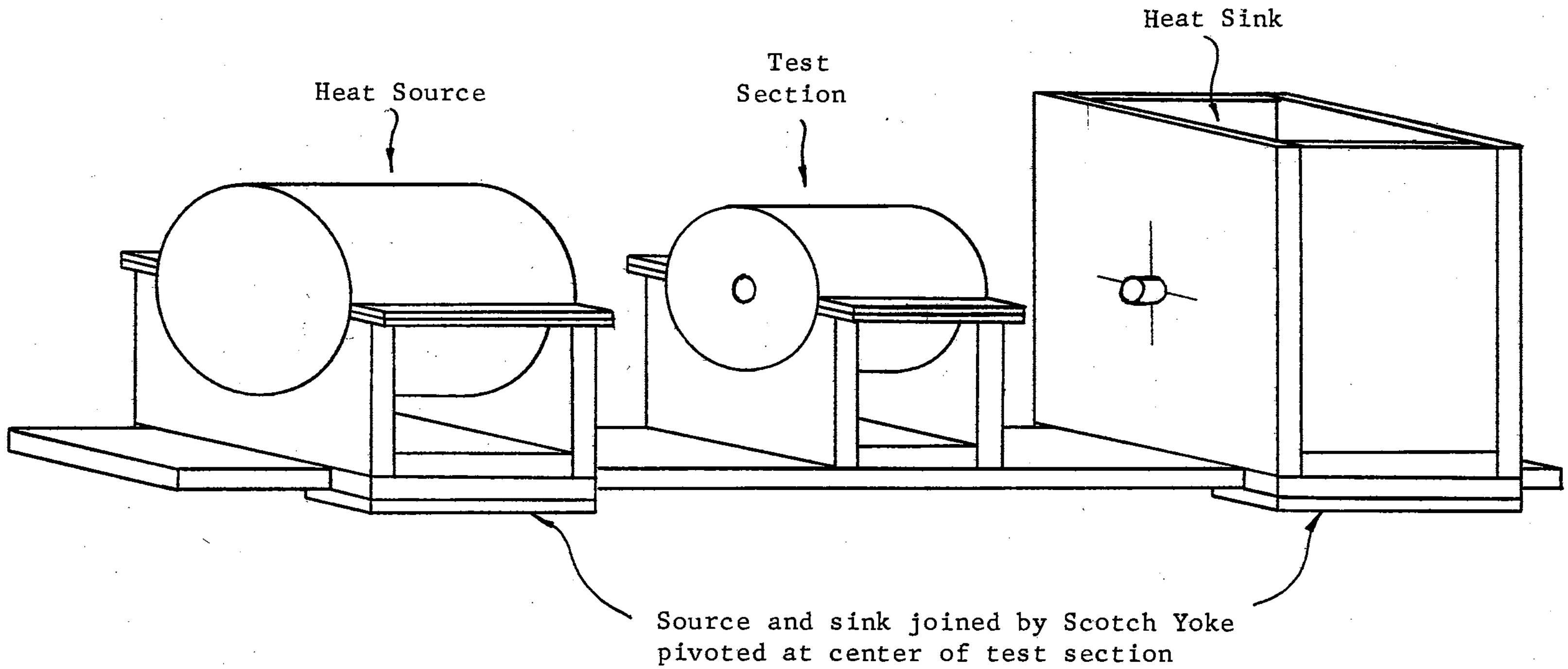


Figure 2. Sketch of Experimental Apparatus

improve these conditions, a high temperature grease was applied to the ends of the test rod before the boundary temperature sources were imposed. This insured much better contact between the k-monel and the copper but was still not sufficient to achieve the ideal boundary conditions. The effect of having time dependent boundary temperatures results in complicating the input data used in the analytical solution.

After completing an experimental test, the temperature histories recorded for the boundaries of the rod were specified as the boundary temperatures of a k-monel wall, 0.5 ft. thick, which was initially at the same temperature as the experimental rod. The transient temperatures were then obtained for this theoretical wall by employing the one-dimensional numerical solution. This allowed the experimentally determined temperature histories to be compared with the corresponding temperatures predicted by the analytical solution. If the two histories compared favorably, a strict verification of both methods would be obtained.

Results From First Apparatus

Unfortunately, the comparison of experimental and analytical data did not indicate any degree of correlation. For a particular point at a particular time, the two methods suggested temperatures which differed by up to 70°F. The continuous, smooth temperature distributions obtained from the analytical solution indicated that the numerical method was basically correct. Since continuous distributions were also obtained from the experimental results, no large error due to faulty thermocouples was apparent. Hence, in order to isolate the difficulties involved, the steady state temperature distributions indicated by both the numerical solution and the experimental tests were compared to the steady state distribution described by the exact steady state solution (see Appendix A). This indicated that the analytic solution was at least valid for the final condition of steady state. For the experimental data, however, this comparison suggested that a large radial heat flow existed along the length of the test rod. Such a loss would violate the assumption of one-dimensional heat transfer and the objective of the program could not be achieved.

Second Apparatus

To correct the experimental apparatus required that the test specimen be modified in such a way as to reduce the radial heat loss from the surface of the rod to zero. This could be achieved by placing a guard heater around the surface of the specimen. However, because such a heater would be extremely difficult to control in lieu of the transient situation, another means for eliminating the loss was employed. The test rod was shielded by a one-inch diameter stainless steel tube having approximately the same cross sectional area as the test rod. The ends of this tube were soldered to copper pieces which were attached to the test rod, Figure 3. This insured that both the shield and the rod would have identical boundary conditions. As the average thermal conductivity of stainless steel is nearly equivalent to that of k-monel, both the shield and the rod would have approximately the same axial temperature distribution at any one time. This would then reduce the possibility for a radial temperature gradient to exist and in so doing would also greatly reduce the heat loss. The new specimen was insulated as shown

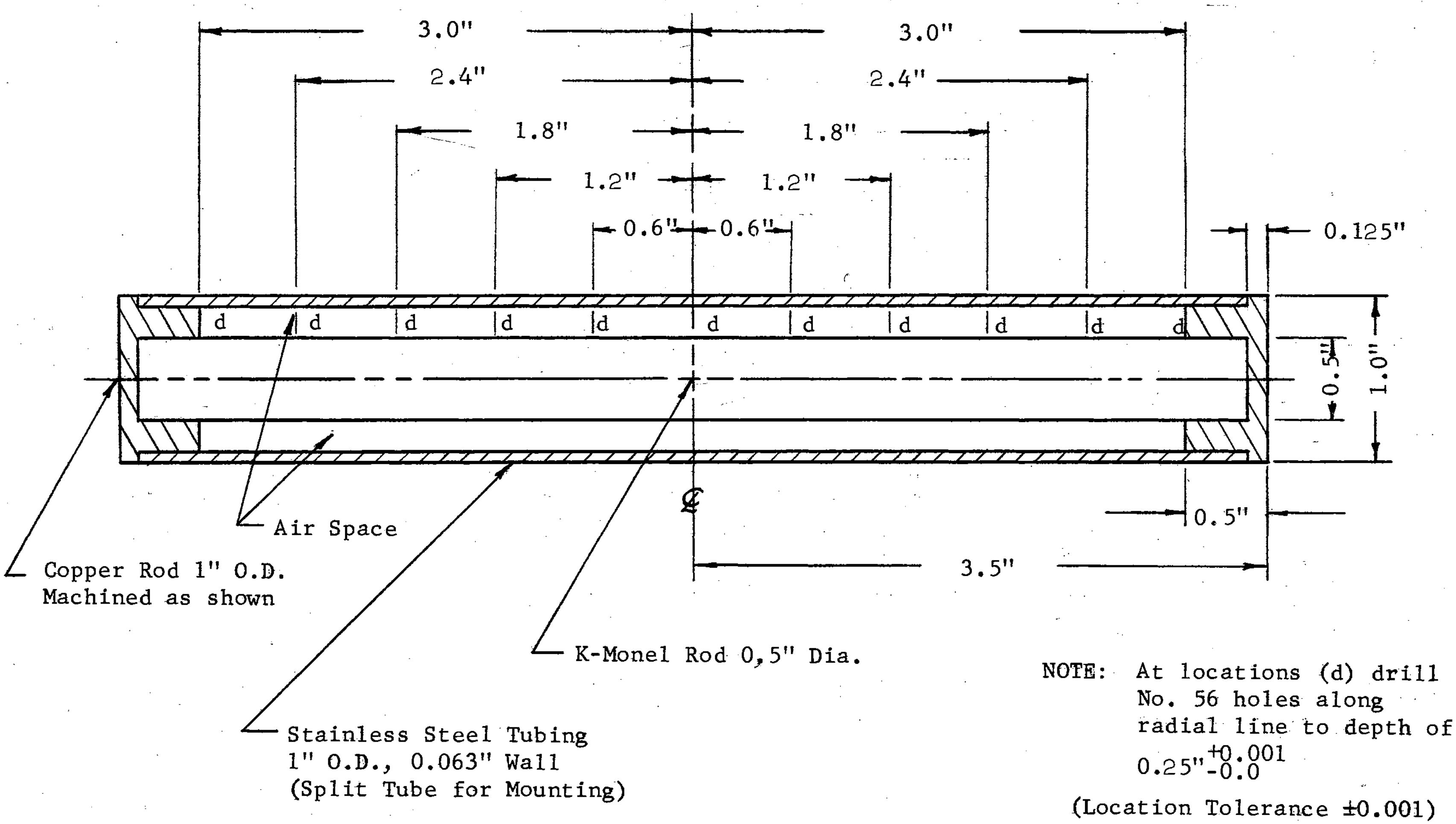


Figure 3. Revised Test Specimen With Heat Shield

in Figure 4. The air gap between the insulation and the shield was used to reduce the radial loss from the shield. Reducing this loss would improve the axial distribution of the shield.

Results From Second Apparatus

In reality, the stainless steel shield did not completely eliminate the radial loss. This is indicated by the comparison of the analytical and experimental results illustrated on Figure 5. The curves presented there define quite well the effect of such a loss. At small times, agreement between the analytical and experimental methods is observed for all values of $\frac{x}{\ell}$. However, for large times, the two methods compare less favorably as the center position is approached from either boundary. Since the effect of the radial loss on the temperature of any particular point is time dependent, these observations are precisely what should be expected.

Assume that for small times the axial distribution of temperature in the shield is slightly different than that in the rod. This is to be expected because of the radial loss from the shield to the ambient air. Hence, a small radial gradient will exist and there will be a small radial heat flow. Because the flow is small, the temperature distribution in the rod is only slightly affected. This accounts for the agreement between the analytical and experimental data at small times. However, at large times, a larger radial gradient between the rod and shield will exist due to the increase in the temperature difference between the shield and the ambient air. This gradient gives rise to a radial heat flow which has an appreciable effect on the temperature

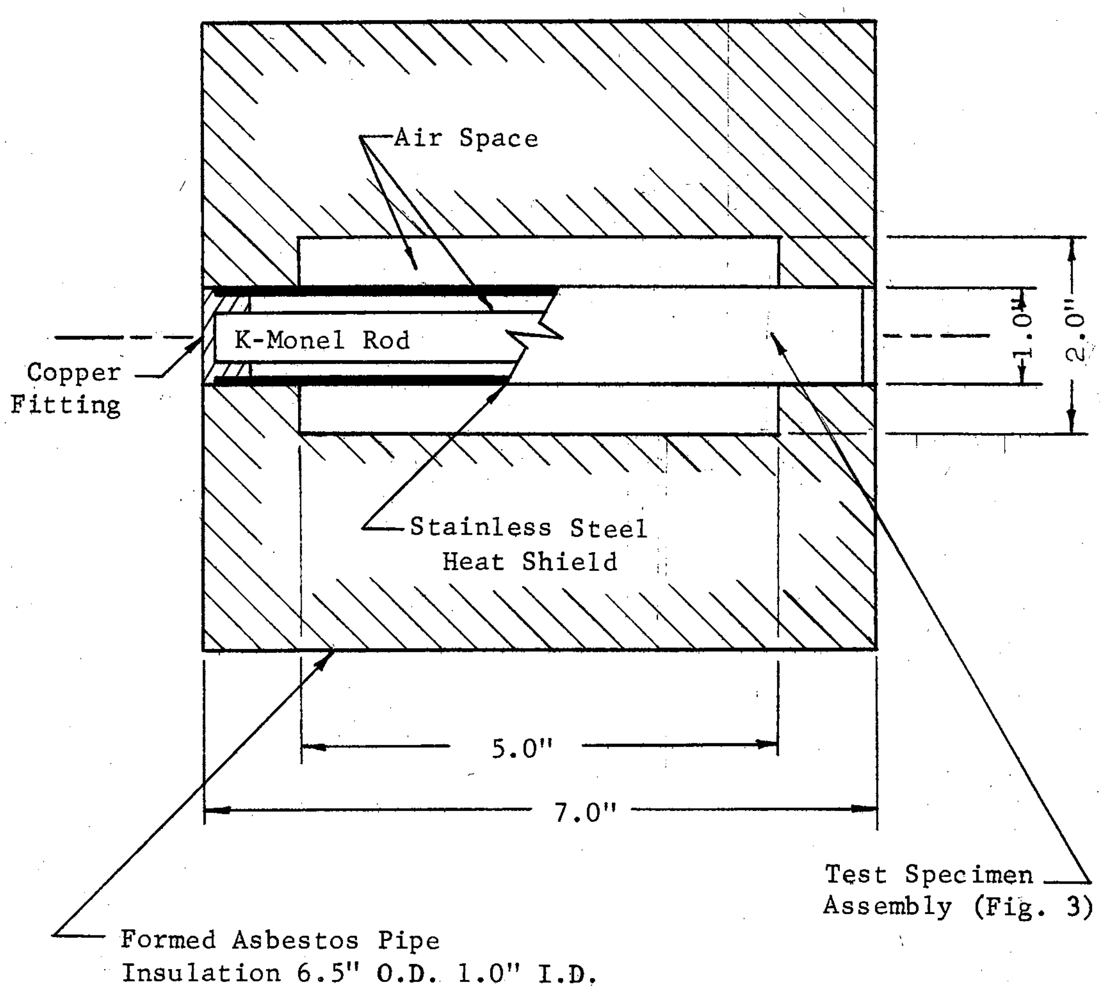


Figure 4. Second Test Section Design

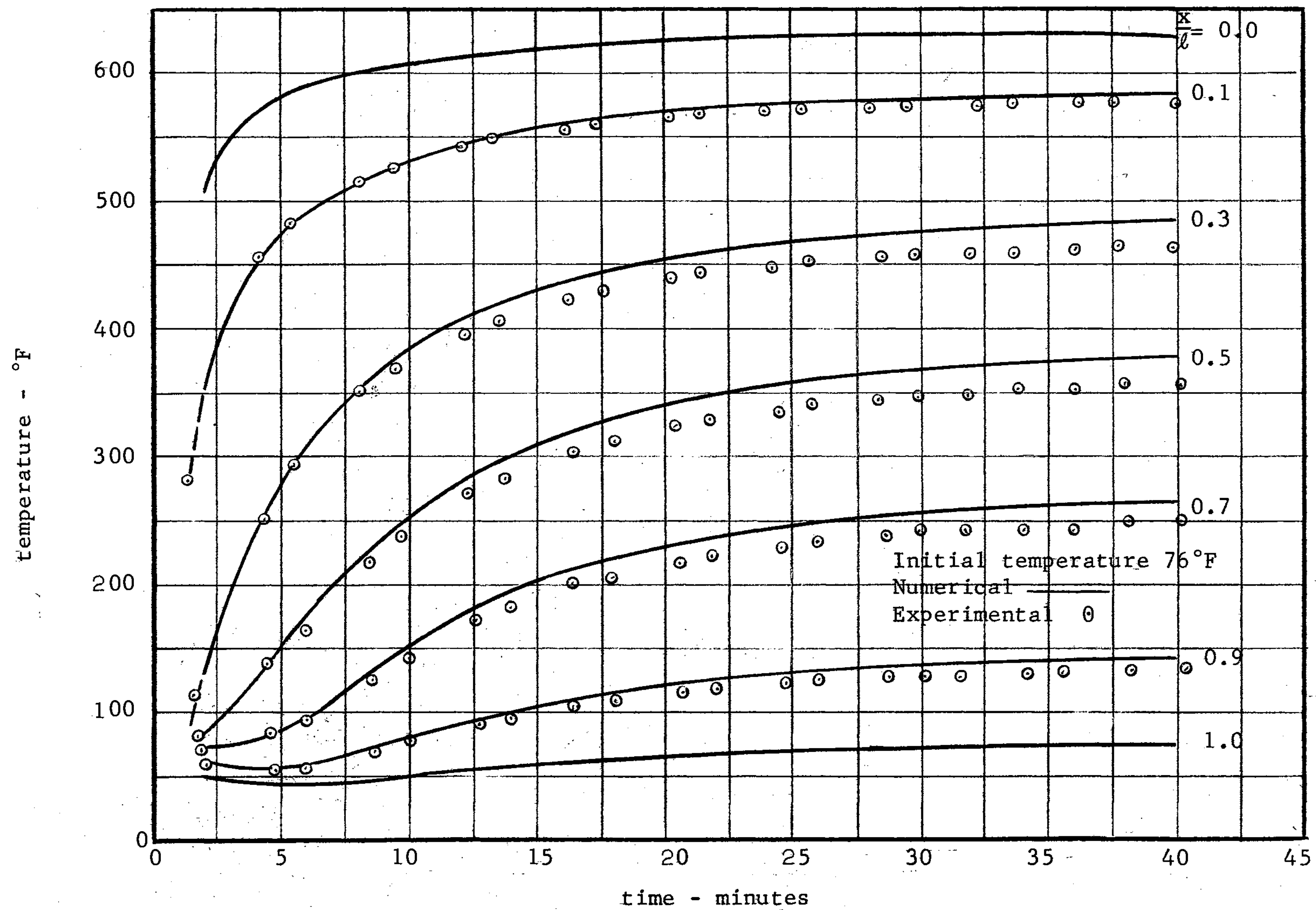


Figure 5. Comparison of Experimental and Numerical Results

distribution of the rod. Hence, the numerical and experimental results fail to agree. The large differences observed at points near the center of the rod simply illustrate the fact that the temperature at a position for a specific time is dependent upon the total radial loss which occurred along that portion of the rod separating the point in question from the controlled boundaries.

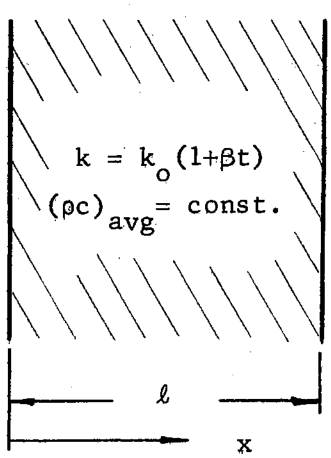
Since the analytical results did compare favorably with the experimental at small times and with the exact steady state solution at a sufficiently large time, no further attempt was made to improve the experimental program. However, improvement is certain if care is taken to construct the test specimen so that the shield completely encloses the rod. This would help reduce any axial temperature distribution differences between the shield and the rod. Also, this difference could be further reduced by constructing the shield from the same material as the rod. In the case considered, the cost of such a shield did not justify its use.

CHAPTER V

GRAPHICAL RESULTS

Due to the generality of the solutions which have been developed, a complete graphical presentation of the problem types which can be solved was quite impossible. However, a specific class of one-dimensional problems was considered to demonstrate how these solutions might be employed to generate practical solution charts.

This class of problems is best defined by its characteristic initial and boundary conditions. Initially, the material which forms the wall of consideration, Figure 6, is at a uniform temperature. At some instant in time, both surfaces of the wall are raised to constant and equal temperatures. The temperature at any point within the wall for any specific time is to be determined.



$t(x,0) = \text{constant} = t_i$

$t(0,\theta) = t(\ell,\theta) = \text{constant} = t_b$

$t_b > t_i$

Figure 6. Problem Definition for Solution Charts

The reader will no doubt recognize this problem as one of the types presented on various charts for materials of constant thermal conductivity. The Heisler charts presented on page 258 of Ref. (21) are representative of such work. Indeed, this available analysis of the constant conductivity problem was one reason for considering that particular class of problems. The Heisler charts provide a convenient means of checking the accuracy of the one-dimensional numerical solution for those situations where the materials involved have constant thermal conductivities ($\beta = 0.0$). In addition, since the reader is already familiar with the constant property solution presented within those charts, the effect of variable conductivity on the temperature distribution will be immediately more meaningful.

In an attempt to follow Heisler's work as closely as possible, the charts on Figures 7 and 8 present the dimensionless centerline temperature,

$$(1 + 0.4\ \beta t_i)\left\{\frac{t_c - t_b}{t_i - t_b}\right\}$$

as a function of the dimensionless time,

$$\Theta = \frac{4\alpha_0 \theta}{\ell^2}$$

for various values of the parameter

$$\psi = 10\ \beta t_b \quad .$$

The symbols used in these expressions are defined below:

β = thermal conductivity temperature coefficient $(^\circ F)^{-1}$

t_i = initial temperature of wall (°F)

t_c = center temperature of wall (°F)

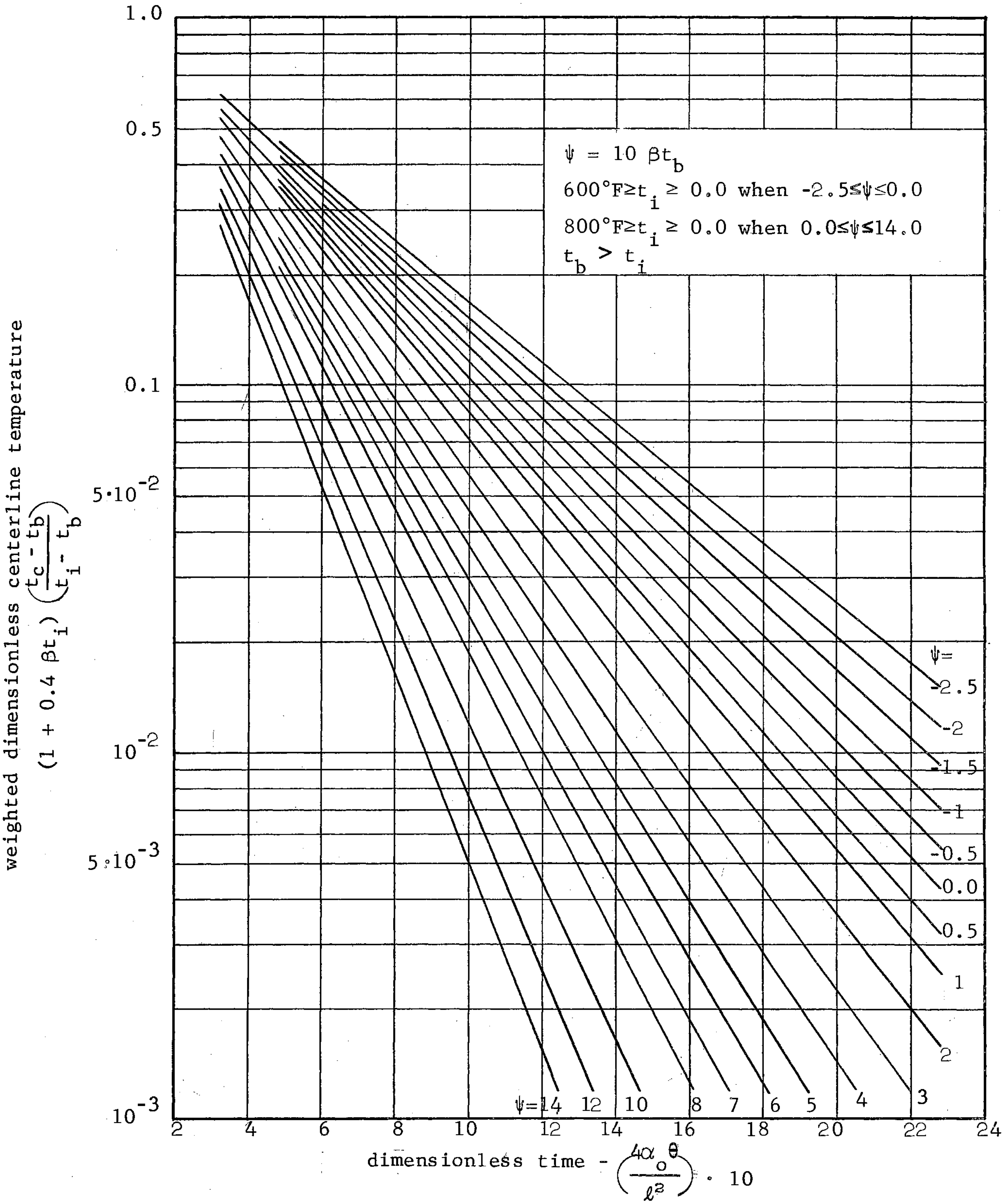


Figure 7. Solution Chart

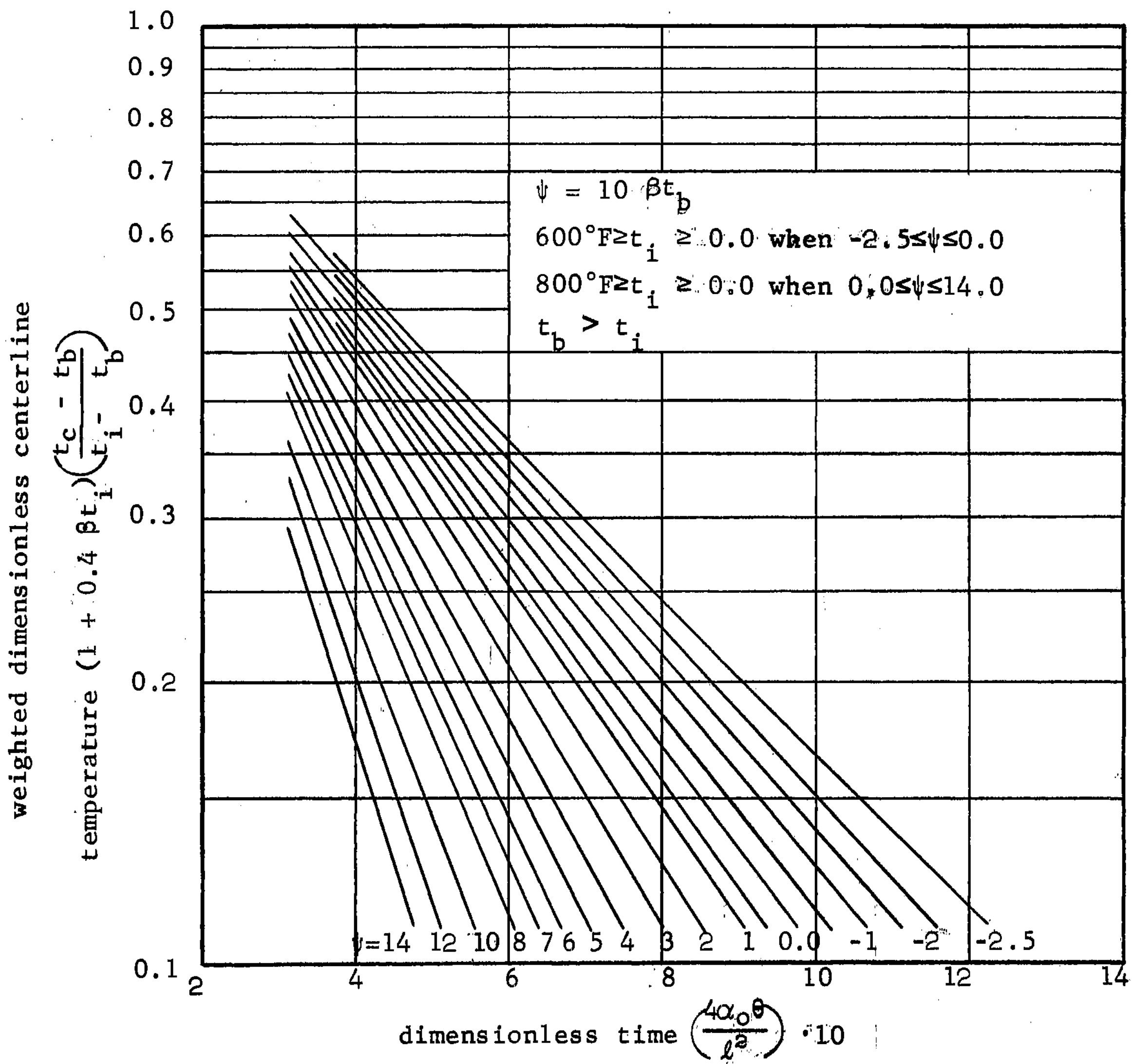


Figure 8. Enlargement of Solution Chart for Small Times

t_b = boundary temperature of wall (°F)

θ = elapsed time of analysis (hrs)

ℓ = thickness of wall (ft)

α_o = representative thermal diffusivity (ft^2/hr) .

The term α_o is calculated for each problem from the ratio of the thermal conductivity at 0°F and the characteristic value of the volumetric specific heat for the particular temperature range considered.

A position correction chart, Figure 9, provides a means of evaluating the temperature at other points within the wall. On this chart, the dimensionless temperature

$$\frac{t_x - t_b}{t_c - t_b}$$

is plotted as a function of the dimensionless group, (ψ/Θ), for various values of the dimensionless location, $\left(\frac{x}{\ell}\right)$. The symbol t_x is used to define the temperature within the wall at location x .

The centerline temperature charts and the position correction chart are empirical in that they employ correlation parameters which approximate the variations indicated by the results obtained for a variety of problems from the numerical solution. For example, the dimensionless centerline temperature is weighted by the factor

$$(1 + 0.4\ \beta t_i)$$

and is plotted as a function of the dimensionless time for various values of the parameter ψ . The ψ group was found to correlate almost exactly the results obtained for problems concerning totally different materials with different boundary temperatures as long as the initial temperature

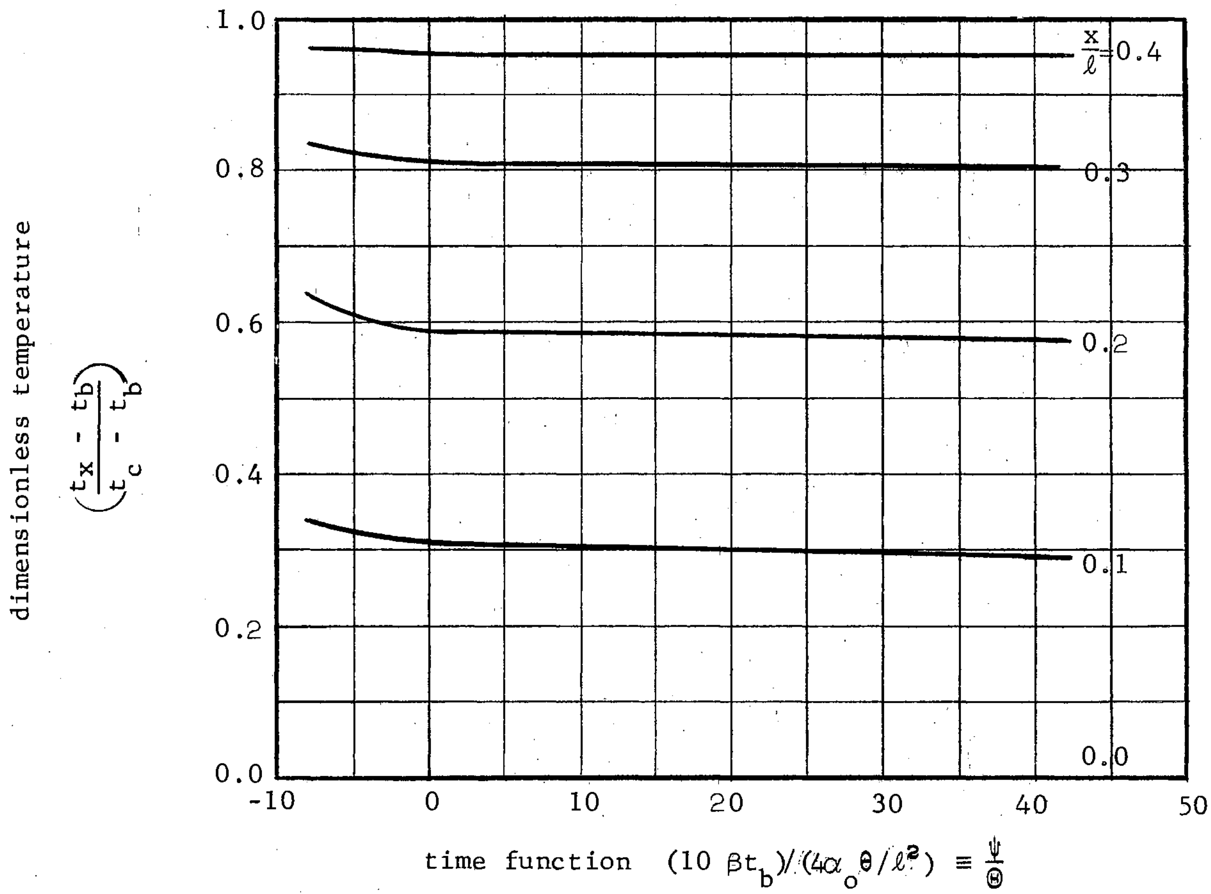


Figure 9. Position Correction Chart

characterizing the entire group of problems was taken to be 0°F. Actually this parameter illustrates the observed tendency that a material of $\beta = 0.0005/°F$ whose boundary temperature is 1000°F will experience the same dimensionless temperature variation with dimensionless time as another material of $\beta = 0.001/°F$ whose boundary temperature is 500°F.

On the other hand, the weighting factor for the centerline temperature is not as exact. A detailed study of problems which involved different initial temperatures as well as different boundary temperatures and different materials indicated that such a term might be employed. This was observed by plotting the ratio of the dimensionless centerline temperature for initial temperatures different from zero to the dimensionless centerline temperature for an initial temperature of zero as a function of βt_i for each material and boundary temperature considered. The ratio was found to be nearly independent of dimensionless time and to depend almost linearly on βt_i. This allowed the weighting group to be specified.

The results from additional problems revealed that the correlation was justly accurate for only certain initial temperatures. Indeed, for centerline temperatures as accurate as the readability of the charts do permit, only problems which involve initial temperatures within the limits defined below can be considered. These limits are:

$$0 \leq t_i \leq 600°F \quad \text{for } \psi \text{ between } -2.5 \leq \psi \leq 0.0$$

$$0 \leq t_i \leq 800°F \quad \text{for } \psi \text{ between } 0.0 \leq \psi \leq 14.0 \; .$$

Under these conditions, a maximum error of four percent exists in the dimensionless centerline temperatures obtained from the charts. This error occurs when $t_i = 600°F$ and $\psi = -2.5$. Also, it is important to note that the charts are valid only for problems which involve an increase in temperature. In fact, to assure accuracy, only problems of

$$t_b \geq t_i + 200°F$$

should be considered. However, problems where $(t_b - t_i)$ is less than 200°F can be treated by assuming a constant conductivity equal to the average conductivity of the material for the temperature range considered. The limitations which have been discussed should be considered as an integral part of the charts.

The position correction chart is in some ways even more limited than the centerline temperature charts. The problem study indicated that the temperature at one location not only varied with time, boundary temperature, and material properties but depended as well on the initial temperature. Since the dependence of temperature on time, boundary temperature and material properties was obviously quite larger than its dependence on the initial temperature, the (ψ/Θ) grouping was formed and the average correlation curves developed. These curves provide correction readings within four percent of the exact as long as the problem of consideration satisfies the above conditions and the dimensionless time is larger than or equal to 0.32.

Within the limits described above, the charts offer a rapid solution to many problems of the class considered. A similar analysis of other general classes may yield very useful charts. However, one should always remember that when extreme accuracy is desired or when a problem of unusual boundary temperature variation is considered, the numerical solution should be employed.

CHAPTER VI

CONCLUSIONS

A general, straightforward, analytical procedure has been presented for accurately describing the transient temperature distribution within materials whose thermal conductivities vary with temperature. The procedures developed are applicable to problems of one and two rectangular dimensions which are characterized by known boundary temperatures. These temperatures, however, can be both constant and varying. Although only the one- and two-dimensional solutions are developed, the same technique can be applied to yield a solution for three-dimensional problems.

To insure extreme accuracy, the materials considered should have thermal conductivities which can be closely approximated by a linear function of temperature for the range of temperatures considered. They should exhibit the character of having a relatively constant volumetric specific heat within that range of temperature. These conditions do not severely limit the solutions because most materials available today do satisfy such requirements.

Since the solutions are numerical, careful consideration should also be given to the stability and convergence criteria as discussed in the text of this report. These conditions are indeed an integral part of the solution. In addition, a digital computer must be available for the analysis if the solutions are to be of significant value. As presented, the solutions are immediately applicable for analysis by machines which will accept the IBM 1620 Fortran notation without Format. However, no great problem exists in rewriting the solutions so that other computers can be employed.

The study includes a graphical presentation of a particular class of one-dimensional problems. The problem type considered is characterized by a uniform initial temperature with constant and equal step function boundary temperatures. Solution charts involving dimensions parameters were developed by employing the analytical solution. These charts enable a rapid analysis to be made of the transient temperature at any position within a wall whose boundaries are subjected to such conditions as described on Figure 6. As the correlation parameters used in constructing these charts are not exact, certain additional limitations must be placed on the problems which can be considered. Only problems which involve an increase in temperature, $t_b > t_i$, and which satisfy the conditions defined on page 41 of Chapter V can be accurately solved. Regardless of the problem considered, the charts will enable the engineer to gain an understanding of the effect a material with variable conductivity will have upon the temperature distributions and the energy transfer rates thus encountered.

To conclude this study, a brief discussion of possible future investigations follows.

With reference to the techniques employed in this study, problems of cylindrical and spherical coordinates should be considered in future studies. The solutions developed should be extended to include problems of general convective boundary conditions and the variation of the volumetric specific heat with temperature should be included within the analysis. The techniques developed might then be used to construct solution charts for the classes of problems which are frequently encountered. With such solutions, the majority of the problems which are of concern could be treated.

However, an effort should be made to use this information to develop more convenient analytical solutions of all particular problems. Indeed, the technique employed here represents only a beginning step toward the optimum analysis of problems involving materials of variable thermal properties.

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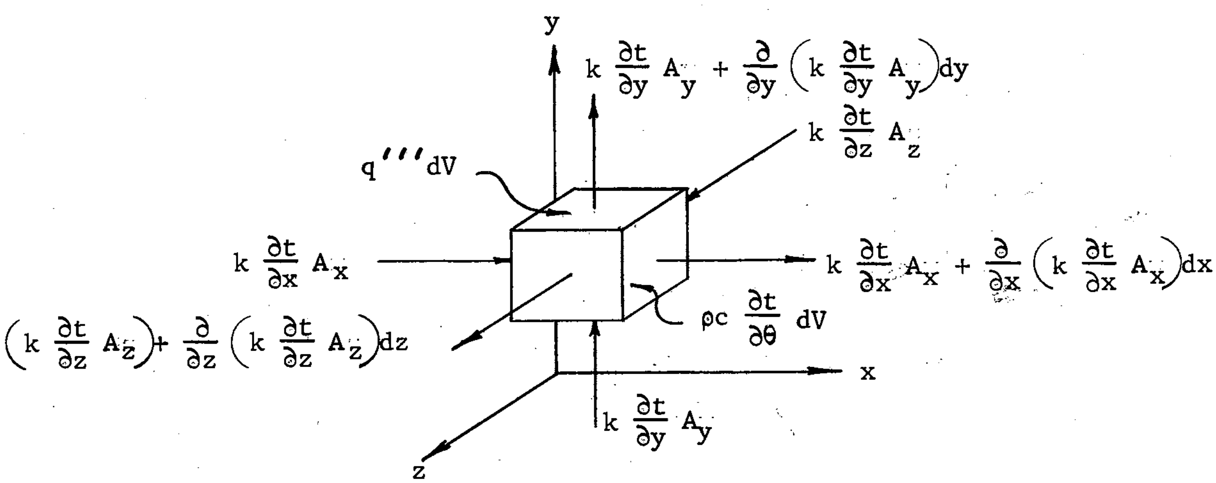
APPENDIX A

DERIVATION OF GOVERNING EQUATIONS

To provide the reader with a complete description of the general equations which are related to this study, a detailed derivation of the heat conduction equation for solid bodies is presented below. The transformation of this equation to one involving the thermal conductivity function is also considered, and an exact steady state solution is generated for one-dimensional problems.

Derivation of the Heat Conduction Equation

With reference to the following diagram, the principle of conservation of energy implies that the net rate at which energy is added to the differential mass must be exactly equal to the net rate at which the internal energy of that mass is increased.



There are two possible ways by which the element can receive energy. The first is by the conduction of energy in the form of heat through the faces of the element and the second is by the generation of energy by the mass of material within the element itself. With regard to the conducted energy, the Fourier-Biot Law states that the rate of energy flow by conduction, q_n, in any specific direction, n, is defined by the expression

$$q_n = -\left(kA_n \frac{\partial t}{\partial n}\right)_n \tag{A-1}$$

where k represents the thermal conductivity of the material, A_n denotes the cross sectional area perpendicular to the direction n and $\frac{\partial t}{\partial n}$ is the temperature gradient taken with respect to the direction n. Denoting the net flow of energy into the element in the x direction by $\dot{Q}_x$, the Fourier-Biot Law applied to the element face at x and x + dx yields

$$\dot{Q}_x = \left[-\,k(dydz)\frac{\partial t}{\partial x}\right]_x - \left[-k(dydz)\frac{\partial t}{\partial x}\right]_{x+dx} \quad . \tag{A-2}$$

For mathematical convenience, the second term on the right hand side of Eq. (A-2) can be written as a function of the first by employing Taylor's Series Expansion,

$$f(x+dx) = f(x) + \frac{\partial f(x)}{\partial x}\,dx + \frac{1}{2!}\frac{\partial^2 f(x)}{\partial x^2}(dx)^2 + \ldots \tag{A-3}$$

Considering terms of the order dx or larger allows Eq. (A-2) to be written as

$$\dot{Q}_x = \left[-\,k(dydz)\frac{\partial t}{\partial x}\right] - \left\{\left[-\,k(dydz)\frac{\partial t}{\partial x}\right] + \frac{\partial}{\partial x}\left[-\,k(dydz)\frac{\partial t}{\partial x}\right]dx\right\}$$

which reduces to

$$\dot{Q}_x = \frac{\partial}{\partial x}\left(k\,\frac{\partial t}{\partial x}\right) dxdydz \quad . \tag{A-4}$$

A similar treatment of the remaining directions will yield the relations

$$\dot{Q}_y = \frac{\partial}{\partial y}\left(k \frac{\partial t}{\partial y}\right) dxdydz \tag{A-5}$$

and

$$\dot{Q}_z = \frac{\partial}{\partial z}\left(k \frac{\partial t}{\partial z}\right) dxdydz \quad . \tag{A-6}$$

Defining the net flow of energy into the element by conduction as $\dot{Q}_c$, it is clear that

$$\dot{Q}_c = \dot{Q}_x + \dot{Q}_y + \dot{Q}_z$$

$$\dot{Q}_c = \left\{\frac{\partial}{\partial x}\left(k \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial t}{\partial z}\right)\right\} dxdydz \ . \tag{A-7}$$

If the mass of material within the element has the ability to generate energy at the rate of q''' which is expressed in terms of (Energy/unit time/ unit volume) then the net rate at which the element receives generated energy, $\dot{Q}_G$, is defined by the equation,

$$\dot{Q}_G = q''' \ dxdydz \quad , \tag{A-8}$$

and hence the element receives both generated and conducted energy at the net rate of $\dot{Q}_c + \dot{Q}_G$.

At any particular time, the internal energy of the elemental mass can be expressed as

$$E_I dxdydz$$

where E_I represents the internal energy per unit volume of the material. Hence, the rate of increase of internal energy is equal to

$$\frac{\partial}{\partial \theta}\left\{E_I\right\} dxdydz \ ,$$

where the symbol θ is used to denote time. If E_I is assumed to be a function of temperature which implies that the density, ρ, and the specific heat, c, are also functions of temperature, then

$$E_I = \int_0^t \rho c \; d\lambda \qquad \text{and} \qquad \frac{\partial E_I}{\partial \theta} = \rho c \frac{\partial t}{\partial \theta}$$

which defines the rate of increase of internal energy as

$$\rho c \frac{\partial t}{\partial \theta} \, dxdydz \quad . \tag{A-9}$$

Therefore, by the principle of conservation of energy,

$$\rho c \frac{\partial t}{\partial \theta} \, dxdydz = \dot{Q}_c + \dot{Q}_G'$$

or

$$\rho c \frac{\partial t}{\partial \theta} = \frac{\partial}{\partial x}\left(k \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial t}{\partial z}\right) + q''' \tag{A-10}$$

which is the Fourier heat conduction equation.

Transformation of the Heat Conduction Equation

It has been mentioned that the thermal conductivity of most engineering materials depends linearly upon the temperature. This means that k can be expressed as

$$k = k_o(1 + \beta t)$$

where k_o is the conductivity at $t = 0°F$ and β is a weighted constant which is proportional to the slope of the k versus t curve.

Considering the change in k with respect to some independent variable, say n, indicates that

$$\frac{\partial k}{\partial n} = k_o \beta \frac{\partial t}{\partial n}$$

or

$$\frac{\partial t}{\partial n} = \left(\frac{1}{k_o \beta}\right) \frac{\partial k}{\partial n} \quad . \tag{A-11}$$

Assuming that the material in question cannot generate heat, the above relation is substituted into Eq. (A-10),

$$\frac{\rho c}{k_o \beta} \frac{\partial k}{\partial \theta} = \frac{1}{k_o \beta} \left\{ \frac{\partial}{\partial x}\left(k \frac{\partial k}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial k}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial k}{\partial z}\right) \right\} .$$

By recognizing that

$$\frac{1}{2} \frac{\partial k^2}{\partial n} \equiv k \frac{\partial k}{\partial n}$$

this equation can be written as

$$\frac{\rho c}{k} \frac{\partial k^2}{\partial \theta} = \frac{\partial^2}{\partial x^2} (k^2) + \frac{\partial^2}{\partial y^2} (k^2) + \frac{\partial^2}{\partial z^2} (k^2) \quad . \tag{A-12}$$

Taking ρc as constant allows a new variable g to be defined as

$$g = \frac{k}{\rho c} \tag{A-13}$$

which reduces the form of the equation to

$$\frac{\partial g^2}{\partial \theta} = g \left\{ \frac{\partial^2 g^2}{\partial x^2} + \frac{\partial^2 g^2}{\partial y^2} + \frac{\partial^2 g^2}{\partial z^2} \right\} \quad . \tag{A-14}$$

Although the heat conduction equation as defined by Eq. (A-12) is generally considered, Eq. (A-14) may offer some mathematical simplification in the grouping of the variable conductivity and the constant volumetric specific heat into one variable.

Steady State Solution--One Dimension

Under the condition of steady state, Eq. (A-12) reduces to the very familiar Laplace equation,

$$\frac{\partial^2}{\partial x^2} (k^2) + \frac{\partial^2}{\partial y^2} (k^2) + \frac{\partial^2}{\partial z^2} (k^2) = 0 \tag{A-15}$$

for which a general solution is known. The assumption of one-dimensional energy transfer permits the equation to be written in yet simpler form as an ordinary differential equation,

$$\frac{d^2}{dx^2}(k^2) = 0 \quad . \tag{A-16}$$

Integration of this equation yields the general solution,

$$k^2 = c_1 x + c_2$$

which can be written in terms of temperature by substituting $k_o(1+\beta t)$ for k,

$$k_o^2(1 + \beta t)^2 = c_1 x + c_2 \tag{A-17}$$

Specifying the value of t at x = 0 and x = ℓ allows the evaluation of c_1 and c_2 to be made. Hence,

$$c_2 = k_o^2(1 + \beta t_1)^2$$

and

$$c_1 = \frac{k_o^2(1 + \beta t_2)^2 - k_o^2(1 + \beta t_1)^2}{\ell}$$

where t_1 = boundary temperature at x = 0

t_2 = boundary temperature at x = ℓ.

This enables the temperature equation to be written as

$$k_o^2(1+\beta t)^2 = k_o^2\left[\frac{(1+\beta t_2)^2 - (1+\beta t_1)^2}{\ell} x + (1+\beta t_1)^2\right] .$$

Therefore,

$$t = \frac{1}{\beta}\left\{\left[\frac{(1+\beta t_2)^2 - (1+\beta t_1)^2}{\ell} x + (1+\beta t_1)^2\right]^{\frac{1}{2}} - 1\right\} \tag{A-18}$$

represents the one-dimensional steady state temperature distribution in a material having a thermal conductivity which varies linearly with temperature.

APPENDIX B

PROGRAM OF ONE- AND TWO-DIMENSIONAL SOLUTIONS

In the following sections, complete listings of the programs for both the one- and two-dimensional solutions are presented in IBM 1620 Fortran notation without Format. Each listing is supplemented with physical definitions of the quantities called for as input data and those received as output data. For a detailed description of the problems which can be solved by either one of these solutions, the reader should refer to Chapter III.

Program of the One-Dimensional Solution

Before reference is made to the program listing which follows, the Fortran quantities defined below should be carefully studied.

(a) RHO = density of the material (lb_m/ft^3).

(b) SPHT = specific heat of the material ($Btu/lb_m{}^\circ F$).

(c) CONDO = thermal conductivity of the material at $0^\circ F$ ($Btu/hr\ ft\ {}^\circ F$).

(d) BETA = thermal conductivity temperature coefficient $({}^\circ F)^{-1}$;
(Note: See Appendix A, page 51.)

(e) DIM = thickness of wall or length of rod considered (ft).

(f) DELT = elapsed time between temperature calculations (hrs).

(g) TEMPM = maximum temperature experienced by the material during the transient process (°F); (Note: TEMPM should be taken equal to 0°F when BETA is negative).

(h) J = number of spatial nodes considered by assuming the first node, node 1, is at one boundary; i.e., counting the nodes from unity instead of zero. (J is a fixed point number; i.e., it is presented without a decimal).

(i) RUTIM = number of time increments to be taken before the machine provides a record of the temperature distribution and other desired quantities.

(j) UM = number of output data sets desired.

(k) BC = type of boundary conditions involved (BC = 0.0 refers to variable boundary temperatures while BC = 1.0 denotes constant boundary temperatures).

(l) TIMO = actual time at which the material experiences the temperature distribution which characterizes the beginning point of the run (hrs).

(m) STAB = stability condition of the numerical solution (must be greater than two or machine will refuse the input data).

(n) ETIME = exact time which has elapsed since the beginning initial temperature distribution at TIMO = 0.0 hrs. was considered (hrs).

(o) TEMP(N)= temperature of nodal points at some specific time (°F). (Note: The read statement involving TEMP(N) calls for the distribution at time TIMO).

(p) DIST = location in (ft) of Node (I) where TEMP(I) occurs (referenced to Node (1) whose spatial dimension has been assumed equal to zero).

(q) COND = actual thermal conductivity at Node(I) where TEMP(I) occurs (Btu/hr ft °F).

With the exception of item (h), all of the above quantities must be defined with a decimal point. To gain a better understanding of how the input data should appear and what to expect as output, please refer to Appendix C where a sample problem is discussed.

In Fortran notation, the one-dimensional solution is as follows on page 57.

Program of the Two-Dimensional Solution

With the exception of the following quantities, the definitions given in the first section of this appendix are valid in the program of the two-dimensional solution.

(a) DIMX = length of wall in the direction of x, (ft).

(b) DIMY = length of wall in the direction of y, (ft).

(c) J = number of divisions in the x-dimension used to define the nodal system. (Note: J is determined by counting the divisions from unity instead of zero).

(d) K = number of divisions in the y-dimension used to define the nodal system. (Note: K is determined in the same manner as J).

(e) TEMP(N,M) = temperature at nodal point where N defines the x displacement from the origin and M represents the y displacement for some specific time (°F). (Note: The read statement involving TEMP(N,M) calls for the distribution at time TIMO).

PROGRAM LISTING OF THE ONE-DIMENSIONAL NUMERICAL SOLUTION

```
      START
      DIMENSION TEMP(101), TEMPN(101)
1     READ, RHO, SPHT, CONDO, BETA, DIM, DELT, TEMPM, J
      READ, RUTIM, UM, BC, TIMO
      ACT=J-1
      FACT=1.0/ACT
      DELX=DIM*FACT
      CONDM=CONDO*(1.0+BETA*TEMPM)
      ALPHA=CONDO/(RHO*SPHT)
      STAB=((DELX**2)*CONDO)/(ALPHA*DELT*CONDM)
      PUNCH, STAB
      IF(STAB-2.0)1,3,3
3     CNST1=(ALPHA*DELT)/(DELX**2)
      TIMT=UM*RUTIM
      TI=0.0
      ETIME=TI*DELT+TIMO
      PUNCH, ETIME
      N=1
5     DIS=N-1
      DIST=DIS*DELX
      READ, TEMP(N)
      COND=CONDO*(1.0+BETA*TEMP(N))
      PUNCH, DIST, TEMP(N), COND
      N=N+1
      IF(N-J)5,5,10
10    TI=TI+1.0
      ETIME=TI*DELT+TIMO
      N=2
15    PART1=TEMP(N)*(1.0-2.0*CNST1*(1.0+BETA*TEMP(N)))
      PART2=CNST1*(TEMP(N+1)+TEMP(N-1))*(1.0+BETA*TEMP(N))
      ULT1=(CNST1*BETA)/4.0
      ULT2=(TEMP(N+1)**2)-2.0*TEMP(N+1)*TEMP(N-1)+(TEMP(N-1)**2)
      TEMPN(N)=PART1+PART2+(ULT1*ULT2)
      N=N+1
      IF(N-J)15,20,20
20    IF(BC-1.0)25,30,30
25    READ, TEMP(1), TEMP(J)
30    N=2
35    TEMP(N)=TEMPN(N)
      N=N+1
      IF(N-J)35,40,40
40    DO 60 I=1,50
      ENC=I
      IF(TI-ENC*RUTIM)60,45,60
45    PUNCH, ETIME
      N=1
50    DIS=N-1
      DIST=DIS*DELX
      COND=CONDO*(1.0+BETA*TEMP(N))
      PUNCH, DIST, TEMP(N), COND
      N=N+1
      IF(N-J)50,50,60
60    CONTINUE
      IF(TI-TIMT)10,65,75
65    PUNCH, ETIME
      N=1
70    PUNCH, TEMP(N)
      N=N+1
      IF(N-J)70,70,75
75    CONTINUE
      GO TO 1
      END
```

(f) DISTX = x location of node in question (ft).

(g) DISTY = y location of node in question (ft).

Since items (c) and (d) are fixed point numbers, they must not be defined with a decimal point. All of the remaining quantities discussed above must be defined in decimal notation.

In Fortran notation, the two-dimensional solution is as follows on pages 59 and 60

PROGRAM LISTING OF THE TWO-DIMENSIONAL NUMERICAL SOLUTION

```
      START
      DIMENSION TEMP(15,15), TEMPN(15,15)
1     READ, RHO, SPHT, CONDO, BETA, DIMX, DIMY, DELT, TEMPM, J, K
      READ, RUTIM, UM, BC, TIMO
      ACT1=J-1
      FACT1=1.0/ACT1
      DELX=DIMX*FACT1
      ACT2=K-1
      FACT2=1.0/ACT2
      DELY=DIMY*FACT2
      CONDM=CONDO*(1.0+BETA*TEMPM)
      ALPHA=CONDO/(RHO*SPHT)
      TAB1=((DELX*DELY)**2)*CONDO/(CONDM*DELT)
      TAB2=ALPHA*((DELX**2)+(DELY**2))
      STAB=TAB1/TAB2
      PUNCH, STAB
      IF(STAB-2.0)1,3,3
3     CNSTX=(ALPHA*DELT)/(DELX**2)
      CNSTY=(ALPHA*DELT)/(DELY**2)
      TIMT=UM*RUTIM
      TI=0.0
      ETIME=TI*DELT+TIMO
      PUNCH, ETIME
      N=1
5     M=1
10    DISX=N-1
      DISTX=DISX*DELX
      DISY=M-1
      DISTY=DISY*DELY
      READ, TEMP(N,M)
      COND=CONDO*(1.0+BETA*TEMP(N,M))
      PUNCH, DISTX, DISTY, TEMP(N,M), COND
      M=M+1
      IF(M-K)10,10,15
15    N=N+1
      IF(N-J)5,5,20
20    TI=TI+1.0
      ETIME=TI*DELT+TIMO
      N=2
25    M=2
30    PART1=(1.0-2.0*(CNSTX+CNSTY)*(1.0+BETA*TEMP(N,M)))*TEMP(N,M)
      PART2=(1.0+BETA*TEMP(N,M))*CNSTX*(TEMP(N+1,M)+TEMP(N-1,M))
      PART3=(1.0+BETA*TEMP(N,M))*CNSTY*(TEMP(N,M+1)+TEMP(N,M-1))
      ULT1=(BETA*CNSTX)/(4.0)
      ULT2=(TEMP(N+1,M)**2)-2.0*TEMP(N+1,M)*TEMP(N-1,M)+(TEMP(N-1,M)**2)
      ULT3=(BETA*CNSTY)/(4.0)
      ULT4=(TEMP(N,M+1)**2)-2.0*TEMP(N,M+1)*TEMP(N,M-1)+(TEMP(N,M-1)**2)
      TEMPN(N,M)=PART1+PART2+PART3+(ULT1*ULT2)+(ULT3*ULT4)
      M=M+1
      IF(M-K)30,35,35
35    N=N+1
      IF(N-J)25,40,40
40    IF(BC-1.0)45,65,65
45    M=1
```

PROGRAM LISTING OF THE TWO-DIMENSIONAL NUMERICAL SOLUTION (Continued)

```
50      READ, TEMP(1,M), TEMP(J,M)
        M=M+1
        IF(M-K)50,50,55
55      N=2
60      READ, TEMP(N,1), TEMP(N,K)
        N=N+1
        IF(N-J)60,65,65
65      N=2
70      M=2
75      TEMP(N,M)=TEMPN(N,M)
        M=M+1
        IF(M-K)75,80,80
80      N=N+1
        IF(N-J)70,85,85
85      DO 110 I=1,50
        ENC=I
        IF(TI-ENC*RUTIM)110,90,110
90      PUNCH, ETIME
        N=1
95      M=1
100     DISY=M-1
        DISX=N-1
        DISTX=DISX*DELX
        DISTY=DISY*DELY
        COND=CONDO*(1.0+BETA*TEMP(N,M))
        PUNCH, DISTX, DISTY, TEMP(N,M), COND
        M=M+1
        IF(M-K)100,100,105
105     N=N+1
        IF(N-J)95,95,110
110     CONTINUE
        IF(TI-TIMT)20,115,135
115     PUNCH, ETIME
        N=1
120     M=1
125     PUNCH, TEMP(N,M)
        M=M+1
        IF(M-K)125,125,130
130     N=N+1
        IF(N-J)120,120,135
135     CONTINUE
        GO TO 1
        END
```

APPENDIX C

REPRESENTATIVE CALCULATIONS

As a supplement to the discussion of the analytical solutions and their subsequent programs, two examples are considered. The first illustrates the use of the one-dimensional numerical solution and the second represents an application of the two-dimensional solution.

One-Dimensional Example

With reference to the discussion presented in Chapter III and Appendix B, consider the problem of describing the temperature distribution which exists at certain specific times within a wall whose surface temperatures are known functions of time. Assume that the wall is a k-monel slab, 0.5 ft. thick, and is initially at a uniform temperature of 400°F. At some time, $\theta = 0$, one surface of the wall is instantaneously lowered to a constant temperature of 0°F while the other surface is maintained at a temperature defined by the relation

$$t = 400(1 + 2.5\,\theta)\,^{\circ}\mathrm{F} \tag{C-1}$$

where θ is given in hours. The temperature distribution within the wall for times from $\theta = 0$ to $\theta = 0.4$ hours separated by increments of 0.02 hours are to be specified.

To begin the analysis, notice that the temperature range considered is from 0°F to 800°F. Therefore, the thermal properties of k-monel for this range can be found from Ref. (22) to be

$$\rho = 523.0\ lb_m/ft^3$$

$$c = 0.1075\ Btu/lb_m{}^\circ F \qquad (C\text{-}2)$$

$$k = 9.5\ [1+(0.00093/{}^\circ F)t]Btu/hr\ ft\ {}^\circ F$$

where ρ and c are presented as averaged values. If the volumetric specific heat at the end temperatures are compared to the average value defined by the above ρ and c, a difference of approximately 10 percent will be found. Hence, the condition that the volumetric specific heat be constant is sufficiently satisfied. The variation in conductivity as described above does closely represent the actual variation. This satisfies the remaining requirement of the solution and indicates that the results can be employed with confidence.

A nodal system for the wall must now be specified. Since the wall is 0.5 ft thick, increments of 0.05 ft which define eleven nodal points are convenient. With this information, a time increment can be determined which will allow the stability condition,

$$\frac{(\Delta x)^2}{\alpha_o(\Delta\theta)(1+\beta t_M)} \geq 2.0 \quad ,$$

to be satisfied.

Solving this expression for the time increment indicates that

$$\Delta\theta \leq \frac{(\Delta x)^2}{(2.0)\alpha_o(1+\beta t_M)}$$

or more specifically

$$\Delta\theta \leq \frac{(0.05)^2\ hrs}{(2.0)\frac{(9.5)}{(523)(0.1075)}[1 + 0.00093(200)]}$$

$$\Delta\theta \leq 0.00424\ hrs.$$

Since the temperatures at time intervals of 0.02 hours are desired, a $\Delta\theta$ of 0.004 hours is convenient. This means that the temperatures should be obtained after consideration of groups or runs of five-time increments $\Delta\theta$ are completed. Also, as the solution is only to be carried to 0.4 hours , twenty of these five interval runs must be made.

Before writing this information in the form of input data for the programmed solution, the time dependent boundary temperature should be considered. This time dependent temperature requires that the boundary temperatures which occur at each $\Delta\theta$ increment considered be supplied to the machine. Equation (C-1) should be used in determining these temperatures. For this particular problem, one hundred increments are considered and therefore one hundred boundary temperature cards should be supplied.

With this understanding, the input data can now be formulated. For the reader's convenience, the Fortran notation defined in Appendix B is given above the actual input values. Hence, for this problem, the input data should appear as follows:

Card 1

RHO	SPHT	CONDO	BETA	DIM	DELT	TEMPM	J
523.0	0.1075	9.5	0.00093	0.5	0.004	800.0	11

Card 2

RUTIM	UM	BC	TIMO
5.0	20.0	0.0	0.0

Card 3 (temperatures at time TIMO)

TEMP(1)	TEMP(2)	TEMP(3)	TEMP(4)	TEMP(5)	TEMP(6)
0.0	400.0	400.0	400.0	400.0	400.0
TEMP(7)	TEMP(8)	TEMP(9)	TEMP(10)	TEMP(11)	
400.0	400.0	400.0	400.0	400.0	

Boundary Temperature Cards (100 Total)

TEMP(1)	TEMP(J)
0.0	404.0
0.0	408.0
0.0	412.0
.	.
.	.
.	.
0.0	800.0

By the definition of the term BC, if constant boundary temperatures had been specified, BC would equal 1.0 and only the temperatures on Card 3 would be required as input data. Also, notice that all the data except that defining the number of nodes, J, are written in decimal notation. This must be followed if the machine is to interpret the data correctly.

The actual locations, in feet, of the nodal points are defined with reference to the node defined by TEMP(1). For this problem, the surface at spatial location $x = 0$ is defined to be at a temperature of 0°F. Therefore, the remaining nodes are located with reference to that surface. For example, the temperature at spatial location $x = 0.5$ ft. is that defined by Eq. (C-1).

When this input information is supplied to the machine, the first set of data which will be received defines the stability condition used and reproduces the initial distribution. If a mistake has been made and the stability condition is not satisfied, after looking at the first data card, the machine will punch the value of the stability condition and will call for new data. If the condition is satisfied, the computer will punch that value, the time TIMO, and the initial distribution. These distribution cards will give the location of the node, the temperature at that node for the time TIMO, and the corresponding conductivity.

The next series of data sets will each be defined by a time card followed by distribution cards which give, for this new time, the same information as the distribution cards defined above. When the final time is reached, a set of cards will be obtained which defines that time and the corresponding temperatures at each node. These temperatures are punched on separate cards in the numerical order of the nodes; i.e., TEMP(1), TEMP(2), etc. This last set of data is only included in the program because of the convenience it offers when an extension of the time of analysis is desired. As should be expected, all output quantities are specified in terms of the same units used in defining the input information.

A study of the listing of the program as presented in Appendix B should now clarify any detail which may be of concern to the reader.

Two-Dimensional Example

To exemplify the application of the two-dimensional solution and program, consider the problem of specifying the temperature history of a long rectangular bar whose surfaces are maintained at constant temperatures. Assume that the bar is k-monel with an x-dimension of 0.4 ft and a y-dimension of 0.6 ft. At some time, $\theta = 0$, when the bar is at a uniform temperature of 800°F, the surfaces defined by $x = 0$ and $y = 0$ are instantaneously lowered to a constant temperature of 0°F. At the same time, the surfaces at $x = 0.4$ ft and $y = 0.6$ ft are forced to maintain the constant initial temperature of 800°F. The temperature distribution of the bar at times from $\theta = 0$ to $\theta = 0.5$ hrs. separated by increments of 0.05 hrs. is to be determined.

Since the same material and the same temperature range as considered in the one-dimensional example are in question here, the first part of the analysis is identical to that of the one-dimensional problem. The two-dimensional nodal system can be specified by considering the geometry of the bar and the imposed boundary temperatures. As the temperature difference from boundary to boundary is the same for both the x- and y-dimensions, the accuracy of the solution will depend more on the spatial increment chosen to represent the x-dimension than that used for the y-dimension. This is because the x-dimension is the smaller of the two. However, the geometry of the bar indicates that an increment of 0.05 ft magnitude for both the x- and y-dimensions would be convenient. Because this particular increment is still quite small compared to the x-dimension of the bar, the accuracy of the solution will not be severely reduced by choosing $\Delta x = \Delta y = 0.05$ ft. With these increments, the x-dimension of the bar is broken into 9 nodes while the y-dimension is separated into 13 nodes.

To determine a convenient time increment, the two-dimensional stability criteria

$$\frac{(\Delta x)^2 (\Delta y)^2}{\alpha_o \Delta\theta (\Delta x^2 + \Delta y^2)(1 + \beta t_M)} \geq 2.0$$

is employed. Substituting for the known quantities and solving for $\Delta\theta$ yields the condition,

$$\Delta\theta \leq 0.00212 \text{ hrs.}$$

Since the temperatures at times separated by intervals of 0.05 hrs. are desired, a convenient value of $\Delta\theta$ would then be 0.002 hrs. This would require the temperatures to be defined after each series of twenty-five time increments $\Delta\theta$ are completed. The total number of these twenty-five interval runs is governed by the final time considered which for this specific problem indicates that ten of these groups must be obtained.

As the boundary temperatures involved in this problem are constant, no additional calculations must be made. If the conditions were time and space dependent, a procedure similar to that used in the one-dimensional example would be followed in generating the input data. For this example, the input data should appear as follows. Please note that the Fortran notation has been presented above the specific values of the quantities involved.

Card 1

RHO	SPHT	CONDO	BETA	DIMX	DIMY	DELT	TEMPM
523.0	0.1075	9.5	0.00093	0.4	0.6	0.002	800.0

J	K
9	13

Card 2

RUTIM	UM	BC	TIMO
25.0	10.0	1.0	0.0

On Card 3 the choice of the temperatures specified for the corners of the bar, defined by (1,1), (1,13), (9,1), and (9,13), is an arbitrary one since these never enter into the calculations. Hence, Card 3 (temperatures at time TIMO)

TEMP(1,1)	TEMP(1,2)	. . .	TEMP(1,12)	TEMP(1,13)
0.0	0.0		0.0	0.0
TEMP(2,1)	TEMP(2,2)	. . .	TEMP(2,12)	TEMP(2,13)
0.0	800.0		800.0	800.0
TEMP(3,1)	TEMP(3,2)	. . .	TEMP(3,12)	TEMP(3,13)
0.0	800.0		800.0	800.0
.	.		.	.
.	.		.	.
.	.		.	.
TEMP(8,1)	TEMP(8,2)	. . .	TEMP(8,12)	TEMP(8,13)
0.0	800.0		800.0	800.0
TEMP(9,1)	TEMP(9,2)	. . .	TEMP(9,12)	TEMP(9,13)
800.0	800.0		800.0	800.0

As in the one-dimensional program, the actual locations, in feet, of the nodal points are defined with reference to the node defined by TEMP(1,1). This node is always to be located at one corner of the bar so that all the coordinates which describe the locations of the other nodes will have positive values. To follow the standard right hand coordinate system, the lower left hand corner of the bar should be taken as Node (1,1).

The results obtained from this program are of exactly the same form as those specified by the one-dimensional program. Of course, to locate a node in this case requires two coordinates. Hence, on the temperature distribution cards, the first value indicates the x-coordinate, the second value denotes the y-coordinate, the third defines the temperature of that node and the fourth provides the corresponding thermal conductivity.

The two-dimensional program listing provided in Appendix B should be referred to if any further explanation of the program is desired.

VITA

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