DETERMINATION OF PARAMETERS AND OPERATIONS IN DYNAMIC SYSTEMS

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NOMENCLATURE

Symbols are defined throughout the text as they are used to describe physical or mathematical quantities. The most important symbols and the quantities they represent are as follows:

a	System parameter or constant coefficient appearing in a differential equation.
c	Exponential series coefficient.
$c(\dot{y})$	Damper force function, dependent on $\dot{y}$.
$f(t)$	Arbitrary time dependent function.
$k(y)$	Spring force function, dependent on y.
m	Mass term.
s	Laplace transform variable.
t	Independent variable, time.
x, $x(t)$	Time dependent system input variable.
$\dot{x}$, $\ddot{x}$	Time derivatives of the system input.
y, $y(t)$	Time dependent system output variable.
$\dot{y}$, $\ddot{y}$	Time derivatives of the system output.
w	Relative weight of a residual.
A	Smoothing operator.
B	Differentiating operator.
$E\left[f(t)\right]$	Error in the bracketed function, $f(t)$.
$F(s)$	Laplace transform of the function, $f(t)$.
$F(\omega)$	Fourier transform of the function, $f(t)$.

$G(a)$ . . . . . Sum of the squares of the residuals or errors.

P . . . . . Differential equation operator.

R . . . . . Residual or error.

W . . . . . Linear differential equation operator.

S . . . . . Scale factor.

λ . . . . . Exponential series coefficient.

σ . . . . . Real part of the complex variable s.

τ . . . . . Particular value of the independent variable time.

ω . . . . . Imaginary part of the complex variable s.

Δ, Δt . . . . . Independent variable time increments.

CHAPTER I

INTRODUCTION

1-1. Statement of the Problem

The primary objective of this study is concerned with the determination of an appropriate mathematical description of an unknown dynamic system, linear or nonlinear, subjected to time varying inputs. Applicable systems include those which are mathematically modeled by ordinary differential equations and classified by the descriptive words as lumped parameter time invariant systems. The method described herein utilizes input-output relationships which are experimentally measured in the form of continuous or discrete plots versus the independent variable time.

The problem may be symbolically represented by the single input-output system shown in Figure 1-1.1.*

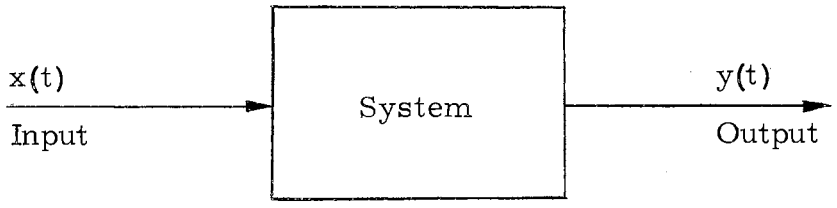

Figure 1-1.1. Single Input-Output System.

The system is subjected to the prescribed input x(t) while simulta-

* The case of multiple inputs and outputs will be treated in Chapter II.

neously observing the output variable y(t). The fundamental problem is then, what set of operations are present such that the system may be characterized by an ordinary differential equation of the form

$$W\left[x(t)\ ,\ y(t)\right] = 0 \quad , \tag{1-1.1}$$

where W represents a sum of differential and multiplicative operations on the variables x(t) and y(t). For example, the operational equation (1-1.1) might take the form

$$a_1\ddot{y} + a_2\dot{y}y + a_3\dot{x}y = x \quad , \tag{1-1.2}$$

where the dot notation implies time derivatives. This thesis presents a method by which the system parameters or constant coefficients a_1, a_2 and a_3 in equation (1-1.2) may be determined with the associated implication that should any of the parameters vanish, the corresponding operation is a non-contributory relation between the input and output. In brief, the method concentrates on the numerical evaluation of the differential and multiplicative operations in the operational equation, i.e., $\ddot{y}$, $\dot{y}y$ and $\dot{x}y$ in equation (1-1.2), at a sufficient number of independent time values such that the problem is reduced to the solution of a set of linear simultaneous equations for the parameters. Because the operations can only be approximately evaluated, numerous evaluations are made and the parameters determined in a minimal or least squares sense.

1-2. Existing Methods of Determination

The vast majority of past effort associated with this problem has been confined to systems which are identified as linear, a linear

system being one which is completely characterized by a set of linear ordinary differential equations. With very little exception, past effort has been centered on the determination of the system transfer function. In linear system theory, a very powerful and elegant relationship is obtained by resorting to the notion of a transfer function. This is accomplished by a mathematical transformation, the most popular being the Laplace transformation. This transformation is a mathematical operation indicated symbolically by $\mathcal{L}[f(t)]$, where the associated operation is

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt = F(s) \quad , \tag{1-2.1}$$

and s is the complex variable $s = \sigma + j\omega$ with $j = \sqrt{-1}$. The symbolic representation of Figure 1-1.1 is identified as a block diagram in linear system theory and the associated set of linear differential equation operations is called the system or Laplace transfer function W(s). The relation between the transformed input X(s) and output Y(s) then takes the form

$$Y(s) = W(s)\, X(s) \quad , \tag{1-2.2}$$

and

$$\frac{Y(s)}{X(s)} = W(s) \quad . \tag{1-2.3}$$

Another transformation ideally suited to the experimental identification problem is the Fourier transform $\mathcal{F}[f(t)]$, where the associated mathematical operation is

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = F(\omega) \quad . \tag{1-2.4}$$

Utilizing this transformation, the system function becomes the Fourier transfer function

$$\frac{Y(\omega)}{X(\omega)} = W(\omega) \quad . \qquad (1\text{-}2.5)$$

The Fourier transfer function is in general complex and is usually resolved into its real and imaginary parts

$$W(\omega) = A(\omega) + jB(\omega) \quad , \qquad (1\text{-}2.6)$$

which are identified as the magnitude and phase components respectively.

The most obvious method for experimentally determining the Fourier transfer function of a given system is to subject the system to a sinusoidal input, $x(t) = \bar{x} \sin \omega t$, and measure the magnitude and phase output components. This is termed a frequency response test and yields the Fourier transfer function directly and intuitively. The prime disadvantage of this test is its length and cost because of the numerous input frequencies ω required to develop the system frequency spectrum. The magnitude and phase components of the Fourier transfer function are dependent only on the forcing frequency and not on the input amplitude. The fact that $A(\omega)$ and $B(\omega)$ are independent is a condition for the linearity of the system. If A and B do depend on the input amplitude $\bar{x}$, the experimental functions $A(\bar{x}, \omega)$ and $B(\bar{x}, \omega)$ constitute a kind of generalized transfer function, which is called the describing function.[1]* The describing function in general yields very little information regarding the differential equation behavior of the system. Gibson[2]

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*Numbers in parentheses refer to references in the Bibliography.

presents an analytic approach to the inverse describing function; however, the system or block diagram is restricted to a single nonlinear element.

The frequency response test has given way in recent years to a controlled transient test in which the input signal is a simulated impulse. An impulse or unit pulse is an input signal x(t) which exists during the small time interval between t = 0 and t = δ. During this time interval the value of x(t) jumps from zero at t = 0 to the large constant value $x(t) = \frac{1}{\delta}$ and then returns to zero at t = δ. If such an input signal can be physically implemented without damaging the system, the output is identified as the system impulse response and the Fourier transfer function takes the particularly simple form

$$W(\omega) = \int_0^\infty y(t)\, e^{-j\omega t}\, dt \quad . \qquad (1\text{-}2.7)$$

This integral is easily resolved into its real and imaginary parts, such that magnitude and phase characteristics can be established. The ideal unit pulse is extremely difficult to achieve in practical situations. For such cases, an approximate pulse is utilized which raises certain practical considerations in the determination of the transfer function.[3, 4]

It is important to recognize that the frequency response and impulse tests only yield the Fourier transfer function. The technique of determining the system parameters and/or linear differential equation behavior from the frequency or impulse response is identified as a synthesis problem in linear system theory. Sanathanan and Koerner[5] have described a method of curve fitting a ratio of two complex polynomials to the Fourier transfer function as an approxi-

mation to the transformed differential equation. Kardashov and Karniushin[6] have presented a method of parameter determination utilizing a complex-plane plot of the system frequency response.

A method for obtaining the impulse response of a linear system by using white noise as the input and obtaining the cross-correlation between the noise input and system output has been described by Truxal.[7] The white noise input signal is defined as a random signal with a flat frequency spectrum over all frequencies. The practical difficulties in achieving such an ideal signal are similar to the ideal unit pulse simulation.

Recent efforts in the parameter determination problem which pursue solutions in the time domain rather than the transform domain include the work of Potts, Ornstein and Clymer[8] who used a steepest descent method to establish the parameters in a mathematical model simulation of a human operator for an airplane. The required derivatives were obtained on an analog computer for an essentially error free output. Kumar and Sridhar[9] have presented an identification method for systems possessing a state model description, i.e., described by a set of first order differential equations. They adjoin to the state model an auxiliary set of first order differential equations in which the derivatives of the desired constant parameters are set equal to zero. Then by observing the state variables at some value of time, the method treats the adjoined state model as a boundary value problem and iteratively determines the parameters by a computational procedure discussed by Bellman, Kagiwada and Kalaba.[10] Published accounts thus far have been confined to completely noise free systems.

The above discussion briefly summarizes the past concentration

of effort on the problem of system identification. As noted earlier, this concentration has been largely directed towards the determination of the system transfer function. The transfer function concept is most appealing; however, it appears limited like transform methods in general to linear system behavior. The method of this thesis concentrates on the differential equation form relating input and output in the time domain, utilizes no variable transformations and addresses itself to either the linear or nonlinear problem. It is believed that by attacking the problem in this manner, while utilizing the best of present day technology and machinery, a more than adequate solution to the problem can be obtained.

CHAPTER II

PRESENTATION OF THE METHOD

2-1. Determination of System Parameters

In this section, the method of parameter determination for multiple input-output systems is presented. In Figure 2-1.1,

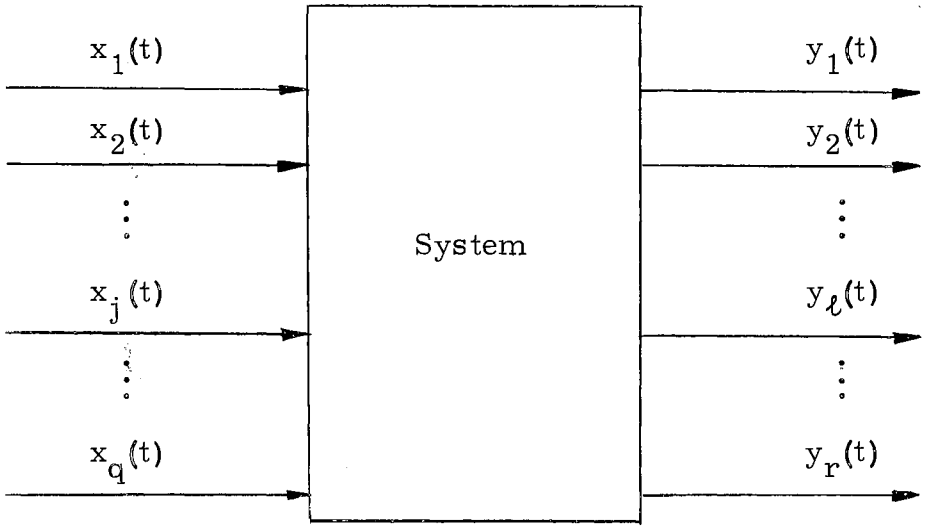

Figure 2-1.1. Multiple Input-Output System.

the set of inputs $x_j(t)$ and outputs $y_\ell(t)$ constitute a measured set of variables in the dynamic process. A measured set of variables is defined as that set which will permit the system to be completely described by a set of ordinary differential equations of the form

$$\begin{aligned} P\left[a_k,\ x_j(t),\ y_\ell(t)\right] &= 0 \quad , \\ j &= 1,\ 2,\ \ldots,\ q \quad , \\ \ell &= 1,\ 2,\ \ldots,\ r \quad , \\ k &= 1,\ 2,\ \ldots,\ m \quad . \end{aligned} \tag{2-1.1}$$

The set of equations (2-1.1) will be hereinafter referred to as simply the system equation. The operator P appearing in the system equation represents a sum of operations on the measured variables $x_j(t)$ and $y_\ell(t)$, admissible operations include derivatives, integrals and products of the measured variables. A constant parameter a_k is associated with each of the m distinct operations included in the system equation.

The unknown operations in the system equation are then determined by either differentiating or integrating the measured variables a sufficient number of times, such that each of the m operations included in the system equation may be numerically represented as a function of the independent variable time at a discrete number of points n.* The system equation then takes the form

$$P\left[a_k,\ x_j(t),\ y_\ell(t)\right]_i = 0 \quad , \quad i = 1,\ 2,\ \ldots,\ n, \qquad (2\text{-}1.2)$$

where n represents the number of distinct evaluations of each of the m operations in the system equation.

Consideration of the following illustrative example will aid in fixing ideas at this point. Let the system equation be represented by the following nonlinear ordinary differential equation

$$a_1\ddot{y}_1 + a_2 y_1 \dot{y}_2 + a_3 \dot{x}_1 y_1^2 + a_4 + x_2 = 0 \quad , \qquad (2\text{-}1.3)$$

where the coefficients a_1, a_2, a_3, a_4 denote the unknown parameters which are to be determined. Integrating the system equation once

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*The subject of obtaining derivatives and integrals from experimental data is discussed in Chapter III.

yields

$$a_1\left[\dot{y}_1(\tau) - \dot{y}_1(\tau_o)\right] + a_2 \int_{\tau_o}^{\tau} y_1\dot{y}_2 dt + a_3 \int_{\tau_o}^{\tau} \dot{x}_1 y_1^2 dt$$

$$+ a_4\left[\tau - \tau_o\right] + \int_{\tau_o}^{\tau} x_2 dt = 0 \quad . \qquad (2\text{-}1.4)$$

In many cases, the value of $y_1(\tau_o)$ is only approximately known and if its value is subject to question, the system equation may be alternately written

$$a_1\dot{y}_1(\tau) + a_2 \int_{\tau_o}^{\tau} y_1\dot{y}_2 dt + a_3 \int_{\tau_o}^{\tau} \dot{x}_1 y_1^2 dt + a_4\left[\tau - \tau_o\right]$$

$$+ a_5 + \int_{\tau_o}^{\tau} x_2 dt = 0 \quad , \quad a_5 = -a_1\dot{y}_1(\tau_o) \quad . \qquad (2\text{-}1.5)$$

Thus the problem is now to determine the five parameters in the above equation. Each of the derivatives and integral operations of equation (2-1.5) are then evaluated at a discrete number of points by performing the indicated operations on the measured time series data. The only task remaining is to select five values of τ along the time series in such a way that the indicated operations constitute an independent set, insert the numerical values of these operations in the system equation (2-1.5) and solve the resulting set of five linear simultaneous equations for the desired parameters.

In the general problem, the numerical values of the operations are only approximately known; hence, numerous evaluations are made, some of which may be dependent. The basic problem is then to determine the parameters a_k in equation (2-1.2) such that the system

equation is satisfied for each of the n discrete evaluations in a minimal sense. In general, the n linear simultaneous equations represented by equation (2-1.2) are highly overdetermined, i.e., n, the number of equations is greater than m, the number of unknown parameters. The most popular technique used in obtaining solutions for overdetermined sets is the method of least squares. This is accomplished in the present problem of parameter determination by defining a weighted residual R_i associated with each of the n discrete equations (2-1.2)

$$R_i = P\left[a_k,\ x_j(t),\ y_\ell(t)\right]_i (w_i) \quad , \quad i = 1,\ 2,\ \ldots,\ n, \qquad (2\text{-}1.6)$$

where w_i denotes the relative weight of the residual R_i; and then minimizing the sum of the squares of the weighted residuals

$$G(a_k) = \sum_{i=1}^{n} R_i^2\left[a_k,\ x_j(t),\ y_\ell(t)\right] \quad , \qquad (2\text{-}1.7)$$

by

$$\frac{\partial G}{\partial a_k} = 0 \quad , \quad k = 1,\ 2,\ \ldots,\ m \quad . \qquad (2\text{-}1.8)$$

The residual weight w_i is generally restricted to a number between zero and one, and associates with the residual R_i a relative value in the formulation of G(a). For example, if near dependent entries are included in the formulation, they can be weighted accordingly or if the numerical value of an evaluated operation is subject to question, it could be assigned a reduced weight.

The above problem is identified by different descriptions in

several disciplines. To the statistician, it represents a problem in linear regression analysis; to the numerical analyst, a best solution to overdetermined linear equations or the hyper-plane of best fit in the m-dimensional vector space and to the systems analyst, the integral error squared problem. The problem is well documented in the literature.[11, 12, 13] The least squares description will suffice for this discussion, primarily because it provides the weighted residual concept and the statistical parameter G(a) given by equation (2-1.7) to aid in the selection of a mathematical model of best fit. For example, if the system equation has several alternate forms, that form which possesses a minimum sum of the squares of the residuals will constitute the solution of best fit in a least squares sense. This point will be demonstrated in Chapter IV.

2-2. Exponential Curve Fitting and Linear System Theory

Quite often concurrent developments and research occur in the physical sciences in seemingly divorced disciplines when in actuality the identical mathematical problem is being attacked. It is this writer's observation that this is precisely the case in the fields of linear system identification and the problem of exponential curve fitting. The necessity of exponential curve fitting has received the attention of investigators in the fields of radio active decay and the biological sciences. The basic problem is to observe a process which has been disturbed by some external influence and then to fit a curve of the form

$$f(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} \qquad (2\text{-}2.1)$$

to the measured data. Perl[14] summarizes some preliminary attempts to fit exponential functions and Worsley and Lax[15] present an extensive bibliography associated with the problem.

The physical analogy of the linear system identification problem and exponential decay problem is quite simple. If the linear system has been subjected to an input disturbance, the resulting output behavior, being the solution of a linear ordinary differential equation, can be completely characterized by the exponential series given by equation (2-2.1). The value of f(t) would constitute the system output y(t) and the coefficients c_i and λ_i would in general be complex. However, because y(t) is real, the constants c_i and exponential coefficients λ_i must occur in conjugate pairs such that the resulting sum is real. The exponential curve fit problem in general represents a decay phenomenon and the exponential coefficients λ_i are ordinarily restricted to real negative numbers. It is well known that a linear time invariant system may be described by a linear ordinary differential equation with constant coefficients[1] of the form

$$a_1 \frac{d^m y}{dt^m} + a_2 \frac{d^{m-1} y}{dt^{m-1}} + \cdots + a_m \frac{dy}{dt} + a_{m+1} y = x \quad . \quad (2\text{-}2.2)$$

Either problem, i.e., linear system identification or exponential curve fitting, can be reduced to the determination of the constant coefficients in equation (2-2.2). Hudson[16] noted this similarity while engaged in an investigation of the exponential curve fit problem. Once these coefficients are known, the Laplace or Fourier transfer function can be easily established. If the differential equation (2-2.2) is

Laplace transformed, the zeros of the resulting characteristic equation are precisely the exponential coefficients of the corresponding exponential series (2-2.1). The coefficients c_i of the exponential series depend upon the particular input signal x(t) and/or the initial conditions of the system.

The differential equation (2-2.2) represents the most general form of the system equation (2-1.1) for the case of a linear system. The linear system equation, unlike the nonlinear case may be integrated as many times as necessary to completely circumvent the necessity of estimating derivatives. For example, integrating equation (2-2.2) once yields

$$a_1 \frac{d^{m-1}y(\tau)}{dt^{m-1}} + a_2 \frac{d^{m-2}y(\tau)}{dt^{m-2}} + \cdots + a_m y(\tau) + a_{m+1} \int_{\tau_o}^{\tau} y\,dt + a_{m+2} = \int_{\tau_o}^{\tau} x\,dt \quad , \tag{2-2.3}$$

where a_{m+2} denotes a constant of integration and is given by

$$a_{m+2} = -a_1 \frac{d^{m-1}y(\tau_o)}{dt^{m-1}} - a_2 \frac{d^{m-2}y(\tau_o)}{dt^{m-2}} - \cdots - a_m y(\tau_o) \quad . \tag{2-2.4}$$

The derivatives appearing in this equation are rarely known; hence, the coefficient a_{m+2} must be treated as an additional unknown parameter in equation (2-2.3). Performing the integration m successive times yields

$$a_1 y(\tau) + a_2 \int_{\tau_o}^{\tau} y dt + a_3 \int_{\tau_o}^{\tau} \int y dt^2 + \ldots + a_{m+1} \int_{\tau_o}^{\tau} \int \ldots \int y dt^m$$

$$+ a_{m+2} \frac{\tau^{m-1}}{(m-1)!} + \ldots + a_{2m} \tau + a_{2m+1} = \int_{\tau_o}^{\tau} \int \ldots \int x dt^m \quad . \tag{2-2.5}$$

Thus if the indicated integral operations and power terms are numerically evaluated at a sufficient number of discrete values of τ along the time series, the problem is again reduced to the best set of parameters in a least squares sense. The numerical evaluation of the integral operations is far superior to the estimation of derivatives from the standpoint of accuracy; however, in most cases the additional parameters $a_{m+2}, a_{m+3}, \ldots, a_{2m+1}$ must also be determined in the minimization process. Hudson[16] noted the advantage of the finite range of integration associated with the integral operations in equation (2-2.5) as opposed to the infinite range when using the Fourier transform, equation (1-2.7).

Certain nonlinear differential equations can be successively integrated to yield forms which also circumvent the need of estimating derivatives. For example, the well known van der Pol equation

$$\ddot{y} + a_1 \dot{y} + a_2 \dot{y} y^2 + a_3 y = 0 \tag{2-2.6}$$

may be alternately written

$$\ddot{y} + a_1 \dot{y} + \frac{a_2}{3} \frac{d}{dt} (y^3) + a_3 y = 0 \quad . \tag{2-2.7}$$

Integrating the alternate form once gives

$$\dot{y} + a_1 y + \frac{a_2}{3} y^3 + a_3 \int y dt + a_4 = 0 \quad , \tag{2-2.8}$$

and by a second integration the final desired mathematical model is obtained

$$y + a_1 \int y dt + \frac{a_2}{3} \int y^3 dt + a_3 \int\int y dt^2 + a_4 t + a_5 = 0 \quad . \tag{2-2.9}$$

2-3. Determination of System Operations

Quite often the exact form of the system equation is unknown to the systems analyst. For such cases, the method presented herein may in some instances prove to be a valuable aid in selecting the correct mathematical model for the system. To demonstrate this, suppose the system of Figure 2-1.1 is exactly characterized by a system equation of the form

$$P\left[a_k, x_j(t), y_\ell(t)\right] = 0 \quad . \tag{2-3.1}$$

If, however, the system equation had been erroneously assumed as

$$P\left[a_k, x_j(t), y_\ell(t)\right] + P_1\left[a_{k1}, x_{j1}(t), y_{\ell 1}(t)\right] = 0 \quad , \tag{2-3.2}$$

then each of the parameters a_{k1} in the assumed differential equation must be identically zero if the system is to be exactly described by equation (2-3.1).

In this connection, if the method described herein is to be used effectively for the purpose of operation identification, it is absolutely necessary to assume the most general differential equation behavior possible to insure that the correct set of operations given by equation (2-3.1) is included in the assumed set of operations. Recognizing that

the numerical evaluation of system operations depends heavily on approximated derivatives and integrals, which are in turn obtained from experimental data; it is not advocated that the method described herein will identically determine the contributory operations in an assumed general differential equation. However, quite often the analyst has some apriori knowledge of the dynamic process and in many cases merely wishes to assess the contributory effect of certain restricted operations, which are superimposed on a mathematically derived system model. It is felt that the method presented herein satisfies this need.

2-4. Adequacy of the Determined Parameters

The above sections have presented a method of determination for system parameters and operations; in general, the method will always yield a solution to either problem. In some cases, the sum of the squares of the residuals will serve as a criterion for the acceptance or rejection of the established mathematical model. The most critical test of the validity of the derived model is obtained by solving the system equation using the parameters and/or operations established by the least squares solution. A comparison of the differential equation solution to the experimentally obtained solution in general will provide a very strong criterion for acceptance or rejection.

CHAPTER III

DATA ANALYSIS

3-1. Smoothing and Differentation

The definition of the derivative as the limit of $\Delta y/\Delta t$ is of little value if the dependent variable y represents experimental data. It is well known that the ratio $\Delta y/\Delta t$ becomes extremely sensitive if Δt becomes very small, particularly if the values of y contain even small errors. The problem is generally resolved by resorting to a least squares method.[3] Two approaches are in current use to establish the derivative of an empirical function; they are approximation in the large and approximation in the small.

Approximation in the large refers to the concept of approximating the entire set or a large portion of the data by an empirical differentiable function and then differentiating this fitted function. Some of the most popular empirical functions are the Fourier series, power series, and, in particular, the orthogonal polynomials.

Personal experience, as well as the reported experiences of other investigators,[3, 17] has led the author to favor approximation in the small or the so called neighborhood technique. Approximation in the small utilizes data only in the neighborhood of a point where the derivative is desired. By considering only local data, the computational problem is considerably reduced both from the standpoint of machine time and storage requirements. In general, it is first

necessary to apply smoothing techniques to raw digital data before differentiation is attempted. However, it is possible to integrate standard numerical differentiation methods with smoothing formulas to yield a single set of formulas for a combined smoothing and differentiation operation. This technique is called a moving arc method and is employed in this thesis to obtain derivatives where required. The method is fully documented by several authors[3, 18] and only general formulas are given here.

It is desired to approximate the derivative $\dot{y}_n$ of the empirical function shown as discrete equidistant data in Figure 3-1.1.

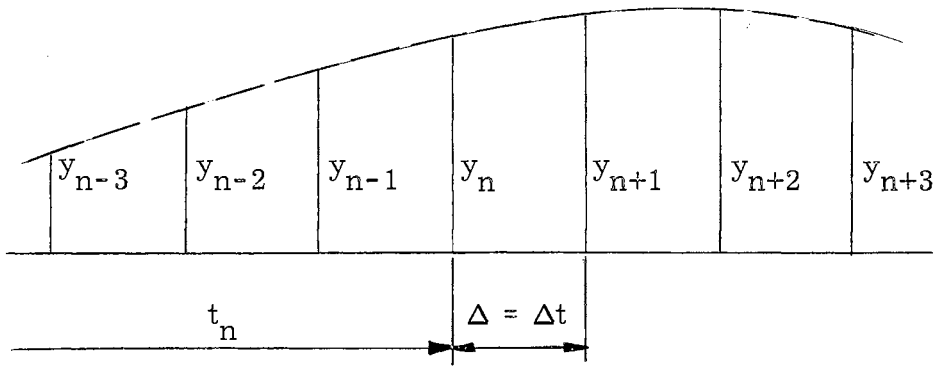

Figure 3-1.1. Equidistant Discrete Data.

This is accomplished by fitting a least squares low order polynomial to the data in the neighborhood of the point and using the derivative of the approximating polynomial as an approximation to the derivative. Two popular functions are a second order polynomial fitted to five points in the neighborhood of t_n and a third order polynomial fitted to seven adjacent points. If the neighborhood is taken to each side of t_n, the approximating function will yield symmetric centrally weighted coefficients.

The second order polynomial may be written

$$y = A + B\tau + C\tau^2 \qquad (3\text{-}1.1)$$

fitted to the following data points

$$\tau = -2\Delta \quad , \quad y = y_{n-2} \quad ;$$

$$\tau = -\Delta \quad , \quad y = y_{n-1} \quad ;$$

$$\tau = 0 \quad , \quad y = y_n \quad ;$$

$$\tau = \Delta \quad , \quad y = y_{n+1} \quad ;$$

$$\tau = 2\Delta \quad , \quad y = y_{n+2} \quad .$$

The coefficient A in the above equation is called a smoothing operator

$$A = \frac{1}{35}\left[17y_n + 12(y_{n+1} + y_{n-1}) - 3(y_{n+2} + y_{n-2})\right] \quad , \qquad (3\text{-}1.2)$$

and the coefficient B a differentiating operator

$$B = \dot{y}_n = \frac{1}{10\Delta}\left[2y_{n+2} + y_{n+1} - y_{n-1} - 2y_{n-2}\right] \quad . \qquad (3\text{-}1.3)$$

Second derivatives are obtained by applying equation (3-1.3) to the approximated derivatives

$$\ddot{y}_n = \frac{1}{10\Delta}\left[2\dot{y}_{n+2} + \dot{y}_{n+1} - \dot{y}_{n-1} - 2\dot{y}_{n-2}\right] \quad . \qquad (3\text{-}1.4)$$

When this formula is expanded, the following result is obtained

$$\ddot{y}_n = \frac{1}{100\Delta^2}\left[4(y_{n+4} + y_{n-4}) + 4(y_{n+3} + y_{n-3}) + (y_{n+2} + y_{n-2})\right.$$

$$\left. - 4(y_{n+1} + y_{n-1}) - 10y_n\right] \quad . \qquad (3\text{-}1.5)$$

The corresponding formulas for a third order polynomial fitted to seven points are

$$A = \frac{1}{21}\left[7y_n + 6(y_{n+1} + y_{n-1}) + 3(y_{n+2} + y_{n-2}) - 2(y_{n+3} + y_{n-3})\right] \quad , \tag{3-1.6}$$

$$\dot{y}_n = \frac{1}{252\Delta}\left[58(y_{n+1} - y_{n-1}) + 67(y_{n+2} - y_{n-2}) - 22(y_{n+3} - y_{n-3})\right] \quad , \tag{3-1.7}$$

$$\ddot{y}_n = \frac{1}{(252\Delta)^2}\left[484(y_{n+6} + y_{n-6}) - 2948(y_{n+5} + y_{n-5}) + 1937(y_{n+4} + y_{n-4}) + 7772(y_{n+3} + y_{n-3}) + 5916(y_{n+2} + y_{n-2}) - 4824(y_{n+1} + y_{n-1}) - 16674y_n\right] \quad . \tag{3-1.8}$$

3-2. Integration and Scaling

This section considers two other requirements necessary for the numerical implementation of this study. These are numerical integration and operation scaling. Numerical integration involves the determination of the area under a given curve where in most cases the curve is a set of discrete data. This subject is amply discussed in most any reference dealing with numerical analysis. Numerical integration unlike differentiation tends to smooth data which contains error; however, it is advisable to first smooth the data before actually performing the integration. The technique used in this study was the well known Simpson formula which determines the area under the second order polynomial uniquely fitted to three points.

The problem of operation scaling can best be described by

means of a simple example. In the differential equation

$$a_1\dot{y} + a_2\dot{y}y + a_3y^2 = x \qquad (3\text{-}2.1)$$

it is assumed that the values of y, $\dot{y}$ and x have been determined for n data points. Thus equation (2-1.7) takes the form

$$G(a) = \sum_{i=1}^{n}(a_1\dot{y}_i + a_2\dot{y}_iy_i + a_3y_i^2 - x_i)^2 \qquad (3\text{-}2.2)$$

for the assumed case of equal weights, i.e., $w_i = 1$. When this expression is minimized with respect to the unknown parameters a_1, a_2 and a_3, the following set of linear simultaneous equations is obtained

$$\begin{bmatrix} \sum_{i=1}^{n}\dot{y}_i^2 & \sum_{i=1}^{n}\dot{y}_i^2y_i & \sum_{i=1}^{n}\dot{y}_iy_i^2 \\ \sum_{i=1}^{n}\dot{y}_i^2y_i & \sum_{i=1}^{n}\dot{y}_i^2y_i^2 & \sum_{i=1}^{n}\dot{y}_iy_i^3 \\ \sum_{i=1}^{n}\dot{y}_iy_i^2 & \sum_{i=1}^{n}\dot{y}_iy_i^3 & \sum_{i=1}^{n}y_i^4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n}\dot{y}_ix_i \\ \sum_{i=1}^{n}\dot{y}_iy_ix_i \\ \sum_{i=1}^{n}y_i^2x_i \end{bmatrix} . \qquad (3\text{-}2.3)$$

If the magnitude of $\dot{y}$ is considerably different from that for y, which is generally the case, the resulting matrix equation (3-2.3) is poorly conditioned for inversion, due to the large variations in the matrix elements. This problem can be partially obviated by introducing scale factors. Replacing y by y/S_1 and $\dot{y}$ by y/S_2 the differential equation becomes

$$(a_1 S_2)\left(\frac{\dot{y}}{S_2}\right) + (a_2 S_2 S_1)\left(\frac{\dot{y}}{S_2}\right)\left(\frac{y}{S_1}\right) + (a_3 S_1^2)\left(\frac{y}{S_1}\right)^2 = x \quad . \qquad (3\text{-}2.4)$$

If the scale factors S_1 and S_2 are properly chosen, the resulting matrix equation will be reasonably well conditioned.

CHAPTER IV

NUMERICAL APPLICATIONS

4-1. Second Order Quasi-Linear System

Determination of the parameters for a second order quasi-linear system is demonstrated in this section. The system studied is shown in Figure 4-1.1.

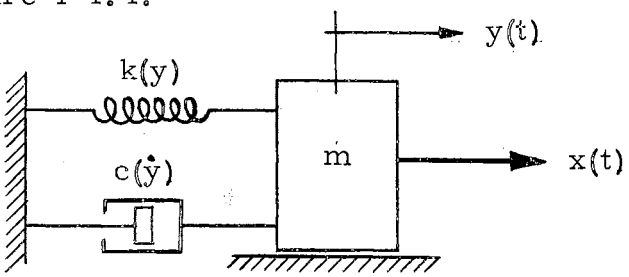

Figure 4-1.1. Second Order Quasi-Linear System.

The governing differential equation for this system is

$$m\ddot{y} + c(\dot{y}) + k(y) = x(t) \quad , \qquad (4\text{-}1.1)$$

where x(t) denotes a known input or excitation and y(t) the system output. The variables and components will be referred to in general system terminology as follows:

y = displacement or output,

$\dot{y}$ = velocity,

$\ddot{y}$ = acceleration,

x = input,

m = mass,

$c(\dot{y})$ = damper force,

k(y) = spring force.

In order to obtain data for the problem, values of the parameters and input were arbitrarily selected as follows

$$
\begin{aligned}
m &= 0.051813 \quad , \\
c(\dot{y}) &= 0.5(\dot{y} + 0.005\dot{y}^3) \quad , \quad \dot{y} \le 0 \quad , \\
c(\dot{y}) &= \dot{y} + 0.010\dot{y}^3 \quad , \quad \dot{y} \ge 0 \quad , \qquad (4\text{-}1.2) \\
k(y) &= 30(y - 0.10y^2) \quad , \\
x(t) &= 60 \sin 4\pi t \quad .
\end{aligned}
$$

Plots showing the above force vs. velocity and force vs. displacement relationships for the damper and spring are displayed in Figures 4-1.2 and 4-1.3 respectively. Data representing the output y(t) was obtained by numerically solving* the differential equation (4-1.1) with the following initial conditions

$$y(0) = 0 \quad , \quad \dot{y}(0) = 0 \quad . \qquad (4\text{-}1.3)$$

These data will hereinafter be referred to as the exact or observed data. Plots of the displacement y, velocity $\dot{y}$, and acceleration $\ddot{y}$ are shown in Figures 4-1.4, 4-1.5 and 4-1.6. Digital data resulting from the numerical solution is given in Appendix A. The calculated values of the output y are rounded to two significant decimals and constitute the exact or observed data used in all subsequent computations. The differential equation solution was not carried past one second as the system, being heavily damped, essentially demonstrates steady state behavior past 0.82 seconds as evidenced by the phase plane portrait displayed in Figure 4-1.7.

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*Numerical solutions in this study were obtained by the fourth order Runge-Kutta method.[19]

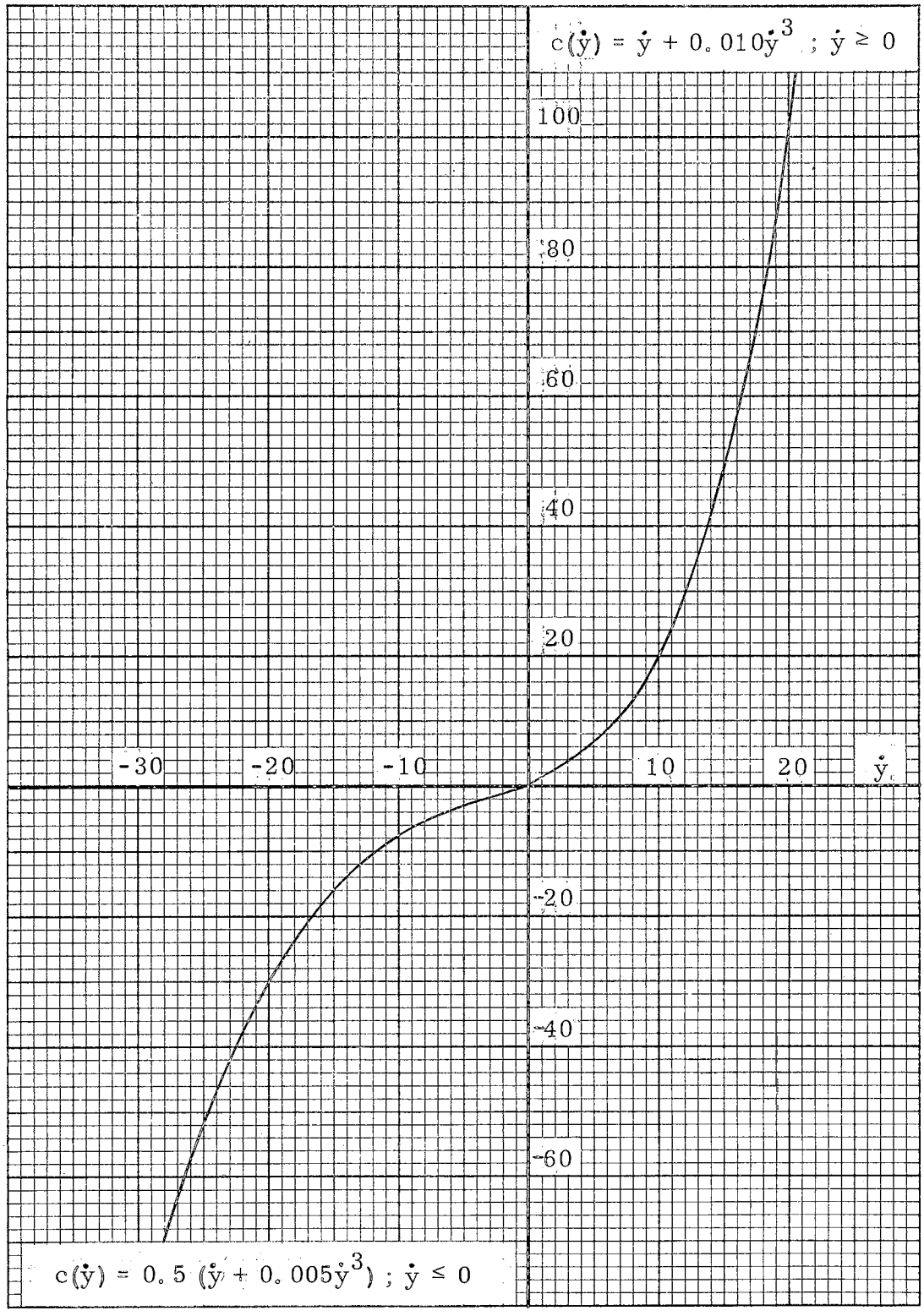

Figure 4-1.2. Damper Force $c(\dot{y})$ vs. Velocity $\dot{y}$.

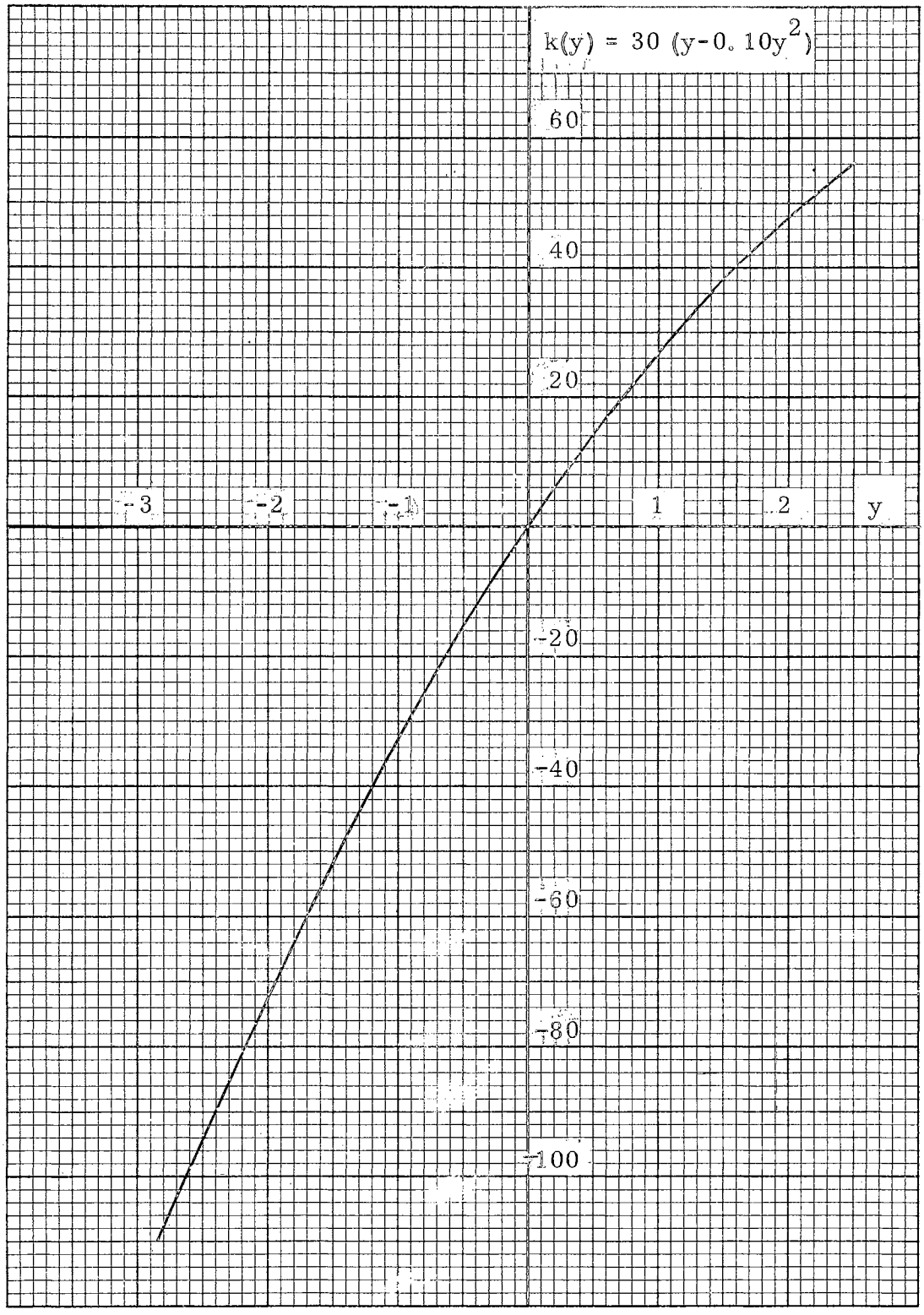

Figure 4-1.3. Spring Force k(y) vs. Displacement y.

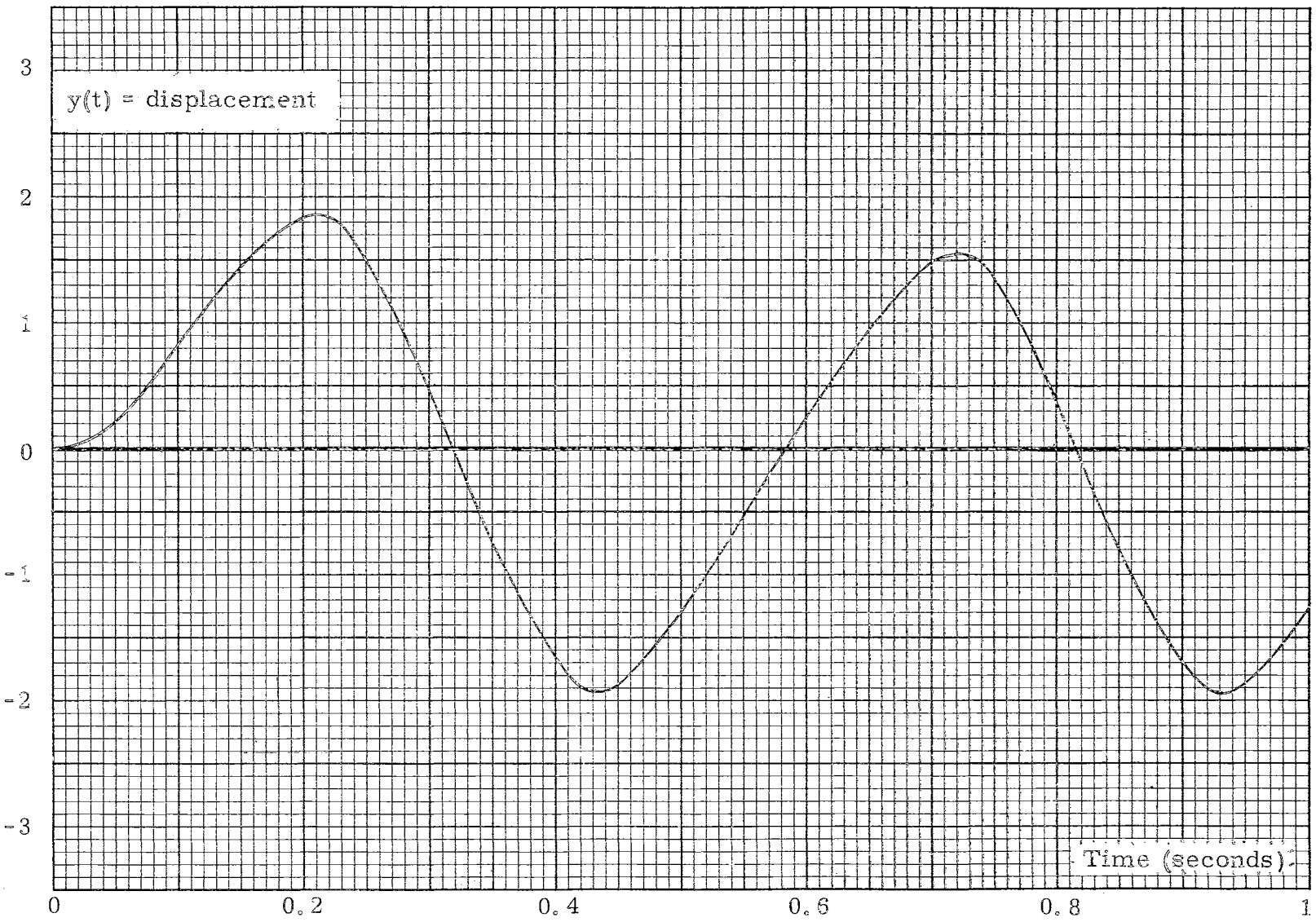

Figure 4-1.4. Displacement y(t) vs. Time.

Figure 4-1.5. Velocity $\dot{y}(t)$ vs. Time.

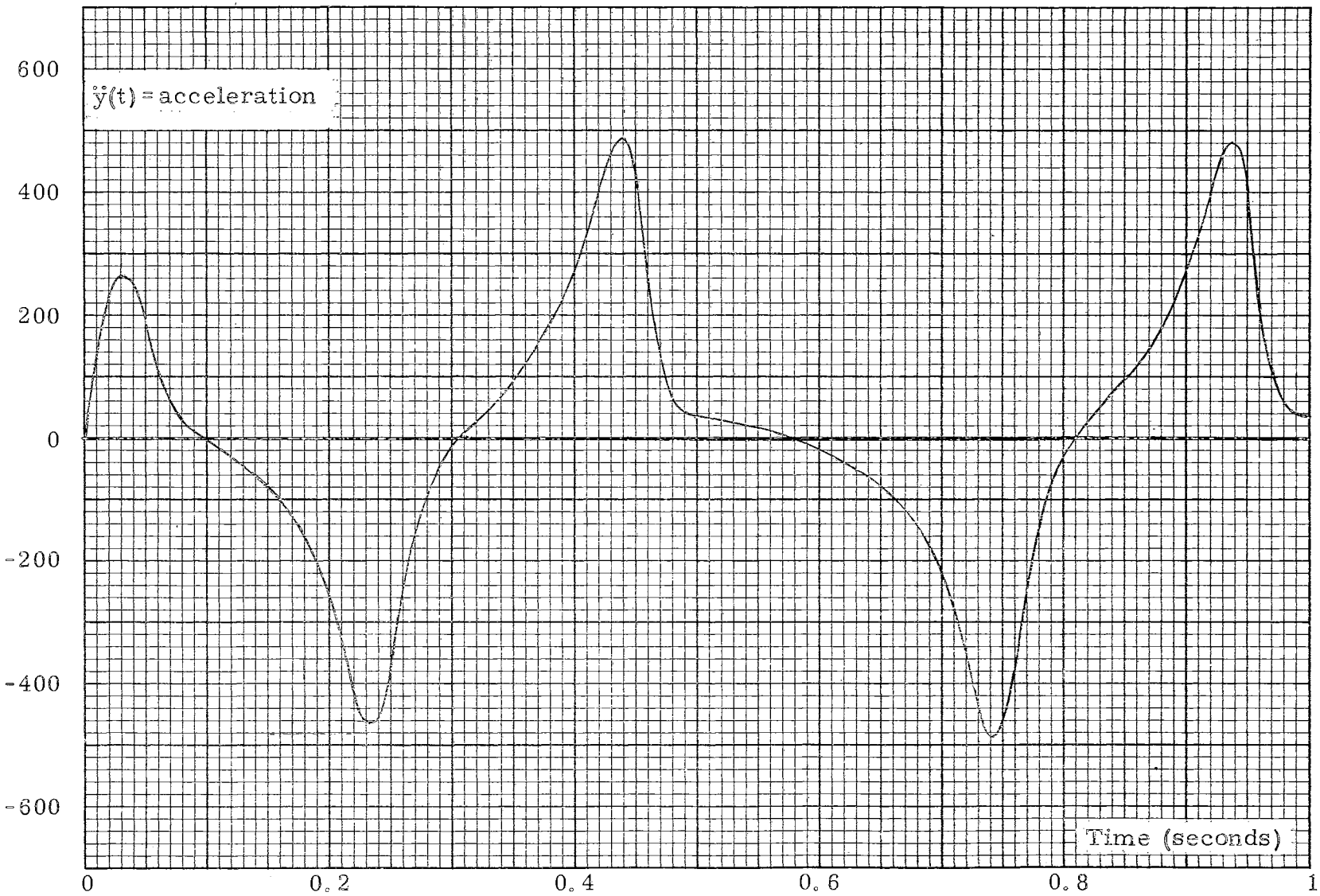

Figure 4-1.6. Acceleration $\ddot{y}(t)$ vs. Time.

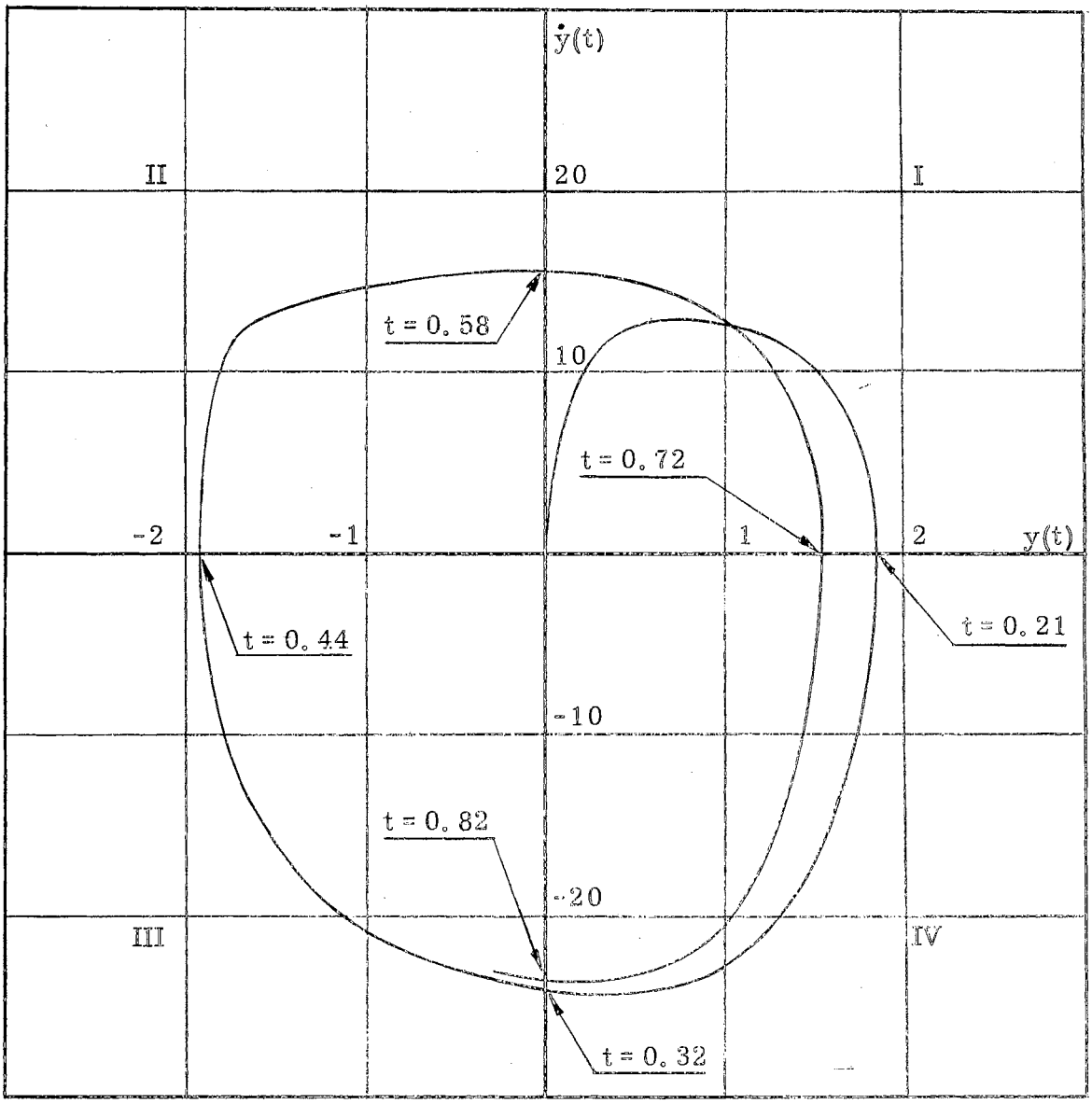

Figure 4-1. 7. Phase Plane Portrait.

In order to gain some appreciation of the accuracy of approximated derivatives and their corresponding effect on parameter determination, two values of the independent variable time increment were studied. These two values were $\Delta = 0.010$ seconds and $\Delta = 0.020$ seconds, and correspond to fifty and twenty-five data points per period of oscillation respectively. Dividing a typical oscillatory wave into twenty-five to fifty increments is fairly common practice in engineering applications and serves as the basis for the selection of the above two time increments. First and second derivatives for the two time increments were obtained in accordance with the differentiating operators (3-1.7 and 3-1.8), i.e., third order polynomial fitted to seven data points.

Error curves for these approximations are given in Figures 4-1.8 and 4-1.9. In this study, error $E\left[f(t)\right]$ is defined

$$E\left[f(t)\right] = \text{observed } f(t) - \text{approximated } f(t) \quad , \qquad (4\text{-}1.4)$$

and percentage error as

$$\%E\left[\pm f(t)\right] = \left| \frac{\text{observed } f(t) - \text{approximated } f(t)}{\text{observed}\left[\pm f(t)\right]_{max.}} \right| (100) \quad . \qquad (4\text{-}1.5)$$

Percentage error as defined by equation (4-1.5) yields a relative quantity for comparative purposes only. This form was adopted over the more conventional definition

$$\%\ \text{error} = \left| \frac{\text{observed} - \text{approximated}}{\text{observed}} \right| (100) \quad , \qquad (4\text{-}1.6)$$

because this study deals with very small and even zero values of the observed quantity. For example, comparing an approximated second derivative error of $E[\ddot{y}] = 1.0$ to an observed value of $\ddot{y} = 0.1$ gives a 1000% error. However, an error of $E[\ddot{y}] = 1.0$ is indeed very small when compared to the errors shown in Figures 4-1.8 and 4-1.9. Maximum percentage errors for the approximated derivatives as defined by equation (4-1.5) are noted in Figures 4-1.8 and 4-1.9. The following maximum values were used as divisors in calculating these percentages,

$$-\dot{y}_{max.} = 24.2 \quad , \quad +\dot{y}_{max.} = 15.6 \quad ,$$

$$-\ddot{y}_{max.} = 488 \quad , \quad +\ddot{y}_{max.} = 492 \quad .$$

Examination of these error curves reveals both a positive and negative distribution of errors. When using a least squares method in the presence of error, such a distribution is intuitively more beneficial than an error distribution of similar signs.

No attempt was made to smooth the approximated derivatives nor evaluate them in the region $0 \leq t < \tau_o$, where τ_o denotes one half of the neighborhood associated with the previously mentioned differentiating operators, i.e., $\tau_o = 3\alpha\Delta$ and α is the order of the desired derivative. It must be recognized that a simple plot of the approximated derivatives would reveal the more obvious larger errors, such that these could be corrected before proceeding further in the analysis.

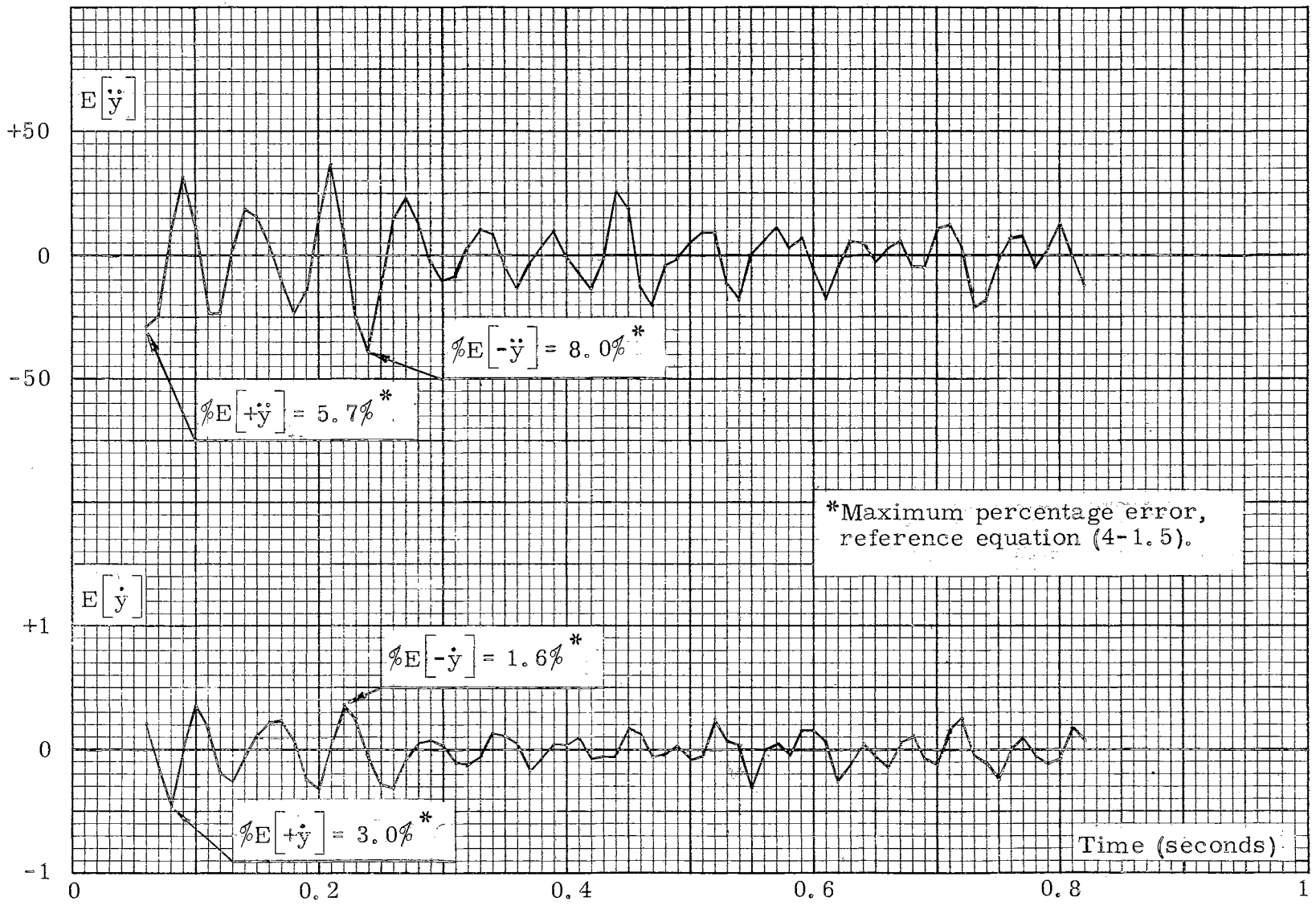

Figure 4-1.8. Error Curves for $E[\dot{y}]$ and $E[\ddot{y}]$; $\Delta = 0.010$.

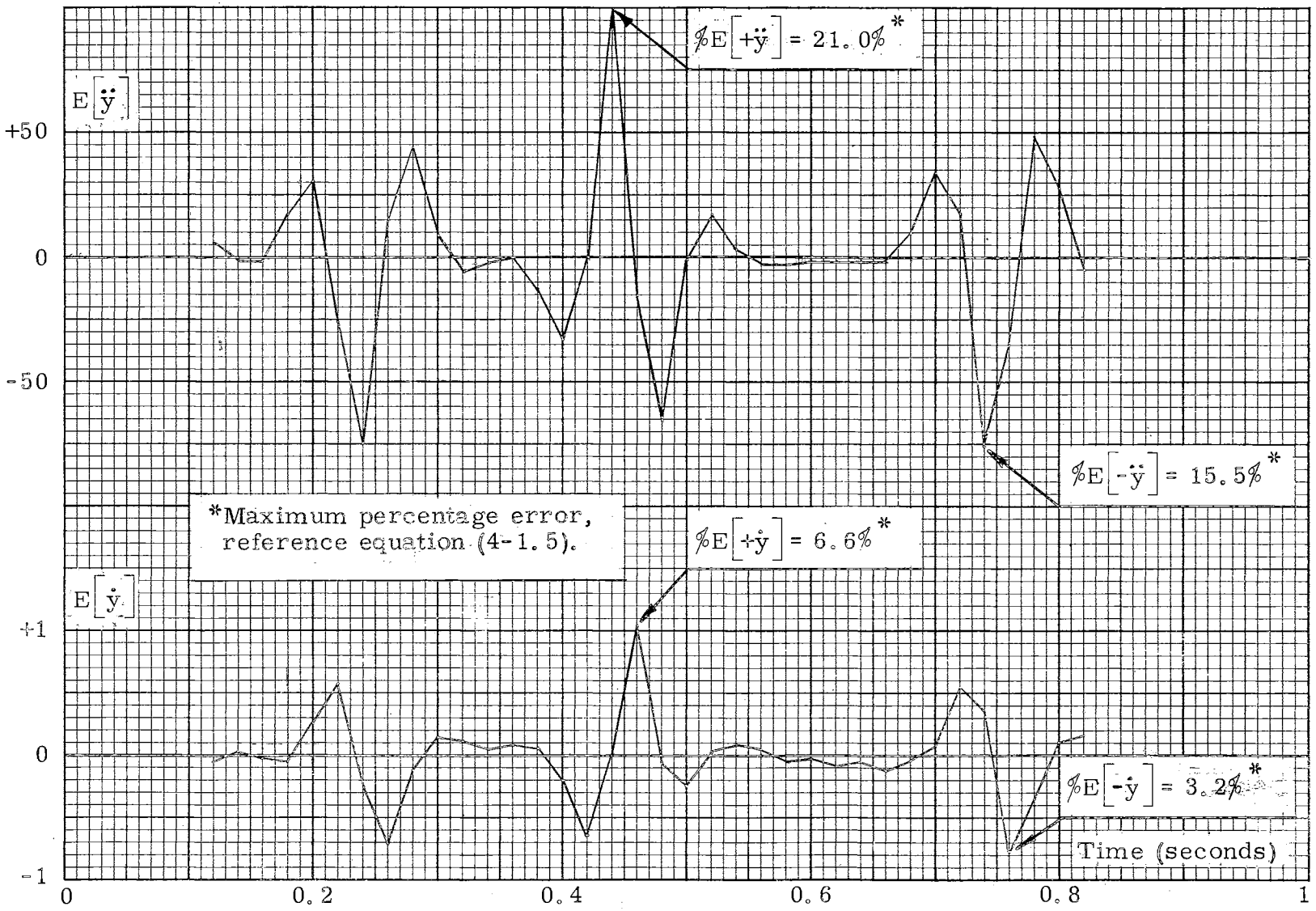

Figure 4-1.9. Error Curves for $E\left[\dot{y}\right]$ and $E\left[\ddot{y}\right]$; $\Delta = 0.020$.

The following apriori knowledge was assumed in connection with the parameter determination problem; the mass term m was considered a constant, the damper $c(\dot{y})$ and spring $k(y)$ were assumed to be continuous functions in each of their regions of definition, i.e., positive and negative regions. To determine a functional relationship for $c(\dot{y})$ and $k(y)$, a sixth order approximating polynomial was selected in each region as follows:

$$
\begin{aligned}
c_1(\dot{y}) &= a_1\dot{y} + a_2\dot{y}^2 + \ldots + a_6\dot{y}^6 \quad , \quad -25 < \dot{y} \leq 0 \quad ; \\
c_2(\dot{y}) &= a_7\dot{y} + a_8\dot{y}^2 + \ldots + a_{12}\dot{y}^6 \quad , \quad 0 \leq \dot{y} < 16 \quad ; \\
k_1(y) &= a_{13}y + a_{14}y^2 + \ldots + a_{18}y^6 \quad , \quad -2 < y \leq 0 \quad ; \\
k_2(y) &= a_{19}y + a_{20}y^2 + \ldots + a_{24}y^6 \quad , \quad 0 \leq y < 2 \quad .
\end{aligned}
\tag{4-1.7}
$$

It is not uncommon in engineering systems to encounter components, i.e., springs, dampers, etc., which exhibit characteristics as shown in Figure 4-1.10.

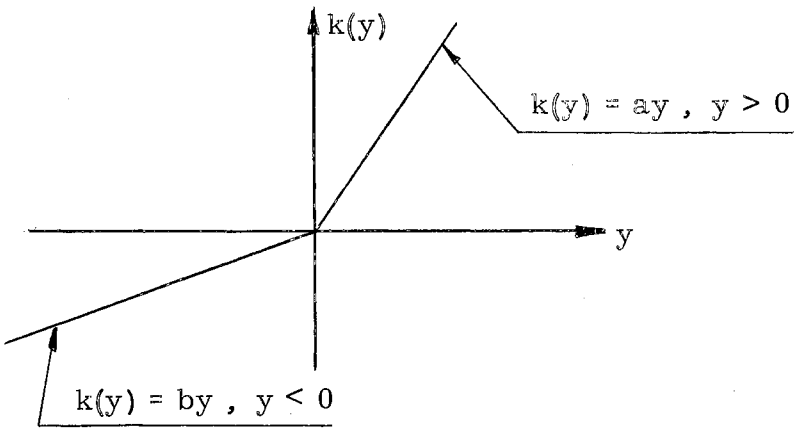

Figure 4-1.10. Typical Quasi-Linear Spring.

The polynomial forms, equation (4-1.7), express the most general behavior possible such that the condition implied by Figure 4-1.10 can easily be determined by merely eliminating the parameters a_{14}, a_{15}, . . . , a_{18} and a_{20}, a_{21}, . . . , a_{24} from the analysis.

The general differential equation (4-1.1) may then be written

$$my + c_1(\dot{y}) + c_2(\dot{y}) + k_1(y) + k_2(y) = x(t) \quad , \qquad (4\text{-}1.8)$$

where it must be explicitly understood that when

$$\begin{aligned} \dot{y} < 0 \quad &, \quad c_2(\dot{y}) = 0 \quad ; \\ \dot{y} > 0 \quad &, \quad c_1(\dot{y}) = 0 \quad ; \\ y < 0 \quad &, \quad k_2(y) = 0 \quad ; \\ y > 0 \quad &, \quad k_1(y) = 0 \quad . \end{aligned} \qquad (4\text{-}1.9)$$

The above four regions are precisely the four quadrants of the phase plane (Figure 4-1.7). The approximate values of time associated with each quadrant are noted on the phase plane portrait and aids in the determination of the total number of independent data points for which equation (4-1.8) may be formed. As noted earlier, the solution essentially exhibits steady state behavior past 0.82 seconds; thus, the total number of independent data points n associated with each time increment is

$$n = \frac{0.82 - 3\alpha\Delta}{\Delta} + 1 \qquad (4\text{-}1.10)$$

where α denotes the order of the highest approximated derivative, i.e., $\alpha = 2$. The total number of data points and the approximate number occurring in each of the four phase plane quadrants is given in Table 4-1.1. Each of these data points was assigned a residual weight of unity.

Table 4-1.1

Distribution of Independent Data Points

Quadrant	$\Delta = 0.010$	$\Delta = 0.020$
I, $y > 0$, $\dot{y} > 0$	30	12
II, $y < 0$, $\dot{y} > 0$	15	8
III, $y < 0$, $\dot{y} < 0$	13	7
IV, $y > 0$, $\dot{y} < 0$	21	9
Total n	77	36

Thus the task is to select among the twenty-five parameters in equation (4-1.8) the best set in a least squares or minimal sense. There are included in the twenty-four polynomial coefficients a multitude of choices. In order to narrow these choices, several representative combinations were selected as a basis for the parameter determination problem. These selected combinations are listed in Table 4-1.2. Parameters were then determined for these eight combinations for each of the two aforementioned time increments. Using these determined parameters, the sum of the squares of the residuals G(a) was evaluated for each of the eight combinations. These residuals are also listed in Table 4-1.2.

Table 4-1.2

Selected Polynomial Combinations and Residuals G(a)

Combination	$\Delta = 0.010$	$\Delta = 0.020$
Odd order polynomial	73	16
Even order polynomial	480	261
First order polynomial	1874	815
Second order polynomial	154	119
Third order polynomial	74	23
Fourth order polynomial	90	17
Fifth order polynomial	1286	24785
Sixth order polynomial	62616	164132

Quite often the notion of standard error* is used to select a mathematical model of best fit in lieu of the residual function G(a). This is a perfectly acceptable criterion when applied to data from the same set. For example, the standard error for the odd order approximating polynomial for the two time increments is

$\Delta = 0.010$, standard error = 0.97 ;

$\Delta = 0.020$, standard error = 0.67 .

---

* Standard error $= \sqrt{\frac{G(a)}{n}}$.

This does not mean that the fitted parameters associated with the larger time increment constitute a better solution to the problem, but rather that the $\Delta = 0.020$ solution fits its data set better than that for the case of $\Delta = 0.010$; the two data sets being essentially independent. The polynomial combination for a particular time increment with the minimum residual function G(a) serves as a criterion for selection with the primary goal being how well the elements of the differential equation, m, $c(\dot{y})$ and k(y) are reconstructed. The large G(a) associated with the higher degree approximating polynomials is partially attributed to a loss of accuracy in the matrix inversion routine. The machine utilized, IBM 1620, has a fixed word length of eight decimal digits and roundoff error in the higher order inversions becomes excessive. The required order of inverse for the sixth degree polynomial being twenty-five and for the fifth degree, twenty-one.

The odd order polynomial, possessing a minimum sum of the squares of the residuals G(a), was selected as the function of best fit for $c(\dot{y})$ and k(y) for both time increments. The odd order approximating polynomials $c_1(\dot{y})$, $c_2(\dot{y})$, $k_1(y)$ and $k_2(y)$ were then evaluated over their regions of definition and compared to the original functions $c(\dot{y})$ and k(y). Error curves for these comparisons are given in Figures 4-1.11 and 4-1.12. The following maximum values were used as divisors in calculating the maximum percentage errors noted in Figures 4-1.11 and 4-1.12 in accordance with equation (4-1.5),

$$c(-\dot{y})_{max.} = -51.6 \quad , \quad c(+\dot{y})_{max.} = 57.0 \quad ,$$

$$k(-y)_{max.} = -72.0 \quad , \quad k(+y)_{max.} = 48.0 \quad .$$

A critical observation of these error curves will reveal that the damper and spring characteristics are substantially reproduced by the odd order approximating polynomials. This is further substantiated by examining Figures 4-1.13 and 4-1.14 which show the odd order approximating polynomial superimposed on the exact damper and spring functions for the time increment containing the greatest error, $\Delta = 0.020$. No attempt was made to plot the approximating spring and damper functions for the time increment $\Delta = 0.010$ due to the almost exact correspondence between the exact and approximating functions. The numerical value of the determined mass parameter m and the corresponding percentage error for each time increment was

$$\Delta = 0.010 \quad , \quad m = 0.050850 \quad , \quad \%E\left[m\right] = 1.9\% \quad ;$$

$$\Delta = 0.020 \quad , \quad m = 0.055993 \quad , \quad \%E\left[m\right] = 8.1\% \quad .$$

Remarkably good agreement was obtained for both time increments; however, it cannot be overemphasized that the odd order approximating polynomials are only valid for the fitted regions, i.e., $-25 < \dot{y} < 16$ and $-2 < y < 2$. As in any nonlinear problem, extreme caution must be exercised in attempting to predict general behavior from the information established in a single test. To test the adequacy of the above determined parameters, the differential equations incorporating the odd order approximating polynomials were solved for the output y(t) and compared to the original data. The maximum error encountered was $E\left[y\right] = 0.03$, i.e., $\%E\left[y\right] = 1.5\%$, and occurred for the larger time increment $\Delta = 0.020$. It is felt that this extremely small error for the essentially "worst" case investigated demonstrates the practicality of the method described herein. Error curves for the output are not included due to the trivial variations in $E\left[y\right]$.

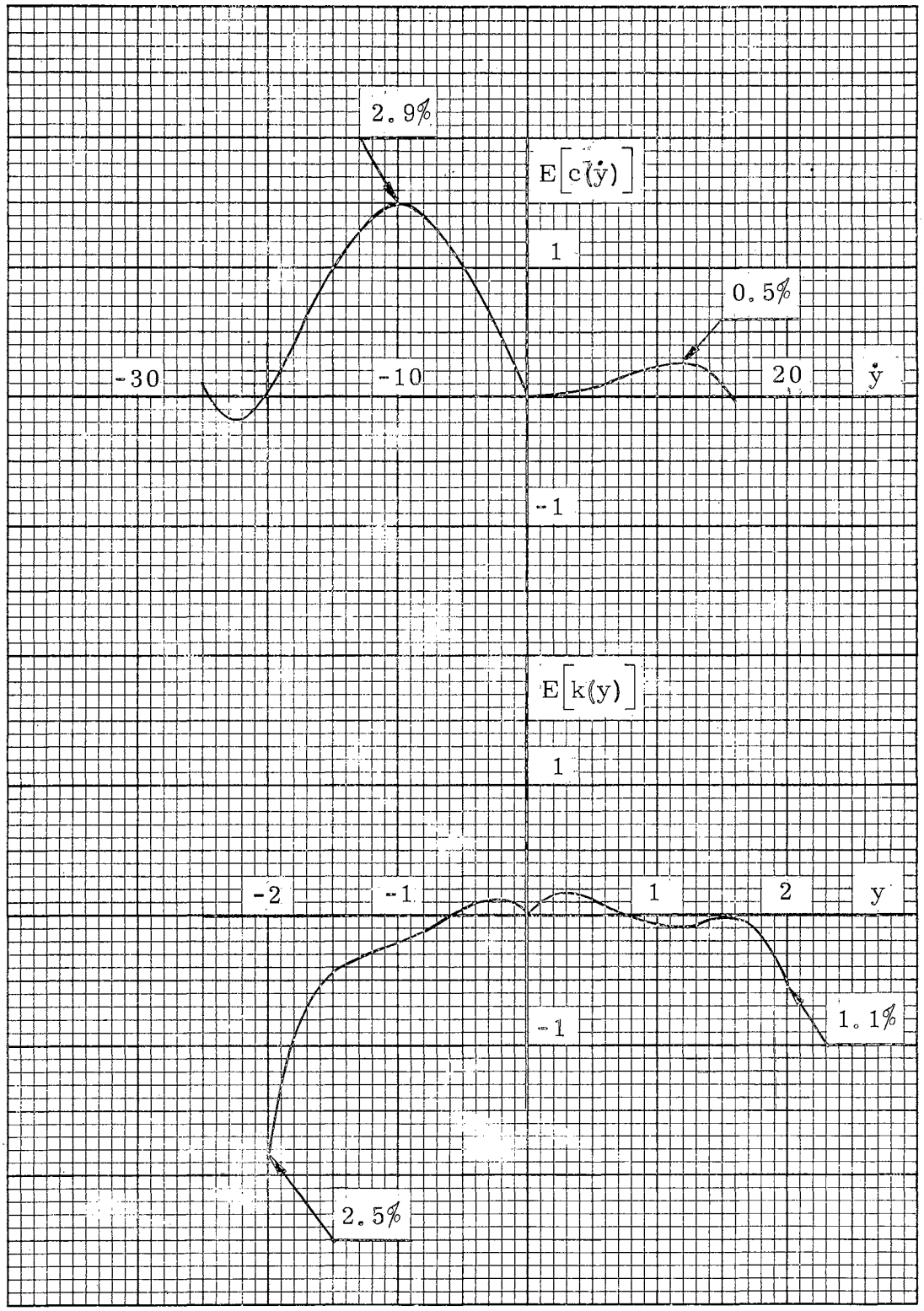

Figure 4-1.11. Error Curves for the Damper and Spring; $\Delta = 0.010$.

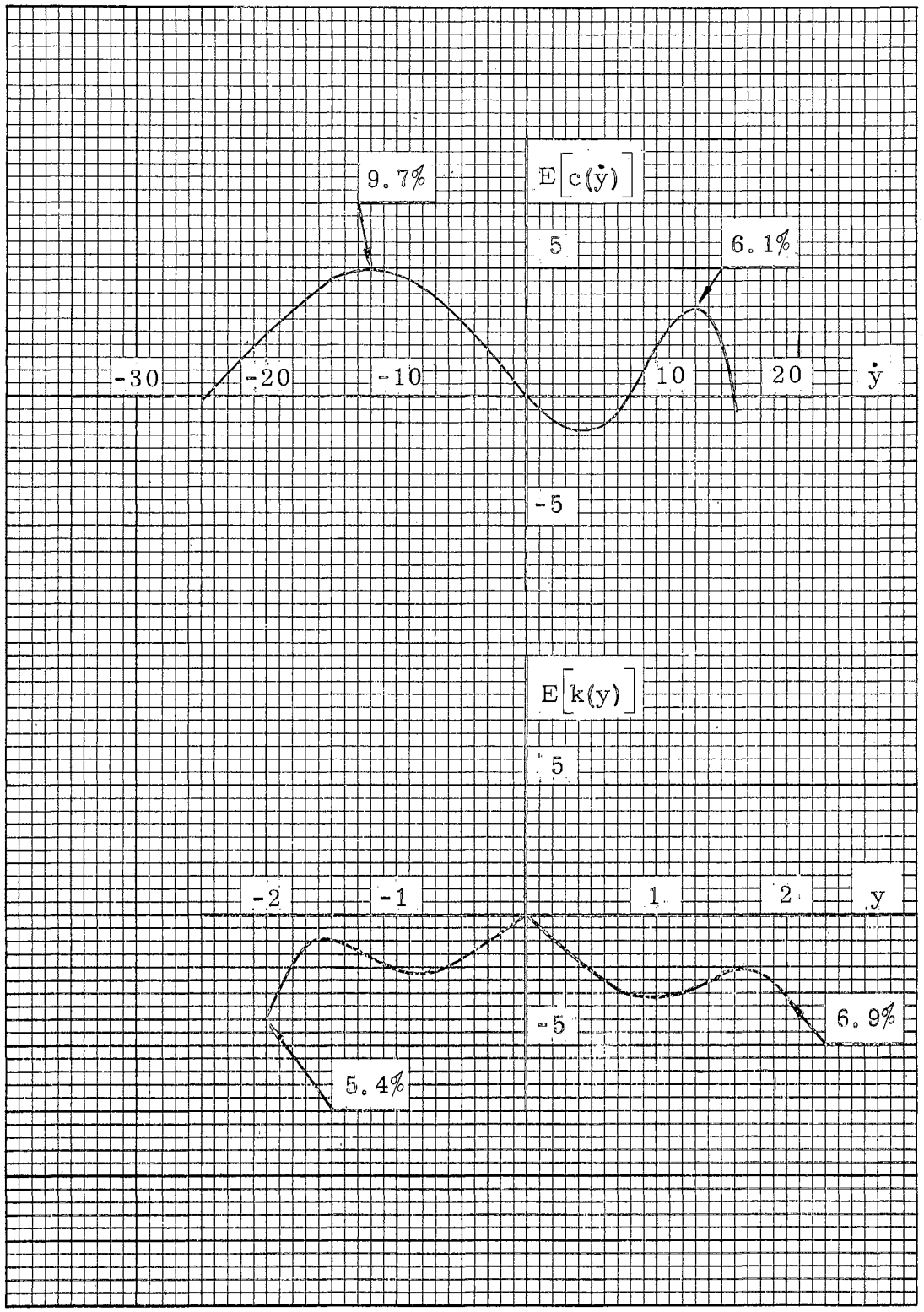

Figure 4-1.12. Error Curves for the Damper and Spring; $\Delta = 0.020$.

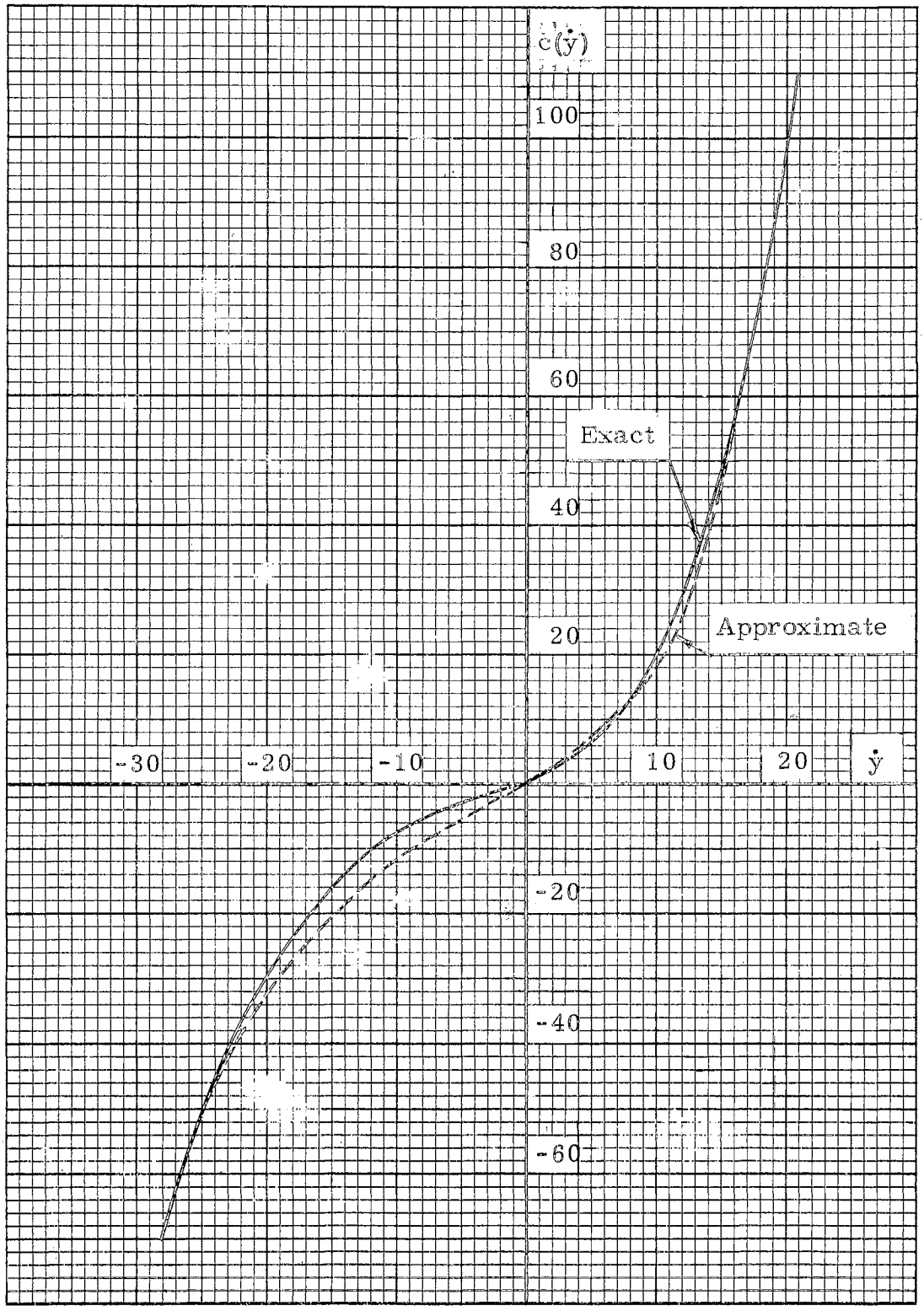

Figure 4-1.13. Damper Approximation ; $\Delta = 0.020$.

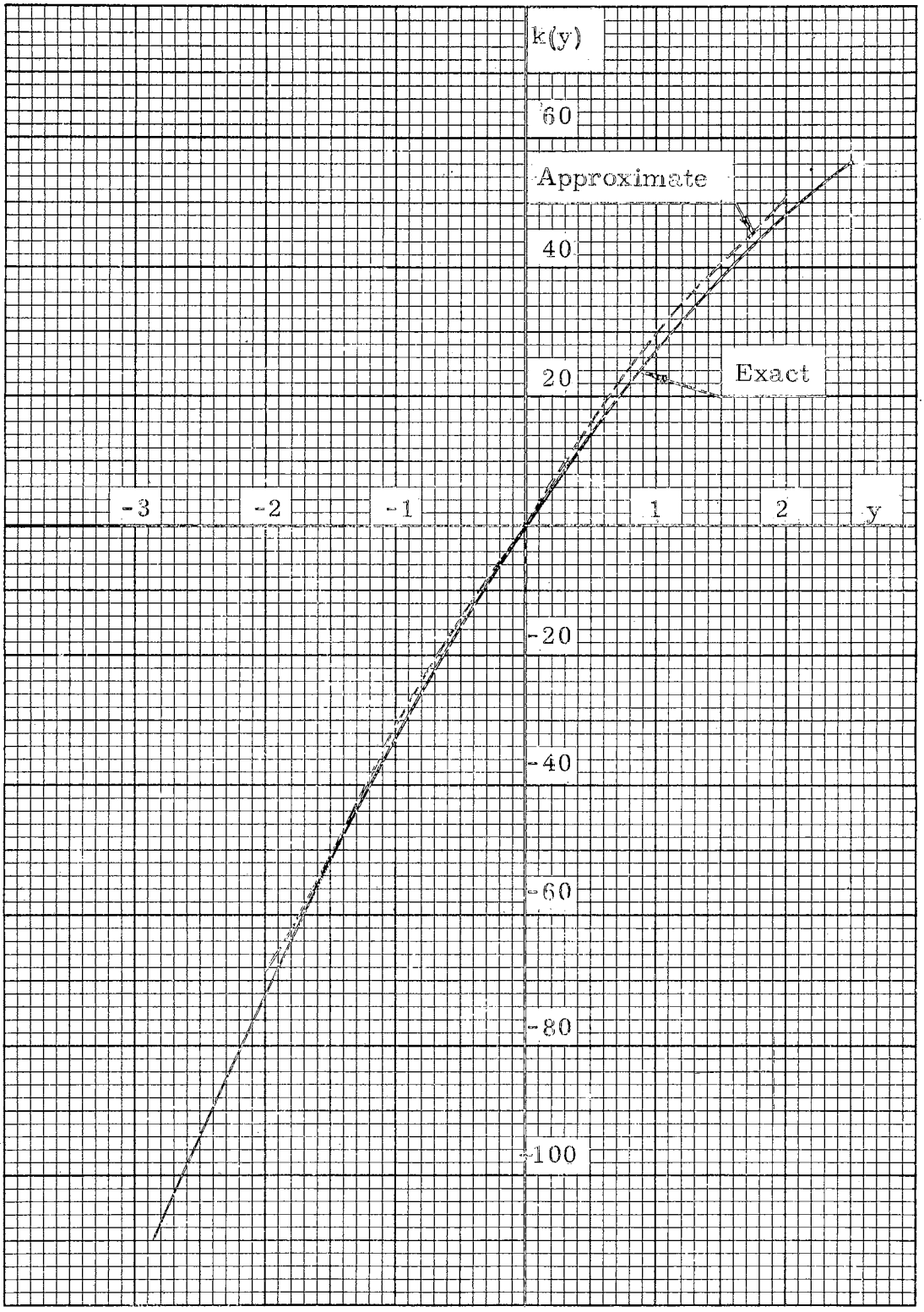

Figure 4-1.14. Spring Approximation ; $\Delta = 0.020$.

A second solution for this problem was obtained by integrating the differential equation (4-1.8) once to obtain the following form

$$m\dot{y}(\tau) + \int_{\tau_o}^{\tau} c_1(\dot{y})dt + \int_{\tau_o}^{\tau} c_2(\dot{y})dt + \int_{\tau_o}^{\tau} k_1(y)dt$$

$$+ \int_{\tau_o}^{\tau} k_2(y)dt + a_{25} = \int_{\tau_o}^{\tau} x(t)dt \quad . \qquad (4\text{-}1.11)$$

This particular form circumvented the necessity of estimating the second derivatives and yielded correspondingly greater accuracies. The damper and spring were replaced by sixth order polynomials as before, with the odd order function again producing a minimum error squared G(a) for both time increments. The residuals G(a), together with the mass parameter and the maximum deviations in the evaluated damper and spring functions, are summarized in Table 4-1.3. Examination of these errors reveals an almost exact reconstruction of the elements m, $c(\dot{y})$ and k(y) of the differential equation.

Table 4-1.3
Parameter and Error Comparisons

	Δ = 0.010		Δ = 0.020	
		%E		%E
G(a)	0.44	—	0.36	—
m	0.051782	0.1	0.051805	0.1
$E\left[c(-\dot{y})\right]_{max.}$	-0.40	0.8	3.43	6.6
$E\left[c(+\dot{y})\right]_{max.}$	-0.91	1.5	4.16	7.3
$E\left[k(-y)\right]_{max.}$	0.75	1.0	-2.94	4.1
$E\left[k(+y)\right]_{max.}$	+0.39	0.8	-2.68	5.6

4-2. Dynamics of a Chemical Process

The dynamic behavior of a chemical process is presented in this section and is symbolically represented by the block diagram of Figure 4-2.1.

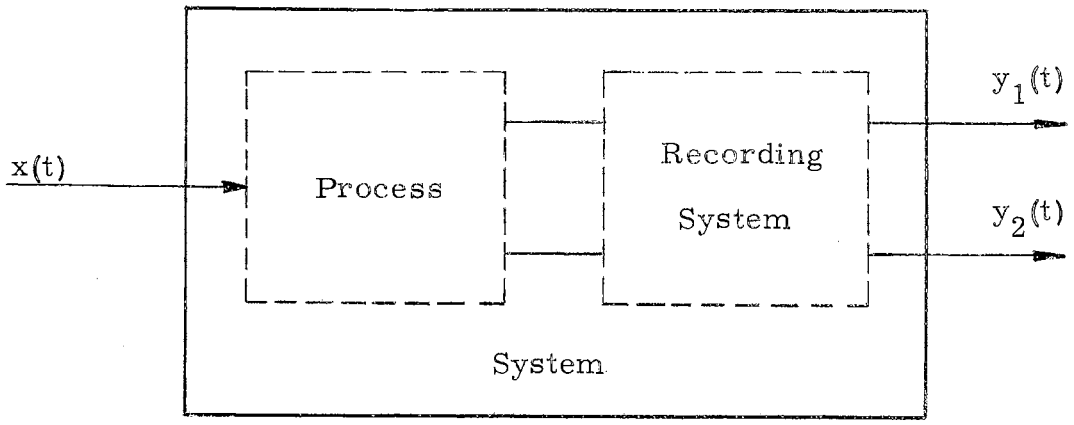

Figure 4-2.1. Block Diagram for the Chemical Process.

Certain details concerning the interrelationship between the process and recording system were not available; hence, the process and recording system are combined and will be hereinafter referred to as simply the system. The system was subjected to a single input x(t), and two output variables $y_1(t)$ and $y_2(t)$ were recorded as discrete digital data in 0.75 minute intervals. Plots of the input and outputs are displayed in Figures 4-2-2, 4-2.3 and 4-2.4.*

* The data for this problem was donated by the Phillips Petroleum Company, Bartlesville, Oklahoma. In order to preserve propriety for the industrial donor's research efforts, the recorded variables as presented herein have been translated so as to disassociate the variable magnitudes with any particular chemical process. However, in the interest of a later discussion concerning error, the recorded variables fall in a general range of two hundred to four hundred units of magnitude.

The basic problem was to determine suitable parameters such that the system could be described by the set of linear differential equations

$$\dot{y}_1 = a_1 y_1 + a_2 y_2 + a_3 x + a_4 \quad ,$$

$$\dot{y}_2 = a_5 y_1 + a_6 y_2 + a_7 x + a_8 \quad , \qquad (4\text{-}2.1)$$

$$y_1(0) = 0.0 \quad , \quad y_2(0) = 50.0 \quad .$$

Ordinarily, the input-output data would be smoothed before attempting the determination of parameters; however, in this case the raw data was used directly to gain some appreciation of the method of this thesis when applied to data containing random errors. Due to the small variations in the output variable $y_2(t)$, the following substitution was made to render equation (4-2.1) more amenable to data analysis

$$\dot{y}_1 = a_1 y_1 + a_2\left[y_2 - y_2(0)\right] + a_3 x + a_9 \quad ,$$

$$\dot{y}_2 = a_5 y_1 + a_6\left[y_2 - y_2(0)\right] + a_7 x + a_{10} \quad , \qquad (4\text{-}2.3)$$

$$a_9 = a_4 + a_2 y_2(0) \quad ,$$

$$a_{10} = a_8 + a_6 y_2(0) \quad .$$

Integrating the above differential equations yields the desired mathematical model for the problem

$$y_1(\tau) - y_1(0) = a_1 \int_0^\tau y_1 dt + a_2 \int_0^\tau \left[y_2 - y_2(0)\right] dt + a_3 \int_0^\tau x dt + a_9 \tau \quad , \qquad (4\text{-}2.4)$$

$$y_2(\tau) - y_2(0) = a_5 \int_0^\tau y_1 dt + a_6 \int_0^\tau \left[y_2 - y_2(0)\right] dt + a_7 \int_0^\tau x dt + a_{10} \tau \quad .$$

The indicated integral operations were evaluated for fifty-seven values of τ along the time series, i.e., $\Delta t = 1.50$ minutes. The unknown parameters were then determined from this set of fifty-seven overdetermined equations. The results of this determination are

$$\dot{y}_1 = -0.169520\, y_1 - 0.150604\, y_2 - 0.013276\, x + 7.607216 \quad , \tag{4-2.5}$$

$$\dot{y}_2 = -0.166036\, y_1 - 0.170566\, y_2 - 0.001100\, x + 8.578805 \quad .$$

The numerical solution of this set of differential equations was next obtained and is shown superimposed on the data in Figures 4-2.5 and 4-2.6. Error curves for the output variables y_1 and y_2 are shown in Figure 4-2.7, together with the arithmetic mean and standard error for the total number of data points considered, $n = 115$. The maximum error in y_1 is 0.28 and represents a 6.1% error when compared to the total deviation of the system variable y_1; similarly, the maximum percentage error in y_2 is 7.8%. As noted earlier, the output variables have a general range of two hundred to four hundred units of magnitude. If percentage errors in y_1 and y_2 are calculated on the basis of variable magnitude, they are indeed very small, approximately 0.1%, which is more than well within the accuracies of present day recording instruments.

The set of determined parameters was judged to be quite satisfactory, particularly those for the variable y_1. It is believed that the discrepancy in y_2 can be partially attributed to the incomplete knowledge of the relationship between the process and recording system. An examination of the error curves, Figure (4-2.7), reveals a generally random distribution of positive and negative error which

tends to further substantiate the adequacy of the determined parameters.

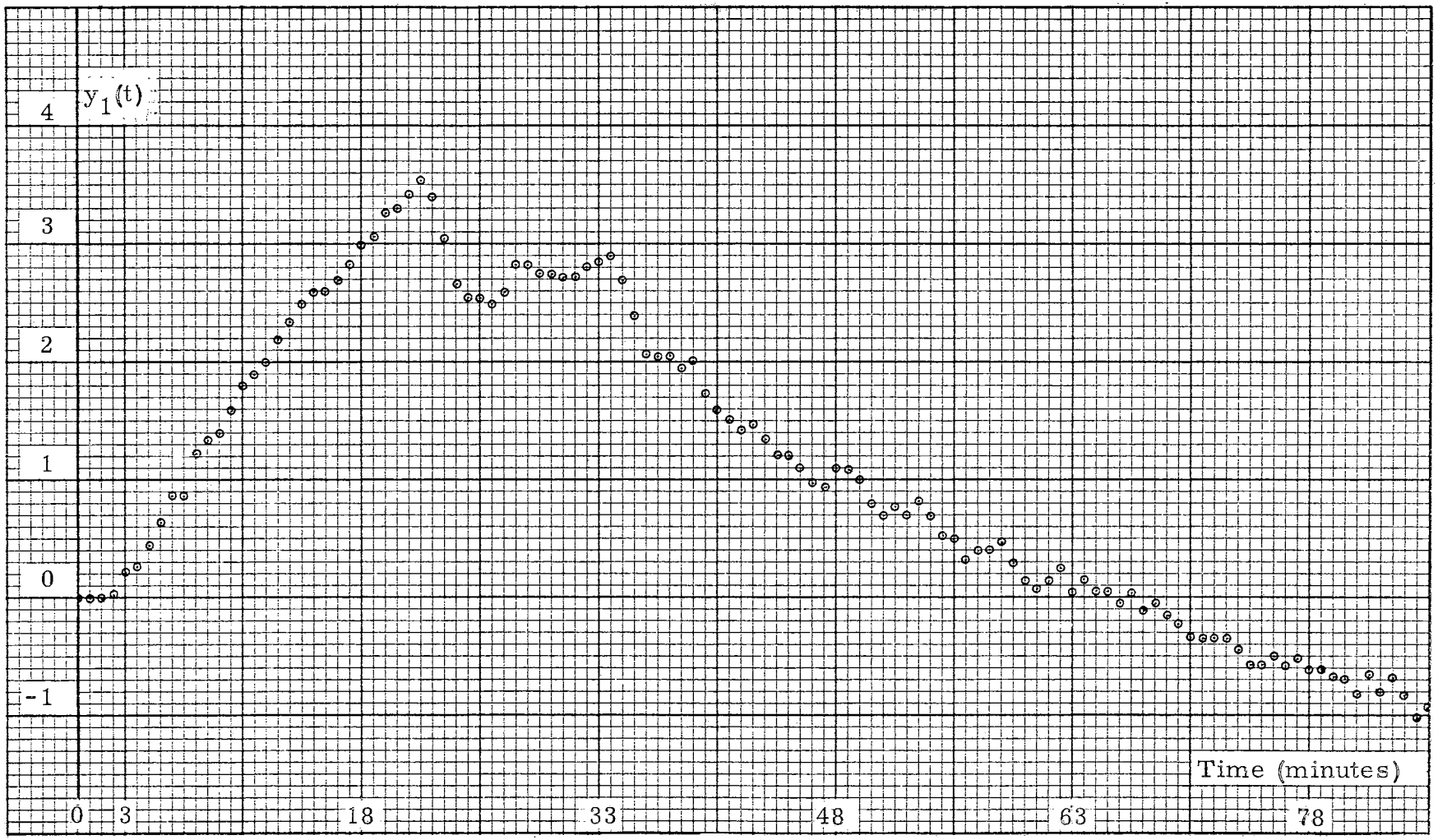

Figure 4-2.2. Recorded Output Variable $y_1(t)$.

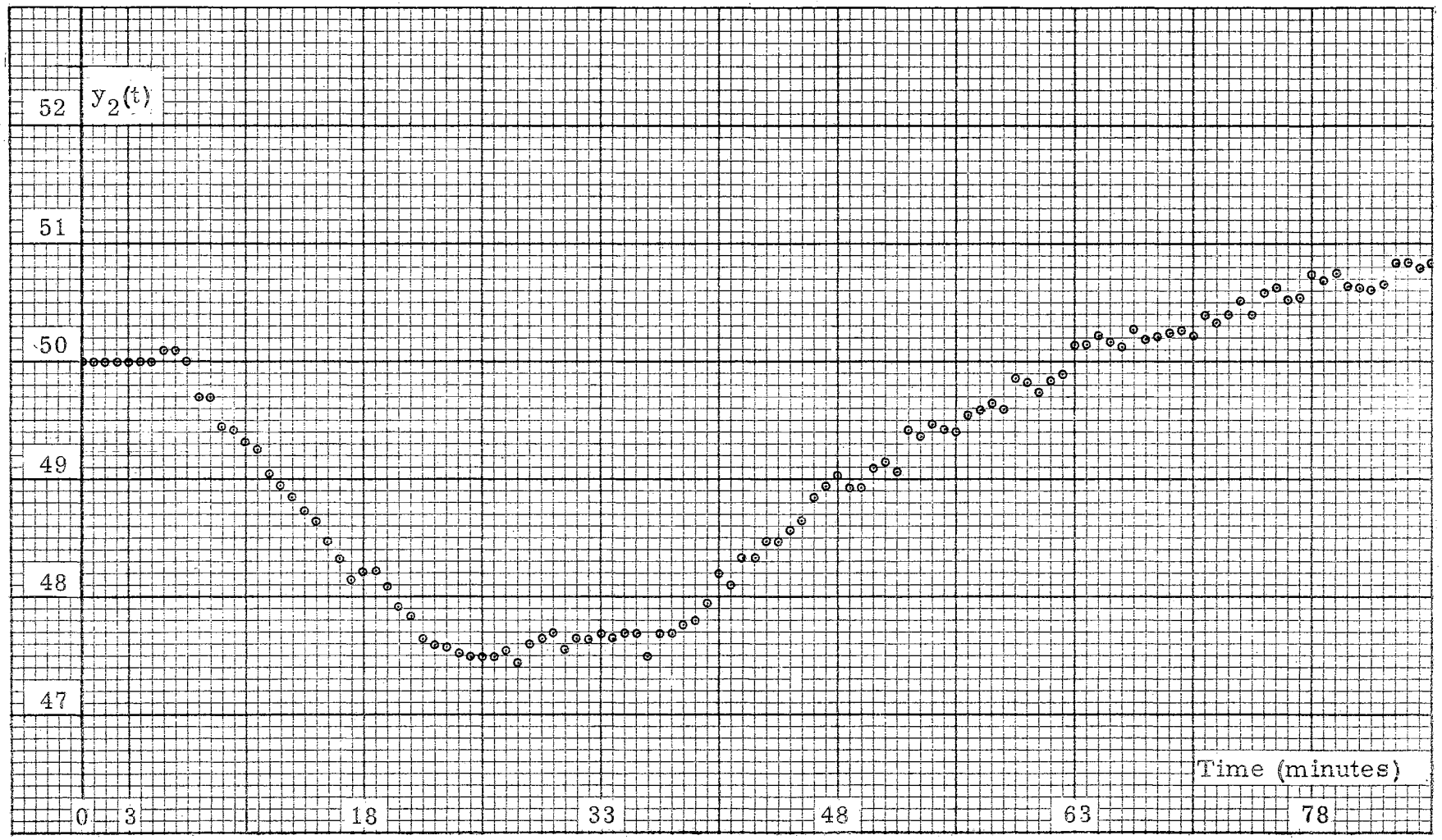

Figure 4-2.3. Recorded Output Variable $y_2(t)$.

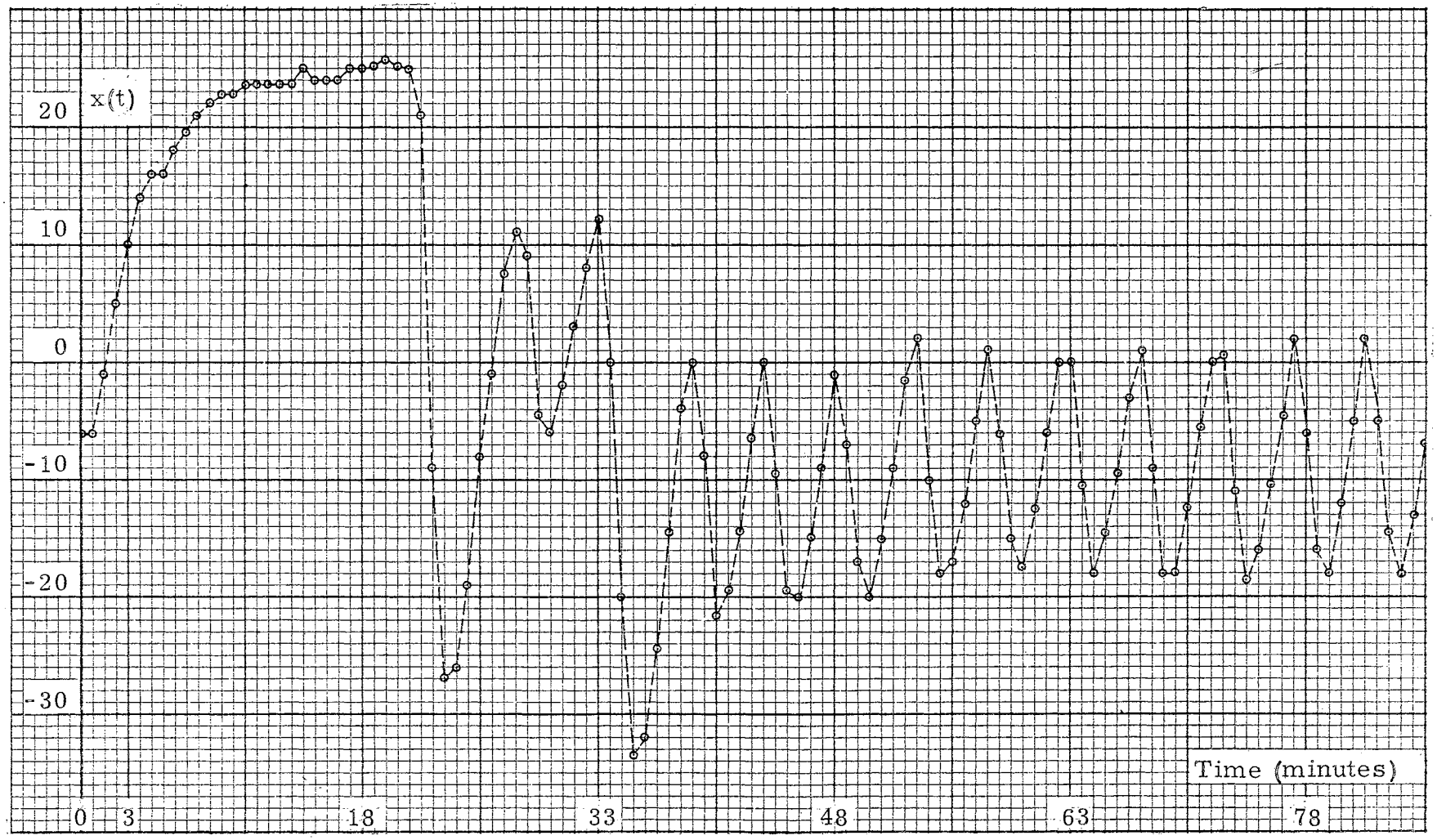

Figure 4-2.4. Recorded Input Variable x(t).

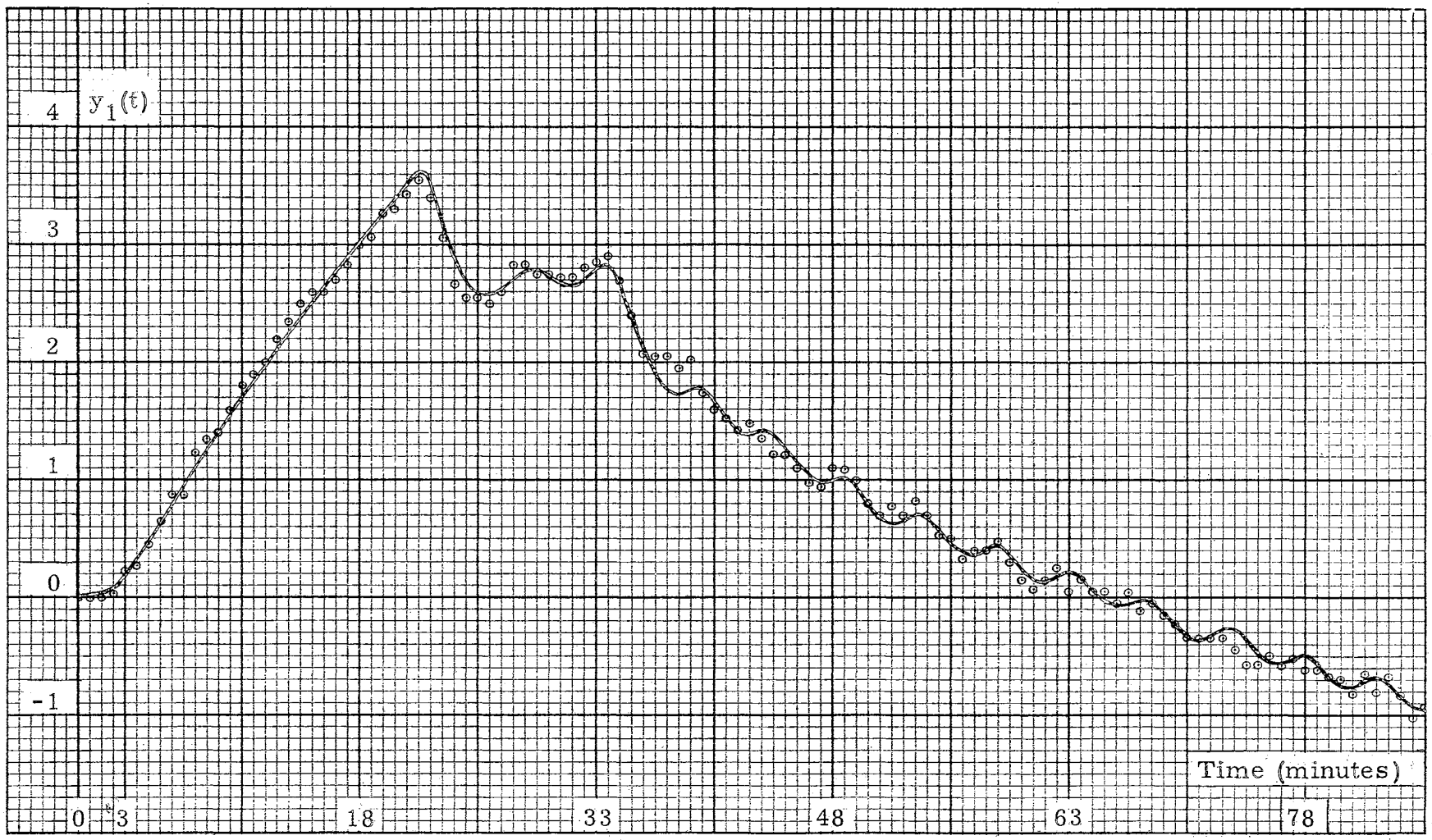

Figure 4-2.5. Differential Equation Solution for $y_1(t)$.

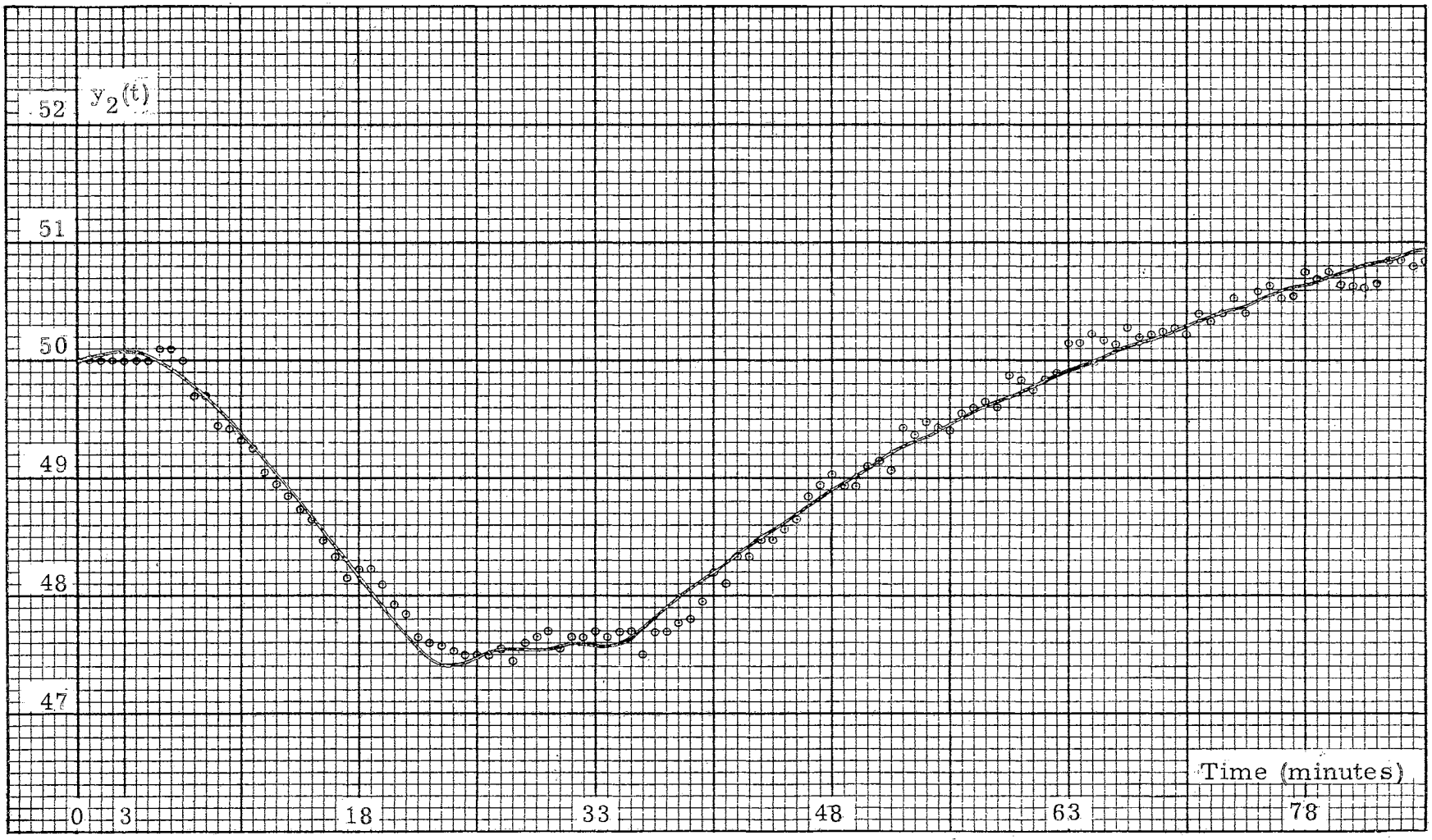

Figure 4-2.6. Differential Equation Solution for $y_2(t)$.

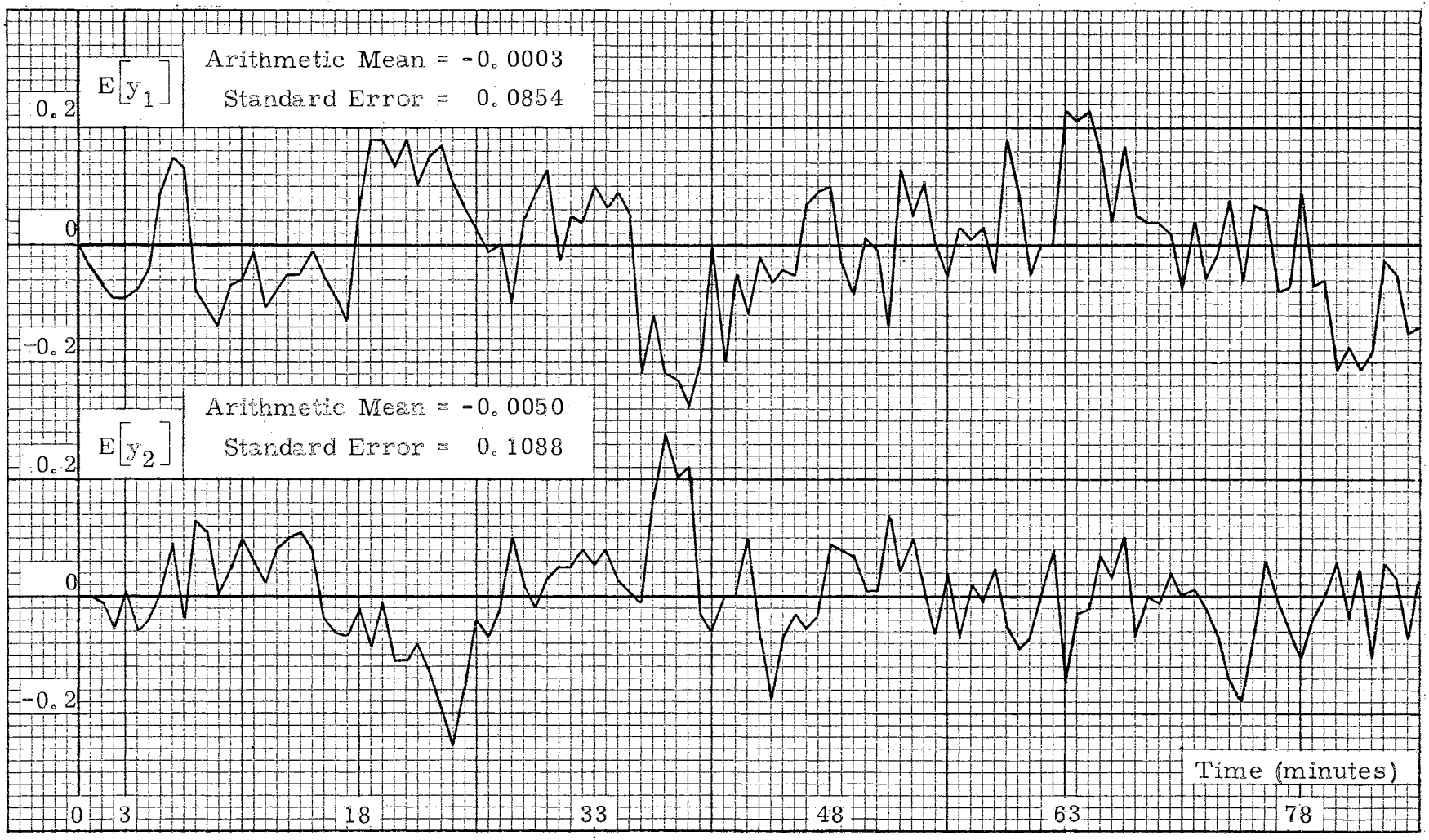

Figure 4-2.7. Error Curves for $E[y_1]$ and $E[y_2]$.

CHAPTER V

SUMMARY AND CONCLUSIONS

A method for determining the parameters in lumped parameter linear or nonlinear time invariant systems is presented in this study. The method depends on the successful evaluation of the integral, differential and multiplicative operations contained in the differential equation governing the behavior of the system. By an application of the general method of least squares, the system parameters are determined in an optimal sense by solving overdetermined sets of linear algebraic equations. In this manner errors in the evaluated operations are minimized such that the differential equation parameters constitute a best set consistent with the numerically evaluated operations.

The method essentially demands the use of automatic computing machinery because of the rather extensive numerical labor involved; however, the required computer routines are not particularily difficult to establish. The successful implementation of the method is only limited by the accuracy of the numerically evaluated operations contained in the differential equation; the most difficult and uncertain being that of differentiation. Once these operations are suitably determined, the method is extremely straightforward. For this reason it seems justifiable for an investigator to spend a suitable amount of time in the approximation of the required operations,

particularily that of differentiation.

The observed data, being continuous time histories or discrete digital data representations of the system input and output, may be selected either from the transient and/or steady state behavior of the process. No restrictions are imposed on the form of the input signal so long as it is recordable in a physical sense. It is noteworthy to mention that the parameters established by the method of least squares are entirely consistent with the recordings of the dynamic process. If extreme care has been exercised in the recording process, one can be assured of a reasonable set of parameters. In this connection, the selected form of the input disturbance for a particular system being tested should be one that produces the smoothest output possible, as well as an output which lends itself to numerical differentiation or other such required operations contained in the differential equation.

It is extremely difficult to predict expected error when using the method of this study. The numerical examples of Chapter IV demonstrate that final errors in the determined parameters are considerably less than the errors in the data which was used to develop the parameters. This in general will always be the case when applying the method of least squares to data which contains random error.[20] Comparing the observed output of the example problems to the output obtained by solving the determined differential equation is particularly noteworthy from the standpoint of error, and in the author's judgment fully demonstrates the practicality of the method presented in this thesis.

The method of parameter determination presented herein can

very easily be extended to systems characterized by time varying parameters provided the parameters are slowly varying. This would be accomplished by determining the parameters in a local neighborhood with the assumption that the time varying parameters remain essentially constant in this region. Assuming that the method would yield the mean values of the parameters in the neighborhood under consideration, this extension appears to have application in the field of adaptive process identification. A most interesting and promising extension in the field of process control is described by Hove.[21] He employs the least squares identification concept to establish the parameters in a set of control operations, which are superimposed on a fixed plant, so as to achieve a predetermined desired output response.

In conclusion, the method will in general always yield a set of parameters for the differential equation under study. The correctness of this set of parameters is, to a large extent, dependent on the quality of sound engineering judgment used to generate the solution. Some factors which are considered influential in determining as well as aiding individual judgment include: the type of system under study, the selected input signal and the variables to be measured, the accuracy of the experimental measurements, the selection of the mathematical model which best portrays the dynamic system, and the all important judgment associated with the accuracy of the numerically evaluated operations contained in the mathematical model.

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APPENDIX A

Input-Output Data for the Second Order Quasi-Linear System

t	y(t)	$\dot{y}(t)$	$\ddot{y}(t)$	x(t)
.00	.00	.00	.0	.00
.01	.00	.67	130.6	7.51
.02	.02	2.49	226.6	14.92
.03	.05	5.03	272.8	22.08
.04	.12	7.70	252.8	28.90
.05	.21	9.90	184.1	35.26
.06	.31	11.38	112.3	41.07
.07	.43	12.24	62.4	46.23
.08	.56	12.70	32.0	50.65
.09	.69	12.92	12.2	54.28
.10	.82	12.96	-3.0	57.06
.11	.94	12.86	-16.9	58.93
.12	1.07	12.62	-30.6	59.88
.13	1.20	12.24	-45.1	59.88
.14	1.32	11.71	-61.0	58.93
.15	1.43	11.01	-79.1	57.06
.16	1.54	10.12	-100.5	54.28
.17	1.63	8.99	-126.6	50.65
.18	1.71	7.56	-159.8	46.23
.19	1.78	5.75	-202.9	41.07
.20	1.83	3.45	-258.9	35.26
.21	1.85	.53	-325.0	28.90
.22	1.84	-3.12	-410.0	22.08
.23	1.78	-7.55	-466.7	14.92
.24	1.68	-12.23	-459.5	7.52
.25	1.54	-16.46	-380.1	.00
.26	1.36	-19.70	-265.8	-7.51
.27	1.15	-21.83	-164.4	-14.92
.28	.93	-23.10	-94.0	-22.08
.29	.69	-23.81	-48.8	-28.90
.30	.45	-24.13	-18.0	-35.26
.31	.21	-24.19	6.0	-41.07
.32	-.03	-24.02	27.8	-46.23
.33	-.27	-23.64	49.2	-50.65
.34	-.50	-23.03	71.5	-54.28
.35	-.73	-22.20	95.4	-57.06
.36	-.95	-21.12	121.7	-58.93
.37	-1.15	-19.75	151.2	-59.88

.38	-1.34	-18.08	185.0	-59.88
.39	-1.51	-16.03	224.5	-58.93
.40	-1.66	-13.56	271.2	-57.06
.41	-1.78	-10.57	326.9	-54.28
.42	-1.87	-6.98	391.5	-50.65
.43	-1.92	-2.73	459.6	-46.23
.44	-1.92	2.11	491.9	-41.07
.45	-1.88	6.76	420.3	-35.26
.46	-1.79	10.20	262.9	-28.90
.47	-1.68	12.12	131.0	-22.08
.48	-1.55	13.07	66.8	-14.92
.49	-1.42	13.61	43.7	-7.52
.50	-1.28	14.00	36.1	-.00
.51	-1.14	14.34	32.7	7.51
.52	-.99	14.66	29.6	14.92
.53	-.85	14.93	25.8	22.08
.54	-.70	15.17	21.2	28.90
.55	-.54	15.36	15.9	35.26
.56	-.39	15.49	9.9	41.07
.57	-.23	15.55	3.3	46.23
.58	-.08	15.55	-3.7	50.65
.59	.08	15.47	-11.5	54.28
.60	.23	15.32	-19.8	57.06
.61	.38	15.07	-29.0	58.93
.62	.53	14.73	-38.9	59.88
.63	.68	14.29	-50.0	59.88
.64	.82	13.73	-62.5	58.93
.65	.95	13.03	-76.8	57.06
.66	1.08	12.18	-93.7	54.28
.67	1.20	11.14	-114.2	50.65
.68	1.30	9.88	-139.8	46.23
.69	1.39	8.32	-172.9	41.07
.70	1.47	6.38	-216.9	35.26
.71	1.52	3.93	-275.1	28.90
.72	1.54	.83	-345.1	22.08
.73	1.53	-3.04	-432.3	14.92
.74	1.48	-7.68	-488.2	7.52
.75	1.38	-12.54	-471.2	.00
.76	1.23	-16.82	-377.3	-7.51
.77	1.05	-19.97	-252.4	-14.92
.78	.83	-21.95	-146.9	-22.08
.79	.61	-23.04	-76.0	-28.90
.80	.38	-23.57	-30.8	-35.26
.81	.14	-23.71	.3	-41.07
.82	-.10	-23.58	25.3	-46.23
.83	-.33	-23.21	48.4	-50.65
.84	-.56	-22.61	71.6	-54.28
.85	-.78	-21.77	96.1	-57.06
.86	-1.00	-20.68	122.9	-58.93
.87	-1.20	-19.30	152.9	-59.88

.88	-1.38	-17.61	187.3	-59.88
.89	-1.55	-15.54	227.4	-58.93
.90	-1.69	-13.03	274.8	-57.06
.91	-1.81	-10.01	331.0	-54.28
.92	-1.89	-6.38	395.5	-50.65
.93	-1.93	-2.09	461.8	-46.23
.94	-1.93	2.72	481.9	-41.07
.95	-1.88	7.21	398.5	-35.26
.96	-1.79	10.42	242.0	-28.90
.97	-1.67	12.18	120.4	-22.08
.98	-1.55	13.07	63.0	-14.92
.99	-1.41	13.58	42.8	-7.52
1.00	-1.28	13.97	36.1	-.00

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