ENERGY SOLUTION FOR CONTINUOUS RECTANGULAR THIN PLATE PROBLEMS

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PREFACE

The application of the Rayleigh-Ritz energy method to the analysis of continuous rectangular thin plate problems is presented in this dissertation. A mathematical model in the form of a polynomial series is adopted to represent the deflection surface of the plate. A set of simultaneous Fredholm Integral Equations of the First Kind for the edge redundant functions is obtained through the use of equations of continuity between panels.

This research is the outgrowth of ideas expressed by Dr. Robert W. Little in the fall of 1963. At that time, Dr. Little suggested that the Rayleigh-Ritz energy method with the use of a polynomial series could be extended to the analysis of continuous rectangular plates.

The writer wishes to take this opportunity to express his indebted-ness and appreciation to the following individuals and organizations:

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William H. Ying

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a, b . . . . Dimensions of Plate.

a_i, b_i . . . . Dimensions of Plate i.

a_{i+1}, b_{i+1} . . . . Dimensions of Plate i+1.

$f\left(\frac{x}{a}, \frac{y}{b}\right)$ . . . . A Polynomial Function Prescribing the Geometrical Boundary Requirements of Plate.

$f_i\left(\frac{x_i}{a_i}, \frac{y_i}{b_i}\right)$ . . . . A Polynomial Function Prescribing the Geometrical Boundary Requirements of Plate i.

$f_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}, \frac{y_{i+1}}{b_i}\right)$ . . . . A Polynomial Function Prescribing the Geometrical Boundary Requirements of Plate i+1.

h . . . . Plate thickness.

h_i . . . . Thickness of Plate i.

h_{i+1} . . . . Thickness of Plate i+1.

j, k . . . . Summation Indices.

m, n . . . . Summation Indices.

m_γ . . . . Coefficient of Series for Edge Moment Function.

p, q . . . . Summation Indices.

q . . . . Intensity of Normal Load.

q_i . . . . Intensity of Normal Load Acting on Plate i.

q_{i+1} . . . . Intensity of Normal Load Acting on Plate i+1.

r, s . . . . Summation Indices.

v_β . . . . Coefficient of Series for Edge Reactive Force Function.

w . . . . Deflection of Plate.

w_b . . . . Deflection of Supporting Beam.

w_i . . . . Deflection of Plate i.

w_{i+1} . . . . Deflection of Plate i+1.

x, y, z . . . . Coordinate Axes.

x_i, y_i, z_i . . . . Coordinate Axes for Plate i.

x_{i+1}, y_{i+1}, z_{i+1} . . . . Coordinate Axes for Plate i+1.

$\left[(x_i y_i)_v\right]_{x_i = a_i}$,

$\left[(x_{i+1} y_i)_t\right]_{x_{i+1}=0}$ . . . . . Matrices with Elements in Polynomials for Plates i and i+1, Respectively.

$\left[(x_{i+1} y_i)'_t\right]_{x_{i+1}=0}$ . . . . Modified Matrix with Elements in Polynomials for a Free-free Plate i+1.

A . . . . Area of Plate.

A_i . . . . Area of Plate i.

A_{i+1} . . . . Area of Plate i+1.

A_{mn} . . . . Coefficient of Series for Deflection of Plate.

$[A_v]$, $[B_t]$ . . . . Matrices with Elements A_{mn} and B_{jk}, Respectively.

B_{jk} . . . . Coefficient of Series for Deflection of Plate i+1.

$[B'_t]$ . . . . Modified Matrix with Elements B_{jk} for a Free-free Plate i+1.

C_{mnrs} . . . . Coefficient of Strain Energy in Bending of Plate.

$[C_{uv}]$, $[D_{ht}]$ . . . . Coefficient Matrices with Elements C_{mnrs} and D_{jkpq} , Respectively.

$[D'_{ht}]$ . . . . Modified Coefficient Matrix with Elements D_{jkpq} for a Free-free Plate i+1.

D . . . . Flexural Rigidity of Plate.

D_i . . . . Flexural Rigidity of Plate i.

D_{i+1} . . . . Flexural Rigidity of Plate i+1.

D_{jkpq} . . . . Coefficient of Strain Energy in Bending of Plate i+1.

E . . . . Modulus of Elasticity of Plate.

E_b . . . . Modulus of Elasticity in Beam.

E^{i+1}_{pq} . . . . A Quantity due to Normal Load Acting on Plate i+1.

E^{i}_{rs} . . . . A Quantity due to Normal Load Acting on Plate i.

$\left[E^{i+1}_{h}\right], \left[E^{i}_{u}\right]$ . . . . Matrices with Elements E^{i+1}_{pq} and E^{i}_{rs}, Respectively.

$\left[E'^{i+1}_{h}\right]$ . . . . Modified Matrix with Elements E^{i+1}_{pq} for a Free-free Plate i+1.

$F^{i+1}_{pq}(\eta)$ . . . . A Quantity due to a Unit Force Acting on the Left Edge of Plate i+1.

$F^{i}_{rs}(\eta)$ . . . . A Quantity due to a Unit Force Acting on the Right Edge of Plate i.

$\left[F^{i+1}_{h}(\eta)\right], \left[F^{i}_{u}(\eta)\right]$ . . . . Matrices with Elements $F^{i+1}_{pq}(\eta)$ and $F^{i}_{rs}(\eta)$, respectively.

$\left[F'^{i+1}_{h}(\eta)\right]$ . . . . Modified Matrix with Elements $F^{i+1}_{pq}(\eta)$ for a Free-free Plate i+1.

G_b . . . . Modulus of Rigidity of Beam.

$G^{i+1}_{pq}(\eta)$ . . . . A Quantity due to a Unit Couple Acting on the Left Edge of Plate i+1.

$G^{i}_{rs}(\eta)$ . . . . A Quantity due to a Unit Couple Acting on the Right Edge of Plate i.

$\left[G^{i+1}_{h}(\eta)\right], \left[G^{i}_{u}(\eta)\right]$ . . . . Matrices with Elements $G^{i+1}_{pq}(\eta)$ and $G^{i}_{rs}(\eta)$, Respectively.

$\left[G'^{i+1}_{h}(\eta)\right]$ . . . . Modified Matrix with Elements $G^{i+1}_{pq}(\eta)$ for a Free-free Plate i+1.

I_b . . . . Moment of Inertia of Beam.

J_b . . . . Torsional Constant of Beam.

M_x, M_y, M_{xy}, M_{yx} . . . . Intensities of Bending Moments and Twisting Moments in a Plate.

M_B . . . . Bending Moment in Edge Beam.

M_T . . . . Twisting Moment in Edge Beam.

$M(\eta)$ . . . . Edge Moment Function.

P . . . . A Concentrated Load.

Q_{xz} , Q_{yz} . . . . Intensities of Shearing Forces in a Plate.

U . . . . Strain Energy in Bending of Plate.

V . . . . Potential Energy of Total External Forces Acting on a Plate.

V_i . . . . Potential Energy of Total External Forces Acting on the Plate i.

V_{i+1} . . . . Potential Energy of Total External Forces Acting on the Plate i+1.

V_q . . . . Potential Energy of Total Load Acting on a Plate.

V_{xz} . . . . Edge Reactive Force Acting on Planes Perpendicular to the x-axis and Parallel to z-axis.

V_M . . . . Potential Energy of Edge Moment.

V_V . . . . Potential Energy of Edge Reactive Force.

$V(\eta)$ . . . . Edge Reactive Force Function.

II . . . . Total Potential Energy of Plate.

II_i . . . . Total Potential Energy of Plate i.

II_{i+1} . . . . Total Potential Energy of Plate i+1.

η . . . . A Dimensionless Coordinate.

ν . . . . Poisson's Ratio.

σ_x, σ_y . . . . Normal Stresses in the x and y Directions, Respectively.

τ_{xy}, τ_{yx} . . . . Shearing Stresses on Planes Perpendicular to the x and y Axes and Parallel to the y and x Axes, Respectively.

$\left[\frac{\partial}{\partial x_i}(x_i y_i)_v\right]_{x_i=a_i}$,

$\left[\frac{\partial}{\partial x_{i+1}}(x_{i+1} y_i)_t\right]_{x_{i+1}=0}$ . . . . Matrices with Elements in Polynomials for Plates i and i+1, Respectively.

$\left[\frac{\partial}{\partial x_{i+1}}(x_{i+1} y_i)'_t\right]_{x_{i+1}=0}$ . . . . Modified Matrix with Elements in Polynomials for a Free-free Plate i+1.

CHAPTER I

INTRODUCTION

1-1. General

The classical plate theory may be divided into three parts according to the thickness and deflection of the plate; namely,

1. small deflection theory of thin plates,
2. large deflection theory of thin plates,
3. thick plate theory.

The analysis of a thick plate problem is considerably involved, because it is necessary to consider it as a three-dimensional problem of elasticity. Only a few particular cases up to this date have been solved. Problems of thin plates, based on the small deflection theory, both in the area of single panel with various edge conditions and of continuous several panels under certain limited conditions have been worked out. In the large deflection theory of thin plates, the strain of an element in the middle plane of the plate should be considered and thus two simultaneous nonlinear partial differential equations are formed.[1] In the small deflection theory, only one linear partial differential equation of order four occurs. Kirchhoff[2] showed that in this case two boundary conditions on each edge are sufficient to determine the deflection surface of the elastic plate.

The work developed in this thesis will be based on the assumption of the small-deflection thin plate theory. Customarily, this theory is often referred to as classical thin plate theory. In this theory, the following assumptions are made:

1. The material of the plate is homogeneous, isotropic and continuous.
2. The thickness of the plate is linearly elastic.
3. The thickness of the plate is constant and is small in comparison with other dimensions.
4. The normals to the middle surface before deflection remains normal after deformation.
5. The middle surface of the plate has no deformation during bending.
6. Stresses normal to the middle surface are negligible.
7. The plate undergoes small deformation which does not alter the geometry of the plate.
8. The loads are lateral loads acting at the middle surface.

Assumptions 4 to 6 listed are equivalent to neglecting the effect of transverse shear deformation on the bending of elastic plates. By taking this deformation into account Reissner's theory of bending of elastic plate yields a sixth-other partial differential equation for which all three boundary conditions can and must be satisfied.[3,4] Kirchhoff's boundary conditions can be sufficiently justified for a thin plate with small corner reactions. For a bent plate with a hole, small in comparison with the thickness of the plate, the magnitude of stress concentration at the edge of hole becoming uncertain, the classical thin plate theory cannot be adopted.

1-2. Historical Study

Since the study of the behavior of elastic plates attracted so many investigators, it is necessary to limit the following historical study only to the energy solution of thin plates with small deflection, and to continuous rectangular thin plates.

The energy method was first developed by Ritz,[5] who considered the problem of a clamped square plate acted upon by a uniform load, and who did not carry through the solution for moments and deflections. Knott[6] carried on this problem and obtained successfully the result of deflection at several points. By considering the same problem, Mesnager[7] using the improper functions, failed to obtain an accurate result.

Later, Pickett[8] developed a general formula and utilized the energy method to compute the unknown coefficients of a double power series for the deflection surface of a clamped plate. Timoshenko and Woinowsky-Krieger[1] were able to obtain through the use of a Fourier's series an exact form of a Navier solution[9] for a simply supported plate. More recently, McInnis, Tsai and Sims[10] used a special power series to solve a square, uniformly loaded, clamped plate. Little[11] applied the general formula for the problem of a concentrated loaded, cantilever plate.

The solution for rectangular plates continuous in one direction over rigid supports was first obtained exactly by Galerkin.[12] Marcus,[13] Jensen[14] and Woinowsky-Krieger[15] also studied the same problem. In Jensen's paper, a case having intermediate elastic supports was also considered. This case actually was studied one year earlier by Weber.[16] Newmark[17] developed a distribution procedure in analyzing rectangular plates that were continuous in one direction over rigid or flexible beams and were simply-supported on the side edges parallel to this direction.

Problems of rectangular plates continuous in two directions over supports were treated by Bittner,[18] and Maugh and Pan,[19] by assuming certain approximate continuity conditions between panels. Engelbreth,[20] and Siess and Newmark,[21,22] obtained an approximate solution through distribution procedures for the same problem. Lechter[23] also analyzed this case by the flexibility approach. Corner supported rectangular plates, that were composed of an infinite number of identical panels acted upon by a uniform load, were considered by Nadai[24] and Galerkin.[25] Sutherland, Goodman and Newmark[26] made analyses of this type of plates supported by elastic beams in some special cases. Recently, Ang,[27] with Newmark,[28] developed a numerical procedure for obtaining an approximate but more general results. In their approach, the finite physical model similar to the finite differences analogue was used. Further extension of this method was made to some plate-beam systems by Ang and Prescott.[29] Without considering the torsional effect of beams, Reddy[30] investigated this problem based on the flexibility methods. Oden[31] also studied this problem through the use of a Fourier's series.

CHAPTER II

SCOPE AND PROCEDURE OF INVESTIGATION

2-1. Statement of the Problem

The study of this thesis is concerned with the problem of continuous rectangular plates subjected to a general loading system of any type of out-of-plane forces and couples. The behavior of the plate elements is strictly confined to the classical thin plate theory of bending and the analysis follows the assumptions made in page 2. Continuous plates may be composed of several basic panels continuous in one or two directions over rigid or flexible beams. Certain type of special plate structures, such as an arbitrarily stable plate with an "overhanging panel," is also considered. The support conditions of a basic panel are general and may have various edge conditions.

2-2. Procedure of Investigation

The general approach to the solution of a continuous plate structure under investigation in this thesis is through the use of the Rayleigh-Ritz energy method. A deflection surface function is assumed in the form of a polynomial series with arbitrary coefficients premultiplied by a function that satisfies the geometrical or essential boundary conditions of a basic rectangular plate. Expressions of the total potential energy of the plate in terms of the assumed deflection surface are formulated for each basic rectangular plate. The potential energy of unknown edge reactive force functions and edge moment functions along the junctions of a continuous plate structure is also taken into account. By applying the principle of minimum potential energy, two infinite sets of infinite simultaneous equations are thus obtained.

The compatibility relationships between the plates must then be established, from which two simultaneous Fredholm integral equations of first kind in terms of unknown edge reactive force functions and edge moment functions are obtained. A technique for solving these unknown functions is utilized by assuming these unknowns in the form of some other finite polynomial series and, thus, these unknown functions can be evaluated by the "matching" process term by term.

In the case of an "overhanging plate," constrained relations for a free-free plate should be introduced from the conditions of static equilibrium.

2-3. Rayleigh-Ritz Energy Method

It can be shown that from the principle of minimum potential energy, of all displacements satisfying the given boundary conditions of a stable structure, those which satisfy the equilibrium conditions make the potential energy an absolute minimum.

A procedure of solving a problem by employing the principle of minimum potential energy is that usually called the Rayleigh-Ritz method. In this method, the solution of a rectangular thin plate may be assumed in the form of an "admissible" series function of the displacements, and the total potential energy Π is expressed in terms of this series function. Let the assumed deflection function be $w(x,y)$ that consists of $m \times n$ undetermined parameters A_{ij} and is in the form of

$$w(x,y) = \sum_i \sum_j A_{ij} X_i(x) Y_j(y) \tag{2-1}$$

in which $i = 1, 2, \ldots\ldots m$, $j = 1, 2, \ldots\ldots n$. Since the potential energy must be a minimum at equilibrium, these parameters can be obtain-ed from the minimizing conditions

$$\frac{\partial \Pi}{\partial A_{ij}} = 0 \qquad (i = 1, 2, \ldots m; \; j = 1, 2, \ldots n) \tag{2-2}$$

If m x n parameters are taken, a set of m x n simultaneous equations is obtained from which these parameters can be solved.

The variational method used requires that the assumed functions $X_i(x)\ Y_j(y)$ must be "admissible"; that is, these functions must satisfy the geometric or essential boundary conditions, but need not satisfy the natural boundary conditions of the plate. When the series considered is a complete sequence, the minimization of the total potential energy enforces the condition of equilibrium that the virtual work vanishes with respect to each virtual displacements.

2-4. Total Potential Energy of a Thin Plate

The total potential energy consists of potential energy of deformation or strain energy of bending and the potential energy of the external forces acting on the body. Following the assumptions of a thin plate, that the stresses normal to the plate and the transverse shear deformations are negligible, the strain energy U stored in the elastic plate can be represented by the following expression,

$$U = \frac{D}{2}\iint_A \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right] dA \tag{2-3a}$$

in which w = the deflection of the plate in the z direction (Fig. 2-1),

A = the area of the plate,

D = flexural rigidity of the plate = $\dfrac{Eh^3}{12(1-\nu^2)}$,

h = plate thickness,

E = modulus of elasticity of the plate,

ν = Poisson's ratio for the plate.

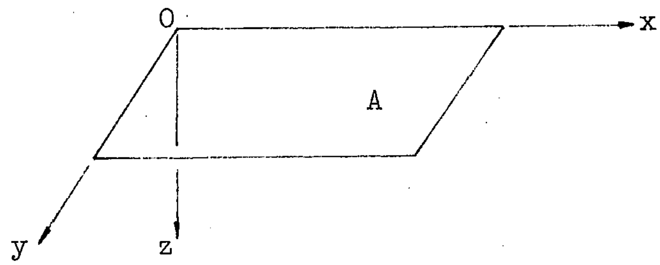


Fig. 2-1.

Coordinate System of a Rectangular Plate

For rectangular plates with zero deflection along all edges, expression (2-3a) may be reduced by the integration by parts into another simpler form:[40]

$$U = \frac{D}{2}\iint_A \left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right]^2 dA \qquad (2\text{-}3b)$$

If the plate is under the action of a general load of intensity q(x,y), the potential energy of the total load may be written as

$$V_q = -\iint_A w\, q\, dA \qquad (2\text{-}4)$$

Then, the total potential energy of a plate becomes

$$\Pi = U + V_q \qquad (2\text{-}5)$$

The expression of strain energy U in equation (2-5), depending upon the geometric boundary conditions of a plate, may be in a form of either expression (2-3a) or (2-3b).

2-5. The Plate Equations and Boundary Conditions

Consider a differential element of the plate shown in Fig. 2-2.

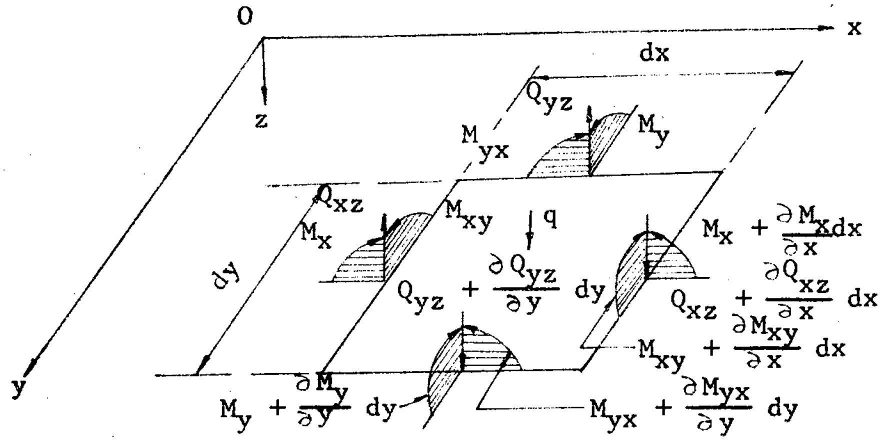


Fig. 2-2.
Differential Element of a Thin Plate

The directions of positive bending moments, M_x and M_y, twisting moments, M_{xy} amd M_{yx}, and shearing forces, Q_{xz} and Q_{yz} are indicated. All moments and shears are expressed per unit length. Applying Hooke's law and the equations of equilibrium leads to the following expressions,

$$M_x = - D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \tag{2-6a}$$

$$M_y = - D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \tag{2-6b}$$

$$M_{xy} = M_{yx} = - D (1-\nu) \frac{\partial^2 w}{\partial x \partial y} \tag{2-6c}$$

$$Q_{xz} = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = - D \left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) \tag{2-6d}$$

$$Q_{yz} = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = - D \left(\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y} \right) \tag{2-6e}$$

Substituting expression (2-6) into the equation of equilibrium, that the summation of the z components of the forces be zero, the differential equation governing the small deflection of a thin plate under bending is obtained,

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \tag{2-7}$$

The stresses may be also expressed in terms of the deflection function by the following relations,

$$\sigma_x = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \tag{2-8a}$$

$$\sigma_y = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \tag{2-8b}$$

$$\tau_{xy} = \tau_{yx} = -\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x \partial y} \tag{2-8c}$$

where σ_x and σ_y are normal stresses in the x and y directions respectively, τ_{xy} is the shearing stress and z is the distance of the point in question measured down from the middle surface of the plate.

The solution of a specific rectangular plate requires a deflection function w that satisfies equation (2-7) and the essential boundary conditions of the plate.

The boundary conditions for a rectangular plate with edges parallel to the x and y axes of a rectangular coordinate system are considered primarily with the following three cases:

a. Simply supported edge. If the edge x = a is simply supported, the deflection and bending moment along the edge must be zero. Furthermore, along this edge the curvature remains zero, arising from the fact that the deflection is zero along the boundary. Hence,

$$w(a, y) = 0, \qquad \frac{\partial^2 w}{\partial x^2}(a, y) = 0 \tag{2-9}$$

b. Clamped edge. If the edge x = a is clamped, along this edge the deflection and the slope of the middle plane are zero. Therefore,

$$w(a, y) = 0, \qquad \frac{\partial w}{\partial x}(a, y) = 0 \tag{2-10}$$

c. Free edge. At a free boundary x = a, there are no bending and twisting moments, and also no shearing forces. That is,

$$M_x (a, y) = 0, \qquad M_{xy} (a, y) = 0, \qquad Q_{xz} (a, y) = 0 \tag{2-11}$$

However, only two conditions are sufficient for the complete determination of the deflection surface of a thin plate. Three is too many. A proof can be obtained that the bending moment and the sum of shearing force and the rate of change of the twisting moment, the so-called "substitute shear force" or "reactive force," are to be prescribed along the boundary. This reactive force V_{xz} may be expressed in terms of the deflection function,

$$V_{xz}(a, y) = - D\left[\frac{\partial^3 w}{\partial x^3}(a, y) + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}(a, y)\right] \tag{2-12}$$

Thus, the boundary conditions for a free edge x = a are

$$\begin{aligned} &\frac{\partial^2 w}{\partial x^2}(a, y) + \nu\frac{\partial^2 w}{\partial y^2}(a, y) = 0 \\ &\frac{\partial^3 w}{\partial x^3}(a, y) + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}(a, y) = 0 \end{aligned} \tag{2-13}$$

CHAPTER III

FUNCTIONS OF THE BASIC PLATE

3-1. General

In the analysis of the plate structures, the determination of the deflection surface of a rectangular plate may be accomplished by obtaining some series function which satisfies the boundary conditions as well as the differential equation of a thin plate. An alternate approach to the solution of this problem can also be given by means of the Rayleigh-Ritz method in assuming a deflection function that needs to satisfy only the essential boundary conditions of a plate. In the former approach, the deflection function that satisfies the differential equation of a plate satisfies also the condition of equilibrium for each element of the plate. While in the latter, the minimization of the total potential energy guarantees each plate element approaches an equilibrium state.

In this investigation, the Rayleigh-Ritz method is adopted to analyze a basic rectangular plate subjected to a general type of lateral loads as well as arbitrary edge moments and arbitrary edge reactive forces for the plate with a free edge.

A function in the form of a double polynomial series in both the x and y directions with undetermined parameters A_{mn} is assumed as a general deflection surface $w(x, y)$ of the plate. This double polynomial series is premultiplied by an appropriate function $f\left(\frac{x}{a}, \frac{y}{b}\right)$ to satisfy the geometrical boundary requirements of the plate. In order to simplify the numerical work, the deflection surface may be expressed in a dimensionless form:

$$w(x, y) = f\left(\frac{x}{a}, \frac{y}{b}\right) h \sum_{m}\sum_{n} A_{mn} \left(\frac{x}{a}\right)^{m} \left(\frac{y}{b}\right)^{n} \quad (3\text{-}1)$$

where a and b are length parameters of the plate in the x and y directions, respectively.

<u>3-2. Formation of Strain Energy for a Basic Rectangular Plate With Various Boundary Conditions</u>

This article is devoted to formulating the strain energy of bending of an elastic plate in terms of the assumed deflection function w(x, y). The origin of a coordinate system is specifically arranged in three different positions. Thirty cases of rectangular plates which are free, simply supported, clamped or corner-supported are considered as shown in Fig. 3-1. The advantages of biaxial symmetry and uniaxial symmetry of the panels for cases 1 to 6 and cases 7 to 20, respectively, are taken into account. For non-symmetrical panels shown in cases 21 through 30, the origin of the coordinates is taken at the left upper corner of the plates.

Substituting the assumed deflection surface into expression (2-3) and integrating over the whole area of the plate, the strain energy for the bending of the rectangular plates may be expressed in the following general form:

$$U = \frac{Dh^2}{2ab} \sum_{m}\sum_{n}\sum_{r}\sum_{s} A_{mn} A_{rs} C_{mnrs} \quad (3\text{-}2)$$

The expressions for the coefficient C_{mnrs} are successively recorded in Tables 1 through 30 for the thirty cases as shown in Fig. 3-1.

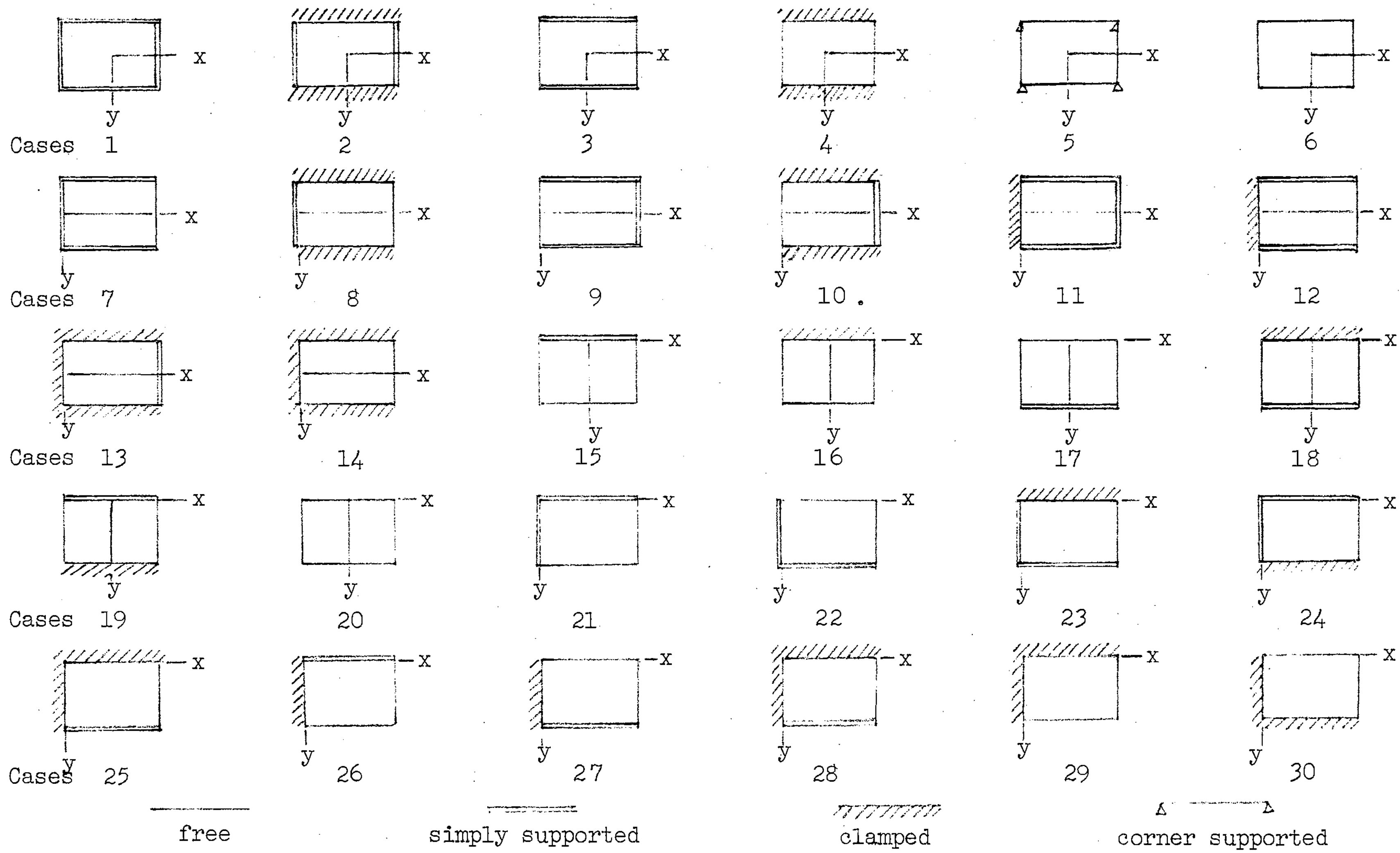
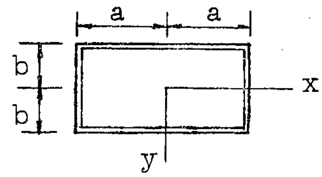


Fig. 3-1.

Cases of Combinations of Edge Boundary Conditions of the Rectangular Plates

TABLE 3-1 COEFFICIENT C_{mnrs} - CASE 1



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left[\left(\frac{x}{a}\right)^m - \left(\frac{x}{a}\right)^{m+2} \right] \left[\left(\frac{y}{b}\right)^n - \left(\frac{y}{b}\right)^{n+2} \right]$

Coefficient $C_{mnrs} = \begin{cases} 0, & \text{if } m+r \text{ or } n+s \text{ is odd;} \\ 4\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2 L_{mnrs}\right], & \text{otherwise.} \end{cases}$

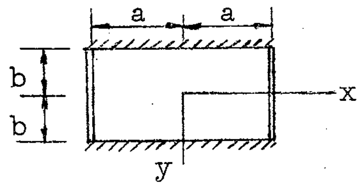
in which

$$J_{mnrs} = \left[\frac{m(m-1)r(r-1)}{m+r-3} - \frac{m(m-1)(r+2)(r+1)+(m+2)(m+1)r(r-1)}{m+r-1} + \frac{(m+2)(m+1)(r+2)(r+1)}{m+r+1}\right] \cdot \left[\frac{1}{n+s+1} - \frac{2}{n+s+3} + \frac{1}{n+s+5}\right]$$

$$K_{mnrs} = \left[\frac{1}{m+r+1} - \frac{2}{m+r+3} + \frac{1}{m+r+5}\right]\left[\frac{n(n-1)s(s-1)}{n+s-3} - \frac{n(n-1)(s+2)(s+1)+(n+2)(n+1)s(s-1)}{n+s-1} - \frac{(n+2)(n+1)(s+2)(s+1)}{n+s+1}\right]$$

$$\begin{aligned}
L_{mnrs} = \frac{1}{2}\Bigg[& \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{(m+r-1)(n+s-1)} - \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{(m+r-1)(n+s+1)} - \frac{m(m-1)(s+2)(s+1)}{(m+r-1)(n+s+1)} \\
& - \frac{r(r-1)(n+2)(n+1)}{(m+r-1)(n+s+1)} + \frac{m(m-1)(s+2)(s+1)+r(r-1)(n+2)(n+1)}{(m+r-1)(n+s+3)} - \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{(m+r+1)(n+s-1)} \\
& - \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{(m+r+1)(n+s-1)} + \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{(m+r+1)(n+s+1)} + \frac{m(m-1)(s+2)(s+1)}{(m+r+1)(n+s+1)} \\
& + \frac{r(r-1)(n+2)(n+1)}{(m+r+1)(n+s+1)} + \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{(m+r+1)(n+s+1)} + \frac{(m+2)(m+1)(s+2)(s+1)}{(m+r+1)(n+s+1)} \\
& + \frac{(r+2)(r+1)(n+2)(n+1)}{(m+r+1)(n+s+1)} - \frac{m(m-1)(s+2)(s+1)+r(r-1)(n+2)(n+1)}{(m+r+1)(n+s+3)} - \frac{(m+2)(m+1)(s+2)(s+1)}{(m+r+1)(n+s+3)} \\
& - \frac{(r+2)(r+1)(n+2)(n+1)}{(m+r+1)(n+s+3)} + \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{(m+r+3)(n+s-1)} - \frac{(m+2)(m+1)s(s-1)}{(m+r+3)(n+s+1)} \\
& - \frac{(r+2)(r+1)n(n-1)}{(m+r+3)(n+s+1)} - \frac{(m+2)(m+1)(s+2)(s+1)+(r+2)(r+1)(n+2)(n+1)}{(m+r+3)(n+s+1)} \\
& + \frac{(m+2)(m+1)(s+2)(s+1)+(r+2)(r+1)(n+2)(n+1)}{(m+r+3)(n+s+3)} \Bigg]
\end{aligned}$$

TABLE 3-2 COEFFICIENT C_{mnrs} - CASE 2



Deflection surface: $$w = h \sum_m \sum_n A_{mn} \left[\left(\frac{x}{a}\right)^m - \left(\frac{x}{a}\right)^{m+2}\right]\left[\left(\frac{y}{b}\right)^n - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+4}\right]$$

Coefficient $$C_{mnrs} = \begin{cases} 0, & \text{if } m+r \text{ or } n+s \text{ is odd;} \\ 4\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2 L_{mnrs}\right], & \text{otherwise.} \end{cases}$$

in which

$$J_{mnrs} = \left[\frac{m(m-1)r(r-1)}{m+r-3} - \frac{m(m-1)(r+2)(r+1)+(m+2)(m+1)r(r-1)}{m+r-1} + \frac{(m+2)(m+1)(r+2)(r+1)}{m+r+1}\right]$$
$$\cdot\left[\frac{1}{n+s+1} - \frac{4}{n+s+3} + \frac{6}{n+s+5} - \frac{4}{n+s+7} + \frac{1}{n+s+9}\right]$$

$$K_{mnrs} = \left[\frac{1}{m+r+1} - \frac{2}{m+r+3} + \frac{1}{m+r+5}\right]\left[\frac{n(n-1)s(s-1)}{n+s-3} - \frac{2n(n-1)(s+2)(s+1)+2(n+2)(n+1)s(s-1)}{n+s-1}\right.$$
$$+ \frac{n(n-1)(s+4)(s+3)+4(n+2)(n+1)(s+2)(s+1)+(n+4)(n+3)s(s-1)}{n+s+1} - \frac{2(n+4)(n+3)(s+2)(s+1)}{n+s+3}$$
$$\left. - \frac{2(n+2)(n+1)(s+4)(s+3)}{n+s+3} + \frac{(n+4)(n+3)(s+4)(s+3)}{n+s+5}\right]$$

$$L_{mnrs} = \frac{1}{2}\left[\frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{(m+r-1)(n+s-1)} - \frac{2m(m-1)(s+2)(s+1)+2r(r-1)(n+2)(n+1)}{(m+r-1)(n+s+1)} - \frac{2m(m-1)s(s-1)}{(m+r-1)(n+s+1)}\right.$$
$$- \frac{2r(r-1)n(n-1)}{(m+r-1)(n+s+1)} + \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{(m+r-1)(n+s+3)} + \frac{4m(m-1)(s+2)(s+1)+4r(r-1)(n+2)(n+1)}{(m+r-1)(n+s+3)}$$
$$+ \frac{m(m-1)(s+4)(s+3)+r(r-1)(n+4)(n+3)}{(m+r-1)(n+s+3)} - \frac{2m(m-1)(s+2)(s+1)+2r(r-1)(n+2)(n+1)}{(m+r-1)(n+s+5)}$$
$$- \frac{2m(m-1)(s+4)(s+3)+2r(r-1)(n+4)(n+3)}{(m+r-1)(n+s+5)} + \frac{m(m-1)(s+4)(s+3)+r(r-1)(n+4)(n+3)}{(m+r-1)(n+s+7)}$$
$$- \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{(m+r+1)(n+s-1)} - \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{(m+r+1)(n+s-1)}$$
$$+ \frac{2m(m-1)(s+2)(s+1)+2r(r-1)(n+2)(n+1)}{(m+r+1)(n+s+1)} + \frac{2(m+2)(m+1)(s+2)(s+1)+2(r+2)(r+1)(n+2)(n+1)}{(m+r+1)(n+s+1)}$$
$$+ \frac{2m(m-1)s(s-1)+2r(r-1)n(n-1)}{(m+r+1)(n+s+1)} + \frac{2(m+2)(m+1)s(s-1)+2(r+2)(r+1)n(n-1)}{(m+r+1)(n+s+1)}$$
$$- \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{(m+r+1)(n+s+3)} - \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{(m+r+1)(n+s+3)}$$
$$- \frac{4m(m-1)(s+2)(s+1)+4r(r-1)(n+2)(n+1)}{(m+r+1)(n+s+3)} - \frac{4(m+2)(m+1)(s+2)(s+1)+4(r+2)(r+1)(n+2)(n+1)}{(m+r+1)(n+s+3)}$$

TABLE 3-2 (CONT'D) COEFFICIENT C_{mnrs} - CASE 2

$$- \frac{m(m-1)(s+4)(s+3)+r(r-1)(n+4)(n+3)}{(m+r+1)(n+s+3)} - \frac{(m+2)(m+1)(s+4)(s+3)+(r+2)(r+1)(n+4)(n+3)}{(m+r+1)(n+s+3)}$$

$$+ \frac{2m(m-1)(s+2)(s+1)+2r(r-1)(n+2)(n+1)}{(m+r+1)(n+s+5)} + \frac{2(m+2)(m+1)(s+2)(s+1)+2(r+2)(r+1)(n+2)(n+1)}{(m+r+1)(n+s+5)}$$

$$+ \frac{2m(m-1)(s+4)(s+3)+2r(r-1)(n+4)(n+3)}{(m+r+1)(n+s+5)} + \frac{2(m+2)(m+1)(s+4)(s+3)+2(r+2)(r+1)(n+4)(n+3)}{(m+r+1)(n+s+5)}$$

$$- \frac{m(m-1)(s+4)(s+3)+r(r-1)(n+4)(n+3)}{(m+r+1)(n+s+7)} - \frac{(m+2)(m+1)(s+4)(s+3)+(r+2)(r+1)(n+4)(n+3)}{(m+r+1)(n+s+7)}$$

$$+ \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{(m+r+3)(n+s-1)} - \frac{2(m+2)(m+1)(s+2)(s+1)+2(r+2)(r+1)(n+2)(n+1)}{(m+r+3)(n+s+1)}$$

$$- \frac{2(m+2)(m+1)s(s-1)+2(r+2)(r+1)n(n-1)}{(m+r+3)(n+s+1)} + \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{(m+r+3)(n+s+3)}$$

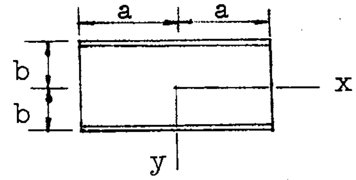
$$+ \frac{4(m+2)(m+1)(s+2)(s+1)+4(r+2)(r+1)(n+2)(n+1)}{(m+r+3)(n+s+3)} + \frac{(m+2)(m+1)(s+4)(s+3)}{(m+r+3)(n+s+3)}$$

$$+ \frac{(r+2)(r+1)(n+4)(n+3)}{(m+r+3)(n+s+3)} - \frac{2(m+2)(m+1)(s+2)(s+1)+2(r+2)(r+1)(n+2)(n+1)}{(m+r+3)(n+s+5)}$$

$$- \frac{2(m+2)(m+1)(s+4)(s+3)+2(r+2)(r+1)(n+4)(n+3)}{(m+r+3)(n+s+5)} + \frac{(m+2)(m+1)(s+4)(s+3)}{(m+r+3)(n+s+7)}$$

$$+ \left. \frac{(r+2)(r+1)(n+4)(n+3)}{(m+r+3)(n+s+7)} \right]$$

TABLE 3-3 COEFFICIENT C_{mnrs} - CASE 3



Deflection surface: $$w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^m \left[\left(\frac{y}{b}\right)^n - \left(\frac{y}{b}\right)^{n+2}\right]$$

Coefficient $$C_{mnrs} = \begin{cases} 0, & \text{if } m+r \text{ or } n+s \text{ is odd;} \\ 4\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}\right], & \text{otherwise.} \end{cases}$$

in which

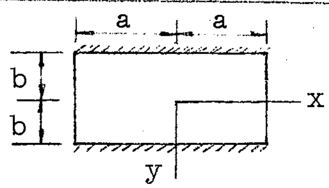
$$J_{mnrs} = \left[\frac{m(m-1)r(r-1)}{m+r-3}\right]\left[\frac{1}{n+s+1} - \frac{2}{n+s+3} + \frac{1}{n+s+5}\right]$$

$$K_{mnrs} = \frac{1}{m+r+1}\left[\frac{n(n-1)s(s-1)}{n+s-3} - \frac{n(n-1)(s+2)(s+1)+(n+2)(n+1)s(s-1)}{n+s-1} + \frac{(n+2)(n+1)(s+2)(s+1)}{n+s+1}\right]$$

$$L_{mnrs} = \frac{1}{2(m+r-1)}\left[\frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{n+s-1} - \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{n+s+1} - \frac{m(m-1)(s+2)(s+1)+r(r-1)(n+2)(n+1)}{n+s+1} + \frac{m(m-1)(s+2)(s+1)+r(r-1)(n+2)(n+1)}{n+s+3}\right]$$

$$M_{mnrs} = \frac{mr}{m+r-1}\left[\frac{ns}{n+s-1} - \frac{n(s+2)+(n+2)s}{n+s+1} + \frac{(n+2)(s+2)}{n+s+3}\right]$$

TABLE 3-4 COEFFICIENT C_{mnrs} - CASE 4



Deflection surface: $w = h \sum_{m} \sum_{n} A_{mn} \left(\frac{x}{a}\right)^{m} \left[\left(\frac{y}{b}\right)^{n} - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+4}\right]$

Coefficient $C_{mnrs} = \begin{cases} 0, & \text{if } m+r \text{ or } n+r \text{ is odd;} \\ 4\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}\right], & \text{otherwise.} \end{cases}$

in which

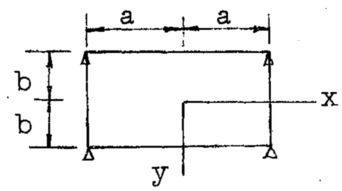
$$J_{mnrs} = \frac{m(m-1)r(r-1)}{m+r-3}\left[\frac{1}{n+s+1} - \frac{4}{n+s+3} + \frac{6}{n+s+5} - \frac{4}{n+s+7} + \frac{1}{n+s+9}\right]$$

$$K_{mnrs} = \frac{1}{m+r+1}\left[\frac{n(n-1)s(s-1)}{n+s-3} - \frac{2n(n-1)(s+2)(s+1)+2(n+2)(n+1)s(s-1)}{n+s-1} + \frac{n(n-1)(s+4)(s+3)+4(n+2)(n+1)(s+2)(s+1)+(n+4)(n+3)s(s-1)}{n+s+1}\right]$$

$$\begin{aligned} L_{mnrs} = \frac{1}{2(m+r-1)}\Bigg[& \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{n+s-1} - \frac{2m(m-1)s(s-1)+2r(r-1)n(n-1)}{n+s+1} \\ & - \frac{2m(m-1)(s+2)(s+1)+2r(r-1)(n+2)(n+1)}{n+s+1} + \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{n+s+3} \\ & + \frac{4m(m-1)(s+2)(s+1)+4r(r-1)(n+2)(n+1)}{n+s+3} + \frac{m(m-1)(s+4)(s+3)+r(r-1)(n+4)(n+3)}{n+s+3} \\ & - \frac{2m(m-1)(s+2)(s+1)+2r(r-1)(n+2)(n+1)}{n+s+5} - \frac{2m(m-1)(s+4)(s+3)+2r(r-1)(n+4)(n+3)}{n+s+5} \\ & + \frac{m(m-1)(s+4)(s+3)+r(r-1)(n+4)(n+3)}{n+s+7}\Bigg] \end{aligned}$$

$$\begin{aligned} M_{mnrs} = \frac{mr}{m+r-1}\Bigg[& \frac{ns}{n+s-1} - \frac{2n(s+2)+2(n+2)s}{n+s+1} + \frac{n(s+4)+4(n+2)(s+2)+(n+4)s}{n+s+3} \\ & - \frac{2(n+2)(s+4)+2(n+4)(s+2)}{n+s+5} + \frac{(n+4)(s+4)}{n+s+7}\Bigg] \end{aligned}$$

TABLE 3-5 COEFFICIENT C_{mnrs} - CASE 5



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left[\left(\frac{x}{a}\right)^m - \left(\frac{x}{a}\right)^{m+2} \right] \left(\frac{y}{b}\right)^n + \left(\frac{x}{a}\right)^m \left[\left(\frac{y}{b}\right)^n - \left(\frac{y}{b}\right)^{n+2} \right]$

Coefficient $C_{mnrs} = \begin{cases} 0, & \text{if } m+r \text{ or } n+s \text{ is odd;} \\ 4\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}\right], & \text{otherwise.} \end{cases}$

in which

$$J_{mnrs} = \left[\frac{m(m-1)r(r-1)}{m+r-3} - \frac{m(m-1)(r+2)(r+1)+(m+2)(m+1)r(r-1)}{m+r-1} + \frac{(m+2)(m+1)(r+2)(r+1)}{m+r+1}\right]\frac{1}{n+s+1}$$

$$+\left[\frac{2m(m-1)r(r-1)}{m+r-3} - \frac{(m+2)(m+1)r(r-1)+m(m-1)(r+2)(r+1)}{m+r-1}\right]\frac{1}{n+s+1} - \left[\frac{2m(m-1)r(r-1)}{m+r-3}\right.$$

$$\left. - \frac{(m+2)(m+1)r(r-1)+m(m-1)(r+2)(r+1)}{m+r-1}\right]\frac{1}{n+s+3} + \frac{m(m-1)r(r-1)}{m+r-3}\left[\frac{1}{n+s+1} - \frac{2}{n+s+3} + \frac{1}{n+s+5}\right]$$

$$K_{mnrs} = \left[\frac{1}{m+r+1} - \frac{2}{m+r+3} + \frac{1}{m+r+5}\right]\frac{n(n-1)s(s-1)}{n+s-3} + \frac{1}{m+r+1}\left[\frac{2n(n-1)s(s-1)}{n+s-3} - \frac{n(n-1)(s+2)(s+1)}{n+s-1}\right.$$

$$\left. - \frac{(n+2)(n+1)s(s-1)}{n+s-1}\right] - \frac{1}{m+r+3}\left[\frac{2n(n-1)s(s-1)}{n+s-3} - \frac{n(n-1)(s+2)(s+1)+(n+2)(n+1)s(s-1)}{n+s-1}\right]$$

$$+ \frac{1}{m+r+1}\left[\frac{n(n-1)s(s-1)}{n+s-3} - \frac{n(n-1)(s+2)(s+1)+(n+2)(n+1)s(s-1)}{n+s-1} + \frac{(n+2)(n+1)(s+2)(s+1)}{n+s+1}\right]$$

$$L_{mnrs} = \left[\frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{2(m+r-1)} - \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{2(m+r+1)} - \frac{(m+2)(m+1)s(s-1)}{2(m+r+1)}\right.$$

$$\left. - \frac{(r+2)(r+1)n(n-1)}{2(m+r+1)} + \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{2(m+r+3)}\right]\frac{1}{n+s-1} + \left[\frac{2m(m-1)s(s-1)}{2(m+r-1)}\right.$$

$$\left. + \frac{2r(r-1)n(n-1)}{2(m+r-1)} - \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{2(m+r+1)} - \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{2(m+r+1)}\right]$$

$$\cdot\left[\frac{1}{n+s-1}\right] - \left[\frac{m(m-1)(s+2)(s+1)+r(r-1)(n+2)(n+1)}{2(m+r-1)} + \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{2(m+r-1)}\right.$$

$$\left. - \frac{(m+2)(m+1)(s+2)(s+1)+(r+2)(r+1)(n+2)(n+1)}{2(m+r+1)} - \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{2(m+r+1)}\right]\frac{1}{n+s+1}$$

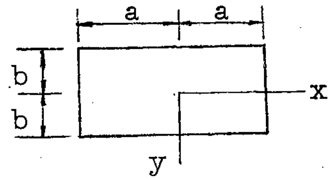
$$+ \frac{1}{m+r-1}\left[\frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{2(n+s-1)} - \frac{m(m-1)(s+2)(s+1)+r(r-1)(n+2)(n+1)}{2(n+s+1)}\right.$$

$$\left. - \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{2(n+s+1)} + \frac{m(m-1)(s+2)(s+1)+r(r-1)(n+2)(n+1)}{2(n+s+3)}\right]$$

$$M_{mnrs} = \left[\frac{mr}{m+r-1} - \frac{m(r+2)+(m+2)r}{m+r+1} + \frac{(m+2)(r+2)}{m+r+3}\right]\frac{ns}{n+s-1} + \left[\frac{2mr}{m+r-1} - \frac{(m+2)r+m(r+2)}{m+r+1}\right]\frac{ns}{n+s-1} - \left[\frac{mr}{m+r-1}\right.$$

$$\left. - \frac{(m+2)r}{m+r+1}\right]\frac{n(s+2)}{n+s+1} - \left[\frac{mr}{m+r-1} - \frac{m(r+2)}{m+r+1}\right]\frac{(n+2)s}{n+s+1} + \frac{mr}{m+r-1}\left[\frac{ns}{n+s-1} - \frac{n(s+2)+(n+2)s}{n+s+1} + \frac{(n+2)(s+2)}{n+s+3}\right]$$

TABLE 3-6 COEFFICIENT C_{mnrs} - CASE 6



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^m \left(\frac{y}{b}\right)^n$

Coefficient $C_{mnrs} = \begin{cases} 0, & \text{if } m+r \text{ or } n+s \text{ is odd;} \\ 4\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu)\ M_{mnrs}\right], & \text{otherwise.} \end{cases}$

in which

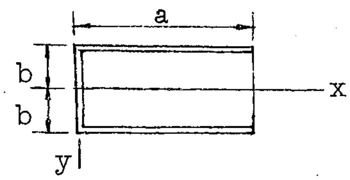
$$J_{mnrs} = \left[\frac{m(m-1)r(r-1)}{m+r-3}\right] \frac{1}{n+s+1}$$

$$K_{mnrs} = \left[\frac{1}{m+r+1}\right] \frac{n(n-1)s(s-1)}{n+s-3}$$

$$L_{mnrs} = \left[\frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{2(m+r-1)}\right] \frac{1}{n+s-1}$$

$$M_{mnrs} = \left[\frac{mr}{m+r-1}\right] \frac{ns}{n+s-1}$$

TABLE 3-7 COEFFICIENT C_{mnrs} - CASE 7



Deflection surface: $w = h \sum_{m} \sum_{n} A_{mn} \left(\frac{x}{a}\right)^{m+1} \left[\left(\frac{y}{b}\right)^{n} - \left(\frac{y}{b}\right)^{n+2}\right]$

Coefficient $C_{mnrs} = \begin{cases} 0, & \text{if } n+s \text{ is odd;} \\ 2\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}\right], & \text{otherwise.} \end{cases}$

in which

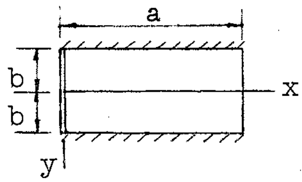
$$J_{mnrs} = \frac{(m+1)m(r+1)r}{m+r-1}\left[\frac{1}{n+s+1} - \frac{2}{n+s+3} + \frac{1}{n+s+5}\right]$$

$$K_{mnrs} = \frac{1}{m+r+3}\left[\frac{n(n-1)s(s-1)}{n+s-3} - \frac{n(n-1)(s+2)(s+1)+(n+2)(n+1)s(s-1)}{n+s-1} + \frac{(n+2)(n+1)(s+2)(s+1)}{n+s+1}\right]$$

$$L_{mnrs} = \frac{1}{2(m+r+1)}\left[\frac{(m+1)ms(s-1)+(r+1)rn(n-1)}{n+s-1} - \frac{(m+1)ms(s-1)+(r+1)rn(n-1)}{n+s+1} - \frac{(m+1)m(s+2)(s+1)+(r+1)r(n+2)(n+1)}{n+s+1} + \frac{(m+1)m(s+2)(s+1)+(r+1)r(n+2)(n+1)}{n+s+3}\right]$$

$$M_{mnrs} = \frac{(m+1)(r+1)}{m+r+1}\left[\frac{ns}{n+s-1} - \frac{n(s+2)+(n+2)s}{n+s+1} + \frac{(n+2)(s+2)}{n+s+3}\right]$$

TABLE 3-8 COEFFICIENT C_{mnrs} - CASE 8



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^{m+1} \left[\left(\frac{y}{b}\right)^{n} - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+4}\right]$

$$\text{Coefficient } C_{mnrs} = \begin{cases} 0, & \text{if } n+s \text{ is odd;} \\ 2\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}\right], & \text{otherwise.} \end{cases}$$

in which

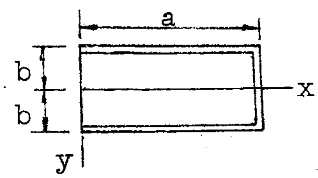
$$J_{mnrs} = \frac{(m+1)m(r+1)r}{m+r-1}\left[\frac{1}{n+s+1} - \frac{4}{n+s+3} + \frac{6}{n+s+5} - \frac{4}{n+s+7} + \frac{1}{n+s+9}\right]$$

$$K_{mnrs} = \frac{1}{m+r+3}\left[\frac{n(n-1)s(s-1)}{n+s-3} - \frac{2n(n-1)(s+2)(s+1)+2(n+2)(n+1)s(s-1)}{n+s-1} + \frac{n(n-1)(s+4)(s+3)}{n+s+1}\right.$$
$$+ \frac{4(n+2)(n+1)(s+2)(s+1)+(n+4)(n+3)s(s-1)}{n+s+1} - \frac{2(n+2)(n+1)(s+4)(s+3)}{n+s+3}$$
$$\left. - \frac{2(n+4)(n+3)(s+2)(s+1)}{n+s+3} + \frac{(n+4)(n+3)(s+4)(s+3)}{n+s+5}\right]$$

$$L_{mnrs} = \frac{1}{2(m+r+1)}\left[\frac{(m+1)ms(s-1)+(r+1)rn(n-1)}{n+s-1} - \frac{2(m+1)ms(s-1)+2(r+1)rn(n-1)}{n+s+1}\right.$$
$$- \frac{2(m+1)m(s+2)(s+1)+2(r+1)r(n+2)(n+1)}{n+s+1} + \frac{(m+1)ms(s-1)+(r+1)rn(n-1)}{n+s+3}$$
$$+ \frac{4(m+1)m(s+2)(s+1)+4(r+1)r(n+2)(n+1)}{n+s+3} + \frac{(m+1)m(s+4)(s+3)+(r+1)r(n+4)(n+3)}{n+s+3}$$
$$- \frac{2(m+1)m(s+2)(s+1)+2(r+1)r(n+2)(n+1)}{n+s+5} + \frac{2(m+1)m(s+4)(s+3)+2(r+1)r(n+4)(n+3)}{n+s+5}$$
$$\left. + \frac{(m+1)m(s+4)(s+3)+(r+1)r(n+4)(n+3)}{n+s+7}\right]$$

$$M_{mnrs} = \frac{(m+1)(r+1)}{m+r+1}\left[\frac{ns}{n+s-1} - \frac{2n(s+2)+2(n+2)s}{n+s+1} + \frac{n(s+4)+4(n+2)(s+2)+(n+4)s}{n+s+3}\right.$$
$$\left. - \frac{2(n+2)(s+4)+2(n+4)(s+2)}{n+s+5} + \frac{(n+4)(s+4)}{n+s+7}\right]$$

TABLE 3-9 COEFFICIENT C_{mnrs} - CASE 9



Deflection surface: $$w = h \sum_m \sum_n A_{mn} \left[\left(\frac{x}{a}\right)^m - \left(\frac{x}{a}\right)^{m+1}\right]\left[\left(\frac{y}{b}\right)^n - \left(\frac{y}{b}\right)^{n+2}\right]$$

Coefficient $$C_{mnrs} = \begin{cases} 0, & \text{if } n+s \text{ is odd;} \\ 2\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}\right], & \text{otherwise.} \end{cases}$$

in which

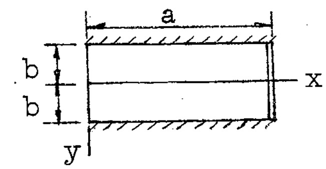
$$J_{mnrs} = \left[\frac{m(m-1)r(r-1)}{m+r-3} - \frac{m(m-1)(r+1)r+(m+1)mr(r-1)}{m+r-2} + \frac{(m+1)m(r+1)r}{m+r-1}\right]\left[\frac{1}{n+s+1} - \frac{2}{n+s+3} + \frac{1}{n+s+5}\right]$$

$$K_{mnrs} = \left[\frac{1}{m+r+1} - \frac{2}{m+r+2} + \frac{1}{m+r+3}\right]\left[\frac{n(n-1)s(s-1)}{n+s-3} - \frac{n(n-1)(s+2)(s+1)+(n+2)(n+1)s(s-1)}{n+s-1} + \frac{(n+2)(n+1)(s+2)(s+1)}{n+s+1}\right]$$

$$\begin{aligned} L_{mnrs} = \frac{1}{2}\Bigg\{ & \left[\frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{n+s-1} - \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{n+s+1} - \frac{m(m-1)(s+2)(s+1)}{n+s+1} \right. \\ & \left. - \frac{r(r-1)(n+2)(n+1)}{n+s+1} + \frac{m(m-1)(s+2)(s+1)+r(r-1)(n+2)(n+1)}{n+s+3}\right]\frac{1}{m+r-1} - \left[\frac{m(m-1)s(s-1)}{n+s-1} \right. \\ & + \frac{r(r-1)n(n-1)}{n+s-1} + \frac{(m+1)ms(s-1)+(r+1)rn(n-1)}{n+s-1} - \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{n+s+1} \\ & - \frac{(m+1)ms(s-1)+(r+1)rn(n-1)}{n+s+1} - \frac{m(m-1)(s+2)(s+1)+r(r-1)(n+2)(n+1)}{n+s+1} - \frac{(m+1)m(s+2)(s+1)}{n+s+1} \\ & - \frac{(r+1)r(n+2)(n+1)}{n+s+1} + \frac{m(m-1)(s+2)(s+1)+r(r-1)(n+2)(n+1)}{n+s+3} + \frac{(m+1)m(s+2)(s+1)}{n+s+3} \\ & \left. + \frac{(r+1)r(n+2)(n+1)}{n+s+3}\right]\frac{1}{m+r} + \left[\frac{(m+1)ms(s-1)+(r+1)rn(n-1)}{n+s-1} - \frac{(m+1)ms(s-1)+(r+1)rn(n-1)}{n+s+1} \right. \\ & \left. - \frac{(m+1)m(s+2)(s+1)+(r+1)r(n+2)(n+1)}{n+s+1} + \frac{(m+1)m(s+2)(s+1)+(r+1)r(n+2)(n+1)}{n+s+3}\right]\frac{1}{m+r+1}\Bigg\} \end{aligned}$$

$$M_{mnrs} = \left[\frac{mr}{m+r-1} - \frac{m(r+1)+(m+1)r}{m+r} + \frac{(m+1)(r+1)}{m+r+1}\right]\left[\frac{ns}{n+s-1} - \frac{n(s+2)+(n+2)s}{n+s+1} + \frac{(n+2)(s+2)}{n+s+3}\right]$$

TABLE 3-10 COEFFICIENT C_{mnrs} - CASE 10



Deflection surface: $w = h \sum_{m} \sum_{n} A_{mn} \left[\left(\frac{x}{a}\right)^{m} - \left(\frac{x}{a}\right)^{m+1} \right] \left[\left(\frac{y}{b}\right)^{n} - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+4} \right]$

$$C_{mnrs} = \begin{cases} 0, & \text{if } n+s \text{ is odd;} \\ 2\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}\right], & \text{otherwise.} \end{cases}$$

in which

$$J_{mnrs} = \left[\frac{m(m-1)r(r-1)}{m+r-3} - \frac{m(m-1)(r+1)r+(m+1)mr(r-1)}{m+r-2} + \frac{(m+1)m(r+1)r}{m+r-1}\right]\left[\frac{1}{n+s+1} - \frac{4}{n+s+3} + \frac{6}{n+s+5} - \frac{4}{n+s+7} + \frac{1}{n+s+9}\right]$$

$$\begin{aligned} K_{mnrs} = {} & \left[\frac{1}{m+r+1} - \frac{2}{m+r+2} + \frac{1}{m+r+3}\right]\left[\frac{n(n-1)s(s-1)}{n+s-3} - \frac{2n(n-1)(s+2)(s+1)+2(n+2)(n+1)s(s-1)}{n+s-1}\right. \\ & + \frac{n(n-1)(s+4)(s+3)+4(n+2)(n+1)(s+2)(s+1)+(n+4)(n+3)s(s-1)}{n+s+1} - \frac{2(n+2)(n+1)(s+4)(s+3)}{n+s+3} \\ & \left. - \frac{2(n+4)(n+3)(s+2)(s+1)}{n+s+3} + \frac{(n+4)(n+3)(s+4)(s+3)}{n+s+5}\right] \end{aligned}$$

$$\begin{aligned} L_{mnrs} = {} & \frac{1}{2(m+r-1)}\left[\frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{n+s-1} - \frac{2m(m-1)s(s-1)+2r(r-1)n(n-1)}{n+s+1} - \frac{2m(m-1)(s+2)(s+1)}{n+s+1}\right. \\ & - \frac{2r(r-1)(n+2)(n+1)}{n+s+1} + \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{n+s+3} + \frac{4m(m-1)(s+2)(s+1)+4r(r-1)(n+2)(n+1)}{n+s+3} \\ & + \frac{m(m-1)(s+4)(s+3)+r(r-1)(n+4)(n+3)}{n+s+3} - \frac{2m(m-1)(s+2)(s+1)+2r(r-1)(n+2)(n+1)}{n+s+5} \\ & \left. - \frac{2m(m-1)(s+4)(s+3)+2r(r-1)(n+4)(n+3)}{n+3+5} + \frac{m(m-1)(s+4)(s+3)+r(r-1)(n+4)(n+3)}{n+s+7}\right] \\ & - \frac{1}{2(m+r)}\left[\frac{m(m-1)s(s-1)+r(r-1)n(n-1)+(m+1)ms(s-1)+(r+1)rn(n-1)}{n+s-1} - \frac{2m(m-1)s(s-1)}{n+s+1}\right. \\ & - \frac{2r(r-1)n(n-1)+2(m+1)ms(s-1)+2(r+1)rn(n-1)}{n+s+1} - \frac{2m(m-1)(s+2)(s+1)+2r(r-1)(n+2)(n+1)}{n+s+1} \\ & - \frac{2(m+1)m(s+2)(s+1)+2(r+1)r(n+2)(n+1)}{n+s+1} + \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{n+s+3} + \frac{(m+1)ms(s-1)}{n+s+3} \\ & + \frac{(r+1)rn(n-1)}{n+s+3} + \frac{4m(m-1)(s+2)(s+1)+4r(r-1)(n+2)(n+1)}{n+s+3} + \frac{4(m+1)m(s+2)(s+1)}{n+s+3} \\ & + \frac{4(r+1)r(n+2)(n+1)}{n+s+3} + \frac{m(m-1)(s+4)(s+3)+r(r-1)(n+4)(n+3)}{n+s+3} + \frac{(m+1)m(s+4)(s+3)}{n+s+3} \\ & + \frac{(r+1)r(n+4)(n+s)}{n+s+3} - \frac{2m(m-1)(s+2)(s+1)+2r(r-1)(n+2)(n+1)}{n+s+5} - \frac{2(m+1)m(s+2)(s+1)}{n+s+5} \end{aligned}$$

TABLE 3-10 (CONT'D) COEFFICIENT C_{mnrs} - CASE 10

$$- \frac{2(r+1)r(n+2)(n+1)}{n+s+5} - \frac{2m(m-1)(s+4)(s+3)+2r(r-1)(n+4)(n+3)}{n+s+5} - \frac{2(m+1)m(s+4)(s+3)}{n+s+5}$$

$$- \frac{2(r+1)r(n+4)(n+3)}{n+s+5} + \frac{m(m-1)(s+4)(s+3)+r(r-1)(n+4)(n+3)}{n+s+7} + \frac{(m+1)m(s+4)(s+3)}{n+s+7}$$

$$+ \left.\frac{(r+1)r(n+4)(n+3)}{n+s+7}\right] + \frac{1}{2(m+r+1)}\left[\frac{(m+1)ms(s-1)+(r+1)rn(n-1)}{n+s-1} - \frac{2(m+1)ms(s-1)+2(r+1)rn(n-1)}{n+s+1}\right.$$

$$- \frac{2(m+1)m(s+2)(s+1)+2(r+1)r(n+2)(n+1)}{n+s+1} + \frac{(m+1)ms(s-1)+(r+1)rn(n-1)}{n+s+3} + \frac{4(m+1)m(s+2)(s+1)}{n+s+3}$$

$$+ \frac{4(r+1)r(n+2)(n+1)}{n+s+3} + \frac{(m+1)m(s+4)(s+3)+(r+1)r(n+4)(n+3)}{n+s+3} - \frac{2(m+1)m(s+2)(s+1)}{n+s+5}$$

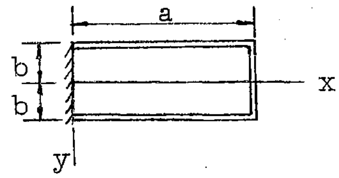
$$- \frac{2(r+1)r(n+2)(n+1)}{n+s+5} - \frac{2(m+1)m(s+4)(s+3)+2(r+1)r(n+4)(n+3)}{n+s+5} + \frac{(m+1)m(s+4)(s+3)}{n+s+7}$$

$$+ \left.\frac{(r+1)r(n+4)(n+3)}{n+s+7}\right]$$

$$M_{mnrs} = \left[\frac{mr}{m+r-1} - \frac{m(r+1)+r(m+1)}{m+r} + \frac{(m+1)(r+1)}{m+r+1}\right]\left[\frac{ns}{n+s-1} - \frac{2n(s+2)+2(n+2)s}{n+s+1} + \frac{n(s+4)+4(n+2)(s+2)}{n+s+3}\right.$$

$$+ \left.\frac{(n+4)s}{n+s+3} - \frac{2(n+2)(s+4)+2(n+4)(s+2)}{n+s+5} + \frac{(n+4)(s+4)}{n+s+7}\right]$$

TABLE 3-11 COEFFICIENT C_{mnrs} - CASE 11



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left[\left(\frac{x}{a}\right)^{m+2} - \left(\frac{x}{a}\right)^{m+3}\right]\left[\left(\frac{y}{b}\right)^{n} - \left(\frac{y}{b}\right)^{n+2}\right]$

Coefficient $C_{mnrs} = \begin{cases} 0, & \text{if } n+s \text{ is odd;} \\ 2\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2 L_{mnrs}\right], & \text{otherwise.} \end{cases}$

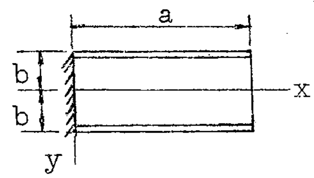
in which

$$J_{mnrs} = \left[\frac{(m+2)(m+1)(r+2)(r+1)}{m+r+1} - \frac{(m+2)(m+1)(r+3)(r+2)+(m+3)(m+2)(r+2)(r+1)}{m+r+2} + \frac{(m+3)(m+2)(r+3)(r+2)}{m+r+3}\right]\left[\frac{1}{n+s+1} - \frac{2}{n+s+3} + \frac{1}{n+s+5}\right]$$

$$K_{mnrs} = \left[\frac{1}{m+r+5} - \frac{2}{m+r+6} + \frac{1}{m+r+7}\right]\left[\frac{n(n-1)s(s-1)}{n+s-3} - \frac{n(n-1)(s+2)(s+1)+(n+2)(n+1)s(s-1)}{n+s-1} + \frac{(n+2)(n+1)(s+2)(s+1)}{n+s+1}\right]$$

$$L_{mnrs} = \frac{1}{2(m+r+3)}\left[\frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{n+s-1} - \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{n+s+1}\right.$$
$$- \frac{(m+2)(m+1)(s+2)(s+1)+(r+2)(r+1)(n+2)(n+1)}{n+s+1} + \frac{(m+2)(m+1)(s+2)(s+1)}{n+s+3}$$
$$\left. + \frac{(r+2)(r+1)(n+2)(n+1)}{n+s+3}\right] - \frac{1}{2(m+r+4)}\left[\frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{n+s-1}\right.$$
$$+ \frac{(m+3)(m+2)s(s-1)+(r+3)(r+2)n(n-1)}{n+s-1} - \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{n+s+1}$$
$$- \frac{(m+2)(m+1)(s+2)(s+1)+(r+2)(r+1)(n+2)(n+1)}{n+s+1} - \frac{(m+3)(m+2)s(s-1)+(r+3)(r+2)n(n-1)}{n+s+1}$$
$$- \frac{(m+3)(m+2)(s+2)(s+1)+(r+3)(r+2)(n+2)(n+1)}{n+s+1} + \frac{(m+2)(m+1)(s+2)(s+1)}{n+s+3}$$
$$\left. + \frac{(r+2)(r+1)(n+2)(n+1)}{n+s+3} + \frac{(m+3)(m+2)(s+2)(s+1)+(r+3)(r+2)(n+2)(n+1)}{n+s+3}\right]$$
$$+ \frac{1}{2(m+r+5)}\left[\frac{(m+3)(m+2)s(s-1)+(r+3)(r+2)n(n-1)}{n+s-1} - \frac{(m+3)(m+2)s(s-1)+(r+3)(r+2)n(n-1)}{n+s+1}\right.$$
$$- \frac{(m+3)(m+2)(s+2)(s+1)+(r+3)(r+2)(n+2)(n+1)}{n+s+1} + \frac{(m+3)(m+2)(s+2)(s+1)}{n+s+3}$$
$$\left. + \frac{(r+3)(r+2)(n+2)(n+1)}{n+s+3}\right]$$

TABLE 3-12 COEFFICIENT C_{mnrs} - CASE 12



Deflection surface: $w = h \sum_{m} \sum_{n} A_{mn} \left(\frac{x}{a}\right)^{m+2} \left[\left(\frac{y}{b}\right)^{n} - \left(\frac{y}{b}\right)^{n+2}\right]$

$$\text{Coefficient } C_{mnrs} = \begin{cases} 0, & \text{if } n+s \text{ is odd;} \\ 2\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}\right], & \text{otherwise.} \end{cases}$$

in which

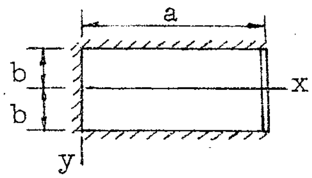
$$J_{mnrs} = \frac{(m+2)(m+1)(r+2)(r+1)}{m+r+1}\left[\frac{1}{n+s+1} - \frac{2}{n+s+3} + \frac{1}{n+s+5}\right]$$

$$K_{mnrs} = \frac{1}{m+r+5}\left[\frac{n(n-1)s(s-1)}{n+s-3} - \frac{n(n-1)(s+2)(s+1)+(n+2)(n+1)s(s-1)}{n+s-1} + \frac{(n+2)(n+1)(s+2)(s+1)}{n+s+1}\right]$$

$$L_{mnrs} = \frac{1}{2(m+r+3)}\left[\frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{n+s-1} - \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{n+s+1}\right.$$
$$- \frac{(m+2)(m+1)(s+2)(s+1)+(r+2)(r+1)(n+2)(n+1)}{n+s+1} + \frac{(m+2)(m+1)(s+2)(s+1)}{n+s+3}$$
$$\left. + \frac{(r+2)(r+1)(n+2)(n+1)}{n+s+3}\right]$$

$$M_{mnrs} = \frac{(m+2)(r+2)}{m+r+3}\left[\frac{ns}{n+s-1} - \frac{n(s+2)+(n+2)s}{n+s+1} + \frac{(n+2)(s+2)}{n+s+3}\right]$$

TABLE 3-13 COEFFICIENT C_{mnrs} - CASE 13



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left[\left(\frac{x}{a}\right)^{m+2} - \left(\frac{x}{a}\right)^{m+3}\right]\left[\left(\frac{y}{b}\right)^{n} - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+4}\right]$

Coefficient $C_{mnrs} = \begin{cases} 0, & \text{if } n+s \text{ is odd;} \\ 2\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2 L_{mnrs}\right], & \text{otherwise.} \end{cases}$

in which

$$\begin{aligned} J_{mnrs} = &\left[\frac{(m+2)(m+1)(r+2)(r+1)}{m+r+1} - \frac{(m+2)(m+1)(r+3)(r+2)+(m+3)(m+2)(r+2)(r+1)}{m+r+2}\right. \\ &\left. + \frac{(m+3)(m+2)(r+3)(r+2)}{m+r+3}\right]\left[\frac{1}{n+s+1} - \frac{4}{n+s+3} + \frac{6}{n+s+5} - \frac{4}{n+s+7} + \frac{1}{n+s+9}\right] \end{aligned}$$

$$\begin{aligned} K_{mnrs} = &\left[\frac{1}{m+r+5} - \frac{2}{m+r+6} + \frac{1}{m+r+7}\right]\left[\frac{n(n-1)s(s-1)}{n+s-3} - \frac{2n(n-1)(s+2)(s+1)+2(n+2)(n+1)s(s-1)}{n+s-1}\right. \\ &+ \frac{n(n-1)(s+4)(s+3)+4(n+2)(n+1)(s+2)(s+1)+(n+4)(n+3)s(s-1)}{n+s+1} - \frac{2(n+2)(n+1)(s+4)(s+3)}{n+s+3} \\ &\left. - \frac{2(n+4)(n+3)(s+2)(s+1)}{n+s+3} + \frac{(n+4)(n+3)(s+4)(s+3)}{n+s+5}\right] \end{aligned}$$

$$\begin{aligned} L_{mnrs} = &\frac{1}{2(m+r+3)}\left[\frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{n+s-1} - \frac{2(m+2)(m+1)s(s-1)+2(r+2)(r+1)n(n-1)}{n+s+1}\right. \\ &- \frac{2(m+2)(m+1)(s+2)(s+1)+2(r+2)(r+1)(n+2)(n+1)}{n+s+1} + \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{n+s+3} \\ &+ \frac{4(m+2)(m+1)(s+2)(s+1)+4(r+2)(r+1)(n+2)(n+1)}{n+s+3} + \frac{(m+2)(m+1)(s+4)(s+3)}{n+s+3} \\ &+ \frac{(r+2)(r+1)(n+4)(n+3)}{n+s+3} - \frac{2(m+2)(m+1)(s+2)(s+1)+2(r+2)(r+1)(n+2)(n+1)}{n+s+5} \\ &- \frac{2(m+2)(m+1)(s+4)(s+3)+2(r+2)(r+1)(n+4)(n+3)}{n+s+5} + \frac{(m+2)(m+1)(s+4)(s+3)}{n+s+7} \\ &\left.+ \frac{(r+2)(r+1)(n+4)(n+3)}{n+s+7}\right] - \frac{1}{2(m+r+4)}\left[\frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{n+s-1}\right. \\ &+ \frac{(m+3)(m+2)s(s-1)+(r+3)(r+2)n(n-1)}{n+s-1} - \frac{2(m+2)(m+1)s(s-1)+2(r+2)(r+1)n(n-1)}{n+s+1} \\ &- \frac{2(m+3)(m+2)s(s-1)+2(r+3)(r+2)n(n-1)}{n+s+1} - \frac{2(m+2)(m+1)(s+2)(s+1)+2(r+2)(r+1)(n+2)(n+1)}{n+s+1} \\ &- \frac{2(m+3)(m+2)(s+2)(s+1)+2(r+3)(r+2)(n+2)(n+1)}{n+s+1} + \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{n+s+3} \\ &+ \frac{(m+3)(m+2)s(s-1)+(r+3)(r+2)n(n-1)}{n+s+3} + \frac{4(m+2)(m+1)(s+2)(s+1)+4(r+2)(r+1)(n+2)(n+1)}{n+s+3} \end{aligned}$$

TABLE 3-13 (CONT'D) COEFFICIENT C_{mnrs} - CASE 13

$$+ \frac{4(m+3)(m+2)(s+2)(s+1)+4(r+3)(r+2)(n+2)(n+1)}{n+s+3} + \frac{(m+2)(m+1)(s+4)(s+3)}{n+s+3}$$

$$+ \frac{(r+2)(r+1)(n+4)(n+3)}{n+s+3} + \frac{(m+3)(m+2)(s+4)(s+3)+(r+3)(r+2)(n+4)(n+3)}{n+s+3}$$

$$- \frac{2(m+2)(m+1)(s+2)(s+1)+2(r+2)(r+1)(n+2)(n+1)}{n+s+5} - \frac{2(m+3)(m+2)(s+2)(s+1)}{n+s+5}$$

$$- \frac{2(r+3)(r+2)(n+2)(n+1)}{n+s+5} - \frac{2(m+2)(m+1)(s+4)(s+3)+2(r+2)(r+1)(n+4)(n+3)}{n+s+5}$$

$$- \frac{2(m+3)(m+2)(s+4)(s+3)+2(r+3)(r+2)(n+4)(n+3)}{n+s+5} + \frac{(m+2)(m+1)(s+4)(s+3)}{n+s+7}$$

$$+ \frac{(r+2)(r+1)(n+4)(n+3)}{n+s+7} + \frac{(m+3)(m+2)(s+4)(s+3)+(r+3)(r+2)(n+4)(n+3)}{n+s+7}\Bigg]$$

$$+ \frac{1}{2(m+r+5)}\Bigg[\frac{(m+3)(m+2)s(s-1)+(r+3)(r+2)n(n-1)}{n+s-1} - \frac{2(m+3)(m+2)s(s-1)+2(r+3)(r+2)n(n-1)}{n+s+1}$$

$$- \frac{2(m+3)(m+2)(s+2)(s+1)+2(r+3)(r+2)(n+2)(n+1)}{n+s+1} + \frac{(m+3)(m+2)s(s-1)+(r+3)(r+2)n(n-1)}{n+s+3}$$

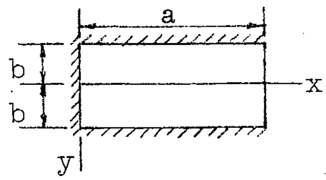
$$+ \frac{4(m+3)(m+2)(s+2)(s+1)+4(r+3)(r+2)(n+2)(n+1)}{n+s+3} + \frac{(m+3)(m+2)(s+4)(s+3)}{n+s+3}$$

$$+ \frac{(r+3)(r+2)(n+4)(n+3)}{n+s+3} - \frac{2(m+3)(m+2)(s+2)(s+1)+2(r+3)(r+2)(n+2)(n+1)}{n+s+5}$$

$$- \frac{2(m+3)(m+2)(s+4)(s+3)+2(r+3)(r+2)(n+4)(n+3)}{n+s+5} + \frac{(m+3)(m+2)(s+4)(s+3)}{n+s+7}$$

$$+ \frac{(r+3)(r+2)(n+4)(n+3)}{n+s+7}\Bigg]$$

TABLE 3-14 COEFFICIENT C_{mnrs} - CASE 14



Deflection surface: $w = b \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^{m+2} \left[\left(\frac{y}{b}\right)^{n} - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+4}\right]$

Coefficient $C_{mnrs} = \begin{cases} 0, & \text{if } n+s \text{ is odd;} \\ 2\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}\right], & \text{otherwise.} \end{cases}$

in which

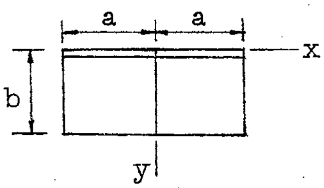
$$J_{mnrs} = \frac{(m+2)(m+1)(r+2)(r+1)}{m+r+1}\left[\frac{1}{n+s+1} - \frac{4}{n+s+3} + \frac{6}{n+s+5} - \frac{4}{n+s+7} + \frac{1}{n+s+9}\right]$$

$$K_{mnrs} = \frac{1}{m+r+5}\left[\frac{n(n-1)s(s-1)}{n+s-3} - \frac{2n(n-1)(s+2)(s+1)+2(n+2)(n+1)s(s-1)}{n+s-1} + \frac{n(n-1)(s+4)(s+3)}{n+s+1}\right.$$
$$+ \frac{4(n+2)(n+1)(s+2)(s+1)}{n+s+1} + \frac{(n+4)(n+3)s(s-1)}{n+s+1} - \frac{2(n+2)(n+1)(s+4)(s+3)}{n+s+3}$$
$$\left. - \frac{2(n+4)(n+3)(s+2)(s+1)}{n+s+3} + \frac{(n+4)(n+3)(s+4)(s+3)}{n+s+5}\right]$$

$$L_{mnrs} = \frac{1}{2(m+r+3)}\left[\frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{n+s-1} - \frac{2(m+2)(m+1)s(s-1)+2(r+2)(r+1)n(n-1)}{n+s+1}\right.$$
$$- \frac{2(m+2)(m+1)(s+2)(s+1)+2(r+2)(r+1)(n+2)(n+1)}{n+s+1} + \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{n+s+3}$$
$$+ \frac{4(m+2)(m+1)(s+2)(s+1)+4(r+2)(r+1)(n+2)(n+1)}{n+s+3} + \frac{(m+2)(m+1)(s+4)(s+3)}{n+s+3}$$
$$+ \frac{(r+2)(r+1)(n+4)(n+3)}{n+s+3} - \frac{2(m+2)(m+1)(s+2)(s+1)+2(r+2)(r+1)(n+2)(n+1)}{n+s+5}$$
$$- \frac{2(m+2)(m+1)(s+4)(s+3)+2(r+2)(r+1)(n+4)(n+3)}{n+s+5} + \frac{(m+2)(m+1)(s+4)(s+3)}{n+s+7}$$
$$\left. + \frac{(r+2)(r+1)(n+4)(n+3)}{n+s+7}\right]$$

$$M_{mnrs} = \frac{(m+2)(r+2)}{m+r+3}\left[\frac{ns}{n+s-1} - \frac{2n(s+2)+2(n+2)s}{n+s+1} + \frac{n(s+4)+4(n+2)(s+2)+(n+4)s}{n+s+3} - \frac{2(n+2)(s+4)}{n+s+5}\right.$$
$$\left. - \frac{2(n+4)(s+2)}{n+s+5} + \frac{(n+4)(s+4)}{n+s+7}\right]$$

TABLE 3-15 COEFFICIENT C_{mnrs} - CASE 15



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^m \left(\frac{y}{b}\right)^{n+1}$

Coefficient $C_{mnrs} = \begin{cases} 0, & \text{if } m+r \text{ is odd;} \\ 2\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}\right], & \text{otherwise.} \end{cases}$

in which

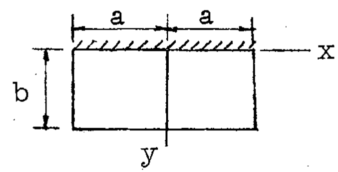
$$J_{mnrs} = \left[\frac{m(m-1)r(r-1)}{m+r-3}\right] \frac{1}{n+s+3}$$

$$K_{mnrs} = \left[\frac{1}{m+r+1}\right] \frac{(n+1)n(s+1)s}{n+s-1}$$

$$L_{mnrs} = \left[\frac{m(m-1)(s+1)s+r(r-1)(n+1)n}{2(m+r-1)}\right] \frac{1}{n+s+1}$$

$$M_{mnrs} = \left[\frac{mr}{m+r-1}\right] \frac{(n+1)(s+1)}{n+s+1}$$

TABLE 3-16 COEFFICIENT C_{mnrs} - CASE 16



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^m \left(\frac{y}{b}\right)^{n+2}$

Coefficient $C_{mnrs} = \begin{cases} 0, & \text{if } m+r \text{ is odd;} \\ 2\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}\right], & \text{otherwise.} \end{cases}$

in which

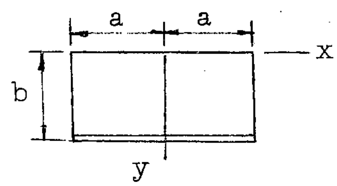
$$J_{mnrs} = \left[\frac{m(m-1)r(r-1)}{m+r-3}\right]\frac{1}{n+s+5}$$

$$K_{mnrs} = \left[\frac{1}{m+r+1}\right]\frac{(n+2)(n+1)(s+2)(s+1)}{n+s+1}$$

$$L_{mnrs} = \left[\frac{m(m-1)(s+2)(s+1)+r(r-1)(n+2)(n+1)}{2(m+r-1)}\right]\frac{1}{n+s+3}$$

$$M_{mnrs} = \left[\frac{mr}{m+r-1}\right]\frac{(n+2)(s+2)}{n+s+3}$$

TABLE 3-17 COEFFICIENT C_{mnrs} - CASE 17



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^m \left[\left(\frac{y}{b}\right)^n - \left(\frac{y}{b}\right)^{n+1}\right]$

$$\text{Coefficient } C_{mnrs} = \begin{cases} 0, & \text{if } m+r \text{ is odd;} \\ 2\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}\right], & \text{otherwise.} \end{cases}$$

in which

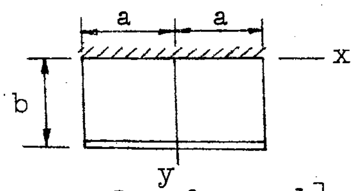
$$J_{mnrs} = \frac{m(m-1)r(r-1)}{m+r-3}\left[\frac{1}{n+s+1} - \frac{2}{n+s+2} + \frac{1}{n+s+3}\right]$$

$$K_{mnrs} = \frac{1}{m+r+1}\left[\frac{n(n-1)s(s-1)}{n+s-3} - \frac{n(n-1)(s+1)s+(n+1)ns(s-1)}{n+s-2} + \frac{(n+1)n(s+1)s}{n+s-1}\right]$$

$$L_{mnrs} = \frac{1}{2(m+r-1)}\left[\frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{n+s-1} - \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{n+s} - \frac{m(m-1)(s+1)s}{n+s} - \frac{r(r-1)(n+1)n}{n+s} + \frac{m(m-1)(s+1)s+r(r-1)(n+1)n}{n+s+1}\right]$$

$$M_{mnrs} = \frac{mr}{m+r-1}\left[\frac{ns}{n+s-1} - \frac{n(s+1)+(n+1)s}{n+s} + \frac{(n+1)(s+1)}{n+s+1}\right]$$

TABLE 3-18 COEFFICIENT C_{mnrs} - CASE 18



Deflection surface: $$w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^m \left[\left(\frac{y}{b}\right)^{n+2} - \left(\frac{y}{b}\right)^{n+3}\right]$$

Coefficient $$C_{mnrs} = \begin{cases} 0, & \text{if } m+r \text{ is odd;} \\ 2\left[\dfrac{b^2}{a^2} J_{mnrs} + \dfrac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}\right], & \text{otherwise.} \end{cases}$$

in which

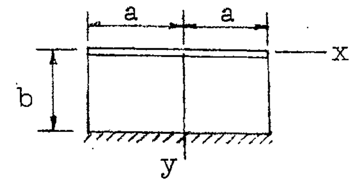
$$J_{mnrs} = \frac{m(m-1)r(r-1)}{m+r-3}\left[\frac{1}{n+s+5} - \frac{2}{n+s+6} + \frac{1}{n+s+7}\right]$$

$$K_{mnrs} = \frac{1}{m+r+1}\left[\frac{(n+2)(n+1)(s+2)(s+1)}{n+s+1} - \frac{(n+2)(n+1)(s+3)(s+2)+(n+3)(n+2)(s+2)(s+1)}{n+s+2} + \frac{(n+3)(n+2)(s+3)(s+2)}{n+s+3}\right]$$

$$L_{mnrs} = \frac{1}{2(m+r-1)}\left[\frac{m(m-1)(s+2)(s+1)+r(r-1)(n+2)(n+1)}{n+s+3} - \frac{m(m-1)(s+2)(s+1)+r(r-1)(n+2)(n+1)}{n+s+4} - \frac{m(m-1)(s+3)(s+2)+r(r-1)(n+3)(n+2)}{n+s+4} + \frac{m(m-1)(s+3)(s+2)+r(r-1)(n+3)(n+2)}{n+s+5}\right]$$

$$M_{mnrs} = \frac{mr}{m+r-1}\left[\frac{(n+2)(s+2)}{n+s+3} - \frac{(n+2)(s+3)+(n+3)(s+2)}{n+s+4} + \frac{(n+3)(s+3)}{n+s+5}\right]$$

TABLE 3-19 COEFFICIENT C_{mnrs} - CASE 19



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^m \left[\left(\frac{y}{b}\right)^{n+1} - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3}\right]$

Coefficient $C_{mnrs} = \begin{cases} 0, & \text{if } m+r \text{ is odd;} \\ 2\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}\right], & \text{otherwise.} \end{cases}$

in which

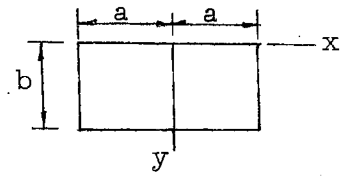
$$J_{mnrs} = \frac{m(m-1)r(r-1)}{m+r-3}\left[\frac{1}{n+s+3} - \frac{4}{n+s+4} + \frac{6}{n+s+5} - \frac{4}{n+s+6} + \frac{1}{n+s+7}\right]$$

$$K_{mnrs} = \frac{1}{m+r+1}\left[\frac{(n+1)n(s+1)s}{n+s-1} - \frac{2(n+1)n(s+2)(s+1)+2(n+2)(n+1)(s+1)s}{n+s} + \frac{(n+1)n(s+3)(s+2)}{n+s+1}\right.$$
$$+ \frac{4(n+2)(n+1)(s+2)(s+1)}{n+s+1} + \frac{(n+3)(n+2)(s+1)s}{n+s+1} - \frac{2(n+2)(n+1)(s+3)(s+2)}{n+s+2}$$
$$\left. - \frac{2(n+3)(n+2)(s+2)(s+1)}{n+s+2} + \frac{(n+3)(n+2)(s+3)(s+2)}{n+s+3}\right]$$

$$L_{mnrs} = \frac{1}{2(m+r-1)}\left[\frac{m(m-1)(s+1)s+r(r-1)(n+1)n}{n+s+1} - \frac{2m(m-1)(s+1)s+2r(r-1)(n+1)n}{n+s+2}\right.$$
$$- \frac{2(m-1)(s+2)(s+1)+2r(r-1)(n+2)(n+1)}{n+s+2} + \frac{m(m-1)(s+1)s+r(r-1)(n+1)n}{n+s+3}$$
$$+ \frac{4m(m-1)(s+2)(s+1)+4r(r-1)(n+2)(n+1)}{n+s+3} + \frac{m(m-1)(s+3)(s+2)+r(r-1)(n+3)(n+2)}{n+s+3}$$
$$- \frac{2m(m-1)(s+2)(s+1)+2r(r-1)(n+2)(n+1)}{n+s+4} - \frac{2m(m-1)(s+3)(s+2)+2r(r-1)(n+3)(n+2)}{n+s+4}$$
$$\left. + \frac{m(m-1)(s+3)(s+2)+r(r-1)(n+3)(n+2)}{n+s+5}\right]$$

$$M_{mnrs} = \frac{mr}{m+r-1}\left[\frac{(n+1)(s+1)}{n+s+1} - \frac{2(n+1)(s+2)+2(n+2)(s+1)}{n+s+2} + \frac{(n+1)(s+3)+4(n+2)(s+2)+(n+3)(s+1)}{n+s+3}\right.$$
$$\left. - \frac{2(n+2)(s+3)+2(n+3)(s+2)}{n+s+4} + \frac{(n+3)(s+3)}{n+s+5}\right]$$

TABLE 3-20 COEFFICIENT C_{mnrs} - CASE 20



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^m \left(\frac{y}{b}\right)^n$

Coefficient $C_{mnrs} = \begin{cases} 0, & \text{if } m+r \text{ is odd;} \\ 2\left[\frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu)\, M_{mnrs}\right], & \text{otherwise.} \end{cases}$

in which

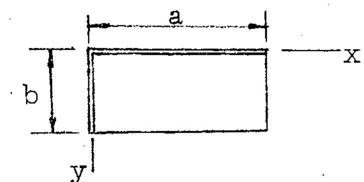
$$J_{mnrs} = \left[\frac{m(m-1)r(r-1)}{m+r-3}\right]\frac{1}{n+s+1}$$

$$K_{mnrs} = \left[\frac{1}{m+r+1}\right]\frac{n(n-1)s(s-1)}{n+s-3}$$

$$L_{mnrs} = \frac{m(m-1)s(s-1)+r(r-1)n(n-1)}{2(m+r-1)(n+s-1)}$$

$$M_{mnrs} = \left[\frac{mr}{m+r-1}\right]\frac{ns}{n+s-1}$$

TABLE 3-21 COEFFICIENT C_{mnrs} - CASE 21



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^{m+1} \left(\frac{y}{b}\right)^{n+1}$

Coefficient $C_{mnrs} = \frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}$.

in which

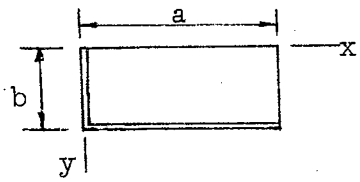
$$J_{mnrs} = \left[\frac{(m+1)m(r+1)r}{m+r-1}\right]\frac{1}{n+s+3}$$

$$K_{mnrs} = \left[\frac{1}{m+r+3}\right]\frac{(n+1)n(s+1)s}{n+s-1}$$

$$L_{mnrs} = \left[\frac{1}{2(m+r+1)}\right]\frac{(m+1)m(s+1)s+(r+1)r(n+1)n}{n+s+1}$$

$$M_{mnrs} = \left[\frac{(m+1)(r+1)}{m+r+1}\right]\frac{(n+1)(s+1)}{n+s+1}$$

TABLE 3-22 COEFFICIENT C_{mnrs} - CASE 22



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^{m+1} \left[\left(\frac{y}{b}\right)^n - \left(\frac{y}{b}\right)^{n+1}\right]$

Coefficient $C_{mnrs} = \frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}$.

in which

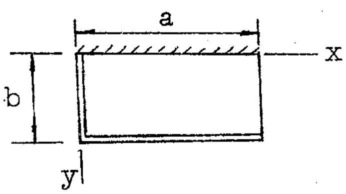
$$J_{mnrs} = \frac{(m+1)m(r+1)r}{m+r-1}\left[\frac{1}{n+s+1} - \frac{2}{n+s+2} + \frac{1}{n+s+3}\right]$$

$$K_{mnrs} = \frac{1}{m+r+3}\left[\frac{n(n-1)s(s-1)}{n+s-3} - \frac{n(n-1)(s+1)s+(n+1)ns(s-1)}{n+s-2} + \frac{(n+1)n(s+1)s}{n+s-1}\right]$$

$$L_{mnrs} = \frac{1}{2(m+r+1)}\left[\frac{(m+1)ms(s-1)+(r+1)rn(n-1)}{n+s-1} - \frac{(m+1)ms(s-1)+(r+1)rn(n-1)}{n+s} - \frac{(m+1)m(s+1)s}{n+s} - \frac{(r+1)r(n+1)n}{n+s} + \frac{(m+1)m(s+1)s+(r+1)r(n+1)n}{n+s+1}\right]$$

$$M_{mnrs} = \frac{(m+1)(r+1)}{m+r+1}\left[\frac{ns}{n+s-1} - \frac{n(s+1)+(n+1)s}{n+s} + \frac{(n+1)(s+1)}{n+s+1}\right]$$

TABLE 3-23 COEFFICIENT C_{mnrs} - CASE 23



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^{m+1} \left[\left(\frac{y}{b}\right)^{n+2} - \left(\frac{y}{b}\right)^{n+3}\right]$

Coefficient $C_{mnrs} = \frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}$.

in which

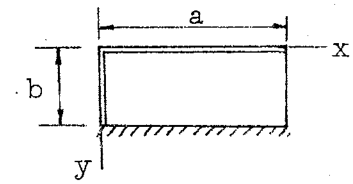
$$J_{mnrs} = \frac{(m+1)m(r+1)r}{m+r-1}\left[\frac{1}{n+s+5} - \frac{2}{n+s+6} + \frac{1}{n+s+7}\right]$$

$$K_{mnrs} = \frac{1}{m+r+3}\left[\frac{(n+2)(n+1)(s+2)(s+1)}{n+s+1} - \frac{(n+2)(n+1)(s+3)(s+2)+(n+3)(n+2)(s+2)(s+1)}{n+s+2} + \frac{(n+3)(n+2)(s+3)(s+2)}{n+s+3}\right]$$

$$L_{mnrs} = \frac{1}{2(m+r+1)}\left[\frac{(m+1)m(s+2)(s+1)+(r+1)r(n+2)(n+1)}{n+s+3} - \frac{(m+1)m(s+2)(s+1)+(r+1)r(n+2)(n+1)}{n+s+4} - \frac{(m+1)m(s+3)(s+2)+(r+1)r(n+3)(n+2)}{n+s+4} + \frac{(m+1)m(s+3)(s+2)+(r+1)r(n+3)(n+2)}{n+s+5}\right]$$

$$M_{mnrs} = \frac{(m+1)(r+1)}{m+r+1}\left[\frac{(n+2)(s+2)}{n+s+3} - \frac{(n+2)(s+3)+(n+3)(s+2)}{n+s+4} + \frac{(n+3)(s+3)}{n+s+5}\right]$$

TABLE 3-24 COEFFICIENT C_{mnrs} - CASE 24



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^{m+1} \left[\left(\frac{y}{b}\right)^{n+1} - 2\left(\frac{y}{b}\right)^{n+2} + \left(\frac{y}{b}\right)^{n+3}\right]$

Coefficient $C_{mnrs} = \frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}$.

in which

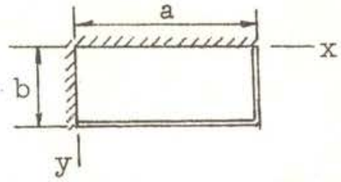
$$J_{mnrs} = \frac{(m+1)m(r+1)r}{m+r-1}\left[\frac{1}{n+s+3} - \frac{4}{n+s+4} + \frac{6}{n+s+5} - \frac{4}{n+s+6} + \frac{1}{n+s+7}\right]$$

$$K_{mnrs} = \frac{1}{m+r+3}\left[\frac{(n+1)n(s+1)s}{n+s-1} - \frac{2(n+1)n(s+2)(s+1)+2(n+2)(n+1)(s+1)s}{n+s} + \frac{(n+1)n(s+3)(s+2)}{n+s+1}\right.$$
$$+ \frac{4(n+2)(n+1)(s+2)(s+1)}{n+s+1} + \frac{(n+3)(n+2)(s+1)s}{n+s+1} - \frac{2(n+2)(n+1)(s+3)(s+2)}{n+s+2}$$
$$\left. - \frac{2(n+3)(n+2)(s+2)(s+1)}{n+s+2} + \frac{(n+3)(n+2)(s+3)(s+2)}{n+s+3}\right]$$

$$L_{mnrs} = \frac{1}{2(m+r+1)}\left[\frac{(m+1)m(s+1)s+(r+1)r(n+1)n}{n+s+1} - \frac{2(m+1)m(s+1)s+2(r+1)r(n+1)n}{n+s+2}\right.$$
$$- \frac{2(m+1)m(s+2)(s+1)+2(r+1)r(n+2)(n+1)}{n+s+2} + \frac{(m+1)m(s+1)s+(r+1)r(n+1)n}{n+s+3}$$
$$+ \frac{4(m+1)m(s+2)(s+1)+4(r+1)r(n+2)(n+1)}{n+s+3} + \frac{(m+1)m(s+3)(s+2)+(r+1)r(n+3)(n+2)}{n+s+3}$$
$$- \frac{2(m+1)m(s+2)(s+1)+2(r+1)r(n+2)(n+1)}{n+s+4} - \frac{2(m+1)m(s+3)(s+2)+2(r+1)r(n+3)(n+2)}{n+s+4}$$
$$\left. + \frac{(m+1)m(s+3)(s+2)+(r+1)r(n+3)(n+2)}{n+s+5}\right]$$

$$M_{mnrs} = \frac{(m+1)(r+1)}{m+r+1}\left[\frac{(n+1)(s+1)}{n+s+1} - \frac{2(n+1)(s+2)+2(n+2)(s+1)}{n+s+2} + \frac{(n+1)(s+3)+4(n+2)(s+2)}{n+s+3}\right.$$
$$\left. + \frac{(n+3)(s+1)}{n+s+3} - \frac{2(n+2)(s+3)+2(n+3)(s+2)}{n+s+4} + \frac{(n+3)(s+3)}{n+s+5}\right]$$

TABLE 3-25 COEFFICIENT C_{mnrs} - CASE 25



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left[\left(\frac{x}{a}\right)^{m+2} - \left(\frac{x}{a}\right)^{m+3}\right]\left[\left(\frac{y}{b}\right)^{n+2} - \left(\frac{y}{b}\right)^{n+3}\right]$

Coefficient $C_{mnrs} = \frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2 L_{mnrs}$.

in which

$$J_{mnrs} = \left[\frac{(m+2)(m+1)(r+2)(r+1)}{m+r+1} - \frac{(m+2)(m+1)(r+3)(r+2)+(m+3)(m+2)(r+2)(r+1)}{m+r+2}\right.$$

$$\left. + \frac{(m+3)(m+2)(r+3)(r+2)}{m+r+3}\right]\left[\frac{1}{n+s+5} - \frac{2}{n+s+6} + \frac{1}{n+s+7}\right]$$

$$K_{mnrs} = \left[\frac{1}{m+r+5} - \frac{2}{m+r+6} + \frac{1}{m+r+7}\right]\left[\frac{(n+2)(n+1)(s+2)(s+1)}{n+s+1} - \frac{(n+2)(n+1)(s+3)(s+2)}{n+s+2}\right.$$

$$\left. - \frac{(n+3)(n+2)(s+2)(s+1)}{n+s+2} + \frac{(n+3)(n+2)(s+3)(s+2)}{n+s+3}\right]$$

$$L_{mnrs} = \frac{1}{2(m+r+3)}\left[\frac{(m+2)(m+1)(s+2)(s+1)+(r+2)(r+1)(n+2)(n+1)}{n+s+3} - \frac{(m+2)(m+1)(s+2)(s+1)}{n+s+4}\right.$$

$$- \frac{(r+2)(r+1)(n+2)(n+1)}{n+s+4} - \frac{(m+2)(m+1)(s+3)(s+2)+(r+2)(r+1)(n+3)(n+2)}{n+s+4}$$

$$\left. + \frac{(m+2)(m+1)(s+3)(s+2)+(r+2)(r+1)(n+3)(n+2)}{n+s+5}\right] - \frac{1}{2(m+r+4)}\left[\frac{(m+2)(m+1)(s+2)(s+1)}{n+s+3}\right.$$

$$+ \frac{(r+2)(r+1)(n+2)(n+1)}{n+s+3} + \frac{(m+3)(m+2)(s+2)(s+1)+(r+3)(r+2)(n+2)(n+1)}{n+s+3}$$

$$- \frac{(m+2)(m+1)(s+2)(s+1)+(r+2)(r+1)(n+2)(n+1)}{n+s+4} - \frac{(m+2)(m+1)(s+3)(s+2)}{n+s+4}$$

$$- \frac{(r+2)(r+1)(n+3)(n+2)}{n+s+4} - \frac{(m+3)(m+2)(s+2)(s+1)+(r+3)(r+2)(n+2)(n+1)}{n+s+4}$$

$$- \frac{(m+3)(m+2)(s+3)(s+2)+(r+3)(r+2)(n+3)(n+2)}{n+s+4} + \frac{(m+2)(m+1)(s+3)(s+2)}{n+s+5}$$

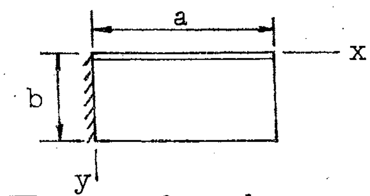
$$\left. + \frac{(r+2)(r+1)(n+3)(n+2)}{n+s+5} + \frac{(m+3)(m+2)(s+3)(s+2)+(r+3)(r+2)(n+3)(n+2)}{n+s+5}\right]$$

$$+ \frac{1}{2(m+r+5)}\left[\frac{(m+3)(m+2)(s+2)(s+1)+(r+3)(r+2)(n+2)(n+1)}{n+s+3} - \frac{(m+3)(m+2)(s+2)(s+1)}{n+s+4}\right.$$

$$- \frac{(r+3)(r+2)(n+2)(n+1)}{n+s+4} - \frac{(m+3)(m+2)(s+3)(s+2)+(r+3)(r+2)(n+3)(n+2)}{n+s+4}$$

$$\left. + \frac{(m+3)(m+2)(s+3)(s+2)+(r+3)(r+2)(n+3)(n+2)}{n+s+5}\right]$$

TABLE 3-26 COEFFICIENT C_{mnrs} - CASE 26



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^{m+2} \left(\frac{y}{b}\right)^{n+1}$

Coefficient $C_{mnrs} = \frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}$.

in which

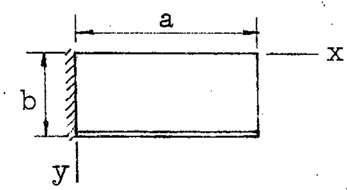
$$J_{mnrs} = \frac{(m+2)(m+1)(r+2)(r+1)}{(m+r+1)(n+s+3)}$$

$$K_{mnrs} = \frac{(n+1)n(s+1)s}{(m+r+5)(n+s-1)}$$

$$L_{mnrs} = \frac{(m+2)(m+1)(s+1)s+(r+2)(r+1)(n+1)n}{2(m+r+3)(n+s+1)}$$

$$M_{mnrs} = \frac{(m+2)(r+2)(n+1)(s+1)}{(m+r+3)(n+s+1)}$$

TABLE 3-27 COEFFICIENT C_{mnrs} - CASE 27



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^{m+2} \left[\left(\frac{y}{b}\right)^n - \left(\frac{y}{b}\right)^{n+1}\right]$

Coefficient $C_{mnrs} = \frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}$.

in which

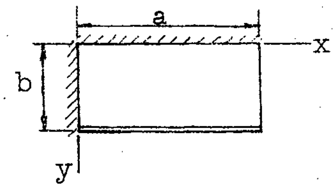
$$J_{mnrs} = \frac{(m+2)(m+1)(r+2)(r+1)}{m+r+1}\left[\frac{1}{n+s+1} - \frac{2}{n+s+2} + \frac{1}{n+s+3}\right]$$

$$K_{mnrs} = \frac{1}{m+r+5}\left[\frac{n(n-1)s(s-1)}{n+s-3} - \frac{n(n-1)(s+1)s+(n+1)ns(s-1)}{n+s-2} + \frac{(n+1)n(s+1)s}{n+s-1}\right]$$

$$L_{mnrs} = \frac{1}{2(m+r+3)}\left[\frac{(m+2(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{n+s-1} - \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{n+s}\right.$$
$$\left. - \frac{(m+2)(m+1)(s+1)s+(r+2)(r+1)(n+1)n}{n+s} + \frac{(m+2)(m+1)(s+1)s+(r+2)(r+1)(n+1)n}{n+s+1}\right]$$

$$M_{mnrs} = \frac{(m+2)(r+2)}{m+r+3}\left[\frac{ns}{n+s-1} - \frac{n(s+1)+(n+1)s}{n+s} + \frac{(n+1)(s+1)}{n+s+1}\right]$$

TABLE 3-28 COEFFICIENT C_{mnrs} - CASE 28



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^{m+2} \left[\left(\frac{y}{b}\right)^{n+2} - \left(\frac{y}{b}\right)^{n+3} \right]$

Coefficient $C_{mnrs} = \frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}$.

in which

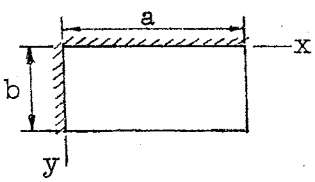
$$J_{mnrs} = \frac{(m+2)(m+1)(r+2)(r+1)}{m+r+1} \left[\frac{1}{n+s+5} - \frac{2}{n+s+6} + \frac{1}{n+s+7} \right]$$

$$K_{mnrs} = \frac{1}{m+r+5} \left[\frac{(n+2)(n+1)(s+2)(s+1)}{n+s+1} - \frac{(n+2)(n+1)(s+3)(s+2)+(n+3)(n+2)(s+2)(s+1)}{n+s+2} + \frac{(n+3)(n+2)(s+3)(s+2)}{n+s+3} \right]$$

$$L_{mnrs} = \frac{1}{2(m+r+3)} \left[\frac{(m+2)(m+1)(s+2)(s+1)+(r+2)(r+1)(n+2)(n+1)}{n+s+3} - \frac{(m+2)(m+1)(s+2)(s+1)}{n+s+4} - \frac{(r+2)(r+1)(n+2)(n+1)}{n+s+4} - \frac{(m+2)(m+1)(s+3)(s+2)+(r+2)(r+1)(n+3)(n+2)}{n+s+4} + \frac{(m+2)(m+1)(s+3)(s+2)+(r+2)(r+1)(n+3)(n+2)}{n+s+5} \right]$$

$$M_{mnrs} = \frac{(m+2)(r+2)}{m+r+3} \left[\frac{(n+2)(s+2)}{n+s+3} - \frac{(n+2)(s+3)+(n+3)(s+2)}{n+s+4} + \frac{(n+3)(s+3)}{n+s+5} \right]$$

TABLE 3-29 COEFFICIENT C_{mnrs} - CASE 29



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^{m+2} \left(\frac{y}{b}\right)^{n+2}$

Coefficient $C_{mnrs} = \frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}$.

in which

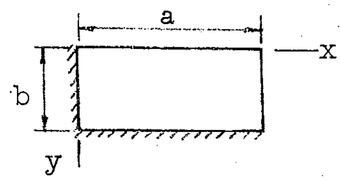
$$J_{mnrs} = \frac{(m+2)(m+1)(r+2)(r+1)}{(m+r+1)(n+s+5)}$$

$$K_{mnrs} = \frac{(n+2)(n+1)(s+2)(s+1)}{(m+r+5)(n+s+1)}$$

$$L_{mnrs} = \frac{(m+2)(m+1)(s+2)(s+1)+(r+2)(r+1)(n+2)(n+1)}{2(m+r+3)(n+s+3)}$$

$$M_{mnrs} = \frac{(m+2)(r+2)(n+2)(s+2)}{(m+r+3)(n+s+3)}$$

TABLE 3-30 COEFFICIENT C_{mnrs} - CASE 30



Deflection surface: $w = h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^{m+2} \left[\left(\frac{y}{b}\right)^n - 2\left(\frac{y}{b}\right)^{n+1} + \left(\frac{y}{b}\right)^{n+2}\right]$

Coefficient $C_{mnrs} = \frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}$.

in which

$$J_{mnrs} = \frac{(m+2)(m+1)(r+2)(r+1)}{m+r+1}\left[\frac{1}{n+s+1} - \frac{4}{n+s+2} + \frac{6}{n+s+3} - \frac{4}{n+s+4} + \frac{1}{n+s+5}\right]$$

$$K_{mnrs} = \frac{1}{m+r+5}\left[\frac{n(n-1)s(s-1)}{n+s-3} - \frac{2n(n-1)(s+1)s+2(n+1)ns(s-1)}{n+s-2} + \frac{n(n-1)(s+2)(s+1)}{n+s-1}\right.$$
$$+ \frac{4(n+1)n(s+1)s}{n+s-1} + \frac{(n+2)(n+1)s(s-1)}{n+s-1} - \frac{2(n+1)n(s+2)(s+1)+2(n+2)(n+1)(s+1)s}{n+s}$$
$$\left. + \frac{(n+2)(n+1)(s+2)(s+1)}{n+s+1}\right]$$

$$L_{mnrs} = \frac{1}{2(m+r+3)}\left[\frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{n+s-1} - \frac{2(m+2)(m+1)s(s-1)+2(r+2)(r+1)n(n-1)}{n+s}\right.$$
$$- \frac{2(m+2)(m+1)(s+1)s+2(r+2)(r+1)(n+1)n}{n+s} + \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{n+s+1}$$
$$+ \frac{4(m+2)(m+1)(s+1)s+4(r+2)(r+1)(n+1)n}{n+s+1} + \frac{(m+2)(m+1)(s+2)(s+1)+(r+2)(r+1)(n+2)(n+1)}{n+s+1}$$
$$- \frac{2(m+2)(m+1)(s+1)s+2(r+2)(r+1)(n+1)n}{n+s+2} - \frac{2(m+2)(m+1)(s+2)(s+1)+2(r+2)(r+1)(n+2)(n+1)}{n+s+2}$$
$$\left. + \frac{(m+2)(m+1)(s+2)(s+1)+(r+2)(r+1)(n+2)(n+1)}{n+s+3}\right]$$

$$M_{mnrs} = \frac{(m+2)(r+2)}{m+r+3}\left[\frac{ns}{n+s-1} - \frac{2n(s+1)+2(n+1)s}{n+s} + \frac{n(s+2)+4(n+1)(s+1)+(n+2)s}{n+s+1} - \frac{2(n+1)(s+2)}{n+s+2}\right.$$
$$\left. - \frac{2(n+2)(s+1)}{n+s+2} + \frac{(n+2)(s+2)}{n+s+3}\right]$$

3-3. Potential Energy of the External Forces for a Basic Rectangular Plate

Consider a rectangular plate acted upon by a load intensity q(x,y) per unit area. The plate deflection may be expressed in the form of expression (3-1). By substituting expression (3-1) into equation (2-4), the potential energy of the total load becomes

$$V_q = -\iint_A q(x, y)\ f\left(\frac{x}{a}, \frac{y}{b}\right) h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^m \left(\frac{y}{b}\right)^n dA \qquad (3\text{-}3)$$

In the case of a concentrated load P acting upon a specific point (x_1, y_1) of the plate, the potential energy of this concentrated load is

$$V_q = -P\ f\left(\frac{x_1}{a}, \frac{y_1}{b}\right) h \sum_m \sum_n A_{mn} \left(\frac{x_1}{a}\right)^m \left(\frac{y_1}{b}\right)^n \qquad (3\text{-}4)$$

For a unit load applied at the point x = a, y = ηb with a differential length $d\eta$, as shown in Fig. 3-2a, the corresponding potential energy of the load is

$$- f(1,\eta)\ h \sum_m \sum_n A_{mn} (\eta)^n\ d\eta$$

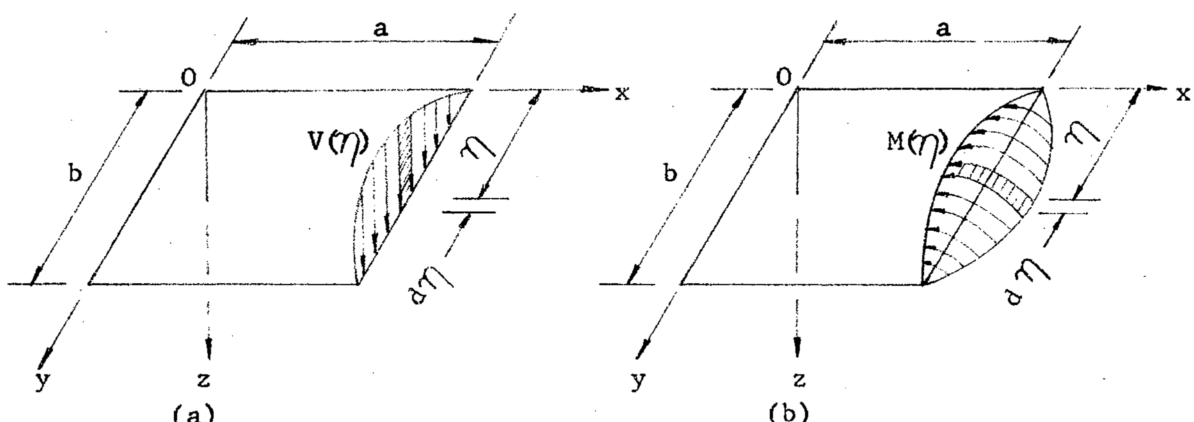


Fig. 3-2.

Edge reactive forces and edge moments

Let $V(\eta)$ be the arbitrary edge reactive force along the edge x = a, the potential energy of the total edge reactive force, V_V, is

$$V_V = -\int_0^1 V(\eta)\ f(1,\eta)\ h \sum_m \sum_n A_{mn}\ (\eta)^n\ d\eta \qquad (3\text{-}5)$$

Similar expressions may be taken for the potential energy of the edge moment, V_M, along the edge x = a, (see Fig. 3-2b)

$$V_M = +\int_0^1 M(\eta) \left\{ \frac{\partial}{\partial x} \left[f\left(\frac{x}{a}, \frac{y}{b}\right) h \sum_m \sum_n A_{mn} \left(\frac{x}{a}\right)^m \left(\frac{y}{b}\right)^n \right] \right\}_{\substack{x = a \\ y = \eta b}} d\eta \qquad (3\text{-}6)$$

Thus, the potential energy of the total external forces, including those of edge reactive force and edge moment, may be written as

$$V = V_q + V_V + V_M \qquad (3\text{-}7)$$

3-4. Strain Energy of the Supporting Beam

Consider a rectangular plate that is supported elastically at the edge x = a by a flexible beam as shown in Fig. 3-3. Let the deflection surface of the plate again be w, and hence the angle of rotation of any section of the beam is $\left(\frac{\partial w}{\partial x}\right)_{x = a}$. By taking the effect of angle of rotation of the plate into account, the deflection of the beam w_b may be expressed as

$$w_b = w - \frac{d}{2}\left(\frac{\partial w}{\partial x}\right)^2_{x = a} \qquad (3\text{-}8)$$

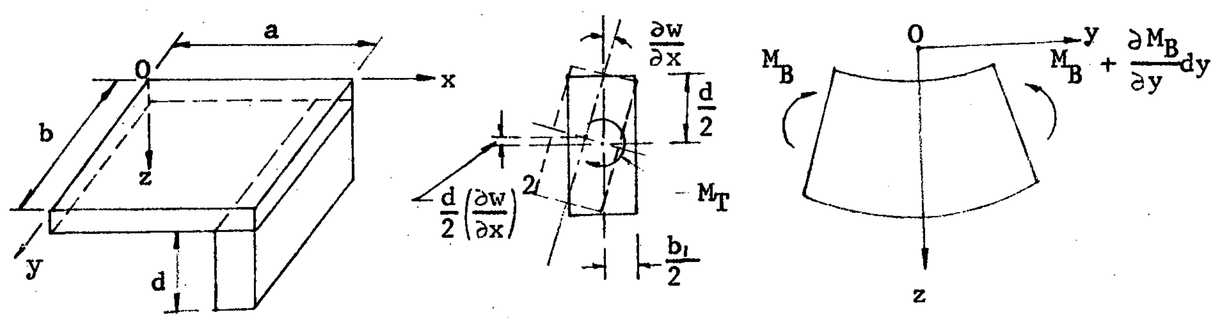


Fig. 3-3.

Twisting and bending of a supporting beam

Let M_T and M_B be the twisting and bending moments in the beam. Taking the direction of the moments shown in Fig. 3-3 as positive, the well-known relations between the deflection and the moments of the beam are obtained:

$$M_T = G_b J_b \frac{\partial^2 w_b}{\partial x \partial y} \tag{3-9}$$

$$M_B = - E_b I_b \frac{\partial^2 w_b}{\partial y^2} \tag{3-10}$$

in which G_b = modulus of rigidity of the beam,

J_b = torsional constant of the beam,

E_b = modulus of elasticity in the beam,

and I_b = moment of inertia of the beam.

By considering only the effect of twisting and bending of the beam, the expression for the strain energy can be represented by

$$U_b = \int_0^b \frac{M_T^2 \, dy}{2G_b J_b} + \int_0^b \frac{M_B^2 \, dy}{2E_b I_b} \tag{3-11}$$

Substituting equations (3-9) and (3-10) into (3-11), the strain energy of the beam U_b can be expressed in terms of the plate deflection w and its derivatives.

For a plate structure with a supporting beam, the strain energy of the beam shown in expression (3-11) should be added to (3-2).

CHAPTER IV

EQUATIONS OF COMPATIBILITY

4-1. Continuous Rectangular Plates with Free Edge at Adjacent Panels

Suppose that a two-span continuous rectangular plate consists of two stable basic panels with free edges at the juncture. These basic panels are isolated from each other and are designated as plates i and i+1 as shown in Fig. 4-1. The corresponding deflection surfaces of the plates

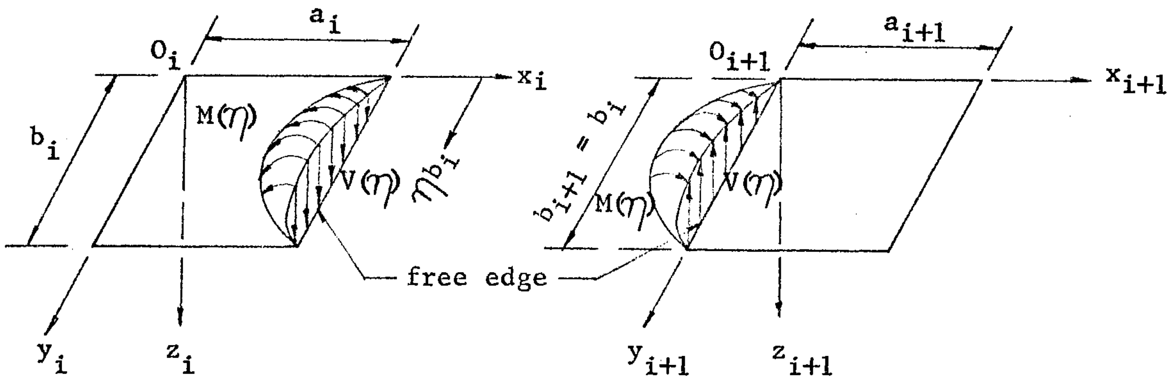


Fig. 4-1.

Stable Basic Plates Freely Supported at the Juncture

i and i+1 are assumed respectively in the forms as

$$w_i = f_i\left(\frac{x_i}{a_i}, \frac{y_i}{b_i}\right) h_i \sum_m \sum_n A_{mn}\left(\frac{x_i}{a_i}\right)^m \left(\frac{y_i}{b_i}\right)^n \tag{4-1a}$$

$$w_{i+1} = f_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}, \frac{y_{i+1}}{b_i}\right) h_{i+1} \sum_j \sum_k B_{jk}\left(\frac{x_{i+1}}{a_{i+1}}\right)^j \left(\frac{y_{i+1}}{b_i}\right)^k \tag{4-1b}$$

In form (4-1b) it has been assumed that the dimensional parameters b_i and b_{i+1} are of equal length.

Following the procedure mentioned in article 3-2, the expressions for the strain energy stored in the plates i and i+1 may be taken in the following forms:

$$U_i = \frac{D_i h_i^2}{2a_i b_i} \sum_m \sum_n \sum_r \sum_s A_{mn} A_{rs} C_{mnrs} \qquad (4\text{-}2a)$$

$$U_{i+1} = \frac{D_{i+1} h_{i+1}^2}{2a_{i+1} b_i} \sum_j \sum_k \sum_p \sum_q B_{jk} B_{pq} D_{jkpq} \qquad (4\text{-}2b)$$

By substituting expressions (3-3), (3-5) and (3-6) into (3-7), the poten -tial energies of the external force for the plates i and i+1, acted upon by the loads, edge reactive force $V(\eta)$ and edge moment $M(\eta)$, are

$$\begin{aligned} V_i = &-\iint_{A_i} q_i(x_i, y_i)\, f_i\left(\frac{x_i}{a_i}, \frac{y_i}{b_i}\right) h_i \sum_r \sum_s A_{rs} \left(\frac{x_i}{a_i}\right)^r \left(\frac{y_i}{b_i}\right)^s dA \\ &-\int_0^1 V(\eta)\, f_i(1,\eta)\, h_i \sum_r \sum_s A_{rs} (\eta)^s\, d\eta \\ &+\int_0^1 M(\eta) \left\{ \frac{\partial}{\partial x_i} \left[f_i\left(\frac{x_i}{a_i}, \eta\right) h_i \sum_r \sum_s A_{rs} \left(\frac{x_i}{a_i}\right)^r (\eta)^s \right] \right\}_{x_i = a_i} d\eta \qquad (4\text{-}3a) \end{aligned}$$

$$\begin{aligned} V_{i+1} = &\iint_{A_{i+1}} q_{i+1}(x_{i+1}, y_{i+1})\, f_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}, \frac{y_{i+1}}{b_i}\right) h_{i+1} \sum_p \sum_q B_{pq} \left(\frac{x_{i+1}}{a_{i+1}}\right)^p \left(\frac{y_{i+1}}{b_i}\right)^q dA \\ &+\int_0^1 V(\eta) \left[f_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}, \eta\right) h_{i+1} \sum_p \sum_q B_{pq} \left(\frac{x_{i+1}}{a_{i+1}}\right)^p (\eta)^q \right]_{x_{i+1}=0} d\eta \\ &-\int_0^1 M(\eta) \left\{ \frac{\partial}{\partial x_{i+1}} \left[f_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}, \eta\right) h_{i+1} \sum_p \sum_q B_{pq} \left(\frac{x_{i+1}}{a_{i+1}}\right)^p (\eta)^q \right] \right\}_{x_{i+1}=0} d\eta \qquad (4\text{-}3b) \end{aligned}$$

In the case when the origin of the coordinate system is arranged at the center of the basic panel or at the mid-point of the edge para-llel to the y-axis, the lower limits of the integral for the potential energies of the edge reactive force and edge moment in expression (4-3) should be replaced by the minus one.

Adding expressions (4-2) and (4-3), the total potential energies of the plates i and i+1, yields

$$
\begin{aligned}
II_i = {} & \frac{D_i h_i^2}{2a_i b_i} \sum_m \sum_n \sum_r \sum_s A_{mn} A_{rs} C_{mnrs} \\
& - \iint_{A_i} q_i(x_i, y_i)\, f_i\left(\frac{x_i}{a_i}, \frac{y_i}{b_i}\right) h_i \sum_r \sum_s A_{rs} \left(\frac{x_i}{a_i}\right)^r \left(\frac{y_i}{b_i}\right)^s dA \\
& - \int_0^1 V(\eta)\, f_i(1,\eta)\, h_i \sum_r \sum_s A_{rs} (\eta)^s\, d\eta \\
& + \int_0^1 M(\eta) \left\{ \frac{\partial}{\partial x_i} \left[f_i\left(\frac{x_i}{a_i}, \eta\right) h_i \sum_r \sum_s A_{rs} \left(\frac{x_i}{a_i}\right)^r (\eta)^s \right] \right\}_{x_i = a_i} d\eta
\end{aligned}
\tag{4-4a}
$$

$$
\begin{aligned}
II_{i+1} = {} & \frac{D_{i+1} h_{i+1}^2}{2a_{i+1} b_i} \sum_j \sum_k \sum_p \sum_q B_{jk} B_{pq} D_{jkpq} \\
& - \iint_{A_{i+1}} q_{i+1}(x_{i+1}, y_{i+1}) f_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}, \frac{y_{i+1}}{b_i}\right) h_{i+1} \sum_p \sum_q B_{pq} \left(\frac{x_{i+1}}{a_{i+1}}\right)^j \left(\frac{y_{i+1}}{b_i}\right)^k dA \\
& + \int_0^1 V(\eta) \left[f_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}, \eta\right) h_{i+1} \sum_p \sum_q B_{pq} \left(\frac{x_{i+1}}{a_{i+1}}\right)^p (\eta)^q \right]_{x_{i+1}=0} d\eta \\
& - \int_0^1 M(\eta) \left\{ \frac{\partial}{\partial x_{i+1}} \left[f_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}, \eta\right) h_{i+1} \sum_p \sum_q B_{pq} \left(\frac{x_{i+1}}{a_{i+1}}\right)^p (\eta)^q \right] \right\}_{x_{i+1}=0} d\eta
\end{aligned}
\tag{4-4b}
$$

From the minimizing conditions

$$\frac{\partial II_i}{\partial A_{rs}} = 0 \qquad (r = 0, 1, 2, \ldots\ldots;\ s = 0, 1, 2, \ldots\ldots)$$

and

$$\frac{\partial II_{i+1}}{\partial B_{pq}} = 0 \qquad (p = 0, 1, 2, \ldots\ldots;\ q = 0, 1, 2, \ldots\ldots)$$

expression (4-4) yields the forms

$$\frac{D_i h_i^2}{a_i b_i} \sum_m \sum_n A_{mn} C_{mnrs}$$

$$= \iint_{A_i} q_i(x_i, y_i)\, f_i\left(\frac{x_i}{a_i}, \frac{y_i}{b_i}\right) h_i \left(\frac{x_i}{a_i}\right)^r \left(\frac{y_i}{b_i}\right)^s dA$$

$$+ \int_0^1 V(\eta)\, f_i(1,\eta)\, h_i\, (\eta)^s\, d\eta$$

$$- \int_0^1 M(\eta) \left\{ \frac{\partial}{\partial x_i} \left[f_i\left(\frac{x_i}{a_i}, \eta\right) h_i \left(\frac{x_i}{a_i}\right)^r (\eta)^s \right] \right\}_{x_i = a_i} d\eta \qquad (4\text{-}5a)$$

$$(r = 0, 1, 2, \ldots\ldots;\ s = 0, 1, 2, \ldots\ldots)$$

$$\frac{D_{i+1} h_{i+1}^2}{a_{i+1} b_i} \sum_j \sum_k B_{jk} D_{jkpq}$$

$$= \iint_{A_{i+1}} q_{i+1}(x_{i+1}, y_{i+1}) f_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}, \frac{y_{i+1}}{b_i}\right) h_{i+1} \left(\frac{x_{i+1}}{a_{i+1}}\right)^p \left(\frac{y_{i+1}}{b_i}\right)^q dA$$

$$- \int_0^1 V(\eta) \left[f_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}, \eta\right) h_{i+1} \left(\frac{x_{i+1}}{a_{i+1}}\right)^p (\eta)^q \right]_{x_{i+1}=0} d\eta$$

$$+ \int_0^1 M(\eta) \left\{ \frac{\partial}{\partial x_{i+1}} \left[f_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}, \eta\right) h_{i+1} \left(\frac{x_{i+1}}{a_{i+1}}\right)^p (\eta)^q \right] \right\}_{x_{i+1}=0} d\eta \qquad (4\text{-}5b)$$

$$(p = 0, 1, 2, \ldots\ldots;\ q = 0, 1, 2, \ldots\ldots)$$

Introducing the following notations for brevity:

$$E^i_{rs} = \iint_{A_i} q_i(x_i, y_i) f_i\left(\frac{x_i}{a_i}, \frac{y_i}{b_i}\right) h_i \left(\frac{x_i}{a_i}\right)^r \left(\frac{y_i}{b_i}\right)^s dA \qquad (4\text{-}6a)$$

$$E^{i+1}_{pq} = \iint_{A_{i+1}} q_{i+1}(x_{i+1}, y_{i+1}) f_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}, \frac{y_{i+1}}{b_i}\right) h_{i+1} \left(\frac{x_{i+1}}{a_{i+1}}\right)^p \left(\frac{y_{i+1}}{b_i}\right)^q dA \qquad (4\text{-}6b)$$

$$F^{i}_{rs}(\eta) = f_i(1,\eta)\ h_i\ (\eta)^s \tag{4-6c}$$

$$F^{i+1}_{pq}(\eta) = -\left[f_{i+1}\left(\frac{x_{i+1}}{a_{i+1}},\eta\right) h_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}\right)^p (\eta)^q\right]_{x_{i+1}=0} \tag{4-6d}$$

$$G^{i}_{rs}(\eta) = -\left\{\frac{\partial}{\partial x_i}\left[f_i\left(\frac{x_i}{a_i},\eta\right) h_i\left(\frac{x_i}{a_i}\right)^r (\eta)^s\right]\right\}_{x_i=a_i} \tag{4-6e}$$

$$G^{i+1}_{pq}(\eta) = \left\{\frac{\partial}{\partial x_{i+1}}\left[f_{i+1}\left(\frac{x_{i+1}}{a_{i+1}},\eta\right) h_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}\right)^p (\eta)^q\right]\right\}_{x_{i+1}=0} \tag{4-6f}$$

expression (4-5) can be simplified as follows:

$$\frac{D_i h_i^2}{a_i b_i}\sum_m\sum_n A_{mn}\ C_{mnrs} = E^i_{rs} + \int_0^1 V(\eta)\ F^i_{rs}(\eta)\ d\eta + \int_0^1 M(\eta)\ G^i_{rs}\ (\eta)\ d\eta$$

$$(r = 0, 1, 2,\ldots\ldots;\ s = 0, 1, 2,\ldots\ldots) \tag{4-7a}$$

$$\frac{D_{i+1} h_{i+1}^2}{a_{i+1} b_i}\sum_j\sum_k B_{jk}\ D_{jkpq} = E^{i+1}_{pq} + \int_0^1 V(\eta)\ F^{i+1}_{pq}(\eta)\ d\eta + \int_0^1 M(\eta)\ G^{i+1}_{pq}(\eta)\ d\eta$$

$$(p = 0, 1, 2,\ldots\ldots;\ q = 0, 1, 2,\ldots\ldots) \tag{4-7b}$$

Interpretation of the above expressions shows that there are two infinite systems of equations for infinitely many unknown parameters A_{mn} and B_{jk}, plus two unknown edge reactive functions $V(\eta)$ and $M(\eta)$. It is impossible to determine all of these infinite unknowns. The indices r and s or p and q may be varied to limited values so that only two finite sets of equations will be formed. For the unknown functions $V(\eta)$ and and $M(\eta)$, two additional equations may be obtained from the conditions of continuity along the juncture of two adjacent panels. The conditions of continuity require

$$\left(w_i\right)_{x_i=a_i} = \left(w_{i+1}\right)_{x_{i+1}=0} \tag{4-8}$$

$$\left(\frac{\partial w_i}{\partial x_i}\right)_{x_i=a_i} = \left(\frac{\partial w_{i+1}}{\partial x_{i+1}}\right)_{x_{i+1}=0} \tag{4-9}$$

These equations are often referred to as the equations of compatibility. Substituting expressions (4-1a) and (4-1b) into the equation (4-8) and observing that $y_i = y_{i+1}$, the following equation is readily obtained:

$$h_i \sum_m \sum_n A_{mn}\left[f_i\left(\frac{x_i}{a_i},\frac{y_i}{b_i}\right)\left(\frac{x_i}{a_i}\right)^m\left(\frac{y_i}{b_i}\right)^n\right]_{x_i=a_i}$$
$$= h_{i+1} \sum_j \sum_k B_{jk}\left[f_{i+1}\left(\frac{x_{i+1}}{a_{i+1}},\frac{y_i}{b_i}\right)\left(\frac{x_{i+1}}{a_{i+1}}\right)^j\left(\frac{y_i}{b_i}\right)^k\right]_{x_{i+1}=0} \tag{4-10}$$

Taking the partial differentials of the expressions (4-1a) and (4-1b) with respect to x_i and x_{i+1}, respectively, and substituting them in the expression (4-9), gives

$$h_i \sum_m \sum_n A_{mn}\left\{\frac{\partial}{\partial x_i}\left[f_i\left(\frac{x_i}{a_i},\frac{y_i}{b_i}\right)\left(\frac{x_i}{a_i}\right)^m\left(\frac{y_i}{b_i}\right)^n\right]\right\}_{x_i=a_i}$$
$$= h_{i+1} \sum_j \sum_k B_{jk}\left\{\frac{\partial}{\partial x_{i+1}}\left[f_{i+1}\left(\frac{x_{i+1}}{a_{i+1}},\frac{y_i}{b_i}\right)\left(\frac{x_{i+1}}{a_{i+1}}\right)^j\left(\frac{y_i}{b_i}\right)^k\right]\right\}_{x_{i+1}=0} \tag{4-11}$$

For the convenience of numerical calculations, expressions (4-5a) and (4-5b) may be arranged in the matrix forms. Let the double indices rs and mn be replaced by the indices u and v, respectively, and observing that mn and rs are really the same thing, the following identities can then be introduced:

$$A_v = A_{mn}$$
$$C_{uv} = C_{mnrs}$$
$$E^i_u = E^i_{rs} \tag{4-12a}$$
$$F^i_u(\eta) = F^i_{rs}(\eta)$$
$$G^i_u(\eta) = G^i_{rs}(\eta)$$

In a similar manner, the indices h and t may replace the double indices pq and jk, respectively,

$$
\begin{aligned}
B_t &= B_{jk} \\
D_{ht} &= D_{jkpq} \\
E_h^{i+1} &= E_{pq}^{i+1} \\
F_h^{i+1}(\eta) &= F_{pq}^{i+1}(\eta) \\
G_h^{i+1}(\eta) &= G_{pq}^{i+1}(\eta)
\end{aligned}
\qquad (4\text{-}12b)
$$

If n terms of A_v are taken, they may be arranged in an orderly column matrix $\left[A_v\right]$. The corresponding algebraic coefficient matrix $\left[C_{uv}\right]$ will have dimensions of (n x n). The matrices $\left[E_u^i\right]$, $\left[F_u^i(\eta)\right]$ and $\left[G_u^i(\eta)\right]$ are all (n x 1) column matrices. The same conditions hold true for the matrices of the plate i+1 if k finite terms of B_t are considered. Thus, the matrix forms for expressions (4-7a) and (4-7b) can be written as

$$\frac{D_i h_i^2}{a_i b_i}\left[C_{uv}\right]\left[A_v\right] = \left[E_u^i\right] + \int_0^1 V(\eta)\left[F_u^i(\eta)\right]d\eta + \int_0^1 M(\eta)\left[G_u^i(\eta)\right]d\eta \qquad (4\text{-}13)$$

$$\frac{D_{i+1} h_{i+1}^2}{a_{i+1} b_i}\left[D_{ht}\right]\left[B_t\right] = \left[E_h^{i+1}\right] + \int_0^1 V(\eta)\left[F_h^{i+1}(\eta)\right]d\eta + \int_0^1 M(\eta)\left[G_h^{i+1}(\eta)\right]d\eta \qquad (4\text{-}14)$$

Multiplying both sides of these matrices by $\frac{a_i b_i}{D_i h_i^2}\left[C_{uv}\right]^{-1}$ and $\frac{a_{i+1} b_i}{D_{i+1} h_{i+1}^2}\left[D_{ht}\right]^{-1}$, expressions for $\left[A_v\right]$ and $\left[B_t\right]$ are obtained:

$$\left[A_v\right] = \frac{a_i b_i}{D_i h_i^2}\left\{\left[C_{uv}\right]^{-1}\left[E_u^i\right] + \int_0^1 V(\eta)\left[C_{uv}\right]^{-1}\left[F_u^i(\eta)\right]d\eta + \int_0^1 M(\eta)\left[C_{uv}\right]^{-1}\left[G_u^i(\eta)\right]d\eta\right\} \qquad (4\text{-}15)$$

$$\left[B_t\right] = \frac{a_{i+1}b_i}{D_{i+1}h_{i+1}^2}\left\{\left[D_{ht}\right]^{-1}\left[E_h^{i+1}\right] + \int_0^1 V(\eta)\left[D_{ht}\right]^{-1}\left[F_h^{i+1}(\eta)\right] d\eta + \int_0^1 M(\eta)\left[D_{ht}\right]^{-1}\left[G_h^{i+1}(\eta)\right] d\eta\right\} \quad (4\text{-}16)$$

in which $\left[C_{uv}\right]^{-1}$ and $\left[D_{ht}\right]^{-1}$ are the inverses of the matrices $\left[C_{uv}\right]$ and $\left[D_{ht}\right]$, respectively.

By converting the assumed deflection surfaces of the plate into the matrix forms and denoting $\left[A_v\right]^T$ and $\left[B_t\right]^T$ as the transposes of the matrices $\left[A_v\right]$ and $\left[B_t\right]$, respectively, the first equation of compatibility may now be expressed as follows:

$$h_i\left[A_v\right]^T\left[(x_iy_i)_v\right]_{x_i=a_i} = h_{i+1}\left[B_t\right]^T\left[(x_{i+1}y_i)_t\right]_{x_{i+1}=0} \quad (4\text{-}17)$$

in which $\left[(x_iy_i)_v\right]$ and $\left[(x_{i+1}y_i)_t\right]$ are column matrices in polynomials. Forming the derivatives of the expressions (4-1a) and (4-1b), and substituting them into the second equation of compatibility, leads to

$$h_i\left[A_v\right]^T\left[\frac{\partial}{\partial x_i}(x_iy_i)_v\right]_{x_i=a_i} = h_{i+1}\left[B_t\right]^T\left[\frac{\partial}{\partial x_{i+1}}(x_{i+1}y_i)_t\right]_{x_{i+1}=0} \quad (4\text{-}18)$$

The transpose theorem of the matrices may be used to obtain $\left[A_v\right]^T$ and $\left[B_t\right]^T$ from expressions (4-15) and (4-16). By substituting $\left[A_v\right]^T$ and $\left[B_t\right]^T$ into equations (4-17) and (4-18), simplifying and collecting terms, two simultaneous integral equations consisting of two unknown functions $V(\eta)$ and $M(\eta)$ are obtained:

$$\int_0^1 V(\eta)\left\{\left[F_u^i(\eta)\right]^T\left[C_{uv}\right]^*\left[(x_i y_i)_v\right]_{x_i=a_i} - \frac{a_{i+1}D_i}{a_i D_{i+1}}\left[F_h^{i+1}(\eta)\right]^T\left[D_{ht}\right]^*\left[(x_{i+1}y_i)_t\right]_{x_{i+1}=0}\right\}d\eta$$

$$+\int_0^1 M(\eta)\left\{\left[G_u^i(\eta)\right]^T\left[C_{uv}\right]^*\left[(x_i y_i)_v\right]_{x_i=a_i} - \frac{a_{i+1}D_i}{a_i D_{i+1}}\left[G_h^{i+1}(\eta)\right]^T\left[D_{ht}\right]^*\left[(x_{i+1}y_i)_t\right]_{x_{i+1}=0}\right\}d\eta$$

$$= \frac{a_{i+1}D_i}{a_i D_{i+1}}\left[E_h^{i+1}\right]^T\left[D_{ht}\right]^*\left[(x_{i+1}y_i)_t\right]_{x_{i+1}=0} - \left[E_u^i\right]^T\left[C_{uv}\right]^*\left[(x_i y_i)_v\right]_{x_i=a_i} \qquad (4\text{-}19)$$

$$\int_0^1 V(\eta)\left\{\left[F_u^i(\eta)\right]^T\left[C_{uv}\right]^*\left[\frac{\partial}{\partial x_i}(x_i y_i)_v\right]_{x_i=a_i} - \frac{a_{i+1}D_i}{a_i D_{i+1}}\left[F_h^{i+1}(\eta)\right]^T\left[D_{ht}\right]^*\left[\frac{\partial}{\partial x_{i+1}}(x_{i+1}y_i)_t\right]_{x_{i+1}=0}\right\}d\eta$$

$$+\int_0^1 M(\eta)\left\{\left[G_u^i(\eta)\right]^T\left[C_{uv}\right]^*\left[\frac{\partial}{\partial x_i}(x_i y_i)_v\right]_{x_i=a_i} - \frac{a_{i+1}D_i}{a_i D_{i+1}}\left[G_h^{i+1}(\eta)\right]^T\left[D_{ht}\right]^*\left[\frac{\partial}{\partial x_{i+1}}(x_{i+1}y_i)_t\right]_{x_{i+1}=0}\right\}d\eta$$

$$= \frac{a_{i+1}D_i}{a_i D_{i+1}}\left[E_h^{i+1}\right]^T\left[D_{ht}\right]^*\left[\frac{\partial}{\partial x_{i+1}}(x_{i+1}y_i)_t\right]_{x_{i+1}=0} - \left[E_u^i\right]^T\left[C_{uv}\right]^*\left[\frac{\partial}{\partial x_i}(x_i y_i)_v\right]_{x_i=a_i} \qquad (4\text{-}20)$$

where $\left[C_{uv}\right]^*$ and $\left[D_{ht}\right]^*$ are the transposes of $\left[C_{uv}\right]^{-1}$ and $\left[D_{ht}\right]^{-1}$, respectively. The integral equations are called the Fredholm equations of the first kind with separately unsymmetric kernels.

The determination of the unknown functions $V(\eta)$ and $M(\eta)$ may be accomplished by a "matching process." An appropriate polynomial series possessing only a number of finite terms is assumed for each unknown functions:

$$V(\eta) = \sum_{\beta=0}^{K} v_\beta\, \eta^\beta \qquad (4\text{-}21)$$

$$M(\eta) = \sum_{\gamma=0}^{N} m_\gamma\, \eta^\gamma \qquad (4\text{-}22)$$

Substituting these series into equations (4-19) and (4-20) and integrating, the matching process that the equations can be satisfied for all values of y only if the coefficients of the corresponding power of y at both sides of the equations are equal is followed. Thus, the approximate solutions of $V(\eta)$ and $M(\eta)$ can be obtained. Substituting these functions into expressions (4-15) and (4-16), the unknown parameters A_{mn} and B_{jk} for the deflection surfaces of the plate may be evaluated.

With the procedure mentioned above, continuous rectangular plates with several panels in one or two directions can be analyzed.

4-2. Continuous Rectangular Plates with Simply Supported Edge at Adjacent Panels

In the case of a continuous-rectangular-plate that consists of two stably basic plates, simply supported at the juncture(see Fig. 4-2), the assumed deflection surfaces of the plate will be expressed in the form of equations (4-1a) and (4-1b). In a similar manner as described in the last article, the following matrix equations may be obtained

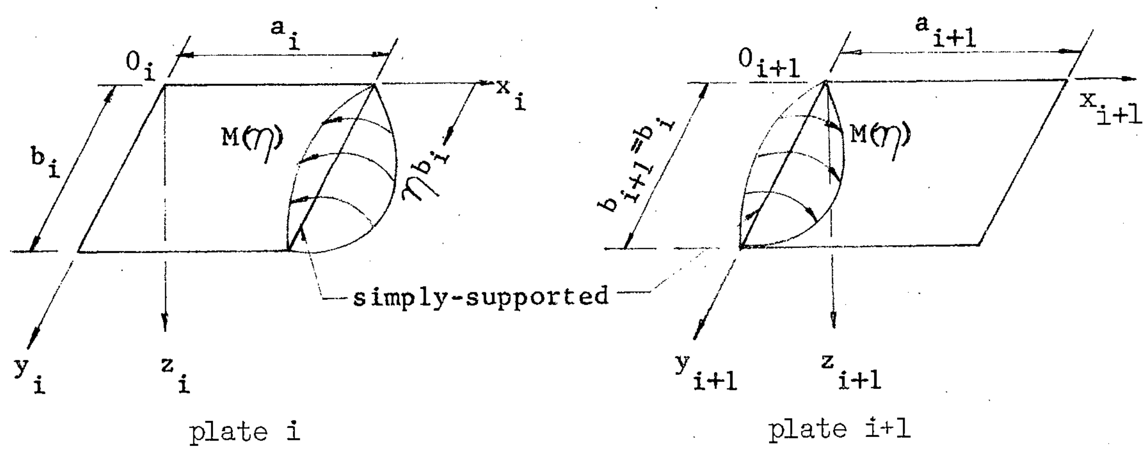


Fig. 4-2.

Stably Basic Plates Simply Supported at the Juncture

from the potential energy expression,

$$\left[A_v\right] = \frac{a_i b_i}{D_i h_i^2}\left\{\left[C_{uv}\right]^{-1}\left[E_u^i\right] + \int_0^1 M(\eta)\left[C_{uv}\right]^{-1}\left[G_u^i(\eta)\right] d\eta\right\} \qquad (4\text{-}23)$$

$$\left[B_t\right] = \frac{a_{i+1} b_i}{D_{i+1} h_{i+1}^2}\left\{\left[D_{ht}\right]^{-1}\left[E_h^{i+1}\right] + \int_0^1 M(\eta)\left[D_{ht}\right]^{-1}\left[G_h^{i+1}(\eta)\right] d\eta\right\} \qquad (4\text{-}24)$$

Note that only the external loads and edge moment function $M(\eta)$ are encountered for the potential energy of the external forces. From the con-dition of continuity, a single Fredholm integral equation of the first kind may be obtained from the use of expressions (4-23) and (4-24),

$$\int_0^1 M(\eta)\left\{\left[G_u^i(\eta)\right]^T\left[C_{uv}\right]^*\left[\frac{\partial}{\partial x_i}(x_i y_i)_v\right]_{x_i=a_i} - \frac{a_{i+1}D_i}{a_i D_{i+1}}\left[G_h^{i+1}(\eta)\right]^T\left[D_{ht}\right]^*\left[\frac{\partial}{\partial x_{i+1}}(x_{i+1}y_i)_t\right]_{x_{i+1}=0}\right\} d\eta$$

$$= \frac{a_{i+1}D_i}{a_i D_{i+1}}\left[E_h^{i+1}\right]^T\left[D_{ht}\right]^*\left[\frac{\partial}{\partial x_{i+1}}(x_{i+1}y_i)_t\right]_{x_{i+1}=0} - \left[E_u^i\right]^T\left[C_{uv}\right]^*\left[\frac{\partial}{\partial x_i}(x_i y_i)_v\right]_{x_i=a_i} \qquad (4\text{-}25)$$

The unknowns $M(\eta)$, A_v and B_t may be determined by the procedure outlined in article 4-1.

A rectangular plate structure that is continuous in one or two directions and is simply supported at intermediate points may also be considered.

4-3. Constraint Equations for Unstable Basic Panels

An investigation of the constant coefficient matrix $\left[D_{ht}\right]$ for a basic panel of the free-free plate i+1 (Fig. 4-1) shows that it is a singular matrix with the elements associated with the unknown parameters B_{00}, B_{01} and B_{10} being zero. Therefore, for a continuous-plate consisting of such an unstable basic panel, equations (4-19) and (4-20) must be modified before solving for the shear and moment functions. However, a procedure analogous to that of obtaining the matrix equation (4-16) may still be followed and an expression for the unknown matrix $\left[B_t'\right]$, after deleting the elements associated with the parameters B_{00}, B_{01} and B_{10}, is obtained:

$$\left[B_t'\right] = \frac{a_{i+1}b_i}{D_{i+1}h_{i+1}^2}\left[D_{ht}'\right]^{-1}\left[E_h'^{i+1}\right] + \int_0^1 V(\eta)\left[D_{ht}'\right]^{-1}\left[F_h'^{i+1}(\eta)\right]d\eta + \int_0^1 M(\eta)\left[D_{ht}'\right]^{-1}\left[G_h'^{i+1}(\eta)\right]d\eta \quad (4\text{-}26)$$

in which $\left[D_{ht}'\right]^{-1}$, $\left[E_h'^{i+1}\right]$, $\left[F_h'^{i+1}(\eta)\right]$ and $\left[G_h'^{i+1}(\eta)\right]$ are the correspondently modified matrices.

The following expressions, similar to the equations (4-19) and (4-20), may be obtained by substitution of equation (4-26) into the equations of compatibility for the adjacent panels i and i+1.

$$\int_0^1 V(\eta)\left\{\left[F_u^i(\eta)\right]^T\left[C_{uv}\right]^*\left[(x_iy_i)_v\right]_{x_i=a_i} - \frac{a_{i+1}D_i}{a_iD_{i+1}}\left[F_h'^{i+1}(\eta)\right]^T\left[D_{ht}'\right]^*\left[(x_{i+1}y_i)_t'\right]_{x_{i+1}=0}\right\}d\eta$$
$$+\int_0^1 M(\eta)\left\{\left[G_u^i(\eta)\right]^T\left[C_{uv}\right]^*\left[(x_iy_i)_v\right]_{x_i=a_i} - \frac{a_{i+1}D_i}{a_iD_{i+1}}\left[G_h'^{i+1}(\eta)\right]^T\left[D_{ht}'\right]^*\left[(x_{i+1}y_i)_t'\right]_{x_{i+1}=0}\right\}d\eta$$
$$+\frac{D_{i+1}h_{i+1}^2}{a_{i+1}b_i}\left[B_{00}\ B_{01}\ B_{10}\right]\begin{bmatrix}1\\ y_i/b_i\\ x_{i+1}/a_{i+1}\end{bmatrix}_{x_{i+1}=0}$$
$$= \frac{a_{i+1}D_i}{a_iD_{i+1}}\left[E_h'^{i+1}\right]^T\left[D_{ht}'\right]^*\left[(x_{i+1}y_i)_t'\right]_{x_{i+1}=0} - \left[E_u^i\right]^T\left[C_{uv}\right]^*\left[(x_iy_i)_v\right]_{x_i=a_i} \quad (4\text{-}27)$$

$$\int_0^1 V(\eta)\left\{\left[F_u^i(\eta)\right]^T\left[C_{uv}\right]^*\left[\frac{\partial}{\partial x_i}(x_iy_i)_v\right]_{x_i=a_i} - \frac{a_{i+1}D_i}{a_iD_{i+1}}\left[F_h'^{i+1}(\eta)\right]^T\left[D_{ht}'\right]^*\left[\frac{\partial}{\partial x_{i+1}}(x_{i+1}y_i)_t'\right]_{x_{i+1}=0}\right\}d\eta$$
$$+\int_0^1 M(\eta)\left\{\left[G_u^i(\eta)\right]^T\left[C_{uv}\right]^*\left[\frac{\partial}{\partial x_i}(x_iy_i)_v\right]_{x_i=a_i} - \frac{a_{i+1}D_i}{a_iD_{i+1}}\left[G_h'^{i+1}(\eta)\right]^T\left[D_{ht}'\right]^*\left[\frac{\partial}{\partial x_{i+1}}(x_{i+1}y_i)_t'\right]_{x_{i+1}=0}\right\}d\eta$$
$$+\frac{D_{i+1}h_{i+1}^2}{a_{i+1}b_i}\left[B_{00}\ B_{01}\ B_{10}\right]\begin{bmatrix}0\\ 0\\ 1/a_{i+1}\end{bmatrix}$$
$$= \frac{a_{i+1}D_i}{a_iD_{i+1}}\left[E_h'^{i+1}\right]^T\left[D_{ht}'\right]^*\left[\frac{\partial}{\partial x_{i+1}}(x_{i+1}y_i)_t'\right]_{x_{i+1}=0} - \left[E_u^i\right]^T\left[C_{uv}\right]^*\left[\frac{\partial}{\partial x_i}(x_iy_i)_v\right]_{x_i=a_i} \quad (4\text{-}28)$$

in which, $\left[(x_{i+1}y_i)'\right]_{x_{i+1}=0}$ and $\left[\frac{\partial}{\partial x_{i+1}}(x_{i+1}y_i)'\right]_{x_{i+1}=0}$, denote the modified polynomial matrices excluding the elements associated with the parameters B_{00}, B_{01} and B_{10} and superscipts T and * stand for the transpose and inverse of matrices, respectively.

Because the parameters B_{00}, B_{01} and B_{10} are introduced in expressions (4-27) and (4-28), the functions $V(\eta)$ and $M(\eta)$ cannot be solved from these expressions alone. Three additional relations must be established to supplement equations (4-27) and (4-28). These relations can be easily obtained from expression (4-7b) by taking the following variations of indices p and q:

$p = 0,\ q = 0;$

$p = 0,\ q = 1;$

$p = 1,\ q = 0.$

Thus,

$$\iint_{A_{i+1}} q_{i+1} dA_{i+1} - \int_0^1 V(\eta)\ d\eta = 0 \tag{4-29a}$$

$$\iint_{A_{i+1}} q_{i+1}\left(\frac{y_i}{b_i}\right) dA_{i+1} - \int_0^1 V(\eta)\,\eta\ d\eta = 0 \tag{4-29b}$$

$$\iint_{A_{i+1}} q_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}\right) dA_{i+1} - \int_0^1 V(\eta)\left(\frac{x_{i+1}}{a_{i+1}}\right)_{x_{i+1}=0} d\eta + \int_0^1 M(\eta)\left(\frac{1}{a_{i+1}}\right) d\eta = 0 \tag{4-29c}$$

A physical interpretation of expression (4-29) shows that these are just the equations of static equilibrium. That means an unstable basic free-free panel must satisfy all the equilibrium conditions.

For an unstable basic panel consisting of three free edges and one simply supported edge, only the first equation of static equilibrium need be satisfied. This equation may be obtained from the minimization of the total energy the condition of $\frac{\partial \Pi_{i+1}}{\partial B_{00}}$.

CHAPTER V

NUMERICAL APPLICATIONS

5-1. Calculation of Constants and Evaluation of Integrals

The numerical computations, such as the calculation of the constants, matrix multiplications and inversions, the evaluation of integrals and the solution of a set of simultaneous equations, for a problem of continuous rectangular plates investigated in this dissertation is laborious and a high-speed electronic digital computer is required. Therefore, the computation for the following example problem was done with the aid of IBM Computer 7090/94.

5-2. Numerical Example

The numerical example considered in the application of the theory is a problem of a two-span continuous plate which consists of two stable basic panels i and i+1 as shown in Fig. 5-1. Only the right panel i+1

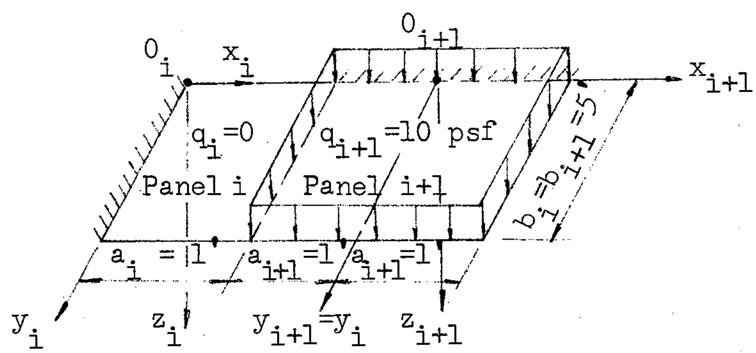


Fig. 5-1.
Two-span Continuous Plates Consisting of Two Basic Cantilever Plates

is considered to be acted upon by a load of intensity q_{i+1} = 10 psf.

The Poisson's ratio for both plates is assumed to be 1/3 and the modulus of elasticity of the plates, 30×10^6 psi. The thickness of the plates is taken to be 0.5 in.

The deflection surface w_i for plate i was not covered in the thirty cases on p.14 and may be assumed to be in the form

$$w_i = h_i \sum_m \sum_n A_{mn} \left(\frac{x_i}{a_i}\right)^{m+2} \left(\frac{y_i}{b_i}\right)^n$$

The corresponding constant C_{mnrs} is

$$C_{mnrs} = \frac{b^2}{a^2} J_{mnrs} + \frac{a^2}{b^2} K_{mnrs} + 2\nu L_{mnrs} + 2(1-\nu) M_{mnrs}$$

$$\begin{pmatrix} m = 0,\ 1,\ \ldots\ldots;\ n = 0,\ 1,\ \ldots\ldots; \\ r = 0,\ 1,\ \ldots\ldots;\ s = 0,\ 1,\ \ldots\ldots \end{pmatrix}$$

in which

$$J_{mnrs} = \frac{(m+2)(m+1)(r+2)(r+1)}{(m+r+1)(n+s-3)}$$

$$K_{mnrs} = \frac{n(n-1)s(s-1)}{(m+r+5)(n+s-3)}$$

$$L_{mnrs} = \frac{(m+2)(m+1)s(s-1)+(r+2)(r+1)n(n-1)}{2(m+r+3)(n+s-1)}$$

$$M_{mnrs} = \frac{(m+2)(r+2)ns}{(m+r+3)(n+s-1)}$$

The deflection surface w_{i+1} and its constant D_{jkpq} may be taken from case 16 on p.14 . The deflection surface is

$$w_{i+1} = h_{i+1} \sum_p \sum_q \left(\frac{x_{i+1}}{a_{i+1}}\right)^p \left(\frac{y_i}{b_i}\right)^{q+2}$$

Before a computer Fortran program is set up for the computation of this problem, the following functions must first be obtained:

$$E^{i}_{rs} = \iint_{A_i} q_i(x_i,y_i)\; h_i\left(\frac{x_i}{a_i}\right)^{r+2}\left(\frac{y_i}{b_i}\right)^{s} dA_i = 0 \qquad \begin{pmatrix} r=0,\ 1,\ldots; \\ s=0,\ 1,\ldots. \end{pmatrix}$$

$$F^{i}_{rs}(\eta) = \left\{ h_i\left(\frac{x_i}{a_i}\right)^{r+2}\left(\frac{y_i}{b_i}\right)^{s} \right\}_{\substack{x_i=a_i \\ y_i=\eta b_i}} \qquad \begin{pmatrix} r=0,1,\ldots; \\ s=0,1,\ldots. \end{pmatrix}$$

$$G^{i}_{rs}(\eta) = -\left\{ \frac{\partial}{\partial x_i}\left[h_i\left(\frac{x_i}{a_i}\right)^{r+2}\left(\frac{y_i}{b_i}\right)^{s} \right] \right\}_{\substack{x_i=a_i \\ y_i=\eta b_i}} = -\, h_i \frac{r+2}{a_i}(\eta)^{s} \qquad \begin{pmatrix} r=0,1,\ldots; \\ s=0,1,\ldots. \end{pmatrix}$$

$$\left\{ (x_i y_i) \right\}_{x_i=a_i} = \left\{ \left(\frac{x_i}{a_i}\right)^{m+2}\left(\frac{y_i}{b_i}\right)^{n} \right\}_{x_i=a_i} = \left(\frac{y_i}{b_i}\right)^{n} \qquad \begin{pmatrix} m=0,1,\ldots; \\ n=0,1,\ldots. \end{pmatrix}$$

$$\left\{ \frac{\partial}{\partial x_i}(x_i y_i) \right\}_{x_i=a_i} = \left\{ \frac{\partial}{\partial x_i}\left[\left(\frac{x_i}{a_i}\right)^{m+2}\left(\frac{y_i}{b_i}\right)^{n} \right] \right\}_{x_i=a_i} = \frac{m+2}{a_i}\left(\frac{y_i}{b_i}\right)^{n} \qquad \begin{pmatrix} m=0,1,\ldots; \\ n=0,1,\ldots. \end{pmatrix}$$

$$E^{i+1}_{pq} = \iint_{A_{i+1}} q_{i+1}\; h_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}\right)^{p}\left(\frac{y_i}{b_i}\right)^{q+2} dA_{i+1}$$

$$= \begin{cases} \dfrac{2\, q_{i+1} h_{i+1} a_{i+1} b_i}{(p+1)(q+3)}, & \begin{pmatrix} p=0,2,\ldots; \\ q=0,1,2,\ldots \end{pmatrix} \\ 0, & \text{otherwise.} \end{cases}$$

$$F^{i+1}_{pq}(\eta) = -\left\{ h_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}\right)^{p}\left(\frac{y_i}{b_i}\right)^{q+2} \right\}_{\substack{x_{i+1}=-a_{i+1} \\ y_i=\eta b_i}}$$

$$= h_{i+1}(-1)^{p+1}\,(\eta)^{q+2} \qquad \begin{pmatrix} p=0,1,\ldots; \\ q=0,1,\ldots. \end{pmatrix}$$

$$G^{i+1}_{pq}(\eta) = \left\{ \frac{\partial}{\partial x_{i+1}}\left[h_{i+1}\left(\frac{x_{i+1}}{a_{i+1}}\right)^{p}\left(\frac{y_i}{b_i}\right)^{q+2} \right] \right\}_{\substack{x_{i+1}=-a_{i+1} \\ y_i=\eta b_i}}$$

$$= h_{i+1}\frac{p}{a_{i+1}}(-1)^{p-1}\,(\eta)^{q+2} \qquad \begin{pmatrix} p=0,1,\ldots; \\ q=0,1,\ldots. \end{pmatrix}$$

$$\left\{(x_{i+1}y_i)\right\}_{x_{i+1}=-a_{i+1}} = \left\{\left(\frac{x_{i+1}}{a_{i+1}}\right)^j \left(\frac{y_i}{b_i}\right)^{k+2}\right\}_{x_{i+1}=-a_{i+1}}$$

$$= (-1)^j \left(\frac{y_i}{b_i}\right)^{k+2} \qquad \left(\begin{matrix} j=0,1,\ldots\ldots; \\ k=0,1,\ldots\ldots. \end{matrix}\right)$$

$$\left\{\frac{\partial}{\partial x_{i+1}}(x_{i+1}y_i)\right\}_{x_{i+1}=-a_{i+1}} = \left\{\frac{j}{a_{i+1}} \left(\frac{x_{i+1}}{a_{i+1}}\right)^{j-1} \left(\frac{y_i}{b_i}\right)^{k+2}\right\}_{x_{i+1}=-a_{i+1}}$$

$$= \frac{j}{a_{i+1}} (-1)^{j-1} \left(\frac{y_i}{b_i}\right)^{k+2} \qquad \left(\begin{matrix} j=0,1,\ldots\ldots; \\ k=0,1,\ldots\ldots. \end{matrix}\right)$$

By varying the indices m,n,r,s and j,k,p,q, respectively, the following matrices were successively formulated and stored in the computer:

$$\left[C_{uv}\right], \left[F_u^i(\eta)\right], \left[G_u^i(\eta)\right], \left[(x_i y_i)_v\right]_{x_i=a_i}, \left[\frac{\partial}{\partial x_i}(x_i y_i)_v\right]_{x_i=a_i}$$

and

$$\left[D_{ht}\right], \left[E_h^{i+1}\right], \left[F_h^{i+1}(\eta)\right], \left[G_h^{i+1}(\eta)\right], \left[(x_{i+1}y_i)_t\right]_{x_{i+1}=-a_{i+1}},$$

$$\left[\frac{\partial}{\partial x_{i+1}}(x_{i+1}y_i)_t\right]_{x_{i+1}=-a_{i+1}}.$$

Among these matrices, those involving variable η or y_i are the ones whose elements are the products of the constant and the order of η or of y_i.

An inspection of the matrices $\left[C_{uv}\right]$ and $\left[D_{ht}\right]$ reveals that they do not possess strong diagonals. Thus, their inversions require a high precision in computation and, thus, a double precision, of 16-digit accuracy, on the IBM Computer 7090/94 is used throughout the entire computation of this problem.

After replacing the matrices $\left[(x_{i+1}y_i)_t\right]_{x_{i+1}=0}$ and $\left[\frac{\partial}{\partial x_{i+1}}(x_{i+1}y_i)\right]_{x_{i+1}=0}$ in the equations (4-19) and (4-20) by

$\left[(x_{i+1}y_i)_t\right]_{x_{i+1}=-a_{i+1}}$ and $\left[\frac{\partial}{\partial x_{i+1}}(x_{i+1}y_i)_t\right]_{x_{i+1}=-a_{i+1}}$, respectively, the evaluation of the integral equations requires a series of matrix multiplications and the integrations of the variable η between limits 0 and 1. Upon collecting all the coefficients of the corresponding power of y_i, a set of simultaneous algebraic equations with 2K unknowns, for the parameters v_β and m_γ of expressions (4-21) and (4-22), respectively, is finally obtained; in which K is the number equal to one plus the highest order of y_i taken from the matrix $\left[(x_iy_i)_v\right]_{x_i=a_i}$ or $\left[(x_{i+1}y_i)_t\right]_{x_{i+1}=-a_{i+1}}$, whichever is the higher. Once these parameters are solved, the moments and edge reactive forces at every $0.1b_i$ point along the juncture of panels are calculated. These values of moments and edge reactive forces, corresponding to the varied indices m, n and j, k, are successively recorded in Tables 5-1 and 5-2, respectively. There are five sets of results given with the variations of the indices in these tables for comparison. Evidently, the moments in the neighborhoods of $\eta = 0.0$, 0.2 and 1.0, and the edge reactive forces between the region $0.2 \leq \eta \leq 0.3$, $\eta = 0.0$ and $\eta = 1.0$ have the characteristics of high sensitivity. At the position $\eta = 0.0$, the value of moment varies from the minimum 383.1255 lbs.-in./in. to the maximum 974.1859 lbs.-in./in. and the value of edge reactive force, the minimum 377.6846 lbs./in. to the maximum 749.7051 lbs./in.

In calculating the deflections of the plates, the resulted moment and edge reactive force functions are first substituted into the equations (4-15) and (4-16). By integrating between the limits 0 and 1, and using matrix multiplications and additions, the matrices $\left[A_v\right]$ and $\left[B_t\right]$ for the parameters of the deflection surfaces w_i and w_{i+1} are obtained. The deflections along the juncture at each interval of $0.1b_i$ are then computed. For checking purposes, these deflections of the referenced points are calculated from both the deflection surfaces w_i and w_{i+1} of the plates. The values of the deflections are given in Table 5-3. Each set of deflections shows that the deflections along the juncture of the plates are highly compatible, particularly the first three sets of the results.

TABLE 5-1. MOMENTS ALONG THE JUNCTURE OF TWO-SPAN CONTINUOUS PLATES

M at x_i=1'-0", y_i=ηb_i= $(5\eta)'$ (lbs.-in./in.)					
$m = m_1$	0→2	0→2	0→2	0→3	0→3
$n = n_1$	0→5	0→6	0→6	0→6	0→6
$m = m_2$	3→5	3→5	3→5	4→6	4→5
$n = n_2$	0→3	0→2	0→3	0→1	0→3
$j = j_1$	0→2	0→2	0→2	0→2	0→2
$k = k_1$	0→7	0→7	0→7	0→7	0→7
$j = j_2$	3→5	3→5	3→4	3→6	3→5
$k = k_2$	0→1	0→1	0→4	0→2	0→3
η = 0.0	503.9302	402.5171	383.1255	962.3057	974.1859
η = 0.1	-246.9957	-248.0053	-245.9946	-222.4730	-224.3460
η = 0.2	-157.4282	-186.0354	-183.5321	-21.4058	-22.7446
η = 0.3	-230.5616	-208.0389	-196.9767	-361.6914	-367.5809
η = 0.4	-384.2971	-380.6185	-379.5351	-428.7691	-427.7859
η = 0.5	-467.9408	-493.4074	-502.3238	-350.3257	-345.3779
η = 0.6	-522.6849	-514.5407	-517.8386	-531.3260	-531.6185
η = 0.7	-635.5659	-603.5773	-601.9476	-748.7958	-749.1111
η = 0.8	-736.5897	-758.7585	-763.0068	-635.4061	-633.3384
η = 0.9	-688.7174	-697.5016	-688.0591	-701.2158	-720.1117
η = 1.0	-1020.4029	-1091.0398	-1137.4317	-563.2472	-458.3841

TABLE 5-2. EDGE REACTIVE FORCES ALONG THE JUNCTURE OF TWO-SPAN CONTINUOUS PLATES

V at x_i=1'-0", $y_i = \eta b_i = (5\eta)'$ (lbs./in.)					
$m = m_1$	0→2	0→2	0→2	0→3	0→3
$n = n_1$	0→5	0→6	0→6	0→6	0→6
$m = m_2$	3→5	3→5	3→5	4→6	4→5
$n = n_2$	0→3	0→2	0→3	0→1	0→3
$j = j_1$	0→2	0→2	0→2	0→2	0→2
$k = k_1$	0→7	0→7	0→7	0→7	0→7
$j = j_2$	3→5	3→5	3→4	3→6	3→5
$k = k_2$	0→1	0→1	0→4	0→2	0→3
$\eta = 0.0$	377.6846	749.1812	749.7051	672.1330	678.9923
$\eta = 0.1$	-159.3382	-166.0139	-166.9687	-159.2142	-159.9862
$\eta = 0.2$	26.7934	104.7508	105.5961	93.2660	94.4932
$\eta = 0.3$	81.9325	-17.3319	-17.0531	0.0434	-1.2107
$\eta = 0.4$	31.3438	16.9011	16.6969	17.1441	17.1190
$\eta = 0.5$	32.4818	133.8782	133.8888	114.6834	115.9564
$\eta = 0.6$	99.2374	84.4642	84.4887	89.1337	88.5163
$\eta = 0.7$	129.9592	31.6814	30.9833	50.4880	49.5534
$\eta = 0.8$	71.5558	150.8504	150.4525	134.2472	135.9791
$\eta = 0.9$	51.9870	43.2204	45.3540	46.9250	44.0626
$\eta = 1.0$	313.4475	693.3558	686.2977	612.5300	629.2485

TABLE 5-3. DEFLECTIONS ALONG THE JUNCTURE OF TWO-SPAN CONTINUOUS PLATES, CALCULATED FROM BOTH THE DEFLECTION SURFACES w_i AND w_{i+1}

$100w_i$ at x_i=1'-0", $y_i=\eta b_i=(5\eta)'$ (in.) upper $100w_{i+1}$ at x_{i+1}=-1'-0", $y_i=\eta b_i=(5\eta)'$ (in.) lower					
$m = m_1$ $n = n_1$	0→2 0→5	0→2 0→6	0→2 0→6	0→3 0→6	0→3 0→6
$m = m_2$ $n = n_2$	3→5 0→3	3→5 0→2	3→5 0→3	4→6 0→1	4→5 0→3
$j = j_1$ $k = k_1$	0→2 0→7	0→2 0→7	0→2 0→7	0→2 0→7	0→2 0→7
$j = j_2$ $k = k_2$	3→5 0→1	3→5 0→1	3→4 0→4	3→6 0→2	3→5 0→3
η = 0.0	.00000000 .00000000	.00000000 .00000000	.00000000 .00000000	.00000000 .00000000	.00000000 .00000000
η = 0.1	.02599834 .02599834	.02610335 .02610328	.02599504 .02599515	.02505969 .02505955	.02521450 .02521453
η = 0.2	.08927906 .08927912	.08952517 .08952475	.08927082 .08927117	.08808052 .08807995	.08843312 .08843235
η = 0.3	.16995067 .16995099	.17018173 .17018053	.16994059 .16994108	.16969244 .16969155	.17001567 .17000994
η = 0.4	.25293341 .25293444	.25297212 .25296951	.25292810 .25292825	.25366891 .25367028	.25373521 .25371195
η = 0.5	.32904361 .32904606	.32883800 .32883313	.32905157 .32905011	.32997463 .32998775	.32975685 .32968642
η = 0.6	.39460798 .39461292	.39426729 .39425921	.39463544 .39462931	.39498727 .39503615	.39466670 .39448906
η = 0.7	.44970908 .44971793	.44941619 .44940421	.44975256 .44973496	.44947332 .44960873	.44925275 .44885814
η = 0.8	.49516287 .49517736	.49502378 .49500820	.49519975 .49515682	.49489845 .49521619	.49473895 .49394123
η = 0.9	.52832947 .52835150	.52829363 .52827697	.52830933 .52821507	.52865245 .52931956	.52817423 .52667537
η = 1.0	.53785816 .53788959	.53791685 .53790598	.53769948 .53750864	.53876912 .54005973	.53767786 .53502229

CHAPTER VI

SUMMARY AND CONCLUSIONS

6-1. Summary

The application of Rayleigh-Ritz energy method to the analysis of continuous rectangular plates is presented in this study. The results of this investigation are limited to thin plates of constant thickness with small deflections, but may be used in general for any type of continuous plates loaded by any system of static loads.

The deflection surface of each plate is represented by a polynomial series with arbitrary coefficients premultiplied by a function that satisfies the geometrical conditions of the plate. Edge moments and reactive forces acting on the plates are taken as redundant functions which are also represented by other power series.

A panel of any type of supported rectangular plate may be selected as a basic panel. Such panels can be basically either a stable or unstable plate. Rayleigh-Ritz solutions are obtained with a set of an infinite number of equations for the deflection coefficients of the basic panels due to applied loading, arbitrary edge moment function, and edge reactive force function if an adjacent edge of such panels is a free one. This set of an infinite number of equations is reduced to an approximate form by only taking a finite number of deflection coefficients. Continuity conditions between panels are established to ensure the compatibility of the continuous plates as a system. A set of two

Fredholm Integral Equations of First Kind for panels of a free adjacent edge, or only one equation for those of a simply-supported adjacent edge, is established. The equations of static equilibrium in addition must be satisfied if the basic panel considered is an unstable one.

The use of a matrix operation provides the solutions of edge redundant functions and the plate deflections in a very systematic approach.

The use of theory is illustrated by a numerical example.

6-2. Conclusions

A method of analysis for continuous rectangular thin plates has been developed. It offers three major advantages over existing methods:

1. The basic panel selected is absolutely general. It may be either a stable panel or an unstable one.
2. Application of a polynomial series for the analysis of the deflection surface of the plate gives a model avoiding a higher mathematical analysis. These polynomial series are particularly useful for the solution of continuous plates consisting of one or more clamped edges or free edges.
3. This method provides the solutions of continuous rectangular plates with general types of edge conditions including simply-supported, clamped, free and corner-supported edges.

Some advantage may be taken in cases involving symmetrically loaded plates with symmetrical edge conditions by only shifting the origins of the rectangular coordinate systems. This usually will offer a more rapidly convergent series.

The reduction of a set of an infinite number of equations to an approximation of the set of a finite number of equations gives reasonably good results.

Because of the tedious numerical calculations involved, a high-speed electronic digital computer must be required. Furthermore, the high sensitivity of the coefficient matrix $\left[C_{uv}\right]$ and also of the result-ing coefficient matrix of the set of equations for the unknown para-meters of edge redundant functions, demands a high multiple-precision accuracy in order to obtain satisfactory results.

A SELECTED BIBLIOGRAPHY

1. Timoshenko, S. and Woinowsky-Krieger, S., Theory of Plates and Shells, 2nd. Ed., McGraw-Hill, New York, 1959, pp. 415-418, 342-346.

2. Kirchhoff, G., "Über das Gleichgewicht und die Bewegung einer elastichen Scheibe," Journal für die reine und angewandte Mathematik (Crelle), vol. 40, 1850, pp. 51-88.

3. Reissner, E., "The Effect of Transverse Shear Deformation on the Bending of Elastic Plates," J. Appl. Mech., Trans. ASME, vol. 67, 1945, pp. A67-A77.

4. Friedrichs, K. O., "The Edge Effect in the Bending of Plates," Reissner Anniversary Volume, Edwards Bros., Inc., Ann Arbor, Mich., pp. 197-210.

5. Ritz, W., "Über eine neue Methodé zur Lösung gewissen Variationprobleme der mathematischen Physik," Journal für die reine und angewandte Mathematik, vol. 135, 1909, pp. 1-61.

6. Knott, C. G., "Comparison of Mr. Crawford's Measurements of the Deflection of a Square Clamped Plate With Ritz's Solution," Proc. of the Royal Society of Edinburgh, vol. 32, 1912, pp. 390-392.

7. Mesnager, M., "Formule de la Plaque mince encastrée sur un contour rectangulaire plan," Comptes Rendus, Paris, vol. 163, 1916, pp. 661-663.

8. Pickett, G., "Solution of Rectangular Clamped Plate With Lateral Load by Generalized Energy Method," J. Appl. Mech., Trans. ASME, vol. 61, 1939, pp. A168-A170.

9. Navier, C. L., Bull. soc. phil-math., Paris, 1823.

10. McInnis, B. C., Tsai, S. L. and Sims, J. R., Discussion of "Laterally Loaded Thin Flat Plates, by W. A. Bradley," Eng. Mech., Proc. ASCE, vol. 2, 1960, pp. 179-182.

11. Little, R. W., "Bending of a Cantilever Plate," MS Thesis, Univ. of Wisconsin, Madison, Wis., 1959.

12. Galerkin, B. G., Elastic Thin Plates (in Russian), Gosstrojsdat, Leningrad, U.S.S.R., 1933.

13. Marcus, H., Die Theorie elastischer Gewebe und ihre Anwedung of die Berechnung beigsamer Platten, 2nd. Ed., Julius Springer, Berlin, 1932.

14. Jensen, V. P., "Solutions for Certain Rectangular Slabs Continuous over Flexible Supports," "Univ. of Ill. Eng. Exp. Sta., Bulletin No. 303, 1938.

15. Woinowsky-Krieger, S., "Beigräge zur Theorie der durchlaufender Platte," Ing.-Arch., No. 9, 1938, p. 396.

16. Weber, E., "Die Berechnung rechteckiger Platten, die durch elastiche Träger unterstützt sing," Ing.-Arch., vol. 8, 1937, pp. 311-325.

17. Newmark, N. M., "A distribution Procedure for the Analysis of Slabs Continuous Over Flexible Beams," Univ. of Ill. Eng. Exp. Sta., Bulletin No. 304, 1938.

18. Bittner, E., Momententafeln und Einflussflächen für Kreuzweis bewehrte Eisenbetonplatten, Springer, Vienna, 1938.

19. Maugh, L. C., and Pan, C. W., "Moments in Continuous Rectangular Slabs on Rigid Supports," Trans. ASCE, vol. 107, 1942, pp. 1118-1142.

20. Engelbreth, K., "En Metode for Tilnaermet Beregning av Kontinuerlige Toveisplater," Betong, vol. 30, No. 2, 1945.

21. Siess, C. P., and Newmark, N. M., "Rational Analysis and Design of Two-way Continuous Slabs," J. Amer. Concr. Inst., vol. 20, 1948, pp. 273-315.

22. Siess, C. P., and Newmark, N. M., "Moments in Two-way Concrete Floor Slabs," Univ. of Ill. Eng. Exp. Sta., Bulletin No. 385, 1950.

23. Lechter, H., "Analysis of Continuous Rectangular Plates on Rigid Supports by Flexibility Methods," MS Thesis, Oklahoma State University, Stillwater, Okla., 1962.

24. Nádai, A., "Über die Biegung durchlaufenden Platten und der rechteckigen Platte mit freien Rändern," Z. angew. Math. Mech., vol. 2, 1922, pp. 1-26.

25. Galerkin, B. G., Collected Papers, vol. 2, Moscow, 1953, pp. 15, 29.

26. Sutherland, J. G., Goodman, L. E., and Newmark, N. M., "Analysis of Plates Continuous Over Flexible Beams," Univ. of Ill. Civil Eng. Studies, Structural Research Series No. 42, Jan., 1953.

27. Ang., A. H-S., "A Distribution Procedure for the Analysis of Continuous Rectangular Plates," Ph.D. Thesis, Univ. of Ill., Urbana, 1959.

28. Ang., A. H-S., and Newmark, N. M., "A Procedure for the Analysis of Continuous Plates," Proceedings, 2nd ASCE Conf. on Electronic Computation," 1960, pp. 379-413.

29. Ang., A. H-S., and Prescott, W., "Equations for Plate-Beam Systems in Transverse Bending," Proceedings, ASCE, Engr. Mech. Div., vol. 87, 1961, pp. 1-15.

30. Reddy, V. A., "Analysis of Continuous Rectangular Plates on Flexible Beam Supports by Flexibility Methods," MS Thesis, Oklahoma State University, Stillwater, Okla., 1963.

31. Oden, J. T., "Analysis of Plate Beam Structures," Ph.D. Thesis, Oklahoma State University, Stillwater, Okla., 1962.

32. Lévy, M., Comptes Rendus, vol. 129, 1899, pp. 535-539.

33. Nádai, A., Die elastische Platten, Berlin, 1925, pp. 78-109, 141-156.

34. Morice, P. B., "Energy Approximations Applied to Problems of Structural Equilibrium, Stability and Vibrations," The Structural Engineer, June, 1955, pp. 173-180.

35. Weinstein, A., and Rock, D. H., "On the Bending of a Clamped Plate," Quarterly of Appl. Mathematics, vol. 2, 1944, pp. 262-266.

36. Trefftz, E., "Ein Gegenstück zum Ritzchen Verfahren," Second International Congress for Appl. Mech., Zurich, Switzerland, 1926, proceedings, 1927, pp. 131-137.

37. Girkman, K., Flächentragwerke, 4th Ed., Springer-Verlag, Vienna, 1956, p. 234.

38. Courant, R., "Variational Methods for the Solution of Problems of Equilibrium and Vibration," Bulletin of the American Mathematical Society, New York, N. Y., vol. 49, 1943, pp. 1-23.

39. Courant, R. and Hilbert, D., Methods of Mathematical Physics, vol. 1, Interscience Publishers, Inc., New York, pp. 112-163.

40. Langhaar, Henry L., Energy Methods in Applied Mechanics, John-Wiley, New York, 1962, pp. 164.

41. Sokolnikoff, I. S., Mathematical Theory of Elasticity, 2nd. Ed., McGraw-Hill, New York, 1956, pp. 409-411.

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