AN INTRODUCTION TO GENERAL INTEGRALS

By

RICHARD E. SHERMOEN

Bachelor of Science
North Dakota State University
Fargo, North Dakota
1953

Master of Science
North Dakota State University
Fargo, North Dakota
1958

Master of Arts
University of Illinois
Urbana, Illinois
1961

Submitted to the faculty of the Graduate School of
the Oklahoma State University
in partial fulfillment of the requirements
for the degree of
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Thesis Approved:

James H. Zant
Thesis Advisor

Kenneth E Wiggins

Wallace Marsden

Robert B. Hoffman

J G Boggs
Dean of the Graduate School

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CHAPTER I

INTRODUCTION

Historical Background

H. Lebesgue's thesis Integrale, Longueur, Aire, appeared in 1902. In his thesis, Lebesgue framed a new integral which, as Van Vleck states

> ... is identical with the integral of Riemann when the latter is applicable but is immeasurably more comprehensive. This new integral of Lebesgue is proving itself a wonderful tool. I might compare it with a modern Krupp gun, so easily does it penetrate barriers which before were impregnable.[1]

In view of the wide applicability of Lebesgue's theory, it would not be unreasonable to assume that further research in this area might be fruitless. This however has not been the case. As Bell remarks,

> We hasten to add that the integral calculus did not come to a sudden and glorious end in 1902. Quite the contrary; integration just began to thrive. Lebesgue's generalization of Riemann integration was but a beginning.[2]

There were perhaps, many factors that led mathematicians to devise even more general theories of integration than that

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[1]E. B. Van Vleck, "Tendencies of Mathematical Research," Bulletin, American Mathematical Society, Vol. 23, 1916, pp. 6,7.

[2]E. T. Bell, The Development of Mathematics, McGraw Hill, New York, 1945, pp. 448.

presented by Lebesgue. E. G. Bliss states his views concerning the motivation for continued research in this area as follows:

> In the field of integration the classical integral of Riemann, perfected by Darboux, was such a convenient and perfect instrument that it impressed itself for a long time upon the mathematical public as being something unique and final. The advent of the integrals of T. J. Stiltjes and H. Lebesgue has shaken the complacency of mathematicians in this respect, and, with the theory of linear integral equations, has given the signal for a re-examination and extension of many of the types of processes which Volterra calls passing from the finite to the infinite.[3]

One of the first men to develop a general integral was P. J. Daniell.[4] In Daniell's work, a linear functional is postulated on a given class of functions. The functional is then extended to a larger class of functions in such a way that certain desireable properties such as additivity and homogeniety are preserved.

A significant aspect of Daniell's theory is that no special properties are assumed on the domain space of the functions. Also, the Lebesgue integral is a special case of Daniell's theory.

Various theories of integration have been developed subsequent to Daniell's work in 1918. Generally speaking, these integrals can be classified either as integrals that

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[3] G. A. Bliss, "Integrals of Lebesgue", Bulletin, American Mathematical Society, Vol. 24, 1917, pp.1.

[4] P. J. Daniell, "A General Form of Integral", Annals of Mathematics, 19 (1918) pp. 279-294.

have been devised for a special purpose, or as integrals that have been devised to increase the generality of the process of integration. In the latter category, the integrals developed by M. H. Stone[5] and E. J. McShane[6] respectively are of special interest because of the wide applicability of their theories, and also because each of these men was motivated by the work of Daniell.

Plan and Limitation of the Study

This paper presents a study of the definition and fundamental properties of general integrals as framed by P. J. Daniell, M. H. Stone, and E. J. McShane respectively.

The original publications of Daniell and Stone have been followed rather closely, but in somewhat more detail. In particular, the original works have been supplemented with specific examples of the theory, elaboration and proof of assertions, and some original theorems of the writer. The discussion here proceeds at a slower rate than the original, for a sincere effort has been made to present the study of general integrals at a level appropriate to an undergraduate mathematics major.

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[5]M. H. Stone, "Notes on Integration I, II, III, IV" *Proceedings National Academy of Science* 34 (1948) pp. 336-342, 447-455, 483-489; 35 (1949) pp. 50-58.

[6]E. J. McShane, "Order Preserving Maps and Integration Processes", *Annals of Mathematics Studies No. 31* Princeton University Press, Princeton, 1953.

The integrals of Daniell and Stone possess a number of additional properties analogous to the classical results of Lebesgue and Riemann integration, but these properties are in general not within the scope of this paper.

E. J. McShane's approach to a general integral is evolved in a rather abstract setting. For this reason his theory is presented on a purely expository basis.

CHAPTER II

THE DANIELL INTEGRAL

Postulates

This chapter presents the definition and elementary properties of a general integral as developed by P. J. Daniell. The reader is referred to Appendix A for an explanation of symbols used in this study.

In this chapter the following two symbols will be used:

$(f \vee g)x = \max \{f(x), g(x)\}$, $x \in D_f \cap D_g$.

$(f \wedge g)x = \min \{f(x), g(x)\}$, $x \in D_f \cap D_g$.

Let X be an arbitrary non-void set of elements and A a non-void class of real valued functions defined on the elements of X such that:

a-1. $f \in A$ and $a \in R \Rightarrow af \in A$.

a-2. $f_1, f_2 \in A \Rightarrow$; $f_1 + f_2 \in A$, $f_1 \vee f_2 \in A$ and $f_1 \wedge f_2 \in A$.

a-3. If $f \in A$, there exists $K_f \in R$ such that $|f(x)| < K_f$ for all $x \in X$.

Let E be a mapping of A into the real numbers such that:

b-1. $f \in A$ and $c \in R \Rightarrow E(cf) = cE(f)$.

b-2. $f_1, f_2 \in A \Rightarrow E(f_1 + f_2) = E(f_1) + E(f_2)$.

b-3. $f_n \in A$, $n = 1,2,\ldots$ and $\{f_n\} \downarrow 0 \Rightarrow \lim E(f_n) = 0$.

b-4. $f \in A$ and $f \geq 0 \Rightarrow E(f) \geq 0$.

b-5. There exists a functional M for all functions of the form $|f|$, where $f \in A$, such that if $f_1 \geq f_2$, $M(f_1) \geq M(f_2)$ and such that $|E(f)| \leq M(|f|)$ for $f \in A$.

The symbol 0 will be used to indicate either the zero function or the real number zero. The context will make it clear which use is intended. Note that the zero function is in A by a-1.

DEFINITION 1.1. An <u>I-integral</u> is a linear functional on A satisfying b-1, b-2, b-3 and b-4.

DEFINITION 1.2. An <u>S-integral</u> is a linear functional on A satisfying b-1, b-2, b-3 and b-5.

EXAMPLE. As an illustration of the theory, it will be shown that the Riemann integral is an I-integral on the class of continuous functions defined on a closed interval.

Let X be the closed interval $[a,b]$ and let A be the class of continuous functions defined on X. From elementary calculus it is clear that A satisfies a-1, a-2, and a-3.

Let E be the Riemann integral $\int_a^b$. Again from the elementary calculus it is clear that $\int_a^b$ satisfies b-1, b-2, and b-4 on A. To show that b-3 is satisfied two results from Riemann integration will be needed.

1. If $\{f_n\} \downarrow 0$ on $[a,b]$, f_n continuous on $[a,b]$, $n = 1,2,\ldots$ then $\{f_n\}$ converges uniformly to 0 on $[a,b]$.

Proof: Let $\epsilon > 0$ and $x \in [a,b]$. Since $\{f_n\} \downarrow 0$ there exists a positive integer N_x (dependent on x and ϵ) such that $n \geq N_x \Rightarrow f_n(x) < \epsilon/2$. f_{N_x} is continuous at x, hence there exists an open set O_{N_x} containing x such that $t \in O_{N_x} \Rightarrow$ $|f_{N_x}(t)| - |f_{N_x}(x)| \le |f_{N_x}(t) - f_{N_x}(x)| < \epsilon/2$. That is, $|f_{N_x}(t)| < \epsilon/2 + \epsilon/2 = \epsilon$. $\{f_n\}\downarrow$, therefore $|f_n(t)| < \epsilon$ for $t \in O_{N_x}$ and $n \ge N_x$. Since this is for arbitrary x in $[a,b]$ the set of all such O_{N_x}, $x \in [a,b]$, are an open covering for $[a,b]$. Thus by the Borel Covering Theorem, a finite number of these open sets, say $O_{N_{x_i}}$, $i = 1,\ldots, k$ are a covering for $[a,b]$. Let $N_0 = \max\{N_{x_1},\ldots,N_{x_k}\}$. Then $|f_n(t)| < \epsilon$, for $n > N_0$ and for all $t \in [a,b]$. That is, $\{f_n\}$ converges uniformly to 0 on $[a,b]$.

2. If $\{f_n\}$ is a sequence of continuous functions on $[a,b]$ converging uniformly to f on $[a,b]$, then $\lim \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

Proof: First, it is well known that uniform convergence preserves continuity, thus the limit function f is a continuous function. It follows that for all n,

$$\int_a^b f_n(x)dx - \int_a^b f(x)dx = \int_a^b [f_n(x) - f(x)]\,dx$$

and therefore $\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| = \left|\int_a^b [f_n(x) - f(x)]\,dx\right| \le \int_a^b |f_n(x) - f(x)|\,dx$.

Let $\epsilon > 0$. Since $\{f_n\}$ converges uniformly to f(x) there exists a positive integer N such that $n > N \Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{b-a}$ for all $x \in [a,b]$. Hence for $n > N$, $\int_a^b |f_n(x) - f(x)|\,dx < \int_a^b \frac{\epsilon}{b-a}\,dx = \epsilon$. That is $\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| < \epsilon$, which by definition, proves 2.

From 1. and 2. it is immediate that if $\{f_n\}\downarrow 0$ on $[a,b]$, f_n continuous on $[a,b]$, $n = 1,2,\ldots$ then $\lim \int_a^b f_n(x)dx = \int_a^b 0\,dx = 0$. That is, b-3 is satisfied.

Thus the Riemann integral is an I-integral on the class of continuous functions defined on a closed interval. The general theory is now continued.

THEOREM 1.3. If f, g $\in$ A and f $\leq$ g, then I(f) $\leq$ I(g).

Proof: Let f, g $\in$ A and f $\leq$ g. Then 0 $\leq$ g - f and g - f $\in$ A. Therefore I(g-f) $\geq$ 0 by b-4. But I(g-f) = I(g) + I(-f) = I(g) - I(f) by b-2 and b-1, therefore I(g) $\geq$ I(f).

THEOREM 1.4. If f $\in$ A, $|f|$ $\in$ A.

Proof: Let f $\in$ A. Then -f $\in$ A and f v (-f) $\in$ A by a-1 and a-2. However, (f v -f)(x) = max $\{f(x), -f(x)\}$ = $|f(x)|$ and so the theorem is proved.

THEOREM 1.5. If f $\in$ A, $|I(f)| \leq I(|f|)$.

Proof: Let f $\in$ A. Then $|f|$ $\in$ A by 1.4 and $-|f| \leq f \leq |f| \Rightarrow$ $I(-|f|) = -I(|f|) \leq I(f) \leq I(|f|)$ by 1.3. That is, $|I(f)| \leq I(|f|)$.

From the preceding theorems, it is clear that the functional I defined on A satisfies the conditions imposed on the functional M of b-5. Since I also satisfies b-1, b-2, and b-3, the following theorem has been proved.

THEOREM 1.6. Any I-integral on A is an S-integral on A.

To show that an S-integral is not necessarily an I-integral, it will suffice to display a functional satisfying b-1, b-2, and b-3 but not b-4. Thus consider the following example.

EXAMPLE. Let A be the set of all constant functions. Clearly, a-1, a-2, and a-3 are satisfied. If f(x) = c define U(f) = -c.

If $a \in R$ then $U(af) = aU(f)$, for let $f \in A$, $a \in R$ and $f(x) = c$. Then $af(x) = ac$ hence $U(af) = -ac = a(-c) = aU(f)$. That is, U satisfies b-1.

If $f_1, f_2 \in A$ then $U(f_1 + f_2) = U(f_1) + U(f_2)$, for let $f_1(x) = c_1$, $f_2(x) = c_2$. Then $(f_1 + f_2)x = c_1 + c_2$ hence $U(f_1 + f_2) = -(c_1 + c_2) = -c_1 + (-c_2) = U(f_1) + U(f_2)$. Thus U satisfies b-2.

Finally, if $f_n \in A$, $\{f_n\} \downarrow 0$ then $\lim U(f_n) = 0$, for let $f_1(x) = c_1$, $f_2(x) = c_2, \ldots f_n(x) = c_n, \ldots$. Since $\{f_n\} \downarrow 0$ it is clear that $c_1 \geq c_2 \geq \ldots \geq c_n \geq \ldots$, $c_j \geq 0$ for all j and $\lim c_n = 0$. It follows that $-c_1 \leq -c_2 \leq \ldots \leq -c_n \leq \ldots$, $-c_j \leq 0$ for all j and $\lim (-c_n) = 0$. That is, $\lim U(f_n) = 0$. Thus U satisfies b-3.

U does not satisfy b-4 however, since for the constant function $f = 1$, $U(f) = -1$.

The above example illustrates a functional that satisfies b-1, b-2, and b-3 but does not satisfy b-4. Thus an S-integral need not be an I-integral.

<u>Definition of an I-Integral</u>

This section illustrates one procedure for defining a functional on A, given any S-integral on A. It is then shown that the defined functional is in fact an I-integral.

DEFINITION 2.1. If $f \in A$ and $f \geq 0$, $I^*(f) = \sup \{S(g) \mid g \in A \text{ and } 0 \leq g \leq f\}$.

I^* is called the <u>positive integral</u> associated with S.

That $I^*(f)$ exists for all $f \in A$, $f \geq 0$, is proved in the following theorem.

THEOREM 2.2. If $f \in A$ and $f \geq 0$ then $I^*(f)$ exists.

Proof: Let $f \in A$, $f \geq 0$, and $g \in A$ be such that $0 \leq g \leq f$. By b-5, $S(g) \leq M(|g|) = M(g) \leq M(f)$. Thus $M(f)$ is an upper bound for all such $S(g)$ and so the supremum of all such $S(g)$ exists.

The following theorems will show that I^* is an I-integral.

THEOREM 2.3. If $f \in A$ and $f \geq 0$, then $I^*(f) \geq 0$.

Proof: Let $f \in A$. $S(f) = S(0 + f) = S(0) + S(f)$ therefore $S(0) = 0$, but 0 serves as a g function in 2.1 therefore $I^*(f) \geq 0$. This shows that I^* satisfies property b-4.

Before proceeding to show that I^* satisfies the remaining postulates for an I-integral, some preliminary results will be presented in the form of lemmas.

LEMMA 2.4. If $f \in A$, $f \geq 0$ and $c \in R$, $c > 0$ then $I^*(cf) = cI^*(f)$.

Proof: Let $f \in A$, $f \geq 0$ and $c \in R$, $c > 0$. Then $\{g \mid g \in A \text{ and } 0 \leq g \leq f\} = \{g \mid 0 \leq cg \leq cf\}$. Thus

$$\begin{aligned} I^*(cf) &= \sup\{S(cg) \mid 0 \leq cg \leq cf\} \\ &= \sup\{cS(g) \mid 0 \leq cg \leq cf\} \\ &= c \sup\{S(g) \mid 0 \leq g \leq f\} \\ &= cI^*(f). \end{aligned}$$

LEMMA 2.5. If $f_1, f_2 \in A$, $f_1 \geq 0$, $f_2 \geq 0$ then $I^*(f_1 + f_2) = I^*(f_1) + I^*(f_2)$.

Proof: Let $f_1, f_2 \in A$, $f_1 \geq 0$ and $f_2 \geq 0$.
First, $I^*(f_1 + f_2) \geq I^*(f_1) + I^*(f_2)$. For suppose $0 \leq g_1 \leq f_1$ and $0 \leq g_2 \leq f_2$. Then $0 \leq g_1 + g_2 \leq f_1 + f_2$ so $I^*(f_1 + f_2) \geq S(g_1 + g_2) = S(g_1) + S(g_2)$. Since this is true for all g_1, g_2 such that $0 \leq g_1 \leq f_1$ and $0 \leq g_2 \leq f_2$, it follows that $I^*(f_1 + f_2) \geq I^*(f_1) + I^*(f_2)$.
Second, $I^*(f_1 + f_2) \leq I^*(f_1) + I^*(f_2)$. Suppose $0 \leq g \leq f_1 + f_2$. Then $g - f_1 \leq f_2$, $0 \leq g \wedge f_1 \leq f_1$ and $0 \leq [(g - f_1) \vee 0] \leq f_2$. Thus $S(g \wedge f_1) \leq I^*(f_1)$, $S((g - f_1) \vee 0) \leq I^*(f_2)$ from which $S(g \wedge f_1) + S((g - f_1) \vee 0) \leq I^*(f_1) + I^*(f_2)$. However, $S(g \wedge f_1) + S((g - f_1) \vee 0) = S(g)$ because $(g \wedge f_1) + (g - f_1) \vee 0 = g$. For if $g(x) \geq f_1(x)$, $[(g \wedge f_1) + ((g - f_1) \vee 0)](x) = f_1(x) + (g - f_1)(x) = g(x)$, while if $g(x) < f_1(x)$, $(g \wedge f_1)(x) + ((g - f_1)(x) \vee 0(x)) = g(x) \vee 0 = g(x)$.
Thus $S(g) \leq I^*(f_1) + I^*(f_2)$ whenever $g \leq f_1 + f_2$, therefore $I^*(f_1 + f_2) \leq I^*(f_1) + I^*(f_2)$. With the first result above, the lemma is proved.

LEMMA 2.6. $f_1, f_2 \in A \Rightarrow (f_1 \vee f_2) + (f_1 \wedge f_2) = f_1 + f_2$ and $-f_1 \vee -f_2 = -(f_1 \wedge f_2)$.

Proof: Let $f_1, f_2 \in A$ and assume $f_1(x) \leq f_2(x)$. Then $(f_1 \vee f_2)(x) = f_2(x)$ and $(f_1 \wedge f_2)(x) = f_1(x)$. In the same fashion, the first result holds if $f_1(x) > f_2(x)$. The second result is proved analogously.

NOTE. By definition, $I^*(f_n) = \sup\{S(g_n) | g_n \in A, 0 \leq g_n \leq f_n\}$, where $f_n \in A$. Thus if $\epsilon > 0$, there exists a $g_n \in A$, $0 \leq g_n \leq f_n$ such that $I^*(f_n) < S(g_n) + 2^{-n}\epsilon$.

LEMMA 2.7. If $I^*(f_{n-1}) < S(h_{n-1}) + \epsilon_{n-1}$, $\epsilon_i > 0$, $i = 2,3,\ldots$, $I^*(f_n) < S(g_n) + 2^{-n}\epsilon$, $\epsilon > 0$, $f_n \leq f_{n-1}$, $0 \leq h_{n-1} \leq f_{n-1}$ and $0 \leq g_n \leq f_n$, then $I^*(f_n) < S(h_{n-1} \wedge g_n) + \epsilon_{n-1} + 2^{-n}\epsilon$.

Proof: $0 \leq h_{n-1} \leq f_{n-1}$ and $0 \leq g_n \leq f_n \leq f_{n-1}$, therefore $0 \leq (h_{n-1} \vee g_n) \leq f_{n-1} \Rightarrow S(h_{n-1} \vee g_n) \leq I^*(f_{n-1}) < S(h_{n-1}) + \epsilon_{n-1}$. Since $h_{n-1} \wedge g_n = (h_{n-1} + g_n) - (h_{n-1} \vee g_n)$ by 2.6,

$$\begin{aligned} S(h_{n-1} \wedge g_n) &= S(h_{n-1}) + S(g_n) - S(h_{n-1} \vee g_n) \\ &> S(h_{n-1}) + S(g_n) - S(h_{n-1}) - \epsilon_{n-1} \\ &= S(g_n) - \epsilon_{n-1}. \end{aligned}$$

By hypothesis $I^*(f_n) < S(g_n) + 2^{-n}\epsilon$ therefore $I^*(f_n) < S(h_{n-1} \wedge g_n) + \epsilon_{n-1} + 2^{-n}\epsilon$ which was to be proved.

THEOREM 2.8. If $\{f_n\} \downarrow 0$ then $\lim I^*(f_n) = 0$.

Proof: Suppose $\{f_n\} \downarrow 0$. Let $\epsilon > 0$, and for each f_n, choose g_n as in 2.7. That is choose g_n so that $I^*(f_n) < S(g_n) + 2^{-n}\epsilon$.

Let $h_1 = g_1$ and $h_n = h_{n-1} \wedge g_n$ $n = 2,3,\ldots$

$\epsilon_1 = \epsilon/2$ and $\epsilon_n = \epsilon_{n-1} + 2^{-n}\epsilon$, $n = 2,3,\ldots$

Now all the conditions of 2.7 are satisfied, therefore

$$\begin{aligned} I^*(f_n) &< S(h_{n-1} \wedge g_n) + \epsilon_{n-1} + 2^{-n}\epsilon \\ &= S(h_n) + \epsilon_n < S(h_n) + \epsilon \text{ since } \epsilon_n = \epsilon(\tfrac{1}{2} + \tfrac{1}{4} + \ldots + \tfrac{1}{2}n). \end{aligned}$$

Since $0 \leq h_n \leq g_n \leq f_n$ and $\lim f_n = 0$, necessarily $\lim h_n = 0$. Also $\{h_n\} \downarrow$ so by b-3, $\lim S(h_n) = 0$, so $\lim I^*(f_n) \leq \epsilon$. ϵ was arbitrary and therefore $\lim I^*(f_n) = 0$. This proves that I^* satisfies property b-3.

LEMMA 2.9. If $f \in A$, then $f = g - h$, where $g, h \in A$ and $g \geq 0$, $h \geq 0$.

Proof: Let $f \in A$. Then $0 \vee f$ and $0 \vee -f$ are non-negative functions in A. Since $f = (0 \vee f) - (0 \vee -f)$ the lemma is proved.

DEFINITION 2.10. If $f = g - h$, where $g \geq 0$, $h \geq 0$ then $I^*(f) = I^*(g) - I^*(h)$.

It is necessary to show that this definition of I^* is independent of the representation for f, and also that it is consistent with I^* on non-negative functions. Thus suppose $f = g_1 - h_1 = g_2 - h_2$ where g_i, h_i, $i = 1,2$ are all non-negative. Then $g_1 + h_2 = g_2 + h_1$ and $I^*(g_1 + h_2) = I^*(g_2 + h_1)$, thus $I^*(g_1) + I^*(h_2) = I^*(g_2) + I^*(h_1)$ by 2.5. Therefore $I^*(f) = I^*(g_1) - I^*(h_1) = I^*(g_2) - I^*(h_2)$ and so the definition is independent of the representation.

If $f \geq 0$ then $f = f - 0$ whence $I^*(f) = I^*(f) - I^*(0) = I^*(f)$. Thus the definition is consistent.

THEOREM 2.11. $I^*(cf) = cI^*(f)$.

Proof: Let $f \in A$. Then $f = g - h$, where $g \geq 0$, $h \geq 0$.

Case I. $c > 0$.

$$\begin{aligned} I^*(cf) &= I^*(cg - ch) \\ &= I^*(cg) - I^*(ch) \text{ by definition} \\ &= cI^*(g) - cI^*(h) \text{ by 2.4} \\ &= cI^*(f). \end{aligned}$$

Case 2. $c < 0$. $cf = -ch - (-cg)$ therefore

$$\begin{aligned} I^*(cf) &= I^*(-ch) - I^*(-cg) \text{ by definition} \\ &= -cI^*(h) + cI^*(g) \text{ by 2.4} \\ &= cI^*(f) \end{aligned}$$

Case 3. $c = 0$. $I^*(0 \cdot f) = I^*(0) = 0 = 0 \cdot I^*(f)$. Thus I^* satisfies property b-1.

THEOREM 2.12. $I^*(f_1 + f_2) = I^*(f_1) + I^*(f_2)$

Proof: Let $f_1, f_2 \in A$, and $f_1 = g_1 - h_1$, $f_2 = g_2 - h_2$ where $g_i, h_i \geq 0$, $i = 1,2$.

$(f_1 + f_2) = (g_1 + g_2) - (h_1 + h_2)$ therefore

$$\begin{aligned} I^*(f_1 + f_2) &= I^*(g_1 + g_2) - I^*(h_1 + h_2) \text{ by definition} \\ &= I^*(g_1) + I^*(g_2) - \left[I^*(h_1) + I^*(h_2)\right] \text{ by 2.5} \\ &= I^*(g_1) - I^*(h_1) + I^*(g_2) - I^*(h_2) \\ &= I^*(f_1) + I^*(f_2). \end{aligned}$$

Thus I^* satisfies property b-2.

Theorems 2.3, 2.8, 2.11 and 2.12 show that I^* is an I-integral.

Extension of the Mapping

Given an I-integral on A the next objective is to extend I to a larger collection of functions, which if possible, contains A as a proper subset. At the same time it is desired that I preserve the properties b-1, b-2, b-3, b-4 in so far as possible on the larger class of functions.

DEFINITION 3.1. If $\{f_n\} \uparrow$, $f_n \in A$, $n = 1,2,\ldots$ then $\lim f_n$ exists (in the extended reals) and the function f defined by $f = \lim f_n$ is said to be a function of class A^*.

THEOREM 3.2. $A \subset A^*$.

Proof: Let $f \in A$ and define $f_n = f$, $n = 1,2,\ldots$ Then $\{f_n\} \uparrow$ and so $\lim f_n = f \in A^*$.

THEOREM 3.3. If $f,g \in A^*$ and $a \in R$, $a \geq 0$, then $af \in A^*$, $f + g \in A^*$, $f \vee g \in A^*$, and $f \wedge g \in A^*$.

Proof: Only the proof that $f \vee g \in A^*$ will be given, as the other proofs are similar. Let $f, g \in A^*$. There exists $\{f_n\} \uparrow f$ and $\{g_n\} \uparrow g$ where $f_n, g_n \in A$, $n = 1,2,\ldots.$ Let $x \in X$ and $\epsilon > 0$. Assume $f(x) > g(x)$. There exists a positive integer N_x such that $n > N_x \Rightarrow f_n(x) \geq g_n(x)$, for if not, $g(x) = \lim g_n(x) \geq \lim f_n(x) = f(x)$. Also there exists a positive integer $N_{\epsilon,x}$ such that $m > N_{\epsilon,x} \Rightarrow \left|f_m(x) - f(x)\right| < \epsilon$. Choose $N = \max\{N_x, N_{\epsilon,x}\}$. Then $n > N \Rightarrow \left|(f_n \vee g_n)(x) - (f \vee g)(x)\right| = \left|f_n(x) - f(x)\right| < \epsilon$. Thus $\lim (f_n \vee g_n) = f \vee g$. Since $f_n \vee g_n \in A$, $n = 1,2,\ldots$ and $\{f_n \vee g_n\}\uparrow$, $f \vee g \in A^*$. If $f(x) = g(x)$ the proof is obvious, while if $f(x) < g(x)$ merely interchange the roles of f and g in the above argument. This completes the proof.

LEMMA 3.4. If $\{f_n\}\uparrow$, $f_n \in A$, $n = 1,2,\ldots$ and if $\lim f_n \geq h$ where $h \in A$, then $\lim I(f_n) \geq I(h)$.

Proof: Let $g_n = f_n \wedge h$. Then $g_n \in A$ and since $\lim f_n \geq h$, $\lim g_n = h$. $f_n \leq f_{n+1}$ therefore $g_n \leq g_{n+1}$, thus $\{g_n\}\uparrow h$, or equivalently, $\{h - g_n\} \downarrow 0$. For all n, $(h - g_n) \in A$, so by b-3 $\lim I(h - g_n) = 0$. That is $I(h) = \lim I(g_n)$. However $f_n \geq g_n \Rightarrow I(f_n) \geq I(g_n)$ for all n, therefore $\lim I(f_n) \geq \lim I(g_n) = I(h)$.

LEMMA 3.5. If $\{f_n\}\uparrow$ and $\{g_n\}\uparrow$, $f_n, g_n \in A$, $n = 1,2,\ldots$. and $\lim f_n \geq \lim g_n$, then $\lim I(f_n) \geq \lim I(g_n)$.

Proof: $\lim f_n \geq \lim g_n$ and $\{g_n\}\uparrow$ therefore $\lim f_n \geq g_m$, $m = 1,2,\ldots.$ Thus each g_m plays the role of the function h in 3.4 so $\lim I(f_n) \geq I(g_m)$, $m = 1,2,\ldots,$ and therefore $\lim I(f_n) \geq \lim I(g_n)$.

THEOREM 3.6. If $\{f_n\}\uparrow$ and $\{g_n\}\uparrow$, $f_n, g_n \in A$, $n = 1,2,\ldots$ and $\lim f_n = \lim g_n$ then $\lim I(f_n) = \lim I(g_n)$.

Proof: It is immediate from 3.5 that $\lim I(f_n) \geq \lim I(g_n)$ and $\lim I(g_n) \geq \lim I(f_n)$, which proves the theorem.

DEFINITION 3.7. If $f \in A^*$ and $f = \lim f_n$ where $\{f_n\}\uparrow$, $f_n \in A$, $n = 1,2,\ldots$ then $I(f) = \lim I(f_n)$.

From 3.6 it is clear that the value of $I(f)$ is independent of the sequence converging to f. Further, if $f \in A$, the value of $I(f)$ by definition 3.7 is consistent with the original value since $f = \lim f_n$ where $f_n = f$, $n = 1,2,\ldots$.

THEOREM 3.8. If $f \in A^*$, and $f \geq 0$, then $I(f) \geq 0$.

Proof: Let $f \in A^*$, $f \geq 0$. There exists $\{f_n\}\uparrow$, $f_n \in A$, $n = 1,2,\ldots$ such that $\lim f_n = f \geq 0$, hence by 3.4 $I(f) = \lim I(f_n) \geq I(0) = 0$.

THEOREM 3.9. If $f, g \in A^*$, and $f \geq g$ then $I(f) \geq I(g)$.

Proof: Let $f, g \in A^*$ and $f \geq g$. There exist $\{f_n\}\uparrow f$ and $\{g_n\}\uparrow g$, $f_n, g_n \in A$, $n = 1,2,\ldots$. Now $f = \lim f_n \geq \lim g_n = g$, therefore $I(f) = \lim I(f_n) \geq \lim I(g_n) = I(g)$ by 3.5.

THEOREM 3.10. If $f, g \in A^*$, $I(f + g) = I(f) + I(g)$.

Proof: Let $f, g \in A^*$. There exist $\{f_n\}\uparrow f$ and $\{g_n\}\uparrow g$ $f_n, g_n \in A$, $n = 1,2,\ldots$ and $\lim (f_n + g_n) = f + g \in A$ by 3.3. Now $I(f_n + g_n) = I(f_n) + I(g_n)$, $n = 1,2,\ldots$ and therefore

$$I(f + g) = \lim I(f_n + g_n) = \lim I(f_n) + \lim I(g_n)$$
$$= I(f) + I(g).$$

From this theorem an immediate result by mathematical induction is:

COROLLARY 3.11. If $f_i \in A^*$, $i = 1,2,\ldots, n$, then
$I(\sum_{i=1}^{n} f_i) = \sum_{i=1}^{n} I(f_i)$.

THEOREM 3.12. If $f \in A^*$ and $c \geq 0$ then $I(cf) = cI(f)$.

Proof: Let $f \in A^*$ and $c \geq 0$. There exists $\{f_n\} \uparrow f$, $f_n \in A$, $n = 1,2,\ldots$ therefore $\{cf_n\} \uparrow cf$ since $c \geq 0$. Thus $I(cf) = \lim I(cf_n) = c \lim I(f_n) = cI(f)$.

The preceding theorems show that I on A^* satisfies b-1 (when c is a non-negative constant), b-2, and b-4.

THEOREM 3.13. If $\{f_n\} \uparrow$ where $f_n \in A^*$, $n = 1,2,\ldots$ then $\lim f_n = f \in A^*$ and $I(f) = \lim I(f_n)$.

Proof: For each n, there exists a sequence $\{g_{nk}\}$ such that $\{g_{nk}\} \underset{k}{\uparrow} f_n$, where $g_{nk} \in A$, $k = 1,2,\ldots$. Let $h_{nk} = g_{1k} \vee g_{2k} \vee \cdots \vee g_{nk}$. A is closed under the operation v, so $h_{nk} \in A$. Now for $k > n$, $g_{nk} \leq h_{nk} \leq h_{kk} \leq f_1 \vee f_2 \vee \cdots \vee f_k = f_k$ therefore $f_n = \lim_k g_{nk} \leq \lim_k h_{kk} \leq \lim_k f_k = f$ and $f = \lim_n f_n \leq \lim_k h_{kk} \leq f$. Clearly $\{h_{kk}\} \uparrow$ so f is the limit of a non-decreasing sequence of functions of A. That is, $\{h_{kk}\} \uparrow f$, thus by definition $f \in A^*$ and $I(f) = \lim I(h_{nn})$. $h_{nn} \leq f_n$ therefore $I(h_{nn}) \leq I(f_n)$ $n = 1,2,\ldots$ by 3.9 $\Rightarrow I(f) = \lim I(h_{nn}) \leq \lim I(f_n)$. From above $f \geq f_n$, $n = 1,2,\ldots$ so $I(f) \geq I(f_n)$ $n = 1,2,\ldots$ by 3.9 $\Rightarrow$ $I(f) \geq \lim I(f_n)$ therefore $I(f) = \lim I(f_n)$.

In the preceding section, A was extended to a larger class of functions A^*. This new class of functions consisted of limits of non-decreasing sequences of functions from A.

Furthermore, the linear functional I, originally assumed on A, was shown to inherit certain important properties on A^*, namely; b-1 (for non-negative c), b-2, and b-4.

What is desired however, is a class of functions that includes A and a linear functional on this class that agrees with the original I-integral on A and also has the properties of b-1, b-2, b-3, and b-4.

Upper and Lower Integrals

DEFINITION 4.1. Let f be any real valued function defined on the elements of X. Then $U(f) = \inf \{ I(g) | g \in A^* \text{ and } g \geq f \}$ is said to be the <u>upper semi-integral</u> of f. If there is no $g \in A^*$ such that $g \geq f$ then define $U(f) = +\infty$.

THEOREM 4.2. If $c > 0$, $U(cf) = cU(f)$.

Proof: Let $c > 0$. Then

$\{g | g \in A^* \text{ and } g \geq f\} = \{g | c\, g \in A^* \text{ and } cg \geq cf\}$

$$\begin{aligned} \text{Thus } U(cf) &= \inf \{I(cg) \mid cg \in A^* \text{ and } cg \geq cf\} \\ &= \inf \{cI(g) \mid g \in A^* \text{ and } g \geq f\} \\ &= cU(f). \end{aligned}$$

THEOREM 4.3. $U(f_1 + f_2) \leq U(f_1) + U(f_2)$.

Proof: Let $g_1, g_2 \in A^*$ such that $f_1 \leq g_1$ and $f_2 \leq g_2$. Then $f_1 + f_2 \leq g_1 + g_2$ therefore

$U(f_1 + f_2) \leq I(g_1 + g_2) = I(g_1) + I(g_2)$ by 3.10.

For fixed g_1 and for all $g_2 \geq f_2$, the inequality holds. Thus $U(f_1 + f_2) \leq I(g_1) + U(f_2)$. Since this inequality holds for all $g_1 \geq f_1$ it follows that $U(f_1 + f_2) \leq U(f_1) + U(f_2)$.

THEOREM 4.4. If $f \leq g$ then $U(f) \leq U(g)$.

Proof: Let $h \in A^*$ and $h \geq g$. Then $h \geq f$ so $\{h | h \in A^*$ and $h \geq f\} \supset \{h | h \in A^*$ and $h \geq g\}$. The theorem follows from the definition of U.

DEFINITION 4.5. $L(f) = -U(-f)$. $L(f)$ is called the <u>lower semi-integral</u> of f.

THEOREM 4.6. $L(f) \leq U(f)$ for all f.

Proof: $0 = U(0) = U(f - f) \leq U(f) + U(-f)$ by 4.3 but $U(-f) = -L(f)$ by definition. Thus, $L(f) \leq U(f)$.

THEOREM 4.7. $U(f \vee g) + U(f \wedge g) \leq U(f) + U(g)$.

Proof: Let $h_1, h_2 \in A^*$, $h_1 \geq f$, $h_2 \geq g$. Then $h_1 \vee h_2 \geq f \vee g$ and $h_1 \wedge h_2 \geq f \wedge g$. Therefore

$$U(f \vee g) + U(f \wedge g) \leq I(h_1 \vee h_2) + I(h_1 \wedge h_2)$$
$$= I(h_1) + I(h_2) \text{ since}$$

$(h_1 \vee h_2) + (h_1 \wedge h_2) = h_1 + h_2$. By the same argument as in the proof of 4.3 the theorem follows.

COROLLARY 4.8. $U(|f|) - L(|f|) \leq U(f) - L(f)$.

Proof: $|f| = f \vee (-f)$ and $-|f| = f \wedge (-f)$

Thus $U(|f|) + U(-|f|) \leq U(f) + U(-f)$ by 4.7. That is,

$$U(|f|) - L(|f|) \leq U(f) - L(f).$$

Definition and Properties of the Integral

This section deals with a class of functions called the summable functions and a linear functional on this class called the Daniell integral.

DEFINITION 5.1. f is said to be <u>summable</u> if and only if

$U(f)$ is finite and $U(f) = L(f)$. In this case define $\int f$ by: $\int f = U(f) = L(f)$. $\int f$ is called the <u>Daniell integral</u>.

THEOREM 5.2. If f is summable, and $f \geq 0$ then $\int f \geq 0$.

Proof: Since $0 \leq f$, $0 = U(0) \leq U(f)$ by 4.4. Therefore $\int f = U(f) \geq 0$.

THEOREM 5.3. If c is a real number and f is summable, then cf is summable and $\int cf = c\int f$.

Proof: Let $c \geq 0$. By 4.2 $U(cf) = cU(f) = c\int f$. Note that 4.2 holds for $c = 0$ when $U(f)$ is finite, as is the case here. Also, $-L(cf) = U(-cf) = cU(-f) = -cLf = -c\int f$. Therefore $L(cf) = c\int f$, and so for $c \geq 0$, $\int (cf) = U(cf) = L(cf) = c\int f$. Let $c < 0$. Then $U(cf) = U(-c)(-f) = -cU(-f) = cL(f) = c\int f$ and $-L(cf) = U(-cf) = -cUf = -c\int f$. That is, $L(cf) = c\int f$. Thus for all real c, $\int cf = L(cf) = U(cf) = c\int f$.

THEOREM 5.4. If f_1 and f_2 are summable, so is $f_1 + f_2$ and $\int (f_1 + f_2) = \int f_1 + \int f_2$.

Proof: $U(f_1 + f_2) \leq U(f_1) + U(f_2)$ by 4.3

$$= \int f_1 + \int f_2$$

$-L(f_1 + f_2) = U(-f_1 - f_2) \leq U(-f_1) + U(-f_2)$ by 4.3

$$= -\int f_1 - \int f_2 \text{ by 5.3}$$

therefore $L(f_1 + f_2) \geq \int f_1 + \int f_2$. However by 4.6 $U(f_1 + f_2) \geq L(f_1 + f_2) \Rightarrow$

$$U(f_1 + f_2) = \int f_1 + \int f_2, \text{ and also}$$

$$L(f_1 + f_2) = \int f_1 + \int f_2,$$

therefore $f_1 + f_2$ is summable and

$$\int (f_1 + f_2) = \int f_1 + \int f_2.$$

THEOREM 5.5. If f_1 and f_2 are summable, so are $f_1 \vee f_2$ and $f_1 \wedge f_2$.

Proof: Suppose f_1 and f_2 are summable. Then

$$U(f_1 \vee f_2) + U(f_1 \wedge f_2) \leq U(f_1) + U(f_2) = \int f_1 + \int f_2,$$

and $U(-f_1 \vee -f_2) + U(-f_1 \wedge -f_2) \leq U(-f_1) + U(-f_2)$ by 4.7.

Now $-f_1 \vee -f_2 = -(f_1 \wedge f_2)$, and $-f_1 \wedge -f_2 = -(f_1 \vee f_2)$

therefore $-L(f_1 \wedge f_2) - L(f_1 \vee f_2) \leq -L(f_1) - L(f_2)$ by 4.5

$$= -\int f_1 - \int f_2.$$

Hence $L(f_1 \wedge f_2) + L(f_1 \vee f_2) \geq \int f_1 + \int f_2$.

$$\geq U(f_1 \vee f_2) + U(f_1 \wedge f_2).$$

That is $\left[U(f_1 \vee f_2) - L(f_1 \vee f_2)\right] + \left[U(f_1 \wedge f_2) - L(f_1 \wedge f_2)\right] \leq 0.$

Since each of these differences is non-negative, each must be zero, and so the theorem is proved.

THEOREM 5.6. If f is summable, $|f|$ is summable and $\left|\int f\right| \leq \int |f|$.

Proof: $U(|f|) - L(|f|) \leq U(f) - L(f)$ by 4.8

$$= \int f - \int f = 0 \text{ but}$$

$U(|f|) - L(|f|) \geq 0$ by 4.6 thus

$U(|f|) = L(|f|)$ and $|f|$ is summable.

Since $-|f| \leq f \leq |f|$

$$-\int |f| \leq \int f \leq \int |f|.$$

That is, $\left|\int f\right| \leq \int |f|$.

THEOREM 5.7. If $\{f_n\} \uparrow f$, where f_n is summable, $n = 1,2,\ldots$ and if $\lim \int f_n$ is finite, then $\lim f_n = f$ is summable and $\int f = \lim \int f_n$. If $\lim \int f_n = +\infty$, $L(f) = +\infty$.

Proof: $\{f_n\} \uparrow f$, so $f_n \leq f$, $n = 1,2,\ldots \Rightarrow -f \leq -f_n$, hence $U(-f) \leq U(-f_n)$ by 4.4 or $-L(f) \leq -L(f_n) = -\int f_n$, $n = 1,2,\ldots$. Thus (1) $L(f) \geq \lim \int f_n$ which proves the last part of the theorem.

Note that for any summable function f and for any $\epsilon > 0$, there exists a function $g \in A^*$ such that $g \geq f$ and $I(g) < \int f + \epsilon$, for if no such g existed, $\int f = U(f) \neq \inf \{I(g) | g \in A^* \text{ and } g \geq f\}$. Since f_n is summable for each n, $f_n - f_{n-1}$ is summable for each n by 5.3 and 5.4.

Let $\epsilon > 0$, and let $g_n \in A^*$, $n = 1,2,\ldots$ such that $g_1 \geq f_1$ and $I(g_1) < \int f_1 + \epsilon/2$,

$g_n \geq f_n - f_{n-1}$ and $I(g_n) < \int (f_n - f_{n-1}) + 2^{-n}\epsilon$ for $n \geq 2$.

It is clear from the above remark that such g_n exist.

$\{f_n\}\uparrow$ so $g_n \geq f_n - f_{n-1} \geq 0$. Let $h_n = \sum_{i=1}^{n} g_i$.

Then $h_n \in A^*$ by 3.3. Clearly $\{h_n\} \uparrow$, therefore $\lim h_n \in A^*$ and $I(\lim h_n) = \lim I(h_n)$ by 3.13.

$$h_n = \sum_{i=1}^{n} g_i \geq f_1 + \sum_{i=2}^{n} (f_i - f_{i-1}) = f_n$$

so $\lim h_n \geq \lim f_n = f$. Thus

$U(f) \leq I(\lim h_n) = \lim I(h_n)$ by 4.1.

$$I(h_n) = I(\sum_{i=1}^{n} g_i) = \sum_{i=1}^{n} I(g_i) \text{ by } 3.11$$

$$< \int f_1 + \sum_{i=2}^{n} \int (f_i - f_{i-1}) + \sum_{i=1}^{n} 2^{-i}\epsilon$$

$$= \int f_1 + \sum_{i=2}^{n} (\int f_i - \int f_{i-1}) + \sum_{i=1}^{n} 2^{-i}\epsilon$$

$$< \int f_n + \epsilon. \text{ Thus}$$

$\lim I(h_n) \leq \lim \int f_n + \epsilon$. That is $U(f) \leq \lim \int f_n + \epsilon$. Since ϵ is any positive number, $L(f) \leq U(f) \leq \lim \int f_n$. However, from (1) above, $U(f) \geq L(f) \geq \lim \int f_n$ therefore $L(f) = U(f) = \lim \int f_n$. Hence if $\lim \int f_n$ is finite, f is summable and $\int f = \lim \int f_n$.

In Lebesgue integration, 5.7 is called the Lebesgue Monotone Convergence Theorem.

THEOREM 5.8. If $\lim f_n = f$, where f_n is summable, $n = 1,2,\ldots$ and if there exists a summable function j such that $|f_n| \leq j$ for all n, then f is summable, $\lim \int f_n$ exists and $\lim \int f_n = \int f$.

Proof: Let $g_{r,s} = f_r \vee f_{r+1} \vee \cdots \vee f_{r+s}$. Then $g_{r,s} \leq g_{r,s+1} \leq \cdots$. f_n is summable for all n, so $g_{r,s}$ is summable by 5.5. Also $f_n \leq j$ so $g_{r,s} \leq j \Rightarrow \int g_{r,s} \leq \int j$ = finite, since j is summable. Thus $\lim_s \int g_{r,s}$ is finite. Letting $g_r = \lim_s g_{r,s}$, g_r is summable by 5.7. Now $g_r \geq g_{r+1} \geq \cdots$ and $\lim g_r = f$ since $\lim f_n = f$. For let $\epsilon > 0$. There exists an N such that $n > N \Rightarrow |f_n - f| < \epsilon/2$. Since $g_n = \lim_s [f_n \vee f_{n+1} \vee \cdots \vee f_{n+s}]$ there exists an $m \geq n$ such that $|g_n - f_m| < \epsilon/2$. Thus for all $n > N$,

$|g_n - f| \leq |g_n - f_m| + |f_m - f| < \epsilon/2 + \epsilon/2 = \epsilon$.

That is $\lim_n g_n = f$. $f_n \geq -j$, $n = 1,2,\ldots$ so $g_{r,s} \geq -j$ or $-g_{r,s} \leq j \Rightarrow -g_r \leq j$ thus $\int(-g_r) \leq \int j$ = finite. Since $-g_r \leq -g_{r+1} \leq \cdots$ and $\lim(-g_r) = -f$, $-f$ is summable and $\lim \int(-g_n) = \int(-f)$ by 5.7. $-f$ is summable, so f is summable by 5.3 and $\lim \int g_n = \int f$. Let $\epsilon > 0$. There exists an r_1

such that $\int g_r < \int f + \epsilon$ whenever $r > r_1$. Now $f_r \leq g_{r,s} \leq g_r$ therefore $\int f_r \leq \int g_r < \int f + \epsilon$. Let $h_{r,s} = f_r \wedge f_{r+1} \wedge \cdots \wedge f_{r+s}$. Then $h_{r,s} \geq h_{r,s+1} \geq \cdots$. Let $\lim h_{r,s} = h_r$. By an argument similar to that above, it can be shown that h_r is summable, $h_r \leq h_{r+1} \leq \cdots$, $\lim h_r = f$ and $\lim \int h_r = \int f$. Hence there exists an r_2 such that $\int h_r > \int f - \epsilon$ when $r > r_2$. Since $f_r \geq h_{r,s} \geq h_r$, $\int f_r \geq \int h_r > \int f - \epsilon$. Thus let $r = \max \{r_1, r_2\}$. Then $\left|\int f_n - \int f\right| < \epsilon$ when $n > r$. That is $\lim \int f_n = \int f$.

Theorems 5.2 through 5.8 show that the summable functions form a vector lattice. Furthermore, $\int$ is a linear functional satisfying b-1, b-2, b-3, and b-4 on the class of summable functions. It should be noted that b-3 follows as a special case of 5.8.

A natural question is whether the class of summable functions includes all the functions of A^*. From the definition of $\int$ it is clear that if $f \in A^*$ and $I(f) = +\infty$, then f is not summable. However, if $I(f)$ is finite, f is summable.

THEOREM 5.9. If $f \in A^*$ and $I(f)$ is finite, f is summable.

Proof: $U(f) = \inf \{I(g) | g \in A^* \text{ and } g \geq f\}$. Since $f \in A^*$, $I(f) \leq I(g)$ for all $g \in A^*$ such that $g \geq f$ by 3.9. It follows that $U(f) = I(f)$. Thus $U(f)$ is finite if and only if $I(f)$ is finite. $f \in A^*$ so there exists $\{f_n\} \uparrow f$, $f_n \in A \subset A^*$, $n = 1,2,\ldots$ and $\lim I(f_n) = I(f)$. For all n, $I(f_n)$ is finite and $I(f_n) \leq I(f)$, thus $-I(f_n) = I(-f_n) \geq -I(f)$. Let $\epsilon > 0$. There exists an N such that $n > N \Rightarrow I(f_n) > I(f) - \epsilon$. That is

$-I(f_n) = I(-f_n) < -I(f) + \epsilon$. Hence $-I(f) \leq I(-f_n) < -I(f) + \epsilon$. ϵ was arbitrary, therefore $\lim I(-f_n) = -I(f)$. Since $\{-f_n\} \downarrow -f$ it follows from the definition that $U(-f) = -I(f)$. Thus, $-L(f) = -I(f)$. Therefore $L(f) = U(f) = I(f)$ is finite and f is summable. Since I is finite on A, and $A \subset A^*$, the following corollary is proved.

COROLLARY 5.10. The class of summable functions includes A.

McShane[7] uses Daniell's approach to introduce the Lebesgue integral. The set X is a closed interval while A is the class of continuous functions on X. The functional I is taken as the Riemann integral over A. A^* is defined as the class of lower semi-continuous functions on X that are bounded below. I is extended to A^* by $I(u) = \sup \{I(g) | g \in A \text{ and } g \leq u\}$. For arbitrary f defined on X, U(f) is defined by $U(f) = \inf \{I(u) | u \in A^* \text{ and } u \geq f\}$. L is defined in an analogous way so that $U(f) = -L(-f)$ is a theorem in McShane's treatment rather than a definition. f is then said to be Lebesgue integrable if and only if $U(f) = L(f)$ and U(f) is finite. Thus the Lebesgue integral is a special case of the Daniell integral.

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[7]Edward James McShane, Integration, Princeton University Press, Princeton, 1944. pp. 52-75.

Summary

Chapter II details the construction of a linear functional called the Daniell integral. A vector space A of bounded real valued functions defined on an arbitrary set X was postulated. In addition, A was assumed closed under the operations v and $\wedge$. An order preserving linear functional I, on A into the reals was also postulated with the additional condition that $I(f_n)$ converges to zero where $\{f_n\}$ is a non-increasing sequence of functions of A whose limit is zero.

A class of functions, A^*, was then defined as the limit functions of non-decreasing sequences of functions of A. The functional I was then extended to A^* in a natural way, that is, if $\{f_n\} \uparrow f$, $I(f) = \lim I(f_n)$.

For arbitrary real valued functions f having X as domain, an upper integral U was defined as $U(f) = \inf\{I(g) \mid g \in A^* \text{ and } g \geq f\}$. A lower integral L was defined as $L(f) = -U(-f)$. f was said to be summable if $U(f) = L(f)$ and $U(f)$ was finite. The Daniell integral $\int$ was defined on summable functions by $\int f = U(f) = L(f)$. It was shown that the Daniell integral satisfies the following conditions:

i) The class of summable functions is a linear lattice space on the set X.

ii) The Daniell integral is an order preserving linear functional on the class of summable functions.

iii) An analogue of the Lebesgue Monotone Convergence

Theorem (for Lebesgue integrals) exists for the Daniell integral over the summable functions.

Finally, it was observed that the Daniell integral includes the Lebesgue integral as a special case.

CHAPTER III

A GENERAL INTEGRAL DEFINED BY A NORM

This chapter presents the definition and elementary properties of a general integral as developed by M. H. Stone.

Postulates

A is a non-empty class of real valued functions called elementary functions, defined on an arbitrary non-void domain X, such that for all real numbers a, and for all f and g in A:

a-1. af in A,

a-2. $f + g$ in A,

a-3. $|f|$ in A.

E is a real valued linear functional defined on A. That is, for all real numbers a, and for all f, g in A:

b-1. $E(af) = aE(f)$,

b-2. $E(f + g) = E(f) + E(g)$,

b-3. $E(|f|) \geq 0$.

The following additional condition is also assumed:

b-4. If f and f_n are in A and $|f| \leq \sum |f_n|$, then $E(|f|) \leq \sum E(|f_n|)$.

Let B be the set of all extended real valued functions defined on X. In B such expressions as $0 \cdot f$, $f + g$ and $f - g$ could be awkward in view of the fact that $+\infty$ and $-\infty$ are permissable function values. For the moment it will be assumed that $0 \cdot f$, $f + g$ and $f - g$ is any function in A that assumes the respective values $0 \cdot f(x)$, $f(x) + g(x)$, $f(x) - g(x)$ for all x where the latter quantities are defined. It is admitted that this convention allows ambiguity, but it will be seen later that this situation raises no serious obstacles.

DEFINITION 1.1. For each $f \in B$, let N be the extended real valued function defined by

$$N(f) = \inf \{g \mid g = \sum E(|f_n|),\ |f| \leq \sum |f_n|,\ f_n \in A\}.$$

If the inequality $|f| \leq \sum |f_n|$ cannot be realized, or the series $\sum E(|f_n|)$ diverges, let $N(f) = +\infty$.

Some of the principal properties of N will now be obtained. Unless stated otherwise, all functions in this section are assumed to be in B.

THEOREM 1.2. If $f \in B$, $0 \leq N(f) \leq +\infty$.

Proof: $E(|f_n|) \geq 0$ for all $f_n \in A$ therefore $\sum E(|f_n|) \geq 0$.

THEOREM 1.3. If $a \in R$, and $f \in B$, then $N(af) = |a| N(f)$ unless $a = 0$ and $N(f) = +\infty$.

Proof: Let $a = 0$ and $N(f)$ be finite. Then $N(0 \cdot f) = N(0) = 0 = 0 \cdot N(f)$ which disposes of this case. Thus assume $a \neq 0$.

$$\begin{aligned} N(af) &= \inf \{g \mid g = \sum E(|af_n|),\ |af| \leq \sum |af_n|,\ f_n \in A\} \\ &= \inf \{g \mid g = |a| \sum E(|f_n|),\ |a|\,|f| \leq |a| \sum |f_n|,\ f_n \in A\} \\ &= |a| \inf \{g \mid g = \sum E(|f_n|),\ |f| \leq \sum |f_n|,\ f_n \in A\} \\ &= |a|\, N(f). \end{aligned}$$

THEOREM 1.4. If $|f| \leq \sum |f_n|$ then $N(f) \leq \sum N(f_n)$.

Proof: For each n,

$N(f_n) = \inf \{g_n | g_n = \sum_j E(|h_{nj}|),\ |f_n| \leq \sum_j |h_{nj}|,\ h_{nj} \in A\}$

hence $\sum N(f_n) = \inf \{\sum_n g_n | \sum_n g_n = \sum_n \sum_j E(|h_{nj}|),$

$\sum_n |f_n| \leq \sum_n \sum_j |h_{nj}|,\ h_{nj} \in A\}.$

but $|f| \leq \sum_n |f_n| \leq \sum_n \sum_j |h_{nj}|$

so $N(f) \leq \sum_n \sum_j E(|h_{nj}|)$ where $\sum |f_n| \leq \sum \sum h_{nj}$.

That is, $N(f) \leq \sum N(f_n)$.

COROLLARY 1.5. $N(f + g) \leq N(f) + N(g)$.

Proof: For all f, g, $|f + g| \leq |f| + |g|$, thus by 1.4, $N(f + g) \leq N(f) + N(g)$.

COROLLARY 1.6. $N(f) \leq N(g)$ whenever $|f| \leq |g|$.

Proof: Direct application of 1.4.

The following theorem is an immediate result of the definition.

THEOREM 1.7. $N(|f|) = N(f)$.

THEOREM 1.8. If $f \in A$, $N(f) = E(|f|)$.

Proof: Since $|f| \leq |f|$ and $f \in A$, $N(f) \leq E(|f|)$ by the definition of N. Since $E(|f|) \leq \sum E(|f_n|)$ whenever $f_n \in A$ and $|f| \leq \sum |f_n|$, it follows that $E(|f|) \leq N(f) = \inf \{g | g = \sum E(|f_n|),\ f_n \in A,\ |f| \leq \sum |f_n|\}$. Thus $N(f) = E(f)$.

Properties of the Norm

Before defining the general integral, an important subset of B will be considered.

DEFINITION 2.1. $F = \{f \mid f \in B \text{ and } N(f) < +\infty\}$.
N restricted to F is a pseudo-norm on F, as is evident from 1.2, 1.3, and 1.5, however, N does not induce a metric since $N(f - g) = 0$ does not imply that $f = g$. The usual procedure for overcoming this difficulty is displayed below.

In this section, all functions are assumed to be in F unless otherwise noted.

THEOREM 2.2. $N(f - g)$ is a pseudo-metric on F.

Proof: Let f, g, and $h \in F$.

1. $N(f - g) = |-1|\, N(g - f) = N(g - f)$ by 1.3 and 1.7.
2. $N(f - f) = N(0) = 0$.
3. $N(f - g) = N((f - h) + (h - g)) \leq N(f - h) + N(h - g)$ by 1.5.

THEOREM 2.3. The relation $N(f - g) = 0$ is an equivalence relation on F.

Proof: 1. Reflexivity follows from 2.2, part 2.

2. Symmetry is a special case of 2.2, part 1.

3. Suppose $N(f - g) = 0$ and $N(g - h) = 0$

Then $N(f - h) \leq N(f - g) + N(g - h) = 0$ by 2.2, part 3. but $N(f - h) \geq 0$ by 1.2. Thus $N(f - h) = 0$.

The relation $N(f - g) = 0$ thus partitions F into equivalence classes.

DEFINITION 2.4. Let A, B be equivalence classes of F. Then $N(A - B) = N(a - b)$ where $a \in A$, $b \in B$.

THEOREM 2.5. $N(A - B)$ is a uniquely defined distance function on the pairs of equivalence classes of F.

Proof: Let $a, a_1 \in A$, $b, b_1 \in B$, where A and B are equivalence classes of F.

$N(a - b) \leq N(a - a_1) + N(a_1 - b) = N(a_1 - b)$

$N(a_1 - b) \leq N(a_1 - b_1) + N(b_1 - b) = N(a_1 - b_1)$.

Thus $N(a - b) \leq N(a_1 - b_1)$. This inequality can be reversed by the same argument, thus $N(a - b) = N(a_1 - b_1)$.

It is the usual practice to refer to $N(g)$ where $g \in F$, when, properly speaking, $N(G)$ is meant, where G is the equivalence class containing g. Adopting this practice, it is now clear that $N(f - g) = 0$ if and only if $f = g$. Thus, it is permissible to say that $N(f - g)$ is a metric on F.

As was stated before, the functional N is a pseudo-norm on F. It is now appropriate to discuss some properties of the so called "null functions" and "null sets".

DEFINITION 2.6. $f \in F$ is called a <u>null function</u> if $N(f) = 0$.

DEFINITION 2.7. Let $C \subset A$. Then C is said to be a <u>null set</u> if and only if the characteristic function of C, X_C, is a null function.

Note, $X_C(x) = \begin{cases} 1 \text{ if } x \in C \\ 0 \text{ if } x \notin C. \end{cases}$

It will be convenient to use the phrase "almost everywhere", to signify "with the exception of the points of a certain null set".

THEOREM 2.8. $f \in F$ is a null function if and only if $f(x) = 0$ almost everywhere.

Proof: First, suppose $f(x) = 0$ almost everywhere. Then $N(X_C) = 0$ where $C = \{x | f(x) \neq 0\}$, and $N(nX_C) = 0$, $n = 1,2,\ldots$ by 1.3. Since $|f| \leq \sum_n nX_C$, $N(f) \leq \sum N(nX_C) = 0$ by 1.4. Thus f is a null function.

Next, suppose $N(f) = 0$, and let $C = \{x | f(x) \neq 0\}$. Then $X_C \leq \sum_n n|f|$ hence by 1.4 $N(X_C) \leq \sum N(n\,(f)) = \sum nN(f) = 0$. Therefore C is a null set and $f(x) = 0$ almost everywhere.

THEOREM 2.9. If $G \subset A$ is covered by a countable family of null sets, G is a null set.

Proof: Let $\bigcup_{n=1}^{\infty} G_n$ be a covering of G, where $N(X_{G_n}) = 0$, $n = 1,2,\ldots.$ Clearly $X_G \leq \sum_n X_{G_n}$, therefore by 1.4 $N(X_G) \leq \sum_n N(X_{G_n}) = 0$. Thus G is a null set.

THEOREM 2.10. If $f \in F$, f is finite almost everywhere.

Proof: Let $f \in F$, and $C = \{x | \, |f(x)| = +\infty\}$. Clearly $X_C \leq \frac{1}{n} |f|$, $n = 1,2,\ldots$ hence, $N(X_C) \leq \frac{1}{n} N(f) = \frac{M}{n}$, $n = 1,2,\ldots,$ where $N(f) = M$ is finite since $f \in F$. It follows that $N(X_C) \leq \lim_n \frac{M}{n} = 0$, so C is a null set, and f is finite almost everywhere.

Properties 2.8, 2.9, and 2.10 indicate that the null sets here play a role analogous to the sets of measure zero in the Lebesgue theory.

At this point it should be noted that such expressions as $0 \cdot f$, $f + g$, and $f - g$ are not ambiguous since, in view of 2.10, each of these expressions is finite almost everywhere. That is, if two functions in F are equal to one of the above expressions, the two functions differ at most on a null set, and therefore both functions are in the same equivalence class.

Clearly, F is a vector space over the reals, indeed, using the norm N, the following important result will be established.

THEOREM 2.11. The normed vector space F is complete.

Proof: Let $\{f_n\}$ be a Cauchy sequence of functions of F. Then there exists an n such that $m > n \Rightarrow N(f_n - f_m) < 2^{-1}$. Since the positive integers are well-ordered, there is a smallest such n, say n_1 such that this inequality holds. Let $g_1 = f_{n_1}$. In general let $g_p = f_{n_p}$, where n_p is the smallest index such that $m > n_p \Rightarrow N(f_{n_p} - f_m) < 2^{-p}$.

Let $g = |g_1| + \sum |g_{n+1} - g_n|$. Then

$$N(g) \leq N(|g_1|) + \sum N(|g_{n+1} - g_n|) \text{ by } 1.4$$
$$= N(g_1) + \sum N(g_{n+1} - g_n) \text{ by } 1.7$$
$$< N(g_1) + \sum 2^{-n} < +\infty.$$

Thus $g \in F$ and so g is finite almost everywhere by 2.10. That is the series $|g_1| + \sum |g_{n_1} - g_n|$ converges almost everywhere, and thus so does the series $g_1 + \sum (g_{n+1} - g_n)$.

$$\text{Let } f(x) = \begin{cases} 0 \text{ if } g(x) = \infty \\ g_1(x) + \sum (g_{n+1}(x) - g_n(x)) \text{ if } g(x) \text{ is finite.} \end{cases}$$

Then $|f| \leq g = |g|$ and so $N(f) \leq N(g) < +\infty$ by 1.6 and therefore $f \in F$. That is, $f = g_1 + \sum (g_{n+1} - g_n)$ almost everywhere, and $f - g_k = \sum_{n=k}^{\infty} (g_{n+1} - g_n)$ almost everywhere.

Thus $|f - g_k| = |\sum_{n=k}^{\infty} (g_{n+1} - g_n)| \leq \sum_{n=k}^{\infty} |g_{n+1} - g_n|$

so $N(f - g_k) \leq \sum_{n=k}^{\infty} N(g_{n+1} - g_n) = 2^{-k+1}$ by 1.4. That is, $\lim_k N(f - g_k) = 0$ almost everywhere.

From the definition of g_n, and the fact that $\{f_n\}$ is a Cauchy sequence it is clear that $\lim_{n,k} N(g_k - f_n) = 0$.

Since $N(f - f_n) \leq N(f - g_k) + N(g_k - f_n)$ by 2.2 it follows that $\lim N(f - f_n) = 0$ as n (and k) $\to \infty$.

Now $N(f) = N(f - f_n + f_n) \leq N(f - f_n) + N(f_n)$ by 1.5 therefore $N(f) - N(f_n) \leq N(f - f_n)$. The other inequality needed to conclude that $|N(f) - N(f_n)| \leq N(f - f_n)$ can be obtained in the same way. Therefore

$\lim |N(f) - N(f_n)| \leq \lim N(f - f_n) = 0$. That is,

$\lim N(f_n) = N(f)$. Thus the Cauchy sequence $\{f_n\}$ converges to $f \in F$, and so F is complete.

Note: In view of 2.11, the normed vector space F is a Banach space.

DEFINITION 2.12. $f^+ = \frac{1}{2}(|f| + f)$, $f^- = \frac{1}{2}(|f| - f)$, where $f \in F$. It is clear that f^+ and f^- are both non-negative, and that both are in F if $f \in F$.

DEFINITION 2.13. If $f \in F$, $G(f) = N(f^+) - N(f^-)$.

An interesting result that will be needed later is the fact that G is a continuous mapping. To prove this, the following two lemmas are needed.

LEMMA 2.14. $|N(f^+) - N(g^+)| \leq N(f^+ - g^+)$ and $|N(f^-) - N(g^-)| \leq N(f^- - g^-)$.

Proof: Only the first statement will be proved, as the proof of the second is identical.

$N(f^+) = N(f^+ - g^+ + g^+) \leq N(f^+ - g^+) + N(g^+)$ by 1.5.

That is, 1) $N(f^+) - N(g^+) \leq N(f^+ - g^+)$

Similarly $N(g^+) - N(f^+) \leq N(g^+ - f^+)$

$= N(f^+ - g^+)$ by 1.7. Thus

$-N(f^+ - g^+) \leq N(f^+) - N(g^+)$. This result with 1) completes the proof.

LEMMA 2.15. $|f^+ - g^+| \leq |f - g|$ and $|f^- - g^-| \leq |f - g|$.

Proof: $|f^+ - g^+| = \left|\frac{1}{2}[|f| + f - (|g| + g)]\right|$. By taking cases the proof is easily finished. Similarly for the second statement.

THEOREM 2.16. G is a continuous mapping of F into R.

Proof: Let $f \in F$, and $\epsilon > 0$. Then for all g such that $N(f - g) < \epsilon/2$,

$$\begin{aligned} |G(f) - G(g)| &= |N(f^+) - N(f^-) - N(g^+) + N(g^-)| \\ &\leq |N(f^+) - N(g^+)| + |N(f^-) - N(g^-)| \\ &\leq N(f^+ - g^+) + N(f^- - g^-) \text{ by 2.14} \\ &\leq 2\,N(f - g) \text{ by 2.15 and 1.6.} \\ &< \epsilon. \end{aligned}$$

An important consideration is whether the functional G is consistent with E on those functions in $A \cap F$. Thus the following theorem.

THEOREM 2.17. $A \subset F$, and if $f \in A$, $G(f) = E(f)$.

Proof: Let $f \in A$. By hypothesis E(f) is finite for all $f \in A$, and since $E(f) = N(f)$ by 1.8, $f \in F$. It is also clear from the hypotheses on A, that $f^+, f^- \in A$ when $f \in A$. Thus $G(f) = N(f^+) - N(f^-) = E(f^+) - E(f^-)$

$= E(f^+ - f^-)$ by b-2

$= E(f)$. This completes the proof.

Definition of the Integral

DEFINITION 3.1. Let A^* be the closure of A in F, and L the contraction of G to A^*. If $f \in A^*$, f is said to be integrable, and its general integral is taken to be $L(f) = N(f^+) - N(f^-)$.

It is evident from the definition, that the integrable functions are those functions that can be approximated by elementary functions in the sense of the norm for F.

In this section it will be shown that A^* and L enjoy the properties assumed for F and E respectively, in a-1, a-2, a-3, b-1, b-2, and b-3.

THEOREM 3.2. A^* is a normed subspace of F with norm $L(|f|)$.

Proof: Let $f \in A^*$. Then

$$L(|f|) = N(|f|^+) - N(|f|^-)$$
$$= N(\tfrac{1}{2}(|f| + |f|)) - N(\tfrac{1}{2}(|f| - |f|))$$
$$= N(|f|) = N(f).$$

Since N has been shown to be a norm on F, it follows that $L(|f|)$ is a norm on A^*.

THEOREM 3.3. $f \in A^*$ and $a \in R \Rightarrow af \in A^*$.

Proof: Let $f \in A^*$, $a \in R$, $a \neq 0$, and $\epsilon > 0$. By 3.1 it follows that there exists a function $h \in A$ such that $N(f - h) < \frac{\epsilon}{|a|}$. Now $ah \in A$ by a-1 and $N(af - ah) = |a|N(f - h) < \epsilon$, thus $af \in A^*$.

THEOREM 3.4. $f, g \in A^* \Rightarrow f + g \in A^*$.

Proof: Let $\epsilon > 0$. There exists $h_1, h_2 \in A$ such that $N(f - h_1) < \epsilon/2$ and $N(g - h_2) < \epsilon/2$. Now $h_1 + h_2 \in A$ by a-2, and $N(f + g - (h_1 + h_2)) \leq N(f - h_1) + N(g - h_2)$ by 1.5

$$< \epsilon/2 + \epsilon/2 = \epsilon$$

thus $f + g \in A^*$.

THEOREM 3.5. If $f \in A^*$, $|f| \in A^*$.

Proof: Let $\epsilon > 0$. There exists an $h \in A$ such that $N(f - h) < \epsilon$. Also $|h| \in A$ by a-3. Now $\big||f| - |h|\big| \leq |f - h|$ therefore $N(|f| - |h|) = N(\big||f| - |h|\big|)$ by 1.7

$\leq N(|f - h|)$ by 1.6

$= N(f - h)$ by 1.7

$< \epsilon$.

therefore $|f| \in A^*$.

THEOREM 3.6. If $f \in A^*$, and $a \in R$, $L(af) = aL(f)$.

Proof: $L(af) = N((af)^+) - N((af)^-)$

$$= N\left[\tfrac{1}{2}(|af| + af)\right] - N\left[\tfrac{1}{2}(|af| - af)\right]$$

$$= N\left[\tfrac{1}{2}(|a|\,|f| + af)\right] - N\left[\tfrac{1}{2}(|a|\,|f| - af)\right]$$

The remainder of the argument will be for $a < 0$.

$$= N\left[\tfrac{1}{2}(-a(|f| - f))\right] - N\left[\tfrac{1}{2}(-a(|f| + f))\right]$$

$$= |-a|\,N\left[\tfrac{1}{2}(|f| - f)\right] - |-a|\,N\left[\tfrac{1}{2}(|f| + f)\right]$$

$$= a\left\{N\left[\tfrac{1}{2}(|f| + f)\right] - N\left[\tfrac{1}{2}(|f| - f)\right]\right\}$$

$= aL(f)$. The argument for $a \geq 0$ is exactly as above.

THEOREM 3.7. If $f, g \in A^*$, $L(f + g) = L(f) + L(g)$.

Proof: L is continuous at $f + g$ therefore if $\epsilon > 0$ there exists a $\delta_1 > 0$ such that $N((f + g) - h) < \delta_1 \Rightarrow |L(f + g) - L(h)| < \epsilon$. That is, 1) $L(h) - \epsilon < L(f + g) < L(h) + \epsilon$. Also L is continuous at f and g, therefore there exists δ_2, δ_3 such that $N(f - h_1) < \delta_2$, $N(g - h_2) < \delta_3 \Rightarrow$
2) $|L(f) - L(h_1)| < \epsilon/2$ and $|L(g) - L(h_2)| < \epsilon/2$.
Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Since f, g are in the closure of A there exists $h_3, h_4 \in A$ such that
$N(f - h_3) < \delta/2$, $N(g - h_4) < \delta/2$. Thus

$$N((f + g) - (h_3 + h_4)) \leq N(f - h_3) + N(g - h_4) \text{ by } 1.5$$
$$< \delta/2 + \delta/2 = \delta.$$

Therefore $h_3 + h_4$ serves as an h function in 1). That is,
3) $L(h_3 + h_4) - \epsilon < L(f + g) < L(h_3 + h_4) + \epsilon$.
Also h_3 serves as an h_1, and h_4 serves as an h_2 in 2) thus $L(h_3) + L(h_4) - \epsilon < L(f) + L(g) < L(h_3) + L(h_4) + \epsilon$. That is, $E(h_3) + E(h_4) - \epsilon < L(f) + L(g) < E(h_3) + E(h_4) + \epsilon$ by 2.17. Therefore
$E(h_3 + h_4) - \epsilon < L(f) + L(g) < E(h_3 + h_4) + \epsilon$ by b-2.
thus 4) $L(h_3 + h_4) - \epsilon < L(f) + L(g) < L(h_3 + h_4) + \epsilon$.
Since ϵ is arbitrary, it follows from 3) and 4) that $L(f + g) = L(f) + L(g)$.

THEOREM 3.8. If $f \in A^*$, $L(|f|) \geq 0$.

Proof: If $f \in A^*$, $|f| \in A^*$, and L is continuous at $|f|$, therefore if $\epsilon > 0$ there exists a $\delta > 0$ such that
$|L(|f|) - L(g)| < \epsilon$ whenever $N(|f| - g) < \delta$.
From the proof of 3.5 it is evident that there exists a

function $|h|$ in A such that $N(|f| - |h|) < \delta$,

thus $|L(|f|) - L(|h|)| < \epsilon$

therefore $L(|f|) > L(|h|) - \epsilon$

$= E(|h|) - \epsilon$ by 2.17

$\geq -\epsilon$ since $E(|h|) \geq 0$. That is,

$L(|f|) > -\epsilon$. Since ϵ was arbitrary, it follows that $L(|f|) \geq 0$.

LEMMA 3.9. If $f \in A^*$ and $f \geq 0$, $L(f) = N(f)$.

Proof: $L(f) = N(f^+) - N(f^-)$

$= N(\frac{1}{2}(|f| + f)) - N(\frac{1}{2}(|f| - f))$

$= N(f)$.

THEOREM 3.10. If $f = \sum f_n$, $f_n \in A^*$, $f_n \geq 0$, $n = 1,2,\ldots$, then f is integrable if and only if $\sum L(f_n)$ converges and in this case $L(f) = \sum L(f_n)$.

Proof: Suppose f is integrable where $f = \sum f_n$, $f_n \in A^*$, and $f_n \geq 0$, $n = 1,2,\ldots$. Necessarily, $f \in F$ so $N(f) < +\infty$.

Now $\sum_{n=1}^{m} L(f_n) = L(\sum_{n=1}^{m} f_n)$ by a generalization of 3.7

$= N(\sum_{n=1}^{m} f_n)$ by 3.9.

Let $|g| = |\sum_{n=1}^{m} f_n| = \sum_{n=1}^{m} f_n \leq f = |f|$. Then

$N(g) \leq N(f)$ by 1.6. That is, $N(\sum_{n=1}^{m} f_n) \leq N(f)$.

Thus for all m, $\sum_{n=1}^{m} L(f_n) \leq N(f) < +\infty$. It follows that

$\sum L(f_n)$ is a convergent series.

Suppose now that $S = \sum L(f_n) < \infty$, where $f = \sum f_n$, $f_n \in A^*$, $f_n \geq 0$, $n = 1,2,\ldots.$

Since $|f| \leq \sum |f_n|$

$$N(f) \leq \sum N(f_n) \text{ by 1.4,}$$
$$= \sum L(f_n) < +\infty \text{ by 3.9}$$

and so $f \in F$.

$$\left|f - \sum_{n=1}^{m} f_n\right| = \left|\sum_{n=m+1}^{\infty} f_n\right| \leq \sum_{n=m+1}^{\infty} |f_n| \text{ and so}$$

1) $$N(f - \sum_{n=1}^{m} f_n) \leq \sum_{n=m+1}^{\infty} N(f_n) \text{ by 1.4}$$

$$= \sum_{n=m+1}^{\infty} L(f_n) \text{ by 3.9.}$$

Since $\sum_{n=m+1}^{\infty} L(f_n) = S - \sum_{n=1}^{m} L(f_n)$; $\lim_{m\to\infty} \sum_{n=m+1}^{\infty} L(f_n) = 0$ and therefore $\sum f_n$ converges in A^* to f. That is $f \in A^*$.

Now $0 \leq (f - \sum_{n=1}^{m} f_n) \in A^*$, so $L(f - \sum_{n=1}^{m} f_n) \geq 0$ by 3.8.

Also $N(f - \sum_{n=1}^{m} f_n) = L(f - \sum_{n=1}^{m} f_n)$ by 3.9

$$= L(f) - \sum_{n=1}^{m} L(f_n) \text{ by a generalization}$$

of 3.7. Thus is view of 1) above,

$$0 \leq L(f) - \sum_{n=1}^{m} L(f_n) \leq \sum_{n=m+1}^{\infty} L(f_n).$$ Since in the limit the right member is zero (see above) it follows that $L(f) = \sum L(f_n)$. This concludes the proof of the theorem.

CHAPTER IV

A GENERAL INTEGRAL DEFINED ON PARTIALLY ORDERED SETS

E. J. McShane's treatment of integration theory is closely related to that of Daniell's, however; McShane develops the integral in a far more general setting then did Daniell and as a result his theory finds considerably wider application.

For example, a special case of McShane's theory yields the Lebesgue - Stieltjes integral, while another case results in an integral similar to the Perron integral, and perhaps identical with it.

In this section, McShane's approach to a general integral will be discussed briefly. The details, although not particularly difficult, are lengthy because of the generality of the setting. Thus a mathematical presentation of the theory will not be included; rather a statement of the problem will be followed by a closer look at the important concepts involved.

Let F and G be two partially ordered sets, and E be a subset of F. Let I be an order preserving mapping of E into G. What conditions are necessary on E, I, F and G to allow I to be extended to a larger subset of F, in such a way that I, on the extended domain, has certain desireable properties of an integral?

To simplify the theory, F is assumed to be a lattice.

This is actually a weak restriction, for every partially ordered set can be embedded in a lattice. Then if the extension of I yields a domain containing points of the lattice not in the original F, these points may be ignored.

G, however, is not assumed to be a lattice, and this fact constitutes one of the chief differences between McShane's treatment and others that have preceded it. In view of the generality of G, it is necessary that a theory of closure, completeness, and convergence be developed for partially ordered sets. Although the details of these concepts are of fundamental importance in a mathematical presentation of the theory, a few definitions will suffice for this discussion.

A lattice F is σ- complete if every countable non-empty subset of F has a supremum and infimum in F.

A set G is directed by a partial ordering $\geq$ if for each pair of elements a,b in G, there exists an element c in G such that $c \geq a$ and $c \geq b$.

A partially ordered set G is Dedekind complete if for every non-empty subset S of G which is directed by $\geq$ and has an upper bound in G, the supremum of S exists in G, and for every non-empty subset S of G which is directed by $\leq$ and has a lower bound in G, the infimum of S exists in G.

In this definition, the restriction to directed subsets of G is significant, for without this restriction, every two elements of G would have a supremum and an infimum. That is, G would necessarily be a lattice.

The concepts of convergence and closure in partially ordered sets are not used in the theory leading to the definition of the integral. However, these ideas play a major role in developing certain properties of the integral such as the Lebesgue dominated convergence theorem, Fatou's lemma, and others. Thus the following definitions are included for their own interest.

If f is a function which assigns to each element b of a directed set A a functional value f(b) in a set M, then f is called a net of elements of M.

A net is a generalization of a sequence; a sequence being a function g which assigns to each positive integer n a functional value g(n). See McShane[8] for a discussion of nets and convergence in partially ordered sets.

Let f(b) (b in a directed set A) be a net of elements of a partially ordered set F. The net f is O-convergent if there exists subsets M, N of F such that:

1. M is directed by $\geq$ and N is directed by $\leq$;
2. sup M and inf N exist and are equal;
3. for each m in M and n in N there exists a b_1 in A such that $m \leq f(b) \leq n$ whenever $b \geq b_1$, b in A.

In this case, f is said to be O-convergent to sup M.

A type of closure can be associated with O-convergence

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[8]E. J. McShane, "Partial Orderings and Moore - Smith Limits", American Mathematical Monthly, 59 (1952) pp. 1-11.

in a natural way.

Let S be a subset of a partially ordered set F. Then S is said to be <u>closed</u> under 0-convergence if it is true that whenever f is a net of elements of S which is 0-convergent to an element f of F, f belongs to S.

For convenience, the following notation will be used in the remainder of this discussion.

Let S be a subset of a partially ordered set F. Then

$\wedge S$ denotes the infimum of S and

$\vee S$ denotes the supremum of S, when they exist.

If S consists of just two members, say $S = \{a,b\}$ then $\wedge S$ will usually be denoted by $a \wedge b$, and $\vee S$ by $a \vee b$.

In applications of the general theory of integration, it is sometimes useful to consider a "strengthening" of a given partial ordering. Even in the case of the real numbers there is often an advantage in considering, along with the ordering $\geq$, the stronger ordering $>$. Note that in a general setting, if $>$ is a partial ordering of a set, so is $\geq$, where $a \geq b$ is defined to mean $a > b$ or $a = b$.

The postulates needed to develop the general integral are the following:

1. F is a σ-complete lattice under the partial ordering $\geq$.
2. G is a Dedekind complete partially ordered set.
3. I is an order preserving mapping of E into G, where E is a subset of F.

4. If S_1 and S_2 are countable subsets of E directed by $>$ and $<$ respectively, and having $\vee S_1 \geq \wedge S_2$ and if $\vee I(S_1)$ and $\wedge I(S_2)$ both exist in G, the inequality $\vee I(S_1) \geq \wedge I(S_2)$ holds.

As an instance of these postulates; F could be taken as the class of all real-valued functions defined on an arbitrary set A, G as the real numbers, E a subset of F, and I an order preserving mapping of E into the real numbers. The partial ordering of F is the natural one induced by the reals, whereas the partial ordering of G is the usual ordering for the reals. Since this instance includes the assumptions of Daniell it is apparent that Daniell's postulates are a special case of McShane's postulates.

As was mentioned previously, the objective is to extend the domain of I to a larger subset of F, in such a way that I, on the enlarged domain, has the desired properties of an integral. As a first step in the extension procedure, two new subsets of F will be defined.

An element u in F is a U-element if there exists a countable subset S of E, directed by $\geq$, such that $\vee S = u$. Each such set S will be said to be associated with u. If such a set exists for which it is also true that $I(S) = \{I(x) \mid x \in S\}$ has an upper bound in G, u is a summable U-element.

An element h in F is an L-element if there exists a countable subset S of E, directed by $\leq$, such that $\wedge S = h$.

Each such set S will be said to be associated with h. If such a set exists for which it is also true that I(S) has a lower bound in G, h is a summable L-element.

A mapping of the U-elements and L-elements into G is defined in the following way:

If u is a summable U-element, and S is associated with u, then $I_1(u) = \vee I(S)$. If h is a summable L-element and S is associated with h, then $I_1(h) = \wedge I(S)$.

The existence of $\vee I(S)$ and $\wedge I(S)$ is assured since G is Dedekind complete. Furthermore, the following theorems can be established, thus proving the consistency of the definition if I_1.

If u is a summable U-element and S_1 and S_2 are associated with u, then $\vee I(S_1) = \vee I(S_2)$. A similar result holds for summable L-elements. Thus the definition of I_1 is consistent.

If f is both a summable U-element and a summable L-element, the value found for $I_1(f)$ when f is regarded as a summable U-element is the same as the value found when f is regarded as a summable L-element. That is, the symbol $I_1(f)$ is not ambiguous.

Two important properties of the mapping I_1 are the following:

If f_1 is a summable U- or L-element and f_2 is a summable U- or L-element, and $f_1 \leq f_2$, then $I_1(f_1) \leq I_1(f_2)$.

If K is a countable collection of summable U-elements directed by $\geq$, then $\vee K$ is a summable U-element if and only if the set $I_1(K)$ has an upper bound in G, and in that case $I_1(\vee K) = \vee I_1(K)$.

Definition of the Integral

If f is in F, $U\{\geq f\}$ is the class of all summable U-elements u satisfying $u \geq f$ and $L\{\leq f\}$ is the class of all summable L-elements h satisfying $h \leq f$.

If f is in F, and both $U\{\geq f\}$ and $L\{\leq f\}$ are non-empty:

$$\overline{\int} f = \wedge I_1(U\{\geq f\})$$

$$\underline{\int} f = \vee I_1(L\{\leq f\})$$

$\overline{\int} f$ and $\underline{\int} f$ are said to be the upper and lower integrals of f respectively.

Some properties of these integrals are the following:

If f is in F and $\overline{\int} f$ and $\underline{\int} f$ exist, then $\underline{\int} f \leq \overline{\int} f$.

If f_1 and f_2 are in F, and both have upper and lower integrals, and if $f_1 \leq f_2$, then $\overline{\int} f_1 \leq \overline{\int} f_2$ and $\underline{\int} f_1 \leq \underline{\int} f_2$.

If f is an element of F for which $\overline{\int} f$ and $\underline{\int} f$ are defined and equal, the integral of f, $\int f$ is defined to be:

$$\int f = \overline{\int} f = \underline{\int} f.$$

In this case f is called a summable element.

The word "summable" was also used in connection with certain U- and L-elements, but since the following statement can be shown true, there is no contradiction between the two uses.

If f is in F and is a U- or L-element, it is a summable U- or L-element if and only if it is a summable element, and in this case $\int f = I_1(f)$.

The following result is immediate from the order preserving property of $\bar{\int}$.

If f_1 and f_2 are summable, and $f_1 \leq f_2$, then $\int f_1 \leq \int f_2$.

In view of the generality of both the domain and range spaces, the properties of the integral are limited. It is the case however that a number of additional results can be obtained under slightly stronger hypotheses.

For example, suppose the range space G is a commutative group under addition. Generalizations of Lebesgue's dominated convergence theorem and Fatou's lemma may now be obtained. In addition certain lattice properties hold on the class of summable elements. By way of illustration, one such result is cited.

If f_1 and f_2 are summable elements of F such that $I(f_1)$ and $I(f_2)$ have a common bound, then $f_1 \vee f_2$ and $f_1 \wedge f_2$ are summable and $I(f_1 \vee f_2) + I(f_1 \wedge f_2) = I(f_1) + I(f_2)$.

As a final example, suppose that both F and G are commutative groups under addition, that both are closed under scaler multiplication by reals and that for all f, g in E, and for each real number c, $I(f + g) = I(f) + I(g)$ and $I(cf) = cI(f)$. Then $\int$ has the following properties. If f_1 and f_2 are summable elements of F, and c is a real number, then $f_1 + f_2$ and cf_1 are summable and

$$\int (f_1 + f_2) = \int f_1 + \int f_2$$
$$\int cf_1 = c\int f_1.$$

CHAPTER V

SUMMARY AND CONCLUSIONS

This paper presents the definition and elementary properties of a general integral as framed by P. J. Daniell, M. H. Stone, and E. J. McShane respectively.

The general integrals as defined by Daniell and McShane depend almost entirely on order properties, with McShane's theory evolving in a more general setting than Daniell's. Stone, on the other hand, uses order properties only to define a norm on the class of elementary functions; the general integral then being defined in terms of this norm.

In each of the three developments, a functional is assumed on a certain class of elementary functions. The functional is then extended to a larger class of functions so as to possess the desireable properties expected of a functional that is to be called an integral.

An important consideration in each of the theories is the fact that the domain of the elementary functions are arbitrary sets, and thus no underlying properties of the domain space are involved. This, as well as the fact that each of the integrals include such classical theories as Riemann and Lebesgue integration, justifies the name "general" integral.

Presently, most of our undergraduate mathematics students learn little of the theories that have developed since Lebesgue's work in 1902. However, with the continuing trend toward teaching more advanced topics in the undergraduate program this writer feels that it is not unreasonable to expect that such topics as an introduction to general integrals may soon play a part in the training of our undergraduates.

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APPENDIX A

Index of Symbols

R	the set of real numbers
$x \in A$	x is an element of the set A
$p \Rightarrow q$	the proposition p implies the proposition q
D_f	domain of the function f
$f \geq g$	$f(x) \geq g(x),\ x \in D_f \cap D_g$
$\{f_n\}\uparrow$	a non-decreasing sequence of functions, that is $f_n \leq f_{n+1}$ $n = 1,2,\ldots$
$\lim f_n$	$\lim_{n\to\infty} f_n$
$\{f_n\}\uparrow f$	a non-decreasing sequence of functions having f as a limit
$\{f_n\}\downarrow f$	a non-increasing sequence of functions having f as a limit
$\sum f_n$	$\sum_{n=1}^{\infty} f_n$
$A \subset B$	the set A is a subset of the set B

VITA

Richard E. Shermoen

Candidate for the Degree of

Doctor of Education

Thesis: AN INTRODUCTION TO GENERAL INTEGRALS

Major Field: Higher Education

Minor Field: Mathematics

Biographical:

Personal Data: Born at Fargo, North Dakota, November 26, 1930, the son of Clarence and Margaret D. Shermoen.

Education: Received the Bachelor of Science degree from North Dakota State University, June, 1953, with a major in Mathematics and a minor in Physical Science; received the Master of Science degree from North Dakota State University, June, 1958, with a major in Mathematics Education; did graduate work at Iowa State University, summer, 1960; received the Master of Arts degree from the University of Illinois, August, 1961, with a major in Mathematics; did graduate work at the University of New Mexico, 1962; completed requirements for the Doctor of Education degree, Oklahoma State University, May, 1965.

Professional Experience: Served in the U. S. Navy, 1948-1949; Mathematics instructor at Harvey, North Dakota, 1953-1955; Mathematics instructor at Hart, Michigan, 1955-1956; Mathematics instructor at North Dakota State University, 1956-1960; Mathematics Instructor at University of New Mexico, 1962-1963.

Professional Organizations: Mathematics Association of America, National Council of Teachers of Mathematics, Pi Mu Epsilon.