THE PRIME NUMBER THEOREM

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Submitted to the Faculty of the Graduate School of
the Oklahoma State University
in partial fulfillment of the requirements
for the degree of
DOCTOR OF EDUCATION
May, 1965

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PREFACE

The prime numbers are irregular in their distribution, and some of the deepest theorems in the theory of numbers have to do with the prime numbers. However, when the large scale distribution of primes is considered, it appears in many ways quite regular and obeys simple laws. The study of these laws falls in the field of analytic number theory. This particular domain of number theory operates with very advanced methods of the calculus and is considered to be technically one of the most difficult fields of mathematics. Its central problem is the study of the function $\pi(x)$, which indicates the number of primes up to a certain number x. It was discovered quite early by means of empirical counts in the prime tables that the function $\pi(x)$ behaves asymptotically like the function $x/\log x$ (see page 7). The following formula is called the Prime Number Theorem:

$$\lim_{x \to \infty} \pi(x)/(x/\log x) = 1$$

Indebtedness is acknowledged to Dr. R. B. Deal for his valuable guidance and assistance, and to Drs. Jeanne L. Agnew, O. H. Hamilton, and J. E. Hoffman for their suggestions and advice during the preparation of this thesis; and to the National Science Foundation for the Faculty Fellowship award which gave me the opportunity to devote full time to the completion of my doctoral program.

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CHAPTER I

INTRODUCTION

The purpose of the dissertation is to analyze the fundamental concepts, organize a logical unit with necessary additional proofs, and present an expository discussion of the prime number theorem and other theorems needed in the development of the proof of the prime number theorem. Prime numbers, distribution of prime numbers, and the sieve of Eratosthenes in relation to the prime number theorem are briefly discussed in this chapter. The history of the prime number theorem is traced through the conjectures of the eighteenth and early part of the nineteenth centuries, the analytic proofs of the theorem in the latter part of the nineteenth century, and the improvements on bounds in the first half of the twentieth century. Chapters two and three are used to present, in detail, a discussion of the modern, so called "elementary" proofs of the prime number theorem. In chapter two proofs of preliminary lemmas and theorems are presented. It culminates with the proof of Selberg's basic formula. The proof of the prime number theorem is completed in chapter three. The elementary proofs of Erdos and Selberg, and the simplifications which have appeared since are discussed in the final chapter.

The natural numbers greater than one may be divided into two classes, prime numbers and composite numbers. A prime number is any natural number (positive integer) greater than one which has exactly two divisors, 1 and the number itself. All other natural numbers greater than 1 are said to be composite numbers. The "fundamental theorem of arithmetic" states that every natural number greater than 1 may be represented uniquely as the product of prime factors. Thus the prime numbers derive their peculiar importance as building stones from which all other natural numbers greater than 1 may be created multiplicatively.

One of the most interesting problems in number theory has to do with the distribution of the primes among the integers. Although there are great irregularities in the occurrence of the primes, the general distribution is found to possess certain features of regularity which can be formulated in precise terms and made the subject of mathematical investigation.

Definition 1.1. If x is any real number, then $\pi(x)$ denotes the number of primes not exceeding x.

The problem of studying the distribution of the prime numbers resolves itself into a study of the function $\pi(x)$. Finding $\pi(x)$ for large values of x is quite a job. In fact, extending the table of primes becomes a formidable task. To decide that a given natural number is prime, one needs to be sure that no natural number less than n divides n, except 1. It is not necessary to try as divisors all natural numbers less than n, for if a given prime will not divide n then no multiple of the prime will divide n. Thus one needs to consider only primes less than n, and not all of these. If no prime less than or equal to $\sqrt{n}$ will divide n, then n must be prime, for if d is any divisor of n such that $\sqrt{n} < d < n$, then n/d also divides n and is less than $\sqrt{n}$. If n/d is not prime, it has prime factors less than $\sqrt{n}$ which must divide any multiple of n/d including n.

There exists an ancient method of finding the primes known as the *sieves of Eratosthenes*. Eratosthenes (276-194 B.C.) was a Greek scholar, chief librarian of the famous library in Alexandria. He is noted for his chronology of ancient history and for his measurement of the meridian between Assuan and Alexandria, which made it possible to estimate the dimensions of the earth with fairly great accuracy [8] .

The primes less than or equal to any real number x may be obtained by the sieves of Eratosthenes. If the natural numbers greater than one and less than or equal to x are listed in their natural order, one may apply the sieve as follows: Underline all multiples of 2 except 2, then consider the first number after 2 that has not been underlined, i.e., 3. The 3 is a prime since 2 does not divide it. In the same manner as for 2, underline all multiples of 3 except 3; then consider the next number after 3 that has not been underlined. It is also a prime. If the process is continued, the first number not underlined after a given prime will also be a prime. If q is the greatest prime $\leq \sqrt{x}$, then the process may stop after the multiples of q (except q) are underlined. Thus, the numbers that have not been underlined are all of the primes less than or equal to x.

If one desires to obtain only the primes greater than the $\sqrt{x}$ and less than or equal to x, then the primes $\leq \sqrt{x}$ are also underlined. For example, consider the case x = 50. The primes less than or equal to $\sqrt{50}$ are 2, 3, 5, and 7. List the natural numbers from 2 to 50; then in consecutive order, underline all multiples of 2, 3, 5, and 7. The sequence looks like this:

$\underline{2}$, $\underline{3}$, $\underline{4}$, $\underline{5}$, $\underline{\underline{6}}$, $\underline{7}$, $\underline{8}$, $\underline{9}$, $\underline{\underline{10}}$, 11, $\underline{\underline{12}}$, 13, $\underline{\underline{14}}$, $\underline{\underline{15}}$,

$\underline{16}$, 17, $\underline{\underline{18}}$, 19, $\underline{\underline{20}}$, $\underline{\underline{21}}$, $\underline{22}$, 23, $\underline{\underline{24}}$, $\underline{25}$, $\underline{26}$, $\underline{27}$,

$\underline{\underline{28}}$, 29, $\underline{\underline{\underline{30}}}$, 31, $\underline{32}$, $\underline{33}$, $\underline{34}$, $\underline{\underline{35}}$, $\underline{\underline{36}}$, 37, $\underline{38}$, $\underline{39}$, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50;

The numbers not underlined

11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47

are the primes greater than or equal to $\sqrt{50}$ and less than or equal to 50.

Definition 1.2. For any real number x, the symbol $[x]$ denotes the greatest integer less than or equal to x.

The number of integers sieved out for any prime $p \le \sqrt{x}$ may be represented by $[x/p]$. One may devise a formula for $\pi(x) - \pi(\sqrt{x})$ by using the bracket function. An expression like

$$[x] - 1 - \sum_{p \le x} \left[\frac{x}{p}\right]$$

will not serve because some numbers are sieved out two or more times. Numbers of the form pq are sieved out twice: once when sieved by p and once when sieved by q. It is necessary to add $\Sigma\, [x/pq]$, where the sum extends over all primes p and q such that $p < q \le \sqrt{x}$. Even this expression,

$$[x] - 1 - \sum_{p \le \sqrt{x}} \left[\frac{x}{p}\right] + \sum \left[\frac{x}{pq}\right] ,$$

is not complete. Numbers of the form $p_1 p_2 p_3$ must be considered. These numbers will be sieved out once each by p_1, by p_2, and by p_3. These numbers are added back three times when the multiples of $p_1 p_2$, $p_1 p_3$, and $p_2 p_3$ are added back. So these numbers have not actually been taken out at all. This situation is remedied by subtracting

$$\sum \left[\frac{x}{p_1 p_2 p_3}\right] ,$$

where the sum extends over all primes p_1, p_2, p_3 such that $p_1 < p_2 < p_3 \le \sqrt{x}$. If this procedure is continued and if $p_1, p_2, \cdots, p_k$ are all the primes $\le \sqrt{x}$, then one obtains the following formula:

$$\pi(x) - \pi(\sqrt{x}) = -1 + [x]$$

$$- \sum_{p \le \sqrt{x}} \left[\frac{x}{p}\right] + \sum \left[\frac{x}{p_j p_r}\right] - \sum \left[\frac{x}{p_i p_j p_r}\right] + \ldots - \ldots ,$$

where the second two sums extend over all primes such that $p_i < p_j < p_r \le \sqrt{x}$. By using the definition of the Mobius function, definition 2.5 of chapter two, this may be expressed in the following precise form:

(1) $$\pi(x) - \pi(\sqrt{x}) = -1 + \sum_d \mu(d) [x/d] ,$$

where the sum is extended over all positive divisors of the product $p_1\ p_2 \ldots p_k$.

It was proved by Euclid (Elements, Book 9, Prop. 20.) around 300 B.C. that there exist an infinite number of primes. In essentials his proof is as follows: Let P be a product of any finite set of primes, and consider P + 1. The integers P and P + 1 can have no prime factor in common, since such a factor would divide 1, which is impossible. Hence P + 1 is either a new prime or it contains a prime factor distinct from those occurring in P. If there were only a finite number of primes altogether, we could take P to be the product of all primes, and a contradiction would result [4] .

In 1737 Euler proved the existence of an infinity of primes by a new method, which shows moreover that

(2) the series, $\sum_{n=1}^{\infty} \frac{1}{p_n}$, is divergent.

Euler's work is based on the idea of using an identity in which the primes appear on one side but not on the other. Stated formally, his identity is

(3) $$\sum_{n=1} n^{-s} = \prod_p (1 + p^{-s} + p^{-2s} + \ldots) = \prod_p (1 - p^{-s})^{-1},$$

where the products are over all primes p. Euler's contribution to the subject is of fundamental importance; for his identity, which may be regarded as an analytical equivalent of the fundamental theorem of arithmetic, forms the basis of nearly all subsequent work [4].

The question of the diminishing frequency of primes was the subject of much speculation before any definite results emerged. The problem assumed a much more precise form with the publication by Legendre in 1808 (after a less definite statement in 1798) of a remarkable empirical formula for the approximate representation of $\pi(x)$. Legendre asserted that for large values of x, $\pi(x)$ is approximately equal to

$$\frac{x}{\log x - B}, \tag{4}$$

where log x is the natural (Napierian) logarithm of x and B a certain numerical constant. A similar, though not identical, formula was prepared independently by Gauss. Gauss's method, which consisted in counting the primes in blocks of a thousand consecutive integers, suggests the function 1/log x as an approximation to the average density of distribution ('numbers of primes per unit internal') in the neighborhood of a large number x, and thus the integral of the density,

$$\mathrm{Li}\,(x) = \int_2^x \frac{du}{\log u} \tag{5}$$

as an approximation to $\pi(x)$. The function (5) is the so-called "integral logarithm of x". Gauss's observations were communicated to Encke in 1849, and first published in 1863; but they appear to have commenced as early as 1791 when Gauss was fourteen years old. In the interval the relevance of the function (5) was recognized independently by other writers [4].

The precise degree of approximation claimed by Gauss and Legendre for their empirical formulae outside the range of the tables used in their construction is not made very explicit by either author, but we may take it that they intended to imply at any rate the "asymptotic equivalence" of $\pi(x)$ and the approximating function $f(x)$, that is to say $\pi(x)/f(x)$ tends to the limit 1 as x tends to infinity. The two theorems which thus arise corresponding to the two forms of $f(x)$ are easily shown to be equivalent to one another and to the simpler relation

$$(6) \qquad \pi(x)/(x/\log x) \to 1 \text{ as } x \to \infty \quad .$$

The distinction between (4) and (5), and the value of B in (4) become important if one inquires more closely into the order of magnitude of the "error" $\pi(x) - f(x)$. The following table indicates the accuracy of $Li(x)$, $x/(\log x - 1)$, and $x/\log x$ as approximations of $\pi(x)$:

x	$\pi(x)$	$Li(x)$	$[x/(\log x - 1)]$	$[x/\log x]$
1,000	168	178	169	144
10,000	1,229	1,246	1,217	1,085
100,000	9,592	9,630	9,512	8,685
1,000,000	78,498	78,628	78,030	72,382
10,000,000	664,579	664,918	661,458	620,420

The proposition (6), which is now known as the "prime number theorem," is the central theorem in the theory of the distribution of primes. The problem of deciding its truth or falsehood engaged the attention of mathematicians for about a hundred years [4] .

The first demonstrated results are due to Tchebychef, who (1850), among other things, proved that the inequalities

$$(7) \qquad \frac{7}{8} \cdot \frac{x}{\log x} < \pi(x) < \frac{9}{8} \cdot \frac{x}{\log x}$$

are valid for all sufficiently large values of x. He also showed that the quotient of the numbers $\pi(x)$ and $\frac{x}{\log x}$ has the limit 1 for increasing x, providing that the limit exists [6]. These results constituted an advance of the first importance, but (as Tchebychef himself was well aware) they failed to establish the essential point, namely, the existence of $\lim \pi(x)/(x/\log x)$. Although the numerical bounds obtained by Tchebychef were successively narrowed by later writers (particularly Sylvester), it came to be recognized in due course that the methods employed by these authors were not likely to lead to a final solution of the problem [4] .

Already Euler had begun applying the methods of the calculus to number-theory problems; however, the German mathematician G. F. B. Riemann (1826-1866) is generally regarded as the real founder of analytic number theory. His personal life was modest and uneventful until his premature death from tuberculosis. According to the wish of his father, he was originally destined to become a minister, but his shyness and lack of ability as a speaker made him abandon this plan in favor of mathematical scholarship. He was unassuming to a fault; yet at present he is recognized as having one of the most penetrating and original mathematical minds of the nineteenth century. In analytic number theory, as well as in many other fields of mathematics, his ideas still have a profound influence [8].

The new ideas which were to supply the key to the solution of the problem on the existence of the $\lim \pi(x)/(x/\log x)$ as $x \to \infty$ were introduced by Riemann in 1859 [9] in a memoir which has become famous, not only for its bearing on the theory of primes, but also for its influence on the development of the general theory of functions. Euler's identity had been used by Euler himself with a fixed value of $s (s = 1)$, and by Tchebychef with s as a real variable. Riemann now introduced the idea of treating s as a complex variable and studying the series on the left of (3) by the methods of the theory of analytic functions. This series converges only in a restricted portion of the plane of the complex variable s, but defines by continuation a single-valued analytic function regular at all finite points except for a single pole at $s=1$. This function is called the "zeta-function of Riemann," after the notation $\zeta(s)$ adopted by its author [4].

Riemann, perceiving the fundamental importance of the zeta function for the study of the distribution of primes, developed the elements of a theory for this function. He also formulated six hypotheses which he could not prove. Especially the position of the imaginary zeros of the function appeared to be of great importance for the applications to prime number theory. According to Riemann's famous but still unproved hypothesis, all the imaginary zeros have the real part $\sigma = 1/2$. All the other hypotheses of Riemann have been proved by later investigators [6]. The problems raised by Riemann's memoir inspired in due course the fundamental researches of Hadamard in the theory of integral functions, the results of which at last removed some of the obstacles which for more than thirty years had barred the way to rigorous proofs of Riemann's theorems. The proofs sketched by Riemann were completed (in essentials), in part by Hadamard himself in 1893, and in part by Van Mongoldt in 1894 [4].

The discoveries of Hadamard prepared the way for rapid advances in the theory of the distribution of primes. The prime number theorem was proved in 1896 by Hadamard himself and by de la Vallée Poussin, independently and almost simultaneously. Of the two proofs Hadamard's is the simpler, but de la Vallée Poussin (in another paper published in

1899) studied in great detail the question of closeness of approximation. His results prove conclusively (what had been foreshadowed by Tchebychef) that, for all sufficiently large values of x, $\pi(x)$ is represented more accurately by $Li(x)$ than by the function (4) (no matter what value is assigned to the constant B), and that the most favorable value of B in (4) is 1. This conflicts with Legendre's original suggestion 1.08366 for B, but this value (based on tables extending only as far as x=400000) had long been recognized as having little more than historical interest.

The theory can now be presented in a greatly simplified form, and de la Vallée Poussin's theorems can (if desired) be proved without recourse to the theory of integral functions. This is due almost entirely to the work of Landau. The results themselves underwent no substantial change until 1921, when they were improved by Littlewood; but Littlewood's refinements lie much deeper and the proofs involve very elaborate analysis [4].

There are two important changes in "depth" of the prime number theorem. First, the zeta function is no longer needed for obtaining the sharpest known error term in the prime number theorem; in fact, the elements of the theory of functions of a complex variable are now sufficient. Second, the prime-number theorem as such (without any estimation of the error term) now falls under the scope of elementary methods. Titchmarsh established the following result:

$$\pi(x) - Li(x) = \varphi\, k\, x \cdot e^{-\omega(x)} \tag{8}$$

where $\omega(x) = \alpha(\log x)^{5/9 - \epsilon}$; it is valid for all sufficiently large values of x; ϵ is a positive number, k and α are certain positive constants, and φ denotes a function of x which varies between the limits -1 and 1. This formula, which was proved in 1938, expresses the best result up to now for the function $\pi(x)$. It is easily seen from the formula that $\pi(x)$ is approximated by $Li(x)$ with great accuracy. It was shown by Littlewood that the left side of (8) assumes both positive and negative values infinitely often [6]. Littlewood's theorem, however, is a pure "existence theorem," and no numerical value of x for which $\pi(x) > Li(x)$ is known.

There was a sensation when an "elementary" proof of the prime number theorem was given in 1948 by P. Erdos and Alte Selberg. Selberg proved the fundamental relation

$$(9) \qquad \sum_{p \leq x} \log^2 p + \sum_{pq \leq x} \log p \log q = 2x \log x + O(x),$$

and he and Erdos, independently, deduced the prime number theorem from it. The resulting proof, while not simple, requires nothing more complicated than the most elementary properties of the logarithmic function.

The so-called "elementary" proofs are discussed in detail in chapter two and chapter three. The proofs of Erdos, Selberg, and simplifications which have appeared since are also discussed in the final chapter.

CHAPTER II

PRELIMINARY LEMMAS AND THEOREMS

As stated in the introduction, the basic intent of this chapter and chapter three is to present, in detail, a discussion of the modern, so-called "elementary," proofs of the prime number theorem. The major contributions were those of Erdos and Selberg, but in presenting detailed proofs in this and the next chapter it was found convenient to rely heavily on the outline and discussion in Nagell [6]. The ultimate goal of this chapter is the proof of Selberg's basic formula and this asymptotic formula is used to deduce the prime number theorem in chapter three. Definitions, lemmas, and other theorems are given as a means of obtaining this objective.

$\pi(x)$ is usually used to denote the function which gives the number of primes less than or equal to the real number x. It may be written as

$$\pi(x) = \sum_{p \le x} 1$$

where p extends over all primes $\le$ x. Rather than consider the function $\pi(x)$, the function $\vartheta(x)$, defined by the following equality, is studied in relation to the prime number theorem:

<u>Definition 2.1</u>. $\vartheta(x) = \sum\limits_{p \le x} \log p$, the sum extending over all primes $p \le x$.

Although the prime number theorem is usually stated as

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\log x} = 1,$$

it may be stated in other forms. In particular, it is equivalent to the proposition

$$\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1.$$

In chapter three the equivalence of the two propositions is proved, and the second relationship is established as a consequence of Selberg's formula (Theorem 2.5),

$$\vartheta(x) \log x + 2 \sum_{p \leq \sqrt{x}} \vartheta(x/p) \log p - 2x \log x = o(x \log x).$$

The term $o(x \log x)$ represents any function of x with the property that

$$\lim_{x \to \infty} \frac{o(x \log x)}{x \log x} = 0.$$

The concept of functions being o of some function or O of some function is discussed later. The formula may also be stated in other forms. In particular, it may be given as

$$\frac{\vartheta(x)}{x} + \frac{2}{x \log x} \sum_{p \leq \sqrt{x}} \vartheta(x/p) \log p - 2 = o(1).$$

This is the form used in chapter three.

Each of the twelve lemmas and each of the first four theorems of this chapter contribute to the proof of Selberg's formula. Lemma 2.12 contributes directly by establishing relationships involving some of the terms in the formula. Lemma 2.9 is also instrumental in the proof by establishing that

$$\sum_{d \leq x} \frac{\mu(d)}{d} \left(\log \frac{x}{d}\right)^2$$

may be approximated by 2 log x. Each of the other lemmas and theorems contributes indirectly by contributing directly to the proof of some other lemma or theorem. For example, Theorem 2.2 establishes that, for $x \leq 2$, $\vartheta(x)/x$ is bounded by two positive constants and this fact is used to prove other lemmas.

Consider the series $[n/p] + [n/p^2] + [n/p^3] + \cdots$, where the brackets have the same meaning as in Definition 1.2. For each term in which $p^k > n$ for some k, the value of the term is zero. Thus one may write

$$[n/p] + [n/p^2] + [n/p^3] + \cdots = \sum_{k=1}^{r} [n/p^k]$$

where r is the highest integral exponent which satisfies the inequality $p^r \leq n$. This series is involved in the proof of Theorem 2.2. Therefore, the following theorem is proved first.

<u>Theorem 2.1.</u> Let n be a natural number, and let p be a prime. Then the exponent of the highest power of p which divides $n! = 1\cdot2\cdot3\cdots n$ is equal to

$$(1) \qquad N = [n/p] + [n/p^2] + [n/p^3] + \cdots .$$

Proof: The series continues so long as the power of p is $\leq n$. If h_v denotes the number of terms in the sequence 1, 2, ..., n which are divisible by p^v, the required exponent N is equal to $h_1 + h_2 + h_3 + \cdots$. The natural numbers $\leq n$ which are divisible by p^v are

$$1\cdot p^v,\ 2\cdot p^v,\ \cdots,\ [n/p^v]\cdot p^v .$$

Thus the term $h_v = [n/p^v]$, and the theorem is proved.

The proof of the following theorem of Tchebychef can now be given:

<u>Theorem 2.2.</u> There exist two positive constants, b and c, such that for all $x \geq 2$

$$(2) \qquad bx < \vartheta(x) < cx .$$

Proof: Let n be an integer ≥ 2. If p^m is the highest power of the prime p which divides the binomial coefficient

$$(3) \qquad \binom{2n}{n} = \frac{(2n)!}{n!\ n!} ,$$

then, by Theorem 2.1,

$$m = \sum_{v=1}^{r} \left(\left[\frac{2n}{p^v}\right] - 2\left[\frac{n}{p^v}\right] \right) ,$$

where r is the highest integral exponent which satisfies the inequality

$$(4) \qquad p^r \leq 2n,$$

and thus

$$r = \left[\frac{\log 2\ n}{\log p}\right] .$$

The difference

$$\left[\frac{2n}{p^v}\right] - 2\left[\frac{n}{p^v}\right]$$

has either the value zero or the value 1, and therefore

$$(5) \qquad m \leq r.$$

Now

$$\binom{2n}{n} = \frac{(n+1)(n+2)\ \cdots\ 2n}{n!} = \prod_{h=1}^{n} \frac{n+h}{h} > 2^n$$

and on the other hand, by (5) and (4),

$$2^n < \binom{2n}{n} = \prod_{p<2n} p^m \leq \prod_{p<2n} p^r ,$$

the products extending over all primes $p < 2n$. Therefore, by taking the logarithm.

$$n \log 2 < \sum_{p<2n} r \log p = \sum_{p<2n} \left[\frac{\log 2n}{\log p}\right] \log p.$$

For every p,

$$\left[\frac{\log 2n}{\log p}\right] \leq \frac{\log 2n}{\log p} ,$$

and for all $p > \sqrt{2n}$,

$$\left[\frac{\log 2n}{\log p}\right] = 1.$$

Thus one obtains

$$n \log 2 < \sum_{p \le \sqrt{2n}} \frac{\log 2n}{\log p} \cdot \log p + \sum_{p > \sqrt{2n}}^{p < 2n} \log p$$

or

$$n \log 2 < \sqrt{2n} \log 2n + \vartheta(2n).$$

So

$$\vartheta(2n) > n \log 2 - \sqrt{2n} \log 2n.$$

The expression $\frac{8}{(\log 2)^2} \frac{(\log 2n)^2}{n-2}$ tends to zero as $n \to \infty$. Hence, for all sufficiently large integers n,

$$1 \ge \frac{8}{(\log 2)^2} \frac{(\log 2n)^2}{n-2}$$

or

$$n - 2 \ge \frac{8}{(\log 2)^2} \cdot (\log 2n)^2 .$$

Thus

$$n^2 - 2n + 1 > 8n \left(\frac{\log 2n}{\log 2}\right)^2 ,$$

and by extracting square roots

$$n - 1 > 2\sqrt{2n} \frac{\log 2n}{\log 2} .$$

It follows that

$$\frac{n}{2} \log 2 - \sqrt{2n} \log 2n > \frac{\log 2}{2}$$

and

$$n \log 2 - \sqrt{2n} \log 2n > \tfrac{1}{2} (n + 1) \log 2.$$

If $2n \le x < 2n + 2$, one obtains

$$\vartheta(x) \ge \vartheta(2n) \ge \tfrac{1}{2}(n + 1) \log 2 > \tfrac{1}{4} x \log 2$$

for all sufficiently large x. Hence, for some N and all $x \geq N$ the above relation holds. Now for all x such that $2 \leq x < N$

$$\vartheta(x) \geq \log 2 > \frac{x}{N} \log 2 = \frac{\log 2}{N} x.$$

This proves the first inequality of Theorem 2.2 with $b = \frac{\log 2}{N}$.

The number $\binom{2n}{n}$ is clearly divisible by all primes p which are > n and < 2n. Since

$$\prod_{p>n}^{p<2n} p \cdot \frac{\Pi\, k}{n!} \geq \prod_{p>n}^{p<2n} p$$

where p is prime and the product, $\Pi\, k$, extends over all composite numbers, k, such that $n < k \leq 2n$ and n! divides $\Pi\, k$, it follows that

$$2^{2n} = (1+1)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} > \binom{2n}{n} \geq \prod_{p>n}^{p<2n} p ,$$

and, by taking logarithms, $2n \log 2 > \vartheta(2n) - \vartheta(n)$. Suppose that x is a power of 2, that is, $x = 2^h$ where h is a positive integer. It follows that

$$\vartheta(x) = \vartheta(2^h) = \sum_{k=1}^{h} \left(\vartheta(2^k) - \vartheta(2^{k-1})\right).$$

Thus

$$\vartheta(2^h) < \log 2 \sum_{k=1}^{h} 2^k < 2^{h+1} \log 2 = 2x \log 2.$$

Further, if $2^{h-1} < x \leq 2^h$, then, for all $x \geq 1$,

$$\vartheta(x) \leq \vartheta(2^h) < 2^{h+1} \log 2 < 4x \log 2,$$

which proves the second inequality with $c = 4 \log 2$.

In proving the prime number theorem and other preliminary lemmas and theorems, one will also need the formula given by the following theorem:

<u>Theorem 2.3.</u> If the sum is extended over all primes $p \leq x$, then

$$(6) \qquad \sum_{p \leq x} \frac{\log p}{p} = \log x + \Theta ,$$

where Θ is a function of x such that $|\Theta|$ is less than a positive constant.

Proof: The proof requires the following relation:

$$\log n! = \sum_{p \le n} \left(\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots \right) \log p,$$

where p extends over all primes $\le n$ and the series continues so long as the power of p is $\le n$. By the definition of the bracket function, it follows that

$$\sum_{p \le n} \left(\left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots \right) \log p$$

$$< \sum_{p \le n} \left(\frac{n}{p^2} + \frac{n}{p^3} + \cdots \right) \log p = n \sum_{p \le n} \frac{\log p}{p(p-1)} ,$$

where the second series is infinite. Further, one sees that

$$n \sum_{p \le n} \frac{\log p}{p} > \sum_{p \le n} \left[\frac{n}{p}\right] \log p >$$

$$\sum_{p \le n} \left(\frac{n}{p} - 1 \right) \log p = n \sum_{p \le n} \frac{\log p}{p} - \vartheta(n).$$

Consequently, one may observe that

$$\sum_{p \le n} \frac{\log p}{p} - \frac{\vartheta(n)}{n} < \frac{1}{n} \sum_{p \le n} \left[\frac{n}{p}\right] \log p <$$

$$\frac{1}{n} \log n! < \sum_{p \le n} \frac{\log p}{p} + \sum_{p \le n} \frac{\log p}{p(p-1)} .$$

Thus

$$\left| \frac{1}{n} \log n! - \sum_{p \le n} \frac{\log p}{p} \right| < \max \left(\frac{\vartheta(n)}{n} , \sum_{p \le n} \frac{\log p}{p(p-1)} \right) .$$

According to Theorem 2.2, $\frac{\vartheta(n)}{n} < c$ where c is a positive constant and $\sum_{p \le n} (\log p)/p(p-1) \le \sum_{m=2}^{n} (\log m)/m(m-1)$ where the right side converges to a positive constant as $n \to \infty$. Therefore,

$$(7) \qquad \frac{1}{n} \log n! \; - \sum_{p \le n} \frac{\log p}{p} = \alpha,$$

where α is a function of n such that $|\alpha|$ is less than a positive constant.

For every integer $h \ge 2$, $\log h = h \log h - (h-1) \log(h-1) - (h-1) \log(1+ \frac{1}{h-1})$, where the last term is less than 1. Thus

$$\sum_{h=2}^{n} \log h = \sum_{h=2}^{n} \left[h \log h - (h-1) \log(h-1) - (h-1) \log (1+\frac{1}{h-1}) \right]$$

$$> n \log n - 1 (n-1) .$$

Hence

$$n \log n - n - 1 < \sum_{h=2}^{n} \log h < n \log n,$$

or

$$\log n - 1 - \frac{1}{n} < \sum_{h=2}^{n} \frac{\log h}{n} = \frac{1}{n} \log n! < \log n .$$

Formula (6) is obtained by combining these inequalities with (7).

The concept of functions being o of some function or O of some function is a useful one in this type of analysis. History has dictated a usage and terminology which is not consistent with modern carefulness. The definitions are given below for the particular case needed in this paper with more care than customary and then translated into standard usage. Let A be a subset of R (reals) such that for every $y \in R$, there exists an $x \in A$ such that $x > y$, and N_∞ be the collection of sets of the form $\{x \mid x \in A \text{ and } x > a \text{ for some } a \in R\}$. If g is a non-zero function defined on A to R, then $o(g)$ and $O(g)$ are defined by

<u>Definition 2.2</u>. $o(g) = \{f \mid \exists \; U \in N_\infty, U \subset \text{dom } f, \lim_{\infty} \frac{f}{g} = 0\}$

and

<u>Definition 2.3</u>. $O(g) = \{f \mid \exists \; U \in N_\infty, k_f > 0 \ni \forall \; x \in U, \left|\frac{f(x)}{g(x)}\right| < k_f\}$.

It should be observed that these definitions work just as well for real or complex valued functions on any topological space X and any point p of X can be used if N_p is the collection of open sets containing p. The following properties are also valid:

(8) For any real constant k,

(a) $h \in o(g)$ implies $kh \in o(g)$

(b) $h \in O(g)$ implies $kh \in O(g)$.

(9) For any positive function h,

(a) $f \in o(g)$ implies $hf \in o(hg)$

(b) $f \in O(g)$ implies $hf \in O(hg)$.

(10) If $0 < f \leq g$ in some $U \in N_\infty$, then

(a) $o(f) \subset o(g)$

(b) $O(f) \subset O(g)$.

If $h \in o(g)$, it is customary to use the following notation:

$$h(x) = o\left(g(x)\right).$$

A similar notation is used for any function $h \in O(g)$, that is,

$$h(x) = O\left(g(x)\right).$$

Thus $x \log x = o(x^2)$, $\cos x = o(\sqrt{x})$, $\log x = o(\sqrt{x})$, $2x = O(x)$, etc. The prime number theorem could also be formulated as follows:

$$\pi(x) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right).$$

Formula (6) may also be written as

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1),$$

or as

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + o(\log x).$$

In the (u,v) plane the area of $\{(u,v) \mid 1 \le u \le y,\ 0 \le v \le 1/u\}$ is $\log y$. This area can be approximated by $\sum_{n\le y} 1/n$. This relation is expressed more precisely by formula (11) of Lemma 2.1. The area of $\{(u,v) \mid 1 \le u \le y,\ 0 \le v \le (\log u)/u\}$ is $\frac{1}{2}(\log y)^2$. This area can be approximated by $\sum_{n\le y} (\log n)/n$. Formula (14) of Lemma 2.2 gives a more precise statement of this fact.

Lemma 2.1. There exists a positive absolute constant γ such that

$$\sum_{n\le y} \frac{1}{n} = \log y + \gamma + O(1/y), \tag{11}$$

where the sum extends over all positive integers $n \le y$.

Proof: Let z be the least integer $> y$, that is, $z = [y + 1]$. Let $\delta_n = 1/n - \log(1 + 1/n)$, then

$$\log z = \sum_{n=1}^{z-1} \Big(\log(n+1) - \log n\Big) = \sum_{n=1}^{z-1} \log(1 + 1/n) \tag{12}$$

$$= \sum_{n=1}^{z-1} \left(\frac{1}{n} - \frac{1}{n} + \log(1 + 1/n)\right) = \sum_{n=1}^{z-1} \frac{1}{n} - \sum_{n=1}^{z-1} \delta_n .$$

Let $f(x) = 1/2x^2 - 1/x + \log(1 + 1/x)$, then $f'(x) = -1/x^3(x + 1) < 0$ for all $x > 0$ and the derivative of $1/x - \log(1 + 1/x)$ is $-1/x^2(x + 1)$ < 0 for all $x > 0$. Thus δ_n and $f(n)$ are both decreasing functions with zero as the limit so $\delta_n > 0$ and $f(n) > 0$. Now, $f(n) = 1/2n^2 - \delta_n$, $\delta_n > 0$, and $f(n) > 0$ imply that the following inequality holds for all n:

$$0 < \delta_n < 1/2n^2 . \tag{13}$$

It follows from this that the infinite series $\Sigma\, \delta_n$ is convergent and has a positive value γ. Further,

$$\sum_{n=z}^{\infty} \delta_n < \frac{1}{2} \sum_{n=z}^{\infty} 1/n^2$$

$$< \frac{1}{2} \sum_{n=z}^{\infty} \frac{1}{n(n-1)} = \frac{1}{2} \sum_{n=z}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right) = \frac{1}{2(z-1)} .$$

From formula (12) and the preceding results, one obtains the following:

$$\sum_{n=1}^{z-1} \frac{1}{n} = \log z + \sum_{n=1}^{z-1} \delta_n = \log z + \sum_{n=1}^{\infty} \delta_n - \sum_{n=z}^{\infty} \delta_n$$

$$= \log z + \gamma + \Theta/z$$

where Θ is a function of z such that $|\Theta|$ is less than a positive constant. Thus

$$\sum_{n \le y} 1/n = \log z + \gamma + O(1/z)$$

$$= \log y + \gamma + \log \frac{z}{y} + O(1/z).$$

But

$$\left| y\left(\log \frac{z}{y} + O(1/z)\right)\right| \le \frac{1}{y} \cdot y + \frac{y}{z} \cdot z \cdot O(1/z)$$

$$= 1 + \Theta ,$$

so $\log z/y + O(1/z)$ is $O(1/y)$. This concludes the proof of formula (11).

<u>Lemma 2.2.</u> There exists an absolute constant c such that

$$\text{(14)} \qquad \sum_{n \le y} \frac{\log n}{n} = \tfrac{1}{2}(\log y)^2 + c + O\left(\frac{\log y}{y}\right),$$

where the sum extends over all positive integers $n \le y$.

Proof: Let $z = [y + 1]$. Clearly

$$(\log z)^2 = \sum_{n=1}^{z-1} \left((\log[n + 1])^2 - (\log n)^2\right)$$

and since $\log(n + 1) = \log n + 1/n - \delta_n$, then

$$\tfrac{1}{2}(\log z)^2 = \tfrac{1}{2} \sum_{n=1}^{z-1} \left((\log n + 1/n - \delta_n)^2 - (\log n)^2\right) =$$

$$\tfrac{1}{2} \sum_{n=1}^{z=1} \left((\log n)^2 + 2\,\frac{\log n}{n} - 2\,\delta_n \log n - \frac{2\delta_n}{n} + \frac{1}{n^2} + \delta_n^{\,2} - (\log n)^2\right)$$

$$= \sum_{n=1}^{z-1} \frac{\log n}{n} - \sum_{n=1}^{z-1} (\delta_n \log n + \delta_n/n - 1/2n^2 - \tfrac{1}{2}\,\delta_n^{\,2}) .$$

By means of (13) one may observe that the latter sum on the right-hand side tends to a finite limit c when $z \to \infty$. Further,

$$\left| \sum_{n=z}^{\infty} (\delta n \log n + \delta_n/n - 1/2n^2 - \tfrac{1}{2}\delta_n^2) \right| < \sum_{n=z}^{\infty} (\log n)/n^2$$

$$< (\log z)/z^2 + \int_z^{\infty} \frac{\log x}{x^2}\, dx = (\log z)/z^2 + 1/z + (\log z)/z.$$

Thus one may conclude that

$$\sum_{n=1}^{z-1} \frac{\log n}{n} = \tfrac{1}{2}(\log z)^2 + O\left(\frac{\log z}{z}\right) + c,$$

where c is an absolute constant. This formula gives (14) in a manner similar to that in Lemma 2.1.

A function f(n) defined for all natural numbers n is called an arithmetical function.

<u>Definition 2.4.</u> The arithmetical function which gives the number of positive divisors of n is denoted by $\tau(n)$.

<u>Lemma 2.3</u>. If the sum is extended over all positive integers $n \le y$, then

$$\sum_{n\le y} \frac{\tau(n)}{n} = \tfrac{1}{2}(\log y)^2 + 2\gamma \log y + \gamma^2 - 2c + O\left(\frac{\log y}{\sqrt{y}}\right),$$

where γ and c are the same absolute constants as in Lemmas 2.1 and 2.2.

Proof: Since $\tau(n)$ is equal to the number of ordered pairs of natural numbers a and b such that ab = n, then $\displaystyle\sum_{ab=n} \frac{1}{ab} = \frac{\tau(n)}{n}$;

and

$$\sum_{n\le y} \frac{\tau(n)}{n} = \sum_{n\le y} \sum_{ab=n} \frac{1}{ab} = \sum_{ab\le y} \frac{1}{ab},$$

where the sum on the right-hand side extends over all natural numbers a and b such that $ab \le y$. Now let S_1 denote the part of this sum in which $a \le \sqrt{y}$, let S_2 denote the part in which $b \le \sqrt{y}$, and S_3 denote the part in which $a \le \sqrt{y}$ and $b \le \sqrt{y}$. Then the value of the required sum is

$S_1 + S_2 - S_3$. By Lemmas 2.1 and 2.2,

$$S_1 = \sum_{a\le\sqrt{y}} \frac{1}{a} \sum_{b\le y/a} \frac{1}{b} = \sum_{a\le\sqrt{y}} \frac{1}{a}\left(\log y/a + \gamma + O(a/y)\right)$$

$$= \sum_{a\le\sqrt{y}} \frac{1}{a}\left(\log y - \log a + \gamma + O(a/y)\right)$$

$$= \log y \sum_{a\le\sqrt{y}} \frac{1}{a} - \sum_{a\le\sqrt{y}} \frac{\log a}{a} + \gamma \sum_{a\le\sqrt{y}} \frac{1}{a} + O(1/y) \sum_{a\le\sqrt{y}} 1$$

$$= (\log y + \gamma)\left(\log\sqrt{y} + \gamma + O(1/\sqrt{y})\right) - \tfrac{1}{2}(\log\sqrt{y})^2$$

$$- c + O\left(\frac{\log\sqrt{y}}{\sqrt{y}}\right) + O(1/\sqrt{y})$$

$$= (\log y + \gamma)\left(\tfrac{1}{2}\log y + \gamma + O(1/\sqrt{y})\right) - 1/8\,(\log y)^2$$

$$- c + O\left(\frac{\log\sqrt{y}}{\sqrt{y}}\right) + O(1/\sqrt{y})$$

$$= 3/8\,(\log y)^2 + 3/2\,\gamma\log y + \gamma^2 - c + O\left(\frac{\log y}{\sqrt{y}}\right).$$

It is plain that $S_2 = S_1$. Further, by Lemmas 2.1 and 2.2,

$$S_3 = \left(\sum_{a\le\sqrt{y}} \frac{1}{a}\right)^2 = \left(\log\sqrt{y} + \gamma + O(1/\sqrt{y})\right)^2$$

$$= \left(\tfrac{1}{2}\log y + \gamma + O(1/\sqrt{y})\right)^2$$

$$= \tfrac{1}{4}(\log y)^2 + \gamma\log y + \gamma^2 + O\left(\frac{\log y}{\sqrt{y}}\right).$$

Hence
$$\sum_{ab\le y} \frac{1}{ab} = S_1 + S_2 - S_3$$

$$= 3/4\,(\log y)^2 + 3\gamma\log y + 2\gamma^2 - \tfrac{1}{4}(\log y)^2$$

$$- 2c - (\log y) - \gamma^2 + O\left(\frac{\log y}{\sqrt{y}}\right)$$

$$= \tfrac{1}{2}(\log y)^2 + 2\gamma\log y + \gamma^2 - 2c + O\left(\frac{\log y}{\sqrt{y}}\right),$$

which proves Lemma 3.2.

Another important arithmetical function is the Mobius function $\mu(n)$, defined as follows:

Definition 2.5. $\mu(1) = 1$; $\mu(n) = 0$, if n is divisible by the square of any prime; $\mu(p_1\, p_2 \cdots p_r) = (-1)^r$, if $p_1, p_2 \cdots$, and p_r are different primes.

Thus $\mu(2) = \mu(3) = \mu(7) = \mu(11) = -1$, $\mu(4) = \mu(8) = 0$, $\mu(6) = \mu(10) = \mu(14) = 1$, etc.

Definition 2.6. An integer is called a square-free number if it is not divisible by any square > 1.

Theorem 2.4. For every natural number $n > 1$,

$$S_n = \sum_{d|m} \mu(d) = 0 ,$$

where the sum extends over all positive divisors d of n.

Proof: One needs only to extend the sum over all positive square-free divisors d of n. The theorem is proved by multiplicative induction. It is true when n is any prime p, since $S_p = \mu(1) + \mu(p) = 0$. Suppose that it is true for $n = m$. Then it can be shown that it is also true for $n = mp$, where p is any prime. If m is divisible by p, it is easily seen that S_{mp} contains the same terms as S_m. Since, by hypothesis, $S_m = 0$, then $S_{mp} = 0$. If m is not divisible by p, then

$$S_{mp} = \sum_{\delta | m} \Big(\mu(\delta) + \mu(p\delta)\Big) ,$$

the sum being extended over all positive square-free divisors δ of m. Since $\mu(p\delta) = -\mu(\delta)$, it follows that $S_{mp} = 0$.

Definition 2.7. For any integer $h \geq 0$, the function $\varphi_h(n)$ is defined by the equation

$$\varphi\ (n) = \sum_{d|n} \mu(d)(\log d)^h ,$$

where the sum extends over all positive divisors d of the natural number n and $(\log d)^0$ means the number 1.

Lemma 2.4. If the natural number n is divisible by more than h different primes, then

$$\varphi_h(n) = 0.$$

Proof: This is true for $h = 0$, according to Theorem 2.4, and so it can be assumed that $h \geq 1$. One can use mathematical induction and suppose that Lemma 2.4 is true for all the functions $\varphi_k(n)$ when $k \leq h-1$. If $n = p^\alpha m$ where $\alpha \geq 1$, and where the integer m is not divisible by the prime p, then for each d that divides n, one may write $d = d_1 \ d_2$ where d_1 divides m and d_2 divides p. Thus one obtains

$$\varphi_h(n) = \sum_d \mu(d)(\log d)^h = \sum_{d_1} \sum_{d_2} \mu(d_1 d_2)(\log d_1 d_2)^h$$

$$= \sum_{d_1} \sum_{d_2} \mu(d_1 d_2)(\log d_1 + \log d_2)^h ,$$

where the outer sum on the right-hand side extends over all positive divisors d_1 of m and the inner sum over all positive divisors d_2 of p. Then

$$\varphi_h(n) = \sum_{s=0}^{h} \binom{h}{s} \sum_{d_1} \mu(d_1)(\log d_1)^s \sum_{d_2} \mu(d_2)(\log d_2)^{h-s}$$

$$= \sum_{s=0}^{h} \binom{h}{s} \varphi_s(m) \varphi_{h-s}(p^\alpha) .$$

Since n has more than h different prime factors, m has more than h-1 different prime factors. Therefore, by hypothesis $\varphi_s(m) = 0$ for $s = 0, 1, \cdots$, and h-1. The remaining term $\varphi_h(m) \varphi_0(p^\alpha)$ is also equal to zero, since its last factor is zero. Thus the lemma is proved.

<u>Lemma 2.5</u>. For any positive number x, let

$$\lambda(d) = \mu(d) \cdot \left(\log \frac{x}{d}\right)^2$$

and

$$f(n) = \sum_{d|n} \lambda(d) ,$$

where the sum extends over all positive divisors d of the positive integer n. Then

$$f(1) = (\log x)^2 ;$$

$$f(p^\alpha) = -(\log p)^2 + 2(\log x)(\log p)$$

when p is a prime and α an integer ≥ 1;

$$f(p^\alpha q^\beta) = 2(\log p)(\log q)$$

when p and q are different primes, and α and β are integers ≥ 1;

$f(n) = 0$ if n is divisible by three or more different primes.

Proof: One may write $\lambda(d)$ in the following way:

$$\mu(d)\left((\log x)^2 - 2 \log x \log d + (\log d)^2\right)$$

$$= (\log x)^2 \quad \mu(d) - 2 \log x \quad \mu(d) \log d + \mu(d)(\log d)^2 .$$

Therefore,

$$f(n) = (\log x)^2 \sum_{d|n} \mu(d) - 2 \log x \sum_{d|n} \mu(d) \log d + \sum_{d|n} \mu(d)(\log d)^2$$

$$= (\log x)^2 \quad \varphi_0(n) - 2 \log x \, \varphi_1 \, (n) + \varphi_2(n).$$

Thus the proof follows from the definitions and Lemma 2.4 (for h = 0, 1, and 2).

<u>Lemma 2.6.</u> For every natural number m,

$$\left| \sum_{n=1}^{m} \frac{\mu(n)}{n} \right| \leq 1.$$

Proof: It follows from Theorem 2.4 that

$$1 = \sum_{n=1}^{m} \sum_{d|n} \mu(d) ,$$

and

$$\sum_{n=1}^{m} \sum_{d|n} \mu(d) = \sum_{r=1}^{m} k_r \mu(r) ,$$

where k_r is the number of multiples of r which are $\leq$ m. However, $k_r = [m/r]$ for $1 \leq r \leq m$. Thus,

$$\sum_{n=1}^{m} \mu(n) [m/n] = 1.$$

Consequently,

$$\left| m \sum_{n=1}^{m} \mu(n)/n - 1 \right| = \left| \sum_{n=1}^{m} \mu(n)m/n - \sum_{n=1}^{m} \mu(n) [m/n] \right|$$

$$= \left| \sum_{n=1}^{m} \mu(n) \left(\frac{m}{n} - \left[\frac{m}{n}\right]\right) \right| \leq \sum_{n=1}^{m} \left(\frac{m}{n} - \left[\frac{m}{n}\right]\right) \leq m-1 .$$

Thus

$$\left| m \sum_{n=1}^{m} \mu(n)/n \right| \leq 1 + m - 1 = m,$$

which proves the lemma.

<u>Lemma 2.7</u>. For every positive real number x,

$$\sum_{k \leq x} \mu(k)/k \log x/k = O(1),$$

where the sum extends over all positive integers $k \leq x$.

Proof: Applying Lemma 2.1, one finds that

$$\sum_{k \leq x} \frac{\mu(k)}{k} \log \frac{x}{k} = \sum_{k \leq x} \frac{\mu(k)}{k} \left(\sum_{n \leq x/k} 1/n - \gamma + \theta_1 \frac{k}{x} \right),$$

where $|\theta_1|$ is less than a positive constant c. For $kn = m$, $n = m/k$, a_k = a function of k, $b_n = b_{m/k}$ = a function of n, then

$$\sum_{k\le x} \frac{a_k}{k} \sum_{n\le x/k} \frac{b_n}{n} = \sum_{\substack{k\le x\\ n\le x/k}} \frac{a_k\, b_n}{kn} = \sum_{\substack{m\le x\\ kn=m}} \frac{a_k\, b_{m/k}}{m}$$

$$= \sum_{m\le x} \frac{1}{m} \sum_{d|m} a_d\, b_{m/d} \;;$$

so

$$\sum_{k\le x} \frac{\mu(k)}{k}\Big(\sum_{n\le x/k} 1/n - \gamma + \Theta_1 \frac{k}{x} \Big)$$

becomes

$$\sum_{m\le x} 1/m \sum_{d|m} \mu(d) - \gamma \sum_{k\le x} \mu(k)/k + \sum_{k\le x} \mu(k)\, \Theta_1/x \;,$$

where d runs through all positive divisors of m. By Theorem 2.4, the first sum has the value 1; and by Lemma 2.6, the second sum has an absolute value $\le \gamma$. The absolute value of the third sum is at most

$$c/x \sum_{k\le x} 1 \le c \,.$$

This gives the desired results.

Lemma 2.8. For every natural number n,

$$\sum_{d|n} \mu(d)\, \tau(n/d) = 1,$$

where the sum extends over all positive divisors d of n.

Proof: Since $\tau(n/k) = \sum_d 1$, the sum extending over all positive divisors d of n/k, then

$$\sum_k \mu(k)\, \tau(n/k) = \sum_k \mu(k) \sum_{d|k} 1 = \sum_d \sum_{d_1} \mu(d_1) \,,$$

where the inner sum on the right-hand side extends over all positive divisors d_1 of n/d. By Theorem 2.4, this inner sum is equal to zero when $d \ne n$ and equal to 1 when $d = n$. Thus the right-hand side is $= 1$.

Lemma 2.9. For every positive real number x,

$$(15) \qquad \sum_{d\le x} \mu(d)/d \ (\log x/d)^2 = 2 \log x + O(1),$$

where the sum extends over all positive integers $d \le x$.

Proof: Applying Lemma 2.3 for $y = x/d$, one may write the left-hand side of formula (15) in the form:

$$2 \sum_{d\le x} \frac{\mu(d)}{d} \left(\sum_{nd\le x} \frac{\tau(n)}{n} - 2\gamma \log \frac{x}{d} - \gamma^2 + 2c \right)$$

$$+ \sum_{d\le x} \frac{\mu(d)}{d} \left(\Theta(d/x)^{\frac{1}{2}} \log \frac{x}{d} \right),$$

where $|\Theta|$ is less than a positive constant c_1. For all sufficiently large x the absolute value of the last sum is smaller than

$$4c_1 \sum_{d\le x} \frac{1}{d} \left(\frac{d}{x}\right)^{\frac{1}{2}} \left(\frac{x}{d}\right)^{\frac{1}{4}} = x^{-\frac{1}{4}}\, O\left(\sum_{d\le x} d^{-3/4} \right)$$

$$= x^{-\frac{1}{4}}\, O\left(\int_1^x z^{-3/4}\, dz \right) = x^{-\frac{1}{4}}\, O(x^{\frac{1}{4}}) = O(1).$$

Further, by letting $k = nd$, one finds that

$$2 \sum_{d\le x} \frac{\mu(d)}{d} \sum_{k\le x} \frac{\tau(n)}{n} = 2 \sum_{k\le x} 1/k \sum_{d|k} \mu(d)\, \tau(k/d),$$

where the inner sum on the right-hand side extends over all positive divisors d of k. Hence, by means of Lemmas 2.8 and 2.1,

$$2 \sum_{d\le x} \frac{\mu(d)}{d} \sum_{k\le x} \frac{\tau(n)}{n} = 2 \log x + O(1).$$

Finally, applying Lemmas 2.7 and 2.6, one sees that the left-hand side of formula (15) is equal to

$$2 \log x + O(1).$$

Thus Lemma 2.9 is proved.

Lemma 2.10. If the sum is extended over all primes $p \le x$, then

$$(16) \qquad \sum_{p\le x} (\log p)(\log x/p) = o(x \log x).$$

Proof: If $y = x/(\log x)$, the sum on the left-hand side is equal to

$$\sum_{p\le y} (\log p)(\log x/p) + \sum_{p>y}^{p\le x} (\log p)(\log x/p)$$

$$< \log x \sum_{p\le y} \log p + (\log \log x) \sum_{p\le x} \log p$$

$$= \log x \; \vartheta(x/\log x) + (\log \log x)\, \vartheta(x).$$

Applying Theorem 2.2, one sees that this function has the order of magnitude

$$O(x \log \log x),$$

which is somewhat better than (16).

In the following lemma the expression "the sum extends over all prime powers $p^\alpha \le x$, where α is a natural number" occurs. This means that there is a term in the sum for each power p^α of a prime p for which $p^\alpha \le x$. For example, if α_p is the greatest integer such that $p^{\alpha_p} \le x$, then $\sum_{p^\alpha\le x} \log p$ is the same as $\sum_{p\le x} \alpha_p(\log p)$, or the same as

$$\sum_{p\le x} \log p + \sum_{p^2\le x} \log p + \sum_{p^3\le x} \log p + \cdots + \sum_{p^k\le x} \log p,$$

where k is the greatest integer such that $2^k \le x$, It follows that

$$\sum_{p^\alpha\le x} \log p = \sum_{r=1}^{k} \vartheta(\sqrt[r]{x}) .$$

Lemma 2.11. If the sum is extended over all prime powers $p^\alpha \le x$, where α is a natural number, then

$$(17) \qquad \sum_{p^\alpha\le x} \log p = O(x) .$$

Proof: The sum on the left-hand side is equal to

$$\vartheta(x) + \vartheta(\sqrt{x}) + \vartheta(\sqrt[3]{x}) + \cdots + \vartheta(\sqrt[k]{x}),$$

where k is the greatest integer such that $2^k \leq x$. This sum is at most equal to

$$\vartheta(x) + k\,\vartheta(\sqrt{x}).$$

From Theorem 2.2 and the fact that $k \leq (\log x)/\log 2$, the order of magnitude of $\vartheta(x) + k\,\vartheta(\sqrt{x})$ is

$$O(x) + \frac{\log x}{\log 2}\, O(\sqrt{x}) = O(x).$$

Lemma 2.12. If $f(n)$ is the function defined in Lemma 2.5, then

$$\sum_{n \leq x} f(n) = (\log x)\vartheta(x) + 2\sum_{p \leq \sqrt{x}} \vartheta(x/p)\log p + o(x \log x),$$

where the sum on the left-hand side extends over all positive integers $n \leq x$, and where the sum on the right-hand side extends over all primes $p \leq \sqrt{x}$.

Proof: It follows from Lemma 2.5 that

$$(18) \quad \sum_{n \leq x} f(n) = (\log x)^2 + \sum \Big(2(\log x)(\log p) - (\log p)^2\Big)$$
$$+ 2\sum (\log p)(\log q).$$

Here the first sum on the right-hand side extends over all prime powers $p^\alpha \leq x$, α being a natural number; the second sum on the right-hand side extends over all primes powers p^α and q^β such that $p^\alpha q^\beta \leq x$ and $p < q$, where α and β are natural numbers.

In the first sum on the right-hand side, the terms with $\alpha \geq 2$ are considered first. Let $g(x)$ denote the number of prime powers $p^\alpha \leq x$

with $\alpha \geq 2$, then the contribution of these terms to the sum is at most equal to

$$2(\log x)^2\, g(x) \;\leq\; 2(\log x)^2 (\sqrt{x} + \sqrt[3]{x} + \cdots + \sqrt[k]{x}),$$

where k is the greatest integer such that $2^k \leq x$. Thus the contribution does not exceed

$$2(\log x)^2\, k\sqrt{x} \;\leq\; 2(\log x)^2 \frac{\log x}{\log 2}\sqrt{x} \;=\; o(x \log x).$$

Consider next the terms with $\alpha = 1$ in the first sum on the right-hand side. The contribution of these terms is equal to

$$\begin{aligned}
\sum_{p\leq x} \Big(2(\log x)(\log p) - (\log p)^2\Big) &= \sum_{p\leq x} \Big(\log x \log p + (\log p)(\log x - \log p)\Big) \\
&= (\log x) \sum_{p\leq x} \log p + \sum_{p\leq x} (\log p)(\log x/p) \\
&= (\log x)\, \vartheta(x) + o(x \log x),
\end{aligned}$$

according to Lemma 2.10. Thus the first sum on the right-hand side in formula (18) is equal to

$$(\log x)\, \vartheta(x) + o(x \log x). \tag{19}$$

Finally, consider the second sum on the right-hand side. Applying Lemma 2.11 for x/q^β instead of for x, then one sees that the contribution of the terms with $\beta \geq 2$ and $\alpha \geq 1$ has the order of magnitude

$$\Sigma (\log q)\, O(x/q^\beta) = O(x)\, \Sigma (\log q)/q^\beta = O(x);$$

for the infinite series, $\sum_2^\infty (\log q)/q^\beta$, extending over all primes q is convergent for $\beta \geq 2$. Thus the second sum on the right-hand side is equal to

$$2\, \Sigma (\log p)(\log q) + O(x), \tag{20}$$

where the sum extends over all primes p and q, such that $pq \leq x$ and $p < q$. The latter sum is equal to

$$\sum_{pq\leq x} (\log p)(\log q) - \sum_{p\leq\sqrt{x}} (\log p)^2$$

$$= \sum_{\substack{p\leq\sqrt{x}\\ pq\leq x}} (\log p)(\log q) + \sum_{\substack{q\leq\sqrt{x}\\ pq\leq x}} (\log p)(\log q)$$

$$- \sum_{\substack{p\leq\sqrt{x}\\ q\leq\sqrt{x}}} (\log p)(\log q) - \sum_{p\leq\sqrt{x}} (\log p)^2 .$$

According to Theorem 2.2, the last two sums have at most the order of magnitude

$$\left(\vartheta(\sqrt{x})\right)^2 = O(x)$$

and

$$(\log\sqrt{x})\ \vartheta(\sqrt{x}) = O(\sqrt{x}\ \log\sqrt{x})$$

respectively. Hence one concludes that expression (20) is equal to

$$\sum_{p\leq\sqrt{x}} (\log p)\ \vartheta(x/p) + \sum_{q\leq\sqrt{x}} (\log q)\ \vartheta(x/p) + O(x)$$

$$= 2 \sum_{p\leq\sqrt{x}} (\log p)\ \vartheta(x/p) + O(x) .$$

Introducing expressions (19) and (20) into formula (18), one finally obtains

$$\sum_{n\leq x} f(n) = (\log x)^2 + (\log x)\ \vartheta(x) +$$

$$o(x \log x) + 2 \sum_{p\leq\sqrt{x}} (\log p)\ \vartheta(x/p) + O(x)$$

$$= o(x \log x) + (\log x)\ \vartheta(x) + o(x \log x) +$$

$$2 \sum_{p\leq\sqrt{x}} (\log p)\ \vartheta(x/p) + o(x \log x)$$

$$= (\log x)\vartheta(x) + 2 \sum_{p\leq\sqrt{x}} (\log p)\ \vartheta(x/p) + o(x \log x).$$

This concludes the proof of Lemma 2.12.

The proof of Selberg's basic formula can now be given.

Theorem 2.5. If the sum is extended over all primes $p \le \sqrt{x}$, then

$$\vartheta(x)(\log x) + 2 \sum_{p\le\sqrt{x}} \vartheta(x/p)(\log p) - 2x \log x = o(x \log x).$$

Proof: According to Lemma 2.12, the left-hand side is equal to

$$\sum_{n\le x} f(n) - 2x \log x + o(x \log x).$$

According to the definition of the function $f(n)$,

$$\sum_{n\le x} f(n) = \sum_{n\le x} \sum_{d|n} \lambda(d) ,$$

where the inner sum extends over all positive divisors d of n. Hence

$$\sum_{n\le x} f(n) = \sum_{d\le x} \lambda(d) [x/d] = \sum_{d\le x} \lambda(d)(x/d - \epsilon d),$$

where $0 \le \epsilon d < 1$. If $z = x/(\log x)^2$, then

$$\sum_{d\le x} \left| \lambda(d)\right| \le \sum_{d\le x} (\log x/d)^2 = \sum_{d\le z} (\log x/d)^2 + \sum_{d>z}^{d\le x} (\log x/d)^2$$

$$\le z(\log x)^2 + \sum_{d>z}^{d\le x} \left(\log \left(\frac{x(\log x)^2}{x}\right) \right)^2$$

$$= z(\log x)^2 + \sum_{d>z}^{d\le x} (2 \log \log x)^2$$

$$\le z(\log x)^2 + 4x (\log \log x)^2$$

$$= O(x) + O\left(x(\log \log x)^2 \right) = o(x \log x).$$

Hence

$$\sum_{n\le x} f(n) = \sum_{d\le x} \lambda(d)\, x/d + o(x \log x)$$

$$= \sum_{d\le x} x\, \mu(d)/d \ (\log x/d)^2 + o(x \log x),$$

and, by Lemma 2.9,

$$\sum_{n \leq x} f(n) = 2x \log x + o(x \log x).$$

This completes the proof of Selberg's basic formula.

CHAPTER III

THE PRIME NUMBER THEOREM

The prime number theorem is usually stated as

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\log x} = 1,$$

it may also be stated in other forms. In particular, it is equivalent to the proposition

$$\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1.$$

In this chapter the equivalence of the two propositions above will be proved, and then the second relationship will be established. Although modified in parts, the proofs given are from Nagell [6] .

Theorem 3.1. The prime number theorem

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\log x} = 1$$

is equivalent to the theorem

$$\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1 .$$

Proof: By definition,

$$\vartheta(x) = \sum_{p \leq x} \log p ,$$

and

$$\pi(x) = \sum_{p \leq x} 1 .$$

So

$$\vartheta(x) \leq \pi(x) \log x .$$

If $3 < y \le x$, then

$$\pi(x) - y \le \pi(x) - \pi(y) = \sum_{y<p\le x} 1 \le \sum_{y<p\le x} \frac{\log p}{\log y}$$

$$= \frac{1}{\log y}\left(\sum_{p\le x} \log p - \sum_{p\le y} \log p\right) = \frac{\vartheta(x) - \vartheta(y)}{\log y} \le \frac{\vartheta(x)}{\log y}$$

Hence,

$$\vartheta(x) \le \pi(x) \log x = y \log x + \log x\,(\pi(x) - y)$$

$$\le y \log x + \frac{\log x}{\log y}\,\vartheta(x) .$$

by choosing $y = x^{\delta}$, where $\delta = 1 - \frac{1}{\log \log x}$, one obtains

$$0 \le \frac{\pi(x)\log x}{\vartheta(x)} - 1 \le \frac{y \log x}{\vartheta(x)} + \frac{\log x}{\delta \log x} - 1$$

$$= \frac{y \log x}{\vartheta(x)} + \frac{1-\delta}{\delta} .$$

Since, according to Theorem 2.2, $\vartheta(x) > bx$, then

$$0 \le \frac{\pi(x)\log x}{\vartheta(x)} - 1 < \frac{x^{\delta}\log x}{bx} + \frac{1}{\log \log x - 1} .$$

Here the right-hand side tends to zero for increasing x; consequently,

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{\vartheta(x)} = 1 ,$$

which proves the theorem.

The proof of the prime number theorem will now be given, stated in the following form:

$$(1) \qquad \lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1.$$

It follows from Theorem 2.2 that for increasing x the quotient $\vartheta(x)/x$ has a positive lower limit a and an upper limit A; then $0 < a \le A$.

Thus to prove (1), one needs to show that

$$(2) \qquad a = A = 1 .$$

The proof is based on Selberg's formula which was proved in the preceding chapter. The formula is used in the form:

$$(3) \qquad \vartheta(x)/x + 2/(x \log x) \sum_{p \leq \sqrt{x}} \vartheta(x/p) \log p - 2 = o(1) ,$$

where the sum extends over all primes $p \leq \sqrt{x}$. The following formula, which was proved in Chapter II, is also useful in the proof:

$$(4) \qquad \sum_{p \leq x} (\log p)/p = \log x + O(1) ,$$

where the sum extends over all primes $p \leq x$.

Lemma 3.1. If $\limsup_{x \to \infty} \vartheta(x)/x = A$ and if $\liminf_{x \to \infty} \vartheta(x)/x = a$, then

$$(5) \qquad a + A = 2.$$

Proof: It is possible to let x tend to infinity in such a way that $\vartheta(x)/x$ tends to A. Let ϵ be a given positive number; then

$$\vartheta(x/p) > (a - \epsilon)\, x/p$$

for every x sufficiently large and for every prime $p \leq \sqrt{x}$, and, therefore,

$$2/(x \log x) \sum_{p \leq \sqrt{x}} \vartheta(x/p) \log p \geq 2/(x \log x) \sum_{p \leq \sqrt{x}} (a-\epsilon)(x/p) \log p$$

$$= 2(a-\epsilon)/\log x \sum_{p \leq \sqrt{x}} (\log p)/p .$$

It follows from (4) that the right-hand side of this inequality tends to $a - \epsilon$ when $x \to \infty$. If formula (3) is applied, one obtains $2 - A \geq a - \epsilon$. The inequality holds for every positive ϵ; consequently,

$$(6) \qquad A + a \leq 2.$$

On the other hand, it is possible to let x tend to infinity in such a way that $\vartheta(x)$ tends to a. If ϵ is a given positive number, then

$$\vartheta(x/p) < (A + \epsilon)(x/p)$$

for every x sufficiently large and for every prime $p \le \sqrt{x}$, and, therefore,

$$2/(x \log x) \sum_{p \le \sqrt{x}} \vartheta(x/p)\log p \le 2/(x \log x) \sum_{p \le \sqrt{x}} (A + \epsilon)(x/p) \log p$$

$$= 2(A + \epsilon)/\log x \sum_{p \le \sqrt{x}} (\log p)/p.$$

It follows from (4) that the right-hand side of this inequality tends to $A + \epsilon$ when $x \to \infty$. If formula (3) is applied, then one obtains $2 - a \le A + \epsilon$. The inequality holds for every positive ϵ; therefore,

$$A + a \ge 2.$$

This inequality together with (6) leads to (5).

In the following proof, let the variable x tend to infinity in such a manner that $\vartheta(x)/x$ tends to A.

<u>Lemma 3.2</u>. If λ is a given number $> a$, and if the sum,

$$S(x) = \Sigma' (\log p)/p ,$$

extends over all primes $p \le x$ and such that $\vartheta(x/p) \ge \lambda x/p$, then the quotient $S(x)/(\log x)$ tends to zero for $x \to \infty$.

Proof: Since $\vartheta(x/p) = \sum_{pq \le x} \log q$, then

$$\sum_{p \le x} \vartheta(x/p) \log p = \sum_{p \le x} \log p \sum_{pq \le x} \log q =$$

$$\sum_{p \le \sqrt{x}} \log p \sum_{pq \le x} \log q + \sum_{q \le \sqrt{x}} \log q \sum_{pq \le x} \log p - \sum_{p \le \sqrt{x}} \log p \sum_{q \le \sqrt{x}} \log q$$

$$= \sum_{p \le \sqrt{x}} \vartheta(x/p) \log p + \sum_{q \le \sqrt{x}} \vartheta(x/p) \log q - \Big(\sum_{p \le \sqrt{x}} \log p\Big)^2$$

$$= 2 \sum_{p \le \sqrt{x}} \vartheta(x/p) \log p - \big(\vartheta(\sqrt{x})\big)^2.$$

Since, by Theorem 2.2 the last term has the order $O(x)$, then Selberg's formula (3) may be written as:

$$(7)\quad \vartheta(x)/x + 1/(x \log x) \sum_{p\le x} \vartheta(x/p) \log p - 2 = o(1).$$

Let ϵ be a positive number. For every x/p exceeding a certain number u which depends on ϵ, then $\vartheta(x/p) > (a - \epsilon)(x/p)$. There exists a positive number b depending on u and so on ϵ such that

$$(8)\qquad \vartheta(x/p) > (a - \epsilon)(x/p) - b$$

for all the primes p such that $x/p \le u$. Thus the latter inequality holds for every $p \le x$.

If the sum Σ' extends over all primes $p \le x$ such that $\vartheta(x/p) \ge \lambda x/p$, then

$$(9)\quad \Sigma' \vartheta(x/p) \log p \ge \lambda x \Sigma' (\log p)/p >$$

$$(\lambda - a)\, x\, \Sigma' (\log p)/p + (a - \epsilon)\, x\, \Sigma' (\log p)/p.$$

If the sum Σ'' extends over all primes $p \le x$ such that $\vartheta(x/p) < \lambda x/p$, then by (8),

$$(10)\qquad \Sigma'' \vartheta(x/p) \log p > (a - \epsilon)\, x\, \Sigma'' (\log p)/p - b\,\vartheta(x)$$

From (9) and (10) one may deduce that

$$\sum_{p\le x} \vartheta(x/p) \log p = \Sigma' \vartheta(x/p) \log p + \Sigma'' \vartheta(x/p) \log p >$$

$$(a - \epsilon)x \sum_{p\le x} (\log p)/p + (\lambda - a)x\, \Sigma' (\log p)/p - b\vartheta(x).$$

Substituting from this result into formula (7), one obtains for x tending to infinity in such a manner that $\vartheta(x)/x$ tends to A,

$$A + a - \epsilon + (\lambda - a) \limsup_{x\to\infty} \frac{\Sigma' (\log p)/p}{\log x} \le 2.$$

Hence, recalling $a + A = 2$,

$$\limsup_{x\to\infty} \frac{\Sigma'(\log p)/p}{\log x} \leq \frac{\epsilon}{\lambda - a} .$$

Since $\lambda - a > 0$, and since ϵ can be chosen arbitrarily small, this gives the desired lemma.

Lemma 3.3. If μ is a given positive number $< A$, and if the sum,

$$R(x) = \Sigma' \left(\frac{\log p}{p}\right)\left(\frac{\log q}{q}\right) ,$$

extends over all the primes p and q which satisfy the following conditions: $p \leq \sqrt{x}$, $q \leq \sqrt{(x/p)}$, and $\vartheta(s/pq) \leq \mu x/pq$, then the quotient

$$\frac{R(x)}{(\log x)^2}$$

tends to zero for $x \to \infty$.

Proof: Replacing x by x/p in Selberg's formula one obtains

$$\vartheta(x/p) = 2x/p + o(x/p) - 2/(\log x/p) \sum_{q\leq\sqrt{x/p}} \vartheta(x/pq) \log q .$$

If this expression for $\vartheta(x/p)$ is introduced into the same formula, then one obtains

$$\vartheta(x) = 2x + o(x) -$$

$$\frac{2}{\log x} \sum_{p\leq\sqrt{x}} \log p \left(\frac{2x}{p} + o(x/p) - \frac{2}{\log x/p} \sum_{q\leq\sqrt{x/p}} \vartheta(x/pq) \log q \right)$$

$$= 2x + o(x) - \frac{2x}{\log x}\sum_{p\leq\sqrt{x}} \frac{\log p}{p}\Big(2 + o(1)\Big) + \frac{4V}{\log x},$$

where

$$V = \sum_{p,q} \vartheta(x/pq) \frac{\log p \log q}{\log x/p} ,$$

the sum extending over all primes p and q such that $p \leq \sqrt{x}$, $q \leq \sqrt{x/p}$.

Since, by formula (4),

$$\sum_{p \leq \sqrt{x}} \log p/p = \tfrac{1}{2} \log x + O(1),$$

it follows that

$$\vartheta(x) = 4V/\log x + o(x).$$

In every term of the sum V, $p \leq \sqrt{x}$ and $q \leq \sqrt{x/p}$, and, therefore, $pq = p^{\frac{1}{2}}(pq^2)^{\frac{1}{2}} \leq x^{\frac{1}{4}}x^{\frac{1}{2}} = x^{3/4}$. Thus, if δ is any positive number, then

$$\vartheta(x/pq) < (A + \delta)(x/pq)$$

for x sufficiently large.

Let

$$V = \Sigma' + \Sigma'' ,$$

where the first sum extends over all primes p and q such that $p \leq \sqrt{x}$, $q \leq \sqrt{x/p}$, and $\vartheta(x/pq) \leq \mu x/pq$, and the second sum extends over all primes p and q such that $p \leq \sqrt{x}$, $q \leq \sqrt{x/p}$, and $\vartheta(x/pq) > \mu x/pq$. Then

$$V \leq \mu x \Sigma' \frac{1}{\log x/p} \frac{\log p}{p} \frac{\log q}{q} +$$

$$(A + \delta)x \Sigma'' \frac{1}{\log x/p} \frac{\log p}{p} \frac{\log q}{q} ,$$

where the sums are taken as above. Further,

$$V \leq (A + \delta) x W - (A + \delta - \mu)x \Sigma' \frac{1}{\log x/p} \frac{\log p}{p} \frac{\log q}{q} ,$$

where

$$W = \sum_{p,q} \frac{1}{\log x/p} \frac{\log p}{p} \frac{\log q}{q}$$

$$= \sum_{p \leq \sqrt{x}} \frac{1}{\log x/p} \frac{\log p}{p} \sum_{q \leq \sqrt{x/p}} \frac{\log q}{q} ,$$

the sums extending over all primes $p \leq \sqrt{x}$ and over all primes $q \leq \sqrt{x/p}$.

Applying formula (4), one obtains

$$W = \sum_{p \le \sqrt{x}} \frac{\log p}{p} \frac{1}{\log x/p} \left(\tfrac{1}{2} \log x/p + O(1)\right)$$

$$= \sum_{p \le \sqrt{x}} \frac{\log p}{p} \left(\tfrac{1}{2} + o(1)\right) = \tfrac{1}{4} \log x + o(\log x).$$

Hence $\vartheta(x) \le$

$$(A + \delta)x - \frac{4}{\log x} (A + \delta - \mu) x \Sigma' \frac{1}{\log x/p} \frac{\log p}{p} \frac{\log q}{q} + \beta x,$$

where β tends to zero for $x \to \infty$. From this, one deduces that

$$\frac{4}{(\log x)^2} (A + \epsilon - \mu) \Sigma' \frac{\log p}{p} \frac{\log q}{q} \le A + \delta - \frac{\vartheta(x)}{x} + \beta ,$$

where the sum is the same as in Lemma 3.3. Hence, for x tending to infinity in such a manner that $\vartheta(x)/x$ tends to A,

$$\limsup_{x \to \infty} \frac{1}{(\log x)^2} \Sigma' \frac{\log p}{p} \frac{\log q}{q} \le \frac{\delta}{4(A-u)} .$$

Since $A - \mu > 0$, and since δ is an arbitrary positive number, this proves the lemma.

By means of Lemmas 3.1, 3.2, and 3.3, one may now prove the following relation:

(11) $$a = A = 1.$$

Suppose that $A > a$, then there exists a positive number $\sigma > 1$ such that $A > \sigma a$ and a positive number δ such that

(12) $$A \ge \sigma a + \delta(\sigma + 2).$$

Further, denote by N a natural number.

Consider the sum

$$S = \Sigma_2 \frac{\log p}{p} \frac{\log q}{q} \Sigma_3 \frac{\log r}{r}$$

where Σ_2 extends over all the primes p and q such that

$$p \leq \sqrt{x}, \quad q \leq \sqrt{x/p}, \quad pq \geq N, \quad \vartheta(x/pq) \geq (A - \delta)\, x/pq ,$$

and where Σ_3 extends over all the primes r such that

$$pq/\sigma < r \leq \sigma\, pq.$$

If there are no primes r, the sum Σ_3 is $= 0$.

For every term in the sum Σ_3, then

$$r \leq \sigma pq = \sigma p^{\frac{1}{2}} (pq^2)^{\frac{1}{2}} \leq \sigma x^{\frac{1}{4}} x^{\frac{1}{2}} = \sigma x^{3/4} \leq x$$

when x is sufficiently large. The following inequality will be proved for the same terms:

$$(13) \qquad \vartheta(x/r) > (a + \delta)\, x/r ,$$

when x is sufficiently large. This inequality is true for all $r \leq pq$, since

$$\vartheta(x/r) \geq \vartheta(x/pq) \geq (A - \delta)\, x/pq > (A - \delta)\, x/\sigma r$$
$$\geq (\sigma a + \sigma\delta)\, x/\sigma r = (a + \delta)\, x/r$$

in virtue of (12).

Consider now the terms with $r > pq$. For these terms, $x/r < x/pq \leq \sigma x/r$. If in Selberg's formula,

$$(\log y)\, \vartheta(y) + 2 \sum_{p \leq \sqrt{y}} \vartheta(y/p) \log p = 2\, y \log y + o(y \log y),$$

one first replaces y by x/pq and then by x/r, one obtains on subtraction

$$(\log x/pq)\, \vartheta(x/pq) - (\log x/r)\, \vartheta(x/r) \leq$$
$$2\,(x/pq)\,(\log x/pq) - (2\, x/r)(\log x/r) + o\left(\frac{x}{r} \log x/r\right).$$

or $\vartheta(X/r) \geq$

$$\frac{\log x/pq}{\log x/r}\, \vartheta(x/pq) - 2\left(\frac{x}{pq} - \frac{x}{r}\right) - 2x/pq\, \frac{\log x/pq - \log x/r}{\log x/r} + o(x/r).$$

Now

$$2x/pq \; \frac{\log x/pq - \log x/r}{\log x/r} = 2x/pq \; \frac{\log r/pq}{\log x/r}$$

$$\leq 2\sigma x/r \; \frac{\log \sigma}{\log x/r} = o(x/r) ,$$

and

$$\frac{\log x/pq}{\log x/r} \geq 1.$$

Thus

$$\vartheta(x/r) > (A - \delta)(x/pq) - 2(x/pq - x/r) + o(x/r)$$

$$= 2x/r - (2 - A + \delta)(x/pq) + o(x/r) ;$$

and, since $a + A = 2$ and $A \geq a\sigma + \sigma\delta + 2\delta$,

$$\vartheta(x/r) > (a + A)(x/r) - (a + \delta)(x/pq) + o(x/r)$$

$$\geq (a + a\sigma + \sigma\delta + 2\delta)\, x/r - (a + \delta)\; \sigma x/r + o(x/r)$$

$$\geq (a + 2\delta)\, x/r + o(x/r).$$

If x is sufficiently large, then

$$\vartheta(x/r) > (a + \delta)(x/r) .$$

This proves inequality (13) for all r.

Consequently,

$$S \leq \sum_r \frac{\log r}{r} \; \Sigma^3 \; \frac{\log p}{p} \; \frac{\log q}{q} ,$$

where the first sum extends over all primes $r \leq x$ and such that $\vartheta(x/r) \geq (a + \delta)(x/r)$, and where the sum Σ^3 extends over all primes p and q such that

$$p \leq \sqrt{x}, \quad q \leq \sqrt{x/p}, \quad r/\sigma \leq pq < \sigma r.$$

Thus

$$\Sigma^3 \frac{\log p}{p} \frac{\log q}{q} \leq \frac{\sigma}{r} \sum_{p \leq \sqrt{x}} \log p \sum_{q \leq \sigma r/p} \log q =$$

$$\sigma/r \sum_{p \leq \sqrt{x}} (\log p)\, \vartheta(\sigma r/p) < c_1 \sum_{p \leq \sqrt{x}} (\log p)/p < c_2 \log x,$$

where c_1 and c_2 are positive constants. Consequently,

$$S \leq c_2 \log x \sum_{r} (\log r)/r ,$$

where the sum extends over all the primes $r \leq x$ and such that $\vartheta(x/r) \geq (a + \delta)(x/r)$. By Lemma 3.2 one then obtains

$$(14) \qquad S = \beta_1 (\log x)^2 ,$$

where β_1 tends to zero for $x \to \infty$.

Now consider the sum

$$T = \Sigma \frac{\log p}{p} \frac{\log q}{q} ,$$

extending over all primes p and q such that

$$p \leq \sqrt{x}, \quad q \leq \sqrt{x/p}, \quad pq \geq N.$$

Then

$$T \geq \left(\sum_{p \geq \sqrt{N}}^{p \leq \sqrt{x}} \frac{\log p}{p} \right) \left(\sum_{q \geq \sqrt{N}}^{q \leq x^{\frac{1}{4}}} \frac{\log q}{q} \right) .$$

Hence by formula (4)

$$T \geq \left(\tfrac{1}{2} \log x - \tfrac{1}{2} \log \sqrt{N} + o(1) \right) \left(\tfrac{1}{4} \log x - \tfrac{1}{2} \log \sqrt{N} + o(1) \right) ,$$

Therefore,

$$(15) \qquad T > c_3 (\log x)^2,$$

where c_3 is a positive constant.

Let

$$T = \Sigma_2 \frac{\log p}{p} \frac{\log q}{q} + \Sigma_2' \frac{\log p}{p} \frac{\log q}{q} ,$$

where the latter sum extends over all the primes p and q satisfying the conditions

$$p \le \sqrt{x}, \quad q \le \sqrt{x/p}, \quad pq \ge N, \quad \vartheta(x/pq) < (A - \delta)(x/pq).$$

This latter sum, in virtue of Lemma 3.3., is equal to $\beta_2(\log x)^2$, where β_2 tends to zero for $x \to \infty$. Hence,

$$\Sigma_2 \frac{\log p}{p} \frac{\log q}{q} = T - \beta_2 (\log x)^2 ,$$

and in virtue of inequality (15),

$$(16) \qquad \Sigma_2 \frac{\log p}{p} \frac{\log q}{q} > \tfrac{1}{2} c_3 (\log x)^2$$

for x sufficiently large.

If in the sum Σ_2, one considers for a fixed value of x, the primes p and q shich have the property that the sum $\Sigma_3 \frac{\log r}{r}$ takes its minimum value μ, μ depends on x only,

Then by (16)

$$S \ge \mu \Sigma_2 \frac{\log p}{p} \frac{\log q}{q} > \tfrac{1}{2} \mu c_3 (\log x)^2 .$$

If one compares this result with inequality (14), one obtains for $x \to \infty$ that

$$\mu = \Sigma_3 \frac{\log r}{r} \to 0.$$

Consequently, to every positive number ϵ and to every natural number N there corresponds a number $t = pq \ge N$ such that

$$\sum_{r\sigma > t}^{r \le \sigma t} \frac{\log r}{r} < \epsilon ,$$

the sum extending over all primes r such that $r > t/\sigma$ and $r \leq \sigma t$.

Hence

$$\sum_{r\sigma > t}^{r \leq \sigma t} \log r < \epsilon \sigma t,$$

and

$$\vartheta(\sigma t) - \vartheta(t/\sigma) < \epsilon \sigma t. \tag{17}$$

If N, and therefore also t, are sufficiently large, then

$$\vartheta(\sigma t) > (a - \epsilon) \sigma t ,$$

and

$$\vartheta(t/\sigma) < (A + \epsilon)(t/\sigma) .$$

Hence, it follows from (17) that

$$(a - \epsilon)\sigma - \frac{A + \epsilon}{\sigma} < \epsilon\sigma .$$

This inequality holds for every positive number ϵ so that one obtains $a\sigma^2 - A \leq 0$. On the other hand, $a\sigma < A$ and $a > 0$.

Thus, every number $\sigma < A/a$ has the property that $\sigma^2 \leq A/a$. If σ tends to A/a, then

$$\left(\frac{A}{a}\right)^2 \leq \frac{A}{a}$$

or

$$\frac{A}{a} \leq 1 .$$

Since $a \leq A$ and $a + A = 2$, it follows that

$$a = A = 1.$$

CHAPTER IV

REMARKS ON THE "ELEMENTARY" PROOFS

The "elementary" proofs of the prime number theorem emphasize once again man's enduring progress in solving problems that seem to have no solution. The telephone, the radio, the television, the airplane, etc. are some of the results of man's success in solving other problems which seemed impossible. As late as 1932, Ingham [4], in discussing the analytic proofs, made the following statement: "The solution just outlined may be held to be unsatisfactory in that it introduces ideas very remote from the original problem, and it is natural to ask for a proof of the prime number theorem not depending on the theory of functions of a complex variable. To this we must reply that at present no such proof is known. We can indeed go further and say that it seems unlikely that a genuinely 'real variable' proof will be discovered, at any rate, so long as the theory is founded on Euler's identity." Thus, it is all the more remarkable that in 1948, Selberg and Erdos were able to give "elementary" proofs of the prime number theorem.

The basic new thing in the "elementary" proof of the prime number theorem is Selberg's asymptotic formula (Chapter II). From this formula evolve several ways to deduce the prime number theorem. The proofs given in Chapter II and Chapter III were based on the outline and discussions of Nagell. The first proof of Selberg dates from 1948. Nagell's proof is related to it; the proof followed an exposition given by van der Carput and based on notes of some lectures held by Erdos in 1948 [6].

In his published paper, Selberg presents another proof. He selected the method of proof because it avoided the concept of lower and upper limits and thus seemed to be the most elementary way. In his introduction he gave a sketch of his first proof in which he made use of the following result by P. Erdos, that for an arbitrary positive fixed number δ, there exist a $K(\delta) > 0$ and an $X_0 = X_0(\delta)$ such that for $X > X_0$, there are more than $K(\delta)\, x/\log x$ primes in the interval from x to $x + \delta x$. Selberg states that the proof of Erdos was obtained without knowledge of his work, except that it is based on his (Selberg's) formula; and after he (Selberg) had the other parts of the proof. He also states that Erdos' proof contains ideas related to his (Selberg's) proof, at which ideas he (Erdos) had arrived independently [10].

In the introduction of his paper, Erdos [2] gives a very interesting account of the prime number theorem. It gives insight into the methods used by other men in this field. The exchange of ideas, as illustrated in his paper, is beneficial in the solving of many problems.

In the second part of his paper, Selberg proves the basic formula. After proving it, he proceeded by writing the basic formula in another form. He introduced a term $R(x)$, by writing $\vartheta(x) = x + R(x)$. In the third part of his paper, he continued by discussing some of the properties of $R(x)$. And in the final part, he proves that

$$\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1$$

by proving

$$\lim_{x \to \infty} \frac{R(x)}{x} = 0 .$$

Since the proofs by Selberg and Erdos, other mathematicians have presented modifications. E. M. Wright [11] has given such a proof in which he uses the elements of calculus to present some of the concepts. He points out that "the use of the calculus makes unnecessary certain complexities of detail and shows more clearly the fundamental ideas (which are, of course, essentially Selberg's)." Robert Breusch [1] proves that for every $\epsilon > 0$, then

$$\Psi(x) = \sum_{p^m \leq x} \log p = x + o(x \log^{-1/6 + \epsilon} x),$$

and thus establishes the proof of the prime number theorem in this form. Other mathematicians have also given "elementary" proofs of the prime number theorem. The details of proofs in languages other than English were not entirely comprehensible by this author. It seems, however, that the other proofs involve the basic ideas of Erdos and Selberg or a variation or simplification of such concepts. Each author has, of course, presented his proof by his own method. Thus Erdos and Selberg have ushered in a new era in the proof of the prime number theorem.

BIBLIOGRAPHY

1. Breusch, Robert. "An Elementary Proof of the Prime Number Theorem With Remainder Term," Pacific Journal of Mathematics, X (1960), 487-497.

2. Erdos, P. "On a New Method in Elementary Number Theory Which Leads to an Elementary Proof of the Prime Number Theorem," Proceedings of National Academy of Science of U.S.A., XXXV (1949), 374-384.

3. Errera, A. "Une Modification de la Demonstration de Landau du Theorem des Nombres Premiers," Mathesis, LXVII (1958), 321-337.

4. Ingham. The Distribution of Prime Numbers (Cambridge Tracts in Mathematics and Mathematical Physics, Number 30), London: Cambridge University Press, 1932.

5. LeVeque, William Judson. Topics in Number Theory. II, Reading, Massachusetts: Addison-Wesley Publishing Co., 1956.

6. Nagell, Trygue. Introduction to Number Theory. New York: John Wiley and Sons, 1951.

7. Niven, Ivan and Herbert S. Zuckerman. An Introduction to the Theory of Numbers. New York: John Wiley and Sons, 1960.

8. Ore, Oystein. Number Theory and Its History. New York: McGraw-Hill Book Co., 1948.

9. Riemann, Beruhard. Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse. Monatsberichte der Berliner Akademic, 1859 (this is found in The Collected Works of Bernhard Riemann, edited by Heinrich Weber, New York: Dover Publications, 1959).

10. Selberg, Atle. "An Elementary Proof of the Prime Number Theorem," Annals of Mathematics, L (1949), 305-313.

11. Wright, E. M. "The Elementary Proof of the Prime Number Theorem," Proceedings of the Royal Society of Edinburg, Section A, LXIII (1952), 257-267.

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