ANALYSIS OF THREE-DIMENSIONAL FRAMEWORKS BY STRING POLYGON AND LINEAR GRAPH THEORY

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PREFACE

This research is the outgrowth of the ideas expressed by Professor Jan J. Tuma in his seminar lectures on "Flexibility Analysis of Continuous Structures" during the Fall of 1963. At that time Professor Tuma also suggested the use of topologic concepts for the formulation of frameworks.

The author wishes to take this opportunity to express his gratitude and sincere appreciation to the following individuals and organizations:

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NOMENCLATURE

A_a^a	Flexibility matrix of bar (a).
A_b, A_c, A	Primitive flexibility matrices of branches, chords, and whole system.
A_{cc}, A_{cJ}, A_{JJ}	Elasto-static matrices of a system.
Superscript a	Indicates local reference system associated with member (a)
Subscript a	Refers to bar (a)
a1	Number of bars positively incident at joint J.
a2	Number of bars negatively incident at joint J.
Prefix B	Indicates due to applied loads.
Subscript b	Refers to branch members.
Subscript c	Refers to chord members.
D	Linear flexibility.
d	Some multiple of 5.
ds	Elemental length along bar.
d_{im}	Position vector from i to m.
$d\overline{P}$, $d\overline{Q}$	Elemental elastic forces and couples.
$d\overline{\overline{W}}$	Elemental elastic weights.
E	Modulus of elasticity.
E (Subscripted)	Mixed carry-over value.
e_{ia}, e_{ja}	Stereo-static matrices of bar (a).
F	Angular flexibility, angular near carry-over value.
f_{ia}, f_{ja}	Stereo-static matrices of bar (a).

G	Modulus of rigidity.
G (Subscripted)	Angular far carry-over value.
H_{ia}, H_{ja}	Internal actions at ends i, j of bar (a)
h, J, k	Joints across a frame.
I	Identity matrix.
Subscripts i, j	Refer to ends i, j of a bar.
Subscripts i, j	Eliminated, residual redundant groups.
Superscript (L)	Read as "due to applied loads".
L_a	Distance between i and j of bar (a).
L_k	Distance between i and k of bar (a).
M	Internal moment.
m	Point of application of loads on bar; group elimination stage number.
mt	Total number of elimination stages.
N	Internal Force.
n	Total number of line segments in a linear graph.
nb1h, nb0h	Numbers of various types of branches.
nc0h, nc2h, nc1h, ncss	Numbers of various types of chords.
nb	Total number of branches in a tree.
nc	Total number of chords in a graph.
ns	Total number of support points.
nk	Total number of k-points.
n_t	Tree subgraphs, segments
Superscript o	Indicates quantity in global system
Subscript o	Origin of global system
P, Q	Applied forces and couples.

$\overline{P}, \overline{Q}$	Elastic forces and couples.
PJ, QJ	Modified loads at joint J.
Q_u	Applied couples corresponding to constraints.
Subscript q	Refers to a section in a bar.
Superscript q	Coordinate system associated with an element at point q.
$r^a_{qj}, rL^a_{qj}, s^a_{qj}, R^a_{qj}$	Stero-static matrices.
$SN^a_{im}, SN^a_{jm}, SM^a_{im}$	Stero-static matrices.
$Sr^a_{im}, SR^a_{jm}, t_{qa,q}{}^a{}_j$	Stero-static matrices.
Superscript (T)	Denotes "due to temperature changes".
T_{bJ}	Node-to-datum matrix of a system.
$T_{b,J}$	Elemental matrix of T_{bJ}.
W	Applied loads.
$\overline{W}$	Elastic weights.
WJ	Modified loads at joint J.
X^a_{iax}	Bar-redundant force
X^a_i, Y^a_i, Z^a_i	Coordinate axes.
X^o_i, Y^o_i, Z^o_i	Coordinate axes.
x, y, z	Coordinates of a point
Subscripts x, y, z	Orientation of the quantity
Y^a_{iay}, Y^a_{iaz}	Bar-redundant moments.
$Y^a_{jax}, Y^a_{jay}, Y^a_{jaz}$	Bar-redundant moments.
Prefix Z	Means "due to bar-redundants".
Z^a_a	Bar-redundants of bar (a).
Z_b, Z_c, Z	Bar-redundants of branches, chords and whole system.

α, β, γ	Direction cosines.
ω, π	Transformation matrices.
$\left[\pi i_{oa}\right]$	Equal $\left[\pi_{oa}\right]\left[f_{ia}\right]$.
$\left[\pi j_{oa}\right]$	Equal $\left[\pi_{oa}\right]\left[f_{ja}\right]$.
$\lambda^{(N)}$	Elemental force flexibility.
$\lambda^{(M)}$	Elemental moment flexibility.
ξ, χ	Stero-static matrices of a system.
μ	Lagrange's undetermined multiplier.
Γ_{bc}	Cutset matrix of a system.
$\Gamma_{b,c}$	Elemental matrix of Γ_{bc}.
Γ_{μ}	Part of Γ_{bc} corresponding to constraints.
η, ϵ	Linear deformations of basic bar.
φ, τ	Angular deformations of basic bar.
∇, σ	End deformations of basic bar.
δ	Joint deflection.
θ	Joint rotation.
Δ	Joint deformations
Σ	Summation.
$[\ ]$	Square matrix.
$\{\ ]$	Rectangular matrix.
$\{\ \}$	Column matrix.
Superscript *	Matrix transposition.

CHAPTER I

INTRODUCTION

1-1. Statement of the Problem

The analysis of three-dimensional frameworks by means of string polygon and linear graph theory is investigated. Bars of general shape and cross-section are rigidly connected and supported in such a manner that the structure so formed is stable under the influence of loads, settlement of supports, temperature and volume changes, and initial distortions (Fig. 1-1).

A minimum number of forces and moments are chosen as primary unknowns in the system. Utilizing equivalent elastic weights applied at member ends, compatibility equations are formulated. Their solution may be obtained using

a) Matrix Inversion

b) Matrix Carry-Over Process[15]

c) Group Elimination Technique.

For the purpose of this study, the problem would be considered solved when the end-conditioning elements of each bar and deformations of all joints are calculated.

1-2. Historical Review

The introduction of the conjugate beam analogy is usually credited to Mohr[1]. Extension of his principles for the calculation of deformations and the solution of indeterminate frames can be seen in the works of Müller Breslau[2], Westergaard[3], Lee[4], Cross[5], Diwan[6], and Kiusalaas[7]. Tuma and his associates [8-15] have conducted extensive research in the use of distributed and equivalent elastic weights leading to flexibility analysis of planar and space structures.

The investigations of Langefors[18], and Wehle and Lansing[19] show a matrix formulation by energy approach. Denke[21] expressed Maxwell-Mohr equations accounting for effects of loads, settlements of supports, initial distorsions and temperature variations. Force methods have been widely employed also for the analysis of piping systems by a number of researchers: Brock[23], Soule[24], and Owens[25] ...

Analysis of space frames by means of the well-known slope deflection method is also possible as shown by Tuma and Tolaba[26], Tsui[27], Baron[28], Martin and Hernandez[29], Feng[30], Monforton and Wu[31], and Shore[32].

The above-mentioned techniques cannot provide the systematic procedure that is very important for establishing compatibility of deformations in complex framed structures. To accomplish this purpose, a matrix formulation by means of linear graph theory has begun to appear recently (1962) in civil engineering literature. This has been primarily a result of the increased capability of dealing with large systems in which abstraction of concepts is a practical necessity. Moreover, the method lends itself to digital computer programming.

Application of topological ideas to the electric circuit theory was initiated by Kirchoff and Maxwell nearly a century ago[35].

Langefors[36, 37, 38], Henderson[39], and Samuelsson[40] studied the extension of these concepts to structural analysis, mainly in the language of theoretical mathematics. Other recent investigators that might be mentioned are Dimaggio[41] and Henderson and Bickley[42], who described the statical indeterminacy of a stable structure as a topological property; Lind[43], who takes the view that structures are only special cases of the class of problems in system theory; and Spillers[45] and Fenves and Branin[46], who have presented a mathematical analogy between the network problem and the linear continuous frame problem.

Electrical analogies have also been employed as shown by Kron[47], Cheng[48], Ryder[49], and others.

Direct solution of large space frames involves voluminous numerical operations besides requiring a computer with huge storage capacity. Langefors[52] proposed the method of piecewise analysis in 1950. Later Kron[50, 51] conducted an extensive research based upon the idea of tearing and interconnecting. His principle found wide application in the area of electrical networks in which the variables are simple scalars. Kron's factorized form and doubling techniques afford a saving in the amount of numerical computation, but do not relieve the problem of storage requirements. Moreover they call for the solution of an interconnection matrix. Recently, Galletly[60], Clough, Wilson and King[34], Eisemann and Namyet[57], and Spillers[54] employed Gauss's principle to eliminate a group of unknowns at a time.

A method for the formulation of compatibility equations for rigidly jointed skeletal structures allowing for releases such as hinges, roller supports and a group elimination technique for the solution of large systems are presented here and can form the basis for a general computer program for the analysis of complex frameworks.

1-3. Scope and Procedure of Investigation

The study is subject to the following limitations.

a) Material is linearly elastic, homogeneous, isotropic and continuous.

b) Deformations are small and elastic.

c) Plane sections remain plane after deformations.

In the development of the analytical approach, the steps considered are:

1) Sign convention for actions and displacements is established;

2) Geometry of bars and associated quantities is defined in terms of coordinates and transformation matrices;

3) Equilibrium and relative deformations of a member are studied;

4) Statics of the entire system is formulated utilizing linear graph theory. Solution for the unknowns is obtained by means of compatibility of deformations.

5) Group elimination technique to solve large structures is discussed.

6) The theory is demonstrated by two numerical examples.

1-4. Coordinate Systems

Two classes of right-handed cartesian systems are required for the analysis of space frameworks. The first, named as the local system, is associated with the member. The origin of such a system is located at the near end (i) and the X_i^a-axis passing through the far end (j) of bar (a) (Fig. 1-2). The second type, referred to as the global system, has its origin at an arbitrary point o. The axes are oriented at convenience (Fig. 1-2). While there are as many local systems as members, there is only one global system for the entire structure. Superscripts other than o are used for member systems.

1-5. Sign Convention and Notation

Forces and deflections are represented by vectors with a single head; couples, moments and angular deformations are denoted by double headed line segments.

Loads consist of three components of forces and three components of couples parallel to X, Y, Z axes of the appropriate reference system (Fig. 1-3). The matrix of these actions $\{W\}$, referred to o-system, is defined as

$$\{W^o_{ma}\} = \left\{\begin{array}{c} P^o_{ma} \\ \hline Q^o_{ma} \end{array}\right\} = \left\{\begin{array}{c} P^o_{max} \\ \hline P^o_{may} \\ \hline P^o_{maz} \\ \hline Q^o_{max} \\ \hline Q^o_{may} \\ \hline Q^o_{maz} \end{array}\right\} \qquad (1.1)$$

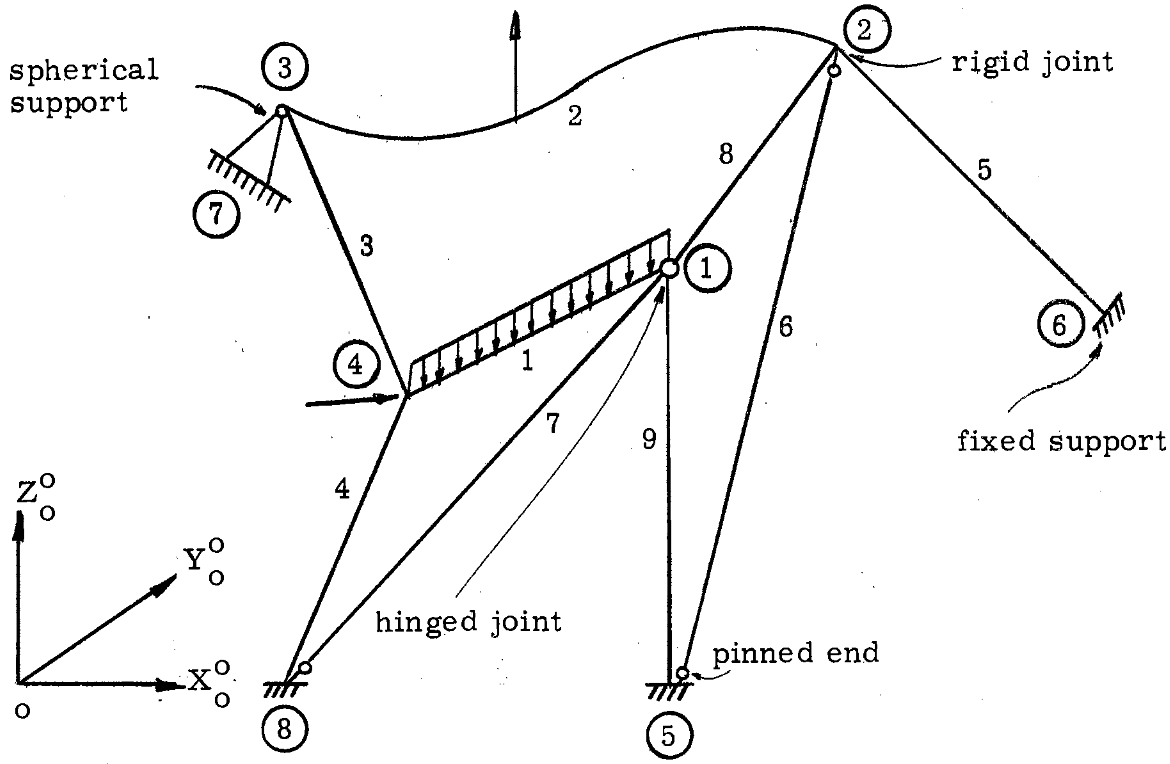


FIG. 1-1. SPACE FRAME

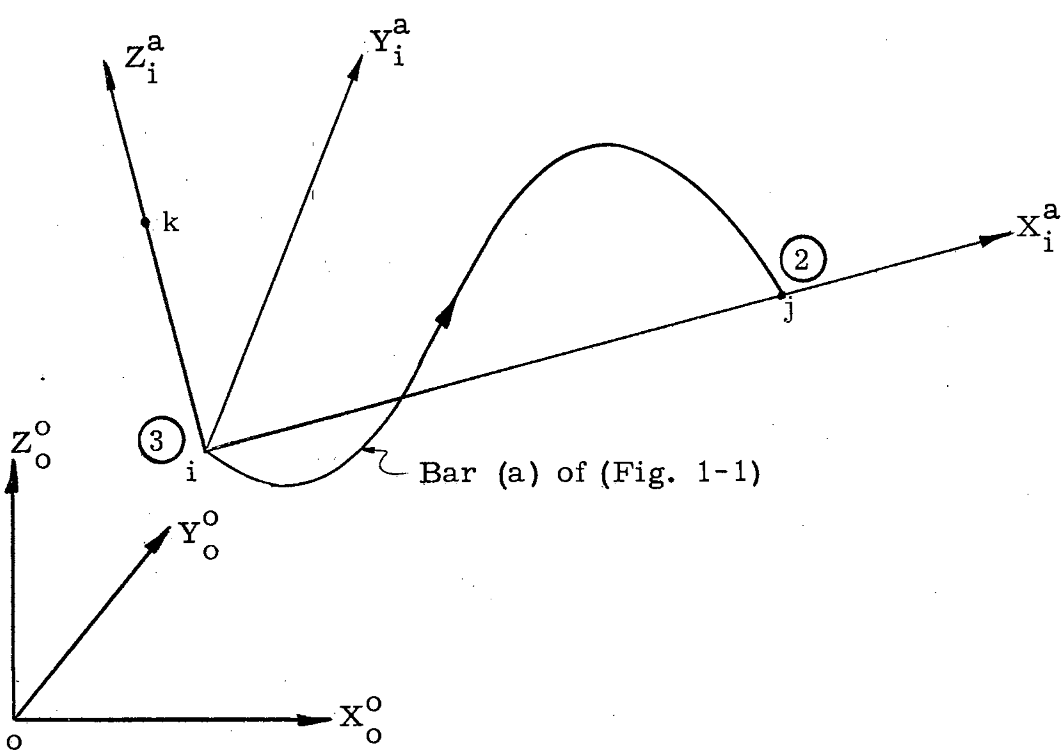


FIG. 1-2. COORDINATE SYSTEMS

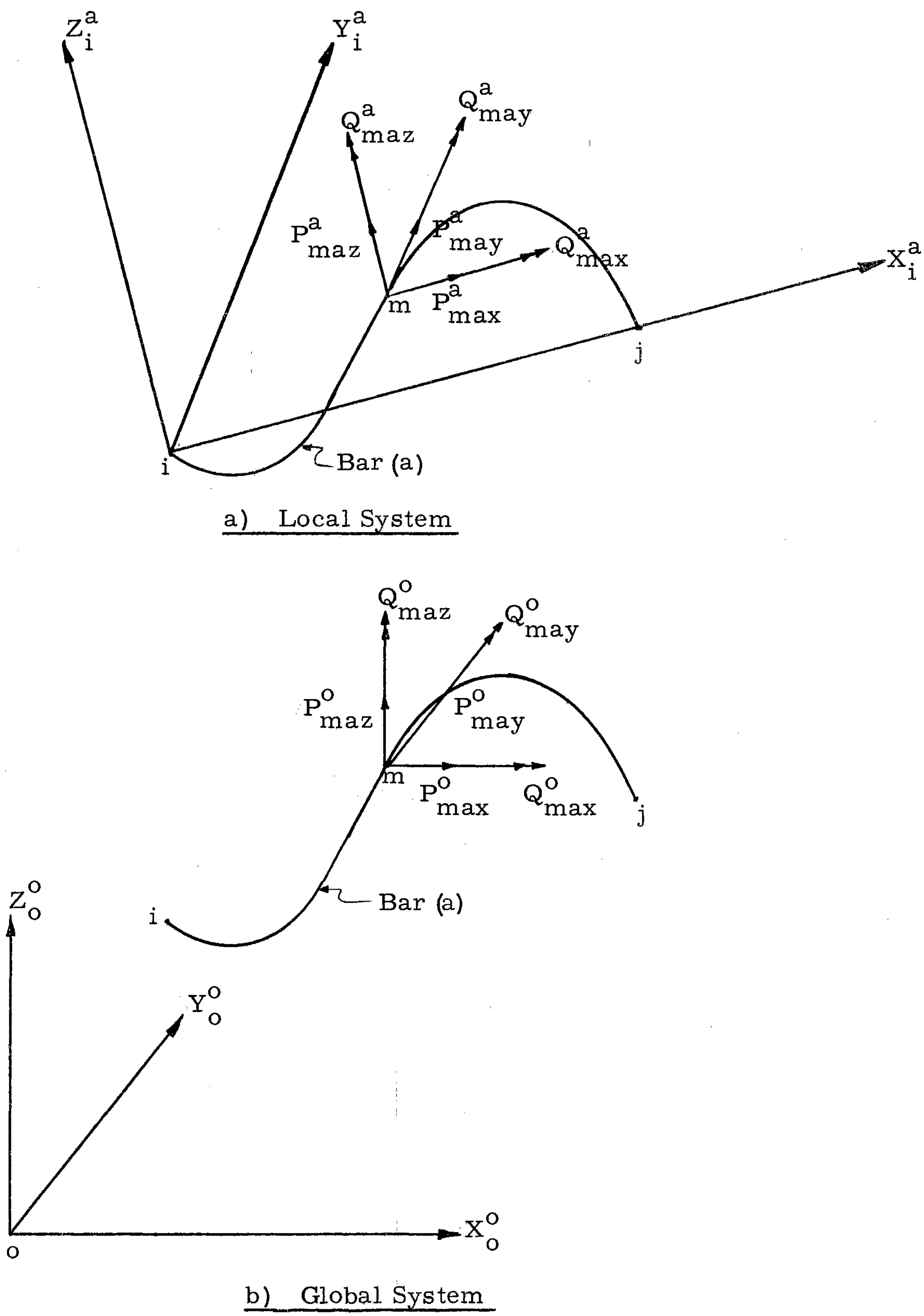


FIG. 1-3. POSITIVE LOAD VECTORS

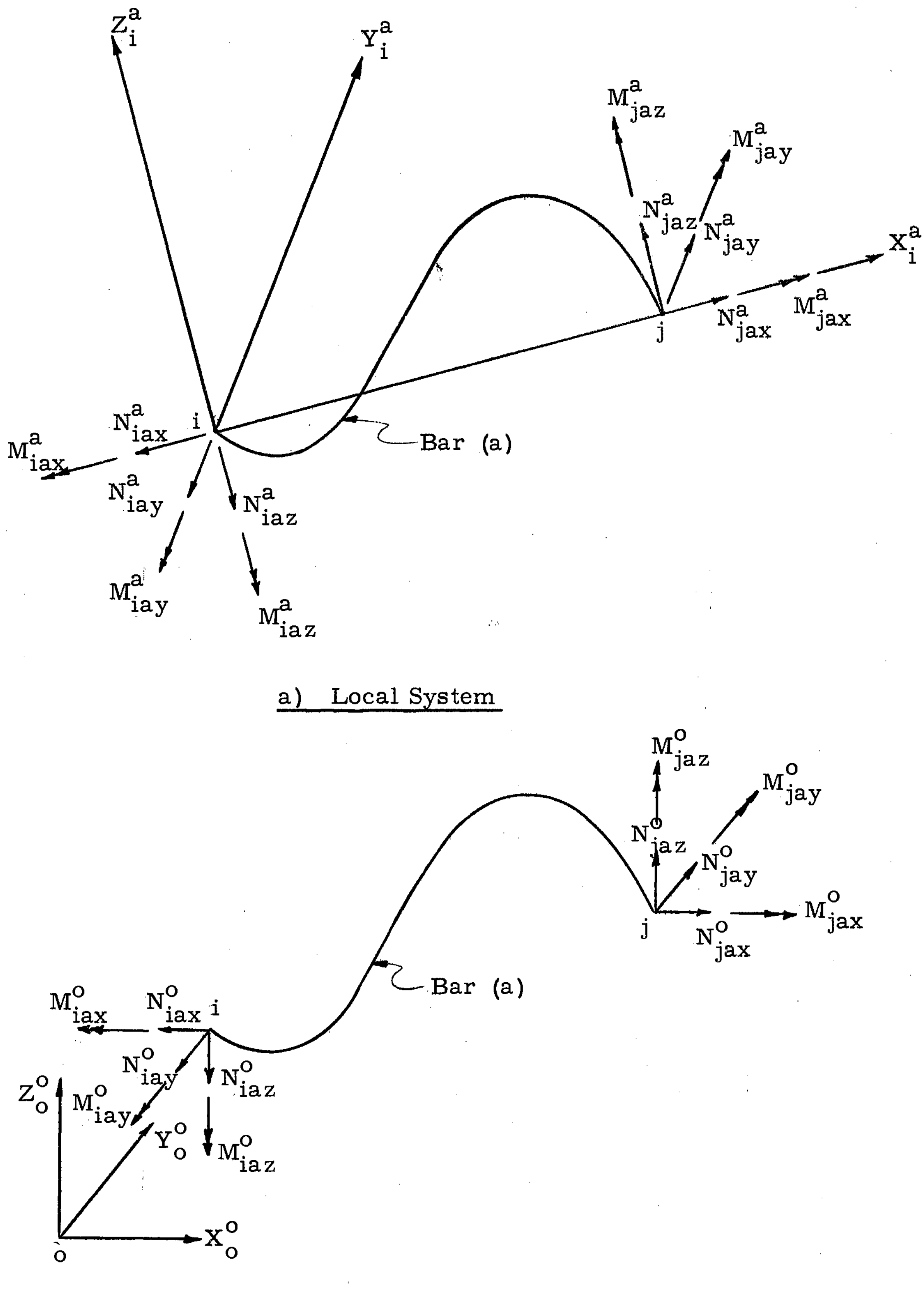
Internal forces and moments are denoted by N and M, respectively. They are positive if acting as shown in Fig. (1-4), and are designated in a-system by the matrix

$$\left\{H^a_{ia}\right\} = \left\{\begin{array}{c} N^a_{ia} \\ \hline M^a_{ia} \end{array}\right\} = \left\{\begin{array}{c} N^a_{iax} \\ \hline N^a_{iay} \\ \hline N^a_{iaz} \\ \hline M^a_{iax} \\ \hline M^a_{iay} \\ \hline M^a_{iaz} \end{array}\right\} \tag{1.2}$$

Joint deflections δ and rotations θ, represented by vectors, are positive when directed in the positive sense of the coordinate axes (Fig. 1-5). These deformations are expressed by the matrix

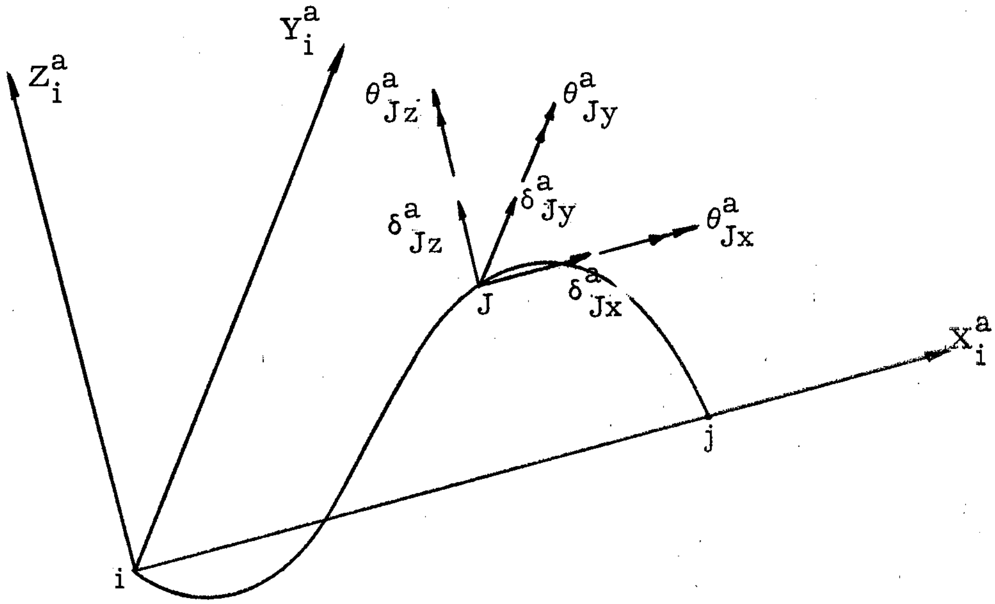
$$\left\{\Delta^o_J\right\} = \left\{\begin{array}{c} \delta^o_J \\ \hline \theta^o_J \end{array}\right\} = \left\{\begin{array}{c} \delta^o_{Jx} \\ \hline \delta^o_{Jy} \\ \hline \delta^o_{Jz} \\ \hline \theta^o_{Jx} \\ \hline \theta^o_{Jy} \\ \hline \theta^o_{Jz} \end{array}\right\} \tag{1.3}$$

Other notations adopted for use in this thesis are explained where they first occur.

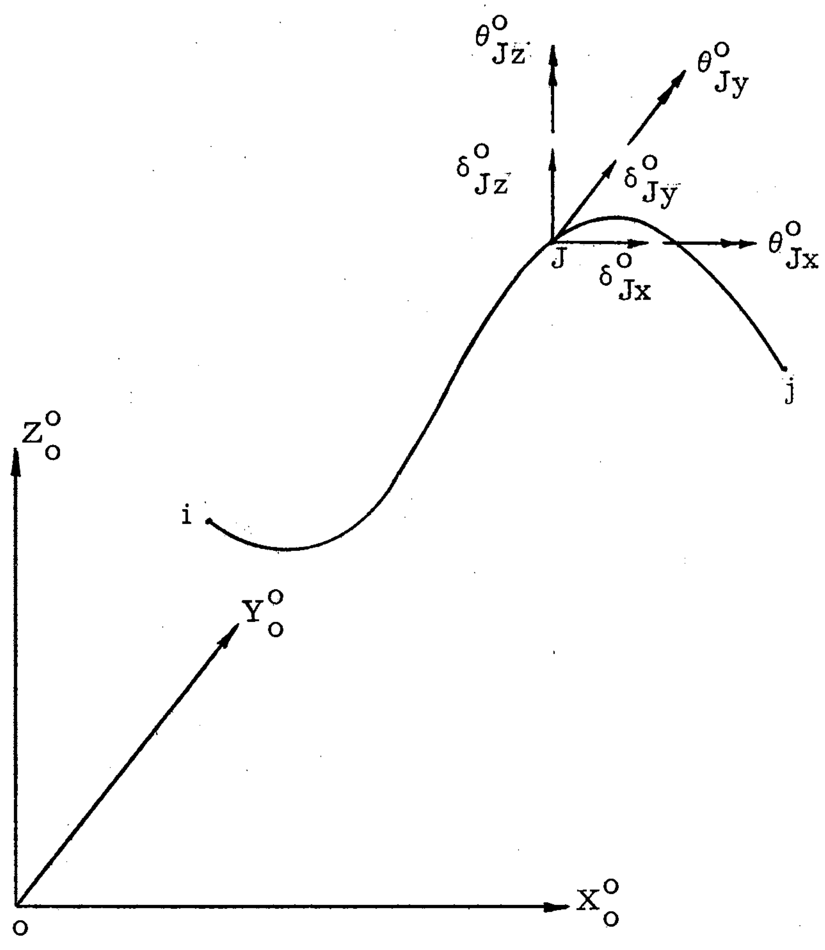


b) Global System

FIG. 1-4. POSITIVE INTERNAL ACTIONS



a) Local System



b) Global System

FIG. 1-5. POSITIVE JOINT DEFORMATIONS

1-6. Stereo-Geometry

Geometry of a space frame can be described by a study of bars, applied loads, and internal actions as viewed in various reference systems.

A member is located in space by means of local and global systems as shown in Fig. (1-6). Consider a point m whose coordinate vector, measured from the near end i in a-system, is denoted by

$$\left\{ d^{a}_{im} \right\} = \left\{ \begin{array}{c} x^{a}_{im} \\ \hline y^{a}_{im} \\ \hline z^{a}_{im} \end{array} \right\} \qquad (1.4a)$$

The same vector, when referred to o-system, becomes

$$\left\{ d^{o}_{im} \right\} = \left[\omega_{oa} \right] \left\{ d^{a}_{im} \right\} \qquad (1.4b)$$

where $\left[\omega_{oa} \right]$ is the matrix of direction cosines between the o-, and a-systems, and is given by

$$\left[\omega_{oa} \right] = \left[\begin{array}{c|c|c} \alpha_{oax} & \alpha_{oay} & \alpha_{oaz} \\ \hline \beta_{oax} & \beta_{oay} & \beta_{oaz} \\ \hline \gamma_{oax} & \gamma_{oay} & \gamma_{oaz} \end{array} \right] \qquad (1.4c)$$

The submatrices resulting from the vertical partitioning of the transformation matrix $\left[\omega_{oa} \right]$ are frequently used in the subsequent chapters. They are

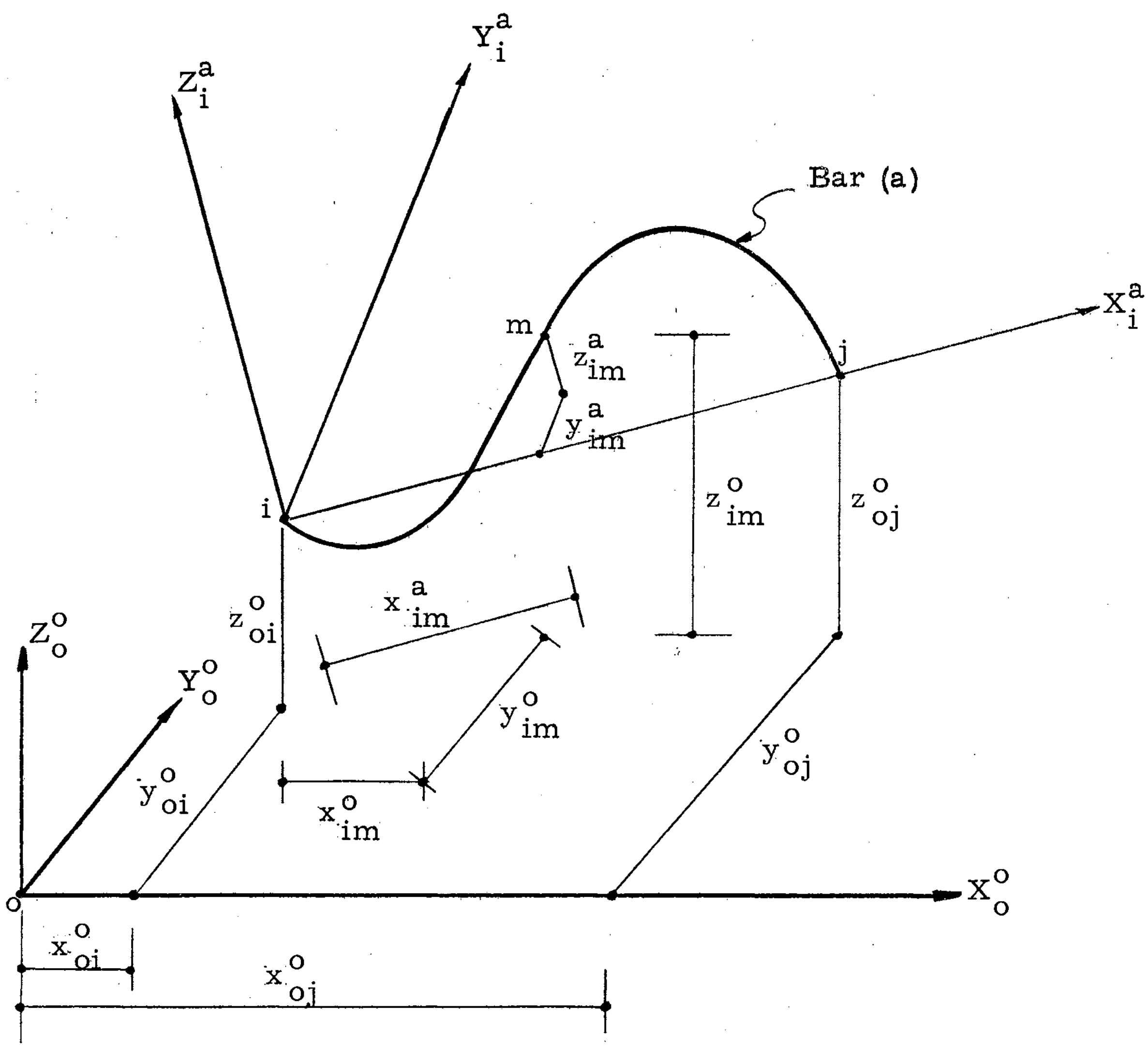


FIG. 1-6. GEOMETRY OF A BAR IN SPACE

$$\left\{\omega_{oax}\right\} = \begin{Bmatrix} \alpha_{oax} \\ \hline \beta_{oax} \\ \hline \gamma_{oax} \end{Bmatrix} ; \left\{\omega_{oay}\right\} = \begin{Bmatrix} \alpha_{oay} \\ \hline \beta_{oay} \\ \hline \gamma_{oay} \end{Bmatrix} ; \left\{\omega_{oaz}\right\} = \begin{Bmatrix} \alpha_{oaz} \\ \hline \beta_{oaz} \\ \hline \gamma_{oaz} \end{Bmatrix} \qquad (1.4d)$$

An important property of the matrix (Eq. 1.4c) is expressed by

$$\left[\omega_{oa}\right]^{-1} = \left[\omega_{oa}\right]^{*} = \left[\omega_{ao}\right] \qquad (1.4e)$$

where the asterisk stands for matrix transposition.

Assuming the coordinates of the end points i and j, and a third point k on Z_i^a-axis known, the direction numbers α, β, γ can be obtained from the following relationships

$$\left\{\omega_{oax}\right\} = \frac{1}{L_a}\left\{d_{ij}^{o}\right\} \qquad (1.5a)$$

$$\left\{\omega_{oaz}\right\} = \frac{1}{L_k}\left\{d_{ik}^{o}\right\} \qquad (1.5b)$$

$$\alpha_{oay} = -\beta_{oax}\gamma_{oaz} + \gamma_{oax}\beta_{oaz} \qquad (1.5c)$$

$$\beta_{oay} = -\gamma_{oax}\alpha_{oaz} + \alpha_{oax}\gamma_{oaz} \qquad (1.5d)$$

$$\gamma_{oay} = -\alpha_{oax}\beta_{oaz} + \beta_{oax}\alpha_{oaz} \qquad (1.5e)$$

in which L_a is the length of the straight line joining i and j, and L_k is measured along Z_i^a-axis between i and k.

Utilizing $\left[\omega_{oa}\right]$, loads, internal actions, and joint deformations can be transformed from one system of reference to the other as shown below:

$$\left\{W_{ma}^{o}\right\} = \left[\pi_{oa}\right]\left\{W_{ma}^{a}\right\};\ \left\{W_{ma}^{a}\right\} = \left[\pi_{ao}\right]\left\{W_{ma}^{o}\right\} \tag{1.6a}$$

$$\left\{H_{ja}^{o}\right\} = \left[\pi_{oa}\right]\left\{H_{ja}^{a}\right\};\ \left\{H_{ja}^{a}\right\} = \left[\pi_{ao}\right]\left\{H_{ja}^{o}\right\} \tag{1.6b}$$

$$\left\{\Delta_{j}^{o}\right\} = \left[\pi_{oa}\right]\left\{\Delta_{j}^{a}\right\};\ \left\{\Delta_{J}^{a}\right\} = \left[\pi_{ao}\right]\left\{\Delta_{J}^{o}\right\} \tag{1.6c}$$

in which

$$\left[\pi_{oa}\right] = \left[\begin{array}{c|c} \omega_{oa} & 0_3 \\ \hline 0_3 & \omega_{oa} \end{array}\right] \quad ; \quad \left[\pi_{ao}\right] = \left[\begin{array}{c|c} \omega_{oa}^{*} & 0_3 \\ \hline 0_3 & \omega_{oa}^{*} \end{array}\right] \tag{1.6d}$$

$$\left[0_3\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.6e}$$

CHAPTER II

STATICS AND END DEFORMATIONS OF A BAR

Equilibrium of a member isolated from a space frame is considered in the first part of this chapter. The second half deals with the end deformations of a simple beam due to bar-redundants, applied loads, temperature changes and support displacements. Special cases arising from the necessity of introducing structural discontinuities such as spherical hinges are also investigated. Equilibrium and deformations of a bar removed from a planar framework are recorded in Appendix A.

2-1. Stereo-Statics

A freebody of a bar is sketched in Fig. 2-1. Six elements — three forces and three moments – are acting at each end. The member is statically indeterminate to the sixth degree. Simple beam and cantilever type basic structures are most commonly employed in structural analysis. The first kind is adopted for this research. Such a basic member (Fig. 2-2) is supported at j by a spherical pin, and the near end i is permitted to displace parallel to line ij and rotate about Y_i^a and Z_i^a axes. Therefore, the bar-redundants, treated as arbitrary loads, consist of the force component in X-direction and the moment components about the Y and Z axes at end i, and three moment components at j (Fig. 2-2). They are

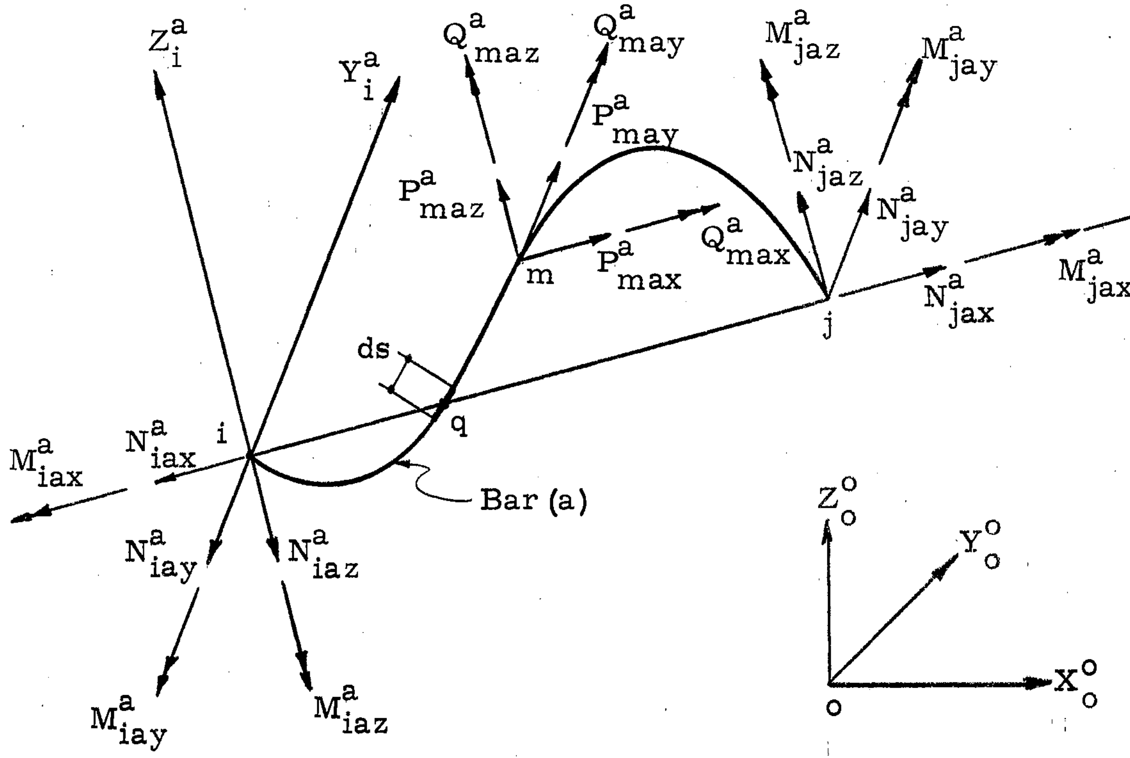


FIG. 2-1. FREEBODY OF BAR (a)

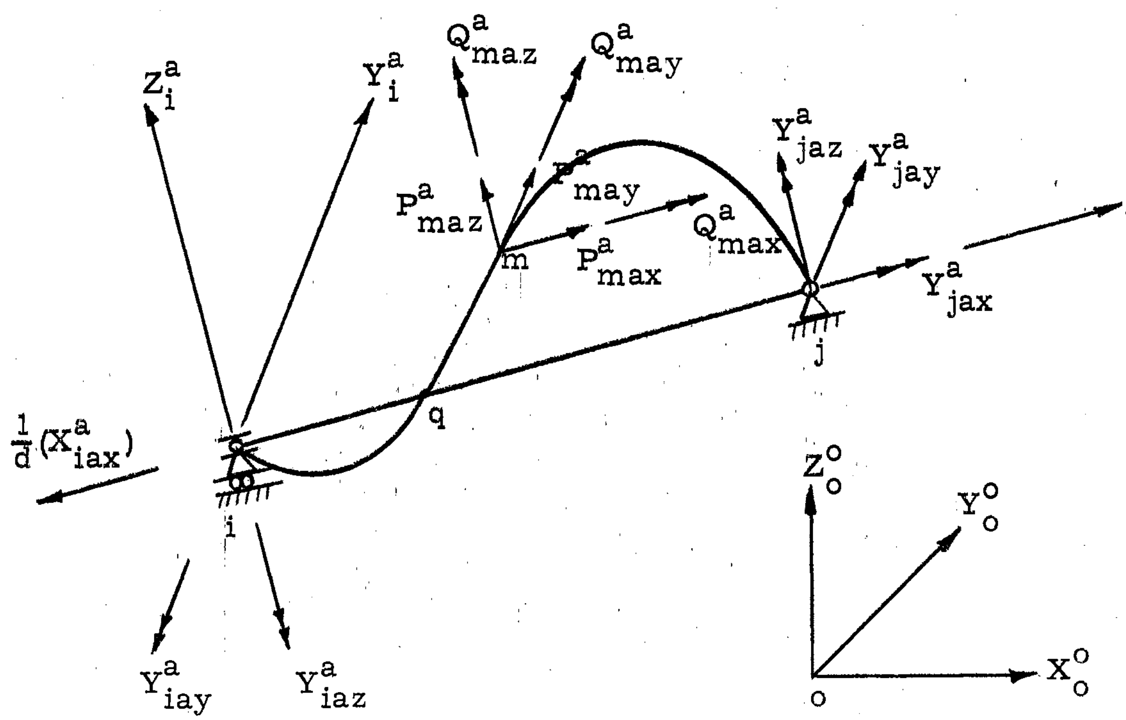


FIG. 2-2. BASIC STRUCTURE – ARBITRARY AND APPLIED LOADS

designated by Z, X, Y as

$$\left\{Z^a_a\right\} = \begin{Bmatrix} Z^a_{ia} \\ \hline Z^a_{ja} \end{Bmatrix} = \begin{Bmatrix} X^a_{iax} \\ \hline Y^a_{iay} \\ \hline Y^a_{iaz} \\ \hline Y^a_{jax} \\ \hline Y^a_{jay} \\ \hline Y^a_{jaz} \end{Bmatrix} = \begin{Bmatrix} (d)N^a_{iax} \\ \hline M^a_{iay} \\ \hline M^a_{iaz} \\ \hline M^a_{jax} \\ \hline M^a_{jay} \\ \hline M^a_{jaz} \end{Bmatrix} \tag{2.1}$$

in which the scale factor d is some multiple of 5. It is found that when d is taken equal to the average length of bars, the resulting flexibility matrix for a system will be better-conditioned.

The end-conditioning* and cross-sectional elements are functions of the loads. Introducing the notation of Table 2-1, the transmission relationships may be stated as

$$\left\{H^a_{ia}\right\} = \left[f_{ia}\right]\left\{Z^a_a\right\} + \left\{BH^a_{ia}\right\} = \left\{ZH^a_{ia}\right\} + \left\{BH^a_{ia}\right\} \tag{2.2a}$$

$$\left\{H^a_{ja}\right\} = \left[f_{ja}\right]\left\{Z^a_a\right\} + \left\{BH^a_{ja}\right\} = \left\{ZH^a_{ja}\right\} + \left\{BH^a_{ja}\right\} \tag{2.2b}$$

$$\left\{H^o_{ia}\right\} = \left[\pi i_{oa}\right]\left\{Z^a_a\right\} + \left\{BH^o_{ia}\right\} = \left\{ZH^o_{ia}\right\} + \left\{BH^o_{ia}\right\} \tag{2.2c}$$

---

* By end-conditioning elements is meant the end reactive forces and moments.

TABLE 2-1. STEREO-STATIC MATRICES OF BAR (a)

$$\left[e_{ia}\right] = \begin{bmatrix} \frac{1}{d} & 0 & 0 \\ 0 & 0 & \frac{1}{L_a} \\ 0 & -\frac{1}{L_a} & 0 \end{bmatrix} ; \quad \left[e_{ja}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{L_a} \\ 0 & \frac{1}{L_a} & 0 \end{bmatrix} ; \quad \left[SN^a_{im}\right] = \frac{1}{L_a}\begin{bmatrix} 0 & 0 & 0 \\ y^a_{im} & x^a_{mj} & 0 \\ z^a_{im} & 0 & x^a_{mj} \end{bmatrix} ; \quad \left[SN^a_{jm}\right] = \frac{1}{L_a}\begin{bmatrix} -L_a & 0 & 0 \\ y^a_{im} & -x^a_{im} & 0 \\ z^a_{im} & 0 & -x^a_{im} \end{bmatrix}$$

$$\left[SM^a_{im}\right] = \begin{bmatrix} 0 & -z^a_{im} & y^a_{im} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \quad \left[\dot{I}_3\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \quad \left[\underset{\cdot\cdot}{I}_3\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad \left[Sr^a_{im}\right] = \begin{bmatrix} SN^a_{im} & e_{ja} \\ SM^a_{im} & \dot{I}_3 \end{bmatrix}$$

$$\left[f_{ia}\right] = \begin{bmatrix} e_{ia} & e_{ja} \\ \underset{\cdot\cdot}{I}_3 & \dot{I}_3 \end{bmatrix} ; \quad \left[f_{ja}\right] = \begin{bmatrix} e_{ia} & e_{ja} \\ 0_3 & I_3 \end{bmatrix} ; \quad \left[Sr^a_{jm}\right] = \begin{bmatrix} SN^a_{jm} & e_{ja} \\ 0_3 & 0_3 \end{bmatrix} ; \quad \left\{0_{3,1}\right\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\left[\omega'_{oa}\right] = \left[\frac{1}{d}\omega_{oax} \;\middle|\; -\frac{1}{L_a}\omega_{oaz} \;\middle|\; \frac{1}{L_a}\omega_{oay}\right] ; \quad \left[\omega''_{oa}\right] = \left[0_{3,1} \;\middle|\; \frac{1}{L_a}\omega_{oaz} \;\middle|\; -\frac{1}{L_a}\omega_{oay}\right] ; \quad \left[\omega'''_{oa}\right] = \left[0_{3,1} \;\middle|\; \omega_{oay} \;\middle|\; \omega_{oaz}\right]$$

$$\left[\omega''''_{oa}\right] = \left[\omega_{oax} \;\middle|\; 0_{3,1} \;\middle|\; 0_{3,1}\right] ; \quad \left[\pi i_{oa}\right] = \begin{bmatrix} \omega'_{oa} & \omega''_{oa} \\ \omega'''_{oa} & \omega''''_{oa} \end{bmatrix} ; \quad \left[\pi j_{oa}\right] = \begin{bmatrix} \omega'_{oa} & \omega''_{oa} \\ 0_3 & \omega_{oa} \end{bmatrix}$$

$$\left\{BH^a_{ia}\right\} = \sum_{(a)}\left[Sr^a_{im}\right]\left\{W^a_{ma}\right\} ; \quad \left\{BH^a_{ja}\right\} = \sum_{(a)}\left[Sr^a_{jm}\right]\left\{W^a_{ma}\right\} ; \quad \left\{BN^a_{ia}\right\} = \sum_{(a)}\left\{SN^a_{im} \;\middle|\; e_{ja}\right] \begin{Bmatrix} P^a_{ma} \\ Q^a_{ma} \end{Bmatrix} ; \quad \left\{BN^o_{ia}\right\} = \left[\omega_{oa}\right]\left\{BN^a_{ia}\right\}$$

$$\left\{BH^o_{ia}\right\} = \left[\pi_{oa}\right]\left\{BH^a_{ia}\right\} ; \quad \left\{BH^o_{ja}\right\} = \left[\pi_{oa}\right]\left\{BH^a_{ja}\right\} ; \quad \left\{BN^a_{ja}\right\} = \sum_{(a)}\left\{SN^a_{jm} \;\middle|\; e_{ja}\right] \begin{Bmatrix} P^a_{ma} \\ Q^a_{ma} \end{Bmatrix} ; \quad \left\{BN^o_{ja}\right\} = \left[\omega_{oa}\right]\left\{BN^a_{ja}\right\}$$

$$\left\{f'_{ia}\right\} = \begin{Bmatrix} e_{ia} \\ I_3 \end{Bmatrix} ; \quad \left\{e'_{ia}\right\} = \begin{Bmatrix} \frac{1}{d} \\ 0 \\ 0 \end{Bmatrix} ; \quad \left\{\pi i'_{oa}\right\} = \begin{Bmatrix} \omega'_{oa} \\ \omega'''_{oa} \end{Bmatrix}$$

$$\left\{s^a_{qj}\right\} = \begin{bmatrix} 0 & -z^a_{qj} & y^a_{qj} \\ z^a_{qj} & 0 & -x^a_{qj} \\ -y^a_{qj} & x^a_{qj} & 0 \end{bmatrix} ; \quad \left[r^a_{qj}\right] = \begin{bmatrix} I_3 & 0_3 \\ s^a_{qj} & I_3 \end{bmatrix} ; \quad \left[R^a_{qj}\right] = \left[r^a_{qj}\right]\left[f_{ja}\right] ; \quad \left\{rL^a_{qj}\right\} = \begin{Bmatrix} I_3 \\ s^a_{qj} \end{Bmatrix}$$

$$\left\{BH^q_{qa}\right\} = \left[r^a_{qj}\right]\left\{BH^a_{ja}\right\} + \sum_{q}^{j}\left[r^a_{qm}\right]\left\{W^a_{ma}\right\} ; \quad \left\{BH^q_{qa}\right\} = \left[\pi_{qa}\right]\left\{BH^a_{qa}\right\}$$

$$\left[t_{qa,q^aj}\right] = \left[\pi_{qa}\right]\left[R^a_{qj}\right]$$

$$\left\{H_{ja}^{o}\right\} = \left[\pi j_{oa}\right]\left\{Z_{a}^{a}\right\} + \left\{BH_{ja}^{o}\right\} = \left\{ZH_{ja}^{o}\right\} + \left\{BH_{ja}^{o}\right\} \tag{2.2d}$$

$$\left\{H_{qa}^{a}\right\} = \left[R_{qj}^{a}\right]\left\{Z_{a}^{a}\right\} + \left\{BH_{qa}^{a}\right\} = \left\{ZH_{qa}^{a}\right\} + \left\{BH_{qa}^{a}\right\} \tag{2.2e}$$

$$\left\{H_{qa}^{q}\right\} = \left[t_{qa,q}{}^{a}{}_{j}\right]\left\{Z_{a}^{a}\right\} + \left\{BH_{qa}^{q}\right\} = \left\{ZH_{qa}^{q}\right\} + \left\{BH_{qa}^{q}\right\} \tag{2.2f}$$

where prefixes Z and B are read as due to arbitrary and applied loads, respectively; and superscript q specifies the coordinate system associated with that cross-section (Fig. 2-1).

2-1.1. Special Cases

a) Bar with a Hinged End

Let the end j of bar (a) (Fig. 2-1) be a spherical pin. Then the redundants consist of those associated with end i. Eqs. (2-2a-d) can be modified to obtain expressions for the member-end reactions. This is accomplished by deleting rows associated with moments at j and columns pertaining to the nonexisting loads. Using the notation of Table 2-1,

$$\left\{H_{ia}^{a}\right\} = \left[f'_{ia}\right]\left\{Z_{ia}^{a}\right\} + \left\{BH_{ia}^{a}\right\} = \left\{ZH_{ia}^{a}\right\} + \left\{BH_{ia}^{a}\right\} \tag{2-3a}$$

$$\left\{N_{ja}^{a}\right\} = \left[e_{ia}\right]\left\{Z_{ia}^{a}\right\} + \left\{BN_{ja}^{a}\right\} = \left\{ZN_{ja}^{a}\right\} + \left\{BN_{ja}^{a}\right\} \tag{2-3b}$$

$$\left\{H_{ia}^{o}\right\} = \left[\pi i'_{oa}\right]\left\{Z_{ia}^{a}\right\} + \left\{BH_{ia}^{o}\right\} = \left\{ZH_{ia}^{o}\right\} + \left\{BH_{ia}^{o}\right\} \tag{2-3c}$$

$$\left\{N^{o}_{ja}\right\} = \left[\omega'_{oa}\right]\left\{Z^{a}_{ia}\right\} + \left\{BN^{o}_{ja}\right\} = \left\{ZN^{o}_{ja}\right\} + \left\{BN^{o}_{ja}\right\} \tag{2-3d}$$

b) Straight Bar with Hinged Ends

A pinned-end straight member is shown in Fig. (2-3). The system of applied loads may be general, subject to the condition that the torsional equilibrium is identically satisfied. X^{a}_{iax} (Fig. 2-4) is the only redundant in this case. As in case (a), expressions for the end-conditioning forces are derived from Eqs. (2-2 a-d), and given below. Nomencluature used in the following relations is recorded in Table 2-1.

$$\left\{N^{a}_{ia}\right\} = \left\{e'_{ia}\right\} X^{a}_{iax} + \left\{BN^{a}_{ia}\right\} = \left\{ZN^{a}_{ia}\right\} + \left\{BN^{a}_{ia}\right\} \tag{2-4a}$$

$$\left\{N^{a}_{ja}\right\} = \left\{e'_{ia}\right\} X^{a}_{iax} + \left\{BN^{a}_{ja}\right\} = \left\{ZN^{a}_{ja}\right\} + \left\{BN^{a}_{ja}\right\} \tag{2-4b}$$

$$\left\{N^{o}_{ia}\right\} = \frac{1}{d}\left\{\omega_{oax}\right\} X^{a}_{iax} + \left\{BN^{o}_{ia}\right\} = \left\{ZN^{o}_{ia}\right\} + \left\{BN^{o}_{ia}\right\} \tag{2-4c}$$

$$\left\{N^{o}_{ja}\right\} = \frac{1}{d}\left\{\omega_{oax}\right\} X^{a}_{iax} + \left\{BN^{o}_{ja}\right\} = \left\{ZN^{o}_{ja}\right\} + \left\{BN^{o}_{ja}\right\} \tag{2-4d}$$

Cross-sectional elements for both cases (a) and (b) may be evaluated from Eqs. (2-2 e, f). Other special situations such as a roller support can be treated similarly.

2-2. Elasto-Geometry

End deformations of the basic bar (Fig. 2-2) are algebraically

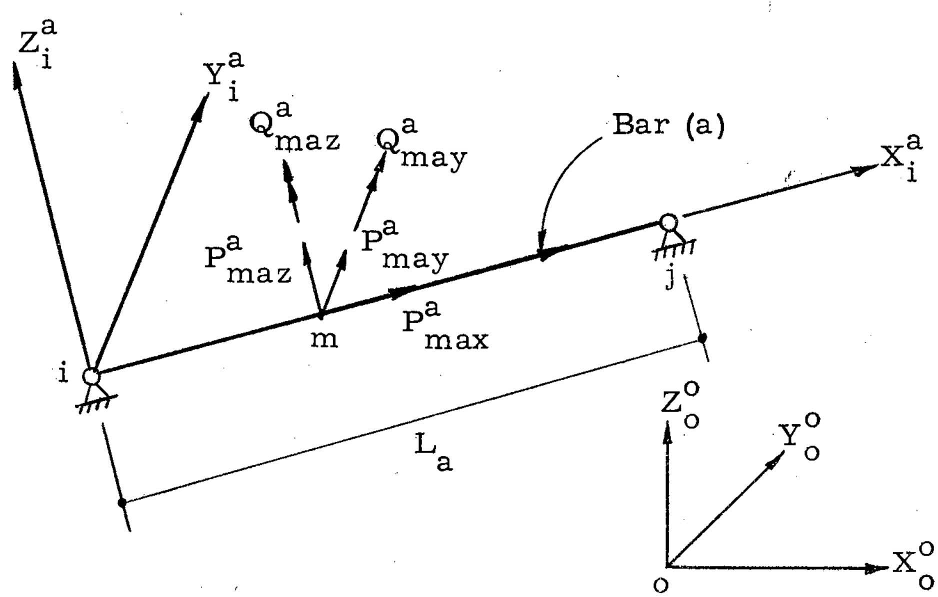


FIG. 2-3. PINNED-END STRAIGHT BAR

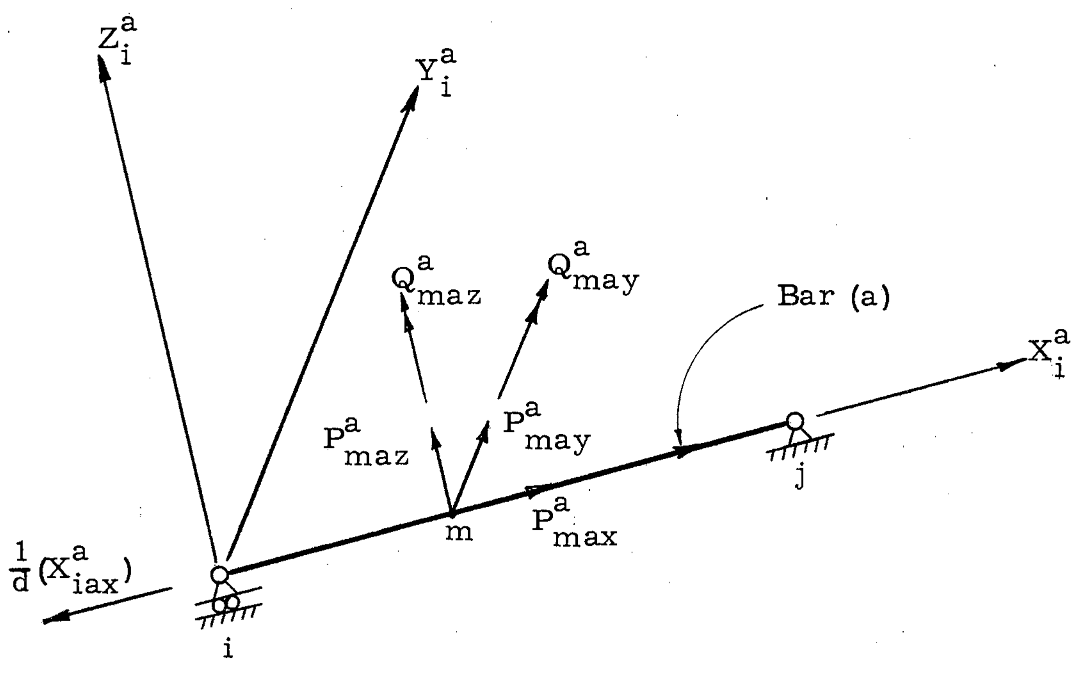


FIG. 2-4. SIMPLE BEAM – ARBITRARY AND APPLIED LOADS

expressed in terms of angular and linear functions, and recorded in Table 2-2. The notation employed is explained below.

All quantities are referred to the member system as specified by superscript a.

η^{a}_{iax} is the displacement of end i of basic beam (a) along X^{a}-axis due to all causes.

φ^{a}_{jax} is the rotation of end j of basic beam (a) about X^{a}-axis due to all causes.

ϵ^{a}_{iax} is the displacement of end i of basic beam (a) along X^{a}-axis due to all causes excluding the bar-redundants.

τ^{a}_{iaz} is the rotation of end i of basic beam (a) about Z^{a}-axis due to all causes excluding the bar-redundants.

D^{a}_{iixx} is the displacement of end i of basic beam (a) along X^{a}-axis due to $N^{a}_{iax} = +1$.

F^{a}_{iiyy} is the rotation at end i of basic beam (a) about Y^{a}-axis due to $Y^{a}_{iay} = +1$.

F^{a}_{jjyz} is the rotation at end j of basic beam (a) about Y^{a}-axis due to $Y^{a}_{jaz} = +1$.

G^{a}_{ijxz} is the rotation at end i of the basic beam (a) about X^{a}-axis due to $Y^{a}_{jaz} = +1$.

E^{a}_{ijxx} is the displacement at end i of the basic beam (a) along X^{a}-axis due to $Y^{a}_{jax} = +1$.

E^{a}_{jizx} is the rotation at end j of the basic beam (a) about Z^{a}-axis due to $N^{a}_{iax} = +1$.

The flexibility matrix denoted by A is symmetrical, as it should be in view of Maxwell's reciprocal theorem for linear systems.

TABLE 2-2 END DEFORMATIONS BASIC BAR (a)

$$
\left\{\begin{array}{c}
\frac{1}{d}\eta^a_{iax} \\
\varphi^a_{iay} \\
\varphi^a_{iaz} \\ \hline
\varphi^a_{jax} \\
\varphi^a_{jay} \\
\varphi^a_{jaz}
\end{array}\right\}
=
\left[\begin{array}{ccc|ccc}
\frac{1}{d^2}D^a_{iixx} & \frac{1}{d}E^a_{iixy} & \frac{1}{d}E^a_{iixz} & \frac{1}{d}E^a_{ijxx} & \frac{1}{d}E^a_{ijxy} & \frac{1}{d}E^a_{ijxz} \\
\frac{1}{d}E^a_{iiyx} & F^a_{iiyy} & F^a_{ijyz} & G^a_{ijyx} & G^a_{ijyy} & G^a_{ijyz} \\
\frac{1}{d}E^a_{iizx} & F^a_{iizy} & F^a_{iizz} & G^a_{ijzx} & G^a_{ijzy} & G^a_{ijzz} \\ \hline
\frac{1}{d}E^a_{jixx} & G^a_{jixy} & G^a_{jixz} & F^a_{jjxx} & F^a_{jjxy} & F^a_{jjxz} \\
\frac{1}{d}E^a_{jiyx} & G^a_{jiyy} & G^a_{jiyz} & F^a_{jjyx} & F^a_{jjyy} & F^a_{jjyz} \\
\frac{1}{d}E^a_{jizx} & G^a_{jizy} & G^a_{jizz} & F^a_{jjzx} & F^a_{jjzy} & F^a_{jjzz}
\end{array}\right]
\cdot
\left\{\begin{array}{c}
X^a_{iax} \\
Y^a_{iay} \\
Y^a_{iaz} \\ \hline
Y^a_{jax} \\
Y^a_{jay} \\
Y^a_{jaz}
\end{array}\right\}
+
\left\{\begin{array}{c}
\frac{1}{d}\epsilon^a_{iax} \\
\tau^a_{iay} \\
\tau^a_{iaz} \\ \hline
\tau^a_{jax} \\
\tau^a_{jay} \\
\tau^a_{jaz}
\end{array}\right\}
$$

or

$$
\left\{\begin{array}{c}
\nabla^a_{ia} \\ \hline
\nabla^a_{ja}
\end{array}\right\}
=
\left[\begin{array}{c|c}
D^a_a & E^a_a \\ \hline
E^{a*}_a & F^a_a
\end{array}\right]
\cdot
\left\{\begin{array}{c}
Z^a_{ia} \\ \hline
Z^a_{ja}
\end{array}\right\}
+
\left\{\begin{array}{c}
\sigma^a_{ia} \\ \hline
\sigma^a_{ja}
\end{array}\right\}
$$

or

$$
\left\{ \nabla^a_a \right\} = \left[A^a_a \right] \left\{ Z^a_a \right\} + \left\{ \sigma^a_a \right\}
$$

Equations (Table 2-2) can be modified for application to bars with other end conditions. This is accomplished by deleting rows and columns associated with the redundants that do not appear for the special case.

The end deformations (Fig. 2-5a) are represented by analogous loads known as elastic weights (Fig. 2-5b). Forces are designated by $\overline{P}$, and couples by $\overline{Q}$. They are defined by the matrix $\left\{\overline{W}\right\}$ as

$$\left\{\overline{W}^a_a\right\} = \left\{\begin{array}{c} \overline{W}^a_{ia} \\ \hline \overline{W}^a_{ja} \end{array}\right\} = \left\{\begin{array}{c} \overline{Q}^a_{iax} \\ \hline \overline{P}^a_{iay} \\ \hline \overline{P}^a_{iaz} \\ \hline \overline{P}^a_{jax} \\ \hline \overline{P}^a_{jay} \\ \hline \overline{P}^a_{jaz} \end{array}\right\} = \left\{\begin{array}{c} \frac{1}{d}\eta^a_{iax} \\ \hline \varphi^a_{iay} \\ \hline \varphi^a_{iaz} \\ \hline \varphi^a_{jax} \\ \hline \varphi^a_{jay} \\ \hline \varphi^a_{jaz} \end{array}\right\} \tag{2.5a}$$

or

$$\left\{\overline{W}^a_a\right\} = \left[A^a_a\right]\left\{Z^a_a\right\} + \left\{\sigma^a_a\right\} \tag{2-5b}$$

2-2.1. Analytical Expressions for Deformation Constants

Let the flexibility matrix for an element of length ds at q (Fig. 2-1) be defined by

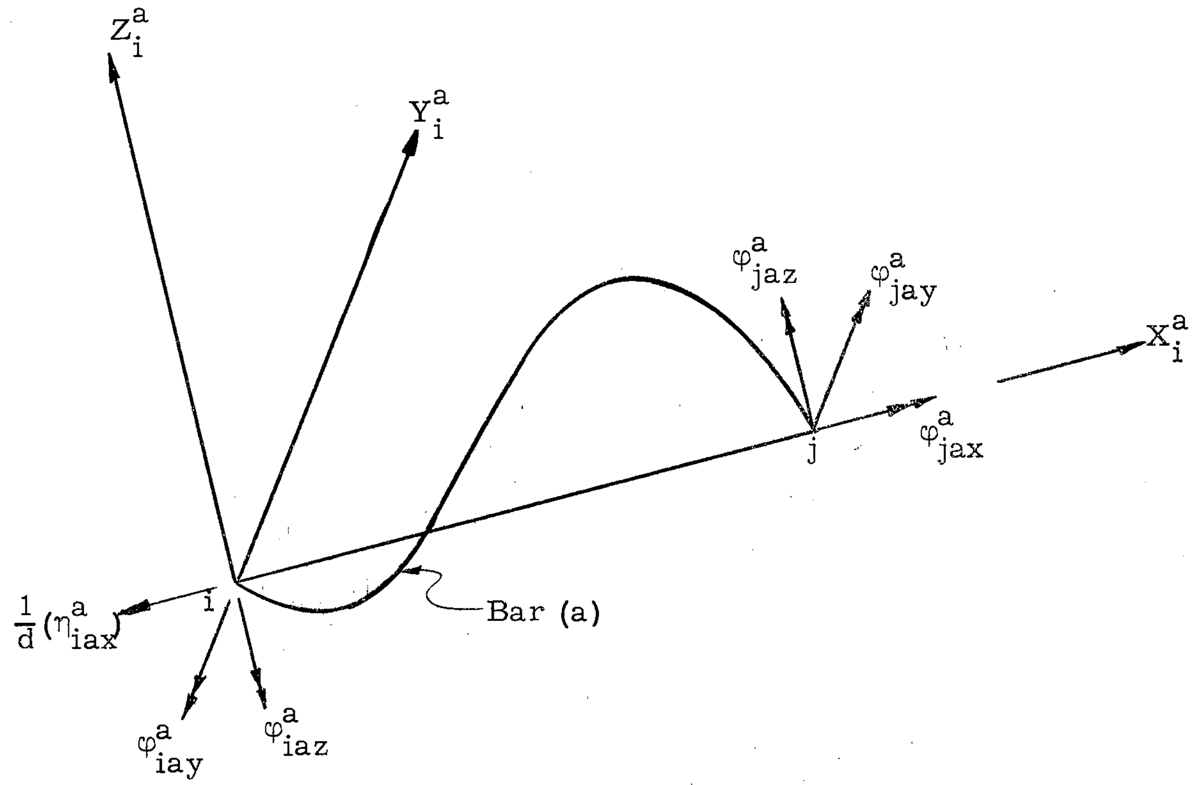


FIG. 2-5a. END DEFORMATION VECTORS

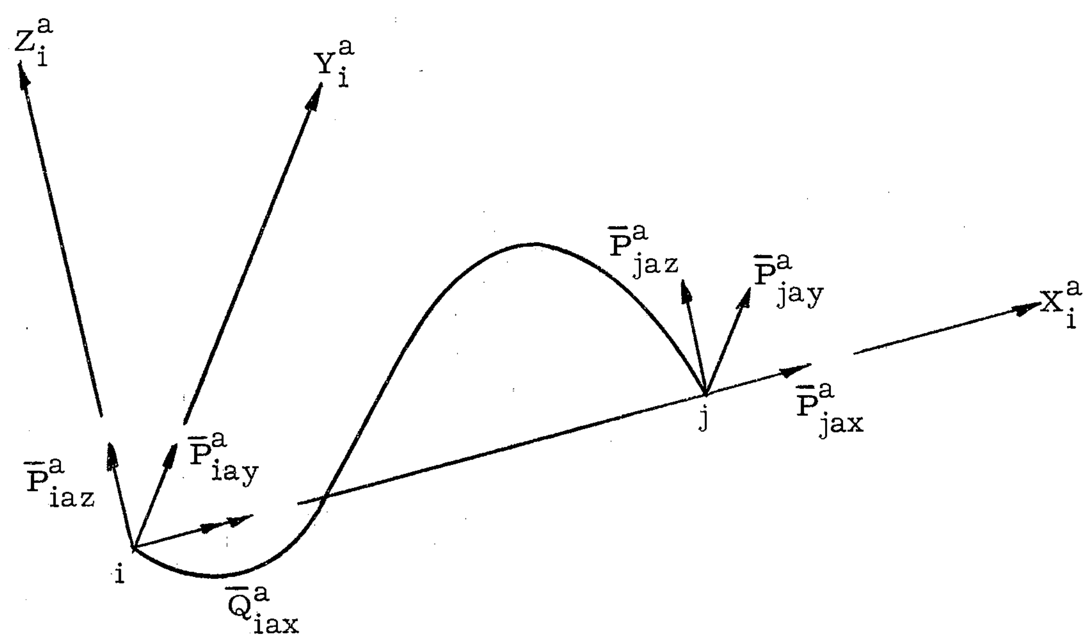


FIG. 2-5b. END ELASTIC WEIGHTS

$$\left[\lambda^{q}_{qa}\right] = \left[\begin{array}{ccc|ccc} \lambda^{q(N)}_{qax} & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^{q(N)}_{qay} & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda^{q(N)}_{qaz} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda^{q(M)}_{qax} & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^{q(M)}_{qay} & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^{q(M)}_{qaz} \end{array}\right] \tag{2.6a}$$

or simply

$$\left[\lambda^{q}_{qa}\right] = \left[\begin{array}{c|c} \lambda^{q(N)}_{qa} & 0_3 \\ \hline 0_3 & \lambda^{q(M)}_{qa} \end{array}\right] \tag{2.6b}$$

in which additional superscripts (N) and (M) denote linear and angular quantities, respectively. The elemental elastic weights are designated by

$$\left\{d\overline{W}^{q}_{qa}\right\} = \left\{\begin{array}{c} d\overline{Q}^{q}_{qax} \\ \hline d\overline{Q}^{q}_{qay} \\ \hline d\overline{Q}^{q}_{qaz} \\ \hline d\overline{P}^{q}_{qax} \\ \hline d\overline{P}^{q}_{qay} \\ \hline d\overline{P}^{q}_{qaz} \end{array}\right\} \tag{2.7a}$$

$$\left\{ d\overline{W}^{q}_{qa} \right\} = \left[\lambda^{q}_{qa} \right] \left\{ H^{q}_{qa} \right\} \tag{2.7b}$$

where cross-sectional elements are available from Eq. (2-2f). By means of conjugate beam analogy, equivalent end elastic weights due to bar-redundants and applied loads can be shown to be

$$\begin{aligned} \left\{ \overline{W}^{a}_{a} \right\} &= \sum_{(a)} \left[t_{qa,\,q}{}^{a}{}_{j} \right]^{*} \left[\lambda^{q}_{qa} \right] \left[t_{qa,\,q}{}^{a}{}_{j} \right] \left\{ Z^{a}_{a} \right\} \\ &+ \sum_{(a)} \left[t_{qa,\,q}{}^{a}{}_{j} \right]^{*} \left[\lambda^{q}_{qa} \right] \left\{ BH^{q}_{qa} \right\} \end{aligned} \tag{2.8}$$

from which the flexibility and load-function matrices are, respectively, given by

$$\left[A^{a}_{a} \right] = \sum_{(a)} \left[t_{qa,\,q}{}^{a}{}_{j} \right]^{*} \left[\lambda^{q}_{qa} \right] \left[t_{qa,\,q}{}^{a}{}_{j} \right] \tag{2.9}$$

$$\left\{ \sigma^{a(L)}_{a} \right\} = \sum_{(a)} \left[t_{qa,\,q}{}^{a}{}_{j} \right]^{*} \left[\lambda^{q}_{qa} \right] \left\{ BH^{q}_{qa} \right\} \tag{2.10}$$

The additional superscript (L) denotes "due to applied loads". Numerical values of these constants (Eqs. 2-9, 10) for various bar shapes are available elsewhere[58].

Let the ends of bar (a) (Fig. 2-1) be displaced by $\left\{ \Delta^{a}_{ia} \right\}$ and $\left\{ \Delta^{a}_{ja} \right\}$ due to yielding of supports. Then the displacement-functions $\left\{ \sigma^{a(\Delta)}_{a} \right\}$ are calculated as

$$\left\{\sigma_a^{a(\Delta)}\right\} = \left\{\begin{array}{c} \sigma_{ia}^{a(\Delta)} \\ \hline \sigma_{ja}^{a(\Delta)} \end{array}\right\} = \left\{\begin{array}{c} \frac{1}{d}\left(\delta_{iax}^{a} - \delta_{jax}^{a}\right) \\ \hline \theta_{iay}^{a} - \frac{1}{L_a}\left(\delta_{iaz}^{a} - \delta_{jaz}^{a}\right) \\ \hline \theta_{iaz}^{a} + \frac{1}{L_a}\left(\delta_{iay}^{a} - \delta_{jay}^{a}\right) \\ \hline \theta_{iax}^{a} - \theta_{jax}^{a} \\ \hline -\theta_{jay}^{a} + \frac{1}{L_a}\left(\delta_{iaz}^{a} - \delta_{jaz}^{a}\right) \\ \hline -\theta_{jaz}^{a} - \frac{1}{L_a}\left(\delta_{iay}^{a} - \delta_{jay}^{a}\right) \end{array}\right\} \tag{2.11}$$

Temperature-functions $\left\{\sigma_a^{a(T)}\right\}$ may be added to Eqs. (2-10, 11) to obtain

$$\left\{\sigma_a^{a}\right\} = \left\{\sigma_a^{a(L)}\right\} + \left\{\sigma_a^{a(\Delta)}\right\} + \left\{\sigma_a^{a(T)}\right\} \tag{2.12}$$

CHAPTER III

FORMULATION AND SOLUTION

In this chapter some basic elements of linear graph theory(59, 62) are reviewed and then used to establish the static equilibrium of skeletal structures. Compatibility equations are then obtained by means of the cotransmission principle of elastic structures. Finally, an algebraic solution for the system unknowns is given.

3-1. Definitions

A linear oriented graph (Fig. 3-1a) is a set of lines and nodes in which each line has two ends called terminals. Orientation of each line is indicated by its associated arrowhead. It is further assumed that

1) The set contains finite number of nodes and lines,

2) Ends of each line coincide with distinct nodes,

3) Each node is a terminal of some line.

A terminal of a line and a node are said to be incident when the terminal and the node coincide.

An end node of a graph is a node at which a terminal of only one line is incident.

A subgraph is any subset of the elements of a graph. When the subgraph contains two and only two distinct end nodes and in which all nonend nodes are incident to only two terminals of distinct lines of the subgraph, it is called a simple open path (2d1b3 in Fig. 3-1a).

A graph is said to be connected if and only if there exists at least one simple open path between every pair of nodes of the graph. A mesh is a connected subgraph such that every node of this subgraph is incident to two and only two terminals of distinct line members of the subgraph (2d1b3c in Fig. 3-1a).

A tree is a connected graph containing no meshes. Thus, it follows that between any two nodes of a tree there exists exactly one simple open path. There are nb line segments in a tree of nb + 1 nodes. Any connected graph has a subgraph which is a tree (indicated

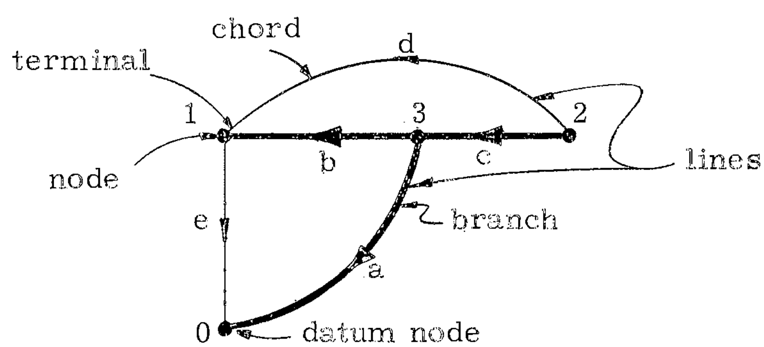


FIG. 3-1a. ORIENTED LINEAR GRAPH-TREE AND CHORDS

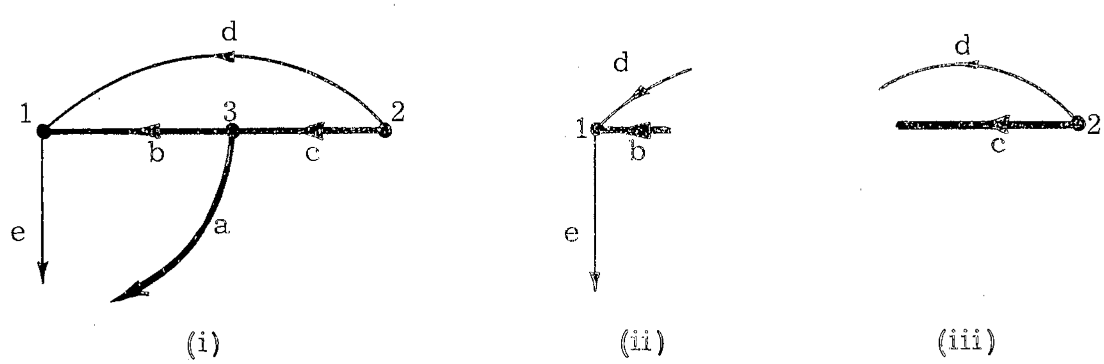


FIG. 3-1b. BASIC CUTSETS FOR THE GRAPH OF FIG. 3-1a

by thick lines in Fig. 3-1a) containing all the nodes of the graph. The lines that are members of the tree are called branches and those that do not belong to the tree are designated as the chords. This tree is, however, not unique. But the number of chords is the same irrespective of the choice of the tree. Each chord corresponds uniquely to a particular mesh (2d1b3c in Fig. 3-1a) consisting of the chord and the simple open path in the tree between the end nodes of the chord. Let n be the number of line elements in a connected graph, nb the number of branches in a tree of the graph; then, n = nb + nc where nc is the number of chords.

For the purpose of this presentation a specific node of the graph is designated as the datum. Its significance and selection will be explained in the next section.

A node-to-datum path is the simple open path in the tree of a graph between the node and the datum (2c3a0 in Fig. 3-1a).

A subgraph, separated from a connected graph by a closed line such that the line severs one and only one branch and none, few or all of the chords of the graph and such that the subgraph does not include the datum, is called a cutset (Fig. 3-1b). Thus, having selected a tree in a connected graph, a cutset can be associated with each branch. It follows then that a total of nb cutsets can be realized in the graph.

3-2. Linear Graph of a Framework

A framework (Fig. 1-1) is a system of bars, each of which (Fig. 1-2) is represented by an oriented line segment in a linear graph (Fig. 3-2a). The arrow specifies the direction of the X-axis of the member system. Joints other than supports, where one or more members meet, are nodes; and all the supports irrespective of their nature are joined by lines meeting at a common point known as the datum. Thus any number of supports are identified by the datum node.

The frame (Fig. 1-1) can be reduced to one or more cantilevers (Fig. 3-2b) by removing a minimum number of redundant bars. Topologically this is equivalent to a tree (called the formulation tree indicated by thick lines in Fig. 3-2a) resulting when certain line segments (chords indicated by thin lines in Fig. 3-2a) are separated from the linear graph. In any connected graph a number of trees can be realized. For formulation purposes only one of them is sufficient.

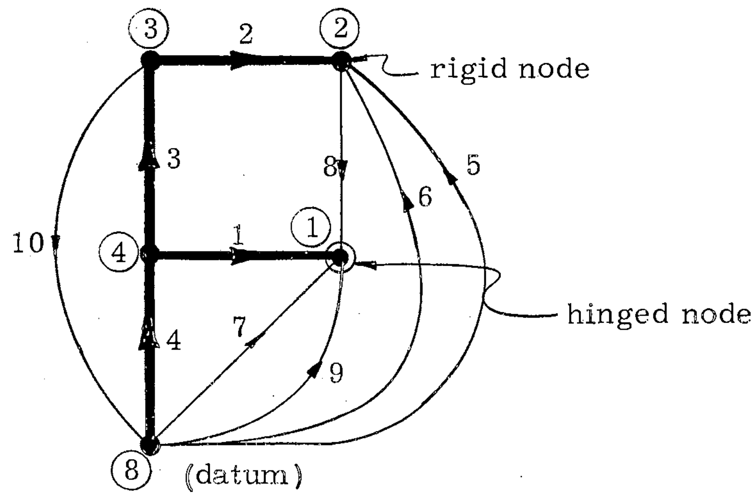


FIG. 3-2a. LINEAR GRAPH OF THE FRAME OF FIG. 1-1

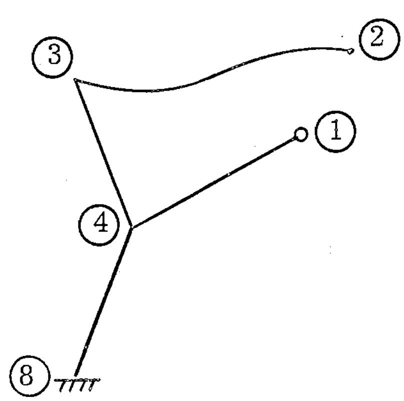
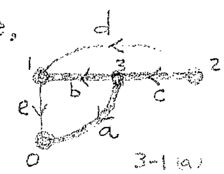


FIG. 3-2b. CANTILEVER BASIC STRUCTURE

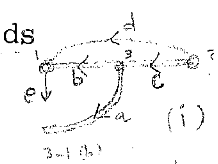
In case all the supports of a structure are other than fixed, one or more as needed may be temporarily fixed. How to obtain the solution of the original structure will be explained in detail in the last section of this chapter.

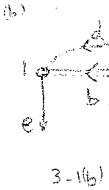
3-3. Sign Convention for Oriented Graphs

A line segment oriented (towards, away from) a node is said to be (positively, negatively) incident upon the node. For example, in Fig. 3-1a, line segment c is positively incident on node 3 and negatively on 2.



A chord is said to be (positively, negatively) intersected in a cutset subgraph, if the associated branch and the chord are (similarly, oppositely) incident on the nodes of the cutset. For example, chords d and e (cutset number (ii) of Fig. 3-1b) are, respectively positively and negatively intersected.





A branch is said to be (positively, negatively) included in a mesh if the branch and the associated chord are (similarly, oppositely) oriented in the mesh. For example, mesh "e0a3b1" (Fig. 3-1a) contains branches b and a positively and negatively, respectively.

A node-to-datum path is said to contain a branch (positively, negatively), if the orientation of the branch is (the same as, opposite to) that of the path. In Fig. 3-1a, path "1b3a0" contains branches a and b positively and negatively, respectively.

3-4. Certain Topological Matrices

Two matrices of utmost importance in this presentation can be established by means of linear graphs. One is the basic cutset matrix designated $\left\{\Gamma_{bc}\right]$, and the other, the node-to-datum matrix $\left[T_{bJ}\right]$.

a) The basic cutset matrix has the following characteristics: Its rows correspond to branches and columns represent chords. The matrix elements which are themselves matrices or vectors may be written down algebraically by an inspection of the basic cutsets.

An element in row b and column c of $\left\{\Gamma_{bc}\right]$ is $(-\Gamma_{b,c}, +\Gamma_{b,c}, 0)$ if the chord c is (positively, negatively, not) intersected in the cutset subgraph corresponding to the branch b. The mathematical structure of an element $\Gamma_{b,c}$ is a function of the geometry and classification of the branch b and the chord c as will be shown later in this chapter.

Following the rules laid down, the cutset matrix for the graphs of Figs. 3-1a, b is given by

Branches ↓ \ Chords →		d	e
a		0	$-\Gamma_{a,e}$
b	$\left\{\Gamma_{bc}\right] =$	$-\Gamma_{b,d}$	$\Gamma_{b,e}$
c		$-\Gamma_{c,d}$	0

$$\left\{\Gamma_{bc}\right] = \left\{\begin{array}{c|c} 0 & -\Gamma_{a,e} \\ \hline -\Gamma_{b,d} & \Gamma_{b,e} \\ \hline -\Gamma_{c,d} & 0 \end{array}\right] \qquad (3.1)$$

Eq. (3.1) can be easily established column by column by means of branch-mesh information. In this case the basic cutsets are not necessary. The fule is:

An element in row b and column c of $\left\{\Gamma_{bc}\right]$ is $(+\Gamma_{b,c}, -\Gamma_{b,c}, 0)$ if the branch b is (positively, negatively, not) included in the mesh associated with the chord c.

The cutset matrix is seldom a square matrix.

b) The node-to-datum matrix has its rows corresponding to branches and columns represented by nodes other than the datum. The matrix can be constructed algebraically by the following rule:

An element in row b and column J of $\left[T_{bJ}\right]$ is $(-T_{b,J}, +T_{b,J}, 0)$ if the branch b is (positively, negatively, not) contained in the J^{th} node-to-datum path. Like $\Gamma_{b,c}$, the mathematical structure of $T_{b,J}$ depends upon the classification and geometry of the branch and the node as will be explained subsequently in this chapter.

For the graph of Fig. 3-1a, the node-to-datum matrix is given by

Branches ↓ \ Nodes →		1	2	3	
b		$T_{b,1}$	0	0	
c	$\left[T_{bJ}\right] =$	0	$-T_{c,2}$	0	(3.2)
a		$-T_{a,1}$	$-T_{a,2}$	$-T_{a,3}$	

Matrix $\left[T_{bJ}\right]$ is always a square matrix unless any columns are deleted for some reason. Proper arrangement of rows and columns yields a triangular form (Eq. (3.2)).

3-5. Statical Indeterminacy of a Space Frame

The system elements are subdivided into:

Group I: nb1h Branches with a hinged end,

Group II: nb0h Branches with rigid ends,

Group III: nc0h Chords with rigid ends,

Group IV: nc2h Chords with both ends hinged,

Group V: nc1h Chords with a hinged end,

Group VI: ncss Chords representing spherical supports at joints where two or more bars are connected together.

The total number of bar-redundants may be shown to be

$$3(nb1h + nc1h + ncss) + 6(nb0h + nc0h) + nc2h,$$

or simply

$$3(n + nb0h + nc0h) - 2(nc2h),$$

where n is the count of line segments in a linear graph (Fig. 3-2a) of a space frame.

Necessary equations for the solution of the member redundants are obtained from statics and compatibility. Excluding the datum, there are nb1h hinged nodes and nb0h rigid ones. (3(nb1h) + 6(nb0h))

Independent equations of equilibrium may conveniently be written by means of (nb1h + nb0h) basic cutsets — freebodies from a structural engineering point of view. Thereby the statical indeterminacy of a skeletal structure may be expressed as

$$3(n + nb0h + nc0h) - 2(nc2h) - (3(nb1h) + 6(nb0h))$$

or simply

$$6(nc0h) + nc2h + 3(nc1h + ncss)$$

The relations of this section for the number of bar-redundants and statical indeterminacy are subject to the condition that a formulation tree can be constructed without the necessity for constraining any supports.

For the frame shown in Fig. 1-1 (corresponding linear graph is given in Fig. 3-2), nb1h = 1; nb0h = 3; nc0h = 1; nc2h = 2; ncss = 1. Correspondingly the bar-redundants and the system unknowns are 38 and 17, respectively.

3-6. Modified Joint Loads

Consider a joint J in a framework at which "a1" bars are positively incident and "a2" bars negatively incident (Fig. 3-3a). The applied loads at the node are given by

$$\left\{W_J^o\right\} = \left\{\begin{array}{c} P_J^o \\ \hline Q_J^o \end{array}\right\} \tag{3.3a}$$

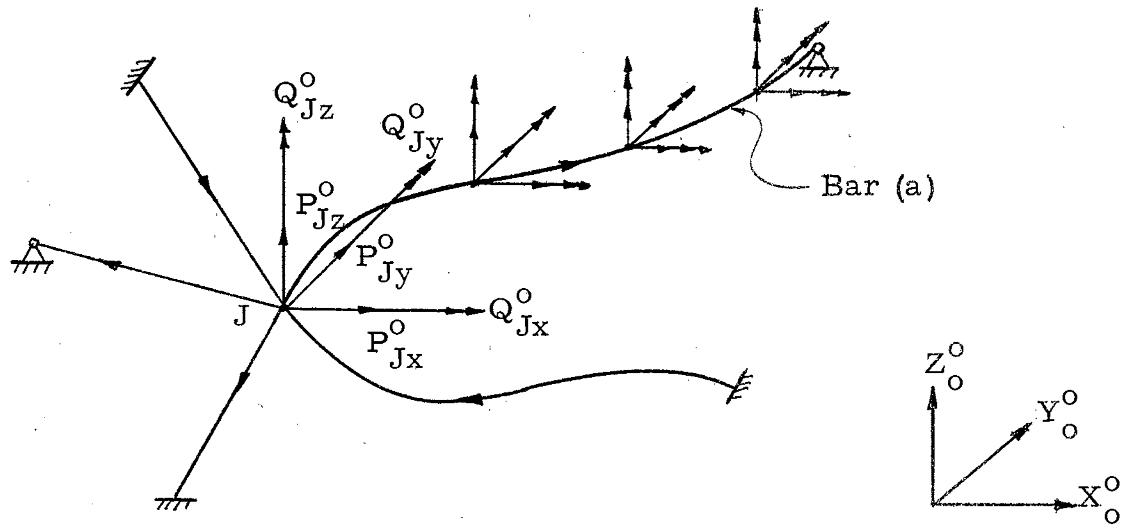


FIG. 3-3a. JOINT J AND ADJOINING BARS

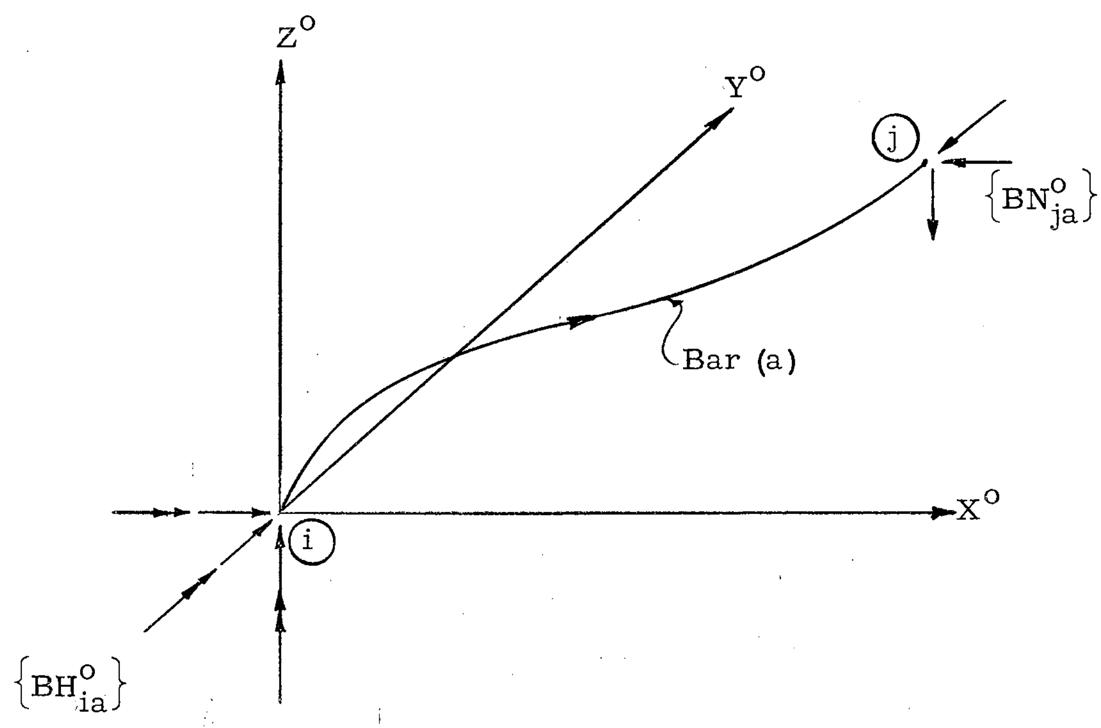


FIG. 3-3b. EQUIVALENT SYSTEM OF LOADS FOR BAR (a) OF FIG. 3-3a

Loads may also be applied at a number of points along the members (Fig. 3-3a). For convenience of formulation, the member loads are replaced by an equivalent system consisting of basic reactions $\left\{BH_{ia}^{o}\right\}$ and $\left\{BN_{ja}^{o}\right\}$ acting at the ends i and j (Fig. 3-3b). They are then added algebraically to obtain the modified joint loads as

$$\left\{PJ^{o}\right\} = \left\{P_{J}^{o}\right\} + \sum_{a2} \left\{BN_{ia}^{o}\right\} - \sum_{a1} \left\{BN_{ja}^{o}\right\} \tag{3.3b}$$

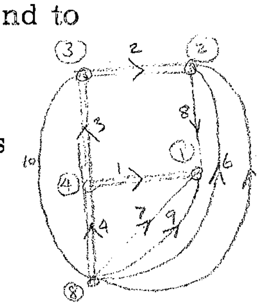
$$\left\{QJ^{o}\right\} = \left\{Q_{J}^{o}\right\} + \sum_{a2} \left\{BM_{ia}^{o}\right\} \tag{3.3c}$$

or simply

$$\left\{WJ^{o}\right\} = \left\{W_{J}^{o}\right\} + \sum_{a2} \left\{BH_{ia}^{o}\right\} - \sum_{a1} \left\{BH_{ja}^{o}\right\} \tag{3.3d}$$

3-7. Equilibrium

Static Equilibrium of an arbitrary system of bars (Fig. 1-1) can be established by means of its linear graph (Fig. 3-2a). Four freebodies of the frame are shown in Fig. 3-4. They correspond to the basic cutsets of Fig. 3-2a.



Force equilibrium of the hinged joint (1) (Fig. 3-4a) is expressed by

$$\left\{N_{j1}^{o}\right\} + \left\{N_{j7}^{o}\right\} + \left\{N_{j8}^{o}\right\} + \left\{N_{j9}^{o}\right\} - \left\{P_{1}^{o}\right\} = \left\{0\right\} \tag{3.4a}$$

Utilizing Eqs. (2.3), (3.3), and notation of Table 3-1 and rearranging terms, Eq. (3.4a) becomes

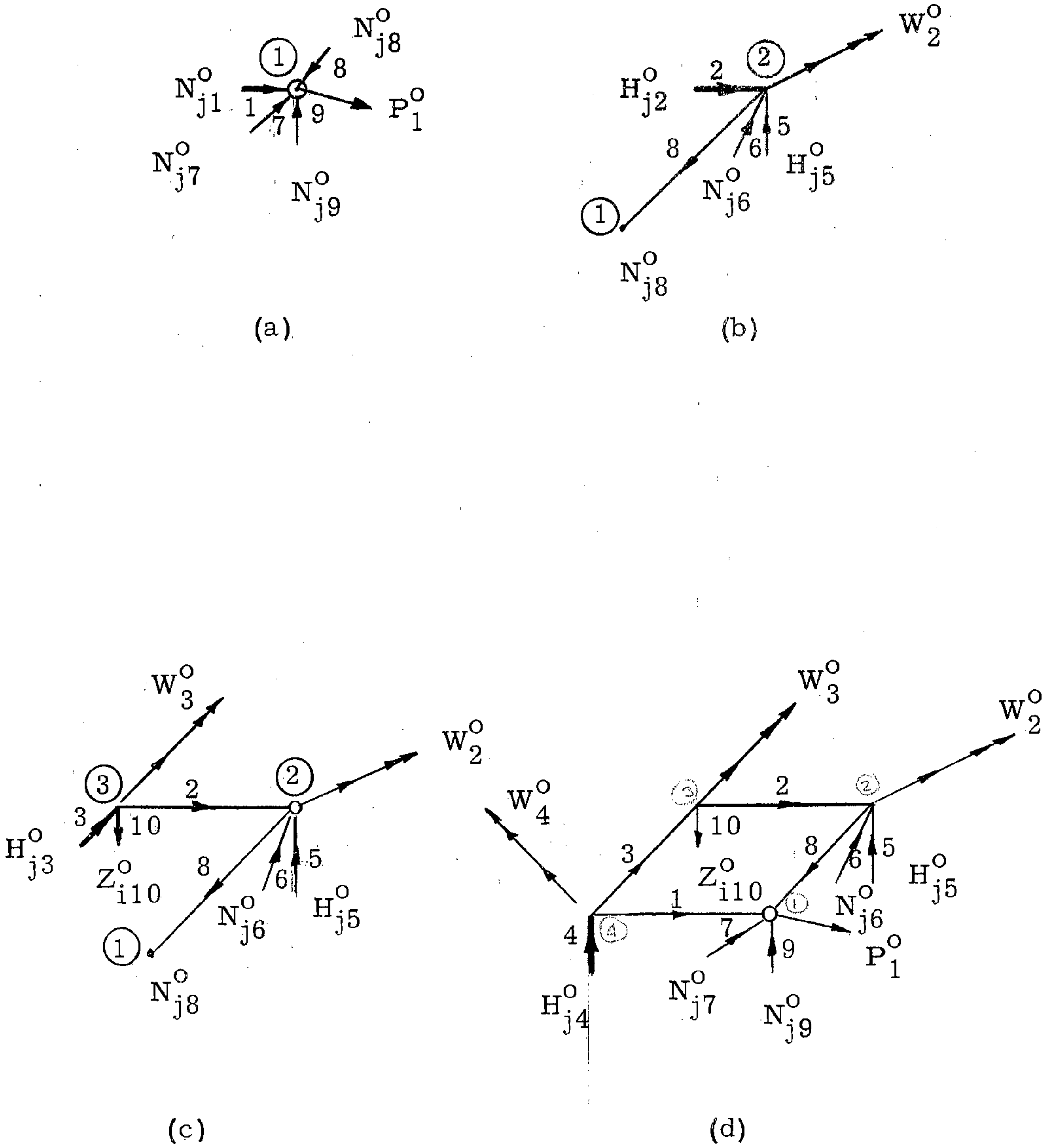


FIG. 3-4. FREEBODIES FOR THE FRAME OF FIG. 1-1

$$\left\{Z_{i1}^{1}\right\} = \left\{0_{3,6} \middle| 0_{3,1} \middle| -\Gamma_{1,7} \middle| -\Gamma_{1,8} \middle| -\Gamma_{1,9} \middle| 0_{3}\right] \begin{Bmatrix} Z_5^5 \\ \hline X_{i6x}^6 \\ \hline X_{i7x}^7 \\ \hline Z_{i8}^8 \\ \hline Z_{i9}^9 \\ \hline Z_{i10}^o \end{Bmatrix} + \left\{T_{1,1}^{o} \middle| 0_{3,6} \middle| 0_{3,6} \middle| 0_{3,6}\right] \begin{Bmatrix} P1^o \\ \hline W2^o \\ \hline W3^o \\ \hline W4^o \end{Bmatrix} \qquad (3.4b)$$

in which, for example, $\left[\Gamma_{1,8}\right]$ and $\left[T_{1,1}\right]$ are given by

$$\left[\Gamma_{1,8}\right] = \left[\omega'_{o1}\right]^{-1}\left[\omega'_{o8}\right] \;, \quad \left[T_{1,1}\right] = \left[\omega'_{o1}\right]^{-1} \qquad (3.4c)$$

and

$$\left\{0_{3,6}\right] = \left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}\right] \;, \quad \left[0_3\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \qquad (3.4d)$$

The isolated part (Fig. 3-4b) yields

$$\left\{H_{j2}^{o}\right\} + \left\{H_{j5}^{o}\right\} + \left\{rL_{2,2}^{o}\right]\left\{N_{j6}^{o}\right\} - \left\{rL_{2,1}^{o}\right]\left\{ZN_{j8}^{o}\right\} - \left\{BH_{i8}^{o}\right\} - \left\{W_{2}^{o}\right\} = \left\{0\right\} \qquad (3.5a)$$

Since the loads applied at intermediate points of the bar number 8 are in equilibrium with basic reactions, only the influence of $\left\{ZN_{j8}^{o}\right\}$ and $\left\{BH_{i8}^{o}\right\}$ is considered in writing the above equation. Substituting from Eqs. (2.2, 3, 4) and introducing the nomenclature of Eq. (3.3)

TABLE 3-1. STEREO-STATIC MATRICES OF A SYSTEM OF BARS

Identification of Branch and Its Details				Type of Elements Associated with the Branch					STEREO-STATIC MATRIX
				Chord Details			Node Details		
No.	Node at which end j is incident	Group classi-fication	Branch Transformation Matrix	No.	Node at which end j is incident	Group classi-fication	Hinged Node No.	Rigid Node No.	
b	b	I	$[\omega'_{ob}]^{-1} = \begin{bmatrix} d\{\omega_{obx}\}^* \\ -L_b\{\omega_{obz}\}^* \\ L_b\{\omega_{oby}\}^* \end{bmatrix}$	c	k	IV	—	—	$\{\Gamma_{b,c}\} = [\omega'_{ob}]^{-1}\{\omega_{ocx}\}\frac{1}{d}$
				c	b	V	—	—	$[\Gamma_{b,c}] = [\omega'_{ob}]^{-1}[\omega'_{oc}]$
				c	b	VI	—	—	$[\Gamma_{b,c}] = [\omega'_{ob}]^{-1}\frac{1}{d}$
				—	—	—	J(=b)	—	$[T_{b,J}] = [\omega'_{ob}]^{-1}$
b	h	II	$[\pi j_{ob}]^{-1} = \begin{bmatrix} d\{\omega_{obx}\}^* & 0_{1,3} \\ -L_b\{\omega_{obz}\}^* & \{\omega_{oby}\}^* \\ L_b\{\omega_{oby}\}^* & \{\omega_{obz}\}^* \\ 0_3 & [\omega_{ob}]^* \end{bmatrix}$	c	k	III	—	—	$[\Gamma_{b,c}] = [\pi j_{ob}]^{-1}[r^o_{hk}][\pi j_{oc}]$
				c	k	IV	—	—	$\{\Gamma_{b,c}\} = [\pi j_{ob}]^{-1}\{rL^o_{hk}\}\{\omega_{ocx}\}\frac{1}{d}$
				c	k	V	—	—	$[\Gamma_{b,c}] = [\pi j_{ob}]^{-1}\{rL^o_{hk}\}[\omega'_{oc}]$
				c	k	VI	—	—	$[\Gamma_{b,c}] = [\pi j_{ob}]^{-1}\{rL^o_{hk}\}\frac{1}{d}$
				—	—	—	J	—	$\{T_{b,J}\} = [\pi j_{ob}]^{-1}\{rL^o_{hJ}\}$
				—	—	—	—	J	$[T_{b,J}] = [\pi j_{ob}]^{-1}[r^o_{hJ}]$

and Table 3-1, Eq. (3.5a) may be written as

$$\left\{Z_2^2\right\} = \left\{-\Gamma_{2,5} \middle| -\Gamma_{2,6} \middle| 0_{6,1} \middle| \Gamma_{2,8} \middle| 0_{6,3} \middle| 0_{6,3}\right] \left\{\begin{array}{c} Z_5^5 \\ \hline X_{i6x}^6 \\ \hline X_{i7x}^7 \\ \hline Z_{i8}^8 \\ \hline Z_{i9}^9 \\ \hline Z_{i10}^o \end{array}\right\} + \left\{0_{6,3} \middle| T_{2,2} \middle| 0_6 \middle| 0_6\right] \left\{\begin{array}{c} P1^o \\ \hline W2^o \\ \hline W3^o \\ \hline W4^o \end{array}\right\} \tag{3.5b}$$

where, for example,

$$\left[\Gamma_{2,5}\right] = \left[\pi j_{o2}\right]^{-1} \left[r_{2,2}^o\right]\left[\pi j_{o5}\right], \quad \left[T_{2,2}\right] = \left[\pi j_{o2}\right]^{-1} \left[r_{22}^o\right] \tag{3.5c}$$

Similar equations can be obtained from the freebodies shown in Figs. 3-4 c, d. All these relations are collected into a single matrix equation

$$\left\{Z_b\right\} = \left\{\Gamma_{bc} \middle| T_{bJ}\right] \left\{\begin{array}{c} Z_c \\ \hline WJ \end{array}\right\} \tag{3.6a}$$

in which

$$\{Z_b\} = \left\{\begin{array}{c} Z_{i1}^{1} \\ \hline Z_2^2 \\ \hline Z_3^3 \\ \hline Z_4^4 \end{array}\right\} \; ; \quad \{Z_c\} = \left\{\begin{array}{c} Z_5^5 \\ \hline X_{i6x}^6 \\ \hline X_{i7x}^7 \\ \hline Z_{i8}^8 \\ \hline Z_{i9}^9 \\ \hline Z_{i10}^{o} \end{array}\right\} \; ; \quad \{WJ\} = \left\{\begin{array}{c} P1^{o} \\ \hline W2^{o} \\ \hline W3^{o} \\ \hline W4^{o} \end{array}\right\} \qquad (3.6b)$$

$\{\Gamma_{bc}]$ and $[T_{bJ}]$ are the cutset and node-to-datum matrices of the linear graph (Fig. 3-2a) of the frame (Fig. 1-1). They are recorded in Table 3-2.

Thus the problem of establishing static equilibrium for frameworks is reduced to the following two simple routine steps

1) Represent a given structure by its linear graph which is a simple plane sketch. Then construct a formulation tree in the same figure indicating the branches by thick lines. Thin lines stand for chords, that is, redundant members.

2) Utilizing the branch-mesh and node-to-datum information of the graph, Eqs. (3.3) and Table 3-1, develop $\{\Gamma_{bc}]$ and $[T_{bJ}]$ as explained in Sections 3-4, 7 of this chapter. It must be noted that this approach eliminates the laborious task of drawing freebody sketches.

Eq. (3.6a) is augmented to obtain

$$\{Z\} = \{\xi \mid \chi] \left\{\begin{array}{c} Z_c \\ \hline WJ \end{array}\right\} \qquad (3,6c)$$

TABLE 3-2. TOPOLOGICAL MATRICES FOR GRAPH OF FIG. 3-2c

CUTSET MATRIX

$\{\Gamma_{bc}\}$ =

Branches ↓ \ Chords →	c = 5	c = 6	c = 7	c = 8	c = 9	c = 10
b = 1	0	0	$-\Gamma_{1,7}$	$-\Gamma_{1,8}$	$-\Gamma_{1,9}$	0
b = 2	$-\Gamma_{2,5}$	$-\Gamma_{2,6}$	0	$\Gamma_{2,8}$	0	0
b = 3	$-\Gamma_{3,5}$	$-\Gamma_{3,6}$	0	$\Gamma_{3,8}$	0	$\Gamma_{3,10}$
b = 4	$-\Gamma_{4,5}$	$-\Gamma_{4,6}$	$-\Gamma_{4,7}$	0	$-\Gamma_{4,9}$	$\Gamma_{4,10}$

NODE-TO-DATUM MATRIX

$[T_{bJ}]$ =

Branches ↓ \ Nodes →	J = 1	J = 2	J = 3	J = 4
b = 1	$T_{1,1}$	0	0	0
b = 2	0	$T_{2,2}$	0	0
b = 3	0	$T_{3,2}$	$T_{3,3}$	0
b = 4	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	$T_{4,4}$

where

$$\{Z\} = \left\{\frac{Z_b}{Z_c}\right\}; \quad \{\xi] = \left\{\frac{\Gamma_{bc}}{I}\right] ; \quad \{\chi] = \left\{\frac{T_{bJ}}{0}\right] \tag{3.6d}$$

$[I]$ and $[0]$ are identity and null matrices.

3-8. Elasto-Statics

In Section 3-5 of this chapter, an expression for the number of redundants in an indeterminate frame was given as

6(nc 0h) + nc2h + 3(nc1h + ncss).

An equal number of deformation equations are required for the complete solution of the problem.

a) System Elastic Weights

Elastic weights of a bar (a) were defined in Chapter II as

$$\{\overline{W}_a^a\} = [A_a^a]\{Z_a^a\} + \{\sigma_a^a\} \tag{3.7a}$$

for the whole system

$$\begin{Bmatrix} \overline{W}_1^1 \\ \cdot \\ \overline{W}_a^a \\ \cdot \\ \overline{W}_n^n \end{Bmatrix} = \begin{bmatrix} A_1^1 & 0 & 0 & 0 & 0 \\ 0 & \bullet & 0 & 0 & 0 \\ 0 & 0 & A_a^a & 0 & 0 \\ 0 & 0 & 0 & \bullet & 0 \\ 0 & 0 & 0 & 0 & A_n^n \end{bmatrix} \begin{Bmatrix} Z_1^1 \\ \cdot \\ Z_a^a \\ \cdot \\ Z_n^n \end{Bmatrix} + \begin{Bmatrix} \sigma_1^1 \\ \cdot \\ \sigma_a^a \\ \cdot \\ \sigma_n^n \end{Bmatrix} \tag{3.7b}$$

or simply

$$\left\{\overline{W}\right\} = \left[A\right]\left\{Z\right\} + \left\{\sigma\right\} \tag{3.7c}$$

in which $\left[A\right]$ is a square matrix whose diagonal elements are either scalars or square matrices, and off-diagonal submatrices consist of zeros. $\left[A\right]$ is known as the primitive flexibility matrix.

Eq. (3.7c) is partitioned corresponding to branches and chords as

$$\left\{\begin{array}{c}\overline{W}_b \\ \hline \overline{W}_c\end{array}\right\} = \left[\begin{array}{c|c}A_b & 0 \\ \hline 0 & A_c\end{array}\right]\left\{\begin{array}{c}Z_b \\ \hline Z_c\end{array}\right\} + \left\{\begin{array}{c}\sigma_b \\ \hline \sigma_c\end{array}\right\} \tag{3.7d}$$

b) System Deformations

Let the deformations in the direction of system unknowns $\left\{Z_c\right\}$ and joint loads $\left\{WJ\right\}$ be denoted by

$$\left\{\Delta\right\} = \left\{\begin{array}{c}\Delta_c \\ \hline \Delta_J\end{array}\right\} \tag{3.8}$$

As a result of the contragredient properties(53) of actions and deformations in elastic structures, the distorsions of the members (Eq. (3.7b)) may be transformed to equivalent displacements $\left\{\Delta\right\}$ at the releases and nodes by the cotransmission matrix $\left\{\xi \mid \chi\right]^{*}$, since $\left\{\xi \mid \chi\right]$ was the matrix which transmitted actions at the releases and joints to the members (Eq. (3.6c)). Thus,

$$\begin{Bmatrix} \Delta_c \\ \hline \Delta_J \end{Bmatrix} = \begin{bmatrix} \Gamma^*_{bc} & I \\ \hline T^*_{bJ} & 0 \end{bmatrix} \begin{bmatrix} A_b & 0 \\ \hline 0 & A_c \end{bmatrix} \begin{bmatrix} \Gamma_{bc} & T_{bJ} \\ \hline I & 0 \end{bmatrix} \begin{Bmatrix} Z_c \\ \hline WJ \end{Bmatrix} + \begin{bmatrix} \Gamma^*_{bc} & I \\ \hline T^*_{bJ} & 0 \end{bmatrix} \begin{Bmatrix} \sigma_b \\ \hline \sigma_c \end{Bmatrix} \tag{3.9}$$

Introducing the notation of Table 3-3, the above equation may be written as

$$\{\Delta_c\} = [A_{cc}]\{Z_c\} + [A_{cJ}]\{WJ\} + \{\sigma_{cc}\} \tag{3.10a}$$

$$\{\Delta_J\} = [A_{cJ}]^*\{Z_c\} + [A_{JJ}]\{WJ\} + \{\sigma_{JJ}\} \tag{3.10b}$$

c) Compatibility and Solution

The solution to system unknowns $\{Z_c\}$ is obtained by postulating that the elastic curve shall be continuous in the direction of the forced releases. This means that $\{\Delta_c\}$ be equal to zero. Thus from Eq. (3.10a),

$$\{Z_c\} = -[A_{cc}]^{-1}\{A_{cJ}]\{WJ\} - [A_{cc}]^{-1}\{\sigma_{cc}\} \tag{3.11a}$$

Eq. (3.11a) is substituted into Eq. (3.10b) to obtain

$$\{\Delta_J\} = [A_J]\{WJ\} + \{\sigma_J\} \tag{3.11b}$$

The nomenclature used in Eqs. (3.11a, b) is recorded in Table 3-3.

Once $\{Z_c\}$ are calculated, determination of other quantities is a simple matter of application of Eqs. (3.3a), (2.2), (2.3), and (2.4).

Deformation matrices in Eqs. (3.10a, b) can be generated element by element by means of the following expressions:

$$\left[A_{c_i, c_i}\right] = \left[A_{c_i}\right] + \sum_{b_k(c_i - c_i)} \left\{\Gamma_{b_k, c_i}\right]^* \left[A_{b_k}\right] \left\{\Gamma_{b_k, c_i}\right] \Bigg|\; i = 1, 2 \cdots nc \tag{3.12a}$$

$$\left\{A_{c_i, c_j}\right] = \sum_{b_k(c_i - c_j)} \left\{\pm\Gamma_{b_k, c_i}\right]^* \left[A_{b_k}\right] \left\{\pm\Gamma_{b_k, c_j}\right] \Bigg|\; \begin{Bmatrix} i = 1, 2 \cdots nc \\ j = i+1 \cdots nc \end{Bmatrix} \tag{3.12b}$$

$$\left\{A_{c_i, J_j}\right] = \sum_{b_k(c_i - J_j)} \left\{\pm\Gamma_{b_k, c_i}\right]^* \left[A_{b_k}\right] \left\{\pm\Gamma_{b_k, J_j}\right] \Bigg|\; \begin{Bmatrix} i = 1, 2 \cdots nc \\ j = 1, 2 \cdots nb \end{Bmatrix} \tag{3.12c}$$

$$\left\{\sigma_{cc_i}\right\} = \left\{\sigma_{c_i}\right\} + \sum_{b_k(c_i - c_i)} \left\{\pm\Gamma_{b_k, c_i}\right]^* \left\{\sigma_{b_k}\right\} \Bigg|\; i = 1, 2 \cdots nc \tag{3.12d}$$

$$\left[A_{J_i, J_j}\right] = \sum_{b_k(J_i, J_j)} \left\{T_{b_k, J_i}\right]^* \left[A_{b_k}\right] \left\{T_{b_k, J_j}\right] \Bigg|\; \begin{Bmatrix} i = 1, 2 \cdots nb \\ j = 1, 2 \cdots nb \end{Bmatrix} \tag{3.12e}$$

$$\left\{\sigma_{JJ_i}\right\} = \sum_{b_k(J_i - J_i)} \left\{T_{b_k, J_i}\right]^* \left\{\sigma_{b_k}\right\} \qquad i = 1, 2 \cdots nb \tag{3.12f}$$

in which

$\sum_{b_k(c_i - c_j)}$ denotes summation over branches common to loops corresponding to chords c_i and c_j,

$\sum_{b_k(c_i - J_j)}$ denotes summation over branches common to loop c_i and node-to-datum path J_j,

$\sum_{b_k(J_i - J_j)}$ denotes summation over branches common to the node-to-datum paths J_i and J_j.

TABLE 3-3. ELASTO-STATIC MATRICES OF A SYSTEM OF BARS

$$\left[A_{cc}\right] = \left\{\Gamma_{bc}\right]^{*}\left[A_{b}\right]\left\{\Gamma_{bc}\right] + \left[A_{c}\right]$$

$$\left\{\sigma_{cb}\right\} = \left\{\Gamma_{bc}\right]^{*}\left\{\sigma_{b}\right\}$$

$$\left\{\sigma_{cc}\right\} = \left\{\sigma_{cb}\right\} + \left\{\sigma_{c}\right\}$$

$$\left\{A_{cJ}\right] = \left\{\Gamma_{bc}\right]^{*}\left[A_{b}\right]\left\{T_{bJ}\right]$$

$$\left[A_{JJ}\right] = \left[T_{bJ}\right]^{*}\left[A_{b}\right]\left[T_{bJ}\right]$$

$$\left[A_{J}\right] = \left[A_{JJ}\right] - \left\{A_{cJ}\right]^{*}\left[A_{cc}\right]^{-1}\left\{A_{cJ}\right]$$

$$\left\{\sigma_{JJ}\right\} = \left[T_{bJ}\right]^{*}\left\{\sigma_{b}\right\}$$

$$\left\{\sigma_{J}\right\} = \left\{\sigma_{JJ}\right\} - \left[A_{cc}\right]^{-1}\left\{\sigma_{cc}\right\}$$

3-9. Analysis of Frameworks with Supports Other Than Fixed

It is not possible to establish a formulation tree when all the supports of a structure (Fig. 3-5a) are other than fixed. This difficulty will be overcome by restraining one or more supports as needed (Figs. 3-5b, c). The solution obtained for the modified structure (Fig. 3-5b) does not generally satisfy the condition that the sum of the end-conditioning moments of all the bars meeting at the fixed support plus the couples applied at that point be equal to zero. That is:

$$\sum_a \left\{M^o_{ja}\right\} - \left\{Q^o_\mu\right\} = \left\{0\right\} \tag{3.13a}$$

or in terms of system unknowns and applied loads,

$$\left\{\Gamma_\mu\right]\left\{Z_c\right\} + \left\{T_\mu\right]\left\{WJ\right\} - \left\{Q^o_\mu\right\} = \left\{0\right\} \tag{3.13b}$$

Eq. (3.13b) may be employed to reduce the degree of statical indeterminacy of the modified structure to that of the original framework. Such a procedure becomes mathematically involved especially in case of complex space frames. In addition the computations cannot easily be adapted to computers.

Lagrange's method of undetermined multipliers is employed to take care of the constraint conditions (Eq. (3.13b)). Let the matrix of the undetermined constants be denoted by $\left\{\mu\right\}$, in which i^{th} constant is μ_i. Their number is equal to that of the constraint equations. The solution equation (Eq. (3.11a)) for the new structure is modified to obtain

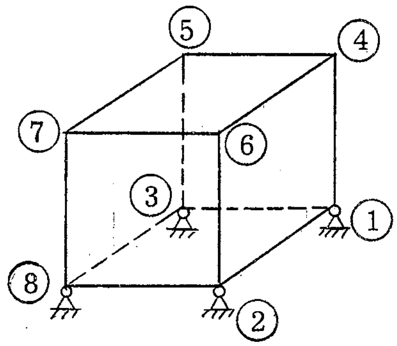


FIG. 3-5a. A FRAME ON SPHERICAL SUPPORTS

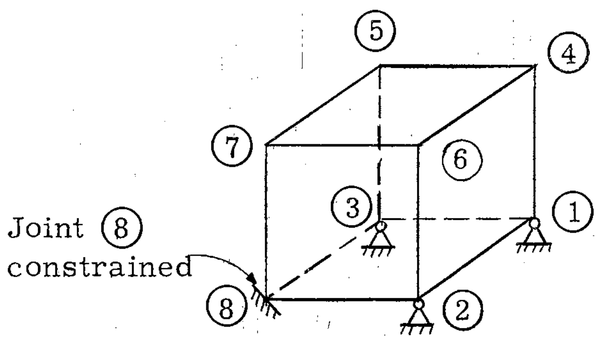


FIG. 3-5b. MODIFIED STRUCTURE

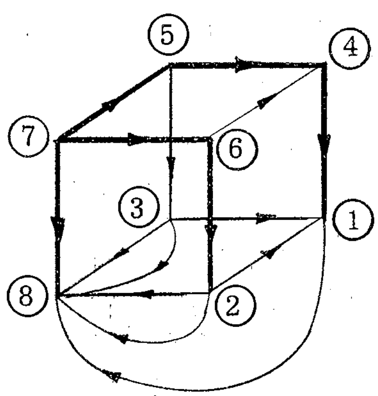


FIG. 3-5c. LINEAR GRAPH OF MODIFIED FRAME

$$\left[A_{cc}\right]\left\{Z_c\right\} + \left\{A_{cJ}\right]\left\{WJ\right\} + \left\{\sigma_{cc}\right\} + \left\{\Gamma_\mu\right]^{*}\left\{\mu\right\} = \left\{0\right\} \tag{3.14}$$

Eqs. (3.13b, 14) are collected and written as

$$\begin{bmatrix} A_{cc} & \Gamma^{*}_{\mu} \\ \hline \Gamma_\mu & 0 \end{bmatrix} \begin{Bmatrix} Z_c \\ \hline \mu \end{Bmatrix} + \begin{Bmatrix} A_{cJ} \\ \hline T_\mu \end{Bmatrix} \left\{WJ\right\} + \begin{Bmatrix} \sigma_{cc} \\ \hline -Q^{o}_{\mu} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \hline 0 \end{Bmatrix} \tag{3.15a}$$

or simply

$$\left[A'_{cc}\right]\left\{Z'_c\right\} + \left\{A'_{cJ}\right]\left\{WJ\right\} + \left\{\sigma'_{cc}\right\} = \left\{0\right\} \tag{3.15b}$$

The correct solution to the original structure (Fig. 3-5a) is given by Eq. (3.15b).

It must be noted that this approach involves unknowns equal to twice the number of constraints plus the redundants of the given frame.

CHAPTER IV

SOLUTION OF HIGHLY REDUNDANT SYSTEMS IN EASY STAGES

Structural analysis consists of formulation and numerical solution, the first of which has been investigated in Chapter III. Many practical problems involve the solution of hundreds, and sometimes thousands of simultaneous equations, requiring development of new techniques to reduce both the volume of numerical operations and the amount of computer storage necessary at any stage of computation. The purpose of this chapter is to describe group elimination method for fairly large frameworks.

4-1. Basis of the Method

The essential feature of the method is the proper arrangement of the system redundants into groups such that the resulting flexibility matrix of the whole structure is of "tridiagonal" or "three and five diagonal" form. In most practical cases this is possible. Two types of grouping are presented in the following.

4-1.1. Subdivision Resulting in "Tridiagonal" Flexibility Matrix

Consider an arbitrary framework shown in Fig. 4-1, for which a formulation tree consisting of n_t separate parts is indicated by thick lines and chords by thin lines (Fig. 4-2). The chords

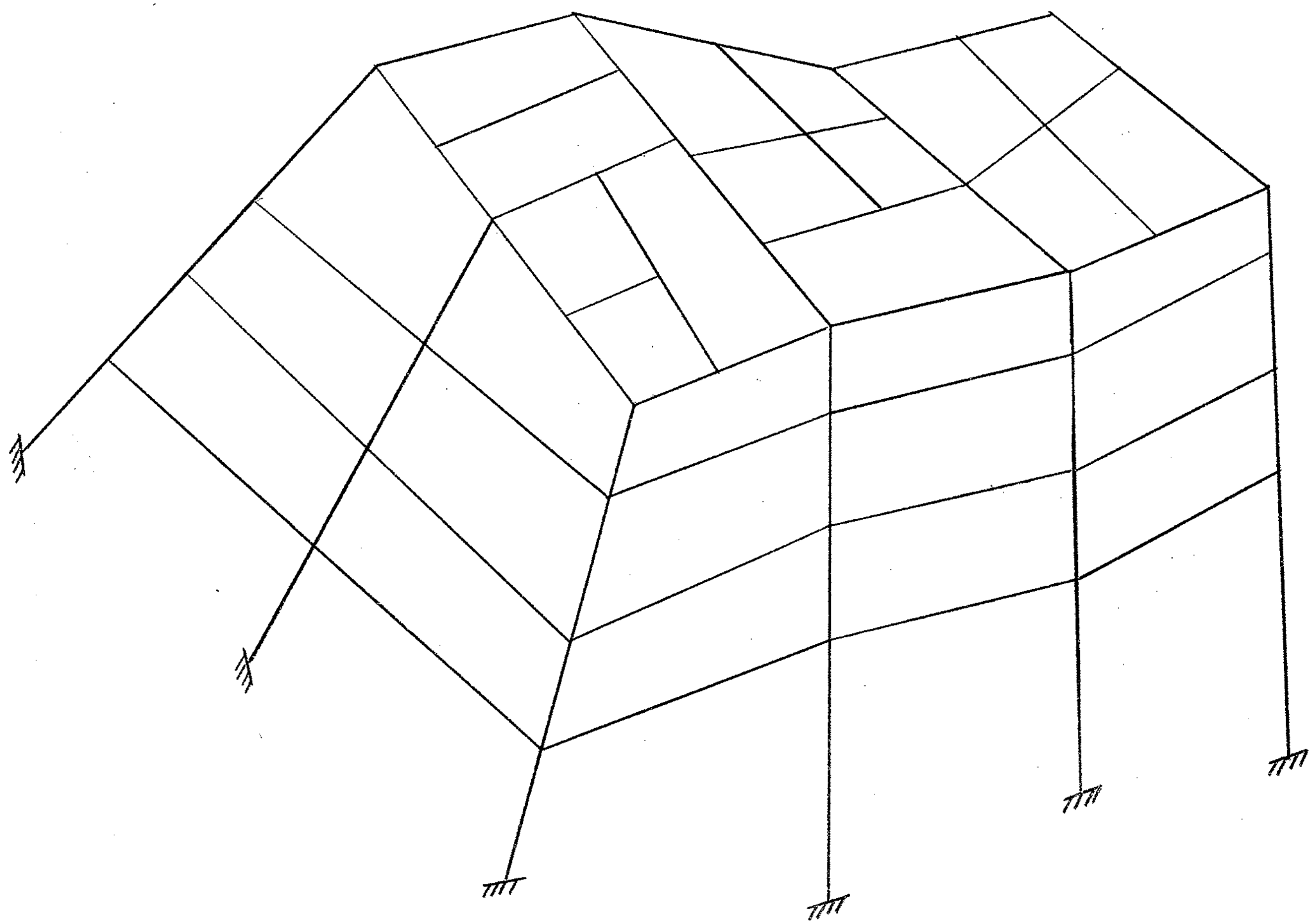


FIG. 4-1. ARBITRARY FRAMEWORK IN SPACE

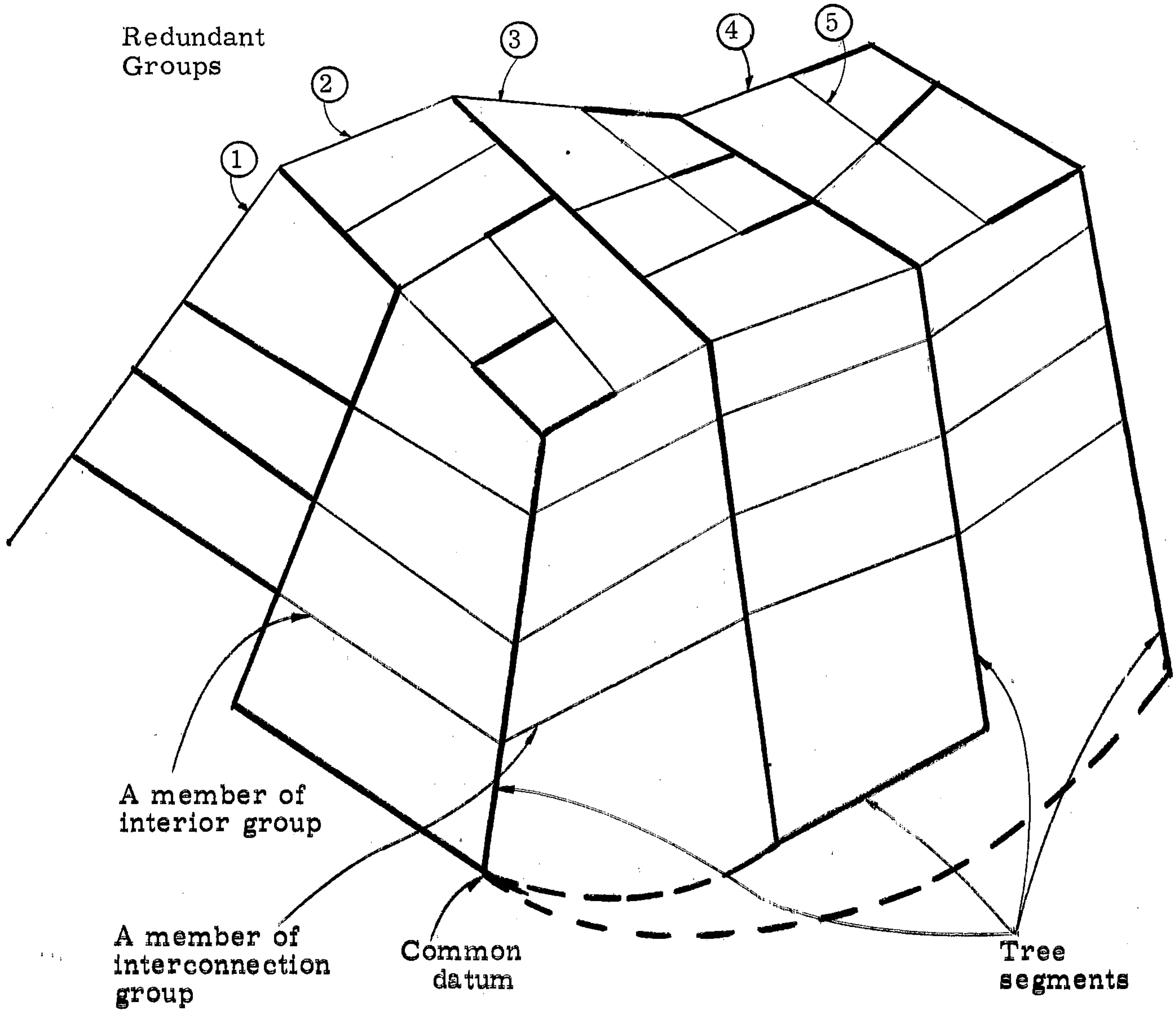


FIG. 4-2. TREE SUBGRAPHS – REDUNDANT GROUPS

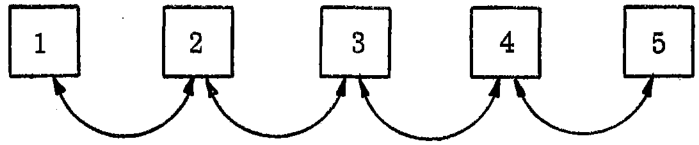


FIG. 4-3. MUTUAL INTERACTION BETWEEN REDUNDANT GROUPS OF FIG. 4-2

contained within a part of the tree make an interior group and those interconnecting two adjacent tree subgraphs constitute an interconnection group. Let the total number of groups be n_g, which for the case under consideration is equal to five. Mutual interaction is graphically represented by Fig. 4-3. The solution matrix for the complete structure is of "tridiagonal" form and is given by

$$\begin{bmatrix} A_{1,1} & A_{1,2} & & & \\ A_{2,1} & A_{2,2} & A_{2,3} & & \\ & A_{3,2} & A_{3,3} & A_{3,4} & \\ & & A_{4,3} & A_{4,4} & A_{4,5} \\ & & & A_{5,4} & A_{5,5} \end{bmatrix} \bullet \begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \end{Bmatrix} = \begin{Bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \\ K_5 \end{Bmatrix} \qquad (4\text{-}1)$$

in which symmetrical submatrices along the diagonal are flexibilities of the individual groups and the nonzero off-diagonal ones represent coupling effects. The matrix is very sparse, as is usually the case with large systems.

This type of grouping is favorable for building frameworks with the number of bays larger than the number of stories.

4-1.2. Sub-Division Resulting in "Three and Five Diagonal" Flexibility Matrix

For the structure of Fig. 4-1, another formulation tree is shown in Fig. 4-4. The tree is divided into n_t segments. The chords with both ends incident within a segment form an interior group and

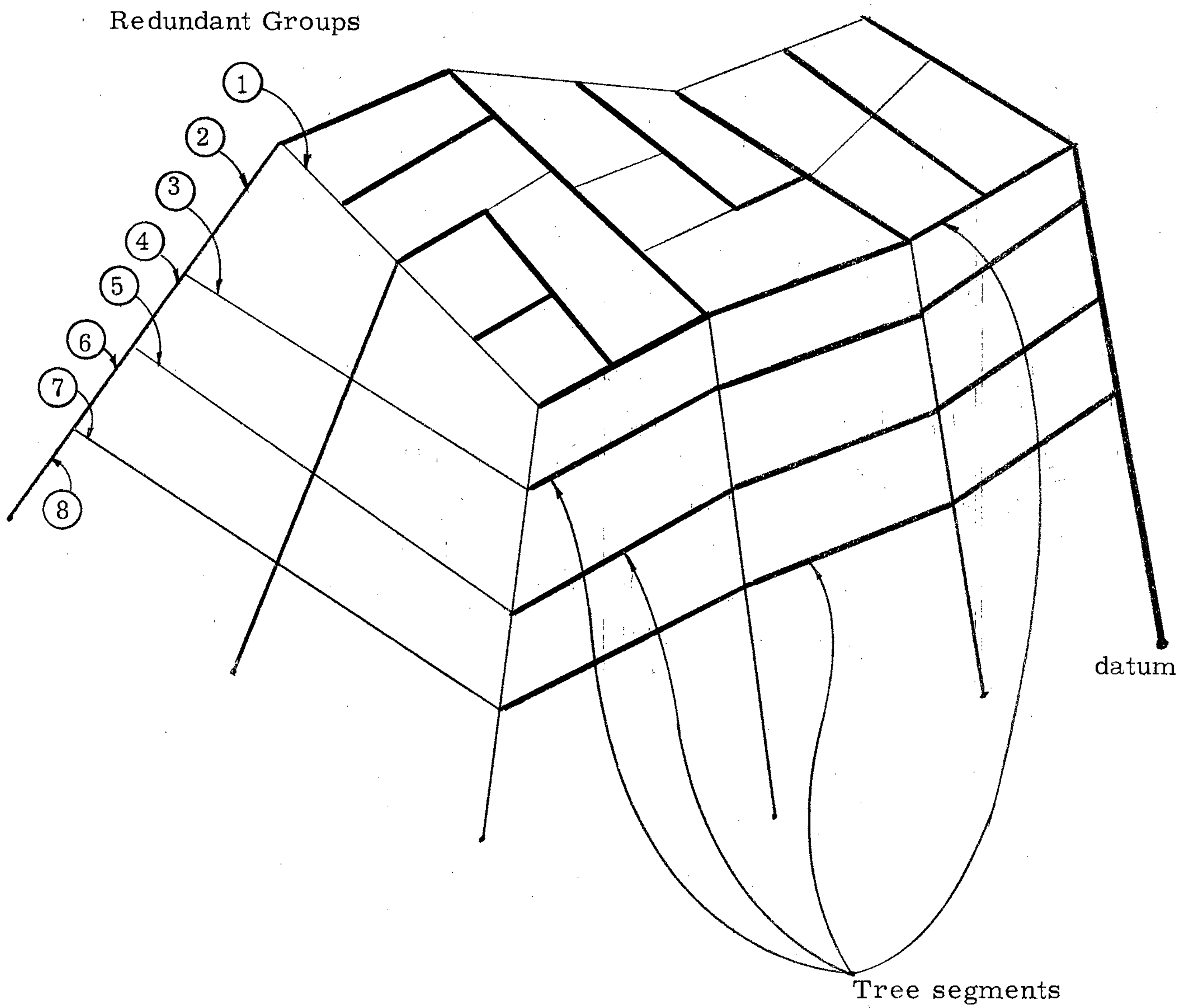


FIG. 4-4. TREE SEGMENTS – REDUNDANT GROUPS

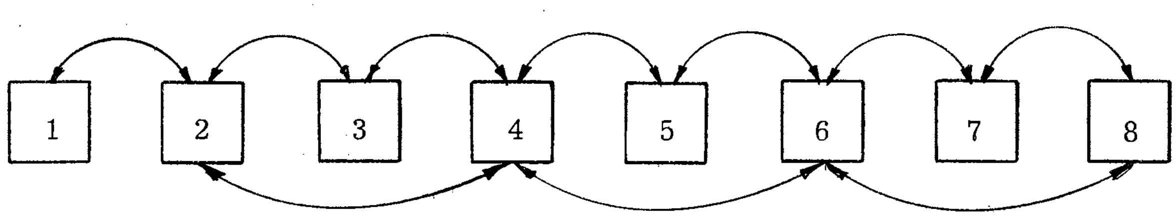


FIG. 4-5. MUTUAL INTERACTION BETWEEN REDUNDANT GROUPS OF FIG. 4-4

those located between two adjacent segments make an interconnection group. As shown in Fig. 4-2, there are a total of $n_g = 8$ redundant groups. Mutual interaction in this case is represented in Fig. 4-5. The solution matrix given by Eq. (4.2) is of "three and five diagonal" form.

This type of subdivision is favorable for tall, slender frameworks.

$$\begin{bmatrix} A_{1,1} & A_{1,2} & & & & & & \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & & & & \\ & A_{3,2} & A_{3,3} & A_{3,4} & & & & \\ & A_{4,2} & A_{4,3} & A_{4,4} & A_{4,5} & A_{4,6} & & \\ & & & A_{5,4} & A_{5,5} & A_{5,6} & & \\ & & & A_{6,4} & A_{6,5} & A_{6,6} & A_{6,7} & A_{6,8} \\ & & & & & A_{7,6} & A_{7,7} & A_{7,8} \\ & & & & & A_{8,6} & A_{8,7} & A_{8,8} \end{bmatrix} \begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \\ Z_6 \\ Z_7 \\ Z_8 \end{Bmatrix} = \begin{Bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \\ K_5 \\ K_6 \\ K_7 \\ K_8 \end{Bmatrix} \tag{4.2}$$

4-2. Group Elimination

Direct matrix inversion is not practical for large scale systems. Gauss's method can be efficiently employed taking advantage of the inherently sparseness of the matrices. The process of elimination for the solution of Eqs. (4.1) and (4.2) is represented graphically in Figs. 4-6 and 4-7, respectively.

In the following development of recursion relationships for the analysis of redundant groups (Fig. 4-4), it is assumed that the

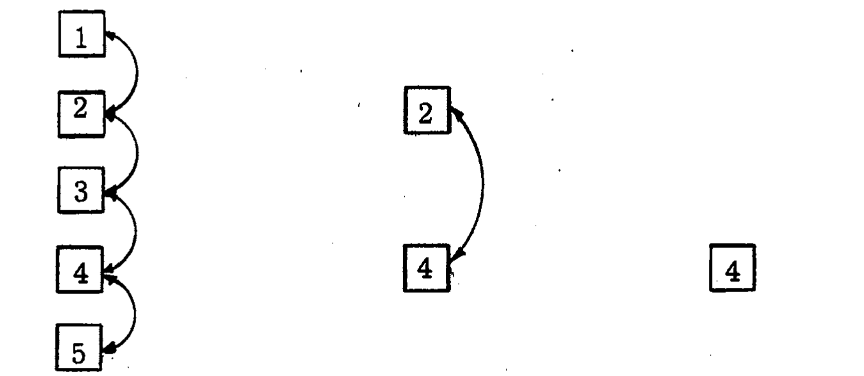


FIG. 4-6. GRAPHICAL REPRESENTATION OF ELIMINATION STAGES FOR EQ. (4.1)

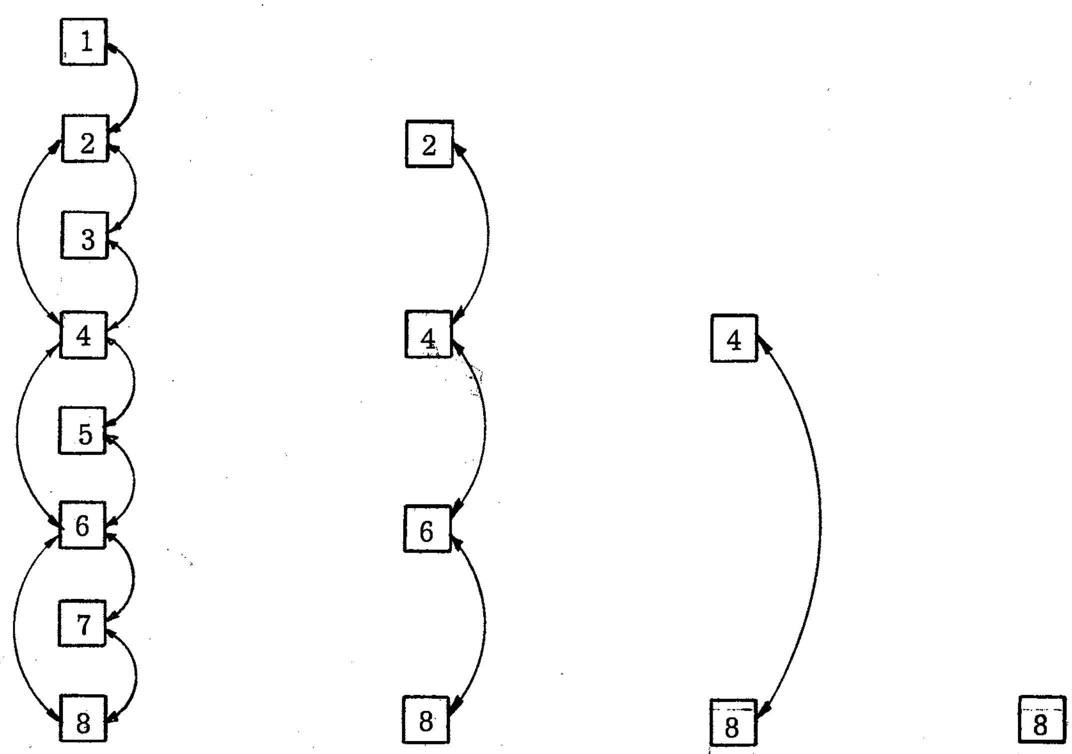


FIG. 4-7. GRAPHICAL REPRESENTATION OF ELIMINATION STAGES FOR EQ. (4-2)

flexibility matrices "A" in Eq. (4.2) are generated first by means of the techniques of Chapters II and III. As shown in Fig. 4-7, the first stage eliminates the alternate groups 1, 3, 5 and 7 corresponding to the tree segments. Thus from Eq. (4.2)

$$\left\{Z_i\right\} = \left[A^m_{i,i-n_m}\right]\left\{Z_{i-n_m}\right\} + \left[A^m_{i,i+n_m}\right]\left\{Z_{i+n_m}\right\} + \left\{K^m_i\right\} \tag{4.3a}$$

in which

$$\left[A^m_{i,i\mp n_m}\right] = -\left[A^{m-1}_{i,i}\right]^{-1}\left[A^{m-1}_{i,i\mp n_m}\right] \tag{4.3b}$$

$$\left\{K^m_i\right\} = \left[A^{m-1}_{i,i}\right]^{-1}\left\{K^{m-1}_i\right\} \tag{4.3c}$$

m denotes the elimination stage number; $n_m = 1$ for $m = 1$; i assumes the numbers 1, 3, 5 and 7 of the groups eliminated; matrices with subscripts less than 1 are considered null; and zero superscript indicates the initial matrix.

The reduced system of equations after the first stage is of "tridiagonal" form and is given by

$$\begin{bmatrix} A^m_{2,2} & A^m_{2,4} & & \\ A^m_{4,2} & A^m_{4,4} & A^m_{4,6} & \\ & A^m_{6,4} & A^m_{6,6} & A^m_{6,8} \\ & & A^m_{8,6} & A^m_{8,8} \end{bmatrix} \begin{Bmatrix} Z_2 \\ Z_4 \\ Z_6 \\ Z_8 \end{Bmatrix} = \begin{Bmatrix} K^m_2 \\ K^m_4 \\ K^m_6 \\ K^m_8 \end{Bmatrix} \tag{4.4a}$$

in which

$$\left[A^{m}_{j,j}\right] = \left[A^{m-1}_{j,j}\right] + \left\{A^{m-1}_{j,j-n_m}\right]\left\{A^{m}_{j-n_m,j}\right] + \left\{A^{m-1}_{j,j+n_m}\right]\left\{A^{m}_{j+n_m,j}\right] \tag{4.4b}$$

$$\left\{A_{j,j+2n_m}\right] = \left\{A_{j+2n_m,j}\right]^{*} = \left\{A^{m-1}_{j,j+2n_m}\right] + \left\{A^{m-1}_{j,j+n_m}\right]\left\{A_{j+n_m,j+2n_m}\right] \tag{4.4c}$$

$$\left\{K^{m}_{j}\right\} = \left\{K^{m-1}_{j}\right\} - \left\{A^{m-1}_{j,j-n_m}\right]\left\{K^{m}_{j-n_m}\right\} - \left\{A^{m-1}_{j,j+n_m}\right]\left\{K^{m}_{j+n_m}\right\} \tag{4.4d}$$

j assumes the numbers 2, 4, 6 and 8 of the remaining groups after the first stage; matrices with subscripts less than one or greater than eight (that is n_g) are considered null; m = 1; n_m = 1 for m = 1; and zero superscript denotes initial matrix.

With m = 2 onwards, $n_m = 2n_{m-1}$, and $\left\{A^{m-1}_{j,j+2n_m}\right] = \left\{0\right]$, Eqs. (4.3) and (4.4) may now be applied repeatedly to eliminate group numbers i = 2, 6 at stage 2, and group number i = 4 at stage 3. Continuing this process ultimately leads to an equation, containing redundants of only one group number j given by

$$\left[A^{mt}_{j,j}\right]\left\{Z_j\right\} = \left\{K^{mt}_{j}\right\} \tag{4.5}$$

where $j = (2)^{mt}$

mt = total number of elimination stages.

Eq. (4.5) can be solved for $\left\{Z_j\right\}$ by any direct method. Redundants in all other groups may then be obtained by back-substitution into Eqs. (4.3).

General recursion relationships for the solution of the type of Eqs. (4.1) and (4.2) are recorded in Table 4-1, in which

m = elimination stage number,

mt = total number of stages,

i denotes group numbers eliminated,

j refers to the residual groups after an elimination stage,

matrices with subscripts less than one or greater than the last group number are considered null, and zero superscript indicates initial matrix.

A relation between mt and the total number of groups ng can be shown to be

$$(2)^{mt} \leq ng \leq (2)^{mt+1} - 1 \qquad (4.6)$$

The process has an interesting physical meaning. Each stage of elimination is equivalent to the closure of gaps corresponding to the redundants eliminated.

Advantages of this technique are:

1) Analysis of large frameworks that previously could have been solved only approximately may now be conducted in easy stages to obtain fairly accurage results.

2) The amount of computation involved in the direct solution is considerably reduced. If several of the groups are identical, the amount of reduction increases proportionately.

3) If a system already solved is altered in any manner, such as changes in some member sizes, the solution of the altered system need not be started from the beginning. Only those parts of the

TABLE 4-1. RECURSION RELATIONSHIPS FOR SOLUTION OF THE TYPE OF EQS. 4.1 AND 4.2

	1st Stage m = 1	2nd Stage m = 2	...	Last Stage m = mt
$\{K_i^m\} = [A_{i,i}^{m-1}]^{-1}\{K_i^{m-1}\}$				
$[A^m_{i,i\pm n_m}] = -[A^{m-1}_{i,i}]^{-1}[A^{m-1}_{i,i\pm n_m}]$	$i = 1, 3, \ldots$	$i = 2, 6, \ldots$	...	$i = (2)^{mt-1}$ or $i = (2)^{mt-1}$, $3(2^{mt-1})$
$\{Z_i\} = [A^m_{i,i-n_m}]\{Z_{i-n_m}\} + [A^m_{i,i+n_m}]\{Z_{i+n_m}\} + \{K_i^m\}$				
$[A^m_{j,j}] = [A^{m-1}_{j,j}] + [A^{m-1}_{j,j-n_m}][A^m_{j-n_m,j}] + [A^{m-1}_{j,j+n_m}][A^m_{j+n_m,j}]$				
$[A^m_{j,j+2n_m}] = [A^m_{j+2n_m,j}]^* = [A^{m-1}_{j,j+2n_m}] + [A^{m-1}_{j,j+n_m}][A^m_{j+n_m,j+2n_m}]$	$j = 2, 4, \ldots$	$j = 4, 8, \ldots$	...	$j = (2)^{mt}$
$\{K^m_j\} = \{K^{m-1}_j\} - [A^{m-1}_{j,j-n_m}]\{K^{m-1}_{j-n_m}\} - [A^{m-1}_{j,j+n_m}]\{K^m_{j+n_m}\}$				

$n_m = 1$	$m = 1$	$[A^{m-1}_{j,j+2n_m}] = [0]$	$m = 1, 2, 3 \ldots mt$ (for Eq. 4.1)
$n_m = 2n_{m-1}$	$m = 2, 3, \ldots mt$	$[A^{m-1}_{j,j+2n_m}] = [0]$	$m = 2, 3, \ldots mt$ (for Eq. 4.2)

analysis which are affected need be repeated.

4) The process can be easily programmed to a digital computer, and the solution efficiently accomplished by means of computer systems such as GISMO (General Interpretive System for Matrix Operation).

CHAPTER V

APPLICATION

5-1. Space Frame

The steel framework (Fig. 5-1) has been completely solved on the IBM 1410 computer. Properties of its members are given in Table 5-1. All bars are prismatic. Modulus of elasticity, E, is equal to 4,320,000 kips per square foot.

Fig. 5-2 shows a formulation tree and corresponding redundant members. The procedure of analysis is given by a detailed flow graph in Appendix B.

Three static effects considered are:

a) Applied loads as shown in Fig. 5-1,

b) Temperature increase of 40°F,

c) Support displacements: $\delta^{o}_{3x} = \delta^{o}_{3y} = -\delta^{o}_{3z} = 0.015'$

Solution of each case is recorded in Table 5-2.

5-2. Planar Frame Loaded out of Plane

The structure shown in Fig. 5-3 has been analyzed by Koepsell[15]. Member properties and joint loads are given in Table 5-3. A formulation tree is sketched in Fig. 5-4.

Utilizing the relationships given in Appendix A and Table 3-1, the topologic matrices recorded in Table 5-4 are calculated. The

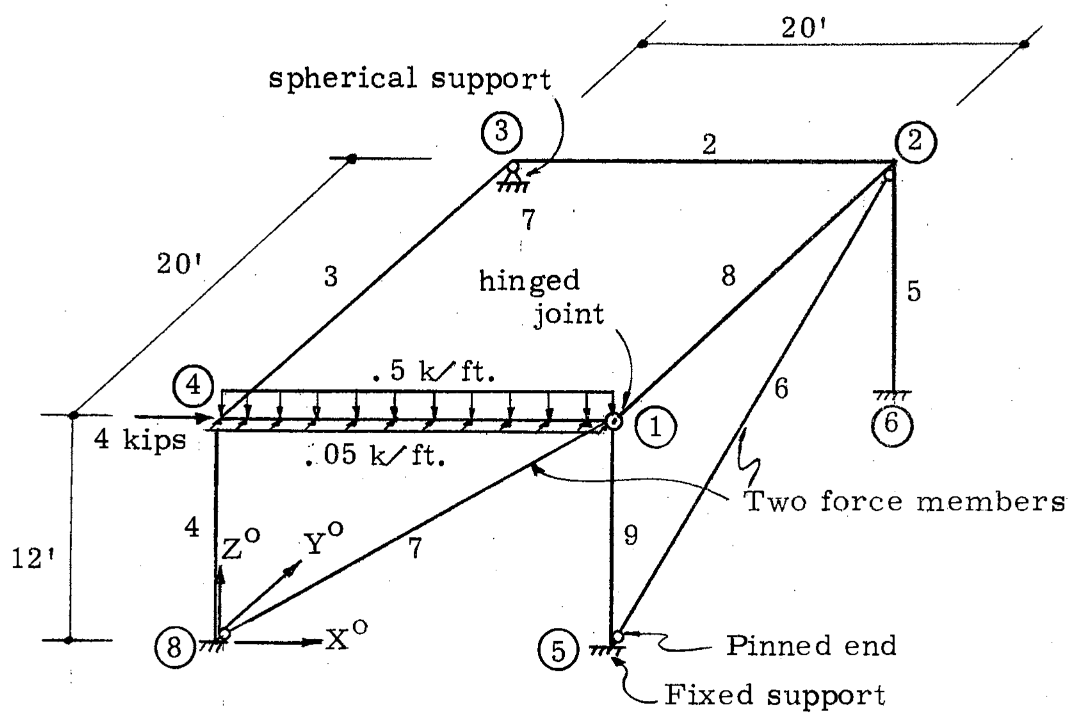


FIG. 5-1. SPACE FRAME

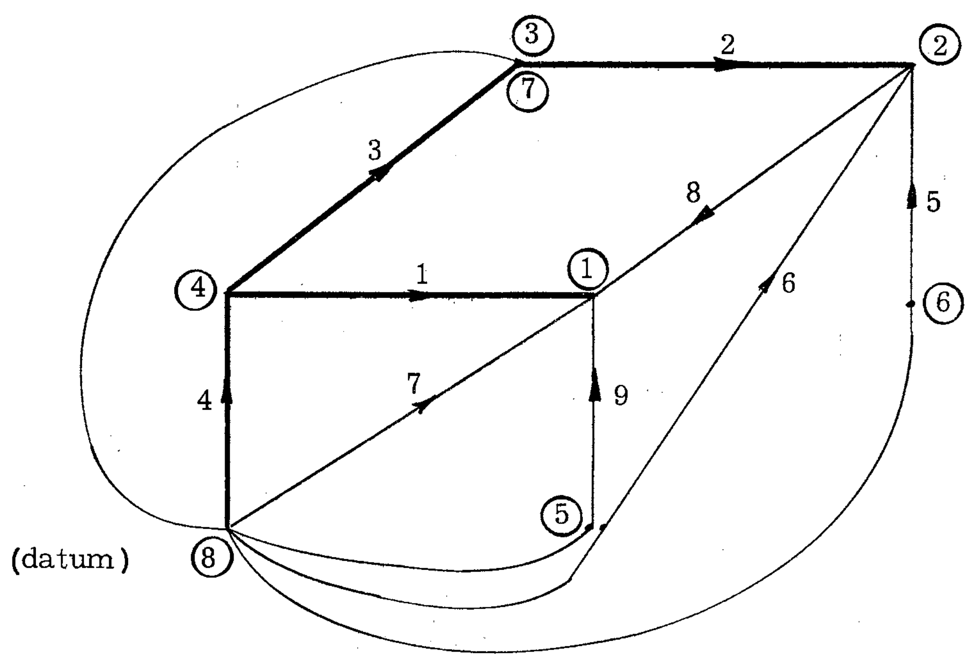


FIG. 5-2. A FORMULATION TREE AND CHORDS FOR THE FRAME OF FIG. 5-1.

TABLE 5-1. PROPERTIES OF SPACE FRAME BARS (UNITS: Kips, Kip-Ft., Ft., Radians)

MEMBER		COORDINATES OF POINTS (Ft.)			FLEXIBILITIES IN MEMBER SYSTEM							
No.	Shape	i	j	k	$(E)D^a_{iixx}$	$(E)F^a_{iiyy}$	$(E)F^a_{iizz}$	$(E)F^a_{jjxx}$	$(E)F^a_{jjyy}$	$(E)F^a_{jjzz}$	$(E)G^a_{IJyy}$	$(E)G^a_{IJzz}$
1	21 WF 142	(0, 0, 12)	(20,0,12)	(0, 0, 24)	68.965517	40.621786	358.22752	—	—	—	—	—
2	21 WF 142	(0,20,12)	(20,20,12)	(0,20,24)	68.965517	40.621786	358.22752	67897.835	40.621786	358.22852	20.310893	179.11376
3	21 WF 142	(0, 0, 12)	(0,20,12)	(0, 0, 24)	68.965517	40.621786	358.22752	67897.835	40.621786	358.22752	20.310893	179.11376
4	14 WF 142	(0, 0, 0)	(0, 0, 12)	(-20,0,0)	41.290322	49.601722	125.65369	43230.020	49.601722	125.65369	24.800861	62.826844
5	14 WF 142	(20,20,0)	(20,20,12)	(0, 20, 0)	41.290322	49.601722	125.65369	43230.020	49.601722	125.65369	24.800861	62.826844
6	8 WF 67	(20, 0, 0)	(20,20,12)	(0, 0, 0)	170.48875	—	—	—	—	—	—	—
7	8 WF 67	(0, 0, 0)	(20,0,12)	(0, 20, 0)	170.48875	—	—	—	—	—	—	—
8	21 WF 142	(20,20,12)	(20,0,12)	(20,20,24)	68.965517	40.621786	358.22752	—	—	—	—	—
9	14 WF 142	(20, 0, 0)	(20, 0, 12)	(0, 0, 0)	41.290322	49.601722	125.65369	—	—	—	—	—

LOAD FUNCTIONS FOR CASE (a)

BAR NO.	LOAD FUNCTIONS					
	$(E)\epsilon^a_{iax}$	$(E)\tau^a_{iay}$	$(E)\tau^a_{iaz}$	$(E)\tau^a_{jax}$	$(E)\tau^a_{jay}$	$(E)\tau^a_{jaz}$
1	0.0	-1015.54	-895.569	—	—	—
2 . . . 9	ALL OTHERS ZERO					

LOAD FUNCTIONS FOR CASES (b) AND (c) ARE ZERO

TABLE 5-2a. COMPUTER SOLUTION FOR THE FRAME OF FIG. 5-1 - CASE (a)

BAR NO.	END	END REACTIONS IN MEMBER SYSTEM					
		NX kips	NY kips	NZ kips	MX kip—ft.	MY kip—ft.	MZ kip—ft.
1	i	-5.13171	0.55946	-5.51995	0.00000	10.39903	1.18929
	j	-5.13171	-0.44054	4.48005	0.00000	0.00000	0.00000
2	i	0.00138	0.00821	0.00020	-0.00011	-0.00878	0.17997
	j	0.00138	0.00821	0.00020	-0.00011	-0.00480	0.01577
3	i	-0.54251	-0.06791	0.00596	-0.00878	-0.11911	-1.17832
	j	-0.54251	-0.06791	0.00596	-0.00878	0.00011	0.17997
4	i	-5.51399	0.01696	1.06296	0.01097	-2.36522	0.08437
	j	-5.51399	0.01696	1.06296	0.01097	10.39025	-0.11911
5	i	-0.24711	0.03046	0.00055	0.00099	-0.00175	0.16504
	j	-0.24711	0.03046	0.00055	0.00099	0.00480	-0.20051
6	i	0.46043	0.00000	0.00000	0.00000	0.00000	0.00000
	j	0.46043	0.00000	0.00000	0.00000	0.00000	0.00000
7	i	5.76536	0.00000	0.00000	0.00000	0.00000	0.00000
	j	5.76536	0.00000	0.00000	0.00000	0.00000	0.00000
8	i	-0.43348	0.00084	-0.01002	0.00000	0.20040	0.01676
	j	-0.43348	0.00084	-0.01002	0.00000	0.00000	0.00000
9	i	-7.43628	0.00705	-0.18628	0.00000	2.23532	0.08462
	j	-7.43628	0.00705	-0.18628	0.00000	0.00000	0.00000

JOINT NO.	JOINT DEFORMATIONS IN GLOBAL SYSTEM					
	δ_x in.	δ_y in.	δ_z in.	θ_x radians	θ_y radians	θ_z radians
1	0.00369594	0.00035250	-0.00085292	——	——	——
2	-0.00000014	0.00027056	-0.00002858	0.00000154	0.00000007	0.00000990
3	0.00000000	0.00000000	0.00000000	0.00000319	0.00000024	-0.00001445
4	0.00467887	0.00010394	-0.00063243	0.00000152	0.00013821	0.00010973

TABLE 5-2b. COMPUTER SOLUTION FOR THE FRAME OF FIG. 5-1 – CASE (b)

BAR NO.	END	END REACTIONS IN MEMBER SYSTEM NX kips	NY kips	NZ kips	MX kip—ft.	MY kip—ft.	MZ kip—ft.
1	i	-0.41824	0.10420	-0.04691	0.00000	0.93819	2.08404
	j	-0.41824	0.10420	-0.04691	0.00000	0.00000	0.00000
2	i	-8.56391	0.13559	2.19126	0.00547	-0.02232	2.19966
	j	-8.56391	0.13559	2.19126	0.00547	43.80282	-0.51216
3	i	-4.51727	-0.21382	-1.27578	-0.02232	25.51021	-2.07667
	j	-4.51727	-0.21382	-1.27578	-0.02232	-0.00547	2.19967
4	i	-1.32269	-4.41307	0.20442	0.00734	-1.53722	-27.44660
	j	-1.32269	-4.41307	0.20442	0.00734	0.91587	25.51021
5	i	-0.86189	3.68843	-8.53980	0.02985	58.67473	24.30140
	j	-0.86189	3.68843	-8.53980	0.02985	-43.80282	-19.95980
6	i	-4.52410	0.00000	0.00000	0.00000	0.00000	0.00000
	j	-4.52410	0.00000	0.00000	0.00000	0.00000	0.00000
7	i	-3.04253	0.00000	0.00000	0.00000	0.00000	0.00000
	j	-3.04253	0.00000	0.00000	0.00000	0.00000	0.00000
8	i	0.05536	-0.02412	-0.99826	0.00000	19.96528	-0.48232
	j	0.05536	-0.02412	-0.99826	0.00000	0.00000	0.00000
9	i	2.61054	-0.04884	3.05131	0.00000	36.61567	-0.58610
	j	2.61054	-0.04884	3.05131	0.00000	0.00000	0.00000

JOINT NO.	JOINT DEFORMATIONS IN GLOBAL SYSTEM δ_x in.	δ_y in.	δ_z in.	θ_x radians	θ_y radians	θ_z radians
1	0.06053333	-0.00246314	0.03773940			
2	0.06076111	0.05998994	0.03734491	-0.00018925	0.00025608	0.00029868
3	0.00000000	0.00000000	0.00000000	-0.00027526	-0.00036142	0.00008882
4	-0.00178449	-0.06153464	0.03728829	0.00008448	-0.00001070	0.00007347

TABLE 5-2c. COMPUTER SOLUTION FOR THE FRAME OF FIG. 5-1 – CASE (c)

BAR NO.	END	END REACTIONS IN MEMBER SYSTEM NX kips	NY kips	NZ kips	MX kip—ft.	MY kip—ft.	MZ kip—ft.
1	i	-0.00452	0.00092	-0.00444	0.00000	0.08885	0.01840
	j	-0.00452	0.00092	-0.00444	0.00000	0.00000	0.00000
2	i	-26.65552	-0.00726	7.13947	0.03686	-0.09013	0.00265
	j	-26.65552	-0.00726	7.13947	0.03686	142.69927	0.14783
3	i	7.98662	-0.00476	1.77284	-0.09013	-35.49371	-0.09255
	j	7.98662	-0.00476	1.77284	-0.09013	-0.03686	0.00265
4	i	1.76840	7.98754	-0.00024	-0.07420	0.00164	60.35683
	j	1.76840	7.98754	-0.00024	-0.07420	-0.00128	-35.49371
5	i	-7.13365	0.01474	-26.65924	-0.07348	177.21158	0.13800
	j	-7.13365	0.01474	-26.65924	-0.07348	-142.69927	-0.03890
6	i	-0.01868	0.00000	0.00000	0.00000	0.00000	0.00000
	j	-0.01868	0.00000	0.00000	0.00000	0.00000	0.00000
7	i	0.00093	0.00000	0.00000	0.00000	0.00000	0.00000
	j	0.00093	0.00000	0.00000	0.00000	0.00000	0.00000
8	i	0.00853	0.00372	-0.00379	0.00000	0.07576	0.07436
	j	0.00853	0.00372	-0.00379	0.00000	0.00000	0.00000
9	i	0.00775	0.00761	-0.000005	0.00000	0.00006	0.09137
	j	0.00775	0.00761	-0.000005	0.00000	0.00000	0.00000

JOINT NO.	JOINT DEFORMATIONS IN GLOBAL SYSTEM δ_x in.	δ_y in.	δ_z in.	θ_x radians	θ_y radians	θ_z radians
1	0.00000083	0.00050972	0.00000083	—	—	—
2	0.17497222	0.00048564	-0.00084336	-0.00000422	0.00059509	-0.00073519
3	0.00000000	0.00000000	0.00000000	-0.00058362	-0.00141644	-0.00075417
4	0.00000169	0.17846997	0.00020283	-0.00108477	0.00000000	-0.00074236

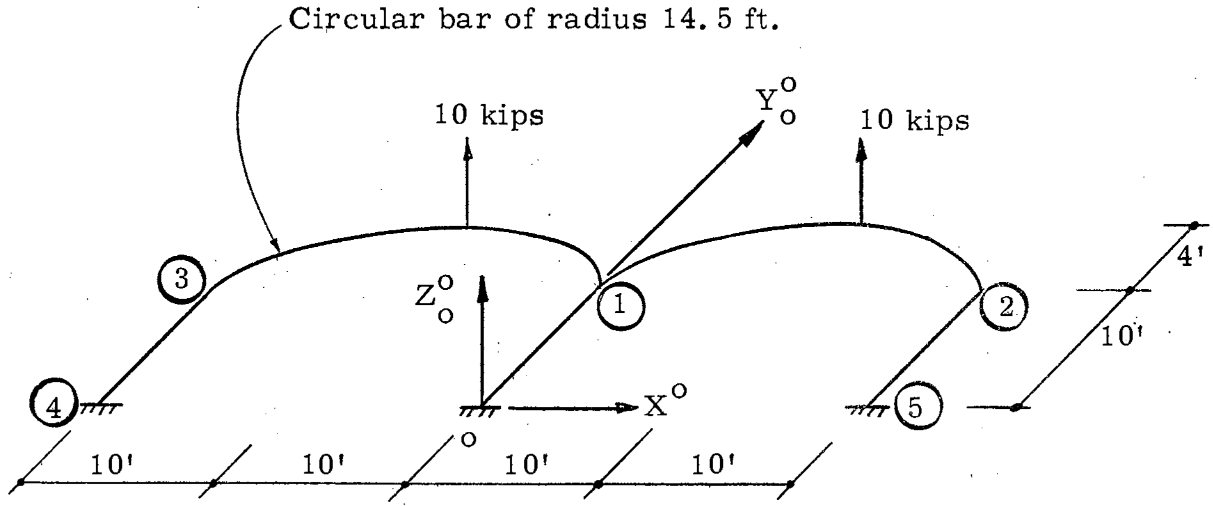


FIG. 5-3. TWO BAY PLANAR FRAME

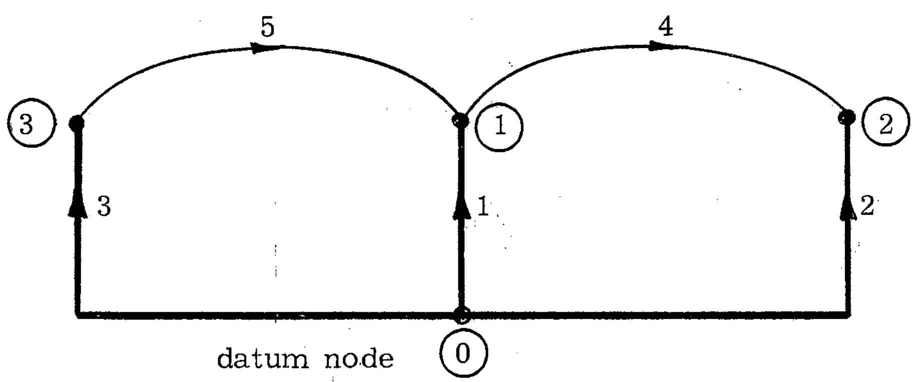


FIG. 5-4. FORMULATION TREE ANDREDUNDANT MEMBERS

TABLE 5-3. MEMBER PROPERTIES AND JOINT LOADS

EI = GJ for all members

$$\left[A_1^1\right] = \left[A_2^2\right] = \left[A_3^3\right] = \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & 1.0 & 0 \\ \frac{1}{6} & 0 & \frac{1}{3} \end{bmatrix} \frac{10}{EI}$$

$$\left[A_4^4\right] = \left[A_5^5\right] = \begin{bmatrix} 0.7975 & 0.2914 & 0.3059 \\ 0.2914 & 2.2069 & -0.2914 \\ 0.3059 & -0.2914 & 0.7975 \end{bmatrix} \frac{10}{EI}$$

$$\left\{\sigma_1^1\right\} = \left\{\sigma_2^2\right\} = \left\{\sigma_3^3\right\} = \begin{Bmatrix} 0. \\ 0. \\ 0. \end{Bmatrix}$$

$$\left\{\sigma_4^4\right\} = \left\{\sigma_5^5\right\} = \begin{Bmatrix} 32.0 \\ 44.15 \\ 20.348 \end{Bmatrix} \frac{10}{EI}$$

$$\left\{W1^o\right\} = \begin{Bmatrix} P1_z^o \\ Q1_x^o \end{Bmatrix} = \begin{Bmatrix} 10 \\ +40 \end{Bmatrix} ; \quad W2^o = P2_z^o = 5$$

$$\left\{W3^o\right\} = \begin{Bmatrix} P3_z^o \\ Q3_x^o \end{Bmatrix} = \begin{Bmatrix} 5 \\ +40 \end{Bmatrix} ; \qquad \left\{WJ\right\} = \begin{Bmatrix} W1^o \\ W2^o \\ W3^o \end{Bmatrix}$$

TABLE 5-4. TOPOLOGICAL MATRICES FOR PLANAR FRAME

$$
\left\{\Gamma_{b\,c}\right] = \left\{\begin{array}{c|c} \Gamma_{1,4} & -\Gamma_{1,5} \\ \hline -\Gamma_{2,4} & 0_3 \\ \hline 0_3 & \Gamma_{3,5} \end{array}\right] = \left[\begin{array}{ccc|ccc} 0.5 & -1 & -0.5 & -0.5 & 1 & 0.5 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ \hline -0.5 & 1 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0.5 & -1 & -0.5 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right]
$$

$$
\left\{T_{b\,J}\right] = \left\{\begin{array}{c|c|c} T_{1,1} & 0_{3,1} & 0_{3,2} \\ \hline 0_{3,2} & T_{2,2} & 0_{3,2} \\ \hline 0_{3,2} & 0_{3,1} & T_{3,3} \end{array}\right] = \left[\begin{array}{cc|c|cc} -10 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -10 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]
$$

following elasto-static matrices are obtained by means of Eqs. (3-12).

$$[A_{cc}] = \begin{bmatrix} 1.9645 & -0.2086 & 0.1389 & -0.0834 & 0.25 & -0.9167 \\ -0.2086 & 4.2069 & 0.2086 & 0.25 & -1 & -0.25 \\ 0.1389 & 0.2086 & 1.9645 & 0.0834 & -0.25 & -0.0834 \\ -0.0834 & 0.25 & 0.0834 & 1.9645 & -0.2086 & 0.1389 \\ 0.25 & -1 & -0.25 & -0.2086 & 4.2069 & 0.2086 \\ -0.9167 & -0.25 & -0.0834 & 0.1389 & 0.2086 & 1.9645 \end{bmatrix} \frac{10}{EI}$$

$$\{A_{cJ}] = \begin{bmatrix} -1.667 & -0.25 & 1.667 & 0 & 0 \\ 5 & 1 & -5 & 0 & 0 \\ 1.667 & 0.25 & -1.667 & 0 & 0 \\ 1.667 & 0.25 & 0 & -1.667 & -0.25 \\ -5 & -1 & 0 & 5 & 1 \\ -1.667 & -0.25 & 0 & 1.667 & 0.25 \end{bmatrix} \frac{10}{EI}$$

$$\{\sigma_{cc}\} = \begin{Bmatrix} 32.0 \\ 44.15 \\ 20.348 \\ 32.0 \\ 44.15 \\ 20.348 \end{Bmatrix} \frac{10}{EI}$$

The flexibility matrix $[A_{cc}]$ is very well-conditioned as can be seen from its strong diagonal. The redundant moments may be solved from the compatibility condition

$$\left[A_{cc}\right]\left\{Z_c\right\} + \left\{A_{cJ}\right]\left\{WJ\right\} + \left\{\sigma_{cc}\right\} = \left\{0\right\}$$

The unknowns as obtained by direct inversion of $\left[A_{cc}\right]$ are given by

$$\left\{Z_c\right\} = \left\{\begin{array}{c} Z_4^4 \\ \hline Z_5^5 \end{array}\right\} = \left\{\begin{array}{c} Y_{i4y}^4 \\ \hline Y_{j4x}^4 \\ \hline Y_{j4y}^4 \\ \hline Y_{i5y}^5 \\ \hline Y_{j5x}^5 \\ \hline Y_{j5y}^5 \end{array}\right\} = \left\{\begin{array}{c} M_{i4y}^4 \\ \hline M_{j4x}^4 \\ \hline M_{j4y}^4 \\ \hline M_{i5y}^5 \\ \hline M_{j5x}^5 \\ \hline M_{j5y}^5 \end{array}\right\} = \left\{\begin{array}{c} -15.06629 \\ \hline -28.47731 \\ \hline -16.98538 \\ \hline -16.98538 \\ \hline -11.52269 \\ \hline -15.06629 \end{array}\right\} \text{(kip-ft.)}$$

Calculation of end-conditioning elements of all the members is a simple matter of using equations of Appendix A.

CHAPTER VI

SUMMARY AND CONCLUSIONS

6.1. Summary

Flexibility method for the analysis of continuous framed structures is generalized in this study. A simple beam in space is selected as the basic element and the end elastic weights due to loads, temperature and volume changes, and support displacements are defined in terms of the deformation functions of the member. Primary unknowns in the system consist of the bar-redundants of the redundant members. Using connectivity properties of linear graphs, equilibrium relationships are developed in matrix notation. Compatibility of deformations yields a minimum set of simultaneous equations. Group elimination technique for solution of large problems is discussed. Two numerical examples are included to demonstrate the theory.

The selection of system unknowns and the formulation technique are believed to be original.

6.2. Conclusions

The formulation presented here is general and is employed with a slight modification to planar structures loaded in or out of plane.

It is observed that the choice of redundants adopted yields a better-conditioned flexibility matrix than that obtained with complete cuts. This can be seen by a comparison of numerical example no. 2 (Chapter V) with the problem solved by Koepsell on page 47 of his doctoral dissertation[15].

A general computer program for the analysis of rigid jointed skeletal structures allowing for internal releases is developed. Group elimination process can easily be programmed by means of computer languages such as GISMO (General Interpretive System for Matrix Operations).

6.3. Extensions

Immediate extensions of this research would be to analyze

1) Semi-rigidly connected frameworks supported by elastic springs;

2) Rigidly jointed structures acted upon by dynamic disturbances;

3) Buckling phenomena of rigidly jointed space frames;

4) Three-dimensional solids such as dams[61];

5) Plate and shell structures as an equivalent assembly of bars; and

6) Investigation of the tree that would result in well-conditioned flexibility matrix.

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APPENDIX A

SPECIAL CASES

The theory presented in Chapters I-V is equally applicable to planar frames. Here the stereo-static and elasto-geometric relations for a bar are given.

A-1. Loads in the Plane

A member isolated from a planar frame loaded in its plane is shown in Fig. (A-1).

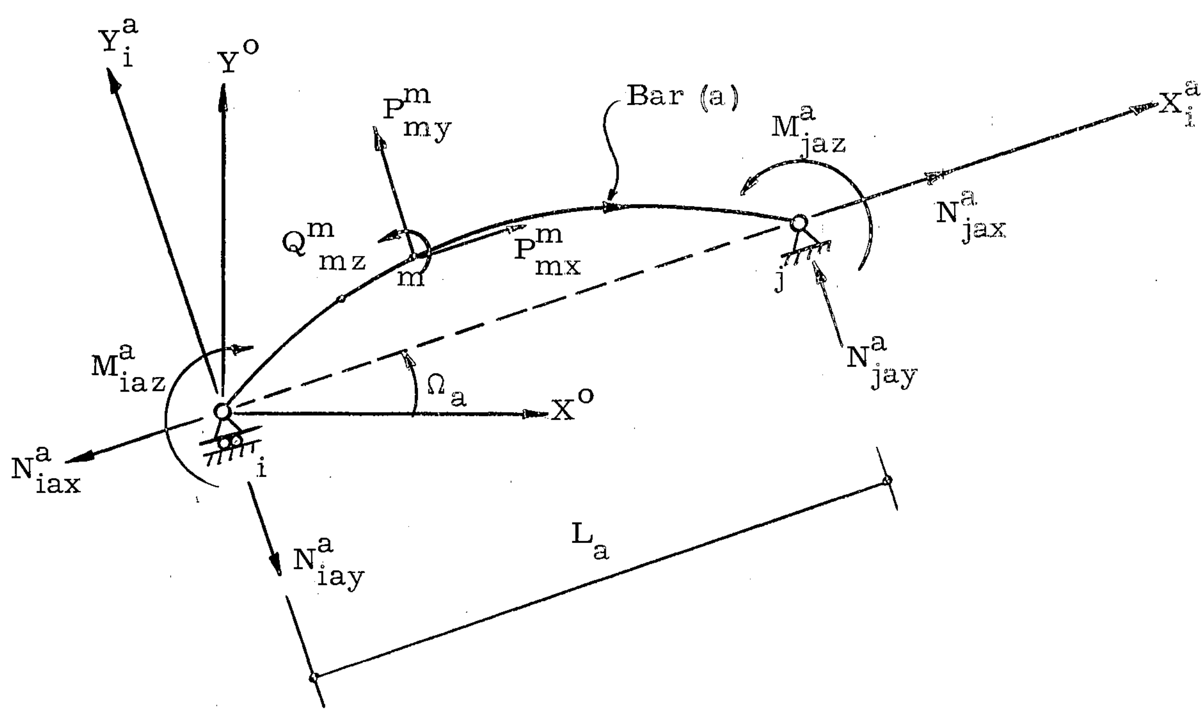

FIG. A-1. BAR LOADED IN PLANE

Choosing (d)N^a_{iax}(=X^a_{iax}), M^a_{iaz}(=Y^a_{iaz}), and M^a_{jaz}(=Y^a_{jaz}) as bar-redundants, the following stereo-static equations can be established.

$$\left\{H^a_{ia}\right\} = \left[f_{ia}\right]\left\{Z^a_a\right\} + \left\{BH^a_{ia}\right\} \quad (A.1)$$

$$\left\{H^o_{ia}\right\} = \left[\pi i_{oa}\right]\left\{Z^a_a\right\} + \left\{BH^o_{ia}\right\} \quad (A.2)$$

$$\left\{H^a_{ja}\right\} = \left[f_{ja}\right]\left\{Z^a_a\right\} + \left\{BH^a_{ja}\right\} \quad (A.3)$$

$$\left\{H^o_{ja}\right\} = \left[\pi_{joa}\right]\left\{Z^a_a\right\} + \left\{BH^o_{ja}\right\} \quad (A.4)$$

in which

$$\left\{H^a_{ia}\right\} = \begin{Bmatrix} N^a_{iax} \\ N^a_{iay} \\ M^a_{iaz} \end{Bmatrix}; \quad \left\{H^o_{ja}\right\} = \begin{Bmatrix} N^o_{jax} \\ N^o_{jay} \\ M^o_{jaz} \end{Bmatrix}; \quad \left\{Z^a_a\right\} = \begin{Bmatrix} X^a_{iax} \\ Y^a_{iaz} \\ Y^a_{jaz} \end{Bmatrix} \quad (A.5a)$$

$$\left[f_{ia}\right] = \begin{bmatrix} \frac{1}{d} & 0 & 0 \\ 0 & \frac{1}{L_a} & -\frac{1}{L_a} \\ 0 & 1 & 0 \end{bmatrix}; \quad \left[f_{ja}\right] = \begin{bmatrix} \frac{1}{d} & 0 & 0 \\ 0 & \frac{1}{L_a} & -\frac{1}{L_a} \\ 0 & 0 & 1 \end{bmatrix} \quad (A.6a)$$

$$\left[\pi i_{oa}\right] = \begin{bmatrix} \frac{1}{d}\cos\Omega_a & -\frac{1}{L_a}\sin\Omega_a & \frac{1}{L_a}\sin\Omega_a \\ \frac{1}{d}\sin\Omega_a & \frac{1}{L_a}\cos\Omega_a & -\frac{1}{L_a}\cos\Omega_a \\ 0 & 1 & 0 \end{bmatrix} \quad (A.7a)$$

$$\left[\pi j_{oa}\right] = \begin{bmatrix} \frac{1}{d}\cos\Omega_a & -\frac{1}{L_a}\sin\Omega_a & \frac{1}{L_a}\sin\Omega_a \\ \frac{1}{d}\sin\Omega_a & \frac{1}{L_a}\cos\Omega_a & -\frac{1}{L_a}\cos\Omega_a \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(A. 8a)}$$

$$\left\{BH^o_{ia}\right\} = \begin{Bmatrix} BN^o_{iax} \\ BN^o_{iay} \\ BM^o_{iaz} \end{Bmatrix}, \qquad \left\{BH^a_{ja}\right\} = \begin{Bmatrix} BN^a_{jax} \\ BN^a_{jay} \\ BM^a_{jaz} \end{Bmatrix} \quad \text{(A. 9a)}$$

If h, k are any two points across the frame, then

$$\left[r^o_{hk}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y^o_{hk} & x^o_{hk} & 1 \end{bmatrix} \quad \text{(A. 10a)}$$

The inverse of $\left[\pi j_{oa}\right]$ is very frequently used in establishing elemental matrices of $\left\{\Gamma_{bc}\right]$ and $\left[T_{bJ}\right]$, and is given by

$$\left[\pi j_{oa}\right]^{-1} = \begin{bmatrix} d\cos\Omega_a & d\sin\Omega_a & 0 \\ -L_a\sin\Omega_a & L_a\cos\Omega_a & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(A. 11a)}$$

Finally the end deformations relation is obtained from Table 2-2 by deleting rows and columns not applicable to this case.

$$\begin{Bmatrix} \bar{Q}^a_{iax} \\ \hline \bar{P}^a_{iaz} \\ \hline \bar{P}^a_{jaz} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{d}\eta^a_{iax} \\ \hline \varphi^a_{iaz} \\ \hline \varphi^a_{jaz} \end{Bmatrix} = \left[\begin{array}{c|c|c} \frac{1}{d^2} D^a_{iixx} & \frac{1}{d} E^a_{iixz} & \frac{1}{d} E^a_{ijxz} \\ \hline \frac{1}{d} E^a_{iizx} & F^a_{iizz} & G^a_{ijzz} \\ \hline \frac{1}{d} E^a_{jizx} & G^a_{jizz} & F^a_{jjzz} \end{array}\right] \begin{Bmatrix} X^a_{iax} \\ \hline Y^a_{iaz} \\ \hline Y^a_{jaz} \end{Bmatrix} + \begin{Bmatrix} \frac{1}{d}\epsilon^a_{iax} \\ \hline \tau^a_{iaz} \\ \hline \tau^a_{jaz} \end{Bmatrix} \tag{A. 12a}$$

or

$$\left\{\bar{W}^a_a\right\} = \left\{\nabla^a_a\right\} = \left[A^a_a\right]\left\{Z^a_a\right\} + \left\{\sigma^a_a\right\} \tag{A. 13a}$$

A-2. Loads Out-Of-Plane

Bar (a) removed from a normally loaded planar frame is shown in Fig. (A-2).

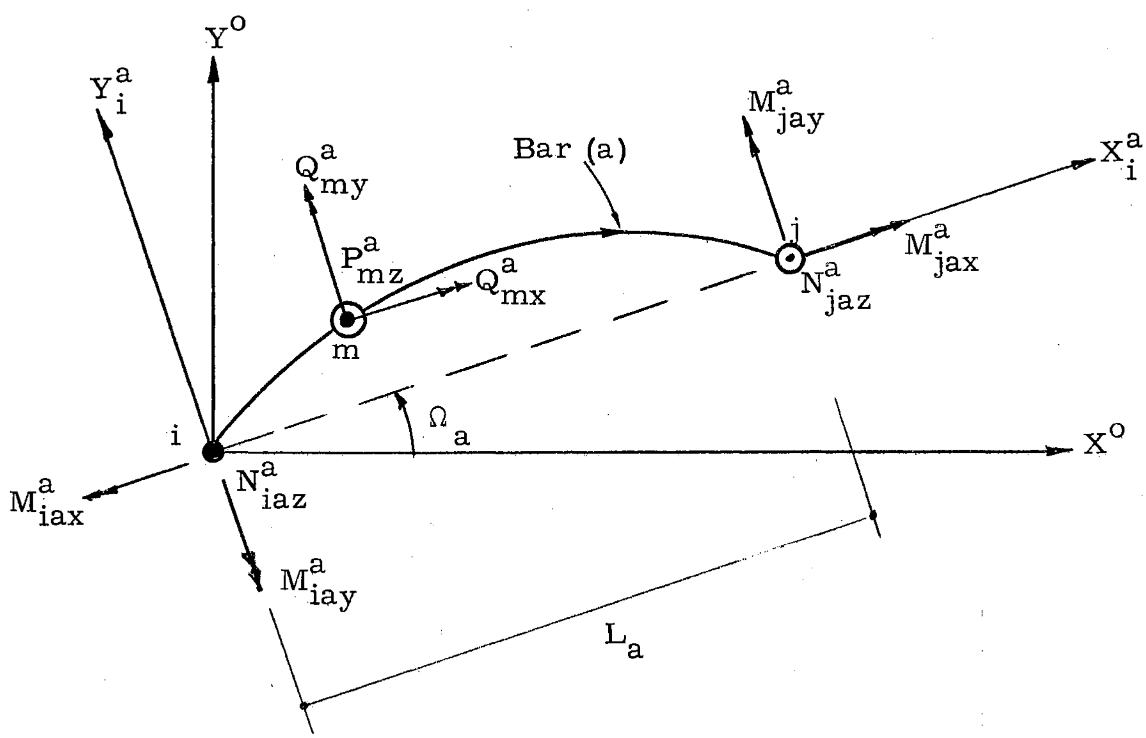

FIG. A-2. BAR LOADED NORMAL TO PLANE

Moments M^a_{iay}, M^a_{jax}, M^a_{jay} are chosen as bar-redundants and expressed by

$$\{Z^a_a\} = \left\{ \begin{array}{c} Y^a_{iay} \\ \hline Y^a_{jax} \\ \hline Y^a_{jay} \end{array} \right\} = \left\{ \begin{array}{c} M^a_{iay} \\ \hline M^a_{jax} \\ \hline M^a_{jay} \end{array} \right\} \qquad (A.14)$$

Stereo-static relations (A. 1 through A. 4), are also applicable to this case. Eqs. (A. 5a) through (A. 13a) are modified in view of the nature of loading as

$$\{H^a_{ia}\} = \left\{ \begin{array}{c} N^a_{iaz} \\ \hline M^a_{iax} \\ \hline M^a_{iay} \end{array} \right\} \; ; \quad \{H^o_{ja}\} = \left\{ \begin{array}{c} N^o_{jaz} \\ \hline M^o_{jax} \\ \hline M^o_{jay} \end{array} \right\} \qquad (A.5b)$$

$$[f_{ia}] = \left[\begin{array}{c|c|c} -\frac{1}{L_a} & 0 & \frac{1}{L_a} \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array} \right] \; ; \quad [f_{ja}] = \left[\begin{array}{c|c|c} -\frac{1}{L_a} & 0 & \frac{1}{L_a} \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array} \right] \qquad (A.6b)$$

$$[\pi i_{oa}] = \left[\begin{array}{c|c|c} -\frac{1}{L_a} & 0 & \frac{1}{L_a} \\ \hline 0 & \cos \Omega_a & 0 \\ \hline 0 & \sin \Omega_a & 0 \end{array} \right] \qquad (A.7b)$$

$$\left[\pi j_{oa}\right] = \left[\begin{array}{c|c|c} -\frac{1}{L_a} & 0 & \frac{1}{L_a} \\ \hline 0 & \cos\Omega_a & -\sin\Omega_a \\ \hline 0 & \sin\Omega_a & \cos\Omega_a \end{array}\right] \qquad \text{(A. 8b)}$$

$$\left\{BH^a_{ia}\right\} = \left\{\begin{array}{c} BN^a_{iaz} \\ \hline BM^a_{iax} \\ \hline BM^a_{iay} \end{array}\right\} , \qquad \left\{BH^o_{ja}\right\} = \left\{\begin{array}{c} BN^o_{jaz} \\ \hline BM^o_{jax} \\ \hline BM^o_{jay} \end{array}\right\} \qquad \text{(A. 9b)}$$

$$\left[r^o_{hk}\right] = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline y^o_{hk} & 1 & 0 \\ \hline -x^o_{hk} & 0 & 1 \end{array}\right] \qquad \text{(A. 10b)}$$

$$\left[\pi j_{oa}\right]^{-1} = \left[\begin{array}{c|c|c} -L_a & -\sin\Omega_a & \cos\Omega_a \\ \hline 0 & \cos\Omega_a & \sin\Omega_a \\ \hline 0 & -\sin\Omega_a & \cos\Omega_a \end{array}\right] \qquad \text{(A. 11b)}$$

$$\left\{\begin{array}{c} \bar{P}^a_{iay} \\ \hline \bar{P}^a_{jax} \\ \hline \bar{P}^a_{jay} \end{array}\right\} = \left\{\begin{array}{c} \varphi^a_{iay} \\ \hline \varphi^a_{jax} \\ \hline \varphi^a_{jay} \end{array}\right\} = \left[\begin{array}{c|c|c} F^a_{iiyy} & G^a_{ijyx} & G^a_{ijyy} \\ \hline G^a_{jixy} & F^a_{jjxx} & F^a_{jjxy} \\ \hline G^a_{jiyy} & F^a_{jjyx} & F^a_{jjyy} \end{array}\right] \left\{\begin{array}{c} Y^a_{iay} \\ \hline Y^a_{jax} \\ \hline Y^a_{jay} \end{array}\right\} + \left\{\begin{array}{c} \tau^a_{iay} \\ \hline \tau^a_{jax} \\ \hline \tau^a_{jay} \end{array}\right\} \qquad \text{(A. 12b)}$$

$$\left\{\bar{W}^a_a\right\} = \left\{\nabla^a_a\right\} = \left[A^a_a\right]\left\{Z^a_a\right\} + \left\{\sigma^a_a\right\} \qquad \text{(A. 13b)}$$

APPENDIX B

COMPUTER ANALYSIS

A computer program was written for the IBM 1410 for complete analysis of space frames composed of straight members.

A macro flow diagram (Fig. B-1) illustrates the basic steps to the solution of a problem. The main program is subdivided into four phases in view of the limited storage capacity of the computer. Required input data are indicated below. The output consists of member end reactions and joint deformations.

INPUT DATA - PHASE I

Number of Bars, Supports, and k-Points

d, nb1h, nb0h, nc0h, nc2h, nc1h, ncss, ns, nk

d average length of bars and some multiple of 5.
Next six numbers indicate numbers of various types of members.
ns, nk are numbers of supports and k-points.

Coordinates in Global System

J x y z

J joint, support, or k-point number
x, y, z indicate coordinates

Connectivity Details

NM i j k

NM bar number
i, j joints at near and far ends of the bar
k point on z-axis of bar

Loop and Node-To-Datum Details

NM J nb1c1, nb0c1, nbnc1

NM redundant bar number or joint number
J joint at far end in case of bar or joint number itself in case of joint
In a loop or node-to-datum path corresponding to bar NM or joint NM
nb1c1 number of branches with a hinged end
nb0c1 number of branches with rigid ends
nbnc1 = nb1c1 + nb0c1

(nbct(I), nbcTJ(I), SGN(I), I=1, nbnc1)

nbct(I) Branch number
nbctJ(I) Joint at far end of the branch
SGN(I) Branch orientation in a loop or node-to-datum path

INPUT DATA - PHASE II

Member Flexibilities in Member System

NM, Eight values of deformation functions

NM bar number

INPUT DATA - PHASE III

Applied Joint Loads in Global System

J P_x P_y P_z Q_x Q_y Q_z

J joint number
P, Q applied forces and couples

Support Displacements in Global System

J δ_x δ_y δ_z θ_x θ_y θ_z

J support number
δ, θ support deflection and rotation

Load Deformation Functions in Member System

> NM ε_{ix} τ_{iy} τ_{iz} τ_{jx} τ_{jy} τ_{jz}
>
> NM bar number
> ε, τ linear and angular load functions

Member Details

> NM NLP IC JC ICS JCS TMPR THERM E
>
> NM bar number
> NLP number of load points
> IC, JC codes (end joint number or 0 if it is a support other than of kind ncss)
> ICS, JCS codes (end support number or (NB + NS + 1), if it is other than any support)
> NB total number of branches
> TMPR, THERM, E are temperature change, thermal coefficient and modulus of elasticity

Member Loads in Global System

> x^a_{im} P_x P_y P_z Q_x Q_y Q_z
>
> x^a_{im} x-coordinate of load point
> P, Q load components

INPUT DATA - PHASE IV

> NI
>
> NI code to direct the flow into Phase I (for new frame), Phase III (for the same frame with new causes), or end.

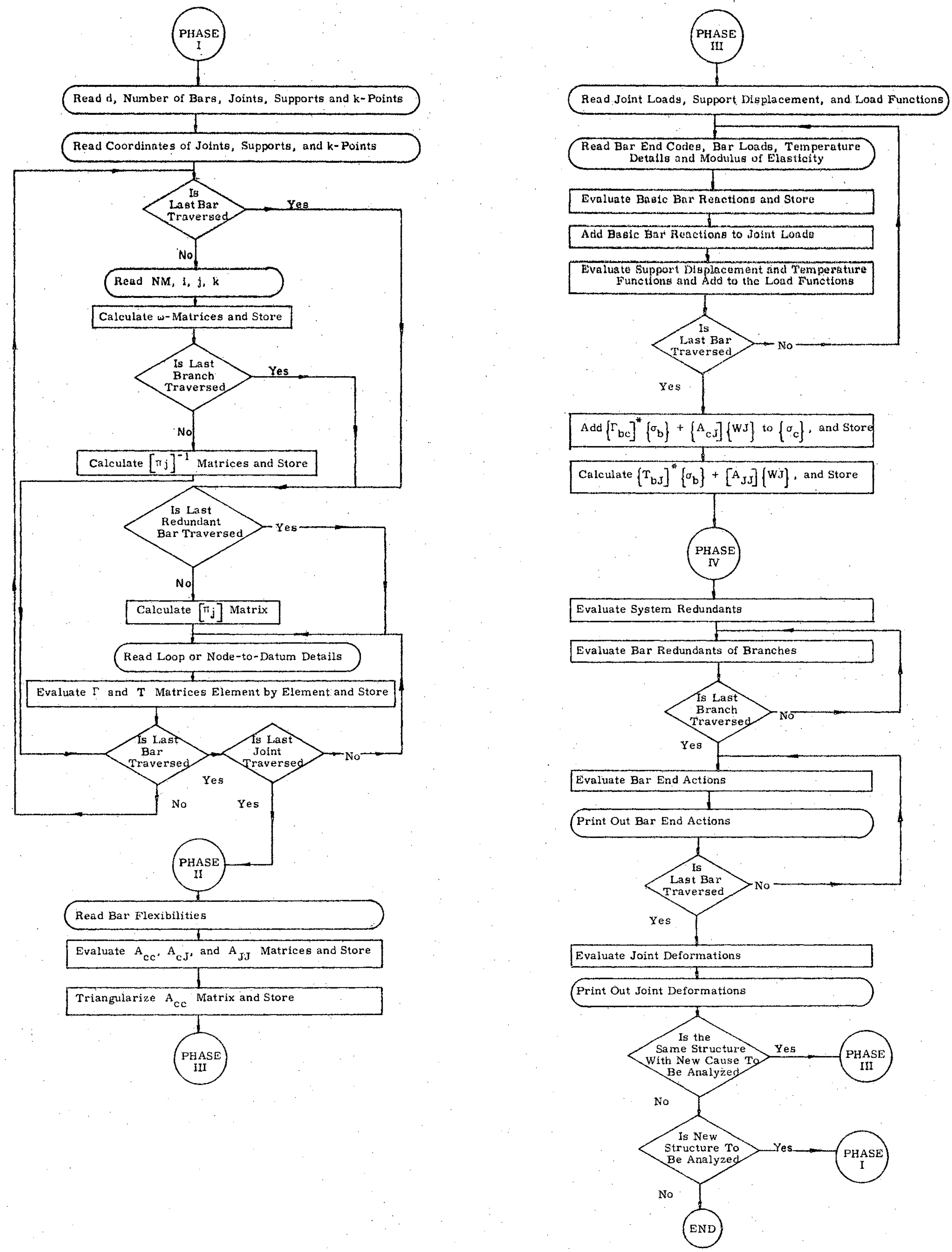

FIG. B-1. FLOW GRAPH OF COMPUTER PROGRAM FOR THE ANALYSIS OF THREE-DIMENSIONAL FRAMES BY FLEXIBILITY METHOD

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