SPACES WITH THE FIXED POINT PROPERTY

FOR PERIODIC TRANSFORMATIONS

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CHAPTER I

THE NATURE AND SIGNIFICANCE OF THE PROBLEM

In the past half century a great many research articles on the fixed point property have appeared in the mathematical journals. Some of these articles involve a periodic homeomorphic transformation of a space into itself. The work that has been done is scattered. No attempt has been made to bring together the ideas expressed by the various mathematicians who have contributed to the development of this area of thought. One purpose of this study is to examine the various contributions, summarize the research on periodic transformations, and show the continuity and pattern of development. Other objectives are to present a brief history of topology and to develop a bibliography for the area of fixed points under a periodic transformation.

The material in this thesis is intended to be readable on senior college or beginning graduate level in mathematics. Discussions and explanations are given in connection with the theorems, and examples and counter-examples are given to illustrate the topological concepts.

Chapter II presents a brief history of topology, some elementary theorems, and some examples suggesting the theorems that are proved later. In Chapter III a brief review of algebraic topology is outlined, and in Chapter IV and V major emphasis is placed on development of the specific topic of the thesis. Chapter VI delineates the educational significance of the thesis and contains a summary.

Need for the Study

The knowledge explosion has made it practical, perhaps necessary, for the mathematician to specialize in a particular area such as topology, geometry, algebra, or analysis. As a result of this specialization articles published in the research journals are frequently written in such specialized language that they cannot be readily understood by a mathematician outside the specialized area. In many cases proofs of theorems in one article are built upon definitions and theorems that have been published in other journals. Every person who attempts to understand these articles must first go back and acquire the necessary background. This is not an easy task since many of the details are missing. Since the journals are the major source of information for continuing developments, it is imperative that someone digest the various contributions and express them in simpler and more complete language. The fixed point property for periodic transformations is one such subject inasmuch as a chronological development has not been included in the textbooks.

There is a definite need for someone to consolidate all the research that has been done on a particular topic and to present it in a way that a well-trained undergraduate or graduate student, or teacher of mathematics can comprehend it. A study of this nature will also be of considerable value to the Ph. D. candidate who is interested in a review of the research in a given area. Some unsolved problems are stated in the thesis and these may be of value to any individual interested in mathematical research.

Scope and Limitations

This thesis will be limited to the properties of fixed points for periodic homeomorphic transformations. Under this limitation we shall omit a survey and analysis of the fixed point properties for noncontinuous, continuous but not periodic, and multi-valued transformations. Also omitted will be the properties of fixed points for periodic transformations where the transformation is an isometry.

It is to be noted that none of the theory in this thesis is original. Most of the proofs are taken from the papers and books listed in the bibliography. The results obtained in this paper are not the most general ones that have been obtained. However, when more general results are available they will be discussed.

Expected Outcomes

As a result of reading this thesis, the mathematics student should gain an awareness of the current and past research in this modern branch of mathematics. He should become acquainted with men who have contributed to its research and development. This thesis should arouse the student's interest and challenge him to read and probe the periodical literature of mathematics. The presentation of unsolved problems should impress upon him the fact that the frontiers of knowledge in this area of mathematics are being pushed back at a steady and continuing rate.

The fact that the reader, who is a potential teacher at either the public school or the college level, may become sufficiently interested in this phase of mathematics to cause him to undertake serious study in this area has great educational potential. He will be confronted with the possibility of contributing to research in mathematics by extending the results given in this thesis and by offering solutions to unsolved problems as well as by developing new properties of fixed points. The bibliography should be a valuable aid to anyone interested in the research of fixed point theory for periodic transformations.

CHAPTER II

BRIEF HISTORY AND INTRODUCTION TO FIXED POINT THEOREMS

Topology, like most new branches of mathematics, had its beginning in already existing fields of study. The basic ideas of homology theory can be found in Riemann's investigation of functions which arise from the integration of total differentials. The theory of sets was developed by Cantor and used in the clarification and solution of problems in function theory. Both ideas are basic concepts in topology.

It was not until the first part of the twentieth century that topology developed into a self-sufficient branch of mathematics. R. L. Wilder, in an address to the Semicentennial History of the American Mathematical Society, said,

> Topology originated in the work of many mathematicians of the past century, including Cantor, Riemann, and Kronecker; it won recognition as a distinct branch of mathematics largely through the writings of Poincaré about the beginning of the present century. Although having many ramifications, it has progressively become a unified subject, and due to its foundations in the theory of abstract spaces has come to collaborate with abstract group theory as a unifying force in mathematics as a whole. It has provided a tool for classification and unification, as well as for extension and generalization, in algebra, analysis, and geometry. Considered as a most specialized and abstract subject in the early 1920's, it is today almost an indispensible equipment for the investigator in modern mathematical theories [23].

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[1]Arabic numerals in brackets indicate references to the Bibliography.

The growth of topology took place generally along two lines; the combinatorial and the point set. The former was that of Riemann and Poincaré, and was distinguished by its finite character. The basic configuration was not a point set, but a polyhedron consisting of a finite number of faces of various dimensions. The point set was that of Schoenflies, Cantor, Brouwer, E. H. Moore, R. L. Moore and others. The basic configuration was a point set, and whereas in the combinatorial approach the properties in the large were the center of interest, in the point set approach the local properties were those naturally studied. However, in more recent years the two lines of topology have been unified to some extent.

A problem in topology that has received much attention is the fixed point property. A topological space X is said to have the fixed point property if, given any continuous function f from X into X, there exists a point p such that $f(p) = p$. The first results on fixed points were obtained by Alexander on topological mappings of a 2-dimensional manifold and by Brouwer for continuous mappings of the n-cell and n-sphere. Brouwer's theorem was introduced in 1912 and is perhaps one of the best known. The next result was obtained by Lefschetz in 1923. He discovered a fixed point formula and the proof of its validity for a self-mapping of a closed manifold. In the period 1925-35 Lefschetz was able to extend his proof to relative manifolds, to general complexes and, finally, to locally connected spaces. Since Lefschetz's theorem was published many fixed point problems have been solved [2]. Yet, today there is no known topological characterization of the fixed point property.

The first fixed point problems were concerned with a continuous mapping from a space into itself. However, some of the more recent results involve non-continuous mappings and periodic mappings. The first fixed point theorem for periodic mappings was proved by H. A. Newman in 1930. The theorem was that if M_n is a locally Euclidean metricized connected n-dimensional space, K_n any domain in it, and p an integer greater than one, there is a positive number d such that no uniform continuous representation of M_n on itself with period p moves every point of K_n a distance less than d [15]. In 1934 P. A. Smith proved that if a compact Hausdorff space was simply connected in some sense then every homeomorphic periodic transformation of prime period would leave at least one point fixed. This theorem by Smith was perhaps stimulated by the following results. Kerekjarto and Eilenberg proved that every periodic transformation of an ordinary sphere into itself is topologically equivalent to a rotation or to the product of a rotation and a reflection across a diametral plane. Lefschetz's theorem was extended to a simplex in 1933. Smith's greatest contributions to this area came in 1937 and 1939 when he completed the topological classification of the set of fixed points for an n-sphere $n \leq 3$ and for an arbitrary n-sphere if the period of the transformation was a power of a prime. In 1950 E. E. Floyd gave examples showing that P. A. Smith's theorem did not hold for an arbitrary period q. The goal of this thesis is to present the results obtained by Eilenberg, Smith, and Floyd.

Introduction

Perhaps one should begin with a discussion of fixed points, with definitions, examples, and theorems that illustrate the concepts to which this thesis is devoted. A minimal background has been assumed. Most of the ideas in topology that have been used are reviewed.

Definition 2.1. A set M, together with a collection of subsets called open sets, is called a <u>topological space</u> if and only if the collection of open sets satisfies the following axioms:

Axiom 1. The union of any collection of open sets is an open set.

Axiom 2. The intersection of any finite collection of open sets is an open set.

The collection of open sets $\{V\}$ is called the topology of the topological space.

Definition 2.2. A topological space M is said to be a <u>Hausdorff</u> space if and only if, given any two distinct points p, q of M, there exist disjoint open sets U and V of M such that p is in U and q is in V.

Definition 2.3. A collection of open sets $\{U\}$ is said to be an open <u>covering</u> of a set M if and only if $M \subset \bigcup U$.

Definition 2.4. A topological space M is said to be <u>compact</u> if and only if every open covering of M contains a finite subcovering of M.

The theorems in Chapters IV and V are in a compact Hausdorff space and the existence of a finite covering is very important in the proofs. We shall also call upon the following definition and theorem.

Definition 2.5. A space M is said to be <u>normal</u> if and only if, given any two disjoint closed subsets F_1 and F_2 of M, there exist disjoint open subsets G_1 and G_2 of M containing F_1 and F_2, respectively.

Lemma 2.1. If M is a compact Hausdorff space, then given any closed subset F of M and any point q of M not in F there exist disjoint open subsets Q and O of M containing F and q, respectively.

Proof. For every open covering $\{U\}$ of F there exists a finite subset of $\{U\}$ that covers F, since a closed subset of a compact space is compact. For every point p in F and for q, a fixed point in $M-F$, there exist open sets U, V where p is in U and q is in V such that $U \cap V = \emptyset$ (M Hausdorff). Let Q be the union of a finite number of the U's that cover F. Let O be the intersection of the finite number of V's that correspond to the U's. Now O and Q are open sets such that $O \cap Q = \emptyset$ and F is in Q and q is in O.

Theorem 2.1. If a Hausdorff space M is compact, then it is normal.

Proof. Let F_1 and F_2 be closed subsets of M. For every p in F_1 there exist open sets U, V such that U contains F_2 and V contains p, $U \cap V = \emptyset$. A finite number of the V's cover F_1 since F_1 is compact. Let Q be the union of these V's. The intersection of the corresponding U's, say O, is an open set containing F_2, $O \cap Q = \emptyset$. Therefore M is normal.

Definition 2.6 through 2.11. define a homeomorphism which is an important concept in this thesis and, indeed, in topology. Other concepts defined are a periodic transformation, an n-sphere, and the fixed point property.

Definition 2.6. Let M and R be spaces; a rule f is called a <u>mapping</u> of M into R $(f\colon M \to R)$ if and only if f associates with each element x of M a unique element y of R.

Definition 2.7. Let M and R be spaces and $f : M \to R$ a mapping. Then f is said to be <u>continuous</u> at a point x of M if and only if, given any open subset G of R where x is in $f^{-1}(G)$, there exists an open set V of M containing x such that for every y in V, $f(y)$ is in G. $\left(f^{-1}(G)\right.$ is the set of all y in M such that $f(y)$ is in $\left.G.\right)$ The function f is continuous on M if and only if f is continuous at every point of M.

Definition 2.8. Let M and R be spaces and $f\colon M \to R$ a mapping. Then f is said to be <u>onto</u> if and only if for every y in R there exists an x in M such that $f(x) = y$.

Definition 2.9. Let M and R be spaces and $f\colon M \to R$ a mapping. Then f is said to be <u>one-to-one</u> if and only if for $x, y \in M$, $f(x) = f(y)$ implies $x = y$.

Definition 2.10. Let M and R be spaces and $f\colon M \to R$ a mapping. Then f is said to be <u>open</u> if and only if for every open set U of M, $f(U) = V$ is an open set in R.

Definition 2.11. Let M and R be spaces and $f\colon M \to R$ a mapping. Then f is said to be a <u>homeomorphism</u> if and only if f is continuous, one-to-one, open, and onto.

Definition 2.12. Let f be a mapping of a space M into itself, f is said to be periodic provided there exists an integer $p > 1$ such that $f^p(x) = x$ for every point x of M, where $f^p(x) = f(f(\ \cdot\ \cdot\ \cdot\ (f(x))\ \cdot\ \cdot\ \cdot))$ (p of them).

Definition 2.13. An n-sphere is any homeomorphic image of the sphere

$$\Sigma_{i=1}^{n+1} X^2 = 1.$$

Definition 2.14. A space M is said to have the fixed point property (fpp) if for every periodic homeomorphic transformation of M into itself there exists a point x such that $f(x) = x$.

We shall give examples to illustrate the above definitions and perhaps the examples will suggest some theorems.

Example 2.1. Let M be the interval $[-1, 1]$ and define $f(x) = -x$ for $x \in M$. Then $f^2(x) = x$, therefore f is periodic of period 2. The set of fixed points is the set $\{0\}$.

Example 2.2. Let M be the set of points (x, y) such that $\sqrt{x^2 + y^2} \leq 1$ and define $f(x, y) = (-x, y)$. Then $f^2(x, y) = (x, y)$, therefore f is periodic of period 2. The set of fixed points is the set $\{(0, y) / |y| \leq 1\}$.

Example 2.3. Let M be the same as in Example 2.2 and define $f(x, y) = (y, -x)$. Then $f^4(x, y) = (x, y)$, therefore f is periodic of period 4. The set of fixed points is the set $\{(0, 0)\}$.

Any number of examples such as the ones above can be found. However, one cannot find a periodic homeomorphic transformation of a connected subspace of the Euclidean space into itself that does not leave a point fixed. In Chapter IV we shall prove that such a space has the fpp.

The next examples are concerned with an n-sphere. Recall that a 0-sphere is homeomorphic to two points, a 1-sphere is homeomorphic to the circumference of a circle, and a 2-sphere is homeomorphic to the hull of an ordinary sphere.

Example 2.4. Let M be a 0-sphere, say the points "a" and "b"; then $f(a) = b$ and $f(b) = a$ is the only possible periodic transformation and it leaves no fixed points.

Example 2.5. Let M be a 1-sphere say the set of points (x, y) such that $\sqrt{x^2 + y^2} = 1$. Define $f(x, y) = (-x, y)$, then $f(x, y)$ is periodic of period two. The set of fixed points is the set $\{(0, 1), (0, -1)\}$, which is a 0-sphere.

Example 2.6. Let M be the same as in Example 2.5. Define $f(x, y) = [\cos(\theta + \pi), \sin(\theta + \pi)]$ where $x = \cos\theta$ and $y = \sin\theta$, then $f(x, y)$ is periodic of period two. The set of fixed points is the null set.

Example 2.7. Let M be a 2-sphere say the set of points (x, y, z) such that $\sqrt{x^2 + y^2 + z^2} = 1$. Define $g(x, y, z) = (x, y, -z)$, then $g(x, y, z)$ is periodic of period two. The set of fixed points is the set of points $(x, y, 0)$ such that $\sqrt{x^2 + y^2} = 1$, which is a 1-sphere.

Example 2.8. Let M be the same as in Example 2.7. Define $h(x, y, z) = [\cos(\theta + \pi), \ \sin(\theta + \pi), z]$ where $x = \cos\theta$ and $y = \sin\theta$, then $h(x, y, z)$ is periodic of period two. The set of fixed points is the set $\{(0, 0, 1), \ (0, 0, -1)\}$ which is a 0-sphere.

Example 2.9. Let M be the same as in Example 2.7. Define $f(x, y, z) = gh(x, y, z)$, where g and h are the same as in Examples 2.7 and 2.8, then f is periodic of period two. The set of fixed points is the null set.

These examples indicate that the set of fixed points when an n-sphere, $n \leq 2$, is mapped into itself by a periodic homeomorphic transformation can be an r-sphere, $r \leq n$, or the null set. In this section we shall show that this is the case. In Chapter V we shall obtain the same result for an arbitrary n-spere if the period of the transformation is a power of a prime.

Elementary Fixed Point Theorems

Definition 2.15. A mapping $f(X) \subset X$ is said to be pointwise almost periodic (p.a.p.) at a point p of a set X provided there exists, for any $\epsilon > 0$, an integer N_p such that $\rho[p, \ f^n(p)] < \epsilon$, where $\rho[p, f^n(p)]$ denotes the distance between p and $f^n(p)$. Note that a periodic transformation is p.a.p..

Lemma 2.2. If $f(X) \subset X$ is a homeomorphism and K is a compact subset of X such that $f(K) \subset K$, f cannot be p.a.p. at any point of $K - f(K)$.

Proof. The relation $f(K) \subset K$ gives $f^2(K) \subset f(K)$, $f^3(K) \subset f^2(K) \subset f(K), \cdots, f^n(K) \subset f^{n-1}(K) \subset \cdots \subset f(K)$, so that $x \in K - f(K)$ implies $f^n(x) \subset f(K)$ for all n. Hence $\rho[x, f^n(x)] \geq \rho[x, f(K)]$ for all n.

Lemma 2.3. If X is an arc, $X = H + K$ is a division of X into H and K such that $H \cap K = p \in X$ and $f(X) \subset X$ is a homeomorphism such that $H \cap f(H) \neq \emptyset \neq K \cap f(K)$ and f is p.a.p. at p, then p is fixed under f.

Proof. Suppose $f(p) \neq p$. Then $f(p)$ belongs either to $H - p$ or $K - p$, say to $K - p$. Since $f(H)$ is connected and intersects both H and K we have $f(H - p) \supset p$. Thus $p \in K - f(K)$. Also, $f(K) \subset K$ since $f(K) \cap K \neq \emptyset$. But by Lemma 2.2, f could not be p.a.p. at p.

Lemma 2.4. If the end points of an arc $\overline{ab}$ are invariant, every element of the arc is invariant.

Proof. Let x be any point of the arc different from a or b. Then x divides $\overline{ab}$ into H and K such that the hypothesis of Lemma 2.3 is satisfied. Hence x is a fixed point.

Definition 2.16. A mapping, $f(x)$ is said to be pointwise periodic at a point x of X provided there exists an integer N_x such that $f^n(x) = x$. (A periodic mapping is pointwise periodic.)

Theorem 2.2. If f is a pointwise periodic mapping of an arc $\overline{ab}$ into itself it must be periodic of period n; furthermore, either

$n = 1$ (all points fixed) or $n = 2$ and there is exactly one interior point p of $\overline{ab}$ which is fixed and f is equivalent to a reflection about p.

Proof. If either a or b is fixed, they both must be fixed since end points map into end points; and every point is fixed by Lemma 2.4, so that $n = 1$. If neither a nor b is fixed, we have $f(a) = b$, $f(b) = a$, $f^2(a) = a$, $f^2(b) = b$ and hence $f^2(x) = x$ for every $x \in \overline{ab}$. Therefore $n = 2$. Also, if p is a fixed point, we have $f(\overline{ap}) = \overline{bp}$ and no $q \neq p$ is fixed.

Theorem 2.3. Any pointwise periodic mapping f of the circle S, $|z| = 1$, into itself is equivalent either to the reflection $w = \bar{z}$ of period 2 with two fixed points or to the rotation $w = z^n$ ($n = 1, 2, \cdots$) all points fixed for $n = 1$, no fixed points for $n > 1$.

Proof. If all points of S are fixed, f is the identity $w = z$. Suppose there exist a fixed point p and a point x with $f(x) \neq x$. Let $\overline{px}$, $\overline{pf(x)}$ and $\overline{xf(x)}$ be the arcs on S each containing only two of the points p, x, and $f(x)$. Then since $\overline{f(px)} = \overline{pf(x)}$, we have $f(\overline{xf(x)}) = \overline{xf(x)}$. Hence from the above there is exactly one fixed point q on $\overline{xf(x)}$ and also $f^2(x) = x$ and since $f(\overline{qxp}) = \overline{qf(x)p}$, f is equivalent to the reflection $w = \bar{z}$ on $|z| = 1$.

Assume there are no fixed points and $p_0 = p \in S$, $p_1 = f(p)$, $\cdots$, $p_{n-1} = f^{n-1}(p)$, $p = f^n(p)$, then $f(\overline{p_0p_1}) = \overline{p_1p_2}$, $f(\overline{p_1p_2}) = \overline{p_2p_3}$, $\cdots$, $f(\overline{p_{n-1}p_0}) = \overline{p_0p_1}$ so that f is equivalent to the mapping $w = z^n$, $n > 1$, on $|z| = 1$.

Notation: K_1 is a topological circle, I_1 its interior, S_1 its circumference, and S_2 is a 2-sphere.

Lemma 2.5. Let $E = K_1$, $a \in L \cap S_1$, $b \in L \cap S_1$, $a \neq b$ and $A = \overline{xy}$ where $A \cap S_1 = \{x, y\}$ and such that A cuts E between a and b. Then there exists an arc $B \subset \bigcup f^i(A)$ with end points on S_2, and $f(B) = B$.

Proof. Let g be a homeomorphism between E and a geometric circle. Extend f to $g(E)$ such that $fg(x) = gf(x)$ for $x \in E$. Form an $S_2 = E + g(E)$ by identification of a point x and $g(x)$ for $x \in S_1$. The curve $S_1' = A + g(A)$ cuts S_2 between a and b. Designate by I_1' the component of $S_2 - S_1'$ that contains a and let I_1'' be the component of $\bigcap f^i I_1'$ that contains a. The closure is a topological circle K_1'', $S_1'' \subset \bigcup f^i(S_1')$, $f(I_1'') = I_1''$ and $f(S_1'') = S_1''$.

The set $S_1'' \cap E$ cuts E between a and b, hence there exists an arc $B \subset S_1'' \cap E$ which cuts E between a and b. Also, $g(B) \subset S_1''$ hence $B + g(B) \subset S_1''$, and $B \cap g(B)$ contains points such that $B + g(B)$ contains a topological circumference or $B + g(B) = S_1''$. Now, $B = S_1'' \cap E$ and the end points of B are on S_1. Also, $f(B) = B$ because $f(S_1'') = S_1''$, and $B \subset \bigcup f^i S_1')$ because $S_1'' \subset \bigcup f^i(S_1')$.

Lemma 2.6. If $E = K_1$ and $S_1 \subset L$, then $n = 1$.

Proof. Let p be any point in $K_1 - S_1$ and $A = \overline{xpy}$ where $A \cap S_1 = \{x, y\}$. There exists an arc $B \subset \bigcup f^i(A)$ such that $B \cap S_1 = \{x', y'\}$, x', y' end points of B and $f(B) = B$ by Lemma 2.5 and by Theorem 2.2 $B \subset L$. Hence $B = A$ and the arbitrary

$p \in L$, therefore $E = L$ and $n = 1$.

Theorem 2.4. Any periodic homeomorphic mapping f of a topological sphere S_2 into itself is of period one all points fixed, of period two a 1-sphere fixed, or of period n two points fixed or no points fixed.

Proof. Assume that L, the set of fixed points, is not null then there exists an $x \in S_2$ such that $f(x) = x$. Let I_1 be any open disk such that $a \in I_1$. Then $\cap f^i(I_1)$ is a collection of open disks one of which, call it I_1', contains a. Now $f(I_1') = I_1'$ and $f(S_2 - I_1') = S_2 - I_1'$. Hence there exists a point $y \in S_2 - I_1'$ such that $f(y) = y$ by Brouwer's Theorem. Therefore if $L \neq \emptyset$, then L contains at least two points.

Let L_1 be the set of fixed points under f, L_2 the set of fixed points under $f^2, \ldots,$ L_{n-1} the set of fixed points under f^{n-1}. Let $A = \bigcup_{i=1}^{n-1} L_i$, a and b points of L_1. Now A either cuts S_2 between a and b or it does not. Consider first the case where A cuts S_2 between a and b. Let S_1' be a 1-sphere in A such that it cuts S_2 between a and b. Suppose $a \in I_1'$ the interior of S_1' and let I_1'' be the component of $\cap f^i(I_1')$ which contains a. Then $b \in S_2 - \cup f^i(I_1')$. Let S_1'' be the boundary of I_1'', then $S_1'' \subset \cup f^i(S_1') \cap A$. Also $f(I_1'') = I_1''$, hence $f(S_1'') = S_1''$. And since more than two points of S_1'' are contained in A then all of S_1'' is in A by Theorem 2.3. Then by Lemma 2.6 K''_1 and S $S_2 - K_1''$ are contained in A. Hence $S_2 \subset A$ and $n = 1$.

Assume next that A does not cut S_2 between a and b, then there exists an arc B such that $f^i(B) \cap B = \{a, b\}$ for all i. Now

B, $f(B)$, $\cdots$, $f^{n-1}(B)$ divides S_2 into n sections R_0, R_1, $\cdots$, R_{n-1} where the boundary of $R_0 = B \cup f(B)$, the boundary of $R_1 = f(B) \cup f^2(B)$ et cetera. Now $f(B_i) = B_{i+1}$ and $f(B_{i+1}) = B_{i+2}$ and the boundary of $f(R_i)$ is the same as the boundary of R_{i+1}. Hence $f(R_i) = R_i$ or $f(R_i) = R_{i+1}$. If $f(R_i) = R_i$ then $n = 2$ and $R_i \cup R_{i+1} = S_2$, and if $f(R_i) = R_{i+1}$ then $L = \{a, b\}$. It remains to show that if $f(R_i) = R_i$ then $L = S_1$. Let $A = \overline{xy}$ such that $A \cap S_{1_i}$, the boundary of R_i, is $\{x, y\}$. Then there exists a $B \subset \bigcup_{i-1}^{2} A_i$ such that $f(B) = B$ and $B \cap S_{1_i}$ is the set consisting of the end points of B. Since $n = 2$, B contains one and only one end point p such that $f(p) = p$. Also, $f(B)$ contains the same fixed point which implies that A contains at least one fixed point. Therefore, there exists an arc C between a and b such that $C \subset L$. Furthermore, there exists a point $c \in S_2 - R_i$ such that $f(c) = c$ and similarly arcs C', $C'' \subset L$ between a and c, and b and c. Let S_1' be the 1-sphere $C \cup C' \cup C'' = L$. The circle K_1' with boundary $B \cup B' \cup B''$, where B' and B'' form the corresponding arc between $\overline{ac}$ and $\overline{bc}$, is mapped by f onto K_1'' with boundary $f(B) \cup f(B') \cup f(B'')$. Hence $L = S_1'$.

The following theorem is a collection of the results in this section. The results presented in this section may be found in [5] and [21].

Theorem 2.5 If an n-sphere $n \leq 2$ is mapped into itself by a periodic homeomorphic mapping, then the set of fixed points is an r-sphere $r \leq n$.

CHAPTER III

HOMOLOGY THEORY

The proof of the existence of fixed points and the classification of the set of fixed points, Chapters IV and V, involve a great deal of homology theory. We shall review the definitions, concepts, and theorems that are used. The theorems are not proved; however, the proofs as well as the definitions given here can be found in [24].

Simplicial Complex

Definition 3.1. An abstract simplicial complex M is a pair (U, Σ) where $\{U\}$ is a set of elements called vertices, and Σ is a collection of finite subsets of $\{U\}$ with the property that each element of $\{U\}$ lies in some element of Σ and, if E is any element of Σ, then every subset of E is again an element of Σ. M is finite if $\{U\}$ is finite.

Definition 3.2. A simplex E is an element of the collection Σ and the dimension of E is one less than the number of vertices in E. Denote a simplex with $m + 1$ vertices by E^m. A face E^k of E^m is a k dimensional subset of E^m.

Definition 3.3. The star of a simplex E (St, E) is the collection of all elements in Σ of which E is a subset.

Note 3.1. Every finite abstract simplicial complex is isomorphic to a geometric simplicial complex.

Example 3.1. Let X be a compact Hausdorff space, and let $\{U\}$ be a finite covering of X by open sets. Define $M = (V, \Sigma)$ by taking $\{V\}$ to be the collection $\{U\}$ and by saying that a subset $U_0, U_1, \cdot \cdot \cdot, U_p$ of elements of $\{U\}$ is a simplex E in Σ if and only if the intersection $\bigcap_{i=0}^{p} U_i$, the Kernel of E, is not empty. Then M is an abstract simplicial complex or the nerve of the covering $\{U\}$. To see this, we need only note that if $\bigcap_{i=0}^{p} U_i$ is not empty, then any subcollection of theorem sets $U_0, \cdot \cdot \cdot, U_p$ also has a nonempty intersection and by definition must constitute an element of Σ.

Oriented Complex

Definition 3.4. An oriented simplex is an abstract simplex with an arbitrary fixed ordering of its vertices.

Definition 3.5. An oriented simplicial complex is an abstract simplicial complex with an arbitrary fixed orientation for each simplex in the complex.

Incidence Number

Given an oriented simplicial complex M, we associate with every pair of simplexes E^m, and E^{m-1}, which differ in dimension by unity, and incident number $[E^m, E^{m-1}]$ defined as follows:

$$[E^m, E^{m-1}] = 0 \quad \text{if} \quad E^{m-1} \text{ is not a face of } E^m;$$

$$[E^m, E^{m-1}] = \pm 1 \quad \text{if} \quad E^{m-1} \text{ is a fact of } E^m.$$

If the orientation of E^m and E^{m-1} agree, then the incidence number is +1. Otherwise it is -1.

Example 3.2. If $+E^2 = (V_0V_1V_2)$ and $+E^1 = (V_1V_2)$, then $[(V_0V_1V_2), (V_2V_1)] = 1$. But if $+E^1 = (V_2V_1)$, then $[(V_0V_1V_2), (V_2V_1)] = -1$.

The oriented simplicial complex M together with the system of incidence numbers $[E^m, E^{m-1}]$ constitutes the basic structure supporting a simplicial homology theory.

Chain

Let M denote an arbitrary oriented simplicial complex, and let G be an arbitrary abelian group.

Definition 3.6. An m-dimensional chain of the finite complex M with coefficients in the group G is a function C_m on the oriented m-simplexes of M with values in the group G such that if $C_m(+E^m) = g$, $g \in G$, then $C_m(-E^m) = -g$.

The collections of all such m-dimensional chains on M will be denoted by the symbol $C_m(M, G)$. We introduce addition of m-chains by means of the usual functional addition. That is, we define

$$(C_m^1 + C_m^2)\ (E^m) = C_m^1\ E^m + C_m^2\ E^m,$$

where the addition on the right is the group operation in G. Under this operation $C_m(M, G)$ is an abelian group, the m-dimensional chain group of M with coefficients in G. An arbitrary m-chain on M can be written as a linear combination $\Sigma g_i\ E_i^m$, where $g_i = C_m(+E_i^m)$. This notation tabulates the function C_m in such a way that addition of such functions is the addition of linear combinations. An elementary m-chain on K is an m-chain such that $C_m(\pm E_0^m) = \pm g_0$ for some particular simplex E_0^m in K and $C_m(E^m) = 0$ for $E^m \neq \pm E_0^m$.

Boundary Operator

The boundary operator F defines a homomorphism of the group $C_m(M,G)$ into the group $C_{m-1}(M,G)$ as follows:

$$Fg_0E_0^m = \Sigma_{E^{m-1}}[E_0^m, E^{m-1}]g_0E^{m-1}.$$

This is extended linearly to arbitrary m-chains by

$$FC_m = F(\Sigma_i\ g_iE_i^m) = \Sigma_i\ F(g_iE_i^m).$$

Also for any chain C_m in $C_m(M, G)$, $F(FC_m) = 0$. That is, $F(FC_m)$ is the (m-2)-chain with value zero on each (m-2) simplex.

Cycle

An m-dimensional cycle on M with coefficients in G is a chain z_m in $C_m(M, G)$ with the property that $F(z_m) = 0$, the (m-1)-chain $\Sigma\, 0 \cdot E_i^{m-1}$. The collection of all m-cycles is precisely the kernel of the homomorphism F in the group $C_m(M, G)$ and hence is a subgroup of $C_m(M, G)$. This subgroup is the m-dimensional cycle group of M with coefficients in G and is denoted by $Z_m(M, G)$.

A 0-dimension chain C_0 is a cycle if and only if $C_0 = \Sigma\, g_iE_i^0$ implies that $\Sigma\, g_i = 0$ in the group.

Boundary

An m-boundary on M with coefficients in G is a b_m chain in $C_m(M, G)$ such that there exists a C_{m+1} chain where $F(C_{m+1}) = b_m$.

The set of all m-boundaries is a subset of $Z_m(M, G)$ since $F(F(C_{m+1})) = 0$. This set is a subgroup denoted by $B_m(M, G)$, the group of m-boundaries of M with coefficients in G.

For FC_0, either of two conventions may be made: (1) Define $FC_0 = 0$, or (2) Augument the complex under consideration by an ideal simplex C_{-1} and for all C_0 of the complex let $FC_0 = C_{-1}$. If case (2) is used, the complex is called augumented.

Homology Group

Since both $B_m(M, G)$ and $Z_m(M, G)$ are abelian groups we can define the difference group $Z_m(M, G) - B_n(M, G)$, which is called the m-homology groups of M over G and is denoted by $H_m(M, G)$. Each element of $H_m(M, G)$ is an equivalence class $[z_m]$ of m-cycles where z_m^1 and z_m^2 are in the same class if and only if the chain $z_m^1 - z_m^2$ is an m-boundary. This equivalence relation is called homology and is written $z_m^1 \sim z_m^2$. If z_m^0 is an m-boundary we write $z_m^0 \sim 0$.

Example 3.3. Let M be the complex consisting of a single 3-simplex E^3 together with all of its faces. This complex is the closure of a simplex E^3 and is denoted by "$Cl(E^3)$." We will orient the complex M by choosing a fixed ordering of its vertices v_0, v_1, v_2, and v_3 and letting this ordering induce the positive orientation of the simplexes. In this way, we have the following list of (representatives of) the oriented simplexes of M:

$$+E_1^1 = (v_2 v_3), \quad +E_3^1 = (v_0 v_3), \quad +E_5^1 = (v_0 v_2),$$

$$+E_2^1 = (v_1v_3), \quad +E_4^1 = (v_1v_2), \quad +E_6^1 = (v_0v_1),$$

$$+E_1^2 = (v_1v_2v_3), \quad +E_3^2 = (v_0v_1v_3), \quad +E^3 = (v_0v_1v_2v_3)$$

$$+E_2^2 = (v_0v_2v_3), \quad +E_4^2 = (v_0v_1v_2).$$

(We omit a consideration of dimension zero.)

Now let G be any abelian group. The only 3-chains on M are the elementary chains gE^3, hence the chain group $C_3(M, G)$ is isomorphic to G. Since there are no 4-simplexes in M, $C_4(M, G) = 0$, and hence $B_3(M, G) = FC_4(M, G) = 0$. It follows that $H_3(M, G) = Z_3(M, G)$.

But let gE^3 be any 3-chain. Computing its boundary we have

$$F(gE^3) = \Sigma_{i=1}^4 [E^3, E_i^2]gE_i^2$$

$$= gE_1^2 - gE_2^2 + gE_3^2 - gE_4^2.$$

It is easy to show that, in the present case, $[E^3, E_i^2] = (-1)^{i+1}$. This chain is the zero 2-chain if and only if $g = 0$. Therefore, the only 3-cycle on M is the trivial 3-cycle $0E^3$. Hence, $Z_3(M, G) = H_3(M, G)$ is trivial. This illustrates one situation in which we obtain a trivial homology group, namely, the case where we have no cycles except the trivial cycle.

Another situation that results in a trivial homology group occurs when every cycle is a boundary, for if $Z_m(M, G) = B_m(M, G)$, then $Z_m - B_m = H_m = 0$. This situation can be illustrated with this same example. Suppose that the 2-chain $\Sigma_{i=1}^4 g_iE_i^2$ is a 2-cycle. Computing its boundary we have

$$F\left(\Sigma_{i=1}^{4} g_i E^2{}_i\right) = \Sigma_{i=1}^{4} F(g_i E_i^2)$$

$$= \Sigma_{i=1}^{4} \Sigma_{j=1}^{6} [E_i^2, E_j^1] g_i E_j^1$$

$$= \Sigma_{j=1}^{6} \left(\Sigma_{i=1}^{4} [E_i^2, E_j^1] g\right) E_j^1.$$

If this is to be the zero 1-chain, then for each fixed index j the sum

$$\Sigma_{i=1}^{4} [E_i^2, E_j^1] g_i$$

must be zero. For instance, if $j = 1$ we have

$$[E_1^2, E_1^1] g_1 + [E_2^2, E_1^1] g_2 + [E_3^2, E_1^1] g_3 + [E_4^2, E_1^1] g_4 = 0.$$

But E_1^1 is not a face of E_3^2 and E_4^2, so the last two terms are zero. Furthermore, $[E_1^2, E_1^1] = +1$ and $[E_2^2, E_1^1] = +1$, hence this equation reduces to nothing more than $g_1 + g_2 = 0$ or $g_2 = -g_1$. Similarly, working with E_6^1 we obtain $g_4 = -g_3$, and working with E_3^1 we show that $g_3 = g_2$. This means that $\Sigma_{i=1}^{4} g_i E_i^2$ can be a 2-cycle only if $g_1 = g_3 = -g_2 = -g_4$; that is, the only 2-cycles are of the form $gE_1^2 - gE_2^2 + gE_3^2 - gE_4^2$. But we have already seen that such a 2-cycle is the boundary of the 3-chain gE^3. Hence, every 2-cycle on M is a 2-boundary, and it follows that $H_2(M, G) = 0$.

By an analogous but much longer method, the reader can prove that $Z_1(M, G) = B_1(M, G)$ and thereby show that $H_1(M, G)$ is also trivial. Geometrically, the complex M is carried by a homeomorph of the 3-cube I^3 and is a 3-cell. Granting that the <u>homology groups are topological invariants</u>, we have found that the homology groups of a 3-cell are trivial for dimensions greater than zero.

Relative Homology Groups

If M is an abstract simplicial complex and L is a closed subcomplex, $L \subset M$. A p-chain C_p on M is called a p-cycle of M modulo L provided FC_p is a chain on L, that is, FC_p has nonzero coefficients only on simplexes of L. A p-boundary of M modulo L is a chain b_n such that there exists a C_{n+1} chain where $FC_{n+1} = b_n + b_n^1$, $b_n^1 \in L$. We form the relative homology group of M mod L as $H_p\left(K/L\right) = Z_p\left(K/L\right) - B_p\left(K/L\right)$. By these definitions, a relative chain z_p is a relative cycle if and only if $Fz_p = 0$. That is, a chain on M represents a relative cycle if and only if its boundary lies in L. Similarly b_p is a relative boundary if and only if there is a chain C_{m+1} such that $b_p = FC_{p+1}$ lies in L, that is, b_p together with some chain in L constitutes the boundary of a chain of M. We also have equivalence classes of p-cycles mod L. Two p-cycles z_p^1 and z_p^2 are in the same class if and only if $z_p^1 - z_p^2$ is a p-boundary mod L. This will be denoted by $z_p^1 \approx z_p^2$ mod L.

Refinement of a Covering

A covering $\{V\}$ of a space M is said to be a refinement of a covering $\{U\}$ if for each element V of $\{V\}$ there is an element U of $\{U\}$ such that $U \supset V$. We write this as $\{U\} < \{V\}$.

Projection

If $\{V\} > \{U\}$ in Σ, then a projection is a simplicial mapping π of $\{V\}$ into $\{U\}$. It is defined by taking $\pi(V)$ to be any fixed element U such that $V \subset U$. There may be several elements of $\{U\}$ containing the set V and hence several choices for $\pi(V)$. This means that there may be many projections of $\{V\}$ into $\{U\}$. However, if $\{U\} < \{V\}$ in Σ, then any two projections π_1 and π_2 of $\{V\}$ into $\{U\}$ induce the same homomorphism of $H_p(V, G)$ into $H_p(U, G)$. That is, π_1 and π_2 coincide.

Cech Cycle

A p-dimensional Cech cycle of a space M is a collection $z_p = \{z_p(U)\}$ of p-cycles $z_p(U)$, one from each and every cycle group $Z_p(U, G)$, $\{U\}$ in $\Sigma(M)$, with the property that if $\{U\} < \{V\}$, then $\pi Z_p(V)$ is homologous to $z_p(U)$. Each cycle $z_p(U)$ in the collection z_p is called a coordinate of the Cech cycle. Hence a Cech cycle has a coordinate on every covering of the space M.

The addition of Cech cycles is defined in a natural way by setting

$$\{z_p(U)\} + \{z_p^1(U)\} = \{z_p(U) + z_p^1(U)\}$$

where the addition on the right is that of chains on the complex $\{U\}$.

The homology relation between Cech cycles is defined as follows: First a Cech cycle $z_p = \{z_p(U)\}$ is homologous to zero on M (or is a bounding Cech cycle) if each coordinate $z_p(U)$ is homologous to zero on the covering $\{U\}$, for all $\{U\}$ in $\Sigma(M)$. In other words, $\{z_p(U)\}$ bounds if and only if there is a (p+1)-chain $C_{p+1}(U)$ on each covering $\{U\}$ in $\Sigma(M)$ such that the coordinate $z_p(U) = FC_{p+1}(U)$. Then two Cech cycles z_p and z_p^1 are homologous Cech cycles if their difference $z_p - z_p^1$ is homologous to zero. The homology relation defined above is an equivalence relation on the set of all Cech p-cycles. The corresponding equivalence classes $[z_p]$ of homologous Cech p-cycles are the elements of the p^{th} Cech homology groups $H_p(M, G)$, the group operation being defined by

$$[z_p] + [z_p^1] = [z_p + z_p^1].$$

Cech homology groups are topological invariants of the space M. If f is a homeomorphism of M to M', then for each covering $\{U\}$ in $\Sigma(M)$ the collection $\{f(U)\}$ of all images of elements of $\{U\}$ is an open covering of M' and conversely. The complexes $\{U\}$ and $\{f(U)\}$ are isomorphic, and the homology groups $H_p(M, G)$ and $H_p(M', G)$ are isomorphic.

Cofinal Family of Coverings of M

A subcollection $\Sigma'(M)$ of $\Sigma(M)$ is called a cofinal family of coverings of M provided that for every covering $\{U\}$ of $\Sigma(M)$ there is some covering $\{U'\}$ in $\Sigma'(M)$ such that $\{U'\} > \{U\}$.

Given such a cofinal family $\Sigma'(M)$, we may go through the development of Cech theory again, restricting the cycles, homologies, et cetera, to be elements of $\Sigma'(M)$. The Cech groups obtained from $\Sigma'(M)$ are isomorphic to the full Cech groups $H_p(M, G)$.

CHAPTER IV

THE EXISTENCE OF FIXED POINTS

The fixed-point theorems for transformations of finite periods which assert that fixed points must exist, if a space M under transformation is simply connected in some sense, seem to be the simplest to prove.

The first theorem of this sort was proved by P. A. Smith in 1934. Since that time Samuel Eilenberg and P. A. Smith have given different proofs, with some generalization, for the same theorem. The proof that is given here follows the second one given by P. A. Smith in [17].

To prove the theorem we first show the existence of a cofinal family of coverings which has desirable properties. This family, being cofinal, is sufficient for the study of the space M. Then different types of chains, cycles, and boundaries are defined. These new chains, called ρ-chains, are shown to have certain properties. We then assume that the space under investigation has no fixed points under a periodic transformation. This assumption leads to a result which contradicts a property of the ρ-chains.

Before proving the existence theorem on fixed-points, we shall give some preliminary definitions and prove some needed theorems. The theorems on the existence of a type refinement are of particular interest.

Preliminaries

For the remainder of this paper M will denote a Hausdorff space,

T a homeomorphic transformation of M into itself. The transformation T will always be periodic of period q. The identity itself will not be considered as being periodic. If A is a subset of M, the sets $A, T(A), \ldots, T^{q-1}(A)$ will be called the T-images of A. Denote $\bigcup_{i=0}^{q-1}(A)$ by σA. The totality of fixed points will be denoted by L.

Definition 4.1. A subset K of M is <u>invariant</u> if $T(K) = K$.

Lemma 4.1. If $A \subset M$, then σA is invariant.

Proof. We have $T(\sigma A) = T(A \cup T(A) \cup \ldots \cup T^{p-1}(A)) = (T(A) \cup T^2(A) \cup \ldots \cup T^p(A))$, therefore $T(\sigma A) = \sigma A$, since $T^p(A) = A$.

Definition 4.2. The transformation T is <u>primitive</u> if each point of M-L has q distinct T-images.

Lemma 4.2. The transformation T is primitive if $q = p$, p a prime.

Proof. Assume $T^i(x) = T^j(x)$, $x \in M$-L, and $i < j < r < p$, where r is any other number such that $T^i(x) = T^r(x)$. Then $T^k(x) = x$, where $j - i = k < p$. But $T^p(x) = x$ since T has period p. Therefore $T^{k\ell}(x) = T^p(x)$, which implies p is not prime. This is true because k is the smallest integer such that $T^k(x) = x$, hence $p = \ell k$ for some ℓ.

Definition 4.3. If M is a closed finite Euclidean simplicial complex whose simplexes are permuted among themselves by T, we say (M, T) is <u>simplicial</u>. The totality of invariant simplexes will be denoted by M_I.

Definition 4.4. A simplicial (M, T) will be called primitive if each simplex in $M-M_I$ has q distinct (hence mutually exclusive) T-images.

Lemma 4.3. A simplicial (M, T) is primitive if $q = p$, p a prime.

Proof. See Lemma 4.2.

Definition 4.5. A simplicial (M, T) is regular if the subcomplex M_I is closed.

Definition 4.6. A system will mean a finite collection $\{U\}$ of point sets in M. The component sets of a system $\{U\}$ are the U-vertices. A system where the vertices are permuted among themselves by T is called a T-system. The vertices of $\{U\}, \{T(U)\}, \cdots, \{T^{q-1}(U)\}$ taken together form a system denoted by $\sigma\{U\}$.

Lemma 4.4. The collection $\sigma\{U\}$ is a T-system.

Proof. Let U_σ be any vertex of $\sigma\{U\}$. Then $U_\sigma = T^i(U)$ where $U \in \{U\}$, and hence $T(U_\sigma) = T^{i+1}(U)$. But $T^{i+1}(U) \in \sigma\{U\}$ by definition of $\sigma\{U\}$. Therefore the vertices of $\sigma\{U\}$ are permuted among themselves.

Let $\{U\}$ be a T-system and write $\{U\} = \{U'\} \cup \{U''\}$, where $\{U'\}$ consists of the invariant U-vertices while $\{U''\}$ denotes the remaining U-vertices.

Note 4.1. The collections $\{U'\}$ and $\{U''\}$ are T-systems.

Definition 4.7. A T-system $\{U\}$ is primitive if each U''-vertex has q mutually exclusive T-images.

Definition 4.8. Let $\{U_i\}$ and $\{U_j\}$ be T-systems with $\{U_i\} > \{U_j\}$. A projection $\pi: \{U_i\} \to \{U_j\}$ is a <u>T-projection</u> if $\pi T = T\pi$.

Lemma 4.5. A T-projection $\pi: \{U_i\} \to \{U_j\}$ carries U'_i-vertices into U'_j-vertices.

Proof. Let U'_i be any U'_i-vertex. Then $\pi T(U'_i) = \pi(U'_i) = T\pi(U'_i)$ since $T\pi = \pi T$. Therefore $U'_j = T(U'_j)$ where $U'_j = \pi(U'_i)$.

Theorem 4.1. Let $\{U_i\}$ and $\{U_j\}$ be T-systems with $\{U_i\} > \{U_j\}$. If $\{U_j\}$ is primitive, there exists a T-projection $\pi: \{U_i\} \to \{U_j\}$.

Proof. Write $\{U_i\} = \{U_i^1\} \cup \{U_i^2\}$ where $\{U_i^2\}$ consists of all U_i-vertices which are contained in U''_j-vertices and $\{U_i^1\}$ consists of all U_i-vertices which are contained in U'_j-vertices. Then $\{U_i^1\}$ and $\{U_i^2\}$ are T-systems and refinements of $\{U'_j\}$ and $\{U''_j\}$, respectively. Moreover, since $\{U_j\}$ is primitive, each U''_j- vertex has q mutually exclusive images and the same is true with U_i^2-vertices since they are contained in the U''_j-vertices. Then $\{U_i^2\}$ can be represented without repetition as consisting of the T-images of a suitably chosen subsystem of its vertices, say, $U^2_{i_1}, U^2_{i_2}, \ldots, U^2_{i_s}$. Let π_2 be a projection of this subsystem into $\{U''_j\}$ and extend π_2 over $\{U_i^2\}$ by the formula $\pi_2 T^q(U^2_{i_1}) = T^q\pi_2(U^2_{i_1})$, $q = (1, 2, \ldots, p-1)$. In this manner π_2 becomes a T-projection, from $\{U_i^2\} \to \{U''_j\}$. Now let π_1 be a projection that takes all T-images of a U_i^1-vertex into the same U'_j-vertex. Thus π_1 is then a T-projection since the U'_j-vertices are invariant. Taken together, π_1 and π_2 define a T-projection from $\{U_i\} \to \{U_j\}$.

We are particularly concerned with systems that are finite open coverings of M. These finite open coverings will be referred to as coverings.

Theorem 4.2. Every covering $\{U\}$ of M is refined by a T-covering.

Proof. Let $\{U\}$ be any covering of M, and $\{V\}$ be the covering made of intersections of $\{U\}$, $\{T(U)\}$, $\cdot\cdot\cdot$, $\{T^{p-1}(U)\}$. Then $\{V\} > \{U\}$ since each V is in a U. If V is any element of $\{V\}$, then $V = U_1 \cap T(U_2) \cap \cdot\cdot\cdot \cap T^{p-1}(U_p)$ where $U_i \in \{U\}$. Now $T(V) = T(U_1) \cap T^2(U_2) \cap \cdot\cdot\cdot \cap T^p(U_p)$ which is another element of $\{V\}$. Therefore $\{V\}$ is a T-covering.

Note 4.2. If $q = p$, p a prime, then for every covering $\{U\}$ of a space M there exists a primitive T-covering $\{V\}$ such that $\{V\} > \{U\}$. This is an accumulation of results up to this point.

Special Systems and Coverings

Let $\{U\}$ be a T-system with X as its nerve. (See Example 1, Chapter II.) Then T induces in X a simplicial transformation T_x which is the identity or else is of period r, r a divisor of q. Denote by X_I the totality of X-simplexes which are invariant under T_x and by X_L the totality of X-simplexes which meet L.

Definition 4.9. A T-system $\{U\}$ satisfies condition L_a if $\{U'\}$ consists precisely of those U-vertices which meet L; $\{U\}$ satisfies L_b if all nonempty intersections of U'-vertices meet L. A T-covering which satisfies L_a and L_b will be called special.

Lemma 4.6. If the covering $\{U\}$ is primitive and special, then $X_I = X_L$ and (X, T_x) is primitive and regular.

Proof. A non-invariant X-simplex E has at least one non-invariant vertex U, the T-images of U are mutually exclusive sets since $\{U\}$ is primitive. Assume that $T_x^q(E) = E$. Then T^q (Kernel E) = Kernel E, which implies $T^q(U) \cap U \neq \emptyset$. Therefore the T-images of E are distinct, hence (X, T_x) is primitive. The vertices of an X_I-simplex E are permuted among themselves by T_x and since as U-vertices they have a non-empty intersection, each U-vertex of E must be invariant by the primitivity of $\{U\}$. The condition L_a implies that E is vertex-wise invariant and (X, T_x) is therefore regular. Moreover, condition L_b implies that E meets L, hence $X_I \subset X_L$. Also, the vertices of an X_L-simplex E_L, since they meet L, are invariant by L_a. Therefore, E_L is invariant and $X_L \subset X_I$. Hence $X_I = X_L$.

Definition 4.10. The set M has dimension less than or equal to m ($\dim M \leq m$) if every covering is refined by a covering, the dimension of whose nerve does not exceed m.

Theorem 4.3. If M is compact, T primitive, and if $\dim M \leq m$, then every covering $\{U_j\}$ is refined by a special primitive covering $\{U_i\}$ such that $\dim(X_i - X_{iI}) \leq k$, $k = pm + p - 1$. (The nerve of $\{U_i\}$ is X_i.)

We shall first prove three lemmas in which it is convenient to say that a T-covering $\{U\}$ satisfies L_c if, among the U-vertices which meet L, each contains a point of L not contained in any other.

Lemma 4.7. If M is compact and T primitive, then every covering $\{U_j\}$ is refined by a primitive T-covering satisfying L_a and L_c.

Proof. Let x be an arbitrary point of L and $U_j(x)$ be any U-vertex containing x. Then $O(x)$, the intersection of the T-images of $U_j(x)$, is an invariant neighborhood of x such that $O(x) \subset U_j(x)$. Since M is compact and L is closed, there exists a finite set of neighborhoods $\{O(x)\}$ say $O_1, O_2, \ldots, O_s$ such that $L \subset \bigcup_i O_i$. Since T is primitive and M is Hausdorff, an arbitrary point y in the closed set $M - \bigcup_i O_i$ possesses a neighborhood $R(y)$ with p mutually exclusive T-images. These images do not meet L, for if they did they would not be mutually exclusive. Now for every y in $M - \bigcup_i O_i$ there exists a $U_j \in \{U_j\}$ such that $T^i(y) \subset T^i U_j$. Choose $R(y)$ so that $T^i R(y) \subset T^i U_j$ and $R(y)$ satisfies the above. This can be done because T is a homeomorphism and M is a compact Hausdorff space. The system of T-images of $R(y)$ is a refinement of $\{U_j\}$. Let $R_1, \ldots, R_t$ be a finite set of the $R(y)$'s such that $M - \bigcup_i O_i \subset \bigcup_j R_j$. The collection $\{O_i\}$ together with the T-images of the collection $\{R(y)\}$ forms a T-covering $\{U_1\}$, such that $\{U_1\} > \{U_j\}$. Thus $\{U_1\}$ is primitive, and satisfies L_a.

Now suppose that $\{U_i\}$ is a covering which is a modification of $\{U_1\}$ obtained by replacing each O_i by an invariant open set Q_i with $Q_i \subset O_i$, $Q_i \cap L \neq \emptyset$. Then $\{U_i\}$, like $\{U_1\}$, is a refinement of $\{U_j\}$, is primitive and satisfies L_a. We shall show that this modification can be carried out in such a way that the resulting $\{U_i\}$ also satisfies L_c. Choose distinct points $a_1, a_2, \ldots, a_s$ with $a_i \in O_i \cap L$. Then choose mutually exclusive invariant neighborhoods $A_1, A_2, \ldots, A_s$ of

$a_1, a_2, \cdots, a_s$ such that for each i, $\bar{A}_i$ is contained in the intersection of these O's which contain a_i. This can be done since a compact Hausdorff space is normal. Now consider the invariant open sets

$$Q_i = O_i - \bigcup \{\bar{A}_j / j \neq i\}.$$

The set Q_i contains a_j if and only if $i = j$ and $\bigcup_i Q_i \subset \bigcup_i O_i$. Also, $\bigcup_i Q_i \supset \bigcup_i O_i$. For a point $x \in O_i$ either is not contained in any A_j $(j \neq i)$ in which case $x \in Q_i$, or it is contained say in A_l $(l \neq i)$ in which case $x \in Q_l$. In either case $x \in \bigcup_i Q_i$. Thus we have shown that $\bigcup_i Q_i = \bigcup_i O_i$, and hence that the system $\{U_i\}$ obtained from $\{U_l\}$ by replacing O_i by Q_i is a covering refinement of $\{U_j\}$. Each vertex Q_i meets L since $a_i \in Q_i$. Hence the passage from $\{U_l\}$ to $\{U_i\}$ is of the type described above and we conclude that $\{U_i\}$ is primitive and satisfies L_a. Moreover, O_i contains a_j if and only if $i = j$, so $\{U_i\}$ satisfies L_c.

Lemma 4.8. If M is compact, T primitive and $\dim M \leq m$, then every primitive T-covering $\{U_j\}$ satisfying L_a and L_c is refined by a covering $\{U_i\}$ of the same sort and such that $\dim X_i \leq k$, $k = pm + p - 1$.

Proof. The hypothesis $\dim M \leq m$ implies the existence of a covering $\{U_j^o\} > \{U_j\}$ with $\dim X_j^o \leq m$. Let $\{U_l\} = \sigma\{U_j^o\}$, then $\dim X_l \leq k$. Moreover, $T\{U_j^o\} > T\{U_j\}$ and $T\{U_j\} = \{U_j\}$ since $\{U_j\}$ is a T-covering. Therefore $\{U_l\} = \sigma\{U_j^o\} > \{U_j\}$. Write $\{U_j\} = \{U'_j\} \cup \{U''_j\}$ where, as always, $\{U'_j\}$ consists of the invariant U_j-vertices. Write $\{U_l\} = \{U_l^1\} \cup \{U_l^2\}$ where $\{U_l\}$ consists of all U_l-vertices which are subsets of U'_j-vertices and $\{U_l^2\}$ consists of the remaining vertices.

Then $\{U_1^1\}$ and $\{U_1^2\}$ are T-systems where $\{U_1^1\} > \{U'_j\}$ and $\{U_1^2\} > \{U''_j\}$. By Theorem 1 there exists a T-projection $\pi\colon \{U_1^2\} \to \{U'_j\}$. Write $\{U'_j\} = \{U'_{j_1}, U'_{j_2}, \ldots, U'_{j_k}\}$. Let O_i be the union of the vertices which constitute $\pi^{-1}U'_{j_i}$. Let $\{U_{1_i}^{11}\} = \{O_i\}$. Then $\{U_1^{11}\} > \{U'_j\}$ since the only sets projected into U'_{j_i} are the ones contained in U'_{j_i}. The U_1^{11}-vertices are open sets since U_1^{11} is the union of sets from $\{U_1^1\}$, and they are invariant since π is a T-projection. The union of all U_1^{11}-vertices is identical with the union of the U_1^1-vertices since all U_1^1-vertices were projected into $\{U'_j\}$. Hence, $\{U_1^{11}\}$ and $\{U_1^2\}$ together form a T-covering $\{U_i\}$ such that $\{U_i\} > \{U_j\}$ and where $\{U'_i\} = \{U_1^{11}\} > \{U'_j\}$ and $\{U''_i\} = \{U_1^2\} > \{U''_j\}$. The primitivity of $\{U_j\}$ and the relation $\{U''_i\} > \{U''_j\}$ imply that $\{U_i\}$ is primitive.

We assert that $\{U_i\}$ satisfies L_a. A U_i-vertex which meets L must be a U'_i-vertex, since each $U''_i \subset U''_j$, which does not meet L. Conversely, every U'_i-vertex meets L. Each U'_i is an O_i, which is $\pi^{-1}U'_{j_i}$ for some U'_{j_i}. The vertex U'_{j_i} meets L (condition L_a for $\{U_j\}$) and U'_{j_i} contains a point a_i of L not contained in U'_{j_j}, $j \neq i$ (condition L_c for $\{U_j\}$). Therefore, $\pi^{-1}U'_{j_i} = U'_i$ contains a_i and U'_i meets L. Thus our assertion is proved.

Now $\{U_i\}$ was formed from $\{U_1\}$ by replacing a number of vertices by the union of those vertices. Since this does not raise the dimension of the nerve, we have dim $X_i \leq X_1 \leq k$. A primitive covering satisfying L_c is obtained in the same way as in Lemma 4.7, by replacing the vertices of the first covering with suitable subsets of themselves. This operation does not raise the dimension of the nerve and hence it is applied to $\{U_i\}$ to yield the required refinement of $\{U_j\}$.

Lemma 4.9. If M is compact, T primitive, and $\dim M \leq m$, then for every primitive T-covering $\{U_j\}$ satisfying L_a, L_c and having $\dim X_j \leq k$, there exists a special primitive covering $\{U_i\}$ such that $\{U_i\} > \{U^*_j\}$, $\dim (X_i - X_{iI}) \leq k$, where $\{U^*_j\}$ denotes the covering $\{St\ U_j\}$.

Proof: Let $\{U_j\} = U_{j_1}, \ldots, U_{j_1}$ and for each i choose a point a_i contained in $L \cap U'_{j_i}$, where $U'_{j_i} \in \{U'_j\}$, but not in U'_{j_j}, $j \neq i$ (condition L_c). Choose invariant open sets $A_1, A_2, \ldots, A_s$ such that $a_i \in A_i \subset U'_{j_i}$, $A_i \cap U'_{j_j} = \emptyset$ for $i \neq j$, and such that no A_i meets any U''_j-vertex. (Recall that the U''_j-vertices do not meet L.) Then $A_i \cap A_j = \emptyset$ when $i \neq j$. For each i choose a set of A's by the following rule: A_j is in the i^{th} set if and only if $U'_{j_i} \cap U'_{j_j} \neq \emptyset$. Let B_i be the union of the A's in the i^{th} set and let

$$O_i = U'_{j_i} \cup B_i. \tag{1}$$

The set O_i together with the U''_j-vertices forms a covering $\{U_i\}$, and since O_i is invariant, then $\{U'_i\} = \{O_i\}$ and $\{U''_i\} = \{U''_j\}$. Hence $\{U_i\}$ is primitive and $\{U_i\} > \{U^*_j\}$. Each O_i meets L because U'_{j_i} does, hence $\{U_i\}$ satisfies L_a.

We assert that $\{U_i\}$ satisfies L_b. Suppose

$$J = O_q \cap O_r \cap \cdots \cap O_t \neq \emptyset.$$

If $U'_{j_q} \cap \cdots \cap U'_{j_t} \neq \emptyset$, we have from (1) and the definition of B_i,

$$A_q \cup A_r \cup \cdots \cup A_t \subset O_q \cap \cdots \cap O_t = J$$

so that J contains $a_q, \ldots, a_t$ and hence meets L. If $U'_{j_q} \cap \cdots \cap U'_{j_t} = \emptyset$, it follows from (1) that J is the intersection of sets of

B's and sets of U''_j-vertices. Since each B is the union of mutually exclusive sets A_i, it follows that J is a union of A's, hence meets L and the assertion is proved.

Now we need to show that $\dim (X_i - X_{iI}) \leq k$. The existence of a non-invariant X_i-simplex E implies a relation of the form

$$\text{(2)} \qquad \text{kernel } E = (U'_{j_0} \cup B_{i_0}) \cap (U'_{j_1} \cup B_{i_1}) \cap \cdots \cap (U'_{j_h} \cup B_{i_h}) \cap S \neq \emptyset$$

where S is an intersection of $\ell U''_j$-vertices, $\ell \geq 1$. The B's do not meet any U''_j-vertices because the A's do not and hence (2) implies

$$(U'_{j_i})_0 \cap \cdots \cap (U'_{j_i})_h \cap S \neq \emptyset.$$

Hence $\ell + h + 1 \leq \dim X_j \leq k$. Therefore $\dim (X_i - X_{iI}) \leq k$.

Lemma 4.10. If M is compact and $\{U_j\}$ is any covering of M, then there exists a covering $\{U_i\}$ such that $\{U^*_i\} > \{U_j\}$.

Proof. Let $\{U\} = \{U_1, U_2, \cdots, U_r\}$ and let it be shrunk to $\{U'\} = \{U'_1, U'_2, \cdots, U'_r\}$, $U'_i \subset U_i$. This shrinking is possible since M is normal. Then $B_i = \{ U_i, M-\overline{U}_i\}$ is a binary open covering and $\{B\} = \{U'\} \cap B_1 \cap \cdots \cap B_r$ is a finite open covering. Let $\{B\} = \{V_1, V_2, \cdots, V_h\}$ and suppose $V_i \cap V_j \neq \emptyset$. The set V_i is contained in a set U'_i of $\{U'\}$, and V_j is in one of the sets of B_i, i.e., $V_j \subset U_i$, or $V_j \subset M-\overline{U'}_i$. The second inclusion is ruled out since V_j meets the subset V_i of U_i. Therefore the first holds. Thus V_i and all sets of $\{B\}$ meeting it are in U_i.

Theorem 4.3 follows from Lemmas 4.7, 4.8, 4.9 and 4.10. Let $\{U_j\}$ be any covering of M. By Lemma 4.10, there exists a $\{U_1\}$ such that

$\{U^*_1\} > \{U_j\}$, and by Lemmas 4.7, 4.8 and 4.9 there exists a $\{U_i\}$ such that $\{U_i\} > \{U^*_1\} > \{U_j\}$ with all the properties required by Theorem 4.3.

Corollary 4.1. If M is compact and T primitive, then every covering is refined by a special primitive covering.

Proof. The proof of this corollary is contained in the proof of Theorem 4.3.

The above theorems are of fundamental importance in the work that follows, because from this point on we only need to consider the special coverings, if $\dim M \leq m$. These coverings form a cofinal family, Σ_k of M; the homology properties of M can be studied solely in terms of this family as long as the period is a prime.

ρ-Chains and Special Homologies in a Complex

Assume throughout this section that (M, T) is simplicial and primitive.

Let G be an abelain coefficient-group for chains and homologies in M. The transformation T induces a chain-mapping which is denoted by T. We may regard T as an additive operator acting on chains over G and permutable with the boundary operator F.

The operators

$$\sigma = 1 + T + \cdot \cdot \cdot + T^{p-1}, \quad \zeta = 1 - T$$

bear useful reciprocal relations to each other and play an important part in the work that follows. We shall denote these operators by ρ and $\bar{\rho}$ and agree that ρ may stand for σ, $\bar{\rho}$ for ζ or vice versa, but the meanings of ρ and $\bar{\rho}$ shall remain fixed in any given discussion.

A chain C is of type ρ if there exists a chain C_1 such that $C = \rho C_1$. The null chain is of both type ρ and type $\bar{\rho}$.

Lemma 4.11. If (M, T) is simplicial and primitive, an h-chain C in $M\text{-}M_I$ is of type ρ if and only if $\bar{\rho}C = 0$.

Proof. Assume C is of type ρ say $C = \rho C_1$. Then

$$
\begin{aligned}
\bar{\rho}C &= \bar{\rho}\,\rho\, C_1 \\
&= \bar{\rho}(C_1 + C_1^1 + \cdots + C_1^{p-1}) \\
&= (C_1 + C_1^1 + \cdots + C_1^{p-1}) - (C_1^1 + C_1^2 + \cdots + C_1^p) \\
&= 0.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\rho C &= \rho\bar{\rho}C_1 \\
&= \rho(C_1 - C_1^1) \\
&= (C_1 - C_1^1) + (C_1^1 - C_1^2) + \cdots + (C_1^{p-1} - C_1^p) \\
&= 0.
\end{aligned}
$$

Now suppose that $\bar{\rho}C = 0$. We may write

$$C = \Sigma_{j=1}^{\alpha(h)} \Sigma_{i=0}^{p-1} g_j^i E_j^i,$$

where g is an element of the coefficient group for chains and homologies, E_j is an h-simplex, and E_j^i is $T^i(E_j)$. This is true since (M, T) is primitive, therefore $E, E^1, \cdots, E^{p-1}$ are distinct. The condition $\bar{\rho}C = 0$ then becomes,

$$\Sigma_{i_j} g_j^i \bar{\rho} E_j^i = 0. \tag{1}$$

Consider the case where $\bar{\rho} = \zeta$. Then we have

$$\Sigma_{ij}\, g_j^i(E_j^i - E_j^{i+1}) = 0$$

where the upper indices are to be reduced mod p. Hence,

$$\Sigma_{ij}\, (g_j^i - g_j^{i-1})\, E_j^i = 0$$

and therefore $g_j^i = g_j^{i-1}$. Then $g_j^0 = g_j^1 = \cdot\ \cdot\ \cdot = g_j^{p-1} = g_j$ so $C = \sigma C_1$. Where

$$C_1 = \Sigma_{j=1}^{\alpha(h)} g_j\, E_j,$$

this concludes the proof for the case $\bar{\rho} = \zeta$.

Suppose $\bar{\rho} = \sigma$. Then (1) becomes

$$\Sigma_{ij}\, g_j^i\, \sigma\, E_j^i = \Sigma_{ij}\, g_j^i\, \sigma\, E_j = \Sigma_{j=1}^{\alpha(h)} (\Sigma_{i=0}^{p-1}\, g_j^i)\, \sigma\, E_j = 0$$

Hence

(2) $\Sigma_{i=0}^{p-1}\, g_j^i = 0$.

Let $\quad C_1 = \Sigma_{j=1}^{\alpha(h)}\ \Sigma_{k=0}^{p-1}\ \Sigma_{i=0}^{k}\ g_j^i\, E_j^k.$

Then $\quad C_1 - C_1^1 = \Sigma_{j=1}^{\alpha(h)}\ \Sigma_{k=0}^{p-1}\ \Sigma_{i=0}^{k}\ g_j^i\ (E_j^k - E_j^{k+1})$

$$= \Sigma_{j=1}^{\alpha(h)} \left[\Sigma_{k=0}^{p-1}\ \Sigma_{i=0}^{k}\ g_j^i\, E_j^k - \Sigma_{k=1}^{p}\ \Sigma_{i=0}^{k-1}\ g_j^i\, E_j^k \right]$$

$$= \Sigma_{j=1}^{\alpha(h)} \left[g_j^0\, E_j^0 + \Sigma_{k=1}^{p-1} (\Sigma_{i=0}^{k}\ g_j^i - \Sigma_{i=0}^{k-1}\ g_i^i) E^k - \Sigma_{i=0}^{p-1}\ g_i^i\, E_j^p \right]$$

$$= \Sigma_{j=1}^{\alpha(h)} \left[\Sigma_{k=0}^{p-1}\ g_j^k\, E_j^k - \Sigma_{i=0}^{p-1}\ g_j^i\, E_j^p \right]$$

$$= \Sigma_{j=1}^{\alpha(h)}\ \Sigma_{k=0}^{p-1}\ g_j^k\, E_j^k. \quad \text{(On account of (2))}$$

Hence $C = \rho\, C_1$ and the proof is complete.

If C is a cycle of type (ρ, M_I) (i.e. $C = \rho C'$ mod M_I) and C is the boundary of a chain of the same type, we write $C \approx 0$ mod M_I. If C_h, C_{h-1} are cycles of type ρ and $\bar{\rho}$ respectively and there exists a chain X_h such that

$$C_h = \rho X_h, \quad C_{h-1} = FX_h,$$

we write $C_h : C_{h-1}$.

We assume during the remainder of this chapter that the coefficient group G is the group p of integers reduced mod p. Denote this group by F_p.

Lemma 4.12. Let C_h, C_{h-1} be cycles of type ρ, $\bar{\rho}$ such that $C_h : C_{h-1}$. If $C_h \cong 0 \bmod M_I$, then $C_{h-1} \cong 0 \bmod M_I$.

Proof. Assume that $C_h \cong 0 \bmod M_I$. Then there exist, by definition of $C_h : C_{h-1}$, chains X_{h+1}, X_h, X_I, with X_{h+1} of type ρ, such that

(1) $$C_h = \rho X_h, \quad C_{h-1} = FX_h, \quad FX_{h+1} = C_h + X_I, \quad X_I \subset M_I.$$

We may write

(2) $$FX_{h+1} = X_h + Z + Z_I \quad (Z_I \subset M_I, \quad Z \subset M - M_I)$$

because $C_h = \rho X_h$ and $\rho X_h = X_h + Z + Z_I^1$.

Then

(3) $$\rho FX_{h+1} = C_h + \rho Z + \rho Z_I = C_h + \rho Z,$$

since $\rho Z_I = 0$.

If $\rho = \sigma$ and $X_{h+1} = \sigma Y_{h+1}$, then $\sigma FX_{h+1} = \sigma F \sigma Y_{h+1} = 0$. From (1) $\sigma FX_{h+1} = \sigma C_h$, therefore $\sigma C_h = 0$. This implies $C_h = 0$ in $M - M_I$. If $\rho = \zeta$ and $X_{h+1} = \zeta Y_{h+1}$, then $\sigma F \zeta Y_{h+1} = 0$ and $C_h = 0$ in $M - M_I$. Therefore from (2), $\rho FX_{h+1} = \rho X_h + \rho Z$ which implies that $\rho Z = 0$ or Z is of type $\bar{\rho}$ by Lemma 4.11. Hence, if we operate on both sides of (2) with F we find that $C_{h-1} \cong 0 \bmod M_I$.

Lemma 4.13. Let E be a vertex of M. If $M_I = \emptyset$, the cycle $\zeta E = E - T(E)$ cannot be $\underset{\sim}{\sim} 0$.

Proof. Suppose there exists a chain X such that $F\zeta X = \zeta E$. Let

(1) $$Z = FX - E.$$

Then $\zeta Z = \zeta FX - \zeta E = 0$ and therefore, since $M_I = \emptyset$, Z is of type σ. Hence there exists a w such that $Z = \sigma w$. Consequently the sum of the coefficients of Z is zero mod p and Z is a cycle. Since FX is a cycle it follows from (1) that $-E$ is a cycle, which is impossible.

The Existence of Fixed Points

Definition 4.11. A set M is acyclic mod p if for every compact set A, $A \subset M$, there exists a compact set $A' \supset A$ such that relative to the coefficient group p cycles in A are ~ 0 in A'. (Example: Euclidean n-space, G arbitrary.)

Definition 4.12. The set M is finite dimensional if there exists an m such that every finite covering by open sets of M has a refinement whose nerve is of dimension $\leq m$. (This condition allows the use of Σ_k from Theorem 4.3.)

Theorem 4.4. Let p be a prime and M a finite dimensional compact space which is acyclic mod p. Then every homeomorphic transformation T of period p of M into itself admits at least one fixed point.

Proof. Let B_0 be a non-empty compact set of M. Then σB_0 is compact because T is a homeomorphism. Also, $\sigma B_0 \supset B_0$, and σB_0 is invariant by Lemma 4.1. Let $A_0 = \sigma B_0$ and choose a compact set B_1 containing A_0 such that cycles in A_0 are ~ 0 in B_1. Let $A_1 = \sigma B_1$. If this process is continued we have

$$\emptyset \neq A_0 \subset A_1 \subset \cdot \cdot \cdot \subset A_n) \quad (n = pm + p)$$

where each A_i is an invariant compact subset of M such that cycles in A_i are ~ 0 in A_{i+1}. Let $N = \bar{A}_m$ and consider it as a subspace of M. T induces a transformation of period p of N into itself and it is sufficient to show that this transformation, denoted by T, admits a fixed point. In the topology of N, where N is regarded as a subspace of M, it is still true that cycles in A_i are ~ 0 in A_{i+1} and since dim $N \leq m$, the homology groups of (N, T) can be based on the family Σ_k. T induces a simplicial transformation in the complexes of N. Assume that $L = \emptyset$. Then by Lemma 6.6 $M_I = \emptyset$ where M_I is the invariant simplexes.

Consider a definite Σ_k covering $\{U_j\}$. If h and k are given positive integers, there exists a Σ_k refinement $\{U_i\}$ of $\{U_j\}$ such that by projecting an h-dimensional U_i-cycle in A_k into $\{U_j\}$ we obtain a U_j-cycle which is the U_j-coordinate of a cycle in A_k and will therefore be ~ 0 in A_{k+1} [3]. Consequently, if $\{U_n\}$ is an arbitrarily chosen Σ_k covering there exist Σ_k coverings $\{U_{n-1}\} \cdot \cdot \cdot \{U_0\}$ such that $\{U_0\} > \{U_1\} > \cdot \cdot \cdot > \{U_n\}$. If π_i is a projection $\{U_i\} \to \{U_{i+1}\}$ and C_i is an i-dimensional U_i-cycle in A_i, then $\pi_i C_i \sim 0$ in A_{i+1}. Choose π_i to be a T-projection by Theorem 4.1.

Let $\rho_0, \rho_1, \cdot \cdot \cdot$ stand alternately for ζ and σ starting with $\rho_0 = \zeta_0$. Let X_0 be a U_0-vertex in A_0. Since A_0 is invariant, TX_0 is also in A_0. Hence $\rho_0 X_0 = X_0 - T(X_0)$ is a 0-dimensional cycle in

A_0, and therefore $\pi_0\rho_0X_0 \sim 0$ in A_1, say $\pi_0\rho_0X_0 = FX_1$, $X_1 \subset A_1$. Then ρ_1X_1 is a cycle because

$$F\rho_1X_1 = \rho_1FX_1 = \rho_1\rho_0\pi_0X_0 = 0.$$

Since $\rho_1X_1 \subset A_1$, we have $\pi_1\rho_1X_1 \sim 0$ in A_2, say $\pi_1\rho_1X_1 = FX_2$. continuing in this manner we obtain chains $X_0, \cdots, X_n$ such that

(1) $$FX_{i+1} = \pi_i\rho_iX_i \quad (i = 0, \cdots, X_{n-1}).$$

Let π_n be the identity projection of $\{U_n\}$ into itself and let $C_i = \rho_i(\pi_n\pi_{n-1}, \cdots, \pi_i)X_i$ $(i = 0, \cdots, n)$. Then $C_n, C_{n-1}, \cdots, C_0$ are U_n-cycles of type $\rho_n, \rho_{n-1}, \cdots, \rho_0$ respectively, and as a consequence of (1) above $C_n:C_{n-1}:\cdots:C_0$. The cycle C_0 is of the form ζE which is a U_n-vertex. Since $\dim U_n < n$, we have $C_n = 0 \simeq 0$ and therefore by Lemma 4.12 (with $M_I = \emptyset$) we have $C_{n-1} \simeq 0, \cdots, C_0 = \zeta E \simeq 0$. By Lemma 4.13, this is impossible, therefore the assumption that $L = \emptyset$ is false.

Theorem 4.6. The set L, which Theorem 4.4 asserts to be non-empty, is acyclic mod p.

Proof. Let B be a compact set containing points of L. It will be sufficient to prove that there exists a compact set $B' \supset B$ such that cycles in $B \cap L$ are ~ 0 in $B' \cap L$. Consider the set $A_0, \cdots, A_n$ in the proof of Theorem 4.4. A_0 can be any non-empty invariant compact set, therefore suppose $A_0 = \sigma B$. Let B' be a bounded open set containing $\bar{A}_n$. It will be shown that B' can be $\bar{A}_n$.

Let $\{U\}$ be an element of Σ_k, $N = \bar{A}_n$. Every cycle in $L_N \cap B$ is ~ 0 in L_N. Since $\dim L_N \leq m$, we need to consider only cycles of dimension $\leq m$. From the special coverings choose $\{U_0\}, \cdots, \{U_n\}$

defined as in the proof of Theorem 4.4, and let π_i, ρ_i also be defined as before. Now let Z_{h-1} be a cycle in $L_N \cap B$ with $h - 1 \leq m$. Since $B \cap L_N \subset A_{h-1}$, we have $Z \sim 0$ in A_h; we may write

$$Z_{h-1}\{U_h\} = FX_h \quad (X_h = X_h\{U_h\} \subset A_h).$$

Since the simplexes of $Z\{U_h\}$ are in L_N they are in U_{hI} and we have $\rho_h\{U_h\} = 0$. Therefore $\rho_h X_h$ is a cycle in A_h. Hence $\pi_h \rho_h X_h \sim 0$ in A_{h-1}. This is the first step in a construction process which leads to the relation

$$C_n : C_{n-1} : \cdot \cdot \cdot : C_0 \quad (C_h = \pi_n \cdot \cdot \cdot \pi_i \rho_i X_i; \quad i = h, \cdot \cdot \cdot, n)$$

in $\{U_n\}$. Since C_n is of the form $\rho_n Z$, it is in $\{U_n\} - \{U_{nI}\}$ because $\rho Z'' = 0$ if $Z'' \subset \{U_{nI}\}$. But all simplexes of $\{U_n\} - \{U_{nI}\}$ are of dimension $\leq m$, therefore $C_n = 0 \simeq 0 \bmod \{U_{nI}\}$. Hence by Lemma 4.12, $C_h \simeq 0 \bmod \{U_{hI}\}$.

Let $Z' = \pi_n \cdot \cdot \cdot \pi_h Z$. From the definition of Cech Cycles $Z'\{U_n\} \sim Z\{U_n\}$ in $B \cap L_N$. Let $\rho = \rho_h$, $X' = \pi_n \cdot \cdot \cdot \pi_h X$. Then

$$\rho X' \simeq 0 \bmod \{U_{nI}\}; \quad FX' = Z'.$$

The first of these relations implies the existence of $\{U_n\}$-chains X_I, Y such that

(1) $$F\rho Y = \rho X' - X_I.$$

Let W_I be that subchain of $FY - X' - X_I$ which is in $\{U_{nI}\}$ and W the remainder. Then

(2) $$FY = X' + X_I + W + W_I \quad (W_I \subset \{U_{nI}\}; \quad W \subset \{U_n\} - \{U_{nI}\}.$$

Operate on both sides of (2) by ρ and take into account the relation (1) and the fact that $\rho X_i = \rho W_I = 0$. Then $\rho W = 0$ in $\{U_n\} - \{U_{nI}\}$. Hence by Lemma 4.11 there exists a D such that $W = \rho D$. Insert this into (2) and operate on both sides of (2) by F to obtain

$$0 = (Z' + FX_I + FW_I) + F\bar{\rho}D.$$

The chain in parenthesis is in $\{U_{nI}\}$, whereas $F\bar{\rho}D$ is in $\{U_n\} - \{U_{nI}\}$. Consequently, both chains are null and Z' is therefore the boundary of a chain in $\{U_{nI}\}$. By the properties of special coverings, chains in $\{U_{nI}\}$ are in L_N. Hence $Z\{U_n\} \sim Z'\{U_n\} \sim 0$ in L_N.

Now return to space M. Let K be a cycle in $B \cap L$. Then since $B \subset A_n$, K may be regarded as being identical to a cycle which belongs to the space N. This cycle is in $B \cap L_N$ and therefore homologous to zero in L_N. Hence $K \sim 0$ in $N \cap L$. Therefore, since $N = \bar{A}_n = B'$, $K \sim 0$ in $B' \cap L$, which completes the proof.

Theorem 4.7. If p is a prime and M a finite dimensional locally compact Hausdorff space which is acyclic mod p, then every homeomorphic transformation T of period p^α of M into itself admits at least one fixed point.

Proof. Assume the theorem to be true for $\beta < \alpha$. Let T be a transformation of period p^α operating in M and let L' be the totality of points which are invariant under T^q where $q = p^{\alpha-1}$. Since T^q is of period p, L' is non-empty by Theorem 4.4, and acyclic mod p by Theorem 4.5. Moreover, L' is transformed into itself by T. The transformation induced in L' by T is the identity or is periodic of period p^β where $\beta < \alpha$. Hence T admits at least one fixed point in L'.

We have now proved that if T is a homeomorphic transformation from n-space into itself and is of period q where q is a prime, then the set L of fixed points is not null. In the next chapter we examine the set L to determine its topological structure.

CHAPTER V

CLASSIFICATION OF THE SET OF FIXED POINTS

It was proved in Chapter II that if an n-sphere, $n \leq 2$, is mapped into itself by a periodic homeomorphic transformation, then the set L of fixed points is an r-sphere, $r \leq n$. The purpose of this chapter is to show that the same is true for an n-sphere in general if the period of the transformation is a power of a prime.

To obtain this result ρ-homology groups are formed from the ρ-cycles and ρ-boundaries of Chapter IV. There is a ρ-homology group of each dimension associated with each special covering of the space. It is shown that the ρ-homology groups can be decomposed into the cross product of two subgroups of which one is associated with the invariant simplexes and the other is associated with the non-invariant simplexes. Next the ρ-homology groups of the space are defined using the concept of inverse systems. Then, based on the properties of the ρ-homologies of the coverings, it is shown that the ρ-homology groups of the space can be decomposed into the cross product of two subgroups. One of these subgroups is the same as the ordinary homology groups of the fixed point set L, and the structure of this subgroup is known to be the same as that of the homology groups of an r-sphere.

Examples are given to show that L is not necessarily an r-sphere if the period is not a power of a prime. But the form of L is not known for an arbitrary period q. Before proving the theorem concerning L we shall first develop several preliminary results.

ρ-Chains in a Complex

We assume throughout this section that (M, T) is simplicial and primitive, G is an abelian coefficient-group, T is a chain-mapping, and that F_p, σ, ζ, and ρ are defined as in Chapter IV. From Lemma 4.11 in Chapter IV $\rho\bar{\rho} = \bar{\rho}\rho = 0$ (the annihilator). Note that in the following definition the ρ-chain is the $\bar{\rho}$-chain of Chapter IV.

Definition 5.1. A chain which is annulled by ρ is called a ρ-chain. A chain which is annulled mod M_I by ρ is called a ρI-chain.

Lemma 5.1. A necessary and sufficient condition for a chain C to be a ρI-chain is that there exist a chain X such that $C = \bar{\rho}X \bmod M_I$.

Proof. The sufficiency is implied by the relation $\rho\bar{\rho} = 0$.

Assume that $\rho C = 0 \bmod M_I$, then by Lemma 4.11 there exists a chain X such that $\bar{\rho}X = C \bmod M_I$.

We shall use the subscript I with the symbol of a chain to show that the chain is in M_I.

Lemma 5.2. If $G = F_p$, all chains in M_I are ρ-chains.

Proof. We have $\zeta C_I = C_I - C_I = 0$ and $\sigma C_I = pC_I = 0$, therefore the Lemma is proved.

Lemma 5.3. If $G = F_p$ and C is a chain, then $\rho C \subset M - M_I$.

Proof. We can write $C = X + X_I$ where $X \subset M - M_I$. Then by Lemma 5.2, $\rho C = \rho X \subset M - M_I$.

Lemma 5.4. If $G = F_p$, a necessary and sufficient condition for C to be a ρ-chain is that C be expressible in the form $\bar{\rho}X + X_I$.

Proof. Assume that $C = \bar{\rho}X + X_I$, then $\rho C = \rho\bar{\rho}X + \rho X_I = 0$ which implies the sufficiency.

Assume that C is a ρ-chain. Write $C = B + B_I$, where $B \subset M-M_I$. Since C and B_I are ρ-chains, so is B. In fact, B is a ρI-chain, hence by Lemma 5.1 $C = \bar{\rho}X + X_I$.

Lemma 5.5. If $G = F_p$, a necessary and sufficient condition for C to be a ρ-chain in $M-M_I$ is that C be of the form $\bar{\rho}X$.

Proof. Lemmas 5.3 and 5.4.

Special Homologies in a Complex

Assume in this section that (M, T) is simplicial, primitive, and regular.

Definition 5.2. A ρ-cycle is a ρ-chain which is a cycle. If a ρ-cycle X is the boundary of a ρ-chain, we write $X \simeq 0 \bmod M_I$.

These homologies, which we shall refer to as ρ and ρI-homologies, have the same algebraic properties as ordinary homologies. A chain which is identically zero may be regarded as a ρ-cycle, and as such, $0 \simeq 0$. Also a chain which is identically zero in $M-M_I$ may be regarded as a ρI-cycle, and as such, $X_I \simeq 0 \bmod M_I$.

Lemma 5.6. The boundaries of chains in M_I are in M_I.

Proof. Regularity implies that M_I is closed.

Lemma 5.7. If $G = F_p$, and if $\bar{\rho}C + C_I$ is a cycle, hence a ρ-

cycle by Lemma 5.4, then $\bar{\rho}C$ and C_I are ρ-cycles. If $\bar{\rho}C + C_I \simeq 0$, then $\bar{\rho}C \simeq 0$ and $C_I \simeq 0$.

Proof. We have, by definition of a cycle and by the fact that $F\rho = \rho F$,

$$0 = F(\bar{\rho}C + C_I) = F\bar{\rho}C + FC_I. \tag{1}$$

But $F\bar{\rho}C = \bar{\rho}FC \subset M\text{-}M_I$ by Lemma 5.3 and $FC_I \subset M_I$ by Lemma 5.6. Hence, by (1) $F\bar{\rho}C = 0$ and $FC_I = 0$. Therefore $\bar{\rho}C$ and C_I are ρ-cycles.

Now suppose that $\bar{\rho}C + C_I \simeq 0$. This implies that $\bar{\rho}C + C_I = F(\bar{\rho}A + A_I)$ or $\bar{\rho}(FA - C) = C_I - FA_I$. The left side of the last equation is in $M\text{-}M_I$ and the right side is in M_I because of Lemma 5.6 and the fact that $\bar{\rho}B_I = 0$ for every $B_I \subset M_I$. Hence both sides vanish. Therefore $\bar{\rho}C \simeq 0$ and $C_I \simeq 0$.

Lemma 5.8. Let $G = F_p$, X be a cycle in M_I, and X' be a ρ-cycle in $M\text{-}M_I$. Then, $X \simeq 0$ if and only if $X \sim 0$ in M_I, and $X' \simeq 0$ if and only if $X' \simeq 0$ in $M\text{-}M_I$.

Proof. Let $X = C_I$ and $X' = \bar{\rho}C$ in Lemma 5.7. Now $X' + X \simeq 0$ because $X \simeq 0$ and $X' \simeq 0$. Also, by the proof of Lemma 5.7, $X' \simeq 0$ in $M\text{-}M_I$ and $X \simeq 0$ in M_I. But every cycle in M_I is a ρ-cycle, hence $X \sim 0$ in M_I. The if part follows by definition of $X \simeq 0$ and $X' \simeq 0$.

Lemma 5.9. Let $G = F_p$ and let $X^h = \bar{\rho}C^h + C_I^h$ and $X^{h-1} = \rho C^{h-1} + C_I^{h-1}$ be ρ and $\bar{\rho}$-cycles such that $X^{h-1} = FC^h$. If $X^h \simeq 0$, then $X^{h-1} \simeq 0$.

Proof. Assume $X^h \simeq 0$. Then by Lemmas 5.7 and 5.8, $\bar{\rho}C^h \simeq 0$ in $M\text{-}M_I$. Hence, there exists a B such that $F\bar{\rho}B = \bar{\rho}C^h$ by Lemma 5.5.

Let $A = C^h - FB$. Then $\bar{\rho}A = \bar{\rho}C^h - \bar{\rho}FB = 0$, so that A is a $\bar{\rho}$-chain. Also, $FA = FC^h = X^{h-1}$. Hence $X^{h-1} \simeq 0$.

Lemma 5.10. Let $G = F_p$, and let C be a cycle. If $C \sim 0$, then $\rho C = FX$ where $\bar{\rho}X = 0$.

Proof. The relation $C \sim 0$ implies there exists an A such that $FA = 0$. Therefore $F\rho A = \rho C$, and ρA is a $\bar{\rho}$-chain since $\rho\bar{\rho} = 0$.

ρ-Homology Groups in a Complex

It is assumed in this section that (M, T) is simplicial, primitive, regular and that $G = F_p$. Denote the additive groups of ρ and ρI-homology classes of (M, T) over G by

$$H^h_\rho(M,T;G) \qquad H^h_{\rho I}(M,T;G).$$

Lemma 5.11. Let X^h_ρ be a ρ-homology class of dimension h. If one ρ-cycle x in X^h_ρ is ~ 0, then every ρ-cycle in X^h_ρ is ~ 0.

Proof. Let y be any ρ-cycle in X^h_ρ, then $y - x = b_n$, b_n a bound. Therefore $y = b_n + s$ is a bound, which implies $y \sim 0$.

Lemma 5.12. The totality of classes X^h_ρ whose ρ-cycles are ~ 0 is a subgroup of H^h_ρ.

Proof. Let X^h_ρ and Y^h_ρ be two classes whose ρ-cycles are ~ 0. Then $x \in X^h$ and $y \in Y^h$ implies that $x = b_h$ and $y = b'_h$ where b_h and b'_h are bounds. Therefore $x - y = b_h - b'_h$, which is a bound. Hence $x - y$ is an element of a class Z^h whose ρ-cycles are ~ 0.

Denote the subgroup in Lemma 5.12 by $B^h_\rho(M,T;G)$. The corresponding subgroup of $H^h_{\rho I}(M,T;G)$ is denoted by $B^h_{\rho I}(M,T;G)$.

Let x^h be a ρ-cycle that is an element of X^h_ρ. Then by Lemma 5.4 we know that $x^h = \bar{\rho}C + C_I$.

Lemma 5.13. The totality of classes X^h_ρ with the property that $x^h = \bar{\rho}C + C_I$, $x^h \in X^h_\rho$, C a cycle, is a subgroup of H^h_ρ.

Proof. Let X^h_ρ, Y^h_ρ be two such classes. We have $x^h - y^h = (\bar{\rho}C + C_I) - (\bar{\rho}C' + C'_I) = \bar{\rho}(C - C') + (C_I - C'_I)$. But $C - C'$ is a cycle and $C_I - C'_I$ is in M_I, therefore $x^h - y^h$ is contained in some Z^h_ρ of the same type.

Denote this subgroup by $K^h_\rho(M,T;G)$. The corresponding subgroup of $H^h_{\rho I}$, denoted by $K^h_{\rho I}$, consists of the ρI-homology classes which contain ρI-cycles of the form $\bar{\rho}C$ where C is a cycle mod M_I.

Let x^h be an ordinary cycle which is an element of an ordinary homology class X^h. Since $\bar{\rho}\rho = 0$, ρx^h is a $\bar{\rho}$-cycle.

Lemma 5.14. The totality of classes X^h with the property that for at least one x^h in X^h, $\rho x^h \simeq 0$, is a subgroup of H^h.

Proof. Let X^h, Y^h be two such classes, then $\rho x^h - \rho y^h = FC_{h+1} - FC'_{h+1}$ where $\bar{\rho}C_{h+1} = \bar{\rho}C'_{h+1} = 0$. Hence $\rho(x^h - y^h) = F(C_{h+1} - C'_{h+1})$ is an element of some Z^h of the same type.

Denote this subgroup by $H^h_{(\rho)}(M,T;G)$ and the corresponding subgroup of $H^h_I(M,T;G)$ by $H^h_{(\rho I)}(M,T;G)$.

Homomorphic mappings g and h of the groups $H^h_\rho(M,T;F_p)$ are defined as follows:

By Lemma 5.4 a ρ-cycle x^h in X^h_ρ has a representation $x^h = \bar{\rho}C^h + C^h_I$. Let $x^{h-1} = FC^h$. Then x^{h-1} is a $\bar{\rho}$-cycle. Because $\bar{\rho}C^h$ is a cycle by Lemma 5.7 and $\bar{\rho}x^{h-1} = F\bar{\rho}C^h = 0$.

Lemma 5.15. The class X^{h-1} containing x^{h-1} is independent of the choice of x^h in X^h and C^h in the representation $\bar{\rho}C^h + C^h_I$ for x^h.

Proof. Suppose $x^h \simeq x'^h$, $x'^h = \bar{\rho}C'^h + C'^h_I$ and $x'^{h-1} = FC'^h$. Then $x^h - x'^h \simeq 0$, $x^h - x'^h = \bar{\rho}(C^h - C'^h) + (C^h_I - C'^h_I)$ and $x^{h-1} - x'^{h-1} = F(C^h - C'^h)$, hence by Lemma 5.9 we have $x^{h-1} - x'^{h-1} \simeq 0$. Thus x^{h-1} and x'^{h-1} are contained in the same class.

Thus the correspondence $g(x^h) = x^{h-1}$, $g: X^h_\rho \to X^{h-1}_{\bar{\rho}}$, is a homomorphic mapping of H^h_ρ into a subgroup $H^{h-1}_{\bar{\rho}}$. Since $x^{h-1} \sim 0$, the image of H^h_ρ under g is a subgroup of $B^{h-1}_{\bar{\rho}}$.

Lemma 5.16. The image of H^h_ρ under g is $B^{h-1}_{\bar{\rho}}$.

Proof. Let $X^{h-1}_{\bar{\rho}}$ be any element of $B^{h-1}_{\bar{\rho}}$ and x^{h-1} be a $\bar{\rho}$-cycle in $X^{h-1}_{\bar{\rho}}$. We have $x^{h-1} \sim 0$ by definition of $B^{h-1}_{\bar{\rho}}$, which implies there exists a C^h such that $FC^h = x^{h-1}$. But $F\bar{\rho}C^h = \bar{\rho}x^{h-1} = 0$. Therefore, $x^h = \bar{\rho}C^h$ is a cycle. In fact, x^h is a ρ-cycle, hence we have $gX^h_\rho = X^{h-1}_{\bar{\rho}}$ where X^h_ρ is the ρ-homology class containing x^h.

Lemma 5.17. The kernel of g is K^h_ρ.

Proof. If $gX^h_\rho = 0$, then $x^h = \bar{\rho}C + C_I$ where $FC \simeq 0$. Therefore, $FC = F(\rho A + A_I)$ since FC is a $\bar{\rho}$-cycle. Then $B = C - \rho A - A_I$ is a cycle. Thus $\bar{\rho}B = \bar{\rho}C$ so that $x^h = \bar{\rho}B + C_I$. Hence $X^h_\rho \subset K^h_\rho$. Conversely, every element of K^h_ρ is carried into the zero of $B^{h-1}_{\bar{\rho}}$ by g. For every element X^h_ρ of K^h_ρ we have $x^h = \bar{\rho}C + C_I$, where C is a cycle, therefore $FC = 0 = x^{h-1}$.

The elements of a ρ-homology class X^h_ρ are contained in a uniquely determined ordinary homology class X^h and the correspondence

$h: X_\rho^h \to X^h$ is a homomorphism of H_ρ^h into H^h. The kernel of h is B_ρ^h by Lemma 5.12.

Lemma 5.18. The image of H_ρ^h under h is $H_{(\rho)}^h$.

Proof. Let $X_\rho^h \subset H_\rho^h$. A ρ-cycle x^h which is a member of X_ρ^h satisfies the relation $\bar{\rho}x^h = 0$, hence trivially $\bar{\rho}x^h \simeq 0$. Therefore $X_\rho^h \subset H_{(\rho)}^h$. Conversely, let $X_{(\rho)}^h$ be an element of $H_{(\rho)}^h$. To demonstrate that $X_{(\rho)}^h$ has a pre-image under h, it is sufficient to show that $X_{(\rho)}^h$ contains a ρ-cycle. In any case, $X_{(\rho)}^h$ contains a cycle x^h such that $\rho x^h \simeq 0$. This implies a relation $F(\rho C + C_I) = \rho x^h$. The relations $\rho x^h \subset M - M_I$ (Lemma 5.2), $F\rho C \subset M - M_I$ (Lemma 5.3), and $FC_I \subset M_I$ (Lemma 5.6) imply $F\rho C = \rho x^h$. Let $x'^h = x^h - FC$. Then $\rho x'^h = 0$ and $x'^h \sim x^h$, thus x'^h is the desired ρ-cycle in $X_{(\rho)}^h$.

Theorem 5.1. For a simplicial regular primitive (M, T),

(a) $H_\rho^h(M,T;F_p) - K_\rho^h(M,T;F_p) = B_{\underline{\rho}}^{h-1}(M,T;F_p)$, and

(b) $H_\rho^h(M,T;F_p) - B_\rho^h(M,T;F_p) = H_{(\rho)}^h(M,T;F_p)$.

These formulas hold for arbitrary G if ρ is replaced by ρI.

Proof. The mapping g is a homomorphism from H_ρ^h onto $B_{\underline{\rho}}^{h-1}$ by Lemma 5.16, and K_ρ^h is the kernel of g by Lemma 5.17. Therefore, by the fundamental theorem of homomorphism of groups, (a) is true. Now h is a homomorphism from H_ρ^h onto $H_{(\rho)}^h$ by Lemma 5.18, and B_ρ^h is the kernel of h. Therefore, (b) is true.

The proof of these formulas for ρI-homology groups depends on the properties of the corresponding homomorphisms g_I, h_I, of $H_{\rho I}^h$.

A Decomposition

It is assumed in this section that (M, T) is simplicial, primitive, regular and that $G = F_p$.

Lemma 5.19. The totality of classes X^h_ρ, $X^h_\rho \subset H^h$, with the property that $x^h \subset M-M_I$ for at least one element x^h in X^h, is a subgroup D^h_ρ of H^h_ρ.

Proof. Let X^h_ρ, Y^h_ρ be two such classes, then there exist $x^h \in X^h$, $y^h \in Y^h$ such that x^h, $y^h \subset M-M_I$. But the difference of two elements in $M-M_I$ is also in $M-M_I$, therefore D^h_ρ is a subgroup.

The totality of classes X^h_ρ with the property that $x^h \subset M_I$ for at least one x^h in X^h_ρ is also a subgroup, $H^h_{0\rho}$, of H^h_ρ.

Lemma 5.20. The subgroup $H^h_{0\rho}$ can be regarded as being identical to $H^h_\rho(M_I, F_p)$.

Proof. See Lemma 5.8.

Lemma 5.21. The homology group H^h_ρ can be decomposed into the subgroups D^h_ρ and $H^h_{0\rho}$.

Proof. Let $X^h_\rho \subset H^h_\rho$ and x an element of X^h_ρ. A ρ-cycle x is a ρ-chain and by Lemma 5.4 $x = \bar{\rho}B + B_I$. Now $\bar{\rho}B$ is a ρ-cycle by Lemma 5.7 and $\bar{\rho}B \subset M-M_{I_h}$. Also, B_I is a ρ-cycle and $B_I \subset M_I$. Therefore, every $X^h_\rho \subset H^h_\rho$ can be expressed as $Y^h_\rho + Z^h_\rho$, where $Y^h_\rho \subset D^h_\rho$ and $Z^h_\rho \subset H^h_{0\rho}$.

Lemma 5.22. The images of D^h_ρ and H^h_ρ under g are equal to $B^{h-1}_{\bar{\rho}}$.

Proof. Let $X^h_\rho \subset H^h_\rho$. Then by Lemma 5.21 $gX^h_\rho = gY^h_\rho + gZ^h_\rho$, where

$Y^h_\rho \subset D^h_\rho$ and $Z^h_\rho \subset H^h_{0\rho}$. Therefore, $gX^h_\rho = gY^h_\rho + 0$ by definition of g. Hence $gH^h_\rho \subset gD^h_\rho$. Conversely, if $X^h_\rho \subset D^h_\rho$, then $X^h_\rho \subset H^h_\rho$. Hence $gD^h_\rho \subset gH^h_\rho$. Also, $gH^h_\rho = B^{h-1}_{\bar\rho}$ by Lemma 5.16.

Lemma 5.23. If f is the projection $D^h_\rho \times H^h_{0\rho} \to H^h_{0\rho}$, and if $k = fg$ (g followed by f), then $kH^h_\rho = kD^h_\rho \subset H^{h-1}_{0\bar\rho}$.

Proof. By Lemma 5.22, $gH^h_\rho = gD^h_\rho$. Hence $kH^h_\rho = kD^h_\rho$. Now $gD^h_\rho = D^{h-1}_{\bar\rho}$ and f is the projection $D^{h-1}_{\bar\rho} \times H^{h-1}_{0\bar\rho} \to H^{h-1}_{0\bar\rho}$, hence kD^h_ρ is contained in $H^{h-1}_{0\bar\rho}$.

For the remainder of this paper no distinction is made between a cycle and its homology class (ρ- or ordinary). We will regard H as being composed of cycles, H_ρ as being composed of ρ-cycles, K_ρ as being composed of ρ-cycles of the form $\bar\rho C + C_I$ where C is a cycle, D_ρ as being composed of ρ-cycles of the form ρC, and $H^h_{(\rho)}$ as being composed of cycles X such that $\rho X \simeq 0$.

Projections of ρ-Homology Classes

Let $\{U\}$ be a T-system with nerve X, and T_x the transformation induced in X by T. Let $\rho_x = 1 - T_x$ or $1 + T_x + \cdots + T_x^{p-1}$ according as $\rho = \zeta$ or σ. We may think of ρ_x as an operator induced in X by ρ. Let $\{V\}$ be a second T-system with nerve Y. Suppose that $\{U\} > \{V\}$ and π is a T-projection $\{U\} \to \{V\}$. The chain-mapping induced by π will also be denoted by π. The relation $\pi T = T\pi$ implies that

$$(5.1) \qquad \pi\rho_x = \rho_x\pi$$

An important consequence of (5.1) is that π carries ρ_x-chains and homologies into ρ_y-chains and homologies.

In general, ρ_x, ρ_y will be denoted by ρ, since the meaning of ρ is clear in the context.

Lemma 5.24. Let $\{U\}$ and $\{V\}$ be T-systems with nerves X and Y respectively and such that $\{V\}$ is primitive and $\{U\} > \{V\}$. Let π_1 and π_2 be T-projections $\{U\} \to \{V\}$. If x is a ρ-cycle in X, $\pi_1 x \cong \pi_2 x$.

Proof. Suppose that the passage from π_1 to π_2 can be effected by re-defining π_1 over the T-images of a single U-vertex. In any case the passage from π_1 to π_2 can be obtained by a finite number of such steps. Suppose then that π_1 differs from π_2 only with respect to the T-images of U. Assume first that U is contained in a non-invariant V-vertex. Then, since $\{V\}$ is primitive, the images $U^q = T^q U$ are mutually exclusive. Let $\pi_i U^q = V_i^q$ $(i = 1, 2)$. Define an additive operator D over X-chains as follows. A simplex E either has just one vertex among the images of U (regarded now as vertices of X), or has none. If none, $DE = 0$. If one, suppose $E = (U^h S)$ where S is a simplex with no vertex U^i, then define $DE = (V_1^h\, V_2^h\, S')$ where $S' = \pi_1 S = \pi_2 S$. Let the definition of D be extended additively to all ρ-chains. Now $FD = \pi_2 - \pi_1 - DF$ and $DT = TD$ for individual simplexes. In the case where E has no vertex among the images of U, then $F(DE) = 0$, and $\pi_2 E - \pi_1 E - D(FE) = 0$ since $\pi_2 E \doteq \pi_1 E$. Also, $DTE = 0 = TDE$. If E has one vertex among the images of U, then

$$F(DEp) = F(V_1^h\, V_2^h\, S')$$

$$= \left[(V_1^h\, V_2^h\, S'),\ (V_2^h\, S')\right] (V_2^h\, S') + \left[(V_1^h\, V_2^h\, S'),\ (V_1^h\, S')\right] (V_1^h S')$$

$$+\Sigma_{i=1}^{p-1}\left[(V_1^h\ V_2^h\ S'),\ S'_i\right]\ (V_1^h\ V_2^h\ S'_i)$$

$$= V_2^h\ \pi_2\ S - V_1^h\ \pi_1\ S + \Sigma_{i=1}^{p-1}\left[(V_1^h\ V_2^h\ S'),\ S'_i\right]\ (V_1^h\ V_2^h\ S'_i),$$

where S'_i is a face of S' with the $i + 2$ vertex removed.

Also,

$$D(FEp) = D\left\{[(U_hS),\ S]\ S + \Sigma_{i=1}^{p-1}[(U_nS),\ S_i]\ S_i\right\}$$

$$= 0 + D\Sigma_{i=1}^{p-1}[(U_nS),\ S_i]\ S_i$$

$$= \Sigma_{i=1}^{p-1}[(U_nS),\ S_i]\ D\,S_i$$

$$= \Sigma_{i=1}^{p-1}[(U_nS),\ S_i]\ (V_1^h\ V_2^h\ S'_i)$$

Therefore, $FD = \pi_2 - \pi_1 - DF$. The relation $DT = TD$ follows because π_1 and π_2 are T-projections.

Since D was extended additively to all chains these formulas hold for all chains. From the first of these formulas it follows that $\pi_2x - \pi_1x = FDx$ and from the second that $\rho Dx = D\rho x = 0$ since x is a ρ-cycle. Hence Dx is a ρ-cycle and $\pi_2x \simeq \pi_1x$.

Now assume that U is not a subset of a non-invariant V-simplex. In this case π_1U and π_2U are invariant, thus $\pi_iU^q = V_i$, $(i = 1,2)$. Suppose the X-simplex E has just one vertex among the images of U, say $E = (U^hS)$. Then in this case take $DE = (V_1V_2S)$ but in all other cases take $DE = 0$. Then $FD = \pi_2 - \pi_1 - DF$ and $DT = TD$ again hold, but in the verification it is necessary to examine the case in which the vertices of E include more than one image of U, say for example $E = (U^hU^kS)$. Here $DE = 0$ by definition, hence $DFE = 0$. Also, π_iE equals $(V_i^hV_i^kS)$ which is degenerate, hence zero, and DFE vanishes by cancellation and definition.

Inverse Systems

The theory of inverse systems which is introduced here is an important concept in this chapter. For the purpose of this thesis it is sufficient to restrict the treatment of inverse systems to Hausdorff spaces and topological groups. (A topological group will be defined later.) The following definitions and theorems can be found in [14].

Definition 5.3. A set S is said to be partially ordered or merely ordered, if certain pairs of elements (a, b) or S satisfy an ordering relation which is denoted by $a < b$ and is subject to the sole condition of transitivity: $a < b$ and $b \subset c$ implies $a < c$.

Definition 5.4. Let S be ordered by $<$. Then S is said to be directed by $<$ (by $>$) whenever given any two elements a, b of S there exists a third element c such that $c < a$ and $c < b$ ($c > a$ and $c > b$). Denote this relation by $S = \{s; <\}$ [$\{s; >\}$].

Definition 5.5. Let $\{M_i\}$ be a system of Hausdorff spaces indexed by a directed set $S = \{i; >\}$ and suppose that whenever $i > j$ there is given a mapping, also known as a projection, $\pi_j^i\colon M_i \to M_j$ such that $i > j > k$ implies $\pi_k^j \pi_j^i = \pi_k^i$. The system $\Sigma = \{M_i; \pi_j^i\}$ of the M_i and the π_j^i is called an inverse mapping system.

Definition 5.6. Let M^π be the product space, and in M^π let M be the set of all the points $x = \{x_i\}$ such that $i > j$ implies $\pi_j^i x_i = x_j$. Then M is called the limit-space of the inverse mapping system Σ. If $i < j$, we have $\pi_j^i x_i = x_j$ and $\pi_i^i = 1$.

As a subset of M^π the limit-space M receives the relative topology and is a Hausdorff space.

Theorem 5.2. If each M_i is compact and not empty then the limit-space is likewise not empty.

The remainder of this section is concerned with systems of topological groups which are indexed by a directed set.

Definition 5.7. Let the group $G = \{g\}$, as a set of elements, be assigned a topology thus turning it into a topological space. Then G thus topologized is called a <u>topological group</u> whenever it is a T_0-space (a Hausdorff space is a T_0-space) and in addition $g - g'$ is a continuous function of $G \times G$ to G in the topology.

Definition 5.8. Let $\{G_i\}$ be a system of topological groups and let the projection π^i_j be a homomorphism. Then $S = \{G_i; \pi^i_j\}$ is said to be an <u>inverse system of groups</u>, or merely an inverse system.

Theorem 5.3. Let $S = \{G_i; \pi^i_j\}$ and $\Sigma = \{H^i; \xi^i_j\}$ be inverse systems both indexed by $A = \{i; >\}$ and with limit-groups $'G$, $'H$. Suppose that for each i there is a homomorphism $f_i\colon G_i \to H_i$ such that $\xi^i_j f_i = f_j \pi^i_j$, $i > j$. Then there exists a homomorphism $f\colon {}'G \to {}'H$ such that if $g = \{g_i\} \in {}'G$, then $fg = \{f_i g_i\}$.

Theorem 5.4. Under the same assumption as in Theorem 5.5 let the G_i be compact. If $H'_j = f_i G_i$, then $\Sigma' = \{H'_i; \xi^i_j\}$ is an inverse system with limit-group, say $'H'$, and f is an open homomorphism of $'G$ onto $'H'$.

ρ-Homology Groups in a Compact Space

It is assumed in this section that M is compact and that T is primitive.

Let $\Sigma = \{U_j\}$ be the totality of primitive special coverings of M, and let T_j be the transformation induced by T in X_j, the nerve of $\{U_j\}$. Each (X_j, T_j) is primitive and regular by Lemma 4.6. Primitivity implies that each T_j is of period p.

By Theorem 4.4 Σ is a directed set relative to ordering by redinement and is cofinal with the totality of all finite open coverings of M. Hence Σ is adequate for carrying the ordinary homology theory of M. It is now shown that Σ carries a ρ-homology theory for (M, T).

Let $\{U_i\}$ and $\{U_j\}$ be coverings in Σ with $\{U_i\} > \{U_j\}$. Since $\{U_j\}$ is primitive, there exists a T-projection π^i_j: $\{U_i\} \to \{U_j\}$ by Theorem 4.1. Since π^i_j is permutable with ρ and F, it carries ρ-cycles into ρ-cycles and preserves ρ-homologies. The projection π^i_j induces a mapping

$$\text{(5.2)} \qquad \bar{\pi}^i_j\colon\ H^h_\rho(X_i,T_i;G) \to H^h_\rho(X_j,T_j;G).$$

It is a consequence of Lemma 5.25 that $\bar{\pi}^i_j$ is independent of the particular choice of the T-projection π^i_j. Thus the groups $H^h_\rho(X_i,T_i;G)$ and associated mappings $\bar{\pi}^i_j$ form an inverse system invariantly related to (M, T).

Let

$$H^h_\rho(M,T;G) = \lim \left\{H^h_\rho(X_i,T_i;G);\ \bar{\pi}^i_j\right\}.$$

The elements of $H^h_\rho(M,T;G)$ may be regarded as a ρ-homology of (M, T).

A ρ-cycle Z^h being a collection $\{Z_i^h\}$ where Z_i^h is a (ρ, X_i)-cycle (ρ-cycle in X_i) and where $\{U_i\} > \{U_j\}$ implies

$$\pi_j^i Z_i^h \simeq Z_j^h. \tag{5.3}$$

We have $Z^h \simeq 0$ if and only if $Z_i^h \simeq 0$ for each i. We call Z^h a Cech ρ-cycle.

As in the simplicial case, we frequently make no distinction between ρ-cycles and their class.

The totality Σ is a topologically definite entity uniquely determined by M and T. It follows that the groups $H_\rho^h(M,T;G)$ are topological invariants of (M, T).

Lemma 5.25. The homomorphism π_j^i carries $B_\rho^h(X_i,T_i;G)$ into a subgroup of $B_\rho^h(X_j,T_j;G)$.

Proof. If $Z_i^h \in B_\rho^h$ then $Z_i^h \sim 0$. Also, $\pi_j^i Z_i^h \simeq Z_j^h$. Hence $Z_j^h \sim 0$, since π_j^i carries ρ-cycles into ρ-cycles and preserves ρ-homologies.

Also, π_j^i carries $K_\rho^h(X_i,T_i;G)$ into a subgroup of $K_\rho^h(X_j,T_j;G)$, and in the same way the homomorphism $H^h(X_i,G) \to H^h(X_j,G)$ induced by π_j^i carries $H_{(\rho)}^h(X_i,T_i;G)$ into a subgroup of $H_{(\rho)}^h(X_j,T_j;G)$. Thus $H_\rho^h(M,T;G)$ and $H^h(M,G)$ admit subgroups

$$B_\rho^h(M,T;G) = \lim \left\{B_\rho^h(X_i,T_i;G);\ \pi_j^i\right\},$$

$$K_\rho^h(M,T;G) = \lim \left\{K_\rho^h(X_i,T_i;G);\ \bar{\pi}_j^i\right\},$$

and

$$H_\rho^h(M,T;G) = \lim \left\{H_{(\rho)}^h(X_i,T_i;G);\ \bar{\pi}_j^i\right\}.$$

The relation $\pi_j^i T = T \pi_j^i$ implies that π_j^i carries invariant X_i-simplexes into invariant X_j-simplexes, hence $\pi_j^i X_{iI} \subset X_{jI}$. Thus $(\rho I,X_j)$-chains are carried by π_j^i into $(\rho I,X_j)$-chains and ρI-homologies are preserved. This leads to inverse systems of groups $H^h_{\rho I}(X_i, T_i; G)$ et cetera. The relation $X_{iI} = X_{iL}$ (see Lemma 4.6) implies that elements in the resulting limit-groups are ρ-cycles modulo L and their limit-groups are therefore properly denoted by $H^h_{\rho L}(M, L; G)$ et cetera.

The following remarks are made for future reference.

Remark (5.1). A topology in the coefficient group G will lead to a topology in H^h_ρ, $H^h_{\rho L}$, et cetera. In what follows, the groups are considered as discrete.

Remark (5.2). Like the groups H^h_ρ, the groups B^h_ρ, K^h_ρ, $H^h_{(\rho)}$ and the corresponding groups $H^h_{\rho L}$ et cetera are topological invariants of (M, T).

Remark (5.3). Suppose that dim $M \leq m$. Then Σ can be replaced in the preceding discussion by $\Sigma_k = \{U_i\}$, the totality of primitive special coverings $\{U_i\}$ such that dim $(X_i - X_{iI}) \leq k$, by Theorem 4.3.

Let h_i, g_i denote the mappings h, g defined as before for (X_i, T_i). The mappings h_i and g_i do exist since (X_i, T_i) is regular and primitive.

Lemma 5.26. If $\{U_i\}$ and $\{U_j\}$ are contained in Σ with $\{U_i\} > \{U_j\}$ and if π^i_j is a T-projection, then $\bar{\pi}^i_j g_i = g_j \bar{\pi}^i_j$ and $\bar{\pi}^i_j h_i = h_j \bar{\pi}^i_j$.

Proof. By definition of g_i, $\bar{\pi}^i_j g_i(Z^h) = \bar{\pi}^i_j Z^{h-1}$ where $Z^{h-1} = FC^h$ and $Z^h = \bar{\rho}C^h + C^h_I$. Hence, since F is permutable with π, we have $\bar{\pi}^i_j Z^{h-1} = \bar{\pi}^i_j FC^h = F\bar{\pi}^i_j C^h$. Now $g_j \bar{\pi}^i_j (Z^h) = g_j \bar{\pi}^i_j (\bar{\rho}C^h + C^h_I) =$

$g_j(\bar{\rho}\,\pi^i_j\, C^h + \bar{\pi}^i_j\, C^h_I) = \bar{\rho}\,\pi^i_j\, C^h$ since $\bar{\rho}$ is permutable with π and by definition of g_i.

The second part holds similarly. If $Z^h = \{Z^h_i\}$ is a ρ-cycle which is an element of $H^h_\rho(M,T;F_p)$, then Lemma 5.26 implies that $\{g_h Z^h_i\}$ is a $\bar{\rho}$-cycle. Denote this $\bar{\rho}$-cycle by gZ^h.

Using π^i_j: $H^h_\rho(X_i,T_i;G) \to H^h_\rho(X_j,T_j;G)$ and the fact that the groups $H^h_\rho(M,T;F_p)$ are finite and compact, it follows from the general theory of inverse systems that g actually covers $B^{h-1}_{\bar{\rho}}$ by Theorem 5.4. Moreover, the fact that the kernel of g_i is $K^h_\rho(X_i,T_i;F_p)$ for every $\{U_i\}$ in Σ implies that the kernel of g is $K^h_\rho(M,T;F_p)$. Similar remarks lead to a homomorphism h: $H^h_\rho(M,T;F_p) \to H^h_{(\rho)}(M,T;F_p)$ with kernel $B^h_\rho(M,T;F_p)$. Therefore the formulas of Theorem 5.1 hold for every compact M and primitive T.

Homologies in L

In this section M is compact, T primitive and $G = F_p$.

Let π^i_j be a T-projection and $\{U_i\}$, $\{U_j\} \in \Sigma$. Also, recall that $\pi^i_j\, X_{iI} \subset X_{jI}$. The elements of $H^h_{0\rho}(X_i,T_i;F_p)$ are cycles in X_{iI} and $H^h_{0\rho}(X_i,T_i;F_p)$ is identical to $H^h(X_{iI}, F_p)$ by Lemma 5.20.

Lemma 5.27. The projection π^i_j carries $H^h_{0\rho}(X_i,T_i;F_p)$ into a subgroup of $H^h_{0\rho}(X_j,T_j;F_p)$.

Proof. There exists an element $Z \in H^h_{0\rho}$ such that $Z \in X_{iI}$ and $\pi^i_j\,(Z) \in X_{jI}$ by Lemma 4.5. Hence the lemma is proved.

Let

$$H^h_{0\rho}(M,T;F_p) = \lim \left\{H^h_{0\rho}(X_i,T_i;F_p);\ \pi^i_j\right\}.$$

The relation $X_{iI} = X_{iL}$ and Lemma 5.20 imply that $H^h_{0\rho}$ is the group of ordinary homology classes of L.

That is

$$(5.4) \qquad H^h_{0\rho}(M,T;F_p) \cong H^h(L,F_p).$$

Lemma 5.28. The image of $D^h_\rho(X_i,T_i;F_p)$ under $\overline{\pi}^i_j$ is contained in $D^h_\rho(X_j,T_j;F_p)$.

Proof. There exists $Z^h \in D^h_\rho$ such that $Z^h \in X_i - X_{iI}$ by definition of D^h_ρ, and by Lemma 5.5 $Z^h = \overline{\rho}A$. Also, $\pi^i_j \,\overline{\rho}\, A = \overline{\rho}\, \pi^i_j A$. Therefore, $\pi^i_j Z^h \in X_j - X_{jI}$.

Let

$$D^h_\rho(M,T;F_p) = \lim \left\{D^h_\rho(X_i,T_i;F_p);\ \overline{\pi}^i_j\right\}.$$

If Z^h is a ρ-cycle that is an element of $D^h_\rho(M,T;F_p)$, then Z^h_i may be taken as a ρ-cycle in $X_i - X_{iI}$. If X_{iJ} is the totality of X_i-simplexes whose kernel meets $M - L$, then $X_i - X_{iI} = X_i - X_{iL} \subset X_iJ$. Lemma 5.8 and the relation $Z^h_i \subset X_i - X_{iI}$ imply that Z^h is a ρ-cycle of $M - L$ or at least is ρ-homologous to such a cycle.

The decomposition of Lemma 5.21, which holds in each (X_i, T_i), implies the decomposition

$$(5.5) \qquad H^h_\rho = D^h_\rho \times H^h_{0\rho} \quad \text{for} \quad (M,T;F_p)$$

and Lemma 5.22 implies

$$(5.6) \qquad gD^h_\rho = gH^h_\rho = F^{h-1}_{\overline{p}} \quad \text{for} \quad (M,T;F_p).$$

Hence, if we denote by f the projection mapping $D^h_\rho \times H^h_{0\rho} \to H^h_{0\rho}$ and by k the mapping fg, then we have by Lemma 5.23

$$(5.7) \qquad kH^h_\rho = kD^h_\rho \subset H^{h-1}_{0\overline{\rho}} \quad \text{for} \quad (M,T;F_p).$$

Denoting by k_i the mapping of $H^h_\rho(X_i, T_i; F_p)$ induced by k, k_i is precisely the mapping k defined for simplicial (M, T) with $M = X_i$ and $T = T_j$. Moreover, Lemma 5.26 implies

$$\pi^i_j \, k_i = k_j \, \pi^i_j, \tag{5.8}$$

where $i > j$ and π^i_j is a T-projection.

Lemma 5.29. If $D^{h-1}_\rho = 0$, then the kernel of k, as applied to H^h_ρ, is K^h_ρ.

Proof. If $D^{h-1}_\rho = 0$, then $gH^h_\rho \subset H^{h-1}_{0\bar\rho}$ since $gH^h_\rho = B^{h-1}_{\bar\rho} = D^{h-1}_{\bar\rho} \times H^{h-1}_{0\bar\rho} = 0 \times H^{h-1}_{0\bar\rho}$. Hence $fgH^h_\rho = gH^h_\rho$, thus k has the same kernel as g, namely K^h_ρ.

Lemma 5.30. If $D^{h-1}_\rho = 0$ and $H^h_{(\rho)} = H^h$, then k transforms D^h_ρ isomorphically.

Proof. The kernel of k acting on D^h_ρ is $D^h_\rho \cap K^h_\rho$ by Lemma 5.29. An element in $D^h_\rho \cap K^h_\rho$ is of the form $Z^h = \{\bar\rho C^h_i\}$ where $C^h = \{C^h_i\}$ is an ordinary cycle. Because $x \in D^h_\rho$ implies $x \in M - M_I$ and by Lemma 5.5, $x = \bar\rho C$. Also, $y \in K^h_\rho$ implies $y = \bar\rho C + C_I$ where C is a cycle. Thus C^h is an element of H^h and is therefore an element of $H^h_{(\rho)}$ by the hypothesis. This implies that $\bar\rho C^h \approx 0$ by definition of $H^h_{(\rho)}$, hence Z^h is the zero of D^h_ρ. Thus the kernel of k is the zero. Therefore k is an isomorphism.

Lemma 5.31. If $H^{h-1} = 0$, then k acting on H^h_ρ is equal to $H^{h-1}_{0\bar\rho}$.

Proof. We need to show that a given cycle Z^{h-1} in $H^{h-1}_{0\bar\rho}$ has a pre-image in H^h_ρ. Write $Z^{h-1} = \{Z^{h-1}_i\}$, where $Z^{h-1}_i \subset X_{iI}$. Since ρ-

homologies imply ordinary homologies, Z^{h-1} may be regarded as an element of H^{h-1}, hence $Z^{h-1} \approx 0$. Thus each element Z_i^{h-1} is contained in $H_{0\bar{\rho}}^{h-1}(X_i,T_i;F_p) \cap B_{\bar{\rho}}^{h-1}(X_i,T_i;F_p)$ and so has a pre-image in $H_\rho^h(X_i,T_i;F_p)$ under k_i by Lemma 5.22. Let R_i denote the totality of these pre-images of Z_i^{h-1}. If π_j^i is a T-projection, the relation (5.8) implies that $\pi_j^i R_i \subset R_j$, since $\pi_j^i k_i (Z_i^{h-1}) = k_i \pi_j^i (Z_i^{h-1}) = k_i (Z_j^{h-1}) \subset R_j$. The sets R_i being finite (hence compact) means that the limit of the inverse system $\{R_i; \pi_j^i\}$ is not empty. (See Theorem 5.2.) Therefore the limit elements are pre-images of Z^{h-1} under k.

Lemma 5.32. Assume that M is finite-dimensional, p a prime. Assume further that $H^h(M,F_p) = 0$ for $h > n$ while $H^n(M,F_p)$ is cyclic of order p. Then

$$H_{(\rho)}^n = H_\rho^n \cong H^n, \quad B_\rho^n = 0; \quad H_\rho^h = 0, \quad h > n, \quad \text{for } (M,T;F_p).$$

Proof. Suppose that the $\dim M \le m$, then $n \le m$. The definition of the ρ-homology groups can be based on Σ_k, $k = pm + p - 1$, in place of Σ by Remark 5.3. The i-coordinate of a cycle Z^h of D_ρ^h is an h-dimensional (ρ,X_i)-cycle in $X_i - X_{iI}$. Therefore, if $h > k$ and $\{U_i\}$ is in Σ_k, then $Z_i^h = 0$ and $D_\rho^h = 0$. Let γ be an integer larger than k such that $\gamma - k$ is even. Then $D_\rho^\gamma = 0$. Now $H^{\gamma-1} = 0$ since $\gamma - 1 > n$, hence $H_{\bar{\rho}}^{\gamma-1} = B_{\bar{\rho}}^{\gamma-1} = gD_\rho^\gamma = g0 = 0$, and $D_{\bar{\rho}}^{\gamma-1} = 0$. From this $H_\rho^{\gamma-2} = D_\rho^{\gamma-2} = 0$, thus $D_{\bar{\rho}}^{n+1} = H_{\bar{\rho}}^{n+1} = 0$, and hence

$$B_\rho^n = gH_{\bar{\rho}}^{n+1} = 0.$$

From Theorem 5.1, part (b), $H_\rho^n = H_{(\rho)}^n$. Now we need to show that $H_{(\rho)}^n \cong H^n$.

Let Z^n be an n-cycle $\not\sim 0$. Then $TZ \sim xZ$ (x a nonzero integer), $T^2Z \sim x^2Z, \cdot \cdot \cdot, Z = T^pZ \sim x^pZ$ since H^n is cyclic of order p. Hence, $x^p = 1 \bmod p$ and since p is prime $x = 1 \bmod p$. Thus for each i, $Tz_i \sim Z_i$, hence $\bar{\rho}Z_i \sim 0$. The collection $\{\bar{\rho}Z_i\}$ is a ρ-cycle since if π^i_j is a T-projection we have $\pi^i_j \, \bar{\rho} \, Z_i \quad \bar{\rho} \, Z_j$. This is true since $\pi^i_j \, Z_i - Z_j \sim 0$ and by Lemma 5.10, $\bar{\rho}(\pi^i_j \, Z_i - Z_i) \sim 0$ as required. Now $\bar{\rho}Z \sim 0$ implies that $\bar{\rho}Z$ is an element of the subgroup B^n_ρ of H^n_ρ. Therefore, since $B^n_\rho = 0$ implies $\bar{\rho}Z \sim 0$, we see that Z^n is a nonzero element of $H^n_{(\rho)}$. Thus $H^n_{(\rho)} \neq 0$ and since it is a subgroup of a cyclic group of prime order, H^n must be identical to $H^n_{(\rho)}$.

Homological Spheres

Definition 5.9. A space M is n-cyclic over G if $H^n(M,G) \cong G$ and $H^i(M,G) = 0, \quad i \neq n$.

Definition 5.10. A space M is said to be augumented if each of its coverings considered as a complex are augumented.

Definition 5.11. A compact finite-dimensional space is called a homological n-sphere over G if, when augumented, it is n-cyclic over G. The empty set is regarded as a homological (-1)-sphere over G.

Theorem 5.5. Let T be a transformation operating in M, of period $p = q^a$ with $a \geq 1$, q a prime. If M is a homological n-sphere over F_q, then the fixed-point set L is a homological r-sphere over F_q, $-1 \leq r \leq n$.

Proof. Assume that $a = 1$, $p = q$, and $L \neq \emptyset$.

Let $\rho_0, \rho_1, \cdot \cdot \cdot$ stand alternately for ζ, σ beginning with

$\rho_0 = \zeta$. Let r be the dimensional index of the first vanishing group in the sequence

(i) $$D^n_{\rho_n},\ D^{n-1}_{\rho_{n-1}},\ \ldots .$$

The definition of r has meaning since $D^0_{\rho_0} = 0$. Consider an element of $D^0_{\rho_0}$. It contains a ρ-cycle of the form $Z^0 = \{\bar{\rho}_0 A^0_i\}$ and since $H^0 = 0$, X_i is connected by section 11, Chapter VII of [14]. Hence if E is an X_i-vertex, then $E \sim 0 \bmod X_{iI}$. It follows that $A^0_i \sim 0 \bmod X_{iI}$, say, $FB'_i = A^0_i + A^0_{iI}$. Then $F\bar{\rho}_0 B'_i = \bar{\rho}_0 A^0_i = Z^0_i$ and so $Z^0_i \sim 0$ for each i which implies $Z^0 \in B^0_{\rho_0}$. Hence our assumption is true.

Next it is shown that $H^r_{0\rho_r}$ is cyclic of order r while $H^i_{0\rho_i} = 0$ for $i \neq r$. This implies that L is a homological r-sphere since the groups $H^h_{0\rho_h}$ are identical to the ordinary homology groups of L by (5.4, page 68). Note that

(ii) $$B^i_{\rho_i} = H^i_{\rho_i},\quad i \neq n,$$

since $H^i = 0$ for $i \neq n$. By Lemma 5.32, $H^n_{\rho_n}$ is cyclic of order p. Assume $r < n$. Then $D^n_{\rho_n} \neq 0$ by definition of r, hence $D^n_{\rho_n} = H^n_{\rho_n}$. We have $gD^n_{\rho_n} = B^{n-1}_{\rho_{n-1}}$ by (5.6, page 68) and consequently $B^{n-1}_{\rho_{n-1}}$ is cyclic, possibly zero. If $r < n - 1$, then $D^{n-1}_{\rho_{n-1}} \neq 0$ and $D^{n-1}_{\rho_{n-1}} \subset H^{n-1}_{\rho_{n-1}}$ so that $B^{n-1}_{\rho_{n-1}} \neq 0$ by (ii). Hence, $B^{n-1}_{\rho_{n-1}}$ is of order p and so is $D^{n-1}_{\rho_{n-1}}$. By repeating this argument we conclude that the groups (i) with dimensional index exceeding r are cyclic of order p. We have also shown that

(iii) $$H^i_{\rho_i} = D^i_{\rho_i},\quad i = n,\ n+1,\ \ldots,\ r+1.$$

Now we need to show that the remaining groups of (i) vanish. We have $D^r_{\rho_r} = 0$ by definition of r and $D^{r-1}_{\rho_{r-1}} \subset H^{r-1}_{\rho_{r-1}} = B^{r-1}_{\rho_{r-1}} = gD^r_{\rho_r} = 0$ by

(ii) and (5.6, page 68). Replacing r by $r - 1$ and so on we have $D^{\ell}_{\rho_\ell} = 0$ for all $\ell \leq r$.

Now by Lemma 5.31 and (iii)

$$\text{(iv)} \qquad H^{i}_{0\rho_i} = kH^{i+1}_{\rho_{i+1}} = kD^{k+1}_{\rho_{i+1}} = k0 = 0 \quad i < r.$$

Using the fact that $H^{i}_{\rho_i} = 0$ for $i > n$, (iv) also holds for $i > n$. Moreover, $H^{i}_{0\rho_i}$ vanishes for $i = r + 1, \cdots, n$, since $H\rho = D_\rho \times H_{0\rho}$ and since $H_\rho = D_\rho$. Therefore $H^{i}_{0\rho_i} = 0$ when $i \neq r$.

It remains to be shown that $H^{r}_{0\rho_i}$ is cyclic of order p. If $r = n$, then $D^{n}_{\rho_n} = 0$. Also, $H_\rho = D_\rho \times H_{0\rho}$, hence $H_\rho = H_{0\rho}$. Now $H^{n}_{\rho_n}$ is cyclic of order p, therefore, $H^{r}_{0\rho}$ is cyclic of order p. If $r < n$, then $D^{r+1}_{\rho_{r+1}}$ is cyclic of order p and $kD^{r+1}_{\rho r+1}$ is isomorphic to $H^{r}_{0\rho_r}$ by Lemmas 5.30 and 5.31; therefore, $H^{r}_{0\rho_r}$ is cyclic of order p. The theorem is proved for $a = 1$.

Assume $a > 1$ and that the theorem has been proved for $p = q^b$, $b < a$. The transformation T^s, $s = q^{a-1}$, is a prime of period q. Hence its fixed point set, L_s, is a homological r-sphere, $r \leq n$. Now T transforms L_s into itself, and the transformation T' induced in L_s is either the identity or it is of period q^c where $c < a$. In the first case $L = L_s$ and the theorem is established. In the second case the fixed point set L' of T' is a homological r'-sphere, $r' \leq r$.

Examples

The results of Theorem 5.5 do not hold for a transformation of period p, p arbitrary. E. E. Floyd constructed examples to show this fact in 1952 [6] and in 1956 [8]. Some of Floyd's examples are given in this section. First we shall state theorems and definitions that are used in constructing the examples.

Definition 5.12. A <u>decomposition</u> of a space M is a partition of M into a family of disjoint subsets of M whose union is M.

Definition 5.13. Let M be a compact Hausdorff space and T a periodic mapping on M, then σx, $x \in M$, is an <u>orbit</u>.

Definition 5.14. The <u>orbit decomposition space</u> M^* of (M, T) is the space whose elements are the sets σx, $x \in M$, with an open set in M^* being the images of an open set in M under the orbit decomposition mapping $f\colon M \to M^*$, where $f(x) = \{\sigma x\}$.

The following theorems are used in the construction of our examples. They can be found in [6] and [7].

Theorem 5.6. Let X be a finite complex and let A be a subcomplex of X. Suppose A is invariant under a simplicial periodic mapping T on X. Let A^* denote the orbit decomposition of the pair $(A, T/A)$ and let $F\colon A \to A^*$ the orbit decomposition mapping. Suppose that the induced homomorphism f^* of the homology groups $H_n(A;G)$ into $H_n(A^*;G)$ are isomorphisms onto for each n, where G is a given coefficient group. Consider the decomposition of X consisting of orbits of points of A and of individual points of $X - A$. Let X^* denote the resulting decomposition space. Then X^* may be triangulated so as to be a finite complex with A^* a subcomples. Moreover, $H_n(X^*;G) \approx H_n(X;G)$.

Theorem 5.7. Let A be an n-dimensional finite complex; let T

be a simplicial periodic homeomorphism of period two of A onto A with exactly one fixed point p and such that if C_a denotes the closed star of a vertex a of A, $a \neq p$, then $T(C_a) \cap C_a$ is either empty or is p. There exists a homeomorphism f of A into Euclidean $(2n + 1)$-space E_{2n+1} such that $fT = Sf$ where $S(x) = -x$ for every $x \in E_{2n+1}$ and such that f is linear on each simplex of A.

Notation. Let n be a positive integer ≥ 2. Let P denote the solid unit circle in the plane, where we use polar coordinates. Consider the decomposition of P which has as its elements the individual points (r, θ) for $r < 1$ and the sets $\{(1, \theta), (1, \theta + 2\pi/n), \cdots, (1, \theta + (2n - 1)\pi/n)\}$ for $r = 1$. Denote the resulting decomposition space by $P(n)$. If $(r, \theta) \in P$ let $(r, \theta)n$ denote the generated point of $P(n)$. An involution R of $P(n)$ onto $P(n)$ is defined by $R((r, \theta)n = (r, \theta + \pi)n$.

Theorem 5.8. If n is odd, then R has a single fixed point, and the orbit decomposition space $P^*(n)$ is homeomorphic to $P(n)$. Also, if $g: P(n) \to P^*(n)$ denotes the orbit decomposition mapping, then $g^*: H_n(P(n);I) \to H_n(P^*(n);I)$ is an isomorphism onto for each n, where I denotes the group of integers.

Theorem 5.9. If X is locally connected and locally simply connected, then so is X^*.

Example 1. Let n be an odd positive integer. There exists a finite complex K and a simplicial mapping T of K onto K of period two such that K is contractable (and thereby homologically trivial over all groups) and such that the fixed point set of T is homeomorphic to $P(n)$.

Proof. Let X denote the 5-cube in Euclidean 5-space consisting of all points whose coordinates are between -1 and 1. Denote by $S: X \Rightarrow X$ the involution $S(x) = -x$ for all $x \in X$. Triangulate $P(n)$ so that the hypothesis of Theorem 5.7 is satisfied with respect to a mapping R. (See notation above.) By Theorem 5.7 we can consider $P(n)$ as being imbedded in the interior of X so that R is equal to S on $P(n)$. Define K to be the decomposition space generated by the decomposition of X whose elements are the individual points of $X - P(n)$ together with the set $\{x, -x\}$, $x \in P(n)$. Let $h: X \to K$ be the natural decomposition mapping. Define $T = h\, s\, h^{-1}$. Then T is a decomposition mapping of K onto K of period two. The transformation T leaves the sets $\{x, -x\}$ fixed. These sets are homeomorphic to $P^*(n)$. Therefore, by Theorem 5.8 they are homeomorphic to $P(n)$. Moreover, $S/P(n) = R$, hence $R: H_n(P(n);I) \to H_n(P^*(n);I)$ is an isomorphism onto for each n. Thus by Theorem 5.6 $h_n(K;I) \approx H_n(X;I)$ and since $H_n(X;I)$ is homologically trivial, $H_n(K;I)$ is also.

We now prove that K is simply connected. It will then follow by [10] that K is contractible. Let X_1 and X_2 be disjoint copies of the 5-cube X. Let S_1 and S_2 denote the involution $x \Rightarrow -x$ in X_1 and X_2 respectively. Let $P_1(n)$ and $P_2(n)$ denote the copies of $P(n)$ in X_1 and X_2 respectively. If $x_1 \in X_2$, let x_2 be its copy in X_2. Form the decomposition of $X_1 \cup X_2$ with elements that are individual points of $X_i - P_i(n)$, for $i = 1, 2$ together with the sets $\{x_1, x_2\}$, for $x_1 \in P_1(n)$. The resulting decomposition space X' is simply connected being the union of two simply connected subcomplexes with a connected intersection. Moreover, there can be defined an involution S' on X' as follows: If $x \in X_1$, define S' on the element of X' determined by x_1 to be the element of X' determined by $S_2(x_2)$; similarly for points $x_2 \in X_2$. Points in the orbit decomposition space of the pair $(X';S')$ are of the form $\{x_1, -x_2\}$ for $s_1 \in X_1 - P_1(n)$ and $[\{x_1, -x_2\}, \{-x_1, x_2\}]$ for $x_1 \in P_1(n)$. Now $h: K \to X'$, where h is a mapping such that $h(x_1) = \{x_1, -x_2\}$ for $x_1 \in X - P(n)$ and $h\{x_1, -x_1\} = [\{x_1, -x_2\}, \{-x_1, x_2\}]$ for $x_1 \in P(n)$, is an homeomorphism fro K to X'. The set S' has fixed points since S_1 and S_2 have fixed points and they must be copies of each other. Therefore the orbit decomposition space of (X', S') is simply connected by Theorem 5.9. Hence K is simply connected.

Example 2. Let G be a non-trivial abelian group. There exists a prime period p such that K is homologically trivial over G but such that the fixed point set L is not homologically trivial over G.

Proof. If $P(3)$ is not homologically trivial over G, then the statement follows from Example 1. If $P(3)$ is homologically trivial over G, then let $K = P(3)$, and define $T((r, \theta)n) = (r, \theta + 2\pi/3)n$. Then T is a periodic mapping of K onto K of period 3 whose fixed point set is the union of a point and a simple closed curve. The point is $(0, 0)$, and the simple closed curve is (r, θ), $r = 1$.

Example 3. Let n be an odd integer. There exist a finite complex K and a periodic cimplicial mapping T of K onto K of period two such that K has the homology groups of a 5-sphere (over the integers) and such that the fixed point set of T is a union of a set M homeomorphic to $P(n)$ and a point not on M.

Proof. Make the following changes in the construction of K in the proof of Example 1. Let X be 5-space compactified with a point at infinity so that X is a 5-sphere. Let S be the same as in Example 1, and consider $P(n)$ as being embedded in X, just as in Example 1. Define K, h, and T as before. Then K has the homology groups of a 5-sphere over the integers by Theorem 5.6. Moreover, T has a fixed point set $h(\infty) \cup (hP(n))$.

Example 4. Let G be the group of integers. There exist a prime number p, a finite complex K, and a periodic simplicial mapping T of K onto K of period p such that K has the homology groups of a sphere of some dimension over G but such that the fixed point set does not have the homology groups of a sphere of any dimension over G.

Proof. If $P(3)$ is not homologically trivial over G, the conclusion follows from Example 3. Suppose, then, that $P(3)$ is homologically trivial over G. Let $P_1(3)$ and $P_2(3)$ be disjoint copies of $P(3)$. Let A_1 be the set of all $(1, \theta)3 \in P_1(3)$ and let A_2 be its copy in $P_2(3)$. In the set $P_1(3) \cup P_2(3)$ identify a point $a_1 \in A_1$ with its copy a_2 in A_2. Call the result K. Then K is the union of two complexes that are homologically trivial over G and whose intersection is a simple closed curve. Hence K has the homology groups over G of a 2-sphere. On each $P_i(3)$ define a mapping of period 3 as follows: $(r, \theta)3 \to (r, \theta + 2\pi/3)3$. These induce a mapping T of K onto K of period 3 whose fixed point set is the union of two points and a simple closed curve, which does not have the homology groups of a sphere.

The above results also hold for G a non-trivial abelian group. E. E. Floyd has shown in [8] that there exist a simply-connected homological 2-sphere, B, and a periodic mapping on B of period six (the lowest period not covered by Smith's Theorem) whose fixed point set is the disjoint union of two points and a simple closed curve.

The structure of L when the period of T is not a power of a prime is one of the unsolved proflems. P. A. Smith originally thought that his results could be extended to an arbitrary period. Recall that most of the results in this thesis are obtained in a compact space. It is not known whether or not the compactness is necessary; however, it plays a major part in the proofs of the theorems.

CHAPTER VI

SUMMARY AND EDUCATIONAL IMPLICATIONS

This thesis presents a collection of mathematical research reports, each concerned with the fixed point property of a space mapped into itself by a periodic homomorphic transformation. A number of research findings are consolidated in this paper so that understandings in this area can be more accessible to students who might not have the skills necessary to read the technical mathematical journals. A brief history of the fixed point problem and topology in general is included for developing background in this general area. Discussions, explanations, and examples which illustrate the theory are given along with some unsolved problems.

Summary

Chapter I contains the statement of the problem and discussions on the justifications, procedures, limitations, and expected outcomes of the thesis. Chapter II, following a brief history of topology, presents definitions of the basic terms such as a homeomorphism, a periodic transformation, a Hausdorff space, and the fixed point property. Some theorems that can be proved by elementary methods are given at this point. Chapter III is a review of homology theory. It includes definitions and some results that are basic in the proofs of the theorems in

Chapters IV and V.

Chapter IV contains a proof of the existence theorem which follows the one given by P. A. Smith in [17]. This theorem is one of the improtant results presented in this thesis. The existence theorem, in somewhat different form, has appeared in the journals at least three times and was one of the first results obtained. In Chapter V the set of fixed points of an n-sphere was classified for a periodic homeomorphic transformation of period q, q a power of a prime. Examples are given to show that the same classification is not possible in general. Many results are obtained in Chapters IV and V that are not of primary importance in this thesis. These results are presented because the proofs of the major theorems are based upon them.

Educational Implications

Since the study of mathematics is becoming increasingly widespread and the body of knowledge in all areas is expanding rapidly, a collection of the research done in any one area is needed, because it is time-consuming for each interested person to do the library research necessary to collect such information. A study such as this one, in addition to consolidating the research, presents the necessary background needed for understanding the problem and therefore brings this collection of knowledge to many students of mathematics.

As a result of reading this thesis, the student should gain an awareness of the current and past research in this modern branch of mathematics. He should become acquainted with men who have contributed to its research and development. It is of great educational significance that the reader, who is a potential teacher at either the public school or the college level, may become sufficiently interested in this phase of mathematics to undertake serious study in this area. He may be challenged by the possibility of contributing to research in mathematics by extending the results given in this thesis and by suggesting solutions to the unsolved problems or by developing new properties of fixed points. The bibliography should be a valuable aid to anyone interested in the research of fixed point theory for periodic transformations.

SELECTED BIBLIOGRAPHY

1. Alexandroff, P. and Hopf, H., Topologie, Berlin, 1935.

2. Bedgood, D. R., Properties of Fixed Point Spaces in Point Set Topology, Doctoral Dissertation, Stillwater, Oklahoma, 1964.

3. Cech, E., Theorie Generale de L'Homologie dans un Espace Quelconque, Fund. Math., Vol. 18 (1932), pp. 149-183.

4. Eilenberg, Samuel, On a Theorem of P. A. Smith Concerning Fixed Points for Periodic Transformations, Duke Math. J., Vol. 6 (1940), pp. 228-237.

5. _____, Sur les Transformations Periodiques de la Surface de Sphere, Fund. Math., Vol. 22 (1934), pp. 28-41.

6. Floyd, E. E., Examples of Fixed Point Sets of Periodic Maps, Ann. of Math., Vol. 55 (1952), pp. 167-172.

7. _____, On Related Periodic Maps, Amer. J. Math., Vol. 74 (1952), pp. 547-554.

8. _____, Examples of Fixed Point Sets of Periodic Maps, II, Ann. of Math., Vol. 55 (1952), pp. 167-172.

9. Haddock, A. G., Fixed Point Theorems, American Math. Monthly, Vol. 71 (1964), pp. 1095-1099.

10. Hurewicz, W., Beitrage Zur Topology der Deformationen. II Homotopie- und Homologie-gruppen, Proc. A. Kad. Amsterdam (1935), pp. 521-528.

11. Kelley, J. L., General Topology, New York: Van Nostrand, 1955.

12. Klee, V. L., Jr., Fixed-Point Sets of Periodic Homeomorphisms of Hilbert Spaces, Ann. of Math., Vol. 64 (1956), pp. 393-394.

13. Lefschetz, S., Introduction to Topology, New York: Princeton University Press, 1949.

14. _____, Algebraic Topology, American Math. Soc., Colloquium Publication, Vol. 27, 1942.

15. Newman, M. H. A., A Theorem on Periodic Transformations of Spaces, Quart. H., Vol. 94. (1925), pp. 39-41.

16. Smith, P. A., A Theorem on Fixed Points for Periodic Transformations, Ann. of Math., Vol. 35 (1934), pp. 572-578.

17. _____, Fixed Points for Periodic Transformations, American J. Vol. 63 (1941), pp. 1-8.

18. _____, Fixed Points for Periodic Transformations, Appendix B in S. Lefschetz Algebraic Topology, American Math. Soc., Colloquium Publications, Vol. 27, New York, 1945.

19. _____, Transformations of Finite Period, Ann. of Math., Vol. 39 (1938), pp. 127-164.

20. _____, Transformations of Finite Period, II, Ann. of Math., Vol. 40 (1939), pp. 690-771.

21. Whyburn, G. T., Analytic Topology, American Math. Soc., Colloquium Publications, Vol. 28, 1942.

22. Wilder, R. L., Topology of Manifolds, American Math. Soc., Colloquium Publications, Vol. 39, 1949.

23. _____, et al., American Mathematical Society Semicentennial Publication, Vol. 2 (1931), pp. 136.

24. Young, G. S. and Hocking, J. G., Topology, Reading, Massachusetts: Addison-Wesley Pub. Co., Inc., 1961.

APPENDIX

INDEX

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