PROPERTIES OF FIXED POINT SPACES

IN POINT SET TOPOLOGY

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CHAPTER I

THE NATURE AND SIGNIFICANCE OF THE PROBLEM

Without a doubt, the modern world is one in which quantitative reasoning and understanding is as essential as reading if the individual is to comprehend the world about him and the events that are taking place in that world. The general problem with which this paper is concerned is an examination of one aspect of mathematics that is an essential part of any college program in mathematics. More specifically, this thesis is concerned with that aspect of point set topology which is concerned with the "fixed point property." The major purpose of this study is to present a brief history of point set topology in America and the men who have contributed to its research and development, and to review and analyze the literature of the mathematical research of the fixed point property in point set topology. It is the intent to present this thesis in a manner which will be comprehensible to the well-trained undergraduate and the beginning graduate student in mathematics. It is also one of the objectives in this thesis to give an extensive bibliography covering the area of fixed points in point set topology.

Need for the Study

The major aim of most sciences today is to increase preciseness of definitions, methods of analysis and synthesis, predictability, and theory. A major tool for helping to achieve this preciseness is mathematics, a specific example of logical reasoning, which itself is seeking to arrive at a more conprehensive and more precise body of knowledge and methodology. In the social sciences, the humanities, and engineering, for instance, all seem to be looking toward and depending upon systematized concepts and the mathematicians to help them provide one major means for improving their particular science.

It becomes obvious that the instruction in mathematics is no longer a luxury or a possibility but is now an imperative. Evidence of the recognition of this necessity is provided by the large amount of work put into the development of mathematics teaching in public schools through such efforts as the School Mathematics Study Group (SMSG), Committee on Undergraduate Program in Mathematics (CUPM), the Ball State Committee on Mathematics, and other committees and groups identified with colleges and/or the Mathematics Association of America.

The need for the improvement of mathematical understanding on the college level has also been recognized. It is the college graduate who will continue into advanced work in a field that is likely to be placing increasing emphasis upon mathematical reasoning (in addition to creative conceptualization) and who will be taking a leadership position in society. The instruction of mathematics on the college level, for all students and especially for those who will become teachers of mathematics in public schools or in institutions of higher education, cannot be left to chance or to yesterday's concepts and knowledge.

One of the newer concepts in mathematics which has not yet found its way into the textbooks is the fixed point property in point set topology. This is a fresh new area of mathematics which has already been shown to have applications in engineering and dynamics as well as in many of the areas of mathematics. Applications will be given throughout the thesis. All of the research concerning the fixed point property has occurred since the beginning of this century, and a majority of it has occurred since 1930. The amount of research concerned with fixed points has been abundant and is widely scattered throughout the periodical literature. A very small number of textbooks, however, give mention to fixed points, and these only briefly. Most undergraduate and beginning graduate students in mathematics are not skilled in reading the professional journals. This, of course, is a result of the fact that many of the research articles are not self-contained. They often use unfamiliar definitions appearing in other articles or periodicals and make references to research scattered throughout the literature. Thus the concepts of fixed points in point set topology have not been readily available to the student and many teachers of mathematics.

Procedure

It is for these reasons that this paper has been written. The general approach employed was a careful survey and analysis of the available literature. This thesis ties together the research showing the continuity and pattern of development of fixed point theory. Discussions and explanations are given in connection with the theorems. Many examples and counterexamples are given to illustrate the topological concepts. Unsolved problems of significant character are strategically located throughout the paper.

Scope and Limitations

We shall limit the scope of this thesis to the properties of fixed points for transformations, both continuous and non-continuous, and the properties of fixed points of multi-valued transformations. However, the research concerning the latter has been quite sparce. With this scope, we shall omit from this thesis a survey and analysis of the fixed point property for periodic transformations. Also omitted will be a study of the fixed point in algebraic topology.

It is to be noted in particular that none of proofs are original but are taken from the papers or treatises listed in the bibliography. It is not claimed that the treatment is complete for the literature on fixed pointsis too extensive to be covered here. However, we hope to have presented an outline of the development of the theory especially as related to point set topology.

Expected Outcomes

It is expected that as a result of reading this thesis, the student will gain an awareness of the current and past research in this modern branch of mathematics. He will become acquainted with men who have contributed to its research and development. It is hoped that this thesis will whet the student's mathematical appetite and will challenge him to read and probe the periodical literature of mathematics. The presentation of unsolved problems will impress upon him the fact that the frontiers of knowledge in this area of mathematics are being pushed back at a steady rate.

The fact that the reader, who is a potential teacher at either the public school level or the college level, comes abreast of this modern phase of mathematics has great educational significance. He will be confronted with the possibility of contributing to the research of mathematics by extending the results given in this thesis and by offering solutions to the unsolved problems as well as by developing new properties of fixed points. The selected bibliography will be a valuable aid to anyone interested in the research of fixed points in point set topology.

Chapter II gives a brief history of topology in America. Also, we introduce the fixed point and give some topological concepts and theorems dealing with fixed points. We state the celebrated fixed point theorem of L. E. J. Brouwer. In Chapter III, we define fixed point property and give some topological concepts of the fixed point property. Chapter IV is concerned with the recent developments of the fixed point property. In this chapter, we give the proof of Brouwer's theorem. Chapter V gives a summary and the educational significance of this thesis.

This paper is an attempt to study point set topology so that students of mathematics may better learn how to live with mathematics in a technological world of change. This pragmatic reason is the major justification for the present study.

CHAPTER II

BRIEF HISTORY AND INTRODUCTION TO FIXED POINT THEOREMS

The output of topology in America during this century seems enormous, even disconcerting, to anyone attempting to survey it all in brief compass. Although the development of mathematics in America has not been radically different from that of any other country, there is one respect in which its progress during this century differs from that in some other countries, and that is the sudden acceleration of research. A contributing factor to this was the opening of the University of Chicago in 1892 and the assembling of a small group of highly trained American mathematicians, most of them having studied abroad. Professor E. T. Bell, in an address to the Semicentennial History of the American Mathematical Society said, "It was the devoted work of one of these men, E. H. Moore, his first rate competence, and his enthusiasm for modern ideas that gave birth to point topology in America. ... In the late 1890's and early 1900's the history of topology in this country is largely an echo of enthusiasms at the University of Chicago." Directly through the work of E. H. Moore and that of one of his students, R. L. Moore, the actual beginnings of topology in this country were made.

Also from the address by Bell, "All of E. H. Moore's work had a constant direction: he strove unceasingly toward the utmost abstractness and generality obtainable." A survey of the literature of mathematics indicates that it is to his (E. H. Moore) influence that much of the development of the first two decades of this century in American mathematics can be traced. But it is R. L. Moore who probably has had more personal influence on point set topology than anyone else, not only in America, but also in Europe. In an address to the Semi-centennial History of the American Mathematical Society, R. L. Wilder said, "Single-handedly, R. L. Moore gathered around him students who have entered the field and have done work which is also of significant character." A literature search strongly indicates that R. L. Moore and the mathematicians who have studied under his direction have more titles in their collected works than any other similarly associated group.

R. L. Moore's influence was beginning to be felt about 1910. There arose subsequently in Poland a second notable mathematical school with similar interest. A striking fact about the advance thus made is the following. As was first seen by H. Poincaré, theoretical dynamics leads immediately to an extraordinary variety of point set theoretic questions. These mathematicians have formulated some of the questions of most interest to dynamics. An instance of this is R. L. Moore's "upper limiting sets" which have been found to be of central importance in the theory of dynamical systems.

Although topology did not emerge as a discipline until the early part of the twentieth century, an analysis of the research shows that its roots lie in the nineteenth century works of Georg Reimann, Georg Cantor, and Henry Poincaré. The topological work of Reimann and Poincaré had a basic geometrical aspect, and as a result of this, topology became known as a type of "position analysis" --- in which the size and shape of configuration are unimportant, the object of study being the "connectivity" of a figure. For example, that an ellipse is oval or that a square has corners is not of interest from a topological point of view. What is important is that each of these figures is so constituted that (i) the omission of a single point leaves each of them connected (a term to be made precise later) and that (ii) the omission of two points will leave the figure disconnected. In topology all such figures, i. e., triangles, polygons, circles, ellipses, are defined as simple closed curves.

The literature reveals that in topology, a change transpired which is evidenced in the dropping of the term "geometrical configuration" in favor of its synonyms "point set" and ultimately "topological space." The notion of configuration gave way to the Cantorian concept that a collection of things, be it a set of integers, functions, or set of sets, can constitute a topological space in some sense or other, the degree of utility of such an interpretation being dependent upon the number of topological properties exhibited by the collection. From this realization stems the fundamental importance of topology and its use as a tool even in such investigation as those of the foundations of logic.

Introduction

We shall begin our discussion of fixed points with a concept that seems to be elementary but is not very obvious to the intuition. In our earlier theorems, a minimal background is assumed. However, it is assumed that the reader is familiar with the basic definitions, notations, and operations of point set theory and has a workable knowledge of transformations and continuity. Topological properties will be defined as they are used throughout this thesis.

Definition 2.1. Let S and T be sets. A rule f is called a mapping of S into T (written $f:S \to T$) if and only if f associates with each element x of S a unique element y of T.

Definition 2.2. Let S and T be spaces and $f:S \to T$ a mapping. Then f is said to be continuous at the point s of S if and only if, given any open subset G of T such that s is in $f^{-1}(G)$, there exists an open subset V of S such that s is in V and V is a subset of $f^{-1}(G)$.

Definition 2.3. Consider a mapping of the type $f:S \to S$, where S is a set. If s is a point of S such that $f(s) = s$, then s is called a fixed-point of the mapping f.

For example, it is known that if D is a circular disk, then every mapping $f:D \to D$ has a fixed-point; and more generally, if E^n is a solid cube, then every mapping $g:E^n \to E^n$ has a fixed point. Theorems on the existence of fixed-points have many applications in mathematics and mechanics.

In addition to such applications as have been cited above, mention could be made of a wide variety of special applications of other fields of mathematics, as for instance the proof of the "fundamental theorem of algebra," which may be deduced from any number of theorems of plane topology; special properties of functions (e.g., existence of real continuous functions having nonderivatives); analytic functions; algebraic geometry; calculus of variations; harmonic integers; etc.

Listed as "A" and "B" are related theorems concerning topological properties of spherical surfaces. Each theorem seems to be a bit of curiosa, an odd fact stated only for its oddity. But the theorems are illustrations of an important body of knowledge.

A. The wind cannot be blowing everywhere on the earth at once. At every instance there must be at least one windless point on the earth's surface. But it is possible for the wind to be blowing everywhere except at one place.

B. If the wind is blowing everywhere in the Northern Hemisphere at any given instant, then given a direction, the wind is blowing in that direction at one or more places on the equator.

Of course, these are mathematical theorems rather than physical actualities. The wind that is mentioned is an ideal wind without discontinuities and is restricted to its surface component. These theorems are simple consequences of a celebrated general "fixed-point" theorem by the Dutch mathematician, L. E. J. Brouwer, first proved in 1911 (5)[1].

During the last decade applied mathematicians and engineers have begun to realize the usefulness of topology in attacking certain types of problems, particularly those involving nonlinear differential equations. Most physical phenomena can be described mathematically by differential equations. In the past it was usually assumed that effects were linear, i.e., directly proportional to causes. The assumption of linearity was made because calculation of nonlinear differential equations was extremely difficult if not impossible. But nature is rarely if ever "linear"; and for many modern applications

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[1]Arabic numerals in parenthesis indicate a reference to the Bibliography.

such as certain problems in electronics and supersonic aerodynamics, the assumption of linearity is misleading.

Topology is coming into use to show what types of solutions of certain nonlinear differential equations are possible. The answers are qualitative, not quantitative. Topology may tell the engineer what general type of circuit can satisfy his requirements, but it will not tell him the values of the circuit elements; these must be determined by other means.

In the applications of topology to other branches of mathematics, fixed point theorems play an important role. A typical example is the celebrated theorem of Brouwer. It is much less obvious to the intuition than most topological facts.

We consider a circular disk in the plane. By this we mean the region consisting of the interior of some circle, together with the circumference. Let us suppose that the points of this disk are subjected to any continuous transformation, which may or may not be one-to-one, although each point remains within the circle. A special case of the theorem of Brouwer now follows:

Theorem 2.1 (12). Let S be a disk in the plane and $f: S \to S$, where f is any continuous transformation. Then there exists some point s in S such that $f(s) = s$; that is, there exists at least one point whose position after the transformation is the same as its original position.

The proof of the existence of a fixed point is typical of the reasoning used to establish many topological theorems.

Proof. Consider the disk before and after the transformation, and assume that no point remains fixed, contrary to the conclusion of the theorem, so that under the transformation each point moves to another point inside or on the circle. To each point p of the original disk attach a "vector" pointing in the direction pp', where p' is the new location or image of p under the transformation. At every point of the disk there is such a vector, for each point is assumed to be moved under the transformation. Now consider the points on the boundary of the disk, with their associated vectors. All of these vectors point into the circle, since by assumption, no points are transformed into a point outside the circle. Let us begin at some point p_1 on the boundary of the disk and travel in a counterclockwise direction around the disk. As we do so, the direction of the vector will change, for the points on the boundary have variously pointed vectors associated with them. We notice that in traversing the circle one time from p_1 to p_1, the vector turns around and returns to its original position. Let us call the number of complete revolutions made by this vector the index of the vectors on the circle; more precisely, we define the index to be the algebraic sum of the various changes in angles of the vectors, so that each clockwise portion of the revolution is taken with a negative sign. The index is the net result, which may <u>a priori</u> be any one of the numbers $0, \pm(1, 2, \cdots)$, corresponding to a total change in angles of $0, \pm(360, 720, \cdots)$ degrees. We now assert that the index equals 1; that is, the total change in the direction of the vector amounts to precisely one positive revolution. To show this, we recall that the transformation vector at any point p on the circle is always directed inside the circle and never along the tangent. Now if this transformation vector turns through a total angle different from the total angle through which the tangent vector turns, then the difference between the total angles through which the tangent vector and the transformation vector turn will be some non-zero multiple of 360 degrees, since each makes an integral number of revolutions. Hence, at least once during the complete circuit from p_1 back to p_1, and since the tangent and the transformation vector turn continuously, at some point on the circumference the transformation vector must point directly along the tangent. But this is impossible, as we have seen.

If we now consider any circle concentric with the circumference of the disk and contained within it, together with the corresponding transformation vector on the circle, then the index of the transformation vector on the circle must be equal to 1. For as we pass continuously from the circumference to any concentric circle, the index must change continuously, since the direction of the transformation vectors varies continuously from point to point with the disk. But the index can assume only integral values and therefore must be constantly equal to its original value 1, since a jump from 1 to some other integer would be a discontinuity in the behavior of the index. Thus we can find a concentric circle as small as we please for which the index of the corresponding transformation vector is 1. But this is impossible, since by the assumed continuity of the transformation the vectors on a sufficiently small circle will all point in approximately the same direction as the vector at the center of the circle. Thus the total net change of their angles can be made as small as we please, less than 10 degrees, say, by taking a small enough circle. Hence, the index, which must be an integer, will be zero. This contradiction shows our initial hypothesis to be false, that is, that there is no fixed point under the transformation is untenable, and completes the proof.

The theorem just proved holds not only for the disk but also for any polygon or elliptic region or any other surface that is the image of the disk under a topological transformation. For if W is any figure correlated with a disk by a one-to-one continuous transformation, then a continuous transformation of W into itself without a fixed point would define a continuous transformation of the disk into itself without a fixed point, which we have just proved to be impossible.

It is easy to see that some geometrical figures do admit continuous fixed point free transformations into themselves. For example, the ring-shaped region between two concentric circles admits as a continuous fixed point free transformation, a rotation through any angle not a multiple of 360 degrees about its center. The surface of a sphere admits a continuous fixed point free transformation that takes each point into its diametrically opposite point. But it may be proved, by reasoning analogous to that which we have used for the disk, that any continuous transformation which carries no point into its diametrically opposite point has a fixed point.

Fixed point theorems such as these provide a powerful method for the proof of many mathematical "existence theorems" which at first sight may not seem to be of a geometrical character. A famous example is a fixed point theorem conjectured by Poincaré in 1912. This theorem has as an immediate consequence the existence of an infinite number of periodic orbits in the restricted problem of three bodies. G. D. Birkhoff succeeded in giving a proof after Poincaré's death.

Since then topological methods have been applied with great success to the study of the qualitative behavior of dynamical systems.

Elementary Fixed Point Theorems

The student is introduced to the "Intermediate Value Theorem" when he first studies analysis of functions, usually elementary calculus. This theorem is very significant in the study of functions at both the elementary and advanced level. We are at this point concerned only with the real number continuum.

Definition 2.4. A set S is <u>connected</u> if it is not the union of two separated sets, i.e., S is not the union of two sets such that neither contains a point or limit point of the other.

Theorem 2.2 (Intermediate Value Theorem). Let X be any closed interval in the real line continuum. Let $f:X \to R$ be continuous and let $f(a) \neq f(b)$, where a and b are the end points of the interval X. Then for each number V between $f(a)$ and $f(b)$, there exists a point v of X such that $f(v) = V$.

Proof. X is connected, hence $f(X)$ is connected since f is continuous and therefore an interval on the real line. Now, $f(a)$ and $f(b)$ are points of $f(X)$. Thus if V is between $f(a)$ and $f(b)$, then V is a point of $f(X)$; that is, there is a v in X such that $V = f(v)$.

Corollary 2.3. Let $f:X \to R$ be continuous. If $f(a) \cdot f(b)$ is a negative number, then there is an x in X such that $f(x) = 0$.

This is a special case of the intermediate value theorem, namely, $V = 0$. Also as a special case of the intermediate value theorem we have a fixed point theorem.

Corollary 2.4. Let $f:S \to S$ be continuous, where S is the closed unit interval. Then there exists a point s in S such that $f(s) = s$.

Proof. In the event that $f(0) = 0$ or $f(1) = 1$, the theorem is true. Thus it suffices to consider the case in which $f(0) > 0$ and $f(1) < 1$. Let $g:S \to R$ be defined by $g(x) = x - f(x)$. (Therefore, if $g(z) = 0$, then $f(z) = z$.) g is continuous and $g(0) = -f(0) < 0$ whereas $g(1) = 1 - f(1) > 0$. Consequently, by Corollary 2.3, there is a z in S such that $g(z) = 0$, whence $f(z) = z$.

Definition 2.5. A set S, together with a collection of subsets called open sets, is called a topological space if and only if the collection of open sets satisfy the following axioms:

Axiom 1. Every open set is a set of points.

Axiom 2. The empty set is an open set.

Axiom 3. For each point p, there is at least one open set containing p.

Axiom 4. The union of any collection of open sets is an open set.

Axiom 5. The intersection of any finite collection of open sets is an open set.

The collection of open sets, v, is called the topology of the topological space, which shall be denoted (S,v).

Definition 2.6. Let X and Y be topological spaces. Then X is said to be homeomorphic to Y if and only if there exists a one-to-one open continuous mapping of X onto Y. The mapping is called a homeomorphism.

One of the convenient facts about fixed point theorems is that if X and Y are homeomorphic topological spaces, and a fixed point theorem is true for X, then it is also true for Y.

Theorem 2.5 (41). Let X and Y be homeomorphic topological spaces. Then the continuous function $h:X \to X$ possesses a fixed point if and only if each continuous function $k:Y \to Y$ possesses a fixed point.

Proof. Let $f:X \to Y$ and $g:Y \to X$ be a pair of continuous inverse functions so that we have the diagram and suppose that each continuous function $h:X \to X$ possesses a fixed point. Then the function $h = gkf:X \to X$ is continuous and there is a point z in X such that $h(z) = z$. Let $w = f(z)$. We have $k(w) = k(f(z)) = fg(k(f(z))) = f(h(z)) = f(z) = w$. Thus, w is a fixed point under k. Since the hypotheses are symmetric with regard to X and Y, it also follows that if each continuous function $k:Y \to Y$ has a fixed point, then so does each continuous function $h:X \to X$.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, "k"] \\ X \arrow[u, "h"] & Y \arrow[l, "g"] \end{tikzcd}$$

Corollary 2.4 is a special case of the Brouwer Fixed Point Theorem, which we shall now state. In R^n, the unit n-cube I^n is defined as the set of points $(x_1, x_2, \cdots, x_n)$ which satisfies the inequality

$$0 \leq x_i \leq 1, \text{ for } i = 1, 2, 3, \cdots, n.$$

Theorem 2.6 (Brouwer's Fixed Point Theorem). Let $f:I^n \to I^n$ be continuous. There is a point z in I^n such that $f(z) = z$.

Let us consider the fixed point theorems applicable to the real line and Euclidean plane. First we give a proof to Corollary 2.4 different from the one given previously, this one self-contained.

Proof (Corollary 2.4). Suppose that f does not have a fixed point. Then the function $g(x) = 1/(x - f(x))$ is continuous at each point x in I. Now $b = g(1) > 0$ and $a = g(0) < 0$ and therefore $g(x)$ must take on all values between a and b, including zero. But $g(x)$

can never equal zero by its definition. This is a contradiction; hence I has a fixed point.

Theorem 2.7 (37). Let R be a real line interval and $f:R \rightarrow R$ be a continuous transformation. If there exists a point p of R such that $f(p) > p$ and a point q of R such that $f(q) < q$, then f has the fixed point property.

Proof. Suppose that f leaves no point of R fixed and for each point x in R define $M = \{x : x > f(x)\}$ and $N = \{x : x < f(x)\}$. Then $R = M + N$ and, since connected, there is a point of M which is a limit point of N or a point of N which is a limit point of M. Let the point r of N be a limit point of M and observe that the proof is the same if the other alternative is taken.

By the definition of N, $r < f(r)$ and we may define in R an open interval $V = \{x : x > (\phi(r) + f(r))/2\}$. Any open interval U containing the point r contains at least one point t of M due to the fact that r is a limit point of M. But $f(t) < t$ and hence cannot lie in V. Thus there is not an open interval U containing r such that f(U) is contained in V contradicting the continuity of f. This implies that f leaves a point of R fixed.

In the plane, there are mappings of continua into themselves which do not have the fixed point property. Consider the mapping of the unit circle into itself such that the mapping is a rotation of $\pi/4$ radians.

Theorem 2.8 (37). If $f:I \rightarrow I$ is continuous, where I is the unit circle and there is a point of I such that $f(1,\alpha) = (1,\alpha')$ where $\alpha < \alpha'$ and another point of I such that $f(1,\beta) = (1,\beta')$, where $\beta > \beta'$, then f leaves a point of I fixed.

Proof. We may consider a point of I to have coordinates $(1,\theta)$, where $0 \leq \theta < 2\pi$. Since f is continuous the image of θ varies continuously and therefore maps the interval $0 \leq \theta < 2\pi$ into itself continuously. Using the hypothesis and Theorem 2.7, there exists a point $(1,\lambda)$ such that $f(1,\lambda) = (1,\lambda)$.

Theorem 2.9 (37). Let I be a simple continuous arc and f a continuous function such that I is contained in f(I). Then f leaves a point of I fixed.

Proof. The conclusion follows directly from the Theorem 2.5 and Corollary 2.4. Another argument is given below. If I is a proper subset of f(I), then

(i) for every point x in I, $f(x) \geq x$ where "$\geq$" means follows or equal to,

(ii) for every x in I, $f(x) \leq x$, or

(iii) there is a point x_1 in I such that $x_1 \geq f(x_1)$ and a point x_2 in I such that $x_2 \leq f(x_2)$.

In case (i) the left end point is fixed and in case (ii) the right end point is fixed. For case (iii) the existence of a fixed point follows from Theorem 2.7.

Consider a system of n linear equations in n unknowns. This system can be written in the form

$$\begin{aligned}
x_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_1 \\
x_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_2 \\
&\;\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_n
\end{aligned} \tag{i}$$

We want to look at the problem of solving this system from an entirely new point of view. For this purpose, we consider the system of n equations

$$
\text{(ii)}\qquad
\begin{aligned}
y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_1 \\
y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
y_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_n
\end{aligned}
$$

For any system of numbers $(x_1,x_2,\cdots,x_n)$ which we substitute into the right members of equations (ii), we obtain certain numbers $(y_1,y_2,\cdots,y_n)$ in the left member. Thus, equations (ii) defines an operation, or as we more frequently say, an operator, which transforms the number complex $(x_1,x_2,\cdots,x_n)$ into the complex $(y_1,y_2,\cdots,y_n)$. In other words, we can say that equations (ii) define an operator which maps points of n-dimensional Euclidean space E^n into points of the same space. With this approach, the problem of solving the system (i) is that of finding a point of the space E^n which is mapped into itself by the operator (ii), i.e., a "fixed point" of the transformation.

It turns out that this treatment of the problem is possible not only with regard to linear algebraic equations, but also with respect to differential equations, integral equation systems of such equations, and many other problems of analysis. A very fruitful method of investigation of all such problems was initiated in this connection. Its theory consists in the investigation of different types of operators and establishing the conditions under which these operators leave definite points fixed.

The Contraction Operator and Banach's Theorem

Definition 2.7. A set S is said to be <u>metric</u> if and only if there is associated with S a mapping $\rho : S \times S \to R$ having the following properties for every x,y,z in S:

$$\text{(i)}\quad \rho(x,y) \geq 0.$$

$$\text{(ii)}\quad \rho(x,y) = 0 \text{ if and only if } x = y.$$

$$\text{(iii)}\quad \rho(x,y) = \rho(y,x).$$

$$\text{(iv)}\quad \rho(x,y) + \rho(y,z) \geq \rho(x,z).$$

Definition 2.8. Let S and S' be metric spaces and let f be a rule which associates some point y in S' with every point x in S. Such a rule is called an <u>operator</u> which is defined in the space S and maps S into S'. If y is in S' and is a point which the operator f assigns to the point x in S, we write $y = f(x)$ and we call y a value of the operator.

Definition 2.9. Let the operator f map the metric space S into itself. If there exists a number q, $0 \leq q < 1$, such that for arbitrary points x and x' of the space S, we have

$$\rho(f(x),\ f(x')) \leq q\cdot\rho(x,x'),$$

then we call f a <u>contraction</u> <u>operator</u>.

In Chapter IV, we define locally contractive and uniformly locally contractive mappings. Theorem 4.24 shows that f has a fixed point for uniformly locally contractive mappings.

Theorem 2.10 (47) (S. S. Banach). If a contraction operator f is defined on a complete metric space S, then there exists a unique point x' in this space for which $f(x') = x'$.

Proof. Take an arbitrary point x_0 in the space S and construct the sequence

$$x_1 = f(x_0),\ x_2 = f(x_1),\ x_3 = f(x_2), \cdots.$$

We shall show that this sequence converges in itself. In fact, for $n \geq 1$, we have $\rho(x_{n+1},\ x_n) = \rho(f(x_n),\ f(x_{n-1}))$

$$\leq q \cdot \rho(x_n, x_{n-1}).$$

From this it follows that $\rho(x_{n+1}, x_n) \leq a \cdot q^n$, where for brevity we set $a \geq \rho(x_1, x_0)$. Having noted this we choose any natural number N and let $n > N$, $m > N$. For definiteness, we shall assume that $m > n$. Then

$$\rho(x_n, x_m) \leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \cdots + \rho(x_{m-1}, x_m).$$

Hence, $\rho(x_n, x_m) \leq a \cdot q^n/(1 - q)$. Since q^N tends to zero with increasing N, the convergence of the sequence $\{x_n\}$ into itself means that the limit $x_n (n \to \infty) = x'$ exists. From Definition 2.4, we have

$$\rho(x_{n+1}, f(x')) = \rho(f(x'), f(x_n)) \leq q \cdot \rho(x_n, x').$$

Therefore, limit $(n \to \infty)$ $\rho(x_{n+1}, f(x')) = 0$, i.e.,

$$\text{limit } (n \to \infty) x_n = f(x')$$

and from this, by virtue of the uniqueness of the limit, $f(x') = x'$ follows. Thus a fixed point exists. It remains to verify that this point is unique. If the operator f had another fixed point x besides x', then it would turn out that $\rho(x', x) = \rho(f(x'), f(x)) \leq q \cdot \rho(x', x)$ for which $\rho(x', x)$ being positive contradicts the condition $q < 1$.

It is interesting to observe that the condition

$$\rho(f(x'), f(x)) < \rho(x', x), \; x' \neq x,$$

is not sufficient for the existence of a fixed point. For example, let E be the set of real numbers with the usual definition of distance. Let $f(x) = x + \pi/2 - \arctan x$. Since $\arctan x < \pi/2$ for every x, the operator f has no fixed point. At the same time, if $x < y$, then

$$f(y) - f(x) = y - x - (\arctan y - \arctan x)$$

and by Lagrange's formula

$$f(y) - f(x) = y - x - (y - x)/(1 + z^2), \; (x < z < y).$$

If we had $|f(y) - f(x)| \geq |y - x|$ this would mean that $|1 - 1/(1 + z^2)| \geq 1$, but this inequality is not satisfied for any z. Therefore, we have $|f(y) - f(x)| < |y - x|$.

Example (47). (Application of Banach's Theorem). Consider the differential equation

$$y' = f(x,y),$$

the right member of which is a function defined and continuous in the whole plane and which satisfies the Lipschitz condition with respect to y

$$|f(x,y) - f(x,y_1)| \leq K|y - y_1|.$$

Functional analysis is that branch of mathematics in which elements of a given class of functions are considered to be points of an appropriate infinite-dimensional space. In this way various theories of real and complex functions may be seen from a point set theoretic view.

Let S be the closed interval from a to b, and

$$f(x) = \int_a^b u(x,y)f(y)dy \tag{i}$$

where u is continuous on the closed square S x S. It can be shown that there is a continuous function f on S which satisfies equation (i) above. Observe that for every continuous function g defined on S, the function

$$h(x) = \int_a^b u(x,y)g(y)dy$$

is continuous so that the right hand side of equation (i) defines a mapping which takes every continuous g into a continuous h. It becomes apparent in the proof that the existence of a solution to (i) amounts to the same thing as a proof that there is a continuous function which is taken into itself by this mapping, i.e., a function which remains fixed under the mapping. Thus, if functions are considered as points of a space, we are interested in a statement regarding the existence of fixed points for certain kinds of mappings.

We consider a more elementary situation. Let R be the real number continuum, i.e., closed and connected, and let f be a mapping which takes R into itself. Then, for every x in R, f(x) is also in R. We suppose there is a positive constant k less than 1 such that, for every x and y in R,

$$|f(x) - f(y)| \leq k|x - y|.$$

This equation implies that f is continuous.

Theorem 2.11 (7). If $f:R \to R$, where R is the set of real numbers, and if there is a k with $0 < k < 1$ such that for every x, y in R with $x \neq y$, $|f(x) - f(y)| \leq k|x - y|$, then there is one and only one x' such that $f(x') = x'$.

The result of this theorem follows readily from Theorem 2.10.

CHAPTER III

FIXED POINT PROPERTIES OF POINT SETS

Since Brouwer's theorem was introduced in 1911, there have been many important generalizations. Yet today there is no known characterization of the fixed point property (hereafter written fpp), even for restricted classes of spaces.

Definition 3.1. A topological space X is said to have the <u>fixed point property</u> (fpp) if, given any continuous function f from X into X, there exists a point p such that $f(p) = p$.

Definition 3.2. Let S be a topological space. Then S is <u>compact</u> provided that, $\{O_\alpha\}$ is any open covering of S, then some finite subcollection of $\{O_\alpha\}$ covers S.

Definition 3.3. A compact connected set S will be called a <u>continuum</u>.

Definition 3.4. A <u>chain</u> is a finite collection of open sets $d_1, d_2, \cdots, d_n$ such that d_i intersects d_j if and only if $|i - j| \leq 1$. If the elements of the chain are of diameter less than a positive number ϵ, that chain is called an <u>ϵ-chain</u>.

Definition 3.5. A compact continuum is said to be <u>chainable</u> if for each positive number ϵ, there is an ϵ-chain covering it.

Cartesian Products

It was conjectured by H. Cohen (10) that if X and Y have the fpp, then X x Y has the fpp. This conjecture was answered in the affirmative in 1956 by E. Dyer (14) for the case where X and Y are compact chainable continua. Such continua were previously known to have the fpp, which will be shown in Chapter IV. In the same year Cohen (10) showed the conjecture to be true if X and Y are compact spaces. He established this result without the use of Brouwer's theorem. The H. Cohen conjecture is false in general, even for metric spaces, as will be shown by an example later in this chapter. Cohen's proof was for the case where X and Y are compact ordered sets with the order topology. It is perhaps worth noting that there is an analogous result due to S. Ginsburg (24) for the similarity transformations on ordered sets with the product ordered lexicographically. It is well known that a compact ordered space has the fpp if and only if it is connected. Hence, Cohen was able to rephrase his results to read as follows: If X and Y are compact connected ordered spaces and Z = X x Y, then Z has the fpp. First we shall state some auxiliary properties of the space Z.

Definition 3.6. A space S is said to be <u>normal</u> if and only if, given any two disjoint closed subsets F_1 and F_2 of S, there exist disjoint open subsets G_1 and G_2 of S containing F_1 and F_2, respectively.

Definition 3.7. A space X is said to be <u>unicoherent</u> if and only if whenever it is the union of two continua A and B, the intersection of A and B is connected.

Theorem 3.1 (10). If X and Y are compact connected ordered spaces and $Z = X \times Y$, then Z is unicoherent.

Theorem 3.2 (10). If C is a connected subset of Z and if $Z - C$ is connected, then the boundary of C, denoted $F(C)$, is also connected.

Proof. If we let $\overline{C}$ and $\overline{Z - C}$ correspond to A and B, respectively, in the proof of Theorem 3.1, then this theorem follows immediately.

The following theorem will be stated without the proof because of its excessive length.

Theorem 3.3 (10). If A is a closed subset of Z, where $Z = X \times Y$ and X and Y are compact connected ordered spaces, then either

(i) there is a component B of $Z - A$ such that $\pi_y(B) = Y$ or

(ii) there is a component K of A such that $\pi_x(K) = X$, (where π_x and π_y are the projection transformations on X and Y).

The next theorem is the culmination of the theorems thus far given in this chapter and is due to H. Cohen.

Theorem 3.4 (10). If X and Y are compact ordered connected spaces and $Z = X \times Y$, then Z has the fpp.

Proof. Let $f: Z \to Z$. Let

$$A = \{(x,y): \pi_y f(x,y) = y\}.$$

Then A is closed; hence by Theorem 3.3, we have either B, a component of $Z - A$, with $\pi_y(B) = Y$, or K, a component of A, with $\pi_x(K) = X$. The first of these possibilities is untenable, however, since $B \cdot \{(x,y): \pi_y f(x,y) > y\}$ and $B \cdot \{(x,y): \pi_y f(x,y) < y\}$ are two nonempty disjoint open sets whose union is B, a connected set. Hence, K exists and since K is connected, either one of the open (in K) and disjoint sets $K \cdot \{(x,y): \pi_x f(x,y) > x\}$ and $K \cdot \{(x,y): \pi_x f(x,y) < x\}$ is empty, or their union cannot be all of K. In either case there is at least one point (x',y') in K such that $\pi_x f(x',y') = x'$. But since K is a subset of A we have $\pi_y f(x',y') = y'$ and therefore (x',y') is a fixed point.

Thus, it has been shown without the use of Brouwer's theorem that if X and Y have the fpp, then X x Y has the fpp if X and Y are compact ordered connected spaces.

One possible method to prove the conjecture true with X and Y possibly satisfying additional properties is to find other topological properties that X and Y must have. If these properties are preserved under the cross product, they may be used in an attempt to show that X x Y has the fpp.

Definition 3.8. A topological space is said to be regular if V is any neighborhood of a point p, there exists a neighborhood U of p such that $\overline{U}$ is a subset of V.

The following theorem, due to E. Connell, establishes that if a topological space possesses the fpp, then if the topology is made stronger (i.e., contains more open sets) for the set of points, this new topological space also possesses the fpp. This theorem is quite useful in establishing other theorems concerning the fpp.

Theorem 3.5 (11). Suppose X is a set and v is a topology for X such that (X,v) is a regular space with the fpp. If u is a stronger topology for X such that if R is an open set in u, its closure, $\overline{R}$, is the same in both topologies, then (X,u) has the fpp.

Proof. Suppose that $f:X \to X$ is continuous in the space (X,u). Suppose p is a point in X and H is any open set of the topology v such that f(p) is in H. Since (X,v) is regular, there exists a set

K open in v such that f(p) is in K and $\overline{K}$ is a subset of H. The closure of K is the same in both topologies u and v since K is an open set of (X,u). $D = f^{-1}(K)$ is an open set in u since K is an open set in (X,u) and f is continuous in (X,u). $\overline{D}$ is the same in both topologies and is closed in each. $C(\overline{D})$, the complement of $\overline{D}$, is open in each. $\overline{C(\overline{D})}$ is the same in both topologies, is closed in each, and contains $C(\overline{D})$. Therefore, $C(\overline{C(\overline{D})})$ is open in both spaces and is contained in $\overline{D}$, and contains p. Now $\overline{D} = f^{-1}(K)$ is a subset of $f^{-1}(\overline{K})$ which is a subset of $f^{-1}(H)$. Thus there is an open set of u which contains p and whose image under f is contained in H. Therefore, f is continuous in (X,v) and thus must have a fixed point.

Definition 3.9. A topological space (X,v) is called a Hausdorff Space if for each pair a and b of distinct points of X, there exist disjoint neighborhoods H and K containing a and b, respectively.

Definition 3.10. A topological space is locally compact if each of its points has a neighborhood whose closure is compact.

Example (11). This is an example of a Hausdorff space which has the fpp yet contains no compact subset except for finite sets.

Let I be the unit interval and u be the collection of all subsets S such that there exist an open set A and a countable (finite or infinite) set B so that S = A - B. It is clear that the intersection of any two such sets is also such a set. Also, it can be shown that an arbitrary union of sets of this type is again a set of this type, and thus that u is a topology for I. In order to show this, note that the union of any collection of sets of this type is equal to the union of a countable subcollection except for a countable number of points.

Let v be the ordinary topology for I and let S_v be the collection of all subsets S in I with the topology v. Also S_u will denote the collection of all subsets S in I with the topology u described above. It must be shown that $\bar{S}_v = \bar{S}_u$. Now suppose p is a limit point of S in (I,v) and S' = A - B contains p, where A is an ordinary open set and B is countable. Since A must intersect S at uncountably many points, (A - B) must intersect S. Thus p is also a limit point of S in (X,u). Thus, $\bar{S}_u$ contains $\bar{S}_v$. But since u is a stronger topology than v, then $\bar{S}_v$ contains $\bar{S}_u$. Hence $\bar{S}_v = \bar{S}_u$.

Now according to Theorem 3.5, the unit interval must have the fpp under this new topology. No countable set can have a limit point under this new topology; therefore the space contains no infinite compact subsets. Thus, there does not exist a point so that the space is locally compact at that point.

Every open set of the space contains infinitely many points and thus no open set has a compact closure. Therefore, there does not exist a point at which the space is locally compact. The space is, in fact, simply the unit interval with a new topology which contains more open sets than the ordinary topology. Theorem 3.5 tells us that every function from the unit interval into itself which is continuous in the ordinary topology is continuous in this new topology. It is not true that every function continuous in the usual topology is continuous in the new topology. Thus the addition of some open sets to the unit interval has created no additional continuous functions. Since intervals have the fpp with the normal topology, this new space also has the fpp.

If K is an open set in the space in the preceding example, its closure in its topology is the same as its closure in the usual topology. Theorem 3.5 states that, in general, if X is a set and v is a topology for X such that (X,v) has the fpp, and u is a stronger topology for X which preserves the closure of open sets, then (X,u) has the fpp. Thus according to this theorem, the space in the preceding example has the fpp.

Definition 3.11. An <u>arc</u> is a homeomorph of the unit interval.

The following definition is analogous for arcs to the one given in Definition 3.4 for open sets.

Definition 3.12. A countable collection of arcs $\{A_n\} = \{[b_n, c_n]\}$ is a <u>chain</u> if $c_n = b_{n+1}$ for all n.

Definition 3.13. A collection of sets $\{S_n\}$ is <u>locally finite</u> if for each point p of the space, there exists an open set containing p and intersecting only finitely many S_n.

The following theorem will be stated without the proof.

Theorem 3.6 (11). If X is a metric space with the fpp, then every locally finite chain of arcs is finite.

Example (11). This is an example of a metric space X that has the fpp and yet the cross product X x X does not have the fpp.

$$\text{Define } f(x) = \sin \pi/(1 - x), \qquad 0 \leq x < 1$$
$$= 1 \qquad x = 1.$$

X is the set of points (x,f(x)) in E^2 with $0 \leq x \leq 1$. The topology for X is the relative topology. Suppose $g: X \rightarrow X$, where g does not have the fpp. Let S_1 be the set of all points whose abscissa moves to the right and S_2 be the set of all points whose abscissa moves to the left. Then $S_1 + S_2 = [0,1]$ and thus one of the sets must contain a limit point p of the other. The point (p, f(p)) must be a fixed point.

Now it will be shown that X x X does not have the fpp. This cross product space has a natural imbedding in E^4 as follows: C is all (x, f(x) + f(z), z) for $0 \leq x \leq 1$ and $0 \leq z \leq 1$. C and X x X are homeomorphic. Consider the one-to-one function h(a, f(a), b, f(b)) = (a, f(a) + f(b), b) from X x X onto C. Suppose $(a_n, f(a_n), b_n, f(b_n))$ is a sequence approaching (a, f(a), b, f(b)) in X x X. Then $a_n \to a$, $f(a_n) \to f(a)$, $b_n \to b$, and $f(b_n) \to f(b)$. Then $(a_n, f(a_n) + f(b_n), b_n) \to (a, f(a) + f(b), b)$ in C. Thus h is continuous. Now it will be shown that h^{-1} is continuous. Suppose $(a_n, f(a_n) + f(b_n), b_n) \to (a, f(a) + f(b), b)$ in C. Then $a_n \to a$, $b_n \to b$, and $f(a_n) + f(b_n) \to f(a) + f(b)$. If a is a point of continuity of f, i.e., if $a \neq 1$, then $f(a_n) \to f(a)$ and so $f(b_n) \to f(b)$. A similar argument holds for $b \neq 1$. Now suppose a = b = 1. In this case, f(a) + f(b) = 2. Thus in any of the above cases, $(a_n, f(a_n), b_n, f(b_n)) \to$ (a, f(a), b, f(b)) in X x X. Hence, h^{-1} is continuous and h is a homeomorphism.

Now an infinite locally finite chain of arcs will be constructed in C, and thus by Theorem 3.6, C will not have the fpp. Note that C is all (x, y, z) where x and z are in the closed interval from zero to one, and y = f(x) + f(z). The construction will be as follows: The first arc will simply be one hump of the sine wave in the xy-plane. It will start at the origin and follow the sine wave in the xy-plane to the first zero. The second arc will start there and follow a sine curve parallel to the yz-plane to the next zero. The third arc will start there and be another hump of a sine wave in the plane parallel to the xy-plane. In this manner the arcs will "approach" in zig-zag manner the vertical line $\{(x, y, z): x = 1, -1 \leq y \leq 1, z = 1\}$ which does not belong to the space.

More specifically, if n is odd,

$$S_n = \{(x, y, z): \frac{n - 1}{n + 1} \leq x \leq \frac{n + 1}{n + 3},$$
$$y = \sin \pi/(1 - x), \text{ and } z = \frac{n - 1}{n + 1}\}.$$

If n is even,

$$S_n = \{(x, y, z): x = \frac{n}{n + 2}, \frac{n - 2}{n} \leq z \leq \frac{n}{n + 2},$$
$$\text{and } y = \sin \pi/(1 - z)\}.$$

This determines an infinite chain of arcs, and it is to be shown that $\{S_n\}$ is locally finite.

If $0 < \epsilon < 1$, there is an N so that $\{S_n\}$ is contained in $\{(x, y, z): x > (1 - \epsilon) \text{ and } z > (1 - \epsilon)\}$ if $n > N$. Thus if (x, y, z) is a point of C where $\{S_n\}$ is not locally finite, $x = 1$ and $z = 1$. In this case, $y = 2$, yet S_n is locally finite at $(1, 2, 1)$ because no S_n has a y-coordinate greater than 1. Therefore, $\{S_n\}$ is locally finite, and C does not have the fpp.

Thus, we have an example of a metric space with the fpp which is not compact. The cross product of this space with itself does not have the fpp. Theorem 3.6 shows a topological property which is implied by the fpp. It states that in a metric space with the fpp, every infinite chain of arcs has a nonempty limiting set. Although the space in the preceding example satisfies this arc-compactness condition, its cross product with itself contains an infinite chain of arcs with no limiting set, and this does not satisfy this condition. Therefore, the cross product space cannot have the fpp.

Definition 3.14. A topological space is called separable if there exists a countable subset H of A which is dense in A, i.e., where A is subset of $\overline{H}$, where A is a subset of the topological space.

Definition 3.15. A topological space is said to be locally connected at a point p if given any neighborhood U of p there exists a connected neighborhood V of p such that V is a subset of U.

The next example is of a separable, locally connected metric space that has the fpp, yet is not compact.

Example (11). Let the space X be a subspace of E^2, $X = \Sigma_{n \geq 0} I_n$, where $I_0 = [(0,0), (1,0)]$, a unit interval on the x-axis, and $I_n = [(1/n,0), (1/n,1)]$, a vertical unit interval extending upward from I_0.

Suppose f is a function from X to X with no fixed points. If f_0 is the restriction of f to the domain I_0, and f_0 is projected onto I_0, a continuous function is obtained from I_0 to I_0. It must have a fixed point at $(p,0)$. Since f_0 has no fixed point, there is an integer k so that $p = 1/k$. Thus $f(1/k,0) = (1/k,y)$ and $0 < y \leq 1$. Let t be the least upper bound of $\{y: f(1/k,y) = (1/k,z) \text{ with } z > y\}$. Then it follows that $(1/k,t)$ is a fixed point.

Note that by making I_n, $n > 0$, of length n instead of unit length, an unbounded subset of the plane with the fpp may be constructed.

It has been shown that under certain conditions, the fpp implies compactness. First, we have a lemma.

Lemma 3.7 (11). If X is a connected, locally connected metric space, M a compact subset of X, and D an open set with M a subset of D and $\overline{D}$ compact, then only a finite number of the components of $C(M)$ intersect $C(\overline{D})$.

Proof. Suppose, to the contrary, that there exists an infinite collection $\{E_i\}$ of distinct components of $C(M)$, each intersecting $C(\overline{D})$. Since X is locally connected and $C(M)$ is open, each E_i is a connected open set. Since the space is connected, each E_i has a limit point in M, and thus intersects D as well as $C(\overline{D})$. Each E_i is connected and thus intersects $\overline{D} - D$ at some point, say p_i. Since $\overline{D} - D$ is closed and compact, and the p_i's are distinct points, they have a limit point p of $\overline{D} - D$. Now p is in some component E of $C(M)$, yet E can be no more than one of the E_i's and thus can intersect no more than one E_i. Thus at most one p_i is in E, and p cannot be a limit point of the p_i's. This is a contradiction and proves the lemma.

The following example, due to W. L. Strother, is a space S which does not have the fpp yet contains a dense subset Y which does have the fpp. The space Y is not compact though its closure is compact, yet its closure does not have the fpp. It is not known what conditions imply that the fpp is preserved under closure.

Example. X is a compact metric space that does not have the fpp although it contains a dense subset Y that does have the fpp.

Let X be a subspace of E^2. Let A be the boundary of the square with $(0,-2)$, $(4,-2)$, $(4,2)$, and $(0,2)$ as its corners. Let $B = \{(x,y): 0 < x \leq 1,\ y = \sin 1/x\}$ and $X = A + B$. X does not have the fpp. Project B onto the vertical line from $(0,-2)$ to $(0,2)$, and then rotate A through an angle of 90 degrees. This determines a continuous transformation from X into X with no points fixed.

Let A' be the set A with the points on the vertical interval (0,-1) to (0,1) removed and Y = A' + B. It will be shown that Y has the fpp. Suppose f is a function from Y into Y without the fpp. Then f(A') intersects B, because if it did not, there would be a fixed point in A'. Since A' is compact and connected, f(A') is compact and connected and thus contained in B. This implies that there exists a point p of B so that f(p) is a point in B. If S is a compact, connected set containing p, f(S) is a subset of B. Thus, f(B) is contained in B, and f(B) + (0,1) is contained in B. However, B + (0,1) is homeomorphic to the space in the example on page 31, and thus must have a fixed point. It is easy to see that Y is dense in X since X is a subset of $\overline{Y}$. This follows from the fact that the points on line (0,-1) to (0,1) are limit points on the set B.

Definition 3.16. Two mappings f and g are said to be homotopic if either one can be continuously deformed into the other.

According to Whyburn (e.g., Theorem 7.4, p. 288) (73), a locally connected, connected and compact metric space X is unicoherent if and only if every mapping of X into the unit circle is homotopic to a constant. It is shown that it is not necessary that a space possessing the fpp be unicoherent.

In the following example, a compact metric space is given which has the fpp, yet is not unicoherent. This is also an example of a space with the fpp that separates the plane. This space is not locally connected.

Example. This is an example of a compact metric space that has the fpp, yet is not unicoherent.

The space X is a subset of E^2. Let

$$A = \{(x,y): 0 < x \leq 1,\ y = \sin \pi/x\} \text{ and}$$

$$B = [(0,1),\ (0,-2)] + [(0,-2),\ (1,-2)] +$$

$$[(1,-2),\ (1,0)],$$

where these are three line segments in E^2. Let $X = A + B$. It is clear that X is not unicoherent. The proof that X has the fpp will be omitted since it is similar to those already presented.

Arc-Wise Connectedness

Definition 3.17. A topological space is arc-wise connected if any two points a and b are contained in some arc with end points a and b.

The proof of the following theorem is omitted because of its excessive length. The proof is given by G. T. Whyburn.

Theorem 3.8 (73). Suppose X is a locally connected metric space in which every connected open set is arc-wise connected. Then if X has the fpp, X is unicoherent.

Definition 3.18. A space X is said to be complete if X is metric and given any Cauchy sequence $\{a_n\}$ of points of X, there exists a point p of X such that $\{a_n\}$ converges to p.

Corollary 3.9 (11). If Y is a locally connected, locally compact metric space with the fpp, then Y is unicoherent.

Corollary 3.10 (11). If Z is a locally connected, complete metric space with the fpp, then Z is unicoherent.

Proof. By theorems 5.3 and 5.5, page 38, of Whyburn (73), each of Y and Z has the property that every connected open set is arc-wise connected. They therefore satisfy the hypothesis of Theorem 3.8.

It is desirable to know what topological properties of fixed point spaces are preserved under cross product. These properties may then be used in an attempt to prove that under certain conditions, the fpp is preserved under the cross product. The following corollary shows that unicoherence is such a property.

Corollary 3.11. If X and Y are compact, locally connected metric spaces with the fpp, then X x Y is unicoherent.

Proof. By Corollary 3.9, each of X and Y is unicoherent. According to Whyburn (73) (e.g., Theorem 7.51, p. 228), if X and Y are unicoherent, locally connected, connected, and compact metric spaces, then X x Y is unicoherent.

In order to make an application of this theorem, it is necessary to prove two lemmas. One lemma states the existence of irreducible separating sets. The second lemma states that irreducible separating sets are connected.

Definition 3.19. The statement that the set M <u>separates</u> the sets A and B in X means that X - M is the union of two separated sets, one containing A and the other containing B.

Definition 3.20. The statement that the closed set M is an <u>irreducible</u> set separating A and B in X means that M separates A and B in X, but no proper closed subset of M separates A and B in X.

Lemma 3.12 (11). If X is a locally connected, compact metric space and M is a closed set separating the two connected sets A and B in X, then there is an irreducible closed subset of M separating A and B in X.

Proof. Suppose there is no closed irreducible subset of M separating A and B. The collection of all closed subsets of M separating A and B forms a partially ordered collection when ordered by inclusion. According to the Hausdorff Maximality Principle as given in Kelley (34), there is a maximal monotonic subcollection $\{M_c\}_{c\in C}$. Let N be their intersection, which is nonempty since the space is compact.

Suppose N does not separate A and B. Let H be the component of X - N containing A. H is open since X is locally connected. $\bar{H}$ - H is nonempty since X is connected, and $\bar{H}$ - H is a subset of N. $\bar{H}$ must intersect H because if not, $\bar{H}$ - H would separate A and B and thus a closed subset of N would separate A and B. If each M_c intersected D, N would intersect D. Therefore, there exists an M_c which does not intersect D and thus which does not separate A and B. This is a contradiction and shows that N separates A and B. The closed set N is then irreducible.

Lemma 3.13 (11). If X is a connected, locally connected, unicoherent metric space, and N is an irreducible closed set separating the two connected sets A and B, then N is connected.

Proof. Let E and F be the components of X - N containing A and B, respectively. E and F are open, and must be disjoint or else N would not separate A and B. Since X is connected, $\bar{E}$ - E and $\bar{F}$ - F are nonempty. Now each of these is a subset of N, and each separates A and B. Since, by assumption, N is irreducible, then we have $\bar{E}$ - E = $\bar{F}$ - F

= N. Thus E and F are disjoint components of X - H having the same boundary, N. Now according to Wilder (77) (e.g., Theorem 4.12, p. 51) this boundary is connected. Hence, this proves the theorem.

The following is a well known theorem and the proof is readily obtained with the use of Lemmas 3.12 and 3.13. Also it is to be observed that this theorem follows as a corollary to Theorem 3.4. We shall omit the proof.

Theorem 3.14 (11). The cross product of the unit interval with itself has the fpp.

One of the unsolved problems concerning fpp is that if X and Y are compact metric spaces with the fpp, must X x Y have the fpp? A special case of this that is also unsolved is the instance when X is locally connected and Y is the unit interval.

CHAPTER IV

RECENT DEVELOPMENTS IN FIXED POINT PROPERTIES

Most of the work concerning the topological properties of fixed point spaces has been done in the last three decades with the exception of that done by three or four men. Of course, the most celebrated of these is L. E. J. Brouwer. Brouwer's famous theorem concerning fixed points was proved in 1911 (5).

Prior to the recent period mentioned above, work of a significant nature concerning fixed points was done by W. Scherrer (53) in 1925 and W. L. Ayres (1) in 1930. From this time, interest became wide spread in the area of topology and, in particular, this branch of topology which deals with fixed points.

It was shown by W. Scherrer (53) that every homeomorphism of an acyclic Peano continuum into a subset of itself has a fixed point. Five years later in 1930, W. L. Ayres (1) gave a generalization of this theorem based on the cyclic structure of a Peano continuum.

Definition 4.1. A compact, connected, and locally connected metric space is a Peano continuum.

Definition 4.2 (Whyburn) (74). If a space has the property that each two distinct points are the endpoints of at most one chain, then it is said to be acyclic.

Definition 4.3. A subset E of a space M will be called a <u>cyclic element</u> of M provided that

(i) E contains two conjugate points a and b, i.e., $a \neq b$ and no point of T separates T between a and b.

(ii) E - (a + b) consists of all points x of T such that x is conjugate to a in T, and x is conjugate to b in T.

Peano Continua

Theorem 4.1 (1). If T is a homeomorphism of the Peano continuum M such that T(M) is a subset of M, then there is a cyclic element E of M such that T(E) is contained in E.

Theorem 4.2 (1). If every cyclic element E of the Peano continuum M has the property that every homeomorphism carrying E into a subset of itself has a fixed point, then the entire continuum M has the fpp.

Proof. Let T be a homeomorphism such that T(M) is a subset of M. By Theorem 4.1, there exists a cyclic element E of M such that T(E) is a subset of E. Then by the assumed property there exists a point p of E such that $T(p) = p$.

The converse of Theorem 4.2 is not true, i.e., there exists a Peano continuum M having the property that every homeomorphism of M into itself has the fpp, but containing cyclic elements not having the fpp. Ayres has given a counterexample to the converse of this theorem. Take, for example, a circle plus an interval having one end point on the circle.

Definition 4.4. An <u>n-simplex</u> is a set which consists of n + 1 linearly independent points $p_0, p_1, \cdots, p_n$ of an Euclidean space of dimension greater than n together with all the points of the type

$$x = c_0 p_0 + c_1 p_1 + \cdots + c_n p_n, \text{ where}$$

$$c_0 + c_1 + \cdots + c_n = 1, \ 0 \leq c_i \text{ for every } i.$$

Theorem 4.3 (1). If every cyclic element of the Peano continuum M is an n-dimensional simplex (n may vary for different elements), then every homeomorphism of M into a subset of itself has a fixed point.

Proof. The conclusion of this theorem follows directly from Theorem 4.2 and Brouwer's fixed point theorem.

The following theorem, due to G. T. Whyburn, is concerned with the cyclic structure of a continuous curve.

Theorem 4.4 (76). In order that the continuous curve M should fail to separate the plane, it is necessary and sufficient that every maximal cyclic curve of M should be a simple closed curve plus its interior.

Proof. The condition is necessary. For suppose M does not separate the plane. Let E be any maximal cyclic curve of M, and let J be the boundary of the unbounded complementary domain of E. Whyburn has shown J is a simple closed curve. Since M does not separate the plane, clearly every point within J belongs to M and also to E. That J plus its interior must be identical with E follows immediately from the fact that J is the boundary of the unbounded complement of E.

The condition is also sufficient. For suppose every maximal cyclic curve of a continuous curve M is a simple closed curve plus its interior, and suppose, contrary to this theorem, that M separates the plane. Then there exists at least one unbounded complementary domain R of M. Let J denote the boundary of the unbounded complementary domain of the boundary of R. R. L. Moore (44) has shown J is a simple closed curve whose interior I contains R. Whyburn has shown that M contains a maximal cyclic curve E which contains J. By hypothesis there exists a simple closed curve K such that if D is the interior of K, then K + D = E. But J is a subset of E and hence must also contain I, the interior of J. But R is a subset of I, a complementary domain of M. Thus the supposition that M separates the plane leads to a contradiction.

Theorem 4.5 (1). If the Peano continuum lies in a plane and does not separate the plane, then every homeomorphism of M into a subset of itself has a fixed point.

Proof. Since M does not separate the plane, it follows from Theorem 4.4 that every maximal cyclic set of M is a simple closed curve plus its interior, a two-dimensional simplex. All other cyclic elements of M are points. Hence, with the result of Theorem 4.3, the conclusion of this theorem follows.

A celebrated unsolved problem that has challenged topologists for decades is whether a general bounded continuum not separating its plane has the fixed point property.

Thus, Ayres has shown that a homeomorphism T which carries a locally connected continuum M into a subset of itself carries some cyclic element C of M into a subset of itself. This result has been shown to yield a fixed point theorem in an acyclic space or under certain other conditions.

Unicoherent and Decomposable Continua

O. H. Hamilton (30) has extended the theorem due to Ayres and this result paved the way for a large amount of research in point set topology.

In particular, Hamilton has shown that a hereditarily unicoherent and hereditarily decomposable continuum has the fpp, i.e., if every subcontinuum of a unicoherent and decomposable continuum is unicoherent and decomposable, then the continuum has the fpp. First we state a lemma that is due to Hamilton.

Lemma 4.6 (30). If, in a metric space, M is a compact non-degenerate hereditarily unicoherent space which is hereditarily decomposable and if there exists a homeomorphism T of M into a subset of itself, then T carries some proper subcontinuum of M into itself.

Definition 4.5. If the space M possesses a countable base for the open sets, i.e., if each open set is a union of sets in the topology, then it is said to be <u>completely separable</u>.

Theorem 4.7 (30). If M is a compact continuum in a metric space, and if M is hereditarily decomposable and hereditarily unicoherent, then every homeomorphism of M into a subset of itself has a fixed point.

Proof. By Lemma 4.6, T carries some proper subcontinuum M_1 of M into itself. Let $\{M_i\}$ be an uncountable, well-ordered sequence β of subcontinua of M having the following properties:

(i) for each ordinal number λ which has an immediate predecessor, M_λ is a proper subcontinuum of $M_{\lambda-1}$ which is carried into itself by T;

(ii) for each ordinal number μ which has no immediate predecessor, M_μ is a proper subcontinuum of M_α which is carried into itself by T, where α is any ordinal which is less than μ;

(iii) if M_γ is any non-degenerate continuum which belongs to the sequence β, $M_{\gamma + 1}$ is a proper subcontinua of M_α belonging to the sequence β. Since M is metric, it is completely separable, and it follows that the sequence β thus defined is countable.

Therefore there exists a countable simple subsequence $M_{n1}, M_{n2}, \cdots$ running through β. That is, if M_γ is any element of β, there is an integer i such that M_{ni} is a subcontinuum of M_γ. Since M is compact, the point set $P = M_{n1} \cdot M_{n2} \cdot M_{n3} \cdot \cdots$, is a compact continuum and is carried into itself by T, since each of the continua M_{ni} is carried into itself by T. But P cannot be a non-degenerate continuum, for if it is, it has a proper subcontinuum which belongs to the sequence β; and this contradicts the definition of P and of the sequences β and M_{ni}. It follows that P is a point p of M such that T(p) = p.

Theorem 4.8 (30). If M is a compact continuum in the plane which is hereditarily decomposable, does not separate the plane, and contains no domain, then every homeomorphism of M into itself has a fixed point.

Proof. It has been shown by Miss Mullikan (45) that a sufficient condition that a compact continuum M separate the plane is that M be hereditarily unicoherent. Thus, if M does not separate the plane and contains no domain, no subcontinuum of M separates the plane. Then by Theorem 4.7, every homeomorphism of M into itself leaves some point fixed.

Ward has extended the result of this theorem, which is Theorem 4.12.

There exist continua in the plane which admit no continuous transformation onto themselves except the identity transformation. Hamilton

(30) gives an example of a compact acyclic continuous curve having this property.

Definition 4.6. If M is a connected set and p is a point of M such that the set M - p is not connected, then p is said to be a <u>cut point</u> of M.

Definition 4.7 (Whyburn). A subcontinuum N of a continuum M is said to be <u>0-th order cyclic element</u> of M, or simply an E_0 set provided N is maximal with respect to the property of being a subcontinuum of M without cut points.

J. L. Kelley (35) proved some significant theorems concerning fixed sets under homeomorphisms which appeared to be related to Hamilton's Theorem 4.7. We shall give Kelley's theorems which give this impression.

Definition 4.8. A set H has the <u>fpp for homeomorphism</u> provided every homeomorphism T(H) that is contained in H leaves at least one point fixed.

Theorem 4.9 (35). If M is a continuum and T is a homeomorphism of M into itself, then there exists a fixed point in M or else a set E_0 contained in M such that $T(E_0)$ is contained in E_0.

Proof. An irreducibly invariant subcontinuum H of M is either a point or is contained in a set E_0 as is shown by Whyburn (73). If H is a non-degenerate subcontinuum, then, since the transformations of the set E_0 containing H is a set of the same type, it follows that $T(E_0)$ is contained in E_0.

Theorem 4.10 (35). If every 0-th ordered cyclic element N, or simply an E_0 set, in a continuum M has the fpp for homeomorphism, so also has M.

Proof. From Theorem 4.9, a homeomorphism on a continuum M such that T(M) is contained in M implies the existence of a set E_0 such that $T(E_0)$ is contained in E_0.

Thus, this theorem tells us the fpp for homeomorphisms in E_0 is extensible.

The result of Theorem 4.10 appears to be related to the theorem of Hamilton (Theorem 4.7) which shows that a hereditarily unicoherent and hereditarily decomposable continuum has the fpp for homeomorphisms. However, his theorem neither includes nor is included in Theorem 4.10, for one can construct a hereditarily unicoherent and hereditarily decomposable continuum which is without cut points. Hamilton's theorem indicates the fpp for homeomorphisms for this continuum, while Theorem 4.10 makes no statement of this. Also, Theorem 4.7 applies necessarily to 1-dimensional continua, while Theorem 4.10 has no restriction of this nature imposed on it.

Pseudo Monotone Mappings

Recently, L. E. Ward, Jr., (69) generalized Theorem 4.8 (Hamilton (30)) in the following manner: If X is a continuum each of whose subcontinua is unicoherent and decomposable, then X has the fpp for each monotone transformation. As a corollary it follows that the same fpp obtains for continua each of whose nondegenerate subcontinua has a cutpoint. However, before we give the theorem it is necessary to have a definition and a lemma.

Definition 4.9. Let X and Y be spaces and $f:X \to Y$ a continuous mapping. We say that f is pseudo monotone if whenever A and B are closed and connected subsets of X and Y, respectively, and B is contained in f(A), it follows that some component of $A \cdot f^{-1}(B)$ is mapped by f onto B.

In general this notion is independent of that of a monotone mapping, but in certain applications of interest every monotone mapping is pseudo monotone. Recall that a continuum (= compact connected Hausdorff space) is hereditarily unicoherent if any two of its subcontinua meet in a connected set.

Lemma 4.11 (69). If X is a hereditarily unicoherent continuum and $f:X \to Y$ is a monotone mapping, then f is pseudo monotone.

Proof. Let A and B be closed and connected subsets of X and Y, respectively, such that B is contained in f(A). Since f is monotone, $f^{-1}(B)$ is a continuum, and since X is hereditarily unicoherent, $A \cdot f^{-1}(B)$ is connected. Hence, f is pseudo monotone.

The following is Ward's theorem which is a generalization of Hamilton's theorem (Theorem 4.8).

Theorem 4.12 (69). If X is a continuum and $f:X \to X$ is a pseudo monotone mapping, then X contains a nonempty subcontinuum Y which is minimal with respect to being invariant under f and Y has no cut points.

Proof. Suppose that X is continuum and that $f:X \to X$ is a continuous mapping. A simple maximality argument establishes the existence of a nonempty subcontinuum Y, which is minimal with respect to being invariant under f. (See Lemma 4.6.) Suppose Y has a cutpoint p, with

$$Y - p = A + B$$

where A and B are disjoint, separated, and nonempty. If $f(p) = p$ then the minimality of Y is contradicted, so we may assume $f(p)$ is a point of A and define $r(Y) = \bar{A}$ by

$$r(x) = x \qquad x \text{ is in } \bar{A}$$

$$r(x) = p \qquad x \text{ is in } \bar{B}.$$

The mapping $g:\bar{A} \to \bar{A}$ defined by $g = rf$ is continuous, and the set

$$K = g^1(\bar{A})\cdot g^2(\bar{A})\cdot g^3(\bar{A})\cdot \cdots$$

is a subcontinuum of $\bar{A}$ which is invariant under g. Thus

$$f(K)\cdot K = rf(K) = g(K) = K$$

and we infer K is contained in $f(K)$. Therefore, if f is pseudo monotone, the set $K\cdot f^{-1}(K)$ has a component K_1 such that $f(K_1) = K$. Inductively, we obtain a sequence of subcontinua, $\{K_n\}$ such that

$$K_n \subset f(K_n) = K_{n-1} \subset \cdots \subset f(K_1) = K.$$

Then the intersection of this sequence is a nonempty subcontinuum invariant under f, and this contradicts the minimality of Y.

From Theorem 4.12, we have two corollaries which readily follow.

Corollary 4.13 (69). If X is a continuum such that each of its nondegenerate subcontinua has a cut point, and if $f:X \to X$ is a pseudo monotone mapping, then there exists a point x_0 of X such that $x_0 = f(x_0)$.

Corollary 4.14 (69). If X is a continuum such that each of its nondegenerate subcontinua has a cut point, and if $f:X \to X$ is a monotone mapping, then there exists a point x_0 of X such that $x_0 = f(x_0)$.

Proof. Ward (67) has shown that if X is a continuum such that each of its nondegenerate subcontinua has a cut point, then X is hereditarily unicoherent. Then by Lemma 4.11, f is a pseudo monotone mapping. Hence there exists a point x_0 of X such that $f(x_0) = x_0$.

The Pseudo-Arc

Let us recall that an arc is a compact nondegenerate continuum that has exactly two non-cut points. In his dissertation directed by R. L. Moore, E. E. Moise (42) has defined what is known as a "pseudo-arc." Using Moise's definition, Hamilton (28) has shown that the pseudo-arc has the fpp with respect to a continuous transformation. Hamilton's proof was an answer to a question raised by F. Burton Jones. Also, Hamilton has shown that a pseudo-arc is a member of a more general class of metric continua which have the fpp.

Before giving these results, we shall define the pseudo-arc. Recall that a chain is a finite collection of open sets (called links) $d_1, d_2, \cdots, d_n$ such that d_i and d_j are identical or consecutive integers and such that $d_i \cdot d_j$ is non-vacuous if and only it $|i - j| \leq 1$. This particular definition is also due to Moise.

Moise defined D to be a refinement of C if C and D are finite collections of disjoint open sets and each element of D is a subset of an element of C. Also, if D is a refinement of C such that for each link c of C, the set of all links of D that lie in C, is a subset of D, then D is said to be straight with respect to C.

Moise has described "very crooked" in the following manner. Let C be a chain from P to Q whose links are $c_1, c_2, \cdots, c_k$.

(a) If C consists of less than five links then a chain D from P to Q is said to be very crooked with respect to C if D is straight with respect to C.

(b) If C is a chain from P to Q which consists of k links, with k greater than 4, then a chain D from P to Q is said to be very crooked with respect to C if D is a refinement of C, and D is the sum of (i) a chain from P to a point x of c_{k-1}, (ii) a chain from x to a point y of c_2, and (iii) a chain from y to Q, such that these chains are very crooked with respect to C - c_k, C - c_1 - c_k, and C - c_1, respectively, and such that no two of them have in common any link that is not an end link of both of them.

Definition 4.10 (Moise). Let Y_1, Y_2, $\cdots$ be a sequence of chains from P to Q such that

(i) $S(Y_1')$ is a compact metric space,

(ii) for each i, Y_{i+1} is very crooked with respect to Y_i, and $S(Y_{i+1}')$ lies in the interior of $S(Y_i')$,

(iii) Y_1 consists of five links,

(iv) if y is a link of Y_i, and X is a subchain of Y_{i+1} which is maximal with respect to the property of being a subchain of Y_{i+1} and a refinement of the chain whose only link is y, then X consists of five links, and

(v) for each i, each link of Y_i has a diameter less than $1/i$.

Let M be the common part of the sets $S(Y_i')$. Then M is said to be a <u>pseudo-arc</u>.

Theorem 4.15 (28). Let Y_1, Y_2, $\cdots$ be a sequence of chains such that:

(i) $C(Y_1)$, the closure of Y_1, is a compact nonempty metric space,

(ii) $C(Y_{i+1})$ is a subset of $C(Y_i)$ for each i,

(iii) limit $(i \to \infty)$ $\rho Y_i = 0$, where ρY_i designates the maximum diameter of a link of chain Y_i.

Let M designate the continuum which is the intersection of the sets $C(Y_i)$. Then if T is a continuous transformation of M into a subset of itself, there is a point p of M such that $T(p) = p$.

Proof. Let ϵ be a positive real number. Let m be a positive integer such that ρY_m is less than ϵ. Let A be the subset of M consisting of all points p of M such that either each of the closed links of Y_m which contain $T(p)$ follows all the closed links of Y_m which contain p or such that $T(p)$ is contained in some closed link of Y_m which contains p. Let B be the subset of M consisting of all points p of M such that either each closed link of Y_m which contains $T(p)$ precedes all the closed links of Y_m which contain p or such that $T(p)$ is contained in some closed link of Y_m which contains p. The points of M in the first closed link of Y_m are in A and the points of M in the last closed link of Y_m are in B, and hence these sets are non-empty. It is easily shown that A and B are closed. Then the compact continuum M is the sum of the two closed sets A and B, which therefore have a point q in common. Hence, q and $T(q)$ lie together in some closed link Z of Y_m, and the distance from q to $T(q)$ is less than ϵ. Since ϵ is an arbitrary positive number, and since T is a continuous transformation and M is closed, it follows that for some point p of M, $T(p) = p$.

A corollary to Theorem 4.15 is an answer to the question posed by F. B. Jones to Professor Hamilton. It follows readily from his theorem.

Corollary 4.16 (28). A pseudo-arc has the fpp for continuous transformations.

Proof. A pseudo-arc is a point set satisfying the definition of the set M of the preceding theorem.

G. S. Young, Jr., defined a type of generalized dendrite and made a study of the topological properties possessed by it. He was particularly interested in deriving certain conditions for the fpp to hold.

Defintion 4.11. A continuum M is said to be a dendrite provided it is locally connected and contains no simple closed curve, i.e., a dendrite is a locally connected continuum S every cyclic element of which reduces to a single point.

Definition 4.12 (Young). By a generalized dendrite is meant a locally connected Hausdorff space T such that if a and b are two points of T, and Y_1 and Y_2 are two simple chains of connected domains from a to b, and Y_1 has more than two links, then some link of Y_1 that does not contain a or b intersects some link of Y_2.

In the following three theorems, which are due to Young, T will denote such a space as described in Definition 4.12.

Theorem 4.17 (79). If T is separable, then there is a continuous transformation of T into a connected subset of a compact, metric dendrite. If in addition the union of every monotone increasing sequence of arcs of T is contained in an arc of T, then the subset is closed.

We are now in a position to prove the fixed point theorem for generalized dendrites. Note that this theorem will be stated for a generalized dendrite which is a Hausdorff space, but the proof will be valid only for arcwise-connected generalized dendrites.

Theorem 4.18 (79). A sufficient condition that a generalized dendrite T have the fpp is that the union of any increasing simple sequence of pseudo arcs of T be contained in a pseudo arc.

Proof. Assume T to be arcwise connected. Let f(T) be a continuous transformation of T into itself. Let a be any point of T. If $f(a) = b$, $a \neq b$, then the image of the arc ab, f(ab), is a compact continuous curve intersecting ab in at least b. Indeed, for each positive integer n, $f^n(ab)$ is a compact continuous curve intersecting $f^{n-1}(ab)$. Hence, the semi-orbit, h of ab is a completely separable connected subset of T, and $\bar{h}$ is a separable subcontinuum of T which is mapped into itself by f. There is a mapping g(h) onto a compact dendrite K satisfying the conditions of Theorem 4.17, since clearly every increasing sequence of arcs of $\bar{h}$ is contained in an arc of $\bar{h}$. The mapping $gfg^{-1}(K)$ is a continuous mapping of K into itself, for suppose that $x_1, x_2, \cdots$ is a sequence of points of K convergent to a point x. Then it is easy to see that no point of $\bar{h}$ separates $g^{-1}(x_1 + x_2 + \cdots)$ from $g^{-1}(x)$ in $\bar{h}$; and hence that no point of $\bar{h}$ separates $fg^{-1}(x_1 + x_2 + \cdots)$ from $gfg^{-1}(x)$ in K, and since this is also true for any subsequence of $x_1, x_2, \cdots$ that $gfg^{-1}(x)$ is a point or limit point of $gfg^{-1}(x_1 + x_2 + x_3 + \cdots)$. From a theorem by W. Scherrer (53), $gfg^{-1}(y) = y$. Then $fg^{-1}(y) = g^{-1}(y)$, which proves the theorem.

Theorem 4.19 (79). If the generalized dendrite T is also arcwise connected, the condition of Theorem 4.18 is also necessary.

Before we give the proof, a definition is necessary.

Definition 4.13. A ray is a non-degenerate locally compact continuous curve M containing a point a such that every compact subcontinuum of M that contains a has only one boundary point with respect to M.

Definition 4.14. If $f(X) = Y$ is a continuous transformation such that Y is contained in X and for each point y of Y, $f(y) = y$, then f is said to be a retraction of X into Y.

Proof (Theorem 4.19). If there is a sequence of arcs of T not satisfying our condition, then it follows that there is a ray R in T. There is a retraction $f(T) = R$ throwing each component of $T - R$ into its boundary point, and there is a translation of R, $g(R) = R'$, where R' is a proper subray of R. Then the transformation $gf(T)$ is continuous and leaves no point fixed. Hence, the theorem is proved.

It will now be shown that certain connected sets which exhibit dendrite properties have the fpp, even though they are not locally connected.

If G is the collection of all arcs of S we shall speak of the arc-topology of S, as opposed to the usual topology where G is the collection of subsets of S called open sets.

Let us consider the class of sets that are arcwise connected sets such that each increasing sequence of arcs is contained in an arc. The following are examples of spaces having this property.

(i) The union of the unit interval and a collection of intervals perpendicular to this interval at the points of irrational abscissa.

(ii) The join of the Cantor set and a point.

(iii) The continuum obtained by joining the points of a hereditarily indecomposable plane continuum to a point not in the plane by straight line intervals.

That (ii) is true follows immediately from Hamilton's general theorem (Theorem 4.7) on fixed points, i.e., if M is a compact continuum in a metric space, and if M does not contain an indecomposable continuum and does not contain a continuum which is the sum of two continua whose intersection is disconnected, then every homeomorphism of M into a subset of itself has the fpp.

Theorem 4.20 (78). Let M be an arcwise connected Hausdorff space which is such that every monotone increasing sequence of arcs is contained in an arc. Then M has the fpp.

Proof. In the usual topology of the space, M contains no simple closed curve and hence, in the arc-topology contains no simple closed curve. Thus, in the arc-topology it is a generalized dendrite such that every monotone increasing sequence of arcs is contained in an arc. Further, by Theorem 4.17, every continuous mapping of M into a subset, H, of itself in the usual topology is continuous with the arc-topology in H. Hence, by Theorem 4.18, M has the fpp.

Interior Transformations

Let us now consider certain types of locally connected continua and fixed point transformations. We shall consider M to be a bounded continuum in the Euclidean plane E^2 and T an interior transformation of M onto a subset of E^2 which contains M.

Definition 4.15. A transformation $f:X \to Y$ of the space X into the space Y is said to be an <u>interior transformation</u> if f is continuous and if the image of every open subset of X is open in Y.

Some mathematicians discuss transformations that carry open sets into open sets but that are not necessarily continuous. Such transformations usually are called <u>open</u>. It is a known theorem that a necessary and sufficient condition that the one-to-one mapping

$f: X \to Y$ of the space X onto the space Y be a homeomorphism is that f be interior.

We shall now consider some fixed point theorems for the interior transformation defined above. Those presented will be the result of Hamilton (27) and will depend on the work of S. Eilenberg (17) in which he was concerned with transformations of circumferences in metric spaces.

M is said to have property (B) provided that every continuous transformation of M into the unit circle S in the Cartesian plane, with center at O, is homotopic to a constant transformation, that is, a transformation which transforms each point of M into a single point of S.

If T is a continuous transformation of a subset A of the plane E^2 onto a subset B of E^2, then for each point x of A let T'(x) be the point y of S such that the directed line segment (Oy) is parallel in direction and sense to the directed line segment x, T(x). Then T' will be referred to as the derived transformation of A into S derived from T.

If T is a continuous transformation of a subset A of E^2 onto a subset B of E^2, A is said to have property (B') with respect to T if T leaves no point of A fixed and the transformation T' of A into S derived from T is homotopic to a constant mapping.

Definition 4.16. An <u>n-dimensional cell</u> (<u>n-cell</u>) is a set which is homeomorphic either with the set of points $(x_1, \cdots, x_n)$ of n-dimensional Euclidean space for which $\Sigma x_i^2 < 1$ (called open n-cell), or with the set for which $\Sigma x_i^2 \leq 1$ (called closed n-cell).

Theorem 4.21 (27). If T is an interior transformation of a locally connected unicoherent bounded plane continuum M onto a topological 2-cell which contains M, then T leaves a point of M fixed.

Proof. Suppose T leaves no point of M fixed. Then the continuous mapping T' of M into S is derived from T exists and by a theorem of Eilenberg (17), since M is unicoherent and locally connected, M has property (B), and hence has property (B') with respect to T. But Hamilton has shown (Lemma 1, (27)) that there does not exist a bounded plane continuum M which has property (B') with respect to an interior transformation T of M onto a topological 2-cell I which contains M. Hence, the supposition is false, i.e., T leaves a point of M fixed.

Corollary 4.22 (27). If T is an interior transformation of a bounded locally connected plane continuum which does not separate the plane onto a topological 2-cell which contains M, then T leaves a point of M fixed.

Proof. The result follows immediately from Theorem 4.21, for a plane continuum is unicoherent provided it does not separate the plane.

Theorem 4.23 (27). If T is an interior transformation of a topological 2-cell I onto a continuum M which contains I, then T leaves a point of I fixed.

Proof. Suppose T leaves no point of I fixed. Since I is unicoherent and the derived mapping T exists, then from the work of Eilenberg (17) it follows that I has property (B) and hence property (B') with respect to T. But T transforms some subset N of I onto I. Then N has property (B') with respect to T. But this is contrary to the fact that there does not exist a bounded plane continuum M which has property (B') with respect to an interior transformation T of M onto a topological 2-cell I which contains M, which was shown by Hamilton (27) to be true.

Applications of this theorem and the preceding one to the theory of functions of a complex variable which is analytic in a region R, defines an interior transformation of R onto a subset of the complex plane E^2.

In Chapter II, we defined (Definition 2.9) a contraction operator for a mapping of a metric space into itself. Also, we have S. S. Banach's theorem (Theorem 2.10) which states that if a contraction operator f is defined on a complete metric space S, then there exists a point x of S such that $f(x) = x$. Recently Michael Edelstein obtained an extension of Banach's contraction principle (15).

Definition 4.17. A mapping f of a metric space X into itself is said to be <u>locally contractive</u> if for every point x of X there exist an $\epsilon > 0$ and λ, $0 \leq \lambda < 1$, which may depend on x such that:

$$p,q \text{ of } S(x,\epsilon) = \{y:\rho(x,y) < \epsilon\} \text{ implies}$$
$$\rho(f(p),f(q)) < \lambda\rho(p,q) \text{ for every } p,q \text{ in } X.$$

Definition 4.18. A mapping f of a metric space X into itself is said to be <u>(ϵ,λ)-uniformly locally contractive</u> if it is locally contractive and both ϵ and λ do not depend on x.

Theorem 4.24 (15). Let X be a complete metric ϵ-chainable space, f a mapping of X into itself which is (ϵ,λ)-uniformly locally contractive, then there exists a unique point x' of X such that $f(x') = x'$.

Proof. Let x be an arbitrary point of X. Consider the ϵ-chain:

$$x = x_0, x_1, x_2, \cdots, x_n = f(x);$$

by the triangle inequality;

$$\rho(x, f(x)) \leq \Sigma_1^n \rho(x_{i-1}, x_i) < n\epsilon.$$

For the pairs of consecutive points of the ϵ-chain, Definition 4.17 is satisfied.

Hence, denoting $f(f^m(x)) = f^{m+1}(x) \quad (m = 1, 2, \cdots)$ we have:

$$\rho(f(x_{i-1}), f(x_i)) < \lambda\rho(x_{i-1}, x_i) < \lambda\epsilon;$$

and, by induction:

(i)
$$\rho(f^m(x_{i-1}), f^m(x_i)) < \lambda\rho(f^{m-1}(x_{i-1}), f^{m-1}(x_i))$$
$$< \lambda^m \epsilon \quad \cdots.$$

From the last inequality we obtain:

$$\rho(f^m(x), f^{m+1}(x) \leq \Sigma_1^n \rho(f^m(x_{i-1}), f^m(x_i))$$
$$< \lambda^m n\epsilon.$$

It follows that the sequence of iterates $\{f^i(x)\}$ is a Cauchy sequence.

If j and k are positive integers ($j < k$), then:

$$\rho(f^j(x), f^k(x)) \leq \Sigma_j^{k-1} \rho(f^i(x), f^{i+1}(x))$$
$$< n\epsilon \frac{\lambda^j}{1 - \lambda} \to 0, \; j \to \infty$$

The completeness of X guarantees the existence of the $\underset{i \to \infty}{\text{limit}} f^i(x)$.

From the continuity of f implied by Definition 4.17, it then follows that:

$$f(\underset{i \to \infty}{\text{limit}} f^i(x)) = \underset{i \to \infty}{\text{limit}} f(f^i(x))$$

$$= \underset{i \to \infty}{\text{limit}} f^{i+1}(x) = \underset{i \to \infty}{\text{limit}} f^i(x).$$

Suppose there exists $\bar{x} \neq x'$ such that $f(\bar{x}) = \bar{x}$. Let $x' = x_0, x_1, \cdots,$ $x_k = f(\bar{x})$ be an ϵ-chain. From equation (i) we obtain:

$$\rho(f(x),f(\bar{x})) = \rho(f^r(x'),f^r(\bar{x}))$$

$$\leq \Sigma_1^k \, \rho(f^r(x_{i-1}),f^r(x_i))$$

$$< \lambda^{r_k} \to 0, r \to \infty$$

which is impossible. Hence, $\bar{x} = x'$ and the proof is complete.

If, in Definitions 4.17 and 4.18, we replace $\lambda < 1$ with $\lambda > 1$, then a corollary follows immediately.

Corollary 4.25 (15). If f is a one-to-one (ϵ,λ)-uniformly locally expansive mapping of a metric space Y onto an ϵ-chainable complete metric space X which contains Y, then there exists a unique point x' such that $f(x') = x'$.

Proof. Since we have f a one-to-one mapping, then all the conditions of the theorem are satisfied for the inverse mapping. Hence the conclusion is immediate.

Hamilton has shown in Theorem 4.15 that every compact chainable continuum has the fpp. Subsequent to the time Hamilton's theorem was made known, Eldon Dyer (14) proved that the Cartesian product of finitely many compact chainable continua has the fpp. Since arcs are compact chainable continua, this is a generalization of Brouwer's fixed point theorem.

An example of a compact chainable continuum is the closure of the set

$$S = \{(x,f(x)):f(x) = \sin(1/x),\ 0 < x \leq 1\}.$$

Another example is the pseudo-arc as defined by Moise (Definition 4.8).

Definition 4.19. A mapping f of a space X into a space Y is said to be an essential mapping of X into Y if it is not homotopic to a constant mapping of X.

Before stating Dyer's theorem, we shall state the following lemmas.

Lemma 4.26 (14). Suppose f and g are continuous transformations of E^n into itself and $g|S^{n-1}$ is an essential mapping onto S^{n-1}. Then there is a point x of E^n such that $f(x) = g(x)$.

Lemma 4.27 (14). Suppose for each integer i, $1 \leq i \leq n$, f_i is a continuous mapping of I onto I such that $f_i(0) = 0$, $f_i(1) = 1$, where I is the unit interval of the real line. For each point $x = (x_1, x_2, \cdots, x_n)$ of I^n let $f(x) = (f_1(x_1), f_2(x_2), \cdots, f_n(x_n))$. Let T^{n-1} denote the set of all points x of I^n such that for each i, x_i is either 0 or 1. Then $f(T^{n-1}) = T^{n-1}$.

Dyer's theorem which follows is the affirmative answer to an inquiry made by A. D. Wallace after he had read a manuscript of Dyer. The proof will be omitted.

Theorem 4.28 (14). Suppose that M is the Cartesian product of n compact chainable continua X_1, X_2, $\cdots$, X_n of a metric space and f is a continuous transformation of M into a subset of itself. Then there is a point x of M such that $f(x) = x$.

Non-Continuous Transformations

The theorems discussed in this chapter have been concerned in general with continuous transformations. However, sets or spaces which possess the fpp are not restricted to transformations that are continuous in nature. For certain noncontinuous transformations, Hamilton (26) has shown that the mapping of the closed n-cell into itself leaves some point invariant. The following definition is due to Hamilton.

Definition 4.20 (Hamilton). If T is a mapping of a subset A of a space S onto a subset B of S, T is said to be <u>peripherally continuous</u> if for each point p of A and each pair of open sets U and V containing T(p) and p, respectively, there is an open set D contained in V, where p is a point of D, such that T transforms F(D) (the boundary of D) into U.

Theorem 4.29 (26). Let T be a peripherally continuous transformation of a closed n-cell I, $n \geq 2$, into itself. Let it be assumed that I is the closed n-cube consisting of the point $(x_1, x_2, \cdots, x_n)$ given by the inequality $0 \leq x_i \leq 1$. Let the faces $x_i = 0$ and $x_i = 1$ be designated by A_i and B_i, respectively. If x is the point $(x_1, x_2, \cdots, x_n)$, let $T(x) = x'$ be the point $(x_1', x_2', \cdots, x_n')$. For each i, $1 \leq i \leq n$, let N_i designate the subset of I for which $x_i' \geq x_i$. Then the components of N_i are closed and there exist a point of I which is fixed under T.

Proof. Let E_i be the component of N_i which contains A_i. Let $q(q_1, q_2, \cdots, q_n)$ be a limit point of E_i and suppose that q does not belong to N_i. Then by definition of N_i, $q_i' = q_i - \epsilon$, for some positive ϵ. Then since T is a peripherally continuous, there is a connected set D of diameter less than $\epsilon/3$ containing q such that

(i) $E_i - (\overline{D} \cdot E_i) \neq \emptyset$, and

(ii) if x is a point of F(D), then $\rho(T(x), T(q))$ is less than $\epsilon/3$.

Then the connected set E_i, since it contains points outside of $\bar{D}$ and within D, must contain a point x of F(D). That is

$$\rho(T(x),T(q)) < \epsilon/3, \ \rho(x,q) < \epsilon/3.$$

Hence,

$$\rho(x_i',q_i') < \epsilon/3 \text{ and } \rho(x_i,q_i) < \epsilon/3.$$

With $q_i' = q_i - \epsilon$, these inequalities give

$$x_i < x_i' - \epsilon/3$$

and this contradicts the fact that x is a point of N_i. Hence E_i is closed.

Let $\{G_a^i\}$ be the collection of all components of $I - E_i$ which contains points of B_i. Let

$$H_i = (\Sigma_a G_a^i) + B_i.$$

Then H_i is connected since B_i is connected and since each G_a^i is connected and contains a point of B_i. Let K_i be the subset of E_i consisting of points in the common boundary between H_i and E_i. Then if L_i designates the subset of I for which $x_i' = x_i$, then by an argument similar to that previously given, K_i is a closed subset of L_i and hence $F_i(D) = K_i + (B_i \cdot L_i)$ is a closed subset of L_i. No component C of $I - F_i(D)$ contains points of both A_i and B_i. For suppose the contrary, that C contains a point α of A_i and a point β of B_i. Then α is in E_i and β in H_i. Hence C contains a point α of K_i, the common boundary of H_i and E_i. This contradicts K_i being a subset of $F_i(D)$ and C is a subset of $I - F_i(D)$.

Thus, let W be a transformation of I into itself defined as follows: for each point $x(x_1,x_2,\cdots,x_n)$ of I let W(x) be designated by $x''(x''_1, x''_2, \cdots, x''_n)$. Let $t_i(x) = \rho(x, F_i(D))$. Then

(i) if x belongs to a component of $I - F_i(D)$ which contains a point of B_i and hence by the argument in preceding paragraph no point of A_i, let $x''_i = x_i - \frac{1}{2}(t_i(x) \cdot x_i)$. Then since $x_i \neq 0$, $x''_i \neq x_i$,

(ii) if x is in $F_i(D)$ let $x''_i = x_i$, and

(iii) if x belongs to a component of $I - F_i(D)$ which contains no point of B_i, let $x''_i = x_i + \frac{1}{2}(t_i(x) \cdot (1 - x_i))$. Since $1 - x_i \neq 0$, we have as in case (i) that $x''_i \neq x_i$.

Then since $t_{i/2}$ is less than 1 we have $0 \leq x''_i < 1$, and $x_i = x''_i$ if and only if $x_i = x'_i$. The transformation W is, by its definition, a continuous mapping of I into a subset of itself and hence, by Brouwer's fixed point theorem for n-cells, leaves some point z of I fixed. That is, for each $i = 1, 2, \cdots, n$, $z''_i = z_i$. The point z then is $\prod_{i=1}^{n} L_i$ and hence by definition of L_i, $z'_i = z$, for $i = 1, 2, \cdots, n$. That is, $T(z) = z$, and T, as required, leaves a point of I fixed.

Definition 4.21 (Nash). A <u>connectivity</u> <u>map</u> from a space A into a space B is said to be a mapping T such that the induced map g of A into A x B, defined by $g(p) = (p \times T(p))$ transforms connected subsets of A onto connected subsets of A x B.

This definition is credited to John Nash (46) who inquired as whether or not every connectivity map of a closed n-cell into itself has a fixed point. Professor Hamilton gave an affirmative answer to this inquiry with the following theorem. First we state a lemma.

Lemma 4.30 (26). If T is a connectivity map of a closed n-cell I, $n \geq 2$, onto a subset B of I, then T is peripherally continuous on I.

Theorem 4.31 (26). If T is a connectivity map of a closed n-cell I into itself, T leaves a point of I fixed.

Proof. If n = 1 and T is a connectivity map of I, $0 \leq x \leq 1$, into itself, g(I), the graph of the mapping is connected by definition of a connectivity map. Furthermore, g(I) contains the points $0 \times T(0)$ and $1 \times T(1)$ of the subsets $0 \times I$ and $1 \times I$, respectively. Hence the connected set g(I) contains a point of the connected set $x \chi x$ of $I \times I$. That is, $T(x) = x$ for some point x of I.

If $n \geq 2$, the theorem follows directly from Lemma 4.30 and Theorem 4.29.

It was not known until recently whether or not the converse of Lemma 4.30 is true, that is, it was not known whether or not a peripherally continuous transformation of a closed n-cell, $n \geq 2$, into itself is necessarily a connectivity map. But Melvin Hagan (25) has shown in his doctoral dissertation that if f is a peripherally continuous transformation of the space S, possessing certain properties, into the space T, where $S \times T$ is completely normal, then f is a connectivity map. It is known that a peripherally continuous transformation of the closed 1-cell, $0 \leq x \leq 1$, into itself is not a connectivity map and also leaves no point fixed. Consider the following example.

Example. Let $T: I \to I$, where I is the closed unit interval and T is a peripherally continuous transformation of the closed 1-cell into itself. Define $T(x) = \sqrt{2}/2$ if x is rational

$= \frac{1}{2}$ is x is irrational.

About three decades after the time of Brouwer's theorem, Shizuo Kakutani (33) gave a generalization of it. First, we need a definition before presenting this theorem.

Definition 4.22. Let $G(S)$ be the family of all closed convex subsets of a space S. A point-to-set mapping $x \rightarrow H(x)$, where x is a point of $G(S)$ and $H:S \rightarrow G(S)$, is called <u>upper semi-continuous</u> if $x_n \rightarrow x_0$, y_n is in $H(x_n)$ and $y_n \rightarrow y_0$ implies y_0 is in $H(x_0)$.

It is seen that this condition is equivalent to saying that the graph of $H(x):\Sigma_{x\epsilon S} x \times H(x)$, the Cartesian product, is a closed subset of S x S. Let us recall that Brouwer's theorem states that if there exist a point-to-point continuous mapping of an n-dimensional closed simplex S into itself, then there exist a point of S that remains fixed under this mapping. Then the generalized fixed point theorem may be stated as follows:

Theorem 4.32 (33). If $H:S \rightarrow G(S)$ is an upper semi-continuous point-to-set mapping of an n-dimensional closed simplex S into $G(S)$, then there exist a point x_0 of S such that x_0 is in $H(x_0)$.

Let us recall that the barycentric coordinates of a point x of an n-cell $(x_1,x_2,\cdots,x_n)$ are the weights $w_1,w_2,\cdots,w_n$ of sum 1 which must be assigned to the respective vertices in order to get x as a centroid. By a barycentric subdivision of a topological realization R we mean the uniquely defined simplical subdivision of R whose vertices are the vertices of R and in addition the barycenters of the cells of R.

Proof. (Theorem 4.32). Let $S^{(n)}$ be the nth barycentric simplicial subdivision of S. For each vertex x^n of $S^{(n)}$ take an arbitrary point y^n from $H(x^n)$. Then the mapping $x^n \rightarrow y^n$ thus defined on all vertices of $S^{(n)}$ will define a continuous point-to-point mapping $x \rightarrow \omega(x)$ of S into itself if it is extended linearly inside each simplex of $S^{(n)}$. Consequently, by Brouwer's theorem there exist an x_n of S such that $x_n = \omega(x_n)$. If we now take a subsequence $\{x_{n_v}\}$ $(v = 1,2,\cdots)$ of $\{x_n\}$

$(n = 1,2,\cdots)$ which converge to a point x_0 of S, then this point x_0 is a required point.

In order to prove this, let D_n be an n-dimensional simplex of $S^{(n)}$ which contains the point x_n. Let $x_0^m, x_1^m, \cdots, x_n^m$ be the vertices of D_n. Then we have the sequence $x_i^{m_v}$ $(v = 1,2,\cdots)$ converges to x_0 for $i = 0, 1,\cdots,n$, and we have $x_n = \Sigma_{i=0}^n c_i^n x_i^n$ for suitable $\{c_i^n\}$. Let us put $y_i^n = \omega(x_i^n)$; then we have y_i^n a point of $H(x_i^n)$ and $x_n = \omega_n(x_n)$. By appropriately selecting subsequences, it follows by the upper semi-continuity of $H(x)$ that y_i^0 is a point of $H(x_0)$ and this implies, by the convexity of $H(x_0)$, that x_0 is a point of $H(x_0)$. (We omit many of the details.) Thus the proof of the theorem is completed.

It is easy to see that Brouwer's theorem is a special case of this theorem where each $H(x)$ consists of only one point $\omega(x)$. In this case, the upper semi-continuity of $H(x)$ is nothing but the continuity of $\omega(x)$.

Thus, Kakutani has proved a fixed point theorem for multi-valued upper semi-continuous transformations of an n-cell into a subset of itself for which the image of a point is a convex continuum. Eilenberg and Montgomery (18) have extended Kakutani's result to transformations for which the images of points are acyclic continua of a more general type. Following this, Hamilton (29), using the definition of upper semi-continuous as given by Kakutani, obtained results which are concerned with special types of upper semi-continuous transformations for which the images are non-acyclic.

Essential Fixed Points

M. K. Fort, Jr., (21) investigated the questions of existence and recognition of fixed points in various cases, always requiring that the space X have the fpp. Let X^X be the space of continuous self-mappings of X furnished with the compact-open topology, that is, X^X denotes the space of continuous functions of X into X topologized by the compact open topology.

Definition 4.23 (Fort). A fixed point x of X where f is in X^X is said to be an <u>essential fixed point</u> if for each neighborhood U of x there is a neighborhood N of f such that if g is in N, then g has a fixed point in U.

If f and g are in X^X, we define

$$\rho(f,g) = \sup(\rho(f(x),g(x)):x \text{ of } X).$$

It is a known theorem that (X^X,ρ) is a complete metric space (21).

Let the points of 2^X be the collection of nonempty compact subsets of X and if A is a point of 2^X and $\epsilon > 0$, we denote by $U(\epsilon,A)$ the set of all points x of X for which there exists y of A such that $\rho(x,y)$ less than ϵ. The set 2^X is metrized by H, where $H(A,B) = \inf(\epsilon: A \subset U(\epsilon,B)$ and $B \subset U(\epsilon,A))$, where A and B are in 2^X. It is also a known theorem that $(2^X,H)$ is a compact metric space.

Fort characterizes an upper semi-continuous function T on a topological space S into 2^X at a point p of S if corresponding to each $\epsilon > 0$ there is a neighborhood V of p such that $T(x)$ is contained in $U(\epsilon,T(p))$ for all x of V. Also, Fort defines a function F on X^X into 2^X defining $F(f)$ to be the set of all fixed points of F whenever f is in X^X. It is a known theorem that F is upper semi-continuous.

Lemma 4.34 (21). Each fixed point of f in X^X is essential if and only if f is a point of continuity of F.

Proof. Assume that each fixed point of f is essential. Let ϵ be any positive number. For each x of $F(f)$ there is a neighborhood $V(x)$ of f such that if g is in $V(x)$ then g has a fixed point in the $(\epsilon/2)$-neighborhood of x. There exists a finite set $x_1, \cdots x_n$ of points of $F(f)$ such that each point of $F(f)$ is within $\epsilon/2$ of some one of the x_k. If we now choose a neighborhood V of f which is contained in the intersection of $V(x_1), \cdots, V(x_n)$ we see that if g is in V then $F(f)$ is contained in $U(\epsilon, F(g))$. We have therefore proved that F is lower semi-continuous at f, where Fort defines lower semi-continuity to parallel upper semi-continuity where $T(p)$ is contained in $U(\epsilon, T(x))$. This fact, in view of F being upper semi-continuous, proves that F is continuous at f.

We now assume that F is continuous at f. Let p be a point of $F(f)$ and let U be a neighborhood of p. Choose $\epsilon > 0$ such that the ϵ-neighborhood of p is contained in U. Now choose $\beta > 0$ such that if $\rho(f,g) < \beta$ for g in X^X, then $H(F(f), F(g)) < \epsilon$. Thus if g is in X^X and $\rho(f,g) < \beta$, then g has a fixed point in U. This proves that p is an essential fixed point of f.

Theorem 4.35 (21). If f is in X^X and $\epsilon > 0$, then there exists g of X^X such that $\rho(f,g) < \epsilon$ and such that every fixed point of g is essential.

Proof. It is known (22) that the points of continuity of an upper semi-continuous set-valued function form a residual set. Since F is upper semi-continuous and X^X is a complete metric space, we see that F is continuous at each point of a set dense in X^X. The result then follows immediately from Lemma 4.34.

It is easy to see that there may exist members of X^X which have no essential fixed point. An example is obtained by taking X to be a

Euclidean n-cell. For this case, every point of X is a fixed point of the identity transformation, but no point is an essential fixed point.

It is interesting to know conditions under which a transformation will have essential fixed points. The following theorem is a sufficient condition for the existence of an essential fixed point of f.

Theorem 4.36 (21). If f has a single fixed point, then this point is an essential fixed point of f.

Proof. Suppose $\epsilon > 0$. By the upper semi-continuity of F, there exists a neighborhood V of f such that if g if in V, then $F(g)$ is contained in $U(\epsilon, F(f))$. Since $F(f)$ contains but a single point, this implies that $F(f)$ is contained in $U(\epsilon, F(g))$. Thus if g is in V, then $H(F(f), F(g)) < \epsilon$. We have proved that F is continuous at f, and hence it follows from Lemma 4.34 that the fixed point of f is an essential fixed point.

Let us state the following theorem due to Fort.

Theorem 4.37 (21). In addition to the previously stated hypothesis, we shall assume that S is a topological n-space. If f is a point of X^X and the set $F(f)$ of fixed points of f is totally disconnected, then f has at least one essential fixed point.

Although this theorem is useful in proving the existence of an essential fixed point of a function f, it is of no value in deciding which fixed points are essential, provided f has more than one. For example, let X be the unit disk on the complex plane and let $f(z) = z^2$. The fixed points of f are 0 and 1. We know by Theorem 4.37 that at least one of these is essential, but we do not know which one is essential and which one is not. Let us consider the following theorem.

Theorem 4.38 (21). If f is in X^X, p in X and p has arbitrarily small neighborhoods V such that $\overline{V}$ has the fpp and $f(\overline{V})$ is contained in V, then p is an essential fixed point of f.

Proof. It is clear that p is a fixed point of f. Let U be any neighborhood of p. Choose a neighborhood V of p such that $\overline{V}$ is contained in U. $\overline{V}$ has the fpp, $f(\overline{V})$ is contained in V, and X - V is nonempty. Let $\epsilon = \inf(\rho(x,y):x \text{ in } f(\overline{V}), y \text{ in } X - V)$. Since $f(\overline{V})$ and X - V are disjoint, nonempty, compact sets, we see that ϵ is some positive number. Now let g be in X^X and assume that $\rho(g,f) < \epsilon$. Then $g(\overline{V})$ is contained in V which is contained in $\overline{V}$, and since $\overline{V}$ has the fpp, g has a fixed point in $\overline{V}$ and hence in U. This proves that p is an essential fixed point of g.

Using this theorem, it is easy to show that, in the preceding example, 0 is an essential fixed point of the function $w = z^2$ for z in the unit complex plane. However, Theorem 4.38 is of no use in some cases. For example, let $f(z) = z/|z|^{1/2}$ defined for all z in the unit complex disk, f(0) being defined to be 0. The behavior of this function on neighborhoods of 0 is fairly typical of the situation we may expect in general, in that the function does not transform small neighborhoods of 0 into themselves.

J. M. Marr (38) has also obtained some interesting results concerning essential fixed points. He proved that if X is a compact Hausdorff space which has the fpp, then there is an f of X^X such that each fixed point of f is essential.

Multi-Valued Transformations

Thus far, we have only mentioned multiple-valued functions in connection with upper semi-continuous transformations. Some interesting results have been obtained concerning the fixed point property for multi-valued functions. Those obtaining the results include L. E. Ward, R. L. Plunkett, and W. Strother.

Definition 4.24. A space X is said to have the Fpp (fixed point property for multi-valued functions) if every continuous multi-valued function F from X to X has a fixed point, i.e., a point x such that x is in F(x).

The question was asked under what conditions on the spaces X and Y will there exist a continuous trace f of F, that is, a continuous function f on X to Y such that f(x) is in F(x) for all x. For some specific multi-valued functions F it is possible to produce a continuous trace. It is by use of these traces that most of the fixed point theorems in the literature for multi-valued functions are proved. W. L. Strother obtained the following results.

Lemma 4.39 (62). Let f be a trace of a multi-valued function F on X to Y and let X be a fixed point of f. Then x is a fixed point of F.

Proof. By definition a trace is a continuous function f on X to Y such that f(x) is in F(x) for all x. But x is a fixed point of f, which implies that we have x to be a fixed point of F.

From the above lemma it follows that a sufficient condition for a continuous multi-valued function F to have a fixed point is that F have a continuous trace which has a fixed point. But this is not a necessary condition. Define F from the unit circle at the origin in the complex plan to itself by $F(z) = \pm\sqrt{z}$. Then one obtains a continuous multi-valued function F with a fixed point and no continuous trace of F.

Theorem 4.40 (62). A bounded interval I of real numbers has the Fpp.

The proof of this theorem, due to Strother, will be omitted. However, Brouwer's theorem assures us that I has the fpp. Hence in view of Lemma 4.39 it is sufficient to prove that every continuous function F on I to I has a continuous trace.

R. L. Plunkett obtained a very significant result regarding fixed points and multi-valued transformations. His work had the restriction that the topological space be continuous, compact, locally connected, metric continua. Recall that a continuum is said to be a dendrite provided it is locally connected and contains no simple closed curve.

Theorem 4.41 (50). Every dendrite has the Fpp. The proof, which is extremely lenghty, will be omitted.

Another significant result concerning Fpp was obtained by Ward, but the proof will not be given. It is to be noted that Ward's theorem is a generalization of Hamilton's theorem (Theorem 4.7).

Theorem 4.42 (67). Each topologically chained, hereditarily unicoherent continuum which contains no indecomposable continuum has the Fpp.

Most theorems on fixed points for multi-valued functions demand either that F(x) be a connected set for every x or that F(x) be a convex set for every x. A survey of the literature shows it to be sparce of fixed point theorems with no condition on the image of a point. Theorem 4.40 is such a theorem.

Fundamental Theorem of Algebra

In the study of polynomials, the student becomes acquainted with the Fundamental Theorem of Algebra. Thus, the utility of this theorem is realized in freshman level courses through graduate level courses. There are many proofs of this theorem, all of them requiring considerable scope. The application of fixed points in point set topology will be used in the proof of this theorem given here.

Theorem 4.43 (80). (The Fundamental Theorem of Algebra). Every polynomial $P(z) = a_0 + a_1 z + \cdots + z^n$, the coefficients a_i being complex numbers, and $n > 0$, has at least one zero.

We may consider P as a mapping $P:E^2 \to E^2$ and if we set $P(\infty) = \infty$, we have a continuous mapping $P:S^2 \to S^2$, where S^2 is the 2-sphere.

Recall, in Definition 3.16, we gave the definition of homotopy of two mappings, where either one can be continuously deformed into the other. To be precise, two mappings f and g of a space X into a space Y are homotopy if there is a mapping $h:X \times I^1 \to Y$ such that for each point x in X,

$$h(x,0) = f(x) \text{ and } h(x,1) = g(x)$$

This is just another way of saying h restricted to $X \times 0$ is f and h restricted to $X \times 1$ is g. The mapping h is called a homotopy between f and g in the product space $X \times I^1$.

Lemma 4.44 (80). The polynomial P(z) is homotopic to the mapping $f(z) = z^n$.

Proof. We define the homotopy explicitly by setting

$$h(x,t) = z^n + (1-t)(a_0 + a_1 z + \cdots + a_{n-1} z^{n-1})$$

for z finite and $h(\infty, t) = \infty$, for all t and all finite z, h is continuous by elementary theorems in function theory. It follows that $\lim_{z \to \infty} h(z,t) = \infty$ for all t and hence that h is continuous on $S^2 \times I$.

Consider two n-spheres S^n and T^n and a continuous mapping $f: S^n \to T^n$. With every such mapping f we associate an integer deg(f), called the degree of f. Intuitively, the degree deg(f) is the algebraic number of times that the image $f(S^n)$ wraps around T^n. It is a known theorem that the degree of $f(z) = z^n$ is n.

Lemma 4.45 (80). If $f: S^n \to E^n$ is a continuous and $\deg(f) \neq 0$, then each point of T^n lies in the image of $f(S^n)$.

We now have the tools for the proof of our theorem.

Proof (Theorem 4.43). From Lemma 4.44 the degree of $P(z) = n$. Then by Lemma 4.45, each point of S^2 is the image of some point of S^2. In particular, there is at least one point z_0 such that $P(z_0) = 0$.

It is incorrect to say that because $\deg(f) = n$, each point is the image of at least n points. The function $f(z) = z^n$ is a counterexample since only zero is mapped onto zero.

It is an unsolved problem whether the set of points x in T^n, such that $f^{-1}(x)$ has at least k points, is nonempty, if $f: S^n \to T^n$ is continuous and $\deg(f) = k$.

Brouwer's Fixed Point Theorem

We first mentioned L. E. J. Brouwer's famous fixed point theorem in Chapter I. Theorem 2.1 is a special case of Brouwer's theorem, that a continuous transformation of a disk in the plane into itself leaves a point fixed under this transformation. We have made numerous appeals to Brouwer's theorem in the preceding theorems. There are many proofs of Brouwer's general theorem appearing in the literature. Since this theorem is the most celebrated fixed point theorem appearing in the literature, we feel that its proof should be presented only after the reader has realized the impact which this theorem has had on mathematics and on point set topology in particular. Hence for this reason its proof has not been presented until now.

We stated Brouwer's theorem in Chapter II (Theorem 2.7). We repeat the theorem here.

(Brouwer's Fixed Point Theorem). For any continuous mapping f of an n-cell into itself, there exists at least one point x_0 for which $f(x_0) = x_0$.

We shall give only one of the many elementary proofs which appear in the literature. First, we need two lemmas. Since the fpp is topological (Theorem 2.6), it suffices to prove this theorem for the convex n-simplex, i.e., the smallest convex set $S = |p_0, p_1, \cdots, p_n|$ in E^n containing the $n + 1$ independent points (vertices) $p_0, p_1, \cdots, p_n$.

Lemma 4.46 (Sperner). Let K be any simplical subdivision of the n-simplex S as described above and v(e) is a mapping of the vertices of K into the vertices of S such that for any vertex e of K, v(e) is a vertex of the carrier side of S, i.e., of the side of least dimension which contains e, then there exists an odd number of n-simplexes in K whose vertices map in one-to-one fashion onto the vertices of S.

Proof. Let us call such an n-simplex in K an E-simplex. Also, let us call an $(n - 1)$-simplex in K an E-side provided its vertices map onto the points $p_0, p_1, \cdots, p_{n-1}$ under v.

This proof, which is given by G. T. Whyburn (73), is by induction on the dimensionality n of S. It is easy to establish the condition holds for $n = 0$, so let us assume it holds for dimensionality $n - 1$. Let

$$d = \text{number of E-simplexes in } K,$$

$$a(T) = \text{number of E-sides on an } n\text{-simplex } T \text{ of } K,$$

$$a = \text{number of E-sides of } K \text{ lying on } F(S), \text{ the boundary of } S.$$

Now if T is an E-simplex of K, then $a(T) = 0$ or 2 according as one of the points $p_0, p_1, \cdots, p_{n-1}$ is or is not missing from the set of images of the vertices of T. Hence

$$d = \Sigma\, a(T), \bmod 2, \qquad T \text{ in } K.$$

On the other hand, an E-side appears exactly twice or exactly once in $\Sigma\, a(T)$ according as it is interior to S or on $F(S)$. Accordingly,

$$\Sigma\, a(T) = a, \bmod 2.$$

Thus

$$a = d, \bmod 2.$$

Now it is seen that any E-side of $F(S)$ must also be on the side $S^{n-1} = |p_0, p_1, \cdots, p_{n-1}|$ of S, since otherwise one of the points $p_0, p_1, \cdots, p_{n-1}$ would fail to be the image of a vertex of this side. Thus a is the number of $(n - 1)$-simplexes in the subdivision of S^{n-1} whose vertices map in one-to-one fashion onto the vertices of S^{n-1} under v. By the induction hypothesis, a is odd and it follows that d is odd.

Lemma 4.47 (73). If $H_0, H_1, \cdots, H_n$ are closed subsets, $S = |p_0, p_1, \cdots, p_n|$ such that for each set of integers $i_0, i_1, \cdots, i_k$ $(0 \leq i_j \leq n)$ the side $|p_{i_0}, p_{i_1}, \cdots, p_{i_n}|$ of S is contained in $A_{i_0} + A_{i_1} + \cdots + A_{i_k}$, then $A_0 \cdot A_1 \cdot \cdots \cdot A_n \neq \emptyset$.

Proof. To show this is equivalent to showing that for every $\epsilon > 0$ there exists a subset of S of diameter less than ϵ intersecting every one of the sets A_i. To this end, let K be a simplical subdivision of S of norm less than ϵ, i.e., each simplex of K is of diameter less than ϵ. For any vertex c of K, let $|p_{i_0}, p_{i_1}, \cdots, p_{i_k}| = S^k$ be the carrier side of S for c, and let A_{i_j} be one of the sets $A_{i_0}, A_{i_1}, \cdots, A_{i_k}$ containing c and define $v(c) = p_{i_j}$. (Note that such an A_{i_j} must exist because S^k is contained in $\Sigma_{j=0}^{k} A_{i_j}$.) By Lemma 4.46 there exists at least one simplex T of K whose vertices map under v onto the vertices of S in one-to-one fashion. Now in order for a vertex c of T to map into p_i it is necessary that c be contained in A_i by definition of $v(c)$. Thus T has a vertex in each of the sets $A_0, A_1, \cdots, A_n$ and since the norm of T is less than ϵ, the result follows.

Proof (Brouwer's Fixed Point Theorem) (73). Let $S = |p_0, p_1, \cdots, p_n|$ be the simplex in E^n with vertices $p_0 = (0,0,\cdots,0)$, $p_1 = (1,0,\cdots,0)$, $\cdots$, $p_n = (0,0,\cdots,1)$. Let

$$f(x_1, x_2, \cdots, x_n) = (x_1', x_2', \cdots, x_n')$$

be any continuous mapping of S into itself. Let A_0 be the set of all $x = (x_1, x_2, \cdots, x_n)$ of S such that

$$\Sigma_1^n x_i' \geq \Sigma_1^n x_i$$

and for each i, $1 \leq i \leq n$, let A_i be the set of all x in S such that

$$x_i' \leq x_i.$$

Now let $S^k = |p_{i_0}, p_{i_1}, \cdots, p_{i_k}|$ be any side of S. It shall be shown that

$$S^k \text{ is a subset of } \Sigma_{j=0}^{k} A_{i_j}.$$

Thus, let $p = (x_1, x_2, \cdots, x_n)$ be the point of S^k. Let us consider the following cases:

(a) If all i_j are different from 0, then

$$\Sigma_{j=0}^{k} x_{i_j} = 1, \ x_i = 0, \ i \neq i_0, \ \cdots \ i_k.$$

Now since

$$\Sigma \ x_1' \leq 1$$

and $x_i' \geq 0 = x_i$ for $i \neq i_0, i_1, \cdots, i_k$, we must have

$$x_{i_j}' \leq x_{i_j} \qquad \text{for some } i_j.$$

This yields

$$p = (x_1, x_2, \cdots, x_n) \text{ in } A_{i_j}.$$

Now consider the case

(b) if some i_j, say i_0, equals zero, then

$$x_i = 0, \text{ for } i \neq i_1, i_2, \cdots i_k.$$

Hence if

$$x_{i_j}' > x_{i_j} \text{ for } j = 1, 2, \cdots, k,$$

we have

$$\Sigma_1^n x_i' \geq \Sigma_{j=1}^{k} x_{i_j}' > \Sigma_{j=1}^{k} x_{i_j} = \Sigma_1^n x_i,$$

and this gives p in $A_0 = A_{i_0}$.

Hence in either case the condition in Lemma 4.44 is satisfied. Accordingly there exist a point

$$p = (x_1, x_2, \cdots, x_n) \text{ of } \overset{n}{\underset{i=0}{\pi}} A_i.$$

Thus for this point conditions (a) and (b) must hold simultaneously; and this requires

$$x_i' \equiv x_i, \; i = 1,2,\cdots,n,$$

so that $f(p) = p$. This completes the proof.

We have endeavored to restrict the scope of this thesis to fixed points in point set topology such that if $f:X \to X$ is continuous, there exist a point x of X such that $f(x) = x$, and a short treatment of the fpp for non-continuous mappings. We have not considered the particular problems of fixed points for periodic functions. In general, the mapping f is said to be periodic provided there exist an integer n such that

$$f^n(x) \equiv x.$$

The least such integer n is called the period of f. In terms of fixed points, a mapping $f:X \to X$ is pointwise periodic at x of X provided x is fixed for some power of f.

P. A. Smith has been most prolific in this area of topology. Also, to contribute richly to periodic functions have been S. Eilenberg and G. T. Whyburn.

CHAPTER V

SUMMARY AND EDUCATIONAL IMPLICATIONS

This thesis gives a collection of mathematical research in an area of modern mathematics which is not readily available to students who are not skilled in reading the mathematical periodicals. Included in the collection are a brief history of point set topology in America; a presentation of the fixed point properties of point set topology, with discussions, explanations, and examples pertaining to these properties; and statements of a number of unsolved problems in this area of mathematics.

Summary

Chapter I states the problem and discusses justifications, procedure, and expected outcomes. In Chapter II, following a brief history of topology, we give the definition of the fixed point in point set topology, along with some applications of the fixed point to the real line continuum; to the plane; to concepts of elementary calculus, of differential equations, and of functional analysis; and to a system of n linear equations in n unknowns. Brouwer's famous fixed point theorem is stated in this chapter, and the fixed point theorem for the disk in the plane, which is a special case of Brouwer's theorem, is proved. In Chapter III, results for conditions under which a fixed point space must be compact are given. The background for this conclusion, which is also exhibited, is an example of a separable, locally connected metric space with the fixed point property that is not compact. We also give an example of a compact metric space with the fixed point property that separates the plane and is not unicoherent. Some of the more recent developments of the fixed point property are discussed in Chapter IV. Of much importance is the research of Hamilton which shows that a hereditarily unicoherent and hereditarily decomposable continuum has the fixed point property. This paved the way for much subsequent research. Also in Chapter IV the fixed point property for multi-valued functions is discussed, and related research is analyzed. The pseudo-arc is shown to possess the fixed point property. The Fundamental Theorem of Algebra is proved using methods of topology. As a climax to this thesis, the proof to Brouwer's theorem is given.

Educational Implications

Since the study of mathematics is becoming increasingly widespread and the body of knowledge in this area is expanding rapidly, a collection of the research done in fixed points is needed in the realm of mathematics education. The selection of materials included and the method in which they are presented, analyzed, and explained can help to bring about an improved understanding of mathematics.

The types of proofs are varied. Examples of these are the direct proofs, construction proofs, mathematical induction, and proof by contradiction. Theorems included in this thesis are, in general, not isolated propositions, but are dependent upon previous research or give implication for subsequent research. Each theorem has associated with it a reference to the bibliography to aid the reader in his search of the literature for additional resource material. The thesis is self contained, i.e., each topological concept is defined in its initial use in the thesis. Known theorems of related concepts are given, with reference, as they are needed. Examples and applications are given where these are relevant to particular concepts and theorems. Discussions and an analysis of the research are often given preceding a collection of theorems.

By reading this thesis, the student, who is a potential mathematics teacher, can come abreast of a branch of modern mathematics. The history of point set topology will make him aware of the vast amount of research currently being done and of the men who have contributed to its development, many of them his contemporaries. Through the reading of this thesis, he will be confronted with the current research and with the possibility of contributing to mathematics by extending the results given here and by offering solutions to the unsolved problems as well as by developing new properties of fixed points. This thesis provides the advanced graduate student who is engaging in research references in the bibliography of a more extensive collection of the research done in this area of mathematics. For anyone interested in the research of fixed points, the bibliography will be a valuable aid.

It is hoped that the person reading this thesis will not only learn of the mathematical research concerning the fixed point property in topology, but will become interested in related areas of topology.

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APPENDIX

INDEX

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