BIVARIATE WARNING-TIME/FAILURE-TIME DISTRIBUTIONS

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PREFACE

In reliability studies, components (or systems of components) are observed to fail randomly in time. As these components fail, they are either repaired or replaced and are then allowed to operate until another failure occurs. If we note the times between successive failure and construct a frequency histogram of these failure-times, we find that a large number of families of probability distributions may then be invoked to describe the frequency histogram. Often a location parameter is used in indexing such distributions. For the most part we shall concern ourselves with probability density functions defined over the entire positive half of the real axis.

If one has under study a given component and desires to ascertain certain of its life-parameters (such as expected life), he may find that the time required for a sample of n components to fail may be excessively long. In such cases, alternative approaches to life-testing are considered; the more profitable alternatives have been two in number: (1) to place the components in a stressed environment, or (2) to make inferences regarding the life-parameters even though only r ($\leq n$) components have failed. The first alternative requires that one possess some knowledge of the relationship between failure-times arising under a stressed environment and those arising under the components' operating environment; the second alternative has been relatively successful whenever the failure-times are assumed to possess an exponential distribution or a Gamma distribution.

In this dissertation we shall discuss a third alternative, applicable to components whose failure-times under normal operating conditions, are distributed according to the generalized gamma distribution. We suppose that, under these same operating conditions, there exists a specific, observable parameter which, for each component, occurs at some (random) time, X, prior to the failure-time, Y, of the component. By specifying a (marginal) distribution of failure-times and relating the warning-times (X) to these failure-times by means of a conditional distribution (of X given Y), one may designate a bivariate probability density function. The warning-times then possess a marginal distribution which is indexed by the parameters both of the failure-time density and of the conditional density.

We shall focus our attention on the family of bivariate distributions whose marginal distribution of failure-times (Y) is the generalized Gamma distribution and whose conditional distribution of X given Y is the Beta distribution. We devote two chapters to the generalized Gamma distribution and to the estimation of its parameters before introducing a further generalization of the Gamma distribution. This latter generalization arises in the next two chapters whenever the aforementioned conditional distribution is assumed to be of the form $p\,x^{p-1}/y^{p}$.

These many generalizations provide, of course, results applicable to less general situations. Since the Weibull distribution, a

special case of the generalized Gamma distribution, has been widely acknowledged in reliability studies, we devote a pair of concluding chapters to our generalizations' implications upon it.

The impetus for undertaking this study can be attributed to a number of persons. Dr. E. W. Stacy, of the IBM Systems Development Division, introduced me to his generalization of the Gamma distribution. Professor H. O. Lancaster, in his lectures at the University of Sydney, provided the motivation for investigating bivariate distributions. Dr. Robert A. Hultquist, whose interests in bivariate distributions have influenced the direction and scope of this study, has been a most patient adviser while reviewing those results, too often hastily presented, which were developed during my year as a Fulbright scholar at the University of Sydney. The many persons, in the Departments of Mathematics and of Mathematical Statistics at the University of Sydney and in the Department of Mathematics and Statistics at Oklahoma State University, who have advised, bolstered, and consoled me during the trials, troubles, and tribulations associated with this research, deserve acknowledgment, but are unfortunately far too many in number to enumerate. Hopefully none will feel offended by this collective expression of my appreciation. In conclusion, acknowledgment of the National Science Foundation's Co-operative Fellowships, under the auspices of which this research was initiated, should be made.

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CHAPTER I

STACY'S GENERALIZED GAMMA DISTRIBUTION

In this chapter we shall present certain fundamental properties of Stacy's three-parameter gamma distribution. After indicating the historical development of the distribution, we define its parameters and examine the effects of assigning certain values to these. We derive an associated distribution, a generalized double exponential distribution. We then examine some of the more interesting properties of these two probability density functions.

1. Introduction

A quite general, three-parameter, univariate probability density function was suggested in September, 1962, by Dr. E. W. Stacy [1]. This distribution was defined, for a positive random variable, Y, as

$$f(y; a, d, p) = pa^{-d} y^{d-1} e^{-(y/a)^p} / \Gamma(\frac{d}{p}) , \qquad (1.1)$$

where each of the parameters a, d, and p is taken to be positive, and where $\Gamma(p)$ is the standard (complete) gamma function given by

$$\Gamma(p) = \int_0^\infty u^{p-1} e^{-u} du.$$

However, a subsequent joint effort by this author and Stacy [2] led to a generalization of this density, one which can be shown to have certain advantages, especially in avoiding difficulties encountered in the estimation of the three parameters. For a positive random variable, Y, we shall consider then

$$h(y; a, b, c) = |c| a^{-bc} y^{bc-1} e^{-(y/a)^c} / \Gamma(b) , \qquad (1.2)$$

for positive parameters a and b and for real parameter c.

We shall be closely concerned throughout this dissertation with density (1.2); and, whenever a random variable Y is distributed according to this density, we shall denote the relationship by the conventional expression

$$Y \sim h(y; a, b, c),$$

and shall refer to this density as the Stacy distribution. On occasion we may write

$$Y \sim h(\cdot ; a, b, c),$$

the dot indicating that the argument which should stand in its place is understood to be y.

2. Parameter Definitions

If one examines the (Stieltjes) probability element associated with density (1.2), the arrangement

$$dH(y) = \left\{ |c| \left(\frac{y}{a}\right)^{bc-1} e^{-(y/a)^c} / \Gamma(b) \quad \left(\frac{dy}{a}\right) \right\} \qquad (1.3)$$

indicates clearly that the parameter a has the effect of scaling the random variable Y. Thus, we refer to this parameter (a) as the scaling parameter.

We refer to the remaining parameters, b and c, as the shaping and exponentiating parameters, respectively. The former (b) receives its designation because its sufficient statistic [See Chapter I, Section 5] is the same as that for the shape parameter of the less general two-parameter Gamma distribution [See Chapter I, Section 3]; however, its role in actually shaping the curve associated with equation (1.2) is one shared frequently with the latter parameter (c) in the product bc. For, $0 < bc < 1$ implies that the curve approaches the ordinate axis asymptotically; $bc = 1$ implies a finite, non-zero ordinate on the ordinate axis; and, $bc > 1$ provides a curve which begins at the origin. Reference to the final parameter, c, as the exponentiating parameter is due to its predominating position in equation (1.2).

3. Special Cases of the Stacy Distribution

In addition to being a valid probability density function on its own merits, the Stacy distribution includes, as special cases, a number of more common probability distributions. We first note the case when $c = 0$. In order for the density (1.2) to have meaning, we must assume that this case represents the degenerate distribution with mean zero.

Whenever we assume that the value of the exponentiating parameter is positive, a large collection of distribution families can each be seen to be merely subfamilies of the family of densities defined by (1.2).

1. The two-parameter Gamma distributions

$$h(y; a, b, 1) = y^{b-1} e^{-(y/a)} / a^{b} \, \Gamma(b).$$

We risk pedantry to list special cases of the two-parameter

Gamma distributions, but these too are familiar families included as subfamilies of the Stacy family:

a. The standard one-parameter Gamma distributions

$$h(y;\ 1,\ b,\ 1) = y^{b-1}\, e^{-y}/\Gamma(b).$$

b. The exponential distributions

$$h(y;\ a,\ 1,\ 1) = e^{-(y/a)}/a.$$

c. The Chi-squared distributions

For n a positive integer ("degrees of freedom"),

$$h(y;\ 2,\ \tfrac{n}{2},\ 1) = y^{n/2\ -1}\, e^{-y/2}/2^{n/2}\,\Gamma(n/2).$$

2. The Weibull distributions

$$h(y;\ a,\ 1,\ c) = cy^{c-1}\, e^{-(y/a)^c}/a^c.$$

Special cases of the Weibull distribution include

a. The exponential distributions. [cf: 1.b]

b. The Rayleigh (circular normal) distributions

$$h(y;\ a\sqrt{2},\ 1,\ 2) = y\, e^{-y^2/2a^2}/a^2.$$

3. The Chi distributions

For n a positive integer,

$$h(y;\ \sqrt{2},\ \tfrac{n}{2},\ 2) = y^{n-1}\, e^{-y^2/2}/2^{\frac{n}{2}-1}\,\Gamma(n/2).$$

4. The Chi distributions with scaling parameter

For n a positive integer,

$$h(y;\ a\sqrt{2},\ \tfrac{n}{2},\ 2) = y^{n-1}\, e^{-y^2/2a^2}/a^n\, 2^{\frac{n}{2}-1}\,\Gamma(n/2).$$

Special cases of note here include

a. The modulus normal distributions

$$h(y;\ a\sqrt{2},\ \tfrac{1}{2},\ 2) = \sqrt{\frac{2}{\Pi a^2}}\quad e^{-y^2/2a^2}.$$

b. The circular normal (Rayleigh) distributions

$$h(y; a\sqrt{2}, 1, 2) = y\, e^{-y^2/2a^2}/a^2.$$

c. The spherical normal distributions

$$h(y; a\sqrt{2}, 3/2, 2) = \frac{1}{a^3}\sqrt{\frac{2}{\Pi}}\, y^2\, e^{-y^2/2a^2}.$$

We shall show [See result (1.8)] that the density (1.2) with $c < 0$, corresponds to that of a random variable $Y = 1/X$, where

$$X \sim h(x; \tfrac{1}{a}, b, c)$$

with $c > 0$. Thus, we see that the densities termed inverse exponential, inverse Chi-squared, etc., are included as special cases of density (1.2).

4. Properties of the Stacy Distribution

Suppose we have a random variable Y with probability density function (1.2). Then the t-th moment of Y is given by

$$E[Y^t] = \begin{cases} a^t\, \Gamma(b + t/c)/\Gamma(b), & \text{if } t/c > -b \\ \infty, & \text{otherwise.} \end{cases} \tag{1.4}$$

Thus, the mean is given by

$$E[Y] = \begin{cases} a\, \Gamma(b + 1/c)/\Gamma(b), & \text{if } 1/c > -b, \\ \infty, & \text{otherwise,} \end{cases} \tag{1.5}$$

and the variance is given by

$$\mathrm{Var}[Y] = \begin{cases} a^2\left[\dfrac{\Gamma(b + 2/c)\,\Gamma(b) - \Gamma^2(b + 1/c)}{\Gamma^2(b)}\right], & \text{if } 2/c > -b, \\ \infty, & \text{otherwise.} \end{cases} \tag{1.6}$$

For $Y \sim h(y; a, b, c)$, the cumulative distribution function may be shown to be

$$H(y) = \int_o^y h(y; a, b, c)\, dy$$

$$= \begin{cases} \Gamma_w(b)/\Gamma(b), & \text{if } c > 0 \\ 1 - \Gamma_w(b)/\Gamma(b), & \text{if } c < 0, \end{cases}$$

where $w = (y/a)^c$ and $\Gamma_w(b)$ is the standard notation for the incomplete Gamma function

$$\Gamma_w(b) = \int_o^w u^{b-1} e^{-u}\, du.$$

We note the following important results regarding the distribution of certain transformed Stacy random variables.

1. Let $Y \sim h(y; a, b, c)$. Then

$$Z = kY \sim h(z; ka, b, c) \qquad (1.7)$$

for any $k > 0$.

2. Let $Y \sim h(y; a, b, c)$. Then for any $t \neq 0$,

$$W = Y^t \sim h(w; a^t, b, c/t). \qquad (1.8)$$

We note in passing that $W' = Y^c \sim h(w'; a^c, b, 1)$, the standard two-parameter Gamma distribution, provided that $c \neq 0$.

3. Let $Y_1 \sim h(y_1; a_1, b_1, c)$ be distributed independently of $Y_2 \sim h(y_2; a_2, b_2, c)$. Then

$$W = 1/\left\{1 + \left[\frac{Y_2 a_1}{Y_1 a_2}\right]^c\right\}$$

has Beta distribution with parameters b_1 and b_2. (See p. 123 of [3].)

4. Let $Y_i \sim h(y_i; a_i, b_i, c_i)$, $i = 1, 2, \ldots, n$, be an

independent set of random variables. Suppose no $c_i = 0$. Then, according to the results listed in (1.7) and (1.8), for each $i = 1, 2, \ldots, n$,

$$Z_i = (X_i/a_i)^{c_i} \sim h(z_i; 1, b_i, 1).$$

Therefore, from the reproductive property of independently distributed one-parameter Gamma variates,

$$Z = \sum_{i=1}^{n} (X_i/a_i)^{c_i} \sim h(z; 1, \sum_{1}^{n} b_i, 1). \qquad (1.9)$$

We note in passing, however, that a correspondingly simple result does not exist for

$$Y = \sum_{i=1}^{n} Y_i .$$

5. Let $Y \sim h(y; a, b, c)$. Then let $Z = \ln Y$. One can then show that

$$Z \sim k(z; a, b, c) = \left| c \right| a^{-bc} e^{bcz} e^{-(e^{cz}/a^c)} / \Gamma(b) \qquad (1.10)$$

for positive parameters a and b, non-zero parameter c, and random variable $Z \in (-\infty, +\infty)$. The density (1.10) we shall refer to as a generalized double exponential distribution, since the double exponential distribution of Gumbel [4] corresponds to $k(z; 1, 1, -1)$.

5. Completeness and Sufficiency

By definition, the density $h(y; a, b, c)$ would be complete if the vanishing of the integral

$$\int_{y=0}^{\infty} k(y)\, h(y; a, b, c)\, dy$$

for every permissible set of values for a, b, and c implied that the function $k(y) = 0$, almost everywhere. If b and c are assumed known (fixed), the completeness property follows immediately from the unicity property of the Laplace transform. Should only c be assumed known, the property of completeness exists since $Z = Y^c$ would have the two-parameter Gamma distribution, known to be complete. However, it is not readily apparent that the more general case, with a, b, and c all unknown, provides a complete density function.

To determine a set of sufficient statistics for the density (1.2), we inspect the function $K(\vec{y}; \vec{y}_o)$ of Lehmann and Scheffé [5], where

$$K(\vec{y}; \vec{y}_o) = L(\vec{y}; a, b, c)/L(\vec{y}_o; a, b, c),$$

where $\vec{y}$ represents the vector, $(y_1, y_2, \ldots, y_n)$, of observations and where $L(\vec{y}; a, b, c)$ is the likelihood function associated with the vector $\vec{y}$. A statistic $T_j(\vec{y})$ is sufficient for the j-th parameter of a distribution if $T_j(\vec{y}) = T_j(\vec{y}_o)$ implies that $K(\vec{y}; \vec{y}_o) \not\equiv 0$ and is independent of this j-th parameter.

Pitman [6] states a necessary condition on the functional form of a distribution in order that a sufficient statistic exist for a parameter. Due to the position of the exponentiating parameter in density (1.2), it is impossible to meet Pitman's factorization criterion, so that one must conclude that no single sufficient statistic exists for this parameter.

However, for the scaling and shaping parameters, Pitman's condition is satisfied and, by use of Lehmann and Scheffé's K-function, we find that the statistic sufficient for the scaling parameter is

$\sum_{i=1}^{n} Y_i^c$ (implying that c must be known) and, for the shaping parameter, $\prod_{i=1}^{n} Y_i$ (or, equivalently, $\sum_{i=1}^{n} \ln Y_i$) .

6. Some Notes on the Generalized Double Exponential Distribution

Since one of our sufficient statistics for the Stacy distribution h(y; a, b, c) involves the random variable $Z = \ln Y$, we shall now investigate further the generalized double exponential distribution, k(z; a, b, c), given by equation (1.10). We note that the moment-generating function associated with density (1.10) is readily given by

$$E[e^{tZ}] = E[e^{t \ln Y}] = E[Y^t];$$

i.e.,

$$E[e^{tZ}] = \begin{cases} a^t \; \Gamma(b + t/c)/\Gamma(b), & t/c > -b \\ \infty & \text{otherwise,} \end{cases} \qquad (1.11)$$

the last equality following as a result of equation (1.4).

We note that, so long as $c > 0$, no problem regarding the existence of our moment-generating function exists for positive values of t. We recall that, should c be negative, we can resolve the matter by concerning ourselves with the random variable $1/Y$ rather that Y; i.e., with $-Z$ rather than Z.

With equation (1.11), then, we may easily ascertain the moments of $Z = \ln Y$. Let us define $Z' = \ln(Y/a)$, so that

$$Z' \sim k(z'; 1, b, c).$$

Then the relationship

$$E[Z']^K = \left\{ \partial^K E[e^{tZ'}]/\partial t^K \Big|_{t=0} \right\}$$

implies that, for $t/c > -b$, and for any $K = 1, 2, \ldots,$

$$E[Z']^K = \frac{\partial^K \Gamma(b + t/c)}{\Gamma(b)\, \partial t^K}\Bigg|_{t=0} .$$

Therefore, for $K = 1, 2, \ldots,$

$$E[\ell n\, (Y/a)]^K = \frac{1}{c^K} \Gamma^{(K)}(b) / \Gamma(b), \tag{1.12}$$

where

$$\Gamma^{(K)}(b) = d^K \Gamma(b)/db^K.$$

Thus, using the additional result that, for any positive constant s,

$$E\left\{\ell n(Y/s) - E[\ell n(Y/s)]\right\}^K = E\left\{\ell n\, Y - E(\ell n\, Y)\right\}^K , \tag{1.13}$$

we have

$$E[\ell n\, Y] = \ell n\, a + \Psi(b)/c, \tag{1.14}$$

$$\mathrm{Var}\,[\ell n\, Y] = \Psi'(b)/c^2, \tag{1.15}$$

and

$$\mu_3\,[\ell n\, Y] = E[\ell n\, Y - E(\ell n\, Y)]^3 = \Psi''(b)/c^3, \tag{1.16}$$

where

$$\Psi(x) = \partial\, \ell n\, \Gamma(x)/\partial x = \Gamma'(x)/\Gamma(x),$$

and

$$\Psi'(x) = d\, \Psi(x)/dx, \quad \text{and} \quad \Psi''(x) = d\, \Psi'(x)/dx,$$

as defined by Edwards [7].

Theorem: Suppose that $Z \sim h(z; 1, b, 1)$. Then, for any $s, t > 0$,

$$E[Z^s(\ell n\, Z)^t] = \Gamma(b + s)\, E[\ell n\, U]^t/\Gamma(b), \tag{1.17}$$

where $U \sim h(u; 1, b + s, 1)$.

Proof: By definition,

$$E[Z^s(\ell n\ Z)^t] = \int_{z=0}^{\infty}\left\{z^s(\ell n\ z)^t z^{b-1}e^{-z}/\Gamma(b)\right\}dz$$

$$= \frac{\Gamma(b+s)}{\Gamma(b)}\int_{u=0}^{\infty}\left\{(\ell n\ u)^t u^{(b+s)-1}e^{-u}/\Gamma(b+s)\right\}\ du$$

$$= \Gamma(b+s)\ E[\ell n\ U]^t/\Gamma(b),$$

where $U \sim h(u;\ 1,\ b+s,\ 1)$. Q. E. D.

Corollary: For $Z \sim h(z;\ 1,\ b,\ 1)$,

$$\mathrm{Var}\ [Z\ \ell n\ Z] = b(b+1)\Psi'(b+2) - b^2\Psi^2(b+1) + b(b+1)\Psi^2(b+2). \quad (1.18)$$

Proof: Direct substitution of the result (1.17), with $s = t = 1$ and $s = t = 2$, into the formula

$$\mathrm{Var}\ [Z\ \ell n\ Z] = E[Z\ \ell n\ Z]^2 - E^2[Z\ \ell n\ Z],$$

provides the result.

7. Lower Bounds for the Variance of Unbiased Estimates of Stacy's Parameters

C. R. Rao [8] showed that, for any unbiased estimate, $\hat{\alpha}$, of a function $\alpha(\theta)$ of a parameter θ associated with a probability density function, $f(x;\ \theta)$, which satisfies certain regularity conditions:

$$\mathrm{Var}\ (\hat{\theta}) \geq [\alpha'(\theta)]^2/n\ \mathrm{Var}\ [\partial\ \ell n\ f(x;\ \theta)/\partial\ \theta].$$

Therefore, letting a^*, b^*, and c^* denote any unbiased estimates of the scaling, shaping, and exponentiating parameters, respectively,

$$\mathrm{Var}\ (a^*) \geq 1/n\ \mathrm{Var}\ [-\frac{bc}{a} + cY^c/a^{c+1}] = a^2/nbc^2, \quad (1.19)$$

$$\mathrm{Var}\ (b^*) \geq 1/n\ \mathrm{Var}\ [-c\ell n\ a - \Psi(b) + c\ell n\ Y] = 1/n\ \Psi'(b), \quad (1.20)$$

and

$$\mathrm{Var}\ (c^*) \geq 1/n\mathrm{Var}\ [-b\ell n\ a + b\ell n Y - \frac{1}{c}(\frac{Y}{a})^c \ell n\ (\frac{Y}{a})^c\ ],$$

or

$$\mathrm{Var}\,(c^{*}) \geq c^{2}/n\left\{1+b\,[\,\Psi'\,(b+1) + \Psi^{2}\,(b+1)]\right\}. \qquad (1.21)$$

These results follow from successive application of the results listed in equations (1.6), (1.7), (1.8), (1.15), (1.17), and/or (1.18). The expressions on the right hand side of inequalities (1.19), (1.20), and (1.21) are called minimum variance bounds. Estimates which attain this variance are termed minimum variance bound estimators.

CHAPTER II

ESTIMATION OF STACY'S PARAMETERS

In this chapter we examine the possibility of utilizing the methods of maximum-likelihood and of moments to obtain non-iterative techniques for the estimation of the three parameters of density (1. 2). Noting the apparent futility of using either method to obtain estimators which are explicite functions of the observations (and not of the remaining, or nuisance, parameters), we then examine the method-of-moments as applied to the generalized double exponential distribution, defined in equation (1. 10).

The resulting estimation technique does not yield the joint probability density function of the estimators; however, our discussion throughout the chapter allows us to indicate many properties of the various estimators which are available.

1. Preliminary Considerations

Suppose we consider the problem of point estimation of the parameters of Stacy's distribution from an unordered and untruncated random sample of n observations, $y_1, y_2, \ldots, y_n$, each from the same parent distribution, $h(y; a, b, c)$, given by equation (1. 2). A first approach to this problem is often that of the method of maximum-likelihood. However, the following equations indicate the difficulties encountered in the attempt to obtain joint estimates by the maximum-likelihood attack:

$$\frac{\partial \ \ell n\, L(\vec{y}\,;\, a,\ b,\ c)}{\partial\, a} = \frac{-nbc}{a} + \frac{c \sum_{i=1}^{n} (y_i/a)^c}{a} = 0, \qquad (2.1)$$

$$\frac{\partial \ \ell n\, L(\vec{y}\,;\, a,\ b,\ c)}{\partial\, b} = -nc\ \ell n a + c \sum_{i=1}^{n} \ell n\ y_i - n\,\Psi(b) = 0, \qquad (2.2)$$

and

$$\frac{\partial \ \ell n\, L(\vec{y}\,;\, a,\ b,\ c)}{\partial\, c} = \frac{\pm n}{|c|} - nb\ell n\, a + b \sum_{i=1}^{n} \ell n\ y_i - \sum_{i=1}^{n} (y_i/a)^c \ell n(y_i/a) = 0, \qquad (2.3)$$

where $L(\vec{y}\,;\, a,\ b,\ c) = \prod_{i=1}^{n} h(y_i;\, a,\ b,\ c)$ is the likelihood function of the sample $\vec{y} = (y_1, y_2, \ldots, y_n)$ and where $\Psi(b)$ is the logarithmic derivative of the Gamma function with respect to its argument b. The plus sign is chosen for the first term of equation (2. 3) if $c > 0$; for $c < 0$, the negative sign is chosen, so that we may write

$$\frac{\partial \ \ell n\, L(\vec{y}\,;\, a, b, c)}{\partial\, c} = \frac{n}{c} - nb\ell n\, a + b \sum_{i=1}^{n} \ell n\ y_i - \sum_{i=1}^{n} (y_i/a)^c \ell n(y_i/a) = 0. \qquad (2.3')$$

Thus, we see that we might solve explicitly equation (2. 1) for either a or b, equation (2. 2) for either a, c, or $\Psi(b)$, and equation (2. 3') for b. Denoting these solutions with carets and indicating their functional dependence on nuisance parameters, we have:

$$\hat{a}_1(b, c) = \left\{ \sum_{i=1}^{n} y_i^{\,c}/nb \right\}^{1/c}, \qquad (2.4)$$

$$\hat{a}_2(b, c) = \exp \left\{ \sum_{i=1}^{n} (\ell n\ y_i)/n - \Psi(b)/c \right\}$$

$$= \left\{ \prod_{i=1}^{n} y_i \right\}^{1/n} \exp \left\{ - \Psi(b)/c \right\}, \qquad (2.5)$$

$$\hat{b}_1(a,\ c) = \sum_{i=1}^{n} (y_i/a)^c / n\,c, \qquad (2.6)$$

$$\hat{b}_2(a, c) = \left\{\sum_{i=1}^{n} [(y_i/a)^c \ell n(y_i/a)^c] - n\right\} \Big/ \left\{\sum_{i=1}^{n} \ell n(y_i/a)^c\right\} \qquad (2.7)$$

$$\hat{\Psi}(a, c) = c\left\{\sum_{i=1}^{n} \ell n\ (y_i/a)\right\}/\ n, \qquad (2.8)$$

$$\hat{c}\ (a, b) = n\ \Psi(b)/\sum_{i=1}^{n} [\ell n\ (y_i/a)]\ . \qquad (2.9)$$

Iterative solutions, using appropriate selections from the preceding six equations, should, if convergent, provide estimates of the three parameters. However, the important statistical properties, such as bias and variance, of the resulting (iterative) estimates would not be available and, consequently, the search for other estimation techniques was undertaken.

Before proceeding to a consideration of other possible estimation techniques, we examine some of the statistical properties of the individual estimators provided in equations (2.4) through (2.9).

First, we note that we may obtain the exact distribution of $\hat{a}_1(b, c)$; for, applying successively the results (1.8), (1.7), (1.9), (1.7), and (1.8), we have, for independent $y_i \sim h\ (y_i;\ a,\ b,\ c),\ i = 1, 2, \ldots, n$:

$$y_i^{\,c} \sim h(y_i^{\,c};\ a^c,\ b,\ 1)$$

$$y_i^{\,c}/a^c \sim h(\ \cdot\ ;\ 1,\ b,\ 1)$$

$$Z = \sum_{i=1}^{n} (y_i^{\,c}/a^c) \sim h(z;\ 1,\ nb,\ 1)$$

$$a^c Z/nb = \left\{\sum_{i=1}^{n} y_i^{\,c}/nb\right\} \sim h(\ \cdot\ ;\ a^c/nb,\ nb,\ 1),$$

and

$$\hat{a}_1(b, c) \sim h(\ \cdot\ ;\ a/(nb)^{1/c},\ nb,\ c) \qquad (2.10)$$

Therefore, from equations (1.5) and (1.6), we see that

$$E[\hat{a}_1(b, c)] = a\,\Gamma(nb+1/c)/\Gamma(nb)(nb)^{1/c},$$

and (2.11)

$$\mathrm{Var}\,[\hat{a}_1(b, c)] = a^2[\Gamma(nb+2/c)\Gamma(nb) - \Gamma^2(nb+1/c)]/\Gamma^2(nb)(nb)^{2/c},$$

so that the following unbiased estimate of the scaling parameter is suggested:

$$\hat{a}_3(b, c) = \Gamma(nb)\,[\sum_{i=1}^{n} y_i^{\,c}]^{1/c}/\Gamma(nb + 1/c). \qquad (2.12)$$

Equivalently, we might write this estimator in the form

$$\hat{a}_3(b, c) = \{\Gamma(nb)\,(nb)^{1/c}/\Gamma(nb+1/c)\}\,\hat{a}_1(b, c). \qquad (2.12')$$

The bracketed quantity in this expression may then be termed the bias correction factor; it may be conveniently approximated, as is shown by Stacy and Mihram [2].

Not only is the estimator $\hat{a}_3(b, c)$ unbiased for the scaling parameter, but also it is sufficient and its density is given by

$\hat{a}_3(b, c) \sim h(\cdot; A, B, C)$, where

$A = a\,\Gamma(nb)/\Gamma(nb + 1/c)$,

$B = nb$,

and

$C = c$.

Therefore, since b and c have been presumed known, reference to section 5 of Chapter I provides the fact that the density of $\hat{a}_3(b, c)$ is complete. Thus, from page 190 of Kendall and Stuart's second volume, $\hat{a}_3(b, c)$ possesses the property of being the unique minimum variance unbiased estimate of the scaling parameter. [9] This variance is

$$\text{Var}\,[\hat{a}_3(b,\ c)] = a^2[\Gamma(nb+2/c)\ \Gamma(nb) - \Gamma^2(nb+1/c)]/\ \Gamma^2\ (nb+1/c). \qquad (2.13)$$

Now, equation (2.5) suggests an estimate, $\hat{a}_2(b,\ c)$, of the scaling parameter. Its expectation may be shown to be

$$E[\hat{a}_2(b, c)] = a\left\{\Gamma(b+1/nc)/\Gamma(b)\right\}^n \exp\left\{-\Psi(b)/c\right\}, \qquad (2.14)$$

and its variance is given by

$$\text{Var}\,[\hat{a}_2(b,\ c)] = \frac{a^2[\exp\{-2\Psi(b)/c\}][\Gamma^n(b+2/nc)\ \Gamma^n(b) - \Gamma^{2n}(b+1/nc)]}{\Gamma^{2n}\ (b)} \qquad (2.15)$$

Therefore, another unbiased estimate (unbiased whenever the shaping and exponentiating parameters are assumed fixed) of the scaling parameter is provided by

$$\hat{a}_4(b,\ c) = \left\{\Gamma(b)/\Gamma(b+1/nc)\right\}^n \left\{\prod_{i=1}^{n} Y_i\right\}^{1/n}, \qquad (2.16)$$

having variance

$$\text{Var}\,[\hat{a}_4(b,\ c)] = a^2\left\{\Gamma^n(b+2/nc)\Gamma^n(b) - \Gamma^{2n}(b+1/nc)\right\}/\Gamma^{2n}(b+1/nc). \qquad (2.17)$$

Of the two shaping parameter estimators, given by equations (2.6) and (2.7), only the first yields readily its statistical properties. From equation (1.9), we recall that

$$Z = \sum_{i=1}^{n} (y_i/a)^c \sim h(z;\ 1,\ nb,\ 1).$$

Therefore, from equation (1.7), we see that

$$\hat{b}_1(a,\ c) \sim h(\cdot\,;\ 1/nc,\ nb,\ 1). \qquad (2.18)$$

Thus, we have immediately the properties

$$\left.\begin{aligned} E[\hat{b}_1(a,\ c)] &= nb/nc = b/c \\ \text{Var}\,[\hat{b}_1(a,\ c)] &= b/nc^2, \end{aligned}\right\} \qquad (2.19)$$

following by reference to equations (1.5) and (1.6).

Therefore, an unbiased estimate of the shaping parameter (with the remaining parameters assumed fixed) is given by

$$\hat{b}_3 (a, c) = c\, \hat{b}_1 (a, c) = \sum_{i=1}^{n} (y_i/a)^c/n . \tag{2.20}$$

From equation (1.7) we see, then, that

$$\hat{b}_3 (a, c) \sim h(\cdot\,; 1/n, nb, 1); \tag{2.21}$$

from equation (1.6), it follows then that the variance of $\hat{b}_3(a, c)$ is

$$\mathrm{Var} [\hat{b}_3(a, c)] = b/n. \tag{2.22}$$

In section 7 of Chapter I, we found the minimum variance bound for unbiased estimates of each of Stacy's parameters. Equation (1.20) provides the result that this bound, for unbiased estimates of the shaping parameter, is $1/n\, \Psi'(b)$. Thus, we may establish the rather interesting result that $\Psi'(b) \geq \frac{1}{b}$, for every positive b.

Equation (2.8) suggests still another technique for estimating the shaping parameter, b. We do not know explicitly the distribution of $\hat{\Psi}(a, c)$, but equation (1.11) allows us to generate its moments; two of interest are

$$E[\hat{\Psi}(a, c)] = \Psi(b) \tag{2.23}$$

and

$$\mathrm{Var} [\hat{\Psi}(a, c)] = \Psi'(b)/n. \tag{2.24}$$

We note then that $\hat{\Psi}(a, c)$ is an unbiased estimate of $\Psi(b)$, which Edwards [7] on page 108 shows to be a monotone increasing function of its argument. Thus, a graphical (or tabular) estimation procedure for b could be organized by calculating $\hat{\Psi}(a, c)$ and referring to the graph (table) of $\Psi(b)$.

From our work in section of Chapter I, we see that $\hat{\Psi}(a, c)$ is a function of the sufficient statistic for the shaping parameter. In addition, the result of Rao [8] would demonstrate that $\hat{\Psi}(a, c)$ attains the minimum variance bound for unbiased estimates of $\Psi(b)$. (Compare the variance given in equation (2. 24) with the result indicated in section 7 of Chapter I.)

As for the estimator (2. 9) of the exponentiating parameter, a discussion of its statistical properties is hampered by the appearance of the expression $\sum_{i=1}^{n} \ln (y_i/a)$, a random variable whose distribution (and whose inverse's distribution) is unknown.

However, suppose we define a parameter d by

$$d = 1/c. \tag{2. 25}$$

Equation (2. 9) would then suggest our considering as its estimator

$$\hat{d}(a, b) = \left\{ \sum_{i=1}^{n} \ln (y_i/a) \right\} / n\, \Psi(b). \tag{2. 26}$$

An inspection of equation (1. 12) would then provide the properties:

$$E[\hat{d}(a, b)] = 1/c = d \tag{2. 27}$$

and

$$\text{Var}[\hat{d}(a, b)] = \Psi'(b)/nc^2\Psi^2(b) = d^2\Psi'(b)/n\Psi^2(b). \tag{2. 28}$$

At this point then we have accumulated a number of estimators which may be employed whenever any pair of the three Stacy parameters are assumed fixed. Since we would aspire to erase such a restriction, we might rightly turn to estimation techniques other than that of maximum-likelihood.

The method-of-moments probably merits attention. In section 4 of Chapter I, we have seen that, for $Y \sim h(y; a, b, c)$ and for $td = t/c > -b$,

$$E(Y^t) = a^t \Gamma(b + t/c)/\Gamma(b).$$

Thus, we have

$$E(Y) = a\,\Gamma(b + 1/c)/\Gamma(b), \tag{2.29}$$

$$E(Y^2) = a^2\,\Gamma(b + 2/c)/\Gamma(b), \tag{2.30}$$

$$\mathrm{Var}\,(Y) = a^2 \left\{ \frac{\Gamma(b + 2/c)\,\Gamma(b) - \Gamma^2(b + 1/c)}{\Gamma^2(b)} \right\}, \tag{2.31}$$

and

$$E(Y^3) = a^3\,\Gamma(b + 3/c)/\Gamma(b), \tag{2.32}$$

whenever the arguments of the indicated Gamma functions are positive. [Recall that, in the event that $t/c \leq -b$, we may consider the random variable $1/Y$ instead of Y and thereby eliminate the complication of having invalid arguments in the Gamma functions.]

We therefore see that, for any $t \neq 0$ and such that $t/c > -b$, we may estimate a^t unbiasedly by

$$\hat{a}^t_5\,(b, c) = \Gamma(b) \sum_{i=1}^{n} Y_i^{\,t}/n\,\Gamma(b + t/c). \tag{2.33}$$

The variance of this estimator would be given by

$$\mathrm{Var}\,[\hat{a}_5^{\,t}\,(b, c)] = \frac{a^{2t}}{n} \left\{ \frac{\Gamma(b + 2t/c)\,\Gamma(b)}{\Gamma^2(b + t/c)} - 1 \right\}. \tag{2.34}$$

However, for general t, the distribution of $\hat{a}_5^{\,t}\,(b, c)$ is not readily available.

With $t = 1$, we may obtain another unbiased estimate of the scaling parameter,

$$\hat{a}_5 (b, c) = \Gamma(b) \sum_{i=1}^{n} y_i/n\ \Gamma(b + 1/c), \tag{2.35}$$

which, then, has variance

$$\text{Var}\ [\hat{a}_5(b, c)] = a^2 \left\{ \Gamma(b+2/c)\Gamma(b) - \Gamma^2(b+1/c) \right\} / n\Gamma^2(b+1/c). \tag{2.36}$$

Explicit solutions, for equations (2.29) through (2.32), for the remaining parameters are not so apparent. Forming ratios of these moments may at first seem promising, since, for $r + s = t$, the expression

$$\frac{E(Y^t)}{E(Y^s)\ E(Y^r)} = \frac{\Gamma(b + t/c)\ \Gamma(b)}{\Gamma(b+s/c)\Gamma(b+r/c)} \tag{2.37}$$

eliminates one nuisance parameter; viz., the scaling parameter. Nevertheless, this procedure does not seem particularly fruitful.

However, we recall that

$$E(Y/a)^c = \Gamma(b + 1)/\Gamma(b) = b. \tag{2.38}$$

Thus, for our random sample, $y_1, y_2, \ldots, y_n$, we have

$$\tilde{b}^n (a, c) = \left\{ \prod_{i=1}^{n} (y_i/a)^c \right\}, \tag{2.39}$$

with expectation

$$E \left\{ \prod_{i=1}^{n} (y_i/a)^c \right\} = b^n \tag{2.40}$$

and with variance

$$\text{Var} \left\{ \tilde{b}^n(a, c) \right\} = b^n(b+1)^n - b^{2n} = b^n[(b+1)^n - b^n]. \tag{2.41}$$

By equation (2.39), we have suggested, as an estimate of the shaping parameter,

$$\tilde{b} (a, c) = \left\{ \prod_{i=1}^{n} (y_i/a)^c \right\}^{1/n}, \tag{2.42}$$

which has expectation

$$E[\tilde{b}(a, c)] = \Gamma^n(b + 1/n)/\Gamma^n(b) \quad (2.43)$$

and variance

$$\mathrm{Var}[\tilde{b}(a, c)] = \left\{\Gamma^n(b + 2/n)\Gamma^n(b) - \Gamma^{2n}(b+1/n)\right\}/\Gamma^{2n}(b). \quad (2.44)$$

Thus, the method-of-moments does not seem effective in providing estimation techniques for Stacy's parameters; the estimates suggested, such as those of equations (2.33) and (2.39), still depend upon one's knowing or presuming values of the nuisance parameters.

2. An Estimation Technique Whenever All Parameters Are Unknown

We have seen, in section 5 of Chapter I, that the statistic sufficient for the shaping parameter, b, is the product of the observations (or, equivalently, the sum of the logarithms of these same observations). We might be led, then, to examine the problem of estimation for the generalized double exponential distribution of equation (1.10), for this distribution is that of the logarithm of a Stacy variate.

Suppose again that we have a random sample $\{y_i\}_{i=1}^{n}$ from density (1.2); thus for $z_i = \ln y_i$, $i = 1, 2, \ldots, n$, $\{z_i\}_{i=1}^{n}$ constitutes a random sample from density (1.10). An examination of the maximum-likelihood equations will provide no new suggestions on estimation. However, as we shall now see, the method-of-moments approach proves most fruitful.

Let us recall the results (1.14), (1.15), and (1.16); viz.,

$$E[\ln Y] = \ln a + \Psi(b)/c \quad (2.45)$$

$$\mathrm{Var}\,[\ell n\ Y] = \Psi'(b)/c^2 \tag{2.46}$$

and

$$\mu_3\,[\ell n\ Y] = E[\ell n\ Y - E(\ell n\ Y)]^3 = \Psi''(b)/c^3, \tag{2.47}$$

where, as previously,

$$\Psi(b) = \partial\ \ell n\ \Gamma(b)/\partial\ b,$$

$$\Psi'(b) = d\ \Psi(b)/db,$$

and

$$\Psi''(b) = d\ \Psi'(b)/db.$$

Consider now the coefficient of skewness of ℓn Y:

$$\frac{\mu_3[\ell n\ Y]}{\{\mathrm{Var}\,[\ell n\ Y]\}^{3/2}} = \pm \frac{\Psi''(b)}{[\Psi'(b)]^{3/2}}, \tag{2.48}$$

which we see to be a function of only one parameter, b. (The plus sign will apply if $c > 0$.) Thus, calculation from a random sample, $z_i = \ell n\ y_i$, $i = 1, 2, \ldots, n$, of $\bar{z} = \sum_{i=1}^{n} z_i/n$ and

$$K_{\ell n\ Y} = n \sum_{i=1}^{n} (z_i - \bar{z})^3/(n-1)(n-2)\, s^3_{\ell n\ Y}, \tag{2.49}$$

where

$$s^2_{\ell n\ Y} = \sum_{i=1}^{n} (z_i - \bar{z})^2/(n-1) \tag{2.50}$$

is an unbiased estimate of Var $[\ell n\ Y]$, should provide a method of estimating the shaping parameter. (Note that the quantity,

$$n \sum_{i=1}^{n} (z_i - \bar{z})^3/(n-1)(n-2),$$

estimates unbiasedly $\mu_3\,[\ell n\ Y]$). Kendall and Stuart [9] on page 244 of volume I, list the variance of $s^2_{\ell n\ Y}$ as

$$\mathrm{Var}\,[s^2_{\ell n\ Y}] = \frac{\mu_4[\ell n\ Y] - [\mathrm{Var}\,(\ell n\ Y)]^2}{n} + \frac{2}{n(n-1)}[\mathrm{Var}\,(\ell n\ Y)]^2,$$

where $\mu_4[\ell n\ Y] = E[\ell n\ Y - E(\ell n\ Y)]^4$. Using the moment-generating function of equation (1.11), we have that

$$\mu_4[\ell n\ Y] = \left\{\Psi'''(b)/c^4\right\} + 3[\mathrm{Var}(\ell n\ Y)]^2,$$

so that

$$\mathrm{Var}[s^2_{\ell n\ Y}] = \left\{\Psi'''(b) + \frac{2n}{n-1}[\Psi'(b)]^2\right\}/nc^4. \qquad (2.51)$$

Now, a plot of the function on the right-hand side of equation (2.48) would provide a double-valued graph, symmetric about the axis of its argument. If, however, we were to plot the function

$$\phi(b) = -\left|\frac{\mu_3(\ell n\ Y)}{[\mathrm{Var}(\ell n\ Y)]^{3/2}}\right| = -\left|\frac{\Psi''(b)}{[\Psi'(b)]^{3/2}}\right|, \qquad (2.52)$$

we would have the single-valued graph appearing in Figure 1, located at the end of this chapter. Thus, calculation of $-\left|K_{\ell n\ Y}\right|$ should provide an estimate of $\phi(b)$ and, hence, via the graph of $\phi(b)$, of the shaping parameter, b.

Let us denote this estimate by $\tilde{b}$. Then equation (2.46) suggests, as an estimator of the exponetiating parameter,

$$\tilde{c}(\tilde{b}) = \pm\left\{\Psi'(\tilde{b})\right\}^{1/2}/s_{\ell n\ Y}, \qquad (2.53)$$

where $s_{\ell n\ Y}$ is the positive root of the expression given in equation (2.50), where the positive sign is selected if $K_{\ell n\ Y} < 0$, the minus if $K_{\ell n\ Y} > 0$, and where the value of $\left\{\Psi'(b)\right\}^{1/2}$ may be taken from the graph of this function presented in Figure 1.

It would remain, then, to select an estimate of the scaling parameter. Selecting the estimator implied by equation (2.45) would prove to be the equivalent of choosing the estimator $\hat{a}_2(\tilde{b}, \tilde{c}(\tilde{b}))$, given by equation (2.5). Since any scaling parameter estimator we may now consider will be a function of our estimates $\tilde{b}$ and $\tilde{c}(\tilde{b})$, we might desire to select that estimate which will have the properties (for these fixed values, $\tilde{b}$ and $\tilde{c}(\tilde{b})$) of being unbiased and of having minimum variance attainable among unbiased estimates. If so, we select $\hat{a}_3(\tilde{b}, \tilde{c}(\tilde{b}))$, given by equation (2.12). If ease of calculability is a criterion deemed important in selecting an estimate, $\hat{a}_5(\tilde{b}, \tilde{c}(\tilde{b}))$ might be suggested.

3. Estimation Techniques Whenever One Parameter Is Known

We have seen how, in the absence of knowledge of the values of any of the three Stacy parameters, we may estimate jointly the entire set. We might now consider relaxing this restriction by considering cases where one, and, for the moment, only one, of our three parameters is assumed known. A number of iterative techniques will surely occur to the reader who refers to our many estimating equations: (2.4), (2.5), (2.6), (2.7), (2.8), (2.9), (2.12), (2.16), (2.20), (2.26), (2.33), (2.35), (2.37), (2.39), (2.42), (2.48), and (2.53). We shall, however, attempt to avoid such techniques and endeavor to present only non-iterative estimators which are explicit functions of the observations.

Suppose we consider first the case in which the exponentiating parameter has a fixed (known)value, c_o. We recall, by reference to result (1.8), that, for $Y \sim h(y; a, b, c_o)$, $W = Y^{c_o} \sim h(w; a^{c_o}, b, 1)$. Thus, for a random sample, $W_i = Y_i^{c_o}$, $i = 1, 2, \ldots, n$, we see that we may estimate unbiasedly $\Psi'(b)$, as suggested by equation (2.46), by

$$\hat{\Psi}'(c_o) = s^2_{\ell n\, W} = c_o^{\,2} s^2_{\ell n\, Y}\,, \tag{2.54}$$

where $s^2_{\ell n\, Y}$ is as defined by equation (2.50). Reference to equation (2.51) provides the variance of $\hat{\Psi}'$ as

$$\mathrm{Var}\,[\,\hat{\Psi}(c_o)] = \left\{\Psi'''(b) + \frac{2n}{n-1}\,[\,\Psi'(b)]^2\right\}/n, \tag{2.55}$$

interesting in that it is independent of the known c_o, as well as the unknown scaling parameter. Edwards [7] on page 100 of Volume II, shows that $\Psi'(b)$ is a monotone, positive-valued function of its positive argument, b, so that, after calculating $s^2_{\ell n\, W}$, we may use the graph of $\Psi'(b)$ to estimate the shaping parameter. (See Figure 2 at the end of this chapter.)

Denoting this estimate by $\hat{b}(c_o)$, we may turn again to any one of the five estimators, $\hat{a}_i\,(\hat{b}\,(c_o),\ c_o)$, for the scaling parameter. The statements at the conclusion of the preceding section are again pertinent in choosing among these estimators.

With the exponentiating parameter assuming a fixed value, c_o, we might consider estimating the remaining parameters by applying the method-of-moments to the random variable, $W = Y^{c_o}$. Appropriate applications of equations (1.8), (1.5), and (1.6) provide the interesting result that

$$\frac{E^2\,(Y^{c_o})}{\mathrm{Var}\,(Y^{c_o})} = \frac{a^2 b^2}{a^2 b} = b.$$

Therefore, one might consider as an estimate of the shaping parameter

$$\tilde{b}\,(c_o) = \frac{\left\{\sum_{i=1}^{n} w_i\right\}^2}{n^2\, s^2_w}\,, \tag{2.56}$$

where s_w^2 is provided analogously to that given by equation (2.50). This procedure for estimating the shaping parameter, however, suffers in that none of its statistical properties are known.

Suppose next that we allow only the shaping parameter to have a fixed value, say, b_o. Then equation (2.46) suggests an unbiased estimate of $d^2 = 1/c^2$; viz.,

$$\tilde{d}^2 (b_o) = s^2_{\ell n\ Y}/\Psi' (b_o), \tag{2.57}$$

where $s^2_{\ell n\ Y}$ is given by equation (2.50) and $\Psi' (b_o)$ may be obtained from Figure 2, located at the end of this chapter. Reference to equation (2.51) provides

$$\text{Var}\ [\tilde{d}^2(b_o)] = \frac{d^4}{n} \left\{ \Psi''(b_o)/[\Psi' (b_o)]^2 \right\} + (\frac{2d^4}{n-1}). \tag{2.58}$$

Thus, we may form the exponentiating parameter's estimator

$$\tilde{c}(b_o) = \sqrt{1/d^2(b_o)} = \pm\ [\Psi'(b_o)]^{1/2}/s_{\ell n\ Y}, \tag{2.59}$$

where $[\Psi'(b_o)]^{1/2}$ may be obtained from Figure 1 at the end of this chapter. The choice of signs will be dependent upon the sign of the following unbiased estimate of $\mu_3(\ell n\ Y)$, defined by equation (2.47),

$$\hat{\mu}_3(\ell n\ Y) = n \sum_{i=1}^{n} (z_i - \bar{z})^3/(n-1)(n-2), \tag{2.60}$$

where $z_i = \ell_n y_i$, $i = 1, 2, \ldots, n$. If $\hat{\mu}_3(\ell n\ Y) < 0$, we select for the right-hand side of equation (2.59) the plus sign; otherwise, we select the minus sign. [See equation (2.53).]

We are again confronted with the estimation of the remaining parameter, a. The reader will be left again to decide his choice among the set of scaling parameter estimates, $\hat{a}_i (b_o, \tilde{c}(b_o))$, $i=1, \ldots, 5$.

Finally, suppose we assume that only the scaling parameter has a fixed value, say, a_o. Then we may estimate the shaping parameter, b, by the method indicated beneath equation (2.52). The resulting estimate we have denoted previously by $\tilde{b}$.

To estimate the remaining parameter, c, the reader may choose between $\tilde{c}(\tilde{b})$, as given by equation (2.53), and $\hat{c}(a_o, \tilde{b})$, as given by equation (2.9). The decision might rest upon several factors. Each of the estimators is calculated from statistics already generated in arriving at the estimate, $\tilde{b}$; thus, ease of computation hardly seems a pertinent factor. Neither of the estimators' statistical properties are known, though the square of the inverse of the first estimator estimates unbiasedly $1/c^2$, while the inverse of the second estimates unbiasedly $1/c$. However, $\hat{c}(a_o, \tilde{b})$ provides automatically the sign of $\hat{c}$, and, for this reason, might be preferred.

4. Estimation Techniques Whenever Two Parameters May Be Assumed Known

If we assume that two of the three Stacy parameters are assumed fixed (known), the task of estimating the third is, in every case, straightforward. In this section we shall list in Table I these estimation techniques, and, wherever possible and applicable, indicate the relative efficiency of our estimates. For the unbiased estimators listed in Table I, we consider, as indicators of efficiency, the concepts of minimum variance bound estimators (MVBE) and of minimum variance unbiased estimators (MVUE), as described respectively on pages 9 and 190 of Volume II of Kendall and Stuart [9]. The minimum variance bounds are presented in section 7 of Chapter I; minimum variance unbiased estimators exist as a result of our discussion in section 5 of Chapter I (i.e., because they are unbiased, are functions of the sufficient statistics, and have probability density functions which are complete.)

In Table I, parenthetical entries, other than those which are obviously functional arguments, refer to equation numbers in Chapters I or II. A 'NO'' entry implies that this property does not exist for the indicated estimator, whereas 'UNK'' implies that the property is unknown or undetermined. An estimator for which the indicated property should not apply is indicated by a dash.

TABLE I

PROPERTIES OF ESTIMATORS

Estimator	Reference	Estimates	Unbiased	Distribution	Variance	MVBE	MVUE
$\hat{a}_1(b, c)$	(2.4)	a	NO	(2.10)	(2.11)	——	——
$\hat{a}_2(b, c)$	(2.5)	a	NO	UNK	(2.15)	——	——
$\hat{a}_3(b, c)$	(2.12)	a	YES	(2.12')	(2.13)	NO *	YES
$\hat{a}_4(b, c)$	(2.16)	a	YES	UNK	(2.17)	NO	NO
$\hat{a}_5(b, c)$	(2.35)	a	YES	UNK	(2.36)	NO *	NO *
$\hat{a}^t_5(b, c)$	(2.33)	a^t	YES	UNK	(2.34)	NO **	NO **
$\hat{b}_1(a, c)$	(2.6)	b	NO	(2.18)	(2.19)	——	——
$\hat{b}_2(a, c)$	(2.7)	b	UNK	UNK	UNK	UNK	UNK
$\hat{b}_3(a, c)$	(2.20)	b	YES	(2.21)	(2.22)	NO	NO
$\hat{\Psi}(a, c)$	(2.8)	$\Psi(b)$	YES	UNK	(2.24)	YES	YES
$\hat{\Psi}'(c)$	(2.54)	$\Psi'(b)$	YES	UNK	(2.55)	NO	UNK
$\tilde{b}(c)$	(2.56)	b	UNK	UNK	UNK	UNK	UNK
$\tilde{b}^n(a, c)$	(2.39)	b^n	YES	UNK	(2.41)	NO	UNK
$\tilde{b}(a, c)$	(2.42)	b	NO	UNK	(2.44)	——	——
$-\lvert K_{\ell n\, Y} \rvert$	(2.49)	$\phi(b)$	UNK	UNK	UNK	UNK	UNK
$\hat{c}(a, b)$	(2.9)	c	UNK	UNK	UNK	UNK	UNK
$\tilde{c}(b)$	(2.53)	c	UNK	UNK	UNK	UNK	UNK
$\hat{d}(a, b)$	(2.26)	$d = 1/c$	YES	UNK	(2.28)	NO	NO
$\tilde{d}^2(b)$	(2.57)	$d^2 = 1/c^2$	YES	UNK	(2.58)	NO	NO

* In general, apparently not. For c = 1, however, "YES."

** In general, apparently not. For t = c, however, "YES."

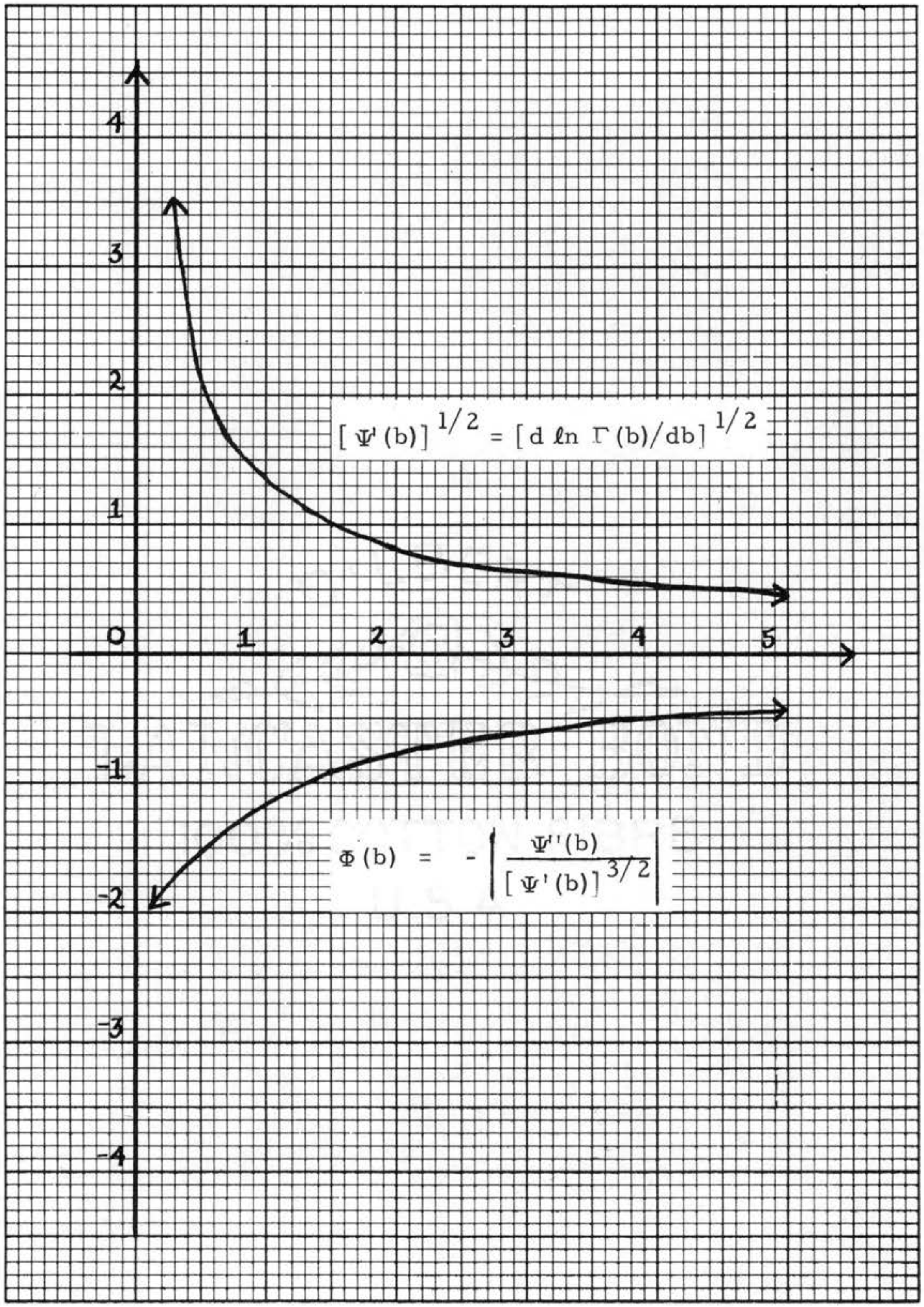

Figure 1. Aids for Estimation

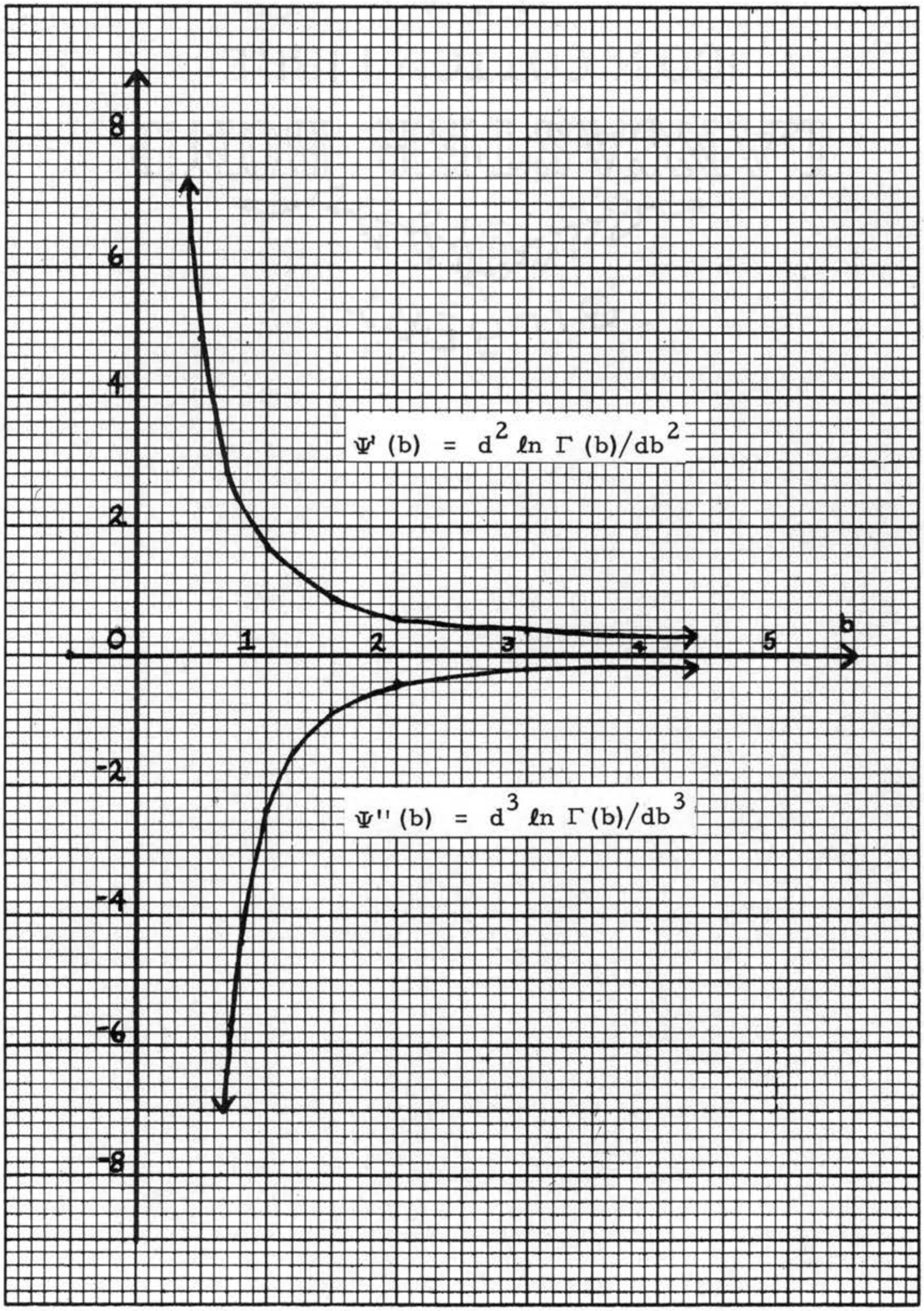

Figure 2. Two Derivatives of the Psi Function

CHAPTER III

A DISTENDED GAMMA DISTRIBUTION

In this chapter we shall discuss a four-parameter univariate probability density function, defined for a positive random variable, X. The density function can be shown to be a generalization of Stacy's three-parameter generalized distribution; furthermore, its behaviour near the ordinate axis is that of Stacy's generalized Gamma distribution. The moments of this distribution can be related to the moments of a corresponding Stacy distribution, as will be shown in Chapters IV and V, where, in discussing certain bivariate density functions, this distended Gamma distribution is found to be one of the marginal densities.

1. Preliminary Considerations

For a positive random variable, X, positive parameters a and p, real parameter $c \neq 0$, and parameter $b > -p/c$, we may define the following univariate probability density function, which we shall term the distended Gamma distribution:

$$g(x; a, b, c, p) = \frac{px^{p-1}}{a^p \Gamma(b + p/c)} \text{ egf } [(\frac{x}{a})^c ; b], \qquad (3.1)$$

where

$$\mathrm{egf}[(\tfrac{x}{a})^c;b] = \begin{cases} \int_{(x/a)^c}^{\infty} u^{b-1} e^{-u} du, & \text{if } c > 0 \\ \int_0^{(x/a)^c} u^{b-1} e^{-u} du, & \text{if } c < 0. \end{cases} \tag{3.2}$$

The cumulative distribution function associated with (3.1) is

$$G(x) = 1 - \frac{1}{\Gamma(b+p/c)} \left\{ \mathrm{egf}[(\tfrac{x}{a})^c; (b+p/c)] - (\tfrac{x}{a})^p \mathrm{egf}[(\tfrac{x}{a})^c; b] \right\} \tag{3.3}$$

and the moments of the random variable X are given by

$$E[X^s] = \frac{a^s p \Gamma(b + \frac{p+s}{c})}{(p+s)\Gamma(b+p/c)}, \tag{3.4}$$

for $b > -(p+s)/c$, $b > -p/c$ and provided that $p \neq -s$.

The properties of the function $\mathrm{egf}[(x/a)^c; b]$ should be of some importance to a discussion of our distended Gamma distribution. We note first that

$$\lim_{x \to 0^+} \mathrm{egf}[(\tfrac{x}{a})^c; b] = \int_0^{\infty} u^{b-1} e^{-u} du = \Gamma(b), \tag{3.5}$$

the Gamma function with parameter b, provided that $b > 0$. (The limit does not exist if $b \leq 0$.) In addition, for any real m,

$$\lim_{t \to \infty} (\tfrac{t}{a})^m \mathrm{egf}[(\tfrac{t}{a})^c; b] = 0, \tag{3.6}$$

as may be seen by noting that, from equation (3.2),

$$\lim_{t \to \infty} \mathrm{egf}[(t/a)^c; b] = 0, \tag{3.7}$$

so that, whenever $m > 0$, repeated application of L'Hopital's Rule provides the desired result. A further result relating to the function (3.2) is that

$$\text{egf}\,[(\tfrac{x}{a})^c;\,1] = \begin{cases} e^{-(x/a)^c}, & \text{if } c > 0 \\ 1 - e^{-(x/a)^c}, & \text{if } c < 0. \end{cases} \tag{3.8}$$

Thus, from (3. 5) we see that

$$\lim_{x \to o^+} g(x;\, a,\, b,\, c,\, p) = \begin{cases} +\infty, & \text{if } 0 < p < 1, \\ \Gamma(b)/a\Gamma(b + 1/c), & \text{if } p = 1, \text{ and} \\ 0, & \text{if } p > 1, \end{cases} \tag{3.9}$$

so that the density (3. 1) behaves, at the ordinate axis, as would a Stacy distribution with shaping parameter, p . [2]

In addition to these properties, we note that certain transformations of distended Gamma variates are again distended Gamma variates; e.g., for $X \sim g(x;\, a,\, b,\, c,\, p)$, $k,\, t > 0$,

$$Y = kX \sim g(y;\, ka,\, b,\, c,\, p), \tag{3.10}$$

and

$$Z = X^t \sim g(z;\, a^t,\, b,\, c/t,\, p/t). \tag{3.11}$$

2. Special Cases of the Distended Gamma Distribution

We have defined and briefly discussed a four-parameter probability density function (3. 1). In order to indicate its merit as an applicable probability density function, one might enumerate, in a manner similar to the listing provided in section 3 of Chapter I for Stacy's generalized Gamma distribution, any familiar families of probability density functions which are subfamilies of it.

First consider the restriction of the parameter space provided whenever c is positive and b = 1. Using the result of equation (3. 8), we see that, in this case, density (3. 1) represents Stacy's three-parameter distribution with positive shaping parameter. For,

$$g(x;\ a,\ 1,\ c,\ p) = c\, x^{p-1} e^{-(x/a)^c} / a^p\, \Gamma(p/c), \tag{3.12}$$

which is identically $f(x;\ a,\ p,\ c)$ as given in equation (1.1). Thus, our distended Gamma distribution includes many families of distributions which we saw in section 3 of Chapter I to be special cases of Stacy's generalized Gamma distribution.

Another interesting sub-family of the family of densities described by equation (3.1) is the one-parameter distribution

$$g(x;\ a,\ \tfrac{1}{2},\ 2,\ 1) = \frac{1}{a} \int_{(x/a)^2}^{\infty} u^{-1/2} e^{-u}\, du, \tag{3.13}$$

which we may write as

$$g(x;\ a,\ \tfrac{1}{2},\ 2,\ 1) = \frac{2}{a^2} \int_{x}^{\infty} e^{-(t/a)^2} dt = \frac{2}{a} \int_{x/a}^{\infty} e^{-y^2}\, dy.$$

From page 77 of Rainville [10], we see that

$$g(x;\ a,\ \tfrac{1}{2},\ 2,\ 1) = \frac{\sqrt{\pi}}{a}\ \operatorname{erfc}\left(\frac{x}{a}\right), \tag{3.14}$$

where erfc (t) is the complement of the error function,

$$\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_{t}^{\infty} e^{-y^2}\, dy = 1 - \frac{2}{\sqrt{\pi}} \int_{0}^{t} e^{-y^2}\, dy. \tag{3.15}$$

A two-parameter generalization of this last probability density function is provided, for a and $p > 0$, by

$$g(x;\ a,\ 1/2,\ 2,\ p) = \frac{p x^{p-1}}{a^p\, \Gamma\left(\frac{p+1}{2}\right)} \int_{(x/a)^2}^{\infty} u^{-1/2} e^{-u} du,$$

or equivalently,

$$g(x;\ a,\ 1/2,\ 2,\ p) = \frac{p x^{p-1} \sqrt{\pi}\, \operatorname{erfc}(x/a)}{a^p\ \Gamma\left(\frac{p+1}{2}\right)}, \tag{3.16}$$

where erfc (t) is defined in equation (3.15). For the sake of later reference, we shall refer to univariate density (3.16) as the error-function distribution.

Having a technique for estimating the parameters of our four-parameter density (3.1) would possibly be of great value. However, since we shall show that this density is, in certain cases, the marginal distribution of the warning-times associated with the bivariate warning-time/failure-time distributions which we discuss at length in the next two chapters, we shall defer our presentation of parameter estimation techniques for the time being.

CHAPTER IV

THE UNIFORM/STACY BIVARIATE DISTRIBUTION

Suppose that one has specified a probability density function, $h(y)$, to describe the population of failure-times, Y (>0), of some component (or system of components). Suppose also that we may measure, or observe, prior to each faulure-time, some property of the system which would warn of, yet not accurately predict the time of, the impending failure of the system. If we denote the time of this warning by X and assume that, for each failure-time Y, there is a unique warning-time, $X(< Y)$, then we could define a bivariate probability density $f(x, y)$ on the range $0 < x < y < \infty$. By assuming a conditional density of X given $Y = y$, we could completely specify this bivariate density function:

$$f(x, y) = g(x \mid y)\, h(y), \tag{4.1}$$

where $g(x \mid y)$ is this conditional density, defined for a random variable X which takes values with positive probability only in the range $0 < x < y$.

In this chapter, we shall concern ourselves with bivariate densities which are capable of factorization as indicated in equation (4.1), such that the marginal density of the failure-times is specified by Stacy's generalized Gamma distribution (1.2), and such that the conditional density is designated by the uniform distribution

$$g(x|y) = 1/y, \quad 0 < x < y. \tag{4.2}$$

1. A Theorem on the Implications of a Uniform Conditional Distribution

We shall then be concerned with the Uniform/Stacy bivariate distribution

$$f(x, y; a, b, c) = h(y; a, b, c)/y = |c| \, a^{-bc} y^{bc-2} e^{-(y/a)^c} / \Gamma(b), \tag{4.3}$$

for $0 < x < y < \infty$, for real parameter c (which we shall continue to refer to as the exponentiating parameter), and for positive parameters a and b (the scaling and shaping parameters, respectively). However, since we may obtain a number of pertinent results relating to general warning-time/failure-time bivariate densities without assuming a functional form for the marginal density of Y, we defer momentarily our discussion of bivariate density (4.3) to present the following theorem.

Theorem: Let $f(x, y)$ be the bivariate probability density function as described by equations (4.1) and (4.2). Then the following results may be established:

(A) The random variable $U = Y - X$ has the same marginal density function as X; furthermore, with $f(x, y)$ as specified by equation (4.3),

$$X \sim g(x; a, b-1/c, c, 1), \tag{4.4}$$

as given by equztion (3.1). Consequently, whenever these moments exist,

$$E(Y-X)^s = E(X)^s. \tag{4.5}$$

(B) Whenever the appropriate moments of Y exist,

$$E[X^s Y^t] = \begin{cases} E[Y^{s+t}]/(s+1), & \text{if } s \neq -1 \\ E[(\ln Y - 1)\, Y^{t-1}], & \text{if } s = -1 . \end{cases} \qquad (4.6)$$

(C) The cumulative marginal distribution function $G(x)$, associated with the random variable X, differs from the cumulative marginal distribution function, $H(y)$, associated with Y, when each is evaluated at t, by

$$G(t) - H(t) = t\, g(t), \qquad (4.7)$$

where $g(t)$ is the marginal density of X (evaluated at t).

(D) Whenever the appropriate moments of Y exist,

$$E(2X) = E(Y), \qquad (4.8)$$

$$\text{Var}(2X) = 4\,\text{Var}(X) = \text{Var}(Y) + \frac{1}{3} E(Y^2) > \text{Var}(Y) \qquad (4.9)$$

and

$$\text{Cov}(X, Y) = \frac{1}{2}\,\text{Var}(Y). \qquad (4.10)$$

(E) Whenever $E(Y^2)$ exists and Y has not a degenerate distribution,

$$0 < \text{Corr}(X, Y) = \sqrt{\frac{\text{Var}(Y)}{4\,\text{Var}(X)}} = \sqrt{\frac{\text{Var}(Y)}{\text{Var}(Y) + \frac{1}{3}E(Y^2)}} < 1. \qquad (4.11)$$

<u>Proof:</u>

(A) Let $U = Y - X$, $V = Y$. Then $X(U, V) = V - U$ and $Y(U, V) = V$. The absolute value of the Jacobian associated with this transformation is unity, so that the joint probability density function of U and V becomes

$$f^*(u, v) = f[x(u, v), y(u, v)] \cdot 1$$

$$= h(v)/v, \quad 0 < u < v < \infty.$$

Thus, $f^*(u, v) = f(u, v)$, so that X and Y have the same bivariate probability density function as U and V (=Y). Therefore, the marginal density of U is that of X; and, whenever we have

$$h(y) = h(y; a, b, c),$$

as specified by equation (1.2), this marginal density becomes

$$g(u; a, b-1/c, c, p) = \int_{y=u}^{\infty} \left\{ |c| \, y^{bc-2} e^{-(y/a)^c} / a^{bc} \Gamma(b) \right\} dy,$$

as given by equation (3.1). The equivalence of the s-th moments of X and U = Y - X then follows immediately.

(B) Now

$$E[X^s Y^t] = \int_{y=o}^{\infty} \int_{x=o}^{y} x^s y^t f(x, y) \, dx \, dy$$

$$= \int_{y=o}^{\infty} \int_{x=o}^{y} x^s y^{t-1} h(y) \, dx \, dy$$

or

$$E[X^s Y^t] = \begin{cases} \dfrac{1}{(s+1)} \displaystyle\int_{y=o}^{\infty} y^{s+t} h(y) \, dy, & s \neq -1 \\ \displaystyle\int_{y=o}^{\infty} [\ell n \, y - 1] \, y^{t-1} h(y) \, dy, & s = -1. \end{cases}$$

(C) Now

$$G(t) = \int_{x=o}^{t} g(x) \, dx,$$

where g(x) represents the marginal density of X; viz.,

$$g(x) = \int_{y=x}^{\infty} f(x, y) \, dy.$$

Thus

$$G(t) = \int_{x=o}^{t} \int_{y=x}^{\infty} f(x, y)\, dy\, dx$$

$$= \left\{ \int_{y=o}^{t} \int_{x=o}^{y} + \int_{y=t}^{\infty} \int_{x=o}^{t} \right\} h(y)/y \quad dx\, dy$$

$$= \int_{y=o}^{t} h(y)\, dy + t \int_{y=t}^{\infty} f(x, y)\, dy.$$

(D) Equation (4. 8) follows by substitution of $s = 1$ into equation (4. 5) and by subsequently applying the linearity property of the expectation operator. Equations (4. 9) and (4. 10) follow from repeated and appropriate applications of equation (4. 6).

(E) Equation (4. 11) follows from the definition of the correlation coefficient and an application of the result (4. 10).

2. Parameter Estimation For the Uniform/Stacy Bivariate Distribution

Suppose we have a random sample of n observations (x_i, y_i), $i = 1, 2, \ldots, n$, each taken from bivariate density (4. 3). Now, since the set of y_i's constitute a random sample from $h(y; a, b, c)$, we see that the parameters of bivariate density (4. 3) could be estimated by utilizing only the observed failure-times, y_i, $i=1, 2, \ldots, n$, and applying the appropriate estimation technique of Chapter II. However, suppose we attempt to estimate these parameters by employing only the warning-times: $x_1, x_2, \ldots, x_n$.

A consideration of the maximum-likelihood equations associated with the univariate density function (4. 4) does not reveal any procedure readily applicable to this estimation problem. The method-of-moments would be concerned with the results obtained from equation (4.6) after setting $t = 0$ and referring to equation (1.4). For convenience, we list

$$E(X) = \begin{cases} a\,\Gamma(b + 1/c)/2\Gamma(b), & \text{if } b + 1/c > 0 \\ +\infty, & \text{otherwise,} \end{cases} \tag{4.12}$$

and

$$\mathrm{Var}(X) = \begin{cases} a^2[\frac{1}{3}\,\Gamma(b+2/c)\Gamma(b) - \frac{1}{4}\,\Gamma^2(b+1/c)]/\Gamma^2(b), & \text{if } b + 2/c > 0 \\ +\infty, & \text{otherwise,} \end{cases} \tag{4.13}$$

but note the futility apparent in any effort to utilize such results jointly in an estimation technique.

However, suppose we consider the application of the method-of-moments to the random variable, $\ell n\, X$. The joint moment-generating function of $\ell n\, X$ and $\ell n\, Y$ is given by

$$E[e^{s\,\ell n\, X + t\,\ell n\, Y}] = E[X^s\, Y^t], \tag{4.14}$$

which, with $s \neq -1$, may be more explicitly presented upon reference to equations (4.6) and (1.4). (With $s = -1$, the explicite representation of (4.14) requires, in addition, reference to equation (1.17).) Equating t to zero in (4.14) provides the moment-generating function for $\ell n\, X$:

$$E[e^{s\,\ell n\, X}] = E[X^s] = E[Y^s]/(s+1) = a^s\,\Gamma(b+s/c)/(s+1)\Gamma(b) \tag{4.15}$$

for all s such that $(b + s/c) > 0$ and such that $s \neq -1$.

Now the i-th moment of $\ell n\, X$ is given by

$$E[\ell n\, X]^i = \partial^i E\,[e^{s\,\ell n\, X}]/\partial\, s^i \Big|_{s=0}\,. \tag{4.16}$$

However, before proceeding directly to the application of equation (4.16), we note that, for any positive constant a,

$$E[\ell n\, X] = \ell n\, a + E[\ell n(X/a)],$$

and that, for any $k = 2, 3, 4, \ldots$,

$$E\left\{\ell n\,(X/a) - E[\ell n\,(X/a)]\right\}^k = E\left\{\ell n\, X - E(\ell n\, X)\right\}^k. \qquad (4.17)$$

Thus, a consideration of the random variable $\ell n\,(X/a)$ should simplify our efforts in obtaining the moments of $\ell n\, X$. For, from equation (4.15), we have, for $s \neq -1$,

$$E[e^{s\,\ell n\,(X/a)}] = E(X/a)^s = \Gamma\,(b+s/c)/(s+1)\,\Gamma\,(b), \qquad (4.18)$$

so that, for $i = 1, 2, \ldots$,

$$E[\ell n\,(X/a)]^i = \sum_{k=o}^{i} \frac{i!\,(-1)^{i-k}\,\Gamma^{(k)}\,(b)}{k!\,c^k\,\Gamma\,(b)}, \qquad (4.19)$$

where $\Gamma^{(k)}(b)$ is the k-th derivative of the Gamma function with respect to its argument. ($\Gamma^{(0)}$ (b) is, of course, taken to be Γ (b).) Thus we readily obtain

$$E[\ell n\,(X/a)] = -1 + \Psi(b)/c, \qquad (4.20)$$

from which we obtain the rather interesting result that

$$E[\ell n\,(X\cdot e)] = \ell n\, a + \Psi(b)/c = E[\ell n\, Y], \qquad (4.21)$$

where $\Psi(b)$ is as defined beneath equation (2.47). In addition, by applying the result of equation (4.17) to the appropriate combination of moments acquired by setting $i = 1, 2,$ and 3 in equation (4.19), we have the central moments

$$\text{Var}\,[\ell n\, X] = 1 + \Psi'(b)/c^2 \qquad (4.22)$$

and

$$\mu_3\,(\ell n\ X) \;=\; -2 \;+\; \Psi''(b)/c^3. \tag{4.23}$$

On page 111 of Volume II, Edwards [7] shows that, for every $b > 0$, $\Psi'(b) > 0$ and $\Psi''(b) < 0$. Thus, from equation (4.22), we have

$$\left| -1 + \mathrm{Var}\,(\ell n\ X) \right| \;=\; \left| \Psi'(b)/c^2 \right| \;=\; \Psi'(b)/c^2 \tag{4.24}$$

and

$$\left| -1 + \mathrm{Var}\,(\ell n\ X) \right|^{3/2} = \begin{cases} [\Psi'(b)]^{3/2}/c^3, & \text{if } c < 0 \\ -[\Psi'(b)]^{3/2}/c^3, & \text{if } c < 0\,. \end{cases} \tag{4.25}$$

In addition, from equation (4.23) we see that the quantity

$$2 + \mu_3\,(\ell n\ X) \;=\; \Psi''(b)/c^3 \tag{4.26}$$

is positive (negative) whenever c is negative (positive).

Therefore we see that

$$\frac{2 + \mu_3\,(\ell n\ X)}{\left| -1+\mathrm{Var}(\ell n\ X) \right|^{3/2}} = \begin{cases} \dfrac{\Psi''(b)}{[\Psi'(b)]^{3/2}} \ (<0) & \text{if } c > 0 \\[2ex] \dfrac{-\Psi''(b)}{[\Psi'(b)]^{3/2}} \ (>0) & \text{if } c < 0, \end{cases} \tag{4.27}$$

and thus we may establish the result

$$-\left| \frac{2 + \mu_3\,(\ell n\ X)}{\left| -1+\mathrm{Var}\,(\ell n\ X) \right|^{3/2}} \right| = -\left| \frac{\Psi''(b)}{[\Psi'(b)]^{3/2}} \right| \;=\; \Phi(b), \tag{4.28}$$

the last equality following as a result of equation (2.52). We may then propose, as an estimate of $\Phi(b)$,

$$\Phi^* \quad - \left| \frac{2 + m_3\,(\ell n\ X)}{\left| -1 + s^2_{\ell n\,X} \right|^{3/2}} \right| , \tag{4.29}$$

where

$$m_3(\ell n\ X) = n \sum_{i=1}^{n} (z_i - \overline{z})^3/(n-1)(n-2), \tag{4.30}$$

for $\overline{z} = \sum_{i=1}^{n} z_i/n$ and $z_i = \ell n\ x_i$, $i = 1, 2, \ldots, n$, is an unbiased estimate of $\mu_3(\ell n\ X)$, and where $s^2_{\ell n\ X}$, an unbiased estimate of Var $(\ell n\ X)$, is given by equation (2.50).

With Φ^*, one may estimate the shaping parameter b by referring to Figure 1. Denoting this estimate by b^*, we may then proceed to estimate the exponentiating parameter by considering equation (4.24), taking as our estimate

$$c^*(b^*) = \pm [\Psi'(b^*)]^{1/2} / \left| -1 + s^2_{\ell n\ X} \right|^{1/2}. \tag{4.31}$$

Since the statistic within the outermost pair of absolute-value signs of the right-hand side of equation (4.29) estimates the expression given in equation (4.27), we see that we may use, as a criterion for determining the sign of c^*, the sign of the quantity, $2 + m_3(\ell n\ X)$; i.e., we take c^* to be positive (negative) if this quantity is negative (positive).

Finally, to estimate the scaling parameter, a, we may consider estimators suggested by equations (4.12) and (4.20); viz.,

$$a_1^*(b^*, c^*) = 2\,\Gamma(b^*) \sum_{i=1}^{n} x_i/n\ \Gamma(b^* + 1/c^*) \tag{4.32}$$

and

$$a_2^*(b^*, c^*) = \left\{ \prod_{i=1}^{n} x_i \right\}^{1/n} \exp\left[1 - \Psi(b^*)/c^*\right]. \tag{4.33}$$

The first estimator has the advantage of being unbiased (whenever b^* and c^* may be considered fixed, no longer random variables), though it is not calculable whenever $b^* + 1/c^*$ is negative [cf: equation (4.12)]. Regardless of the sign of c^*, the second estimator is calculable; however, for fixed b^* and c^*, the resulting estimate is biased for a.

However, reference to equation (4.15) indicates that a bias correction factor may be obtained for $a_2^*(b^*, c^*)$; i.e., for fixed b^* and c^*, we may form the unbiased estimate

$$a_3^*(b^*, c^*) = \frac{\Gamma^n(b^*)(1+1/n)^n \left\{\prod_{i=1}^{n} x_i\right\}^{1/n}}{\Gamma^n(b^* + 1/nc^*)} . \qquad (4.34)$$

This estimator may, like $a_1^*(b^*, c^*)$, suffer the disadvantage of being incalculable in certain cases when $c^* < 0$; viz., whenever $b^* + 1/nc^*$ is negative.

Let us consider b and c as being fixed values. Then, with the variances of our two unbiased estimates of the scaling parameter given by

$$\sigma_1^2 = \mathrm{Var}[a_1^*(b, c)] = \frac{a^2}{n}\left\{\frac{4\Gamma(b+2/c)\Gamma(b)}{3\Gamma^2(b+1/c)} - 1\right\} \qquad (4.35)$$

and

$$\sigma_3^2 = \mathrm{Var}[a_3^*(b, c)] = a^2\left\{\frac{(1+1/n)^{2n}\Gamma^n(b+2/nc)\Gamma^n(b)}{(1+2/n)^n\Gamma^{2n}(b+1/nc)} - 1\right\}, \qquad (4.36)$$

one might consider the formation of the unbiased estimator

$$a_4^*(b, c) = Ba_1^*(b, c) + (1 - B)a_3^*(b, c), \qquad (4.37)$$

where

$$B = (\sigma_1^2 - \sigma_{13})/(\sigma_1^2 + \sigma_3^2 - 2\sigma_{13}) \qquad (4.38)$$

with

$$\sigma_{13} = \mathrm{Cov}[a_1^*(b, c), a_3^*(b, c)]$$

or

$$\sigma_{13} = a^2 \left\{ \frac{2(1+1/n)\ \Gamma(b)\ \Gamma[b+(n+1)/nc]}{(2+1/n)\ \Gamma(b+1/c)\ \Gamma(b+1/nc)} - 1 \right\} . \tag{4.39}$$

By construction, B is selected so that the variance of $a_4^*(b, c)$,

$$\sigma_4^2 = [(\sigma_1^2)(\sigma_3^2) - (\sigma_{13})^2]/[\sigma_1^2+\sigma_3^2-2\sigma_{13}], \tag{4.40}$$

is the minimum variance for all unbiased estimators which are linear combinations of $a_1^*(b, c)$ and $a_3^*(b, c)$; thus, σ_4^2 is not greater than the minimum of the two quantities σ_1^2 and σ_3^2. [See page 323 of Wilks [3].) Though $a_4^*(b, c)$ is more tedious to calculate than either of the estimators given by equations (4.32) and (4.34), the resulting reduction in variance may well justify its use.

It is conceivable that one (or two) of the three parameters of the bivariate density (4.3) might, for specific situations, be assumed known. For example, with h(y) taken to be the Weibull distribution (a case we shall examine more closely in Chapter VII), we need estimate only the scaling and exponentiating parameters, for the shaping parameter is assumed to be unity (b = 1). Such cases could be handled individually and are, with the exception of the case where the marginal density of failure-times is assumed to be Weibull, left for the reader. Procedures for such cases would be derived in a manner analogous to that of sections 3 and 4 of Chapter II.

CHAPTER V

THE BETA/STACY BIVARIATE DISTRIBUTION

Having seen that a uniform distribution may serve as a conditional distribution for the warning times, one might be led to inquire of the possibility of using more general distributions in this role. In this chapter we examine the employment of a Beta distribution (normalized over the interval between 0 and the given y) and, noting the difficulties encountered in attempting to estimate the additional parameters thus introduced, we then consider possible restrictions of the parameter space of this Beta/Stacy distribution.

1. Preliminary Results

Suppose we now consider a bivariate density, defined over the range $0 < x < y < \infty$, such that

$$f(x, y) = g(x \mid y; p, q)\, h(y), \tag{5.1}$$

where $h(y)$ is the marginal distribution of Y (defined over $0 < y < \infty$) and

$$g(x \mid y; p, q) = x^{p-1}(y-x)^{q-1}/B(p, q)\, y^{p+q-1} \tag{5.2}$$

is the Beta distribution, indexed by positive parameters p and q and defined over the range $0 < x < y$. The function $B(p, q)$ is the Beta-function, equal to $\Gamma(p)\,\Gamma(q)/\Gamma(p+q)$.

The marginal distribution of X (Again, this variable will refer to our warning times.) we may denote by

$$g(x; p, q) = \int_{y=x}^{\infty} f(x, y)\, dy, \tag{5.3}$$

though it is also indexed by the parameters specifying h(y). Before circumscribing h(y) more specifically, however, we note that a number of pertinent results may be summarized without such a restriction. This summarization is the content of the following theorem.

Theorem: Let f(x, y) be the bivariate probability density function as described by equations (5.1) and (5.2). Then we have the following results.

(A) For $X \sim g(x; p, q)$, as given in equation (5.3), the random variable $U = Y - X \sim g(u; q, p)$, provided that h(y) is not indexed by the parameters p and q in other than a symmetrical manner.

(B) Whenever the (s + t)-th moment of Y exists,

$$E[X^s Y^t] = B(p + s, q)\, E[Y^{s+t}]/B(p, q), \tag{5.4}$$

provided that $s > -p$.

(C) The cumulative distribution function associated with the random variable X can be expressed, at point t, as

$$G(t) = H(t) + \int_{y=t}^{\infty} \left\{ B_v(p, q) h(y)/B(p, q) \right\} dy \tag{5.5}$$

where $v = t/y$ and where

$$B_v(p, q) = \int_0^v u^{p-1}(1-u)^{q-1}\, du. \tag{5.6}$$

(D) Whenever the appropriate moments of Y exist,

$$E[X^s] = B(p+s, q)\, E[Y^s]/B(p, q), \qquad \text{for } s > -p, \tag{5.7}$$

$$E[X] = p\,E(Y)/(p+q), \tag{5.8}$$

$$\mathrm{Var}\,[(p+q)X/p] = \mathrm{Var}\,[Y] + qE(Y^2)/p(p+q+1), \tag{5.9}$$

$$\mathrm{Cov}\,[X, Y] = \left(\frac{p}{p+q}\right) \mathrm{Var}\,(Y) \tag{5.10}$$

(E) Whenever $E(Y^2)$ exists and whenever Y has not a degenerate distribution,

$$0 < \mathrm{Corr}\,(X, Y) = \sqrt{\frac{p^2\,\mathrm{Var}\,(Y)}{(p+q)^2\mathrm{Var}\,(X)}} < 1 \tag{5.11}$$

Proof:

(A) Letting $U = Y - X$, $V = Y$, we transform to obtain the bivariate density

$$f^*(u, v) = \left\{u^{q-1}(v-u)^{p-1}/B(p, q)\, v^{p+q-1}\right\} h\,(v),$$

over the range $0 < u < v < \infty$. The Beta function $B(p, q)$ is symmetric in its arguments, so that we see that the conditional distribution of U given V (i.e., given Y) is $g(u \mid y; q, p)$. Thus we have the marginal distribution of $U = Y - X$ given by

$$\begin{aligned} g^*(u) &= \int_{v=u}^{\infty} f^*(u, v)\,dv \\ &= \int_{v=u}^{\infty} g(u \mid v; q, p)h\,(v)\,dv, \end{aligned}$$

Therefore, whenever $h(v)$ depends neither upon p nor upon q (unless p and q are symmetric in this indexing of $h(v)$), we have upon reference to equation (5.3), the desired result; viz., that the marginal distribution of $U = Y - X$ is $g\,(u; q, p)$.

(B) By definition,

$$E[X^sY^t] = \int_{y=o}^{\infty}\int_{x=o}^{y} x^{s+p-1}(y-x)^{q-1}y^{-p-q+t+1}h(y)dxdy/B(p, q),$$

Upon performing the interior integration, we have, for $s > -p$,

$$E[X^sY^t] = \int_{y=o}^{\infty} h(y)y^{s+t}B(p+s, q)dy/B(p, q)$$

$$= \frac{B(p+s, q)}{B(p, q)} E[Y^{s+t}],$$

whenever the latter moment exists.

(C) By definition,

$$G(t) = \int_{x=o}^{t} g(x; p, q)dx = \int_{x=o}^{t}\int_{y=x}^{\infty} f(x, y)\, dy\, dx.$$

Reconsidering the area of integration, we have

$$G(t) = \int_{y=o}^{t}\int_{x=o}^{y} f(x, y)\, dx\, dy + \int_{y=t}^{\infty}\int_{x=o}^{t} f(x, y)\, dx\, dy$$

$$= \int_{y=o}^{t} h(y) \int_{x=o}^{y} g(x\,|y; p, q)\, dx\, dy$$

$$+ \int_{y=t}^{\infty} h(y) \int_{x=o}^{t} g(x\,|y; p, q)\, dx\, dy$$

$$= H(t) + \int_{y=t}^{\infty} B_v\,(p, q)\, h(y)/B(p, q)\, dy.$$

(D) The result stated in equation (5.7) follows as a corollary to part (B) of this theorem; similarly for the result in equation (5.8). As for that in equation (5.9), we note that

$$Var\,[\,(p + q)\,X/p] = \frac{(p+q)^2}{p^2} Var\,(X)$$

$$= \frac{(p+q)^2}{p^2}\left[\frac{(p)(p+1)}{(p+q)(p+q+1)} E(Y^2) - \frac{p^2}{(p+q)^2}E^2(Y)\right]$$

$$= Var\,(Y) + \frac{q}{p(p+q+1)} E(Y^2),$$

whenever $E(Y^2)$ exists.

Then,

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

$$= \left(\frac{p}{p+q}\right) E(Y^2) - \left(\frac{p}{p+q}\right) E^2(Y) = \left(\frac{p}{p+q}\right) \text{Var}(Y).$$

(E) Finally, providing $0 < \text{Var}(Y) < \infty$,

$$\text{Corr}(X, Y) = \sqrt{\frac{p^2 \text{Var}(Y)}{(p+q)^2 \text{Var}(X)}} .$$

The positive correlation of X and Y is assured by the positiveness of each of the factors beneath the radical; the right-hand inequality follows upon a consideration of (5.9). Q.E.D.

Suppose we take the marginal distribution, h(y), to be Stacy's generalized Gamma distribution, given by equation (1.2). We are then considering the bivariate density

$$f(x, y; a, b, c, p, q) = |c| x^{p-1}(y-x)^{q-1} y^{bc-p-q} e^{-(y/a)^c} / a^{bc} \Gamma(b) B(p,q), \tag{5.12}$$

for positive parameters a, b, p, and q, for real parameter c, and for positive random variables X and Y such that $x < y < \infty$. (With $p = q = 1$, we have the Uniform/Stacy bivariate density, discussed at length in the preceding chapter.) Using, as the marginal distribution of Y, Stacy's generalized gamma distribution, allows us, by referring to equations (1.4), (1.5), (1.6), and (1.7), to specify the moments and distributions mentioned in the preceding theorem.

In addition, we note that we may obtain the joint moment-generating function of $Z = \ell n\, Y$ and $W = \ell n\, X$ as

$$E[e^{sW + tZ}] = E[X^s Y^t],$$

which, for $s > -p$, becomes

$$E[e^{sW+tZ}] = B(p+s, q)E[Y^{s+t}]/B(p, q),$$

or

$$E[e^{sW+tZ}] = a^{s+t} \frac{\Gamma(p+s)\,\Gamma(p+q)\Gamma[b+(s+t)/c]}{\Gamma(p)\,\Gamma(p+q+s)\Gamma(b)}, \quad (5.13)$$

provided $(s+t)/c > -b$, and $s > -p$. From equation (5.13), we may determine the following moments

$$E(\ell n\, X) = \ell n\, a + \Psi(p) - \Psi(p+q) + \Psi(b)/c, \quad (5.14)$$

or

$$E(\ell n\, X) = E(\ell n\, Y) + \Psi(p) - \Psi(p+q),$$

and

$$\mathrm{Var}\,(\ell n\, X) = \Psi'(p) - \Psi'(p+q) + \Psi'(b)/c^2, \quad (5.15)$$

or

$$\mathrm{Var}\,(\ell n\, X) = \mathrm{Var}\,(\ell n\, Y) + \Psi'(p) - \Psi'(p+q),$$

and

$$\mu_3(\ell n\, X) = \Psi''(p) - \Psi''(p+q) + \Psi''(b)/c^3, \quad (5.16)$$

or

$$\mu_3(\ell n\, X) = \mu_3(\ell n\, Y) + \Psi''(p) - \Psi''(p+q),$$

where the Ψ-function and its derivatives are described in Chapter XXIV of Edwards [7]. We note that, since $q > 0$ implies that $\Psi(p) < \Psi(p+q)$, $\Psi'(p) > \Psi'(p+q)$, and $\Psi''(p) < \Psi''(p+q)$, we are able to establish the inequalities

$$\left.\begin{array}{l} E(\ell n\, X) < E(\ell n\, Y), \\ \mathrm{Var}\,(\ell n\, X) > \mathrm{Var}\,(\ell n\, Y), \text{ and} \\ \mu_3(\ell n\, X) < \mu_3(\ell n\, Y). \end{array}\right\} \quad (5.17)$$

2. Parameter Estimation for the Beta/Stacy Bivariate Distribution

Suppose now that we have a random sample of size n taken from density (5. 12). Denoting these observarions by (x_i, y_i), $i = 1, \ldots, n$, we may derive the maximum-likelihood equations associated with parameters p and q:

$$\frac{\partial \ln L}{\partial p} = -n\,\Psi(p) + n\,\Psi(p+q) + \sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \ln y_i = 0 \quad (5.18)$$

$$\frac{\partial \ln L}{\partial q} = -n\,\Psi(q) + n\Psi(p+q) + \sum_{i=1}^{n} \ln (y_i - x_i) - \sum_{i=1}^{n} \ln y_i = 0. \quad (5.19)$$

The remaining maximum-likelihood equations are identically those presented previously as equations (2. 1), (2. 2), and (2. 3). The resulting set of five equations does not readily suggest, for any one of the parameters, an estimation technique which uses only the observed warning times, x_i, $i = 1, 2, \ldots, n$. In addition, any attempt to solve explicitly the maximum-likelihood equations associated with the marginal distribution of X would apparently be thwarted by the complexity of their expressions.

Nevertheless, it is possible to establish the Cramer-Rao lower bounds for the variances of unbiased estimates of the parameters of density (5. 12). The lower bounds for unbiased estimates of a, b, and c are those presented in section 7 of Chapter I; for p^* and q^* unbiased estimates of p and q, respectively,

$$\mathrm{Var}\,(p^*) \geq 1/n[\Psi'(p) - \Psi'(p+q)] \quad (5.20)$$

and

$$\mathrm{Var}\,(q^*) \geq 1/n[\Psi'(q) - \Psi'(p+q)]. \quad (5.21)$$

In order to ascertain the statistics sufficient for each of the parameters of density (5. 12), we consider again the K-function of Lehmann and Scheffe. [5]. For brevity we merely list these results, noticing that

$\prod_{i=1}^{n}(x_i/y_i)$, or, equivalently, $\sum_{i=1}^{n} \ell n\, (x_i/y_i)$, is sufficient for p,

$\prod_{i=1}^{n}[(y_i-x_i)/y_i]$, or, equivalently, $\sum_{i=1}^{n} \ell n[1-(x_i/y_i)]$, is sufficient for q,

$\prod_{i=1}^{n} y_i$, or, equivalently, $\sum_{i=1}^{n} \ell n\, y_i$, is sufficient for b,

and, whenever c is known, $\sum_{i=1}^{n} y_i^{\,c}$ is sufficient for a. Again, the form of density (5. 12) is not proper to admit a single statistic sufficient for c. [Pitman, 6]

Thus, with our random sample (x_i, y_i), $i = 1, 2, \ldots, n$, it would appear that our estimation technique would be to

(1) Estimate a, b, and c by the appropriate technique of Chapter II,

(2) Solve iteratively equations (5. 18) and (5. 19) for p and q.

However, it would seem feasible in many applications to restrict the parameter space associated with density (5. 12) by assuming one or more of the parameter values to be known. Assigning values to one (or two) of the parameters of the set a, b, c would be entirely reasonable in cases where the distribution of failure times can be assumed to be one of the subfamilies listed in section 3 of Chapter I; in such cases, the estimation of the remaining parameters in this set is straightforward, [See sections 3 and 4 of Chapter II.], and the estimation of p and q would then follow from the iterative solution of equations (5. 18) and (5.19).

Suppose, however, that one is willing to specify the conditional distribution (5.2) by assigning values to p and q. [For example, p = q = 1 specifies a uniform conditional distribution, thereby designating the bivariate density of the preceding chapter.]

Letting

$$\Delta_1(p, q) = \Psi'(p) - \Psi'(p+q) \tag{5.22}$$

and

$$\Delta_2(p, q) = -\Psi''(p+q) + \Psi''(p), \tag{5.23}$$

we see that equations (5.14), (5.15), and (5.16), with p and q each known, suggest a method for estimating those parameters indexing the marginal distribution of Y by using only the observed warning-times $x_1, x_2, \ldots, x_n$:

(1) Estimate $\phi(b)$ by computing

$$\tilde{\phi} = -\left| \frac{m_3(\ell n\, X) - \Delta_2(p, q)}{\left| s_{\ell n\, X}^2 - \Delta_1(p, q) \right|^{3/2}} \right|, \tag{5.24}$$

where $\Delta_1(p, q)$ and $\Delta_2(p, q)$ are given by equations (5.22) and (5.23) [c.f. Figure 2], where $m_3(\ell n\, X)$ is as given by equation (4.30), and where $s_{\ell n\, X}^2$ is as given by equation (2.50). Using the graph of $\phi(b)$ in Figure 1 at the end of Chapter II, we obtain the estimate of b, say, $\tilde{b}$.

(2) Estimate c by computing

$$\tilde{c} = \pm [\Psi'(\tilde{b})]^{1/2} / \left| s_{\ell n\, X}^2 - \Delta_1(p, q) \right|^{1/2}, \tag{5.25}$$

taking the positive (negative) sign if $m_3(\ell n\, X) - \Delta_2(p, q)$, used in calculating $\tilde{\phi}$, is negative (positive). The quantity $[\Psi'(\tilde{b})]^{1/2}$ is readily obtainable from Figure 1 at the end of Chapter II; the denominator of $\tilde{c}$ is available from the calculations used in obtaining $\tilde{\phi}$.

(3) Estimate a by any one of the four estimators:

$$\tilde{a}_1(\tilde{b}, \tilde{c}, p, q) = (p+q)\,\Gamma(\tilde{b}) \sum_{i=1}^{n} x_i / np\,\Gamma(\tilde{b} + 1/\tilde{c}), \quad (5.26)$$

$$\tilde{a}_2(\tilde{b}, \tilde{c}, p, q) = \left\{\prod_{i=1}^{n} x_i\right\}^{1/n} \exp - \left\{\Psi(p) + \Psi(p+q) - \Psi(\tilde{b})/\tilde{c}\right\}, \quad (5.27)$$

$$\tilde{a}_3(\tilde{b}, \tilde{c}, p, q) = \frac{\Gamma^n(p)\,\Gamma^n(p+q+\frac{1}{n})\,\Gamma^n(\tilde{b})\left\{\prod_{i=1}^{n} x_i\right\}^{1/n}}{\Gamma^n(p+q)\,\Gamma^n(p+\frac{1}{n})\,\Gamma^n(\tilde{b}+\frac{1}{n\tilde{c}})}, \quad (5.28)$$

or

$$\tilde{a}_4(\tilde{b}, \tilde{c}, p, q) = A\,\tilde{a}_1(\tilde{b}, \tilde{c}, p, q) + (1-A)\tilde{a}_3(\tilde{b}, \tilde{c}, p, q). \quad (5.29)$$

For fixed b and c, each of these estimators, save $\tilde{a}_2(b, c, p, q)$, is unbiased for a. The coefficient A, used in calculating $\tilde{a}_4(b, c, p, q)$, is given by

$$A = (\sigma_1^2 - \sigma_{13})/[\sigma_1^2 + \sigma_3^2 - 2\sigma_{13}], \quad (5.30)$$

where

$$\sigma_1^2 = \mathrm{Var}[\tilde{a}_1(b, c, p, q)] = \frac{a^2}{n}\left\{\frac{(p+q)(p+1)}{(p+q+1)(p)}\;\frac{\Gamma(b+2/c)\,\Gamma(b)}{\Gamma^2(b+1/c)} - 1\right\}, \quad (5.31)$$

$$\sigma_3^2 = \mathrm{Var}[\tilde{a}_3(b, c, p, q)], \quad (5.32)$$

or

$$\sigma_3^2 = a^2\left\{\frac{\Gamma^n(p)\,\Gamma^n(b)\,\Gamma^n(b+2/nc)\,\Gamma^n(p+2/n)\,\Gamma^{2n}(p+q+1/n)}{\Gamma^n(p+q)\,\Gamma^{2n}(b+1/nc)\,\Gamma^{2n}(p+1/n)\,\Gamma^n(p+q+2/n)} - 1\right\},$$

and

$$\sigma_{13} = \mathrm{Cov}[\tilde{a}_1(b, c, p, q),\ \tilde{a}_3(b, c, p, q)] \quad (5.33)$$

$$= a^2\left\{\frac{(p+q)(np+1)\,\Gamma(b)\,\Gamma(b+(n+1)/nc)}{np(np+nq+1)\,\Gamma(b+1/c)\,\Gamma(b+1/nc)} - 1\right\}.$$

By this construction, Wilks on page 323 shows that $\tilde{a}_4(b, c, p, q)$ has variance not greater than the minimum of σ_1^2 and σ_3^2; viz.,

$$\sigma_4^2 = \mathrm{Var}[\tilde{a}_4(b, c, p, q)] = \frac{\sigma_1^2\sigma_3^2 - (\sigma_{13})^2}{\sigma_1^2 + \sigma_3^2 - 2\sigma_{13}} \tag{5.34}$$

3. A Beta/Gamma Distribution

We have seen that, in the absence of failure-time (Y) data, we encounter difficulty in estimating the five parameters of density (5.12). Furthermore, we have seen how complete specification of the conditional density (5.2) allows us to estimate the parameters of the failure-time distribution, using only warning-times in so doing.

We now consider another interesting restriction of the parameter space associate with bivariate density (5.12). Suppose we assume that $c = 1$, so that, for the marginal distribution of our failure-times, we restrict our attention to the subfamily of two-parameter Gamma distributions. If, in addition, we assume that $(p+q) = b$, we are focusing attention on the bivariate density

$$f(x, y; a, b, 1, p, b-p) = \frac{x^{p-1}(y-x)^{b-p-1}}{y^{b-1}B(p, b-p)} \; \frac{y^{b-1}e^{-y/a}}{a^b\Gamma(b)},$$

or, equivalently,

$$f(x, y; a, b, 1, p, b-p) = x^{p-1}(y-x)^{b-p-1}e^{-y/a}/a^b\Gamma(p)\Gamma(b-p) \tag{5.35}$$

for $0 < x < y < \infty$, and for $a > 0$, $0 < p < b$.

One of the most interesting properties of this bivariate density is that the marginal density of the warning-times becomes

$$g(x; p, b-p) = h(x; a, p, 1), \tag{5.36}$$

a two-parameter Gamma distribution. Thus, by part (A) of the theorem in section 1 of this chapter, we see that, with $b = p + q$, a symmetric function of p and q, the random variable $U = Y - X$ has two-parameter Gamma density:

$$U = Y - X \sim h(u; a, b-p, 1). \qquad (5.37)$$

In addition, since the product of the densities (5.36) and (5.37) is the same as the joint probability density function of X and U, we see that X and $U = Y - X$ are independently distributed Gamma variates with common scale parameter. Thus, from result (1.9), we would have the marginal density of Y:

$$Y \sim h(y; a, b, 1). \qquad (5.38)$$

We have then established the following theorem:

Theorem: Let X and Y be two random variables defined over the range $0 < x < y < \infty$. Let $Y \sim h(y; a, b, 1)$, $U = Y - X$, and $(p + q) = b$. Then $X \sim h(x; a, p, 1)$ and is independent of $U \sim h(u; a, q, 1)$ if and only if the conditional density of X given Y is $g(x \mid y; p, q)$.

This result is most pleasing in that it allows us to discuss warning-times and failure-times which are each distributed according to a common and familiar family of probability density functions. In addition, their joint probability density function is dependent upon only three parameters; viz., a, b, and p. The marginal density of warning-times is indexed by two: a and p, so that, if one utilizes the appropriate procedure outlined in section 3 of Chapter II, estimation of a and p, using only the sample of warning-times, would follow directly. Estimation of b would require information provided only by the failure-times (or, say a truncated subset of these, as is indicated by Chapman [11]).

4. A Simplified Conditional Distribution

In this section, we propose another restriction of the parameter space associated with bivariate density (5.12). We note from equation (5.8) that the parameters p and q are directly proportional; viz., $p\,E(Y-X) = q\,E(X)$. Thus, assigning any fixed value, say q_o, to q might not severely restrict our bivariate density (5.12); especially not if the proportional relationship between the mean of X and that of Y is deemed important.

Let us consider bivariate density (5.12) with $q = 1$, i.e.,

$$f(x, y; a, b, c, p, 1) = |c|p\, x^{p-1} y^{bc-p-1} e^{-(y/a)^c} / a^{bc}\Gamma(b), \tag{5.39}$$

for $0 < x < y < \infty$ and for parameters $a > 0$, $b > 0$, $c \neq 0$, and $p > 0$. The conditional distribution of X given Y becomes

$$g(x|y;\ p,\ 1) = px^{p-1}/y^p, \qquad \text{for } 0<x<y, \text{ and } p>0, \tag{5.40}$$

which we see to be a one-parameter generalization of the conditional distribution used in the preceding chapter.

Though this restriction on q limits the shape of the resulting conditional density (5.2), its use leads to several simplifications whenever one considers the properties of bivariate density (5.39). For example, results (5.4), (5.5), (5.8), and (5.9) become

$$E[X^s Y^t] = pE[Y^{s+t}]/(p+s); \text{ for } s \neq -p, \tag{5.41}$$

$$G(t) = H(t) + t^p \int_{y=t}^{\infty} y^{-p}\, h(y)\, dy, \tag{5.42}$$

or, whenever h(y) is as defined by equation (1.2),

$$G(t) = H(t) + \frac{t}{p} g(t; a, b-p/c, c, p) \qquad \text{[See (3.1).]},$$

$$E[X] = p\, E[Y]/(p+1), \tag{5.43}$$

and

$$\text{Var}[X] = \frac{p^2}{(p+1)^2} \text{Var}(Y) + \frac{p\, E(Y^2)}{(p+1)^2(p+2)}. \tag{5.44}$$

Similarly the moments given by equations (5.14), (5.15), and (5.16) are simplified:

$$E(\ln X) = [\ln a + \Psi(b)/c] - 1/p, \tag{5.45}$$

$$\text{Var}(\ln X) = [\Psi'(b)/c^2] + 1/p^2, \tag{5.46}$$

and

$$\mu_3(\ln X) = [\Psi''(b)/c^3] - 2/p^3. \tag{5.47}$$

We note also that the marginal distribution of warning-times, as implied by equation (5.42) is a distended Gamma distribution, as described in Chapter III; viz.,

$$X \sim g(x; a, b-p/c, c, b). \tag{5.48}$$

With the distribution of failure-times specified by Stacy's generalized Gamma distribution, equation (5.43) becomes

$$E[X] = ap\, \Gamma(b+1/c)]/(p+1)\, \Gamma(b). \tag{5.49}$$

Together with the logarithmic moments presented in equations (5.45), (5.46), and (5.47), this moment should allow us to estimate the four parameters of our bivariate density (5.39) by using only the warning-times $x_1, \ldots, x_n$. Since explicit solutions for the parameters in these four equations (with sample moments replacing theoretical moments therein) are not apparent, an iterative technique is suggested. Unfortunately, statements regarding the statistical properties of the resulting estimators would be primarily speculative in nature.

CHAPTER VI

AN EXTENSION OF THE UNIVARIATE WEIBULL DISTRIBUTION

In Chapter I we noted that the univariate density (1. 2) was a generalization of Stacy's original generalized gamma distribution (1. 1). This generalization was made possible by noting the permissibility of the exponentiating parameter's assuming any real value. In this chapter we intend to display some of the effects, some of the advantages, and some of the possible disadvantages of extending similarly the parameter space of the Weibull distribution.

1. Preliminary Remarks

The univariate Weibull [12] distribution has been mentioned (Chapter I) as that special case of density (1. 2) wherein the shaping parameter has value $b = 1$ and the exponentiating parameter, c, assumes only positive values; i.e., a positive-valued random variable, Y, has been said to possess the Weibull distribution if its probability density function were given by

$$f(y; a, c) = cy^{c-1}e^{-(y/a)^c}/a^c, \tag{6.1}$$

for positive parameters a and c, termed now, respectively, the scale and shape parameters.

Suppose that we have a random, untruncated, unordered sample

y_i, $i = 1, 2, \ldots, n$, taken from density (6.1) and that we seek to estimate the parameters a and/or c. The current techniques for the parameter estimation from such a sample include iterative methods derived from the maximum-likelihood equations [13], a least-squares method which can be facilitated by the use of special probability graph paper [14], a method implied on page 22 by D. R. Cox [15], who utilizes the method-of-moments, and a method presented by M. V. Menon [16], whose conclusions followed an application of the method-of-moments to the random variable $\ell n\, Y$.

Before presenting our extension of the Weibull distribution, let us pause momentarily to discuss the last two estimation techniques. For $Y \sim f(y; a, c)$, we have

$$\left.\begin{aligned} E(Y) &= a\Gamma(1 + 1/c) \\ \text{and} \quad E(Y^2) &= a^2\Gamma(1 + 2/c). \end{aligned}\right\} \qquad (6.2)$$

Thus we see that the monotone function

$$R(d) = E^2(Y)/E(Y^2) = dB(d, d)/2\,, \qquad (6.3)$$

where

$$B(u, v) = \Gamma(u)\,\Gamma(v)/\Gamma(u+v)$$

and $d = 1/c$, is independent of the scale parameter. Thus, calculation of

$$\hat{R} = \left(\sum_{i=1}^{n} y_i\right)^2 / \left(n \sum_{i=1}^{n} y_i^2\right) \qquad (6.4)$$

would provide an estimate of $R(d)$, the graph of which would, in turn, yield an estimate, $c_R = d_R^{-1}$, of the Weibull shape parameter.

This graph, for the reader's convenience, is provided in Figure 3, located at the end of this chapter.

Once the Weibull shape parameter has been estimated, one could then invoke any of the estimators for Stacy's scaling parameter, listed in Table I. We note, however, that the invocation of

$$\hat{a}_5(1, c_R) = \sum_{i=1}^{n} y_i/n\ \Gamma(1 + 1/c_R)$$

would provide an estimate with the interesting property that its variance (with c_R considered constant, no longer a random variable) becomes expressible as

$$\mathrm{Var}[\hat{a}_5(1, c_R)] = a^2[1-R(d_R)]/nR(d_R) = a^2(1-\hat{R})/n\hat{R}. \quad (6.5)$$

Now, in Menon's estimation techniques we note that, in the case where the scale parameter is assumed fixed, the procedure may lead to an undesirable negative estimate of the shape parameter. Menon very adeptly suggests an alternative (though essentially the same) method for avoiding this possible embarrassment. [16] We shall demonstrate that, by extending the parameter space associated with the Weibull distribution, we may avoid being concerned about obtaining any such negative estimate.

2. Extending the Weibull Distribution Parameter Space

Consider the probability density function

$$h(y; a, c) = |c|\, y^{c-1}\, e^{-(y/a)^c}/a^c, \quad (6.6)$$

for a positive random variable Y, positive scale parameter a, and real shape parameter c. This distribution, it may be noted, corresponds to h(y; a, 1, c), as given by equation (1.2).

We may therefore establish the results that, for $Y \sim h(y; a, c)$,

$$Z = Y/k \sim h(z; a/k, c) \tag{6.7}$$

for any positive k, and

$$W = Y^p \sim h(w; a^p, c/p) \tag{6.8}$$

for any $p \neq 0$. Now, for $c \neq 0$, the cumulative distribution function becomes:

$$H(y; a, c) = \begin{cases} 1 - \exp[-(y/a)^c] & \text{if } c > 0 \\ \exp[-(y/a)^c] & \text{if } c < 0, \end{cases} \tag{6.9}$$

and the moments, for $K = 1, 2, \ldots$, are given by

$$\mu_K'(Y) = E(Y^K) = \begin{cases} a^K \Gamma(1+K/c), & \text{if } K/c > -1, \text{ and} \\ +\infty, & \text{if } -K \leq c < 0. \end{cases} \tag{6.10}$$

An important property associated with any reliability distribution, such as density (6.6), is the age-specific failure rate. [Cox, 15] This function becomes, for density (6.6),

$$\phi(y) = \begin{cases} cy^{c-1}/a^c, & \text{if } c > 0 \\ \dfrac{-cy^{c-1}e^{-(y/a)^c}}{a^c[1-e^{-(y/a)^c}]}, & \text{if } c < 0. \end{cases} \tag{6.11}$$

Whenever $0 < c < 1$, the function $\phi(y)$ is monotone decreasing; but, $c > 1$ implies $\phi(y)$ is monotone increasing, and $c = 1$ provides the result that $\phi(y) = 1/a$, constant. However, an appropriate analysis of $\phi(y)$, for negative values of c, leads to the conclusion that this function cannot be monotone. This fact, coupled with the fact, implied by equation (6.10), that only a finite number of moments exist whenever $c < 0$, will probably hamper proposed applications of density (6.6) to reliability studies.

Suppose now that we have a random sample of n unordered, untruncated observations, y_i, $i = 1, 2, \ldots, n$ from density (6.6). We may define three pertinent parameter system states (P.S.S.):

P.S.S. I: c assumed known (fixed), a to be estimated
P.S.S. II: a assumed known (fixed), c to be estimated
P.S.S. III: a, c to be estimated jointly.

Now let us consider Menon's approach to the estimation of the Weibull parameters. [16] Analogous to his procedure, we find the first three moments of the random variable $Z = \ell n\, Y$ to be

$$\left.\begin{aligned} \mu_1'(\ell n\, Y) &= E(\ell n\, Y) = \ell n\, a + \lambda_1/c\,, \\ \mu_2(\ell n\, Y) &= E[\ell n\, Y - E(\ell n\, Y)]^2 = (\lambda_2 - \lambda_1^2)/c^2, \text{ and} \\ \mu_3(\ell n\, Y) &= E[\ell n\, Y - E(\ell n\, Y)]^3 = (\lambda_3 - 3\lambda_2\lambda_1 + 2\lambda_1^3)/c^3, \end{aligned}\right\} \quad (6.12)$$

where the

$$\lambda_i = \int_0^\infty (\ell n\, x)^i e^{-x}\, dx, \quad i = 1, 2, 3,$$

are defined by Mennon [16]; the pertinent values are

$$\left.\begin{aligned} \lambda_1 &= -0.5772, \\ (\lambda_2 - \lambda_1^2) &= \Pi^2/6, \text{ and} \\ (\lambda_3 - 3\lambda_2\lambda_1 + 2\lambda_1^3) &= -2.4036. \end{aligned}\right\} \quad (6.13)$$

We note that the quantity $\mu_3(\ell n\, Y)$ is negative (positive) according to whether c is positive (negative).

Now, for P.S.S. I, our estimation technique for the scale parameter may be selected from any of the estimators $\hat{a}_i[1, c], i = 1, 2, \ldots, 5$, as listed in Table I. For P.S.S. II, one might estimate c by considering estimator $\hat{c}(a, 1)$, as given by equation (2.9); here we note, as did

Menon [16], the possibility of obtaining a negative estimate of the Weibull shape parameter, but this possibility no longer concerns us, since negative values of this parameter are deemed permissible under the definition of density (6.6).

Estimation in the event that P.S.S. III exists is precisely as Menon [16] suggests, except that, as in section 2 of Chapter II, we choose the sign of the estimate of c to be positive (negative) if the sign of the third central sample moment,

$$\sum_{i=1}^{n} [\ell n\ Y_i - (\frac{1}{n} \sum_{1}^{n} \ell n\ Y_i)]^3$$

is negative (positive). Once the estimate $\tilde{c}(1)$ [cf: equation (2.53)], of c is obtained, one may select any of the scale parameter estimates, $\hat{a}_i[1, \tilde{c}(1)]$, $i = 1, 2, \ldots, 5$, given in Table I. (Note: one must take care, however, in the event that $\tilde{c}(1) < 0$; for, difficulty in evaluating some of the Gamma functions, involved in the expressions for certain scale parameter estimators, may be encountered.)

3. A Note on Estimation for the Exponential Distribution

If the Weibull shape parameter c can be assumed fixed, we note that, for $Y \sim h(y; a, c)$, the random variable $W = Y^c \sim h(w; \gamma, 1)$, the exponential distribution with mean $\gamma = a^c$. One may establish that, for a random sample y_i, $i = 1, 2, \ldots, n$, taken from density (6.6) with c fixed, the unique minimum variance unbiased estimate of γ is $\sum_{i=1}^{n} w_i/n$, for $w_i = y_i^c$, $i = 1, 2, \ldots, n$. Our results pertinent to the more general Stacy distribution reveal that estimator $\hat{a}_3[1, c]$ [See Table I] is the unique minimum variance unbiased estimator of the Weibull scale parameter, a, whenever the shape parameter may be assumed known.

In examining the estimators $\hat{a}_i(1, 1)$, i = 1, 2, . . . 5, provided by the appropriate entries in Table I with b = c = 1, we see that $\hat{a}_3(1, 1) = \hat{a}_5(1, 1)$, so that we have only two unbiased estimates for the scale parameter of the exponential distribution. The second, though its variance is not less than $\hat{a}_3(1, 1)$, possibly merits an individual display, along with its variance:

$$\hat{a}_4(1, 1) = \left\{ \prod_{i=1}^{n} w_i \right\}^{1/n} / \Gamma^n(1 + 1/n), \qquad (6.14)$$

and

$$\mathrm{Var}\,[\hat{a}_4(1, 1)] = a^2 \left\{ \Gamma^n(1 + 2/n) - \Gamma^{2n}(1 + 1/c) \right\} / \Gamma^{2n}(1 + 1/c). \qquad (6.15)$$

Though not so efficient as $\hat{a}_3(1, 1)$, the fact that the geometric mean of the observations from an exponential distribution may be employed to estimate unbiasedly the mean of the distribution is itself of possible interest.

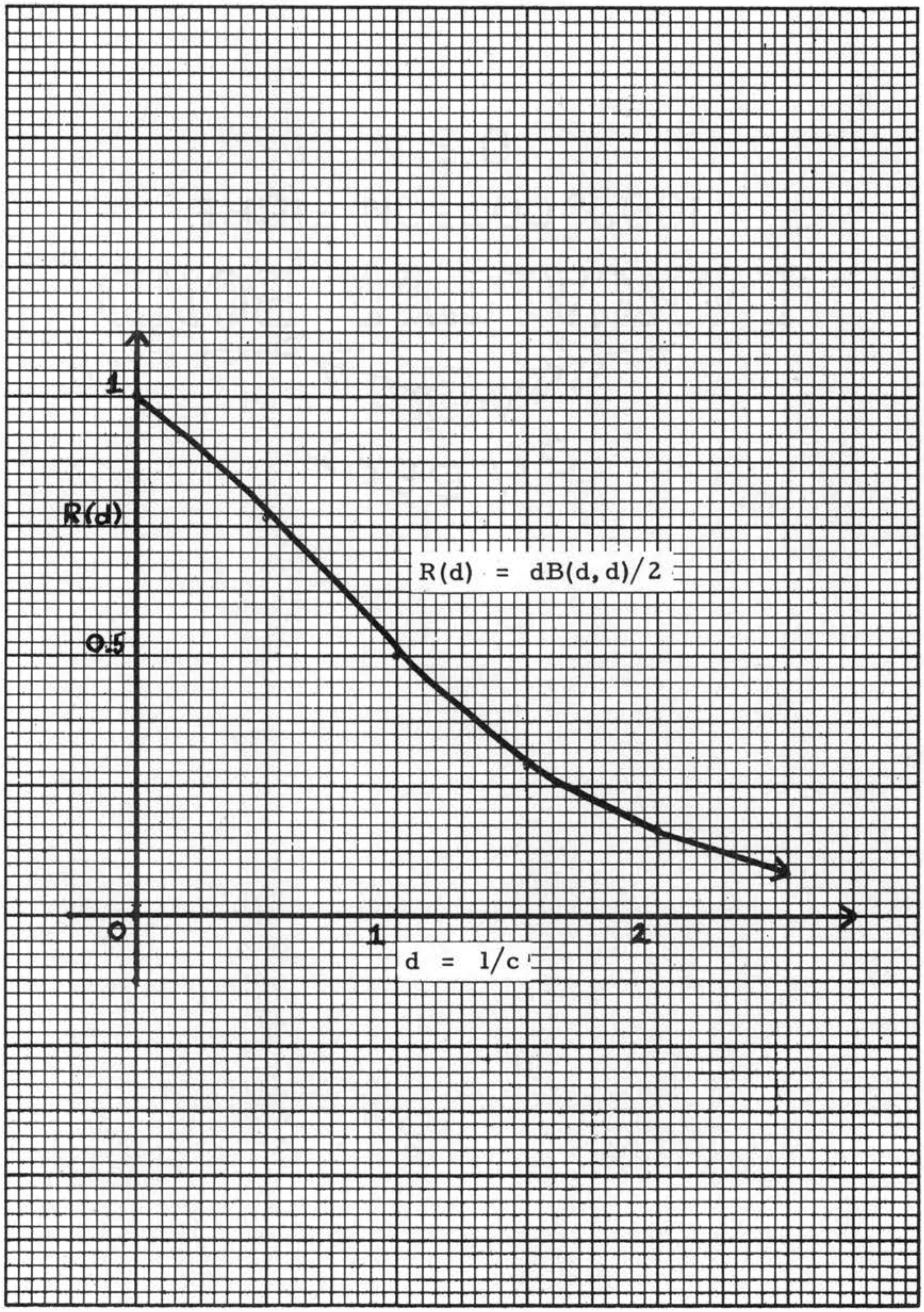

Figure 3. An Aid for the Estimation of the Weibull Shape Parameter

CHAPTER VII

BIVARIATE WEIBULL DISTRIBUTIONS

The Weibull distribution has been discussed both directly and indirectly in the preceding chapters. With positive shape parameter, it has proved a quite powerful tool in the description of data arising from reliability studies. (See, e.g., Gumbel [4] and Kao [13].) We now turn our attention again to a study of the bivariate warning-time/failure-time distributions defined in Chapters IV and V, restricting ourselves to those cases wherein the marginal density of the failure-times is presumed to be the extended Weibull distribution of Chapter VI.

1. Definitions

We shall then define the Uniform/Weibull bivariate probability density function by

$$f(x,\ y;\ a,\ c) = \left|c\right|\ y^{c-2} e^{-(y/a)^c} / a^c, \tag{7.1}$$

for random variables X and Y such that $0 < x < y < \infty$, for positive parameter a, and for real parameter c. The more general Beta/Weibull bivariate probability density function we shall define as

$$f(x,\ y; a, c, p, q) = \left|c\right| x^{p-1} (y-x)^{q-1} y^{c-p-q} e^{-(y/a)^c} / a^c B(p, q), \tag{7.2}$$

where $0 < x < y < \infty$, a, p, and $q > 0$, and c is real. We note that bivariate density (7. 1) is that of equation (7. 2) with $p = q = 1$; i.e., $f(x, y; a, c) = f(x, y; a, c, 1, 1)$.

The marginal density of the failure-times is, of course, in each case, the extended Weibull distribution presented in equation (6. 6), whereas the marginal density of the warning times becomes

$$g(x) = \int_{y=x}^{\infty} f(x, y; a, c, p, q)\, dy. \qquad (7.3)$$

2. Parameter Estimation for the Beta/Weibull Distribution

As stated in Chapter V, we see that, in the absence of knowledge about the values of p and q, we must somehow utilize both sample warning-times and sample failure-times in order to estimate these parameters. Since we are quite desirous of acquiring estimation schemes which will employ only sample warning-times, we find that we still must presume some "a priori" knowledge of the parameters p and q.

In Chapter IV, we saw that the three parameters of the more general Uniform/Stacy distribution (4. 3) may be jointly estimated from the sample of warning-times: $x_1, x_2, \ldots, x_n$. Proceeding analogously, we see that, for $X \sim g(x; a, 1-1/c, c, 1)$, [See equation (3. 1).]

$$E[\ln X] = \ln a + \frac{1}{c}\Psi(1) - 1, \qquad (7.4)$$

and

$$\mathrm{Var}[\ln X] = 1 + \Psi'(1)/c^2. \qquad (7.5)$$

We note that the right-hand side of equation (7. 5) is a function of c only. Thus, as in Chapter IV, an estimate of this Weibull shape parameter is

$$c^*(1) = \pm [\Psi'(1)]^{1/2} / \left| s^2_{\ell n X} - 1 \right|^{1/2}, \quad (7.6)$$

where the sign is selected according to the criterion established for ascertaining the sign of $c^*(b^*)$ in equation (4.31), where $s^2_{\ell n\, x}$ is as given by equation (2.50) and where

$$[\Psi'(1)]^{1/2} = (1.6449)^{1/2} = 1.2825,$$

as provided by Menon. [16] Estimation of the remaining parameter (a) may be effected by selecting one of the estimators $a_i^*(1, c^*(1))$, $i = 1, 2, 3, 4,$ as defined in the closing section of Chapter 4.

If a conditional distribution (of X given Y = y) more general than the uniform density is desired, one might consider that conditional density given by equation (5.40). In this event, we recall that

$$E(\ell n X) = \ell n\, a + \frac{1}{c}\Psi(1) - 1/p, \quad (7.7)$$

$$Var(\ell n X) = (1/p^2) + \Psi'(1)/c^2, \quad (7.8)$$

and

$$E(X) = ap\,\Gamma(1 + 1/c)/(p + 1), \quad (7.9)$$

as may be seen by reference to equations (5.45), (5.46), and (5.49), respectively. Substitution of the corresponding sample moments into these last three results, followed by the simultaneous solution of the resulting equations, will provide estimates of the parameters a, c, and p.

3. Some Specific Distributions

Since the Weibull distribution, especially with positive shape parameter, is often used to describe the distribution of failure-times, we list a few notable cases of bivariate density (7.2) by providing Table II. The product of the entries of the first two columns completely specify a warning-time/failure-time bivariate density which has, as its warning-times' marginal density, the corresponding entry of the third column. Functions of the form $h(y; r, s)$ are provided more explicitly by equation (6.6); those denoted by $h(u; r, s, t)$ are described by equation (1.1); while $g(u; r, s, t, v)$ is specified, in general, by equation (3.1) though, more particularly, $g(u; r, \frac{1}{2}, 2, v)$ is presented in equation (3.16).

TABLE II

BETA/WEIBULL BIVARIATE DISTRIBUTIONS

Marginal of Y	Conditional of X| Y	Marginal of X
Exponential	Uniform	Distended Gamma
h (y; a, 1)	$1/y$	g(x; a, 0, 1, 1)
Exponential	Beta (q = 1)	Distended Gamma
h (y; a, 1)	$p\,x^{p-1}/y^p$	g(x; a, 1-p, 1, p)
Exponential	Beta (p + q = 1)	Gamma
h (y; a, 1)	$x^{p-1}(y-x)^{q-1}/\Gamma(p)\Gamma(q)$	h (x; a, p, 1)
Rayleigh	Uniform	Error-function
h (y; a, 2)	$1/y$	g(x; a, 1/2, 2, 1)
Rayleigh	Beta (q = 1)	Distended Gamma
h (y; a, 2)	$p\,x^{p-1}/y^p$	g(x; a, 1-p/2, 2, p)
Weibull	Beta (q = 1)	Distended Gamma
h (y; a, c)	px^{p-1}/y^p	g(x; a, 1-p/c, c, p)

CHAPTER VIII

SUMMARY

In this dissertation we have discussed a number of probability density functions and have indicated for each a number of their more important properties. The families of univariate distributions discussed in Chapters I and II, as generalizations of the already important Gamma, Weibull, Chi, and double exponential families, will surely find direct application not only in reliability studies but also in other related schools of thought. The extension of the parameter space of the Weibull distribution, as introduced in Chapter VI and as employed in Chapter VII, will provide effectively a means of discussing the distribution of an "inverse-Weibull" random variable.

The distended Gamma distribution, with its four parameters, may prove too cumbersome for facile manipulation, too general for immediate application, too unpleasant in form for the more artistic eye, to stand on its own merits as a useful tool for the reliability engineer. However, its generation as the marginal distribution of warning-times, as indicated in Chapters IV and V, should augment the opportunities for its applicability.

This study has perhaps uncovered as many problems as it has managed to solve. Among the properties which serve to aid in characterizing the distributions we have discussed, we have generally bypassed such characterizations as their associated generating functions. An intensive study, probably via the Monte Carlo method, of the nature of the distributions of the many statistics arising in connection with our diverse estimation techniques, could prove valuable. In addition, an examination of the property of completeness for the distributions of any sufficient statistics could assist in one's selecting among unbiased estimators, especially if minimum variance is desirable.

It is conceivable that the reliability engineer, utilizing the concept of employing warning-times to make inferences about certain parameters of life-time, might seek to make these inferences whenever he has accumulated information only from some portion of his n sample warning-times; viz., information from $r(\leq n)$ warning-times acquired either as a set of order statistics or from a truncated life-test. In either case, the information available might also include the failure-times of $s(\leq r)$ components, so that statistics which are functions both of the observed warning-times and of the observed failure-times might be constructed.

If one assumes each of a system's components is selected from one of Stacy's distributions, then he may seek to infer the system's life-parameters from available information concerning the life-parameters of the individual components. Such a study inevitably leads to an examination of order statistics; again, the value of ordered statistics, obtained from our distributions, is apparent.

The problem of point estimation has been rather extensively investigated in the present study. What, though, might be said of providing interval estimates? The area of hypothesis testing, for conjectures formulated about the parameters describing our many distributions, has been left virtually unexplored.

Thus, though this study has resolved a number of interesting problems, it has produced as many unsolved ones.

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