A STATE-VARIABLE APPROACH TO CONTROL SYSTEM DESIGN

By

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A STATE-VARIABLE APPROACH TO

CONTROL SYSTEM THEORY

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

1.1 Statement of the Problem. Currently, there exists a trend in engineering education to augment the methods of analysis and design of linear systems. For some time the standard methods have depended heavily upon the Laplace transformation, and a number of techniques have evolved which are based principally in the s-plane. Some of these techniques are the root locus, the Bode plot, and the Nyquist plot, as well as others. A relatively new approach, representing quite a departure from conventional techniques, is presently being investigated.

This new approach is based upon a state model of the system. In this state model the system is characterized by a set of first-order differential equations

$$\frac{dx_i}{dt} = f_i\,(x_1, x_2, \ldots, x_n; r_1, r_2, \ldots, r_m, t) \qquad (1.1.1)$$

$$i = 1, 2, 3, \ldots, n$$

where the x_i's are the state variables,

r_i's are the forcing functions, and

t is the independent time variable.

Specifically, for the class of linear systems, the state equations take the form

$$\frac{d}{dt} \underline{X}(t) = \underline{A}\,\underline{X}(t) + \underline{B}\,\underline{R}(t) \;, \quad \underline{X}(0) = \underline{X}_o \qquad (1.1.2)$$

$$\underline{Y}(t) = \underline{C}\,\underline{X}(t) + \underline{D}\,\underline{R}(t)$$

where

$\underline{X}(t)$, n by 1, is the vector state variable,

$\underline{R}(t)$, m by 1, is the vector forcing function,

$\underline{Y}(t)$, k by 1, is the vector output function, and

$\underline{A}$, $\underline{B}$, $\underline{C}$, and $\underline{D}$ are constant matrices

with the appropriate dimensions.

This thesis is concerned with a single input, single output linear system having the state model description

$$\frac{d}{dt} \underline{X}(t) = \underline{A}\,\underline{X}(t) + \underline{B}r(t) \;, \quad \underline{X}(0) = \underline{X}_o \qquad (1.1.3)$$

$$c(t) = \underline{T}\,\underline{X}(t)$$

where

$\underline{X}(t)$, n by 1, is the vector state variable,

$\underline{A}$, n by n, is a real constant matrix,

$\underline{B}$, n by 1, is a real constant column matrix,

$\underline{T}$, 1 by n, is a real constant row matrix,

r (t) is the scalar input forcing function, and

c (t) is the scalar output.

Appendix B provides an extension of the methods explained in the body of the thesis to systems which have a portion of the input included along with the state variables in the output.

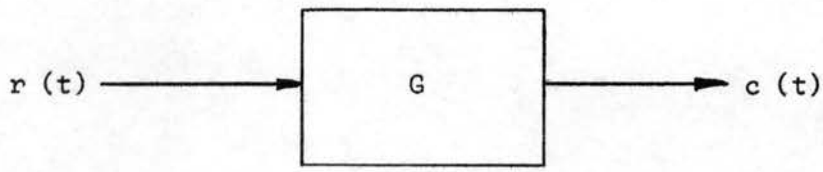

Figure 1.1.1. Linear System G

The problem is to determine the characteristics of the system G by an analysis of the state equations, and to provide methods for compensating the system if necessary. Use will be made of knowledge of conventional control system theory in the derivation of methods, but this knowledge will not necessarily be assumed in the application of the methods.

1.2 Previous Work in This Area. A good source of information about the relationship between linear systems, state variables, and Laplace theory is Zadeh and Desoer (1). The treatment is at the advanced level and is quite rigorous. The fourth chapter of this book provides a foundation for much that has been done in this general area.

The state model of interest in this thesis is described by Zadeh and Desoer (1) on page 254 as a strictly proper time-invariant system of finite order. The authors make the point that this assumed model of the system is equivalent to a differential system which is characterized by a single differential equation

$$L(p)\,c(t) = M(p)\,r(t) \qquad (1.2.1)$$

where L and M are linear differential operators with constant coefficients. Consequently, the problem deals either directly or indirectly with a system described by just such an equation.

While Zadeh and Desoer (1) provide a rigorous foundation for the problem, the level of their treatment precludes a great deal of usefulness at the engineering level.

A reference that is closer to this level is Tou (2). Here the third chapter is closely related to the problem. Many of the techniques involving state variables that have evolved recently are summarized in this chapter.

A good treatment of the applications of matrix theory is provided in a series of articles by Frame (3) in the I.E.E.E. Spectrum. These articles provide a review of matrix theory and treat the solution of systems of differential equations.

The three references cited constitute a compact summary of most of the work that has been done in the general area of state-variable analysis of systems. In each of the two books, at the end of the chapter mentioned, there are more detailed reference lists. Similarly, there is a bibliography after each of the five articles by Frame (3).

The problem stated in Section 1.1 is examined in this thesis in greater detail than in any of the references.

1.3 Outline of the Method of Solution. The system defined by Equation 1.1.3 is observed to have $n^2 + 2n$ constants which distinguish it from another n-th order system. The n^2 constants are in the $\underline{A}$ matrix, and there are n constants in both the $\underline{B}$ and $\underline{T}$ matrices.

It is possible to draw an analog computer diagram for a system that is described by Equation 1.1.3. The diagram would be quite complicated because there would be n integrators and $n^2 + 2n$ amplifiers. While this would represent a physical system described by Equation 1.1.3, the diagram would be undesirable because of the $n^2 + 2n$ arbitrary constants.

Therefore, it is desired that something be done to reduce this number of arbitrary constants, and still retain a system that is equivalent to the original system in the sense that the same input to the original and the modified systems yields the same output.

It is shown in Chapter II that a convenient intermediate step in this reduction process is to find G (s), the Laplace transfer function of the system, which relates the transform of the input and output. Two methods of accomplishing this will be provided.

The first method is a fairly standard one and involves the inversion of the characteristic matrix of $\underline{A}$. This procedure is straightforward but rather lengthy if done by hand.

For this reason the author has developed a secondary method for finding the transfer function. The secondary method consists of two steps. The first step in this procedure is to make a change of variables

$$\underline{X}(t) = \underline{P}\,\underline{Y}(t) \qquad (1.3.1)$$

where

$\underline{P}$, n by n, is a nonsingular real constant matrix, and

Y, n by 1, is the new vector state variable.

After this change of variables has been made, the state equations have the matrix $\underline{P}^{-1}\,\underline{A}\,\underline{P}$ in place of the old $\underline{A}$ matrix. The matrix $\underline{P}$ in Equation 1.3.1 is chosen such that $\underline{P}^{-1}\,\underline{A}\,\underline{P}$ is the transpose of the rational canonical form of $\underline{A}$.

This canonical form of a matrix is not as well known as the Jordan canonical form, but has some advantages over the Jordan form in this particular application that is discussed at length in Chapter II.

One source of information about the rational canonical form is Browne (4). The fifteenth chapter contains a brief discussion defining the rational canonical form of a matrix. The seventeenth chapter contains a more detailed discussion of the rational canonical form and, in particular, the procedure whereby the matrix $\underline{P}$ is determined which will reduce $\underline{A}$ to the transpose of the rational canonical form.

Other sources of information about the rational canonical form are Browne (5) and the fifth part of Frame (3).

Once the matrix $\underline{P}$ has been chosen, the transformed state equations are in a more useful form. The number of arbitrary constants in the state equations have been reduced from $n^2 + 2n$ to $3n$ by this initial transformation.

The second step in the secondary method is now used. This step consists of using an algorithm to produce the transfer function. The algorithm was developed by the author and uses the $3n$ constants which are in the transformed state equations.

The algorithm is provided in order to cover the general case where the order of the system is n. Appendix A contains tables which permit the transfer function to be obtained directly by substitution of the $3n$ constants for the specific cases where $n = 1, 2, 3, 4, 5,$ or 6.

At this point G (s), the Laplace transfer function of the system, has been determined either by the general method or the secondary method. The next major step in the procedure is to write the state equations in the final form directly by inspection of the Laplace transfer function. The author proves a theorem stating that this is always possible. This final form of the state equations is called the F canonical form of the state equations and contains only 2n arbitrary constants.

The balance of the thesis discusses operations with the canonical state equations and examples illustrating the usefulness in having the state equations in the F canonical form. The examples illustrate common ideas in system theory such as cascade combinations, feedback, compensation, error, and stability.

CHAPTER II

REDUCTION OF THE STATE EQUATIONS TO CANONICAL FORM

2.1 General Method of Obtaining the Transfer Function. As mentioned in Chapter I, a convenient intermediate step in the problem solution is to find G (s), the Laplace transfer function of the system. Section 2.1 provides a general method for finding G (s).

Once again the state equtions are assumed to be of the form

$$\frac{d}{dt} \underline{X}(t) = \underline{A}\,\underline{X}(t) + \underline{B}\, r(t), \quad \underline{X}(0) = \underline{X}_o \qquad (2.1.1)$$

$$c(t) = \underline{T}\,\underline{X}(t)$$

where the various quantities are defined in Equation 1.1.3.

At this point the Laplace transform will be applied to the quantities in Equation 2.1.1, yielding

$$s\,\underline{X}(s) - \underline{X}(o) = \underline{A}\,\underline{X}(s) + \underline{B}\,R(s)$$

$$C(s) = \underline{T}\,\underline{X}(s) \qquad (2.1.2)$$

where

s is the Laplace variable,

$\underline{X}(s)$ is the Laplace transform of $\underline{X}(t)$,

$X\ (o)$ is the value of $\underline{X}\ (t)$ at $t = 0$,

R (s) is the Laplace transform of r (t), and

C (s) is the Laplace transform of c (t).

Following common practice, it is assumed that

$$\underline{X}\ (o) = \underline{X}_o = \underline{0} \tag{2.1.3}$$

or that all initial conditions are zero.

This leaves the equations in the form

$$\begin{aligned} s\ \underline{X}\ (s) &= \underline{A}\ \underline{X}\ (s) + \underline{B}\ R\ (s) \\ C\ (s) &= \underline{T}\ \underline{X}\ (s) \end{aligned} \tag{2.1.4}$$

At this point the upper equation can be solved for $\underline{X}$ (s) and substituted into the lower equation, yielding

$$G\ (s) = \frac{C\ (s)}{R\ (s)} = \underline{T}\ [s\ \underline{U} - \underline{A}]^{-1}\ \underline{B} \tag{2.1.5}$$

for the transfer function of the system. As mentioned in Section 1.3, this procedure is a standard method for finding G (s).

From elementary matrix theory, it is known that the inverse of a matrix is the product of the reciprocal of the determinant of the matrix and the adjoint of the matrix. Similarly, the determinant of $[s\ \underline{U} - \underline{A}]$ is an n-th degree polynomial in s and is called the characteristic function of A, within a multiplicative factor of -1. This polynomial in s will be called A (s)

$$\det\ [s\ \underline{U} - \underline{A}] = A\ (s) \tag{2.1.6}$$

Next, it is observed that the adjoint of [s $\underline{U}$ - $\underline{A}$] is an n by n matrix whose elements are polynomials of degree at most n - 1. This is true since the elements of adj. [s $\underline{U}$ - $\underline{A}$] are determinants of dimension n - 1 of submatrices of [s $\underline{U}$ - $\underline{A}$].

Now Equation 2.1.5 may be expressed as

$$G\,(s) = \frac{C\,(s)}{R\,(s)} = \frac{1}{A\,(s)}\,\underline{T}\ \text{adj.}\ [s\ \underline{U} - \underline{A}]\ \underline{B} \qquad (2.1.7)$$

Recalling that $\underline{T}$ is a 1 by n matrix, adj. [s $\underline{U}$ - $\underline{A}$] is an n by n matrix and $\underline{B}$ is an n by 1 matrix, it is observed that

$$\underline{T}\ \text{adj.}\ [s\ \underline{U} - \underline{A}]\ \underline{B} = M\,(s) \qquad (2.1.8)$$

is a 1 by 1 matrix, or simply a scalar. This scalar is a polynomial in s of degree at most n - 1. This polynomial will be called M (s).

Finally, the Laplace transfer function of the system has been derived

$$G\,(s) = \frac{M\,(s)}{A\,(s)} \qquad (2.1.9)$$

It should be mentioned that the denominator polynomial A (s) depends only upon the $\underline{A}$ matrix, while the numerator polynomial M (s) depends upon the $\underline{A}$, $\underline{B}$, and $\underline{T}$ matrices.

In conventional control system terminology this means that the poles of the system depend only upon the $\underline{A}$ matrix, while the zeros depend upon the $\underline{A}$, $\underline{B}$, and $\underline{T}$ matrices.

At this point in the development it should be mentioned that the determination of G (s) for the system allows the designer to use all the knowledge of conventional control theory that has G (s) as a basis. This includes a large amount of information such as root locus, Nyquist plots, Bode plots, etc.

2.2 Secondary Method of Obtaining the Transfer Function. The method of obtaining G (s) described in Section 2.1 is quite general in that it applies to any system described by equations similar to Equation 2.1.1. While the method is general, it is quite laborious if done by hand, since it involves taking the inverse of an n by n matrix which is a function of s. It would be desirable if an alternative method were available which would not have this drawback.

For this reason a secondary method is provided. Much of the repetitious type of work need be performed only once and the results tabulated. The tables may then be used to assist in determining the transfer function. An algorithm is provided for the general case.

The first step in this procedure is to make a matrix change of variables

$$\underline{X}(t) = \underline{P}\,\underline{Y}(t) \tag{2.2.1}$$

where

$\underline{P}$, n by n, is a nonsingular real constant matrix, and

$\underline{Y}(t)$, n by 1, is the new vector state variable.

Equation 2.2.1 is substituted into

$$\frac{d}{dt}\underline{X}(t) = \underline{A}\,\underline{X}(t) + \underline{B}\,r(t), \quad \underline{X}(o) = \underline{X}_o$$
$$c(t) = \underline{T}\,\underline{X}(t) \tag{2.2.2}$$

yielding

$$\underline{P} \frac{d}{dt} \underline{Y}(t) = \underline{A}\,\underline{P}\,\underline{Y}(t) + \underline{B}\, r(t)$$

$$c(t) = \underline{T}\,\underline{P}\,\underline{Y}(t), \quad \underline{X}(o) = \underline{P}\,\underline{Y}(o) \tag{2.2.3}$$

Now, since $\underline{P}$ is nonsingular, Equation 2.2.3 is multiplied on the left by the inverse of $\underline{P}$

$$\frac{d}{dt} \underline{Y}(t) = \underline{P}^{-1}\,\underline{A}\,\underline{P}\,\underline{Y}(t) + \underline{P}^{-1}\,\underline{B}\, r(t)$$

$$c(t) = \underline{T}\,\underline{P}\,\underline{Y}(t), \quad \underline{Y}(o) = \underline{P}^{-1}\,\underline{X}(o) \tag{2.2.4}$$

At this point $\underline{P}$ should be selected such that the matrix $\underline{P}^{-1}\,\underline{A}\,\underline{P}$ has a particular form. The form that is chosen is $\underline{F}$, the transpose of the rational canonical form of $\underline{A}$.

Hence in Equation 2.2.4, the additional definitions are made that

$\underline{P}^{-1}\,\underline{A}\,\underline{P} = \underline{F}$, an n by n real constant matrix which is the transpose of the rational canonical form of $\underline{A}$, (2.2.5)

$\underline{P}^{-1}\,\underline{B} = \underline{E}$, an n by 1 real constant column matrix, and (2.2.6)

$\underline{T}\,\underline{P} = \underline{D}$, a 1 by n real constant row matrix. (2.2.7)

These defining equations may be substituted into Equation 2.2.4, yielding

$$\frac{d}{dt} \underline{Y}(t) = \underline{F}\,\underline{Y}(t) + \underline{E}\, r(t)$$

$$c\,(t) = \underline{D}\,\underline{Y}\,(t)\ ,\ \underline{Y}\,(o) = \underline{P}^{-1}\,\underline{X}\,(o) \qquad (2.2.8)$$

This is the form of the state equations after completion of the initial change of variables. The state equations are now expressed in terms of the new state vector $\underline{Y}$ (t), where each element of $\underline{Y}$ (t) is a linear combination of the elements of the original state vector $\underline{X}$ (t), according to

$$Y\,(t) = \underline{P}^{-1}\,\underline{X}\,(t) \qquad (2.2.9)$$

The matrix $\underline{P}$ is chosen to reduce $\underline{A}$ to the transpose of the rational canonical form according to Equation 2.2.5. This canonical form was selected rather than the more familiar Jordan canonical form for a number of reasons:

(1) If the matrix $\underline{A}$ has complex eigenvalues, then the Jordan canonical form will have complex elements. This is a distinct disadvantage which does not arise in the case of $\underline{F}$. The elements of $\underline{F}$ are +1's, 0's, and the negative of the coefficients in the characteristic function of $\underline{A}$. All these elements are real numbers.

(2) In some cases where $\underline{A}$ has repeated eigenvalues, the Jordan canonical form is not a diagonal matrix. This results in the loss of much of the advantage inherent in the Jordan form. Consequently, a procedure is developed which does not demand a diagonal matrix and, therefore, will not be adversely affected when applied to a matrix which is not similar to a diagonal matrix.

(3) It is easier to find the matrix $\underline{P}$ which reduces $\underline{A}$ to the transpose of the rational canonical than it is to find the matrix $\underline{P}$ that reduces $\underline{A}$ to the Jordan canonical form.

The rational canonical form $\underline{R}$ of a matrix is a diagonal block matrix of the form

$$\underline{R} = \begin{bmatrix} \underline{R}_1 & \underline{0} & \cdot & \cdot & \cdot & \underline{0} \\ \underline{0} & \underline{R}_2 & \cdot & \cdot & \cdot & \underline{0} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \underline{0} & \underline{0} & \cdot & \cdot & \cdot & \underline{R}_k \end{bmatrix} \qquad (2.2.10)$$

where each of the square submatrices $\underline{R}_i$ is of the form

$$\underline{R}_i = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ -a_m & -a_{m-1} & -a_{m-2} & \cdot & \cdot & \cdot & -a_2 & -a_1 \end{bmatrix} \qquad (2.2.11)$$

where there are +1's on the first superdiagonal, and the a_i's are the coefficients of the characteristic polynomial $f_i\ (\lambda)$ of $\underline{R}_i$

$$f_i\ (\lambda) = \lambda^m + a_1\ \lambda^{m-1} + \ldots + a_{m-1}\ \lambda + a_m \qquad (2.2.12)$$

This topic is discussed more fully on page 170 in Browne (4).

In the case of distinct eigenvalues of $\underline{A}$, the matrix $\underline{R}$ in Equation 2.2.10 only has one such block matrix of the form in Equation 2.2.11. In this case

$$\underline{R} = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdot & \cdot & \cdot & -a_2 & -a_1 \end{bmatrix} \qquad (2.2.13)$$

and the characteristic function f (λ) of $\underline{R}$ is

$$f(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1} \lambda + a_n \qquad (2.2.14)$$

Thus $\underline{F}$ may be found by simply transposing $\underline{R}$

$$\underline{F} = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -a_n \\ 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -a_{n-1} \\ 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 & -a_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 & -a_2 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & -a_1 \end{bmatrix} \qquad (2.2.15)$$

where there are +1's on the first subdiagonal and the negative of the coefficients in the characteristic function in the last column. The characteristic function is the same for $\underline{F}$ as for $\underline{R}$.

In case that $\underline{A}$ has repeated eigenvalues, it is possible that $\underline{F}$ would then be a diagonal block matrix such as Equation 2.2.10 where each of the $\underline{R}_i$'s would be in the form of Equation 2.2.15.

The procedure whereby $\underline{P}$ is selected such that $\underline{A}$ is reduced to $\underline{F}$ is discussed on page 210 of Browne (4).

For the following discussion $\underset{\sim}{A}$ is assumed similar to an $\underset{\sim}{F}$ matrix having only one block such as that in Equation 2.2.15. The case of an $\underset{\sim}{F}$ matrix having more than one block will be discussed in the next section.

At this point it is assumed that the change of variables has been made, leaving state equations in the form of Equation 2.2.8, which will be rewritten in a more illustrative form, specifically

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -a_n \\ 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & -a_{n-1} \\ 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 & -a_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 & -a_2 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ \cdot \\ e_{n-1} \\ e_n \end{bmatrix} r(t) \tag{2.2.16}$$

$$c(t) = [d_1 \quad d_2 \quad d_3 \quad \cdot \quad \cdot \quad \cdot \quad d_{n-1} \quad d_n] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ y_{n-1} \\ y_n \end{bmatrix}$$

It is observed that there are 3n arbitrary constants that distinguish this system from another system of order n. There are n of the a_i's, n of the e_i's, and finally n of the d_i's for the total of 3n. This demonstrates that the original $n^2 + 2n$ arbitrary constants of the original system described by Equation 2.1.1 have been reduced to 3n by this change of variables.

The procedure discussed in Section 2.1 is now applied to the transformed state equations given in Equation 2.2.16 to yield an algorithm which gives the transfer function.

Taking the Laplace transform of Equation 2.2.16 and solving for G (s) in the prescribed manner

$$G(s) = \frac{C(s)}{R(s)} = \underline{D}\,[s\,\underline{U} - \underline{F}]^{-1}\,\underline{E} \qquad (2.2.17)$$

it is seen that the transfer function can be obtained by inverting the characteristic matrix of $\underline{F}$.

From the elementary matrix theory, it is known that the inverse of a matrix can be found by dividing the adjoint of the matrix by the determinant of the matrix. This property is used in the calculation of $[s\,\underline{U} - \underline{F}]^{-1}$.

$$[s\,\underline{U} - \underline{F}]^{-1} = \frac{1}{\text{det. }[s\,\underline{U} - \underline{F}]}\ \text{adj. }[s\,\underline{U} - \underline{F}] \qquad (2.2.18)$$

Because $\underline{F}$ is the transpose of the rational canonical form of $\underline{A}$, the determinant may be written by inspection as

$$\text{det. }[s\,\underline{U} - \underline{F}] = s^n + a_1\,s^{n-1} + \ldots + a_n = A(s) \qquad (2.2.19)$$

This leaves the calculation of the adjoint matrix which is performed with the aid of an algorithm developed by the author. This algorithm is stated in the form of a theorem.

Theorem 2.2.1. If $\underline{F}$ is an n by n matrix of the form given in

Equation 2.2.15 and if b_{jk} is the element in the j-th row and k-th column of adj. [s $\underline{U}$ - $\underline{F}$], then b_{jk} is given by

$$b_{jk} = \begin{cases} \sum_{i=0}^{n-j} a_i\, s^{n-i-j+k-1} & \text{for } j \geq k \\ -\sum_{i=n-j+1}^{n} a_i\, s^{n-i-j+k-1} & \text{for } j < k \end{cases} \qquad (2.2.20)$$

where a_o is defined to be unity.

Proof of Theorem 2.2.1. The proof of this theorem consists of showing that if the matrix defined by b_{jk} is multiplied on the left by [s $\underline{U}$ - $\underline{F}$], then the product will be det. [s $\underline{U}$ - $\underline{F}$] $\underline{U}$.

The first row of [s $\underline{U}$ - $\underline{F}$] has the form [s 0 0 ... 0 a_n]. The product of this with the first column of [b_{jk}] yields $sb_{11} + a_n b_{n1}$ which reduces to

$$\sum_{i=0}^{n} a_i\, s^{n-i} = A\,(s) ,$$

and is the 1, 1 entry in the product matrix. The product of the first row with the q-th column of [b_{jk}], where $1 < q \leq n$, yields $sb_{1q} + a_n b_{nq}$ which reduces to zero.

The p-th row of [s $\underline{U}$ - $\underline{F}$], where $1 < p < n$, has the form [0 0 ... 0 -1 s 0 ... 0 a_{n-p+1}] where the -1 is in the p-1st column, and the s is in the p-th column. This row is multiplied by the p-qth column of [b_{jk}], where $1 \leq q \leq p - 1$, to yield the entries below the diagonal in the product matrix. This product yields

$$-b_{p-1,\, p-q} + sb_{p,\, p-q} + a_{n-p+1}\, b_{n,\, p-q}$$

which reduces to zero for all below diagonal terms.

The p-th row of [s $\underline{U}$ - $\underline{F}$] is now multiplied by the p-th column of $[b_{jk}]$, where $1 < p < n$, to obtain all the diagonal entries in the product matrix except the 1, 1 and the n, n entries. This product yields

$$-b_{p-1,\, p} + sb_{p,\, p} + a_{n-p+1}\, b_{n,\, p}$$

which reduces to

$$\sum_{i=0}^{n} a_i\, s^{n-i} = A\,(s)$$

for all these diagonal entries.

The p-th row of [s $\underline{U}$ - $\underline{F}$] is now multiplied by the p + qth column of $[b_{jk}]$, where $1 \leq q \leq n - p$, to yield the entries above the diagonal in the product matrix. This product yields

$$-b_{p-1,\, p+q} + sb_{p,\, p+q} + a_{n-p+1}\, b_{n,\, p+q}$$

which reduces to zero for all above diagonal terms.

Next, the n-th row of [s $\underline{U}$ - $\underline{F}$], which has the form [0 0 ... 0 -1 s + a_1], is multiplied by the q-th column of $[b_{jk}]$, where $1 \leq q < n$, to obtain all the entries in the last row except the n, n entry. This product gives $-b_{n-1,\, q} + (s + a_1)\, b_{n,\, q}$ which reduces to zero for all these entries.

Finally, the n-th row of [s $\underline{U}$ - $\underline{F}$] is multiplied by the n-th column of $[b_{jk}]$ to obtain the n, n entry in the product matrix. This product yields $-b_{n-1,\, n} + (s + a_1)\, b_{n,\, n}$ which reduces to

$$\sum_{i=0}^{n} a_i s^{n-i} = A(s) .$$

This completes the proof that all the diagonal entries in the product matrix are A (s) = det. [s $\underline{U}$ - $\underline{F}$], and that all the off-diagonal entries are zero. Hence

$$[s\,\underline{U} - \underline{F}]\,[b_{jk}] = \text{det.}\,[s\,\underline{U} - \underline{F}]\,\underline{U} \qquad (2.2.21)$$

and $[b_{jk}]$ does in fact define the adjoint matrix of [s $\underline{U}$ - $\underline{F}$].

Theorem 2.2.1 provides an algorithm for the calculation of adj. [s $\underline{U}$ - $\underline{F}$] when $\underline{F}$ is of the form given in Equation 2.2.15. The adjoint matrix is observed to contain polynomials in s as elements. Furthermore, the highest degree of any of these polynomials is n - 1. This allows adj. [s $\underline{U}$ - $\underline{F}$] to be expressed as

$$\text{adj.}\,[s\,\underline{U} - \underline{F}] = \underline{M}_1\, s^{n-1} + \underline{M}_2\, s^{n-2} + \ldots + \underline{M}_{n-1}\, s + \underline{M}_n \qquad (2.2.22)$$

where the $\underline{M}_i$'s are constant matrices. Recalling that a_o is defined to be unity, Equation 2.2.20 can be used to obtain an expression for the general $\underline{M}_p$ in Equation 2.2.22 for an n by n adjoint matrix. Thus, in Equation 2.2.22 $\underline{M}_p$ is the partitioned matrix

$$\underline{M}_p = [\underline{R}_p\ \underline{S}_p] \qquad (2.2.23)$$

where

$$\underline{R}_p = \begin{bmatrix} a_{p-1} & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ a_{p-2} & a_{p-1} & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ a_{p-3} & a_{p-2} & a_{p-1} & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{p-1} \\ a_1 & a_2 & a_3 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_1 & a_2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \qquad (2.2.24)$$

which is an n by n - p + 1 matrix, and

$$\underline{S}_p = \begin{bmatrix} -a_n & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ -a_{n-1} & -a_n & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ -a_{n-2} & -a_{n-1} & -a_n & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -a_p & -a_{p+1} & -a_{p+2} & \cdot & \cdot & \cdot & \cdot & -a_n \\ 0 & -a_p & -a_{p+1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & -a_p & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -a_{p+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -a_p \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix} \qquad (2.2.25)$$

which is an n by p - 1 matrix.

Combining Equations 2.2.17 and 2.2.22, it can be seen that

$$G\ (s) = \frac{1}{A\ (s)} \sum_{p=1}^{n} \underline{D}\ \underline{M}_p\ \underline{E}\ s^{n-p} \tag{2.2.26}$$

where $\underline{M}_p$ is defined in Equations 2.2.23, 2.2.24, and 2.2.25.

Making the definitions that

$$m_p = \underline{D}\ \underline{M}_p\ \underline{E} \tag{2.2.27}$$

and

$$M\ (s) = \sum_{p=1}^{n} m_p\ s^{n-p} \tag{2.2.28}$$

then the transfer function is

$$G\ (s) = \frac{M\ (s)}{A\ (s)} \tag{2.2.29}$$

Theorem 2.2.1 can be used in two ways in determining the transfer function. The first way is the straightforward evaluation of the adjoint matrix. The adjoint matrix is then multiplied on the left by $\underline{D}$ and on the right by $\underline{E}$. This gives the numerator polynomial of G (s). The denominator polynomial is given in Equation 2.2.19.

The second way of determining the transfer function involves finding the coefficients in the numerator polynomial individually. This is accomplished by the multiplication of constant matrices according to Equation 2.2.27. For this reason the second method would be better from the standpoint of computer solution.

These matrix products have been calculated in general terms and are listed in Appendix A for systems up through sixth order. This allows the transfer function to be obtained directly from the transformed state equations by substitution of the 3n numbers into the expressions in Appendix A.

2.3 Extension of the Secondary Method. In Section 2.2 the case was discussed where the $\underline{F}$ matrix contained only one block matrix of the form given in Equation 2.2.15. Not all matrices are similar to this type of matrix, however. The $\underline{F}$ matrix for some $\underline{A}$ matrices is of the form given in Equation 2.2.10, where each of the $\underline{R}_i$'s are of the form discussed. This requires a modification in the procedure. A procedure for finding the transfer function where $\underline{F}$ includes two such blocks is now given. The procedure can readily be extended to cover the general case.

Assume that the initial transformation of variables has been made, and that the state equations result in the form

$$\frac{d}{dt} \underline{Y} (t) = \underline{F}\, \underline{Y} (t) + \underline{E}\, r (t)$$

$$c (t) = \underline{D}\, \underline{Y} (t) \qquad (2.3.1)$$

Also, assume that the transformation of variables resulted in an $\underline{F}$ matrix consisting of two block matrices, each of the form given in Equation 2.2.15. Specifically

$$\underline{F} = \begin{bmatrix} \underline{F}_1 & \underline{0} \\ \underline{0} & \underline{F}_2 \end{bmatrix} \qquad (2.3.2)$$

where

$\underline{F}_1$ is a k by k matrix of the prescribed form,

$\underline{F}_2$ is an m by m matrix of the same form, and

k + m = n, the order of the system.

In this case, partition $\underline{Y}$ (t), $\underline{E}$, and $\underline{D}$ such that Equation 2.3.1 may be written in partitioned form as

$$\frac{d}{dt} \begin{bmatrix} \underline{Y}_1 \\ \underline{Y}_2 \end{bmatrix} = \begin{bmatrix} \underline{F}_1 & \underline{0} \\ \underline{0} & \underline{F}_2 \end{bmatrix} \begin{bmatrix} \underline{Y}_1 \\ \underline{Y}_2 \end{bmatrix} + \begin{bmatrix} \underline{E}_1 \\ \underline{E}_2 \end{bmatrix} r(t)$$

$$c(t) = [\underline{D}_1 \quad \underline{D}_2] \begin{bmatrix} \underline{Y}_1 \\ \underline{Y}_2 \end{bmatrix} \tag{2.3.3}$$

This may be accomplished by letting, $\underline{Y}_1$, $\underline{E}_1$, and $\underline{D}_1$ be the first k elements in $\underline{Y}$, $\underline{E}$, and $\underline{D}$, respectively. Then $\underline{Y}_2$, $\underline{E}_2$, and $\underline{D}_2$ will be the last m elements in $\underline{Y}$, $\underline{E}$, and $\underline{D}$, respectively.

Now observe that Equation 2.3.3 may be separated into two independent state equations

$$\frac{d}{dt} \underline{Y}_1 (t) = \underline{F}_1 \underline{Y}_1 (t) + \underline{E}_1 r(t)$$

$$c_1 (t) = \underline{D}_1 \underline{Y}_1 (t) \tag{2.3.4}$$

and

$$\frac{d}{dt} \underline{Y}_2 (t) = \underline{F}_2 \underline{Y}_2 (t) + \underline{E}_2 r(t)$$

$$c_2(t) = \underline{D}_2 \, \underline{Y}_2 (t) \qquad (2.3.5)$$

where

$$c(t) = c_1(t) + c_2(t) \qquad (2.3.6)$$

Since $\underline{F}_1$ is in the form of Equation 2.2.15, the procedure previously developed may be applied to Equation 2.3.4 to find

$$G_1(s) = \frac{C_1(s)}{R(s)} \qquad (2.3.7)$$

where $C_1(s)$ is the Laplace transform of $c_1(t)$.

Similarly, it is possible to find

$$G_2(s) = \frac{C_2(s)}{R(s)} \qquad (2.3.8)$$

where $C_2(s)$ is the Laplace transform of $c_2(t)$.

Because of Equation 2.3.6, and the fact that the Laplace transformation is a linear operator

$$C(s) = C_1(s) + C_2(s) \qquad (2.3.9)$$

After division of each side of Equation 2.3.9 by R (s), it is apparent that

$$G(s) = \frac{C(s)}{R(s)} = G_1(s) + G_2(s) \qquad (2.3.10)$$

Thus it has been shown that whenever $\underline{F}$ consists of two block matrices of the form given in Equation 2.2.15, a suitable partition of the matrices involved will lead to the solution of the problem by solving two smaller problems. Specifically, two transfer functions, G_1 (s) and G_2 (s), are derived. The algebraic sum of these two transfer functions yields the desired transfer function.

Because $\underline{F}$, in general, has the property that the block matrices are square and are on the diagonal, and the off-diagonal block matrices are all zero, it is apparent that a similar partitioning will always be possible. Hence, if there are k of the block matrices on the diagonal in $\underline{F}$, then a proper partition of the matrices in the transformed state equation yields k independent systems of state equations. In each system of equations the $\underline{F}_i$ matrix is of the form necessary for solution using the methods described in Section 2.2. Thus, a total of k of the G_i (s) are determined independently. Then the transfer function for the original system is

$$G(s) = \sum_{i=1}^{k} G_i(s) \qquad (2.3.11)$$

This observation extends the range of application of the secondary method to systems of any finite order.

A property of the rational canonical form should be mentioned at this point. If the block matrices in $\underline{F}$ are ordered properly and if the i-th diagonal block is called $\underline{F}_i$ and if f_i (λ) is the characteristic function of $\underline{F}_i$, then f_i (λ) divides f_{i+1} (λ). This property is discussed at length on page 172 in Browne (4).

Because the characteristic function of $\underline{F}_i$ is the same as the denominator polynomial in G_i (s), this means that when the sum is taken of all the G_i (s), the lowest common denominator is simply the denominator of the G_i (s) which has the highest degree denominator.

This implies that whenever $\underline{F}$ has more than one block matrix of the form given in Equation 2.2.15, then the degree of the denominator of G (s) will be less than n. Specifically, the degree of the denominator polynomial of G (s) will not exceed the degree of the highest order denominator of all the G_i (s).

Thus, a state equation of order n can very well lead to a transfer function of order less than n. The next section will develop a canonical form for the state equations, assuming a known transfer function. This will make possible the reduction of a state equation of order n to a canonical state equation with order less than n in some cases.

2.4 $\underline{F}$ Canonical Form of the State Equations. Sections 2.1 and 2.2 have presented two methods whereby G (s), the Laplace transfer function of the system G, can be determined from the original state equation given by Equation 1.1.3. Section 2.4 gives a procedure whereby a canonical form for the state equations is derived from the transfer function.

It should be mentioned that the state equations that describe a system are not unique. This can easily be seen by examining Equation 2.2.2 and Equation 2.2.4. If $\underline{P}$ is not the identity matrix, then the equations are not the same, even though the equations describe the same system.

For this reason a canonical form for the state equation is assumed by the author. Then the author shows that this state equation corresponds to the given Laplace transfer function which is a unique representation of the system. Zadeh and Desoer (1) discuss the uniqueness of the transfer function on page 113.

The Laplace transfer function is assumed to be

$$G(s) = \frac{M(s)}{A(s)} = \frac{m_1 s^{n-1} + m_2 s^{n-2} + \dots + m_{n-1} s + m_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n} \qquad (2.4.1)$$

Theorem 2.4.1. The following state equation

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \cdot \\ z_{n-2} \\ z_{n-1} \\ z_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & -a_n \\ 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & -a_{n-1} \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & -a_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 & 0 & -a_3 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 & -a_2 \\ 0 & 0 & 0 & \cdot & & \cdot & 0 & 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \cdot \\ z_{n-2} \\ z_{n-1} \\ z_n \end{bmatrix} + \begin{bmatrix} m_n \\ m_{n-1} \\ m_{n-2} \\ \cdot \\ m_3 \\ m_2 \\ m_1 \end{bmatrix} r(t) \qquad (2.4.2)$$

$$c(t) = [0 \;\; 0 \;\; 0 \;\; \cdot \;\; \cdot \;\; \cdot \;\; 0 \;\; 0 \;\; 1] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \cdot \\ z_{n-2} \\ z_{n-1} \\ z_n \end{bmatrix} = z_n$$

has the Laplace transfer function given by Equation 2.4.1.

Proof of Theorem 2.4.1. Assuming $\underline{Z}(o) = \underline{0}$, take the Laplace transform of Equation 2.4.2, yielding

$$s\,\underline{Z}(s) = \underline{F}\,\underline{Z}(s) + \underline{M}\,R(s)$$

$$C(s) = [0 \;\; 0 \;\; . \;\; . \;\; . \;\; 0 \;\; 1]\,\underline{Z}(s) \qquad (2.4.3)$$

Using the same procedure as in Section 2.1, the transfer function is found to be

$$\frac{C(s)}{R(s)} = [0 \;\; 0 \;\; . \;\; . \;\; . \;\; 0 \;\; 1]\,[s\,\underline{U} - \underline{F}]^{-1}\,\underline{M} \qquad (2.4.4)$$

Because $\underline{F}$ is the transpose of a rational canonical form, det. $[s\,\underline{U} - \underline{F}]$ can be written by inspection as

$$\text{det. } [s\,\underline{U} - \underline{F}] = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = A(s) \qquad (2.4.5)$$

Consequently,

$$\frac{C(s)}{R(s)} = \frac{1}{A(s)}\,[0 \;\; 0 \;\; . \;\; . \;\; . \;\; 0 \;\; 1]\;\text{adj. } [s\,\underline{U} - \underline{F}]\,\underline{M}\,. \qquad (2.4.6)$$

Now $[0 \;\; 0 \;\; . \;\; . \;\; . \;\; 0 \;\; 1]$ adj. $[s\,\underline{U} - \underline{F}]$ is simply the last row in adj. $[s\,\underline{U} - \underline{F}]$.

At this point Theorem 2.2.1 is used to find the last row in adj. $[s\,\underline{U} - \underline{F}]$. The k-th element in the n-th row is

$$b_{nk} = a_o\, s^{k-1} = s^{k-1} \qquad (2.4.7)$$

so that

$$[0 \quad 0 \quad . \quad . \quad . \quad 0 \quad 1] \text{ adj. } [s \underline{U} - \underline{F}] = [1 \quad s \quad . \quad . \quad . \quad s^{n-2} \quad s^{n-1}] \tag{2.4.8}$$

This is now substituted into Equation 2.4.6

$$\frac{C(s)}{R(s)} = \frac{1}{A(s)} [1 \quad s \quad s^2 \quad . \quad . \quad . \quad s^{n-2} \quad s^{n-1}] \begin{bmatrix} m_n \\ m_{n-1} \\ m_{n-2} \\ . \\ m_3 \\ m_2 \\ m_1 \end{bmatrix} \tag{2.4.9}$$

yielding the final expression

$$G(s) = \frac{C(s)}{R(s)} = \frac{m_1 s^{n-1} + m_2 s^{n-2} + \ldots + m_{n-1} s + m_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} \tag{2.4.10}$$

which completes the proof of Theorem 2.4.1.

The significance of Theorem 2.4.1 is that it permits a canonical form for the state equation called the $\underline{F}$ canonical form to be written directly from inspection of the transfer function. M (s), the numerator of G (s), determines the entries in the $\underline{M}$ matrix, while A (s), the denominator of G (s), determines the entries in the $\underline{F}$ matrix. The output is simply the n-th state variable.

An analog computer diagram will now be given which is described by Equation 2.4.2 and consequently has the transfer function given in Equation 2.4.10. This diagram is Figure 2.4.1.

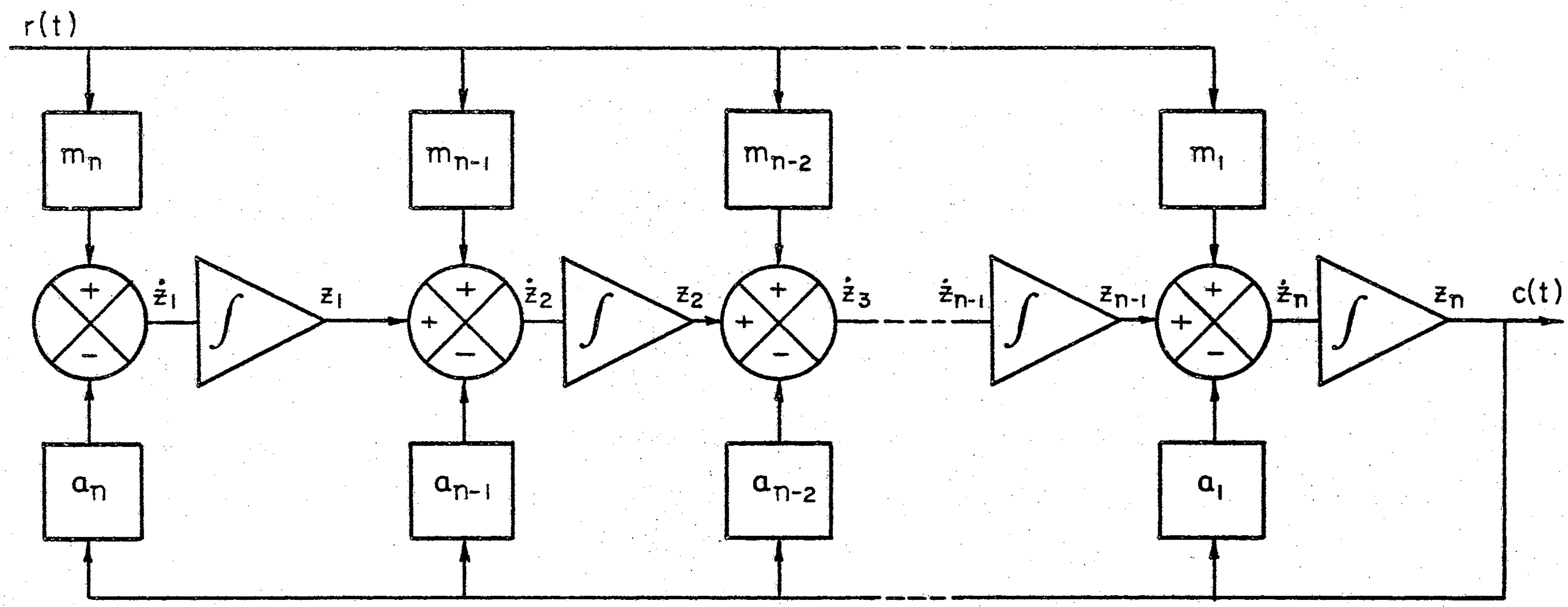

Figure 2.4.1. Analog Computer Diagram of Equation 2.4.2

The analog computer diagram in Figure 2.4.1 is in a convenient form for programming an analog computer. The a_i and m_i gains required can be obtained with two amplifiers and 2 (n - 1) potentiometers due to the common inputs.

Given a transfer function, the analog computer diagram in Figure 2.4.1 can then be formulated directly. This allows a great deal of interchangeability between transfer functions, canonical state equations, and analog computer diagrams. This property proves useful in the next two chapters.

CHAPTER III

OPERATIONS WITH THE $\underline{F}$ CANONICAL STATE EQUATIONS

3.1 Cascade Combination of Systems. A very common connection between two linear systems is the cascade combination of the two systems. This means that the output of one of the systems serves as the input for the other, as in Figure 3.1.1.

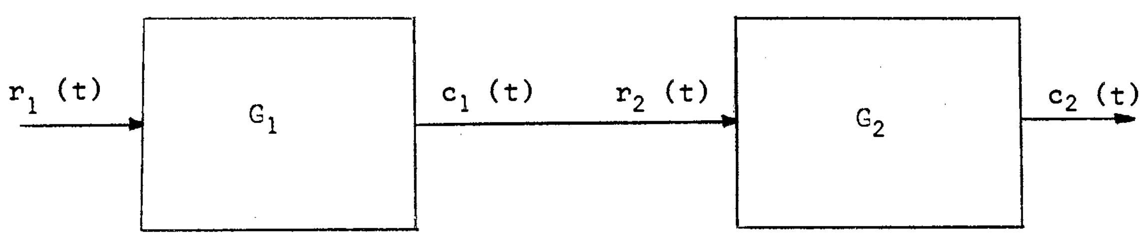

Figure 3.1.1. Cascade Combination of G_1 and G_2

Assume that the $\underline{F}$ canonical state equation for G_1 is known and is

$$\frac{d}{dt} \underline{Z}_1 (t) = \underline{F}_1 \underline{Z}_1 (t) + \underline{M}_1 r_1 (t)$$
$$c_1 (t) = [0 \;\; 0 \;\; . \;\; . \;\; . \;\; 0 \;\; 1] \underline{Z}_1 (t) \qquad (3.1.1)$$

Further, assume that the canonical state equation for G_2 is

$$\frac{d}{dt} \underline{Z}_2 (t) = \underline{F}_2 \underline{Z}_2 (t) + \underline{M}_2 r_2 (t)$$

(3.1.2)

$$c_2 (t) = [0 \quad 0 \quad . \quad . \quad . \quad 0 \quad 1] \underline{Z}_2 (t)$$

The problem is to obtain the state equation for this cascade combination of systems, where the input is r_1 (t) and the output is c_2 (t).

The first step is to write an equation by adjoining Equations 3.1.1 and 3.1.2, yielding the partitioned equation

$$\frac{d}{dt} \begin{bmatrix} \underline{Z}_1 \\ \underline{Z}_2 \end{bmatrix} = \begin{bmatrix} \underline{F}_1 & 0 \\ \underline{0} & \underline{F}_2 \end{bmatrix} \begin{bmatrix} \underline{Z}_1 \\ \underline{Z}_2 \end{bmatrix} + \begin{bmatrix} \underline{M}_1 \, r_1 (t) \\ \underline{M}_2 \, r_2 (t) \end{bmatrix}$$

(3.1.3)

$$c_2 (t) = [0 \quad 0 \quad . \quad . \quad . \quad 0 \quad 1] \begin{bmatrix} \underline{Z}_1 \\ \underline{Z}_2 \end{bmatrix}$$

Now, observe that

$$r_2 (t) = c_1 (t) = [0 \quad 0 \quad . \quad . \quad . \quad 0 \quad 1] \underline{Z}_1 (t) \qquad (3.1.4)$$

and consequently

$$\underline{M}_2 r_2 (t) = \underline{M}_2 [0 \quad 0 \quad . \quad . \quad . \quad 0 \quad 1] \underline{Z}_1 (t) \qquad (3.1.5)$$

Now, Equation 3.1.5 can be substituted into Equation 3.1.3, yielding the state equation

$$\frac{d}{dt} \begin{bmatrix} \underline{Z}_1 \\ \underline{Z}_2 \end{bmatrix} = \begin{bmatrix} \underline{F}_1 & \underline{0} \\ \underline{M}_2 [0 \; 0 \; . \; . \; . \; 0 \; 1] & \underline{F}_2 \end{bmatrix} \begin{bmatrix} \underline{Z}_1 \\ \underline{Z}_2 \end{bmatrix} + \begin{bmatrix} \underline{M}_1 \\ \underline{0} \end{bmatrix} r_1 (t)$$

$$c_2\ (t) = [0 \quad 0 \quad . \quad . \quad . \quad 0 \quad 1] \begin{bmatrix} \underline{Z}_1 \\ \underline{Z}_2 \end{bmatrix} \qquad (3.1.6)$$

This is a state equation with the desired input, r_1 (t), and the desired output, c_2 (t), but it is not in $\underline{F}$ canonical form. Therefore, if the $\underline{F}$ canonical form is desired, the procedure described in the first two chapters may be applied to Equation 3.1.6 to yield the $\underline{F}$ canonical state equation. Obviously, more than two systems in cascade may be handled by combining two at a time by the procedure given.

3.2 Parallel Combination of Systems. Another common connection between two linear systems is the parallel combination of the two systems. This means that the two systems share a common input, and the two outputs are summed to yield the over-all output. This is shown in Figure 3.2.1.

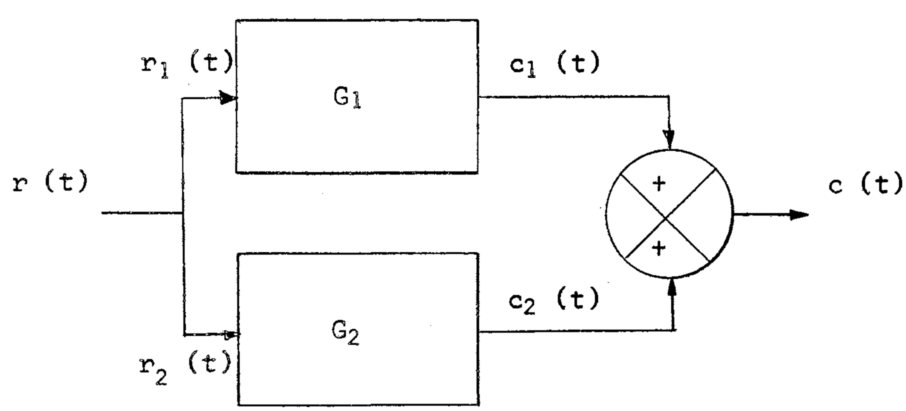

Figure 3.2.1. Parallel Combination of G_1 and G_2

The canonical state equations for G_1 and G_2 are once again assumed to be Equation 3.1.1 and Equation 3.1.2, respectively. The problem now is to obtain the state equation for this combination of systems where the input is r (t) and the output is c (t).

As before, the first step is to write an equation in partitioned form by adjoining the individual state equations

$$\frac{d}{dt}\begin{bmatrix}\underline{Z}_1\\ \underline{Z}_2\end{bmatrix} = \begin{bmatrix}\underline{F}_1 & \underline{0}\\ \underline{0} & \underline{F}_2\end{bmatrix}\begin{bmatrix}\underline{Z}_1\\ \underline{Z}_2\end{bmatrix} + \begin{bmatrix}\underline{M}_1\, r_1\,(t)\\ \underline{M}_2\, r_2\,(t)\end{bmatrix} \tag{3.2.1}$$

$$c\,(t) = [0\ \ 0\ \ .\ \ .\ \ .\ \ 0\ \ 1]\,\underline{Z}_1 + [0\ \ 0\ \ .\ \ .\ \ .\ \ 0\ \ 1]\,\underline{Z}_2$$

Now, observe that

$$r_1\,(t) = r_2\,(t) = r\,(t) \tag{3.2.2}$$

and consequently

$$\frac{d}{dt}\begin{bmatrix}\underline{Z}_1\\ \underline{Z}_2\end{bmatrix} = \begin{bmatrix}\underline{F}_1 & \underline{0}\\ \underline{0} & \underline{F}_2\end{bmatrix}\begin{bmatrix}\underline{Z}_1\\ \underline{Z}_2\end{bmatrix} + \begin{bmatrix}\underline{M}_1\\ \underline{M}_2\end{bmatrix} r\,(t) \tag{3.2.3}$$

$$c\,(t) = [0\ \ 0\ \ .\ \ .\ \ .\ \ 0\ \ 1\ \ 0\ \ 0\ \ .\ \ .\ \ .\ \ 0\ \ 1]\begin{bmatrix}\underline{Z}_1\\ \underline{Z}_2\end{bmatrix}$$

where the +1's are in positions such that c (t) is the sum of the last state variable in $\underline{Z}_1$ and the last state variable in $\underline{Z}_2$.

Equation 3.2.3 is now in the form that r (t) is the input and c (t) is the output as desired. At this point the procedure given in the first two chapters can be applied to Equation 3.2.3 to obtain the

$\underline{F}$ canonical form for the state equation if desired. More than two systems can be comined in parallel by combining two at a time in this fashion.

3.3 Introduction of Feedback into a System. When the input to a system is the sum of a forcing function r (t) and some function of c (t), the output of the system, then the over-all system is said to incorporate feedback. The case where this function of c (t) is simply -c (t) is now considered. Such a system is given in Figure 3.3.1. A simple amplifier whose output is K times the input is also included.

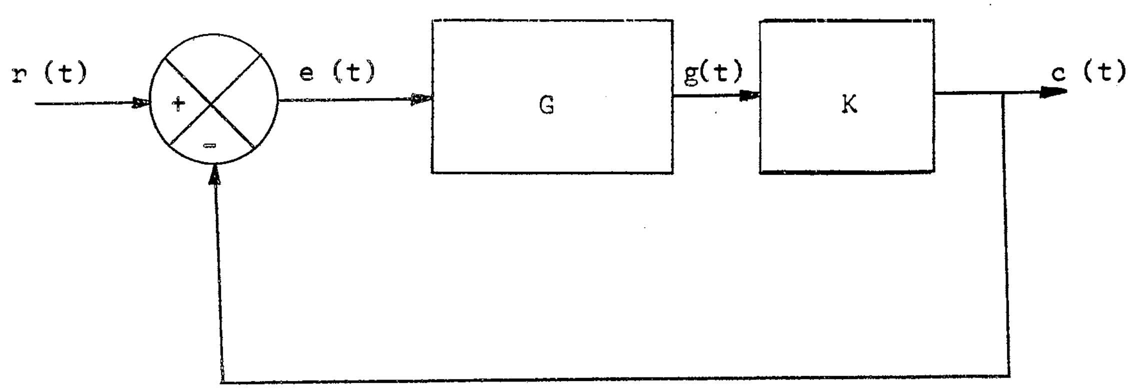

Figure 3.3.1. Feedback System

The $\underline{F}$ canonical state equation for G is assumed to be

$$\frac{d}{dt}\underline{Z} = \underline{F}\,\underline{Z} + \underline{M}\,e\,(t)$$

$$g\,(t) = [0 \quad 0 \quad . \quad . \quad . \quad 0 \quad 1]\,\underline{Z} \tag{3.3.1}$$

It is now desired that the state equation be found where the input is r (t) and the output is c (t).

Observing that

$$e\,(t) = r\,(t) - c\,(t) = r\,(t) - K\,g\,(t) \tag{3.3.2}$$

and thus

$$e\,(t) = r\,(t) - K\,[0 \quad 0 \quad . \quad . \quad . \quad 0 \quad 1]\,\underline{Z} \tag{3.3.3}$$

Equation 3.3.1 may be rewritten

$$\frac{d}{dt}\underline{Z} = \underline{F}\,\underline{Z} + \underline{M}\,r\,(t) - \underline{M}\,[0 \quad 0 \quad . \quad . \quad . \quad 0 \quad K]\,\underline{Z} \tag{3.3.4}$$

$$c\,(t) = K\,g\,(t)$$

and thus

$$\frac{d}{dt}\underline{Z} = [\underline{F} - \underline{M}\,[0 \quad 0 \quad . \quad . \quad . \quad 0 \quad K]]\,\underline{Z} + \underline{M}\,r\,(t)$$

$$c\,(t) = [0 \quad 0 \quad . \quad . \quad . \quad 0 \quad K]\,\underline{Z} \tag{3.3.5}$$

Equation 3.3.5 is seen to have r (t) as the input and c (t) has the output as specified. This equation is in a convenient form for determining the stability of the over-all system. Letting

$$\underline{A} = \underline{F} - \underline{M}\,[0 \quad 0 \quad . \quad . \quad . \quad 0 \quad K] \tag{3.3.6}$$

the characteristic function for the system is seen to be

$$f\,(\lambda) = \text{det.}\,[\underline{A} - \lambda\,\underline{U}] \tag{3.3.7}$$

In general, some of the coefficients of λ^i in f (λ) are functions of K. It is known that the stability of a system depends upon the real part of the eigenvalues. This point is discussed on page 375 in Zadeh and Desoer (1). Consequently, the value of K might be selected to make all the eigenvalues have negative real parts. This might be possible in some cases and impossible in others.

In cases where no real K would make the system stable, or where K would have to be unreasonably large or small, it is necessary to modify the system by more involved methods. This introduces the general topic of compensation which is discussed in Section 3.5.

3.4 Steady-State Error. Ordinarily, the purpose of a control system is to make some variable behave like some input variable as closely as possible. In Figure 3.4.1 the output c (t) of the system G is desired to follow the input r (t) as closely as possible. A measure of the accuracy of the control system is the error signal

$$e\,(t) = r\,(t) - c\,(t) \tag{3.4.1}$$

Of interest in conventional control techniques is the limiting value of e (t) for a system G when the input r (t) is a unit step function, a unit ramp function, etc.

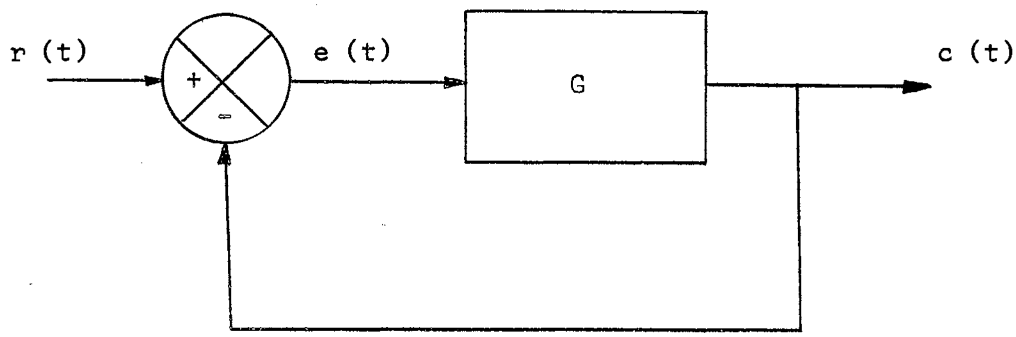

Figure 3.4.1. Feedback Control System

Truxal (7) gives a good discussion of steady-state error on page 81. For a system like that in Figure 3.4.1, Truxal explains how the steady-state error can be calculated from a knowledge of G (s) with the aid of the final-value theorem.

It is assumed that the transfer function G (s), or equivalently the $\underline{F}$ canonical form of the state equations, is known, and is

$$G(s) = \frac{m_1 s^{n-1} + m_2 s^{n-2} + \ldots + m_{n-1} s + m_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} \qquad (3.4.2)$$

where there are no common factors in the numerator and denominator. In conventional control terminology the type of a system is defined to be the order of the pole at the origin in G (s). Thus, if $a_n = 0$ and $a_{n-1} \neq 0$, then G is type 1 because s to the first power could be factored in the denominator. Similarly, if $a_n = a_{n-1} = 0$ and $a_{n-2} \neq 0$, then G is type 2 because s to the second power could be factored in the denominator. Thus, it is seen that the type of G is the number of consecutive zeros in the sequence $\{a_n, a_{n-1}, a_{n-2}, \ldots\}$.

Given the $\underline{F}$ canonical form of the state equations which is formed from the transfer function after any common factors in the numerator and denominator are canceled, and defining the type of G as in the preceding paragraph, the treatment of Truxal (7) can be extended to yield the steady-state error in terms of the a_i and m_i coefficients.

The results of this extension are summarized in the form of Figure 3.4.2.

Input, r (t)	Type 0	Type 1	Type 2
Unit Step, u_{-1} (t)	$\frac{a_n}{a_n + m_n}$	0	0
Unit Ramp, t	∞	$\frac{a_{n-1}}{m_n}$	0
Unit Parabolic, $\frac{1}{2} t^2$	∞	∞	$\frac{a_{n-2}}{m_n}$

Figure 3.4.2. Steady-State Error

From the $\underline{F}$ canonical state equations, the type of system can be determined by inspection of the number of consecutive zeros in the last column of the $\underline{F}$ matrix, beginning with the upper element. Then, for one of the possible inputs shown, the steady-state error for the system can be obtained from Figure 3.4.2.

3.5 Compensation Schemes. The general purpose of compensation is to modify a given system in some manner such that the compensated system has more desirable properties than the original system. One property that is often considered is the stability of a system. This property is now discussed.

A general form for compensation is assumed. The compensating system is placed in cascade with the given system, and feedback is connected around this cascade combination. This arrangement is shown in Figure 3.5.1.

In Figure 3.5.1 the system to be compensated is G, and the compensating system is H. It is assumed that the canonical state equation for G is known and is given by

$$\frac{d}{dt} \underline{Z} = \underline{F}\, \underline{Z} + \underline{M}\, g\,(t)$$

$$c\,(t) = [0 \quad 0 \quad . \quad . \quad . \quad 0 \quad 1]\, \underline{Z} \qquad (3.5.1)$$

Assume that G is unstable, or equivalently, assume that at least one eigenvalue of $\underline{F}$ has a positive real part. This may be determined by factoring f (λ), the characteristic function of $\underline{F}$, or possibly by applying the Routh-Hurwitz criteria to this polynomial. The Routh-Hurwitz test is discussed on pages 197-201 in Gardner and Barnes (6). It is desired that H be selected so that the system in Figure 3.5.1 is stable.

The general procedure is to assume the canonical form of the state equation for H where some or all of the parameters in this equation are to be determined later. Then the state equation for the HG combination is derived by the method given in Section 3.1. At this point the state equation for the entire system is obtained using methods given in Section 3.3. Then the parameters of H are chosen to make the over-all system stable, if this is possible. If not, another form for H may be assumed. The stability can be determined by applying the Routh-Hurwitz test to the characteristic function for the over-all system. If possible, the parameters of H are selected so that the characteristic function has only stable roots. An example demonstrating these concepts is given in the next chapter.

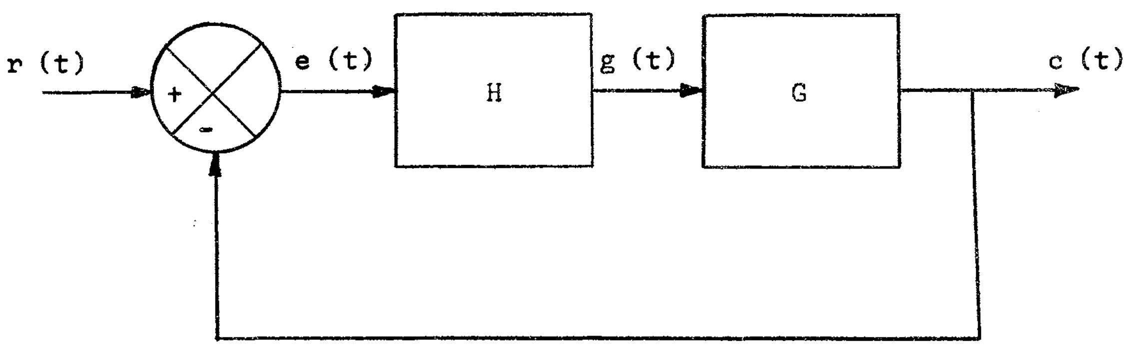

Figure 3.5.1. Compensation Scheme

After the parameters in H have been determined, then the final form for the canonical state equation for H is known. This allows an analog computer diagram to be drawn which represents a system that can be used for H. Similarly, the transfer function for H can be written by inspection, and methods of network synthesis can be applied to the realization of this transfer function.

While the real part of the roots of the characteristic equation determine the stability of a system, the relationship between the real and imaginary parts of the roots also influences the behavior of a system. An example is discussed in Section 4.8 where the magnitudes of the real and imaginary parts are made to be equal.

CHAPTER IV

ILLUSTRATIVE EXAMPLES

4.1 Derivation of the Transfer Function by the Secondary Method.

In this section the procedure discussed in Section 2.2 is used in finding the transfer function for a system G. Part of the secondary method involves the application of Theorem 2.2.1.

Assume that the state equation for the system G is known and is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 9 \\ 2 & -9 & -15 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} r(t) \tag{4.1.1}$$

$$c(t) = [1 \quad -2 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Using the methods given on pages 210-215 in Browne (4), the matrix $\underline{P}$ which reduces $\underline{A}$ to $\underline{F}$ is found to be

$$\underline{P} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{4.1.2}$$

and consequently

$$\underline{P}^{-1} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \tag{4.1.3}$$

Now, using Equations 4.1.1, 4.1.2, and 4.1.3, it is seen that

$$\underline{P}^{-1}\ \underline{A}\ \underline{P} = \underline{F} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -8 \\ 0 & 1 & -6 \end{bmatrix} \tag{4.1.4}$$

$$\underline{P}^{-1}\ \underline{B} = \underline{E} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} \tag{4.1.5}$$

and

$$\underline{T}\ \underline{P} = \underline{D} = [1 \quad -4 \quad 1] \tag{4.1.6}$$

Thus, the matrix substitution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{4.1.7}$$

into Equation 4.1.1 yields the transformed state equation

$$\frac{d}{dt} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -8 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} r\ (t)$$

$$c(t) = [1 \quad -4 \quad 1] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{4.1.8}$$

which is the appropriate form for the application of Theorem 2.2.1.

Using this theorem, it is seen that

$$\text{adj. } [s\,\underline{U} - \underline{F}] = \begin{bmatrix} s^2 + 6s + 8 & 0 & 0 \\ s + 6 & s^2 + 6s & -8s \\ 1 & s & s^2 \end{bmatrix} \tag{4.1.9}$$

and consequently

$$\underline{D} \text{ adj. } [s\,\underline{U} - \underline{F}]\,\underline{E} = 11\,s^2 + 7s - 15 = M(s) \tag{4.1.10}$$

and

$$A(s) = s^3 + 6s^2 + 8s \tag{4.1.11}$$

which yields the transfer function

$$G(s) = \frac{11\,s^2 + 7s - 15}{s^3 + 6\,s^2 + 8s} \tag{4.1.12}$$

4.2 An Example of the Extension of the Secondary Method. In this section the procedure discussed in Section 2.3 is used in finding the transfer function for a system G which requires the extension of the secondary method.

Assume that the state equations for G are known and are

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 4 & -2 \\ 3 & -2 & 2 & -1 \\ 3 & -2 & 2 & -1 \\ 6 & -6 & 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} r\,(t) \qquad (4.2.1)$$

$$c\,(t) = [1 \quad -1 \quad 1 \quad -1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Making the change of variables $\underline{X} = \underline{P}\ \underline{Y}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \qquad (4.2.2)$$

where

$$\underline{P}^{-1} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix} \qquad (4.2.3)$$

it is seen that

$$\underline{P}^{-1}\ \underline{A}\ \underline{P} = \underline{F} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad (4.2.4)$$

$$\underline{P}^{-1}\,\underline{B} = \underline{E} = \begin{bmatrix} -1 \\ 1 \\ -2 \\ 2 \end{bmatrix} \qquad (4.2.5)$$

and

$$\underline{T}\,\underline{P} = \underline{D} = [-1 \quad -1 \quad -1 \quad -1] \qquad (4.2.6)$$

Consequently, the matrix transformation given by Equation 4.2.2, upon substitution into Equation 4.2.1, yields the transformed state equation

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ -2 \\ 2 \end{bmatrix} r\ (t)$$

$$c\ (t) = [-1 \quad -1 \quad -1 \quad -1]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \qquad (4.2.7)$$

The $\underline{F}$ matrix in Equation 4.2.7 is observed to contain two of the block matrices that were discussed in Sections 2.2 and 2.3. Consequently, Equation 4.2.7 is separated into two state equations

$$\frac{d}{dt}[y_1] = [0]\,[y_1] + [-1]\ r\ (t)$$

$$c_1\ (t) = [-1]\ [y_1] \tag{4.2.8}$$

and

$$\frac{d}{dt}\begin{bmatrix} y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} r\ (t)$$

$$c_2\ (t) = [-1 \quad -1 \quad -1]\begin{bmatrix} y_2 \\ y_3 \\ y_4 \end{bmatrix} \tag{4.2.9}$$

with

$$c\ (t) = c_1\ (t) + c_2\ (t) \tag{4.2.10}$$

With the application of Theorem 2.2.1, Equation 4.2.8 yields

$$\text{adj.}\ [s\ \underline{U} - \underline{F}_1] = [1] \tag{4.2.11}$$

$$\underline{D}_1\ \text{adj.}\ [s\ \underline{U} - \underline{F}_1]\ \underline{E}_1 = 1 = M_1\ (s) \tag{4.2.12}$$

$$A_1\ (s) = s \tag{4.2.13}$$

and consequently

$$G_1\ (s) = \frac{C_1\ (s)}{R\ (s)} = \frac{1}{s} \tag{4.2.14}$$

With the application of Theorem 2.2.1, Equation 4.2.9 yields

$$\text{adj. } [s\,\underline{U} - \underline{F}_2] = \begin{bmatrix} s^2 - s & 0 & 0 \\ s - 1 & s^2 - s & 0 \\ 1 & s & s^2 \end{bmatrix} \qquad (4.2.15)$$

$$\underline{D}_2 \text{ adj. } [s\,\underline{U} - \underline{F}_2]\,\underline{E}_2 = -s^2 = M_2\,(s) \qquad (4.2.16)$$

$$A_2\,(s) = s^3 - s^2 \qquad (4.2.17)$$

and consequently

$$G_2\,(s) = \frac{C_2\,(s)}{R\,(s)} = \frac{-s^2}{s^3 - s^2} = \frac{-1}{s - 1} \qquad (4.2.18)$$

Because of Equation 4.2.10 and the fact that the Laplace transformation is a linear operator

$$C\,(s) = C_1\,(s) + C_2\,(s) \qquad (4.2.19)$$

Dividing both sides of Equation 4.2.19 by R (s), it is seen that

$$G\,(s) = G_1\,(s) + G_2\,(s) \qquad (4.2.20)$$

and consequently

$$G\,(s) = \frac{1}{s} - \frac{1}{s - 1} = \frac{-1}{s\,(s - 1)} \qquad (4.2.21)$$

for the transfer function of the system defined by Equation 4.2.1.

In summary, the secondary method involves less work than the general method. The secondary method requires the inversion of a constant matrix, while the general method requires the inversion of a matrix whose elements are functions of s. With the extension of the secondary method described in this section, the secondary method applies to all systems covered by the general method.

4.3 Derivation of the F Canonical State Equations from the Transfer Function. Canonical state equations for the systems discussed in Sections 4.1 and 4.2 are shown in this section. As mentioned in Section 2.4, this is simply a matter of inspection.

The transfer function of the system G in Section 4.1 was shown to be

$$G(s) = \frac{C(s)}{R(s)} = \frac{11s^2 + 7s - 15}{s(s^2 + 6s + 8)} \tag{4.3.1}$$

Consequently, by Theorem 2.4.1, the F canonical state equation having this transfer function is

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -8 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} -15 \\ 7 \\ 11 \end{bmatrix} r(t)$$

(4.3.2)

$$c(t) = [0 \quad 0 \quad 1] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = z_3$$

Similarly, the transfer function of the system G in Section 4.2 was shown to be

$$G(s) = \frac{C(s)}{R(s)} = \frac{-1}{s(s-1)} \tag{4.3.3}$$

Consequently, by Theorem 2.4.1, the F canonical state equation having this transfer function is

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} r\,(t)$$

$$c\,(t) = [0 \quad 1] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_2 \tag{4.3.4}$$

The system of Equation 4.3.4 exhibits the unusual property that the original state equation for the system, Equation 4.2.1, is a fourth-order equation, while the F canonical state equation is a second-order equation. However, the system as described by the fourth-order equation is equivalent to the system as described by the second-order equation in the sense that a common input to the two systems produces a common output. This phenomena provides some justification for the step in the procedure when the transfer function is determined. If a reduction in order is possible, then it can be seen as a common factor in the numerator and denominator polynomials of G (s) and can be eliminated by division.

It can now be seen that the transfer function and the canonical state equation are interchangeable concepts in the sense that one can be obtained from the other by inspection. This allows for a great deal of flexibility in working with state equations.

4.4 State Equations of Two Systems in Cascade. In this section it is assumed that two systems, G_1 and G_2, are connected in cascade. Furthermore, it is assumed that the canonical state equations are known for both systems. The problem is to determine the state equation for the over-all system.

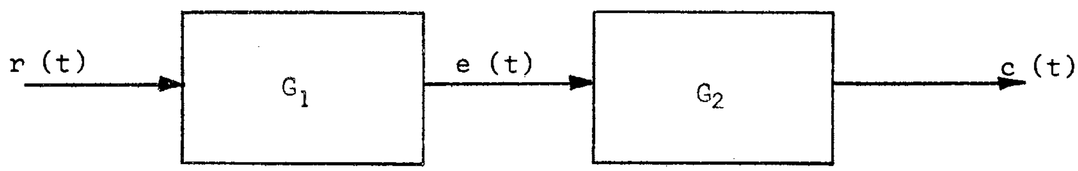

Figure 4.4.1. Cascade Systems

Assume that for system G_1

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} r\,(t)$$

$$e\,(t) = x_2 \qquad (4.4.1)$$

and that for system G_2

$$\frac{d}{dt}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e\,(t)$$

$$c\,(t) = x_4 \qquad (4.4.2)$$

In the method explained in Section 3.1 it is observed that

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & -4 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} r(t)$$

$$c(t) = x_4 \qquad (4.4.3)$$

Making the change of variables $\underline{X} = \underline{P}\,\underline{Y}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \qquad (4.4.4)$$

and consequently

$$\underline{P}^{-1} = \begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & -7 & 9 \\ 0 & 0 & -5 & 6 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad (4.4.5)$$

it is observed that

$$\underline{P}^{-1}\,\underline{A}\,\underline{P} = \underline{F} = \begin{bmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & -12 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & -6 \end{bmatrix} \qquad (4.4.6)$$

$$\underline{P}^{-1}\,\underline{B} = \underline{E} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \qquad (4.4.7)$$

and

$$\underline{T}\ \underline{P} = \underline{D} = [0 \quad 0 \quad 1 \quad -5] \tag{4.4.8}$$

This transformation yields

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & -12 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} r\ (t)$$

$$c\ (t) = [0 \quad 0 \quad 1 \quad -5]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \tag{4.4.9}$$

Thus

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 13 \\ 12 \\ 4 \end{bmatrix}, \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -5 \end{bmatrix} \tag{4.4.10}$$

and substitution of these numbers into the proper expressions in Appendix A yields

$$M\ (s) = 2s^2 + 3s + 1 = (2s + 1)\ (s + 1) \tag{4.4.11}$$

and

$$A\ (s) = s^4 + 6s^3 + 13\ s^2 + 12\ s + 4 = (s + 1)^2\ (s + 2)^2 \tag{4.4.12}$$

Thus

$$G\ (s) = \frac{M\ (s)}{A\ (s)} = \frac{2s + 1}{(s + 1)\ (s + 2)^2} \qquad (4.4.13)$$

or

$$G\ (s) = \frac{2s + 1}{s^3 + 5s^2 + 8s + 4} \qquad (4.4.14)$$

The $\underline{F}$ canonical state equation is then written by inspection as

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & -8 \\ 0 & 1 & -5 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} r\ (t)$$

$$c\ (t) = [0 \quad 0 \quad 1]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \qquad (4.4.15)$$

Thus, it is seen that for the two systems in cascade the resultant system is third order. Since G_1 and G_2 were each second-order systems, this demonstrates that the order of a cascade combination may be less than the sum of the orders of the individual systems. Due to the way the state equations were adjoined in Section 3.1, it can be seen that the order of the combination will not exceed this sum.

4.5 State Equations of Two Systems in Parallel. In this section the systems G_1 and G_2 of Section 4.4 are connected in parallel. The state equations describing G_1 and G_2 are Equations 4.4.1 and 4.4.2, respectively. The problem is to determine the state equations for the over-all system.

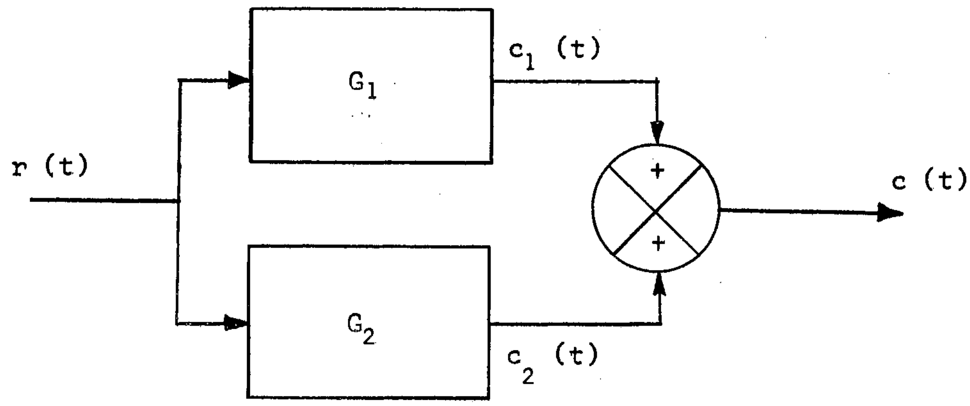

Figure 4.5.1. Parallel Systems

In the method explained in Section 3.2 it is observed that

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} r\,(t) \tag{4.5.1}$$

$$c\,(t) = [0 \quad 1 \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Equation 4.5.1 is a state equation describing the parallel combination of systems shown in Figure 4.5.1. It is observed that the $\underline{A}$ matrix in Equation 4.5.1 is already the transpose of a rational canonical form, where there are two 2 by 2 block matrices in $\underline{F}$, each of the form given in Equation 2.2.15. This means that it is not necessary to make a transformation of variables. Instead, Equation 4.5.1 is split into two state equations in the method explained in Section 2.3. These two state equations are the same as Equation 4.4.1 and Equation 4.4.2. This means that the transfer function for G_1 should be determined from Equation 4.4.1 and the transfer function for G_2 determined from Equation 4.4.2. The over-all transfer function G (s) is simply the algebraic sum of the individual transfer functions. Thus, from Equation 4.4.1

$$G_1(s) = \frac{2s + 1}{s^2 + 2s + 1} \qquad (4.5.2)$$

by inspection, and from Equation 4.4.2

$$G_2(s) = \frac{s + 1}{s^2 + 4s + 4} \qquad (4.5.3)$$

by inspection. Consequently

$$G(s) = G_1(s) + G_2(s) = \frac{3s^3 + 12 s^2 + 15 s + 5}{s^4 + 6s^3 + 13 s^2 + 12 s + 4} \qquad (4.5.4)$$

and the $\underline{F}$ canonical state equation may be written by inspection as

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & -12 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 5 \\ 15 \\ 12 \\ 3 \end{bmatrix} r(t)$$

$$c(t) = [0 \quad 0 \quad 0 \quad 1] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = z_4 \qquad (4.5.5)$$

Thus Equation 4.5.5 has been shown to be the $\underline{F}$ canonical state equation for the parallel combination of systems shown in Figure 4.5.1.

4.6 Feedback System. This section considers a feedback system where the basic system G is unstable. It is desired that the system be made stable by the addition of feedback and an amplifier. The system takes the form given in Figure 4.6.1

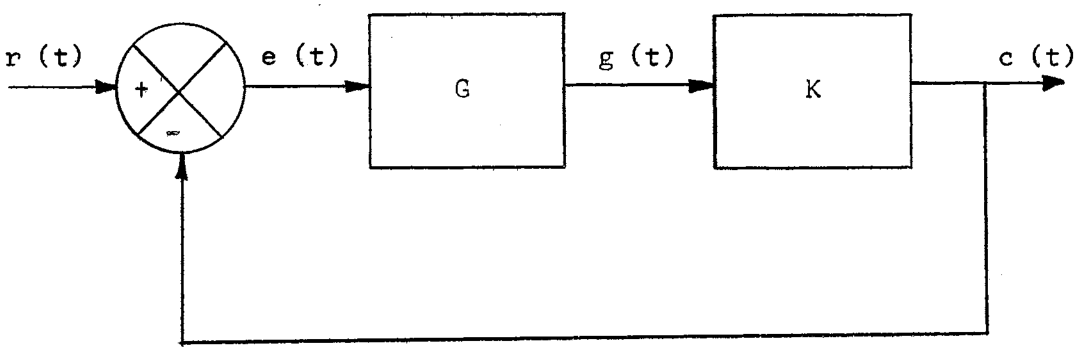

Figure 4.6.1. Feedback System

The $\underline{F}$ canonical state equation for G is assumed to be

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} e\ (t) \tag{4.6.1}$$

$$g\ (t) = [0 \quad 0 \quad 1]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and K is a simple amplifier, where

$$c\ (t) = K\ g\ (t) \tag{4.6.2}$$

Using Equation 3.3.5, it is seen that

$$\underline{F} - \underline{M}\ [0 \quad 0 \quad K] = \begin{bmatrix} 0 & 0 & -2K \\ 1 & 0 & 3 - 3K \\ 0 & 1 & -2 - K \end{bmatrix} \tag{4.6.3}$$

and consequently the state equation for the feedback system is

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2K \\ 1 & 0 & 3 - 3K \\ 0 & 1 & -2 - K \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} r\ (t)$$

$$c\ (t) = Kx_3 \tag{4.6.4}$$

For this system to be stable there can be no eigenvalues with positive real parts. For Equation 4.6.4 the characteristic function, $f\ (\lambda)$, is

$$f\ (\lambda) = -\lambda^3 - \lambda^2\ (K + 2) - \lambda\ (3K - 3) - 2K \tag{4.6.5}$$

The characteristic equation is seen to be

$$\lambda^3 + \lambda^2\ (K + 2) + \lambda\ (3K - 3) + 2K = 0 \tag{4.6.6}$$

At this point the Routh array is formed

$$\begin{array}{cc} 1 & 3K - 3 \\ K + 2 & 2K \\ \dfrac{3K^2 + K - 6}{K + 2} & 0 \\ 2K & 0 \end{array} \tag{4.6.7}$$

Considering only K > 0, it is observed that for 1.257 > K > 0, there are two eigenvalues with positive real parts. However, for K > 1.257 all eigenvalues have negative real parts. This means that the feedback system in Figure 4.6.1 is stable for K > 1.257 and unstable for 1.257 > K > 0. After a value of K is selected, this value can be substituted into Equation 4.6.4, yielding the state equation for the feedback system.

4.7 Compensation of a Feedback System. Two common compensation networks are the lead network and the lag network. The transfer function for these networks is

$$H(s) = \frac{s + b}{s + a} \qquad (4.7.1)$$

where a > b > 0 for a lead network, and b > a > 0 for a lag network.

Observe that

$$H(s) = \frac{s + b}{s + a} = 1 + \frac{b - a}{s + a} = \frac{C(s)}{R(s)} \qquad (4.7.2)$$

This system can be represented by an analog computer diagram given in Figure 4.7.1.

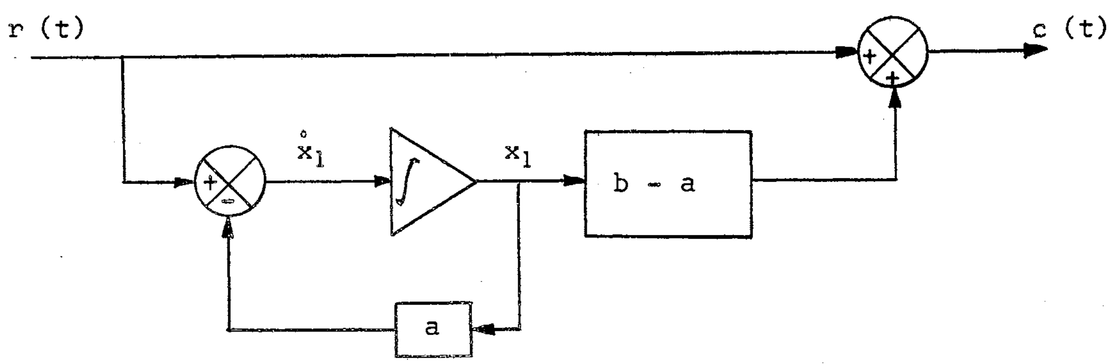

Figure 4.7.1. Analog Computer Diagram for H (s)

Consequently, a state equation can be obtained from Figure 4.7.1

$$\frac{d}{dt} x_1 = - ax_1 + r (t)$$
$$c (t) = (b - a) x_1 + r (t) \qquad (4.7.3)$$

which is a first-order equation. This state equation is of a different form, however. The output c (t) depends upon the state variable x_1 and the input r (t). This is the type of compensating system that will be considered in this section. Appendix B includes a discussion of this type of system.

Assume that a system G is given and that G is unstable. The problem is to compensate G with feedback and a lead or lag system. Assume that the over-all system is given in Figure 4.7.2.

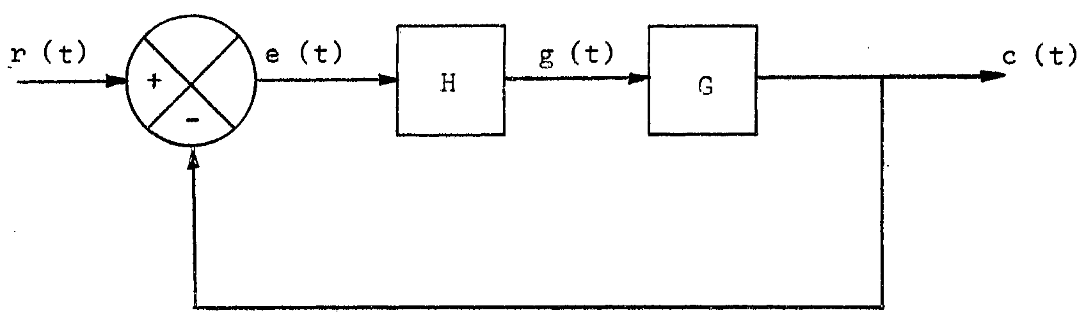

Figure 4.7.2. Feedback Compensation

Assume the state equation for H is

$$\frac{d}{dt} x_1 = - ax_1 + e (t)$$
$$g (t) = (b - a) x_1 + e (t) \qquad (4.7.4)$$

where a and b are determined later.

Assume that the state equation for G is

$$\frac{d}{dt}\begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 4 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 18 \\ 9 \\ 0 \end{bmatrix} g\ (t)$$

(4.7.5)

$$c\ (t) = [0 \quad 0 \quad 1] \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4$$

At this point, Equations 4.7.4 and 4.7.5 are adjoined in a manner explained in Section 3.3, yielding

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -a & 0 & 0 & -1 \\ 18(b - a) & 0 & 0 & -18 \\ 9(b - a) & 1 & 0 & -5 \\ 0 & 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 18 \\ 9 \\ 0 \end{bmatrix} r\ (t)$$

(4.7.6)

$$c\ (t) = x_4$$

as the state equation for the over-all system.

The eigenvalues are known to determine the stability of the system. Consequently, it is necessary to find the characteristic function for the system, which is

$$f\ (\lambda) = \lambda^4 + (a + 3)\ \lambda^3 + (3a + 5)\ \lambda^2 + (9b - 4a + 18)\ \lambda + 18b$$

(4.7.7)

The problem now is to choose a and b such that f $(\lambda) = 0$ has no root with a positive real part. Also, it is desired that a and b are each positive. One way to do this is to choose a value for a and then use the Routh array to see if it is possible to choose a value for b which will result in a stable system. For this reason choose a = 4. The characteristic equation now becomes

$$\lambda^4 + 7 \lambda^3 + 17 \lambda^2 + (9b + 2) \lambda + 18b = 0 \qquad (4.7.8)$$

The Routh array then becomes

$$\begin{array}{ccc} 1 & 17 & 18b \\ 7 & 9b + 2 & 0 \\ 13 - b & 14b & 0 \\ \dfrac{-9b^2 + 17b + 26}{13 - b} & 0 & 0 \\ 14b & 0 & 0 \end{array} \qquad (4.7.9)$$

The third and fifth rows imply

$$13 > b > 0 \qquad (4.7.10)$$

The function $-9b^2 + 17b + 26$ is positive only for $2.89 > b > -1$, which indicates that a possible range for b is

$$2.89 > b > 0 \qquad (4.7.11)$$

Consequently, for a = 4, b can have any value in that range and the feedback system given in Figure 4.7.2 will be stable. Observe that a > b, which indicates that H must be a lead compensation system.

4.8 Design of a Control System. A control system is to be designed which accepts the angular position θ_i of a shaft as the input control function. The output of the system is the angular position θ_o of a shaft which is connected through a gearbox with the shaft of a DC motor. The difference between these two angles is the input to a DC amplifier which drives an amplidyne supplying current to the motor. This interconnection is shown in Figure 4.8.1.

The DC amplifier is governed by the equation

$$v_f = K\, v_1 - 10^4\, i_f \tag{4.8.1}$$

The amplidyne is governed by the state equation

$$\frac{d}{dt}[x_1] = [-12]\,[x_1] + [1/150]\, v_f$$

$$i_f = x_1 \tag{4.8.2}$$

$$v_a = 5\cdot 10^4\, x_1 - 50\, i_a$$

The DC motor is governed by the state equation

$$\frac{d}{dt}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2/3 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2\cdot 10^3 \end{bmatrix} i_a$$

$$\theta = x_2 \tag{4.8.3}$$

$$v_a = 30\, i_a + 3/5\, x_3$$

The gearbox is governed by the equation

$$\theta_o = 1/36\ \theta \tag{4.8.4}$$

Initially, the control system is in the state that K = 17.5 and

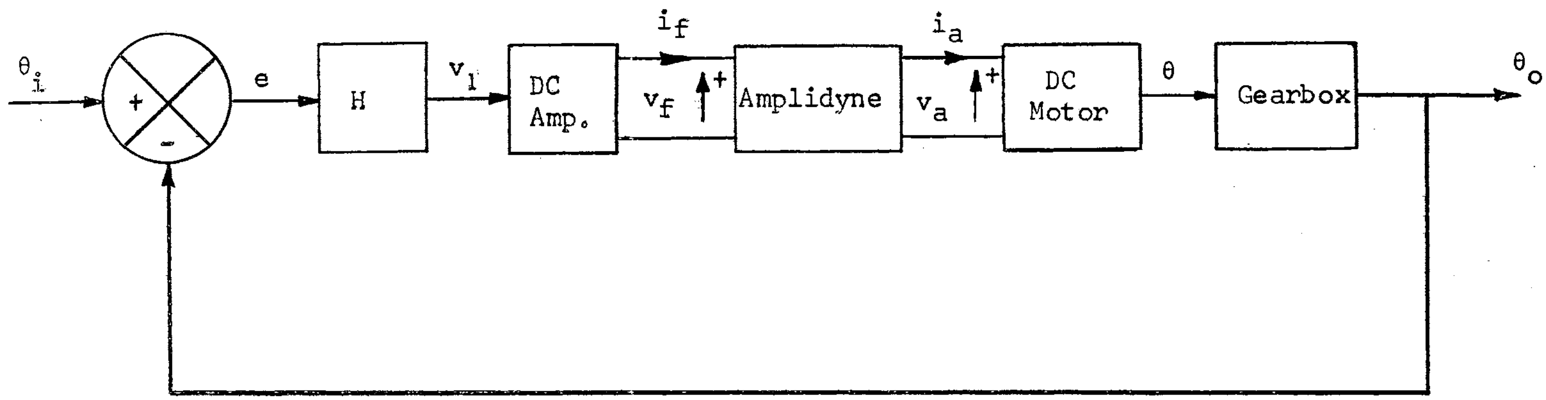

Figure 4.8.1. DC Motor Control System

H, the compensating system, is a simple unity gain network, $v_1 = e$.

The problem is to determine the eigenvalues for the open-loop system, the closed-loop system, and the value for K that would allow the real and imaginary parts of a complex pair of eigenvalues to be equal.

Adjoining Equations 4.8.1, 4.8.2, and 4.8.3, and eliminating v_a from Equations 4.8.2 and 4.8.3, it is seen that

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -236/3 & 0 & 0 \\ 0 & 0 & 1 \\ 125 \cdot 10^4 & 0 & -47/3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} K/150 \\ 0 \\ 0 \end{bmatrix} v_1$$

$$\theta_o = 1/36 \; x_2 \tag{4.8.5}$$

which is the open-loop state equation. The eigenvalues are seen to be

$$\lambda = 0 \;, -47/3 \;, \text{ and } -236/3 \tag{4.8.6}$$

for the open-loop system.

If the open-loop system that has v_1 as the input and θ_o as the output is called G, then the state equation for G is given by Equation 4.8.5.

Using the procedure given in Section 2.2, the matrix change of variables $\underline{X} = \underline{P}\;\underline{Y}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -78.67 & 6190 \\ 0 & 0 & 1.25 \cdot 10^6 \\ 0 & 1.25 \cdot 10^6 & -118 \cdot 10^6 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{4.8.7}$$

where

$$\underline{P}^{-1} = \begin{bmatrix} 1 & .987 \cdot 10^{-3} & 63 \cdot 10^{-6} \\ 0 & 75.5 \cdot 10^{-6} & .8 \cdot 10^{-6} \\ 0 & .8 \cdot 10^{-6} & 0 \end{bmatrix} \qquad (4.8.8)$$

is made in Equation 4.8.5, leaving

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1240 \\ 0 & 1 & -94.3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} K/150 \\ 0 \\ 0 \end{bmatrix} v_1$$

$$\theta_o = 34,700\ y_3 \qquad (4.8.9)$$

as the transformed state equation. The tables in Appendix A are now used to find the transfer function

$$\frac{\theta_o\ (s)}{V_1\ (s)} = G\ (s) = \frac{231\ K}{s\ (s^2 + 94.3\ s + 1240)} \qquad (4.8.10)$$

for the open-loop system G.

Theorem 2.4.1 and Equation 4.8.10 yield the $\underline{F}$ canonical state equation

$$\frac{d}{dt}\begin{bmatrix} z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1240 \\ 0 & 1 & -94.3 \end{bmatrix}\begin{bmatrix} z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 231\ K \\ 0 \\ 0 \end{bmatrix} v_1$$

$$\theta_o = z_4 \qquad (4.8.11)$$

for the open-loop system G.

For the uncompensated system, $v_1 = e$, Figure 3.4.2 shows that the steady-state error for a unit step input is zero, since $a_3 = 0$ in

Equation 4.8.11 indicates a type one system. Figure 3.4.2 yields the steady-state error

$$e_{ss} = \frac{a_2}{m_3} = \frac{1240}{231\ K} = 0.307 \text{ radians} \tag{4.8.12}$$

for a unit ramp input of 1 radian per second when K = 17.5 as specified.

Using the method explained in Section 3.3, the closed-loop state equation is obtained from Equation 4.8.11, and is

$$\frac{d}{dt}\begin{bmatrix} z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -231\ K \\ 1 & 0 & -1240 \\ 0 & 1 & -94.3 \end{bmatrix}\begin{bmatrix} z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 231\ K \\ 0 \\ 0 \end{bmatrix} \theta_i$$

$$\theta_o = z_4 \tag{4.8.13}$$

The characteristic equation is observed to be

$$\lambda^3 + 94.3\ \lambda^2 + 1240\ \lambda + 231\ K = 0 \tag{4.8.14}$$

and for the value of K = 17.5 the eigenvalues are

$$\lambda = -5.2\ ,\ -9.8\ ,\ \text{and } -79.3 \tag{4.8.15}$$

for the closed-loop system.

Next, it is desired that the value of K be found which will make Equation 4.4.14 have complex roots where the real and imaginary parts are equal. If this is the case, and calling $\lambda = a \pm ia$ the desired roots, then $\lambda^2 - 2a\ \lambda + 2a^2$ must divide the expression in Equation 4.8.14. Carrying this out by long division, and equating the remainder to zero, it is seen that

$$2a^2 + 188.6a + 1240 = 0 \qquad (4.8.16)$$

and

$$231\ K - 4a^3 - 188.6\ a^2 = 0 \qquad (4.8.17)$$

are the equations necessary for there to be complex eigenvalues with equal real and imaginary parts. Considering only positive values for K, the solution of Equations 4.8.16 and 4.8.17 yields the values

$$a = -7.1 \qquad (4.8.18)$$

and

$$K = 34.7 \qquad (4.8.19)$$

For this value of K, the eigenvalues of the closed-loop system are

$$\lambda = -80.1\ ,\ \text{and}\ -7.1 \pm i\ 7.1 \qquad (4.8.20)$$

and the steady-state error for a unit ramp input is

$$e_{ss} = \frac{1240}{231\ K} = 0.155\ \text{radians} \qquad (4.8.21)$$

using the same expression as in Equation 4.8.12. Thus, simply increasing K from 17.5 to 34.7 decreases the steady-state error for a unit ramp input from 0.307 to 0.155 radians.

At this point it is desired that the system be compensated by changing H to a lead or a lag system as described in Section 4.7. The state equation for this system is

$$\frac{d}{dt}[z_1] = [-a]\ [z_1] + [1]\ e$$

$$v_1 = [b - a]\,[z_1] + [1]\,e \tag{4.8.22}$$

where the input to H is e and the output of H is v_1 as shown in Figure 4.8.1. The compensation is to be selected so that complex roots will equal real and imaginary are retained, but the magnitude of the imaginary part, corresponding to the oscillation frequency, is to be decreased.

The closed-loop state equation for the compensated system is formed by combining Equations 4.8.11 and 4.8.22 in the method described in Section 3.3, yielding

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} -a & 0 & 0 & -1 \\ 231\ K(b - a) & 0 & 0 & -231\ K \\ 0 & 1 & 0 & -1240 \\ 0 & 0 & 1 & -94.3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 231\ K \\ 0 \\ 0 \end{bmatrix} \theta_i \tag{4.8.23}$$

$$\theta_o = z_4$$

The characteristic equation for this system is

$$\lambda^4 + (94.3 + a)\,\lambda^3 + (1240 + 94.3a)\,\lambda^2 + (1240a + 231\ K)\,\lambda + 231\ K\ b = 0 \tag{4.8.24}$$

At this point a and b are selected, and long division is used to see if K can be chosen to result in the desired pair of complex roots of Equation 4.8.24 having equal real and imaginary parts. Consequently, it is assumed that a = 5 and b = 18, and these values are substituted into Equation 4.8.24, yielding

$$\lambda^4 + 99.3\ \lambda^3 + 1712\ \lambda^2 + (6200 + 231\ K)\ \lambda + 4170\ K = 0 \qquad (4.8.25)$$

A root $\lambda = c \pm ic$ of Equation 4.8.25 is assumed. This means that $\lambda^2 - 2c\ \lambda + 2c^2$ must divide the function. Carrying out the long division, it is seen that

$$198.6\ c^2 + 3424\ c + 231\ K + 6200 = 0 \qquad (4.8.26)$$

and

$$4c^4 + 397.2\ c^3 + 3424\ c^2 - 4170\ K = 0 \qquad (4.8.27)$$

for the remainder to be zero. Considering only $K > 0$ and $|c| < 7.1$, the solution of these two equations is

$$c = -2.37 \qquad (4.8.28)$$

and

$$K = 3.48 \qquad (4.8.29)$$

For $K = 3.48$ the compensated system has eigenvalues

$$\lambda = -15.9\ ,\ -78.7\ ,\ \text{and}\ \ -2.37 \pm i\ 2.37$$

as desired. Thus, by adding the lag compensation network H, the oscillation frequency is decreased from 7.1 to 2.37.

CHAPTER V

SUMMARY AND CONCLUSIONS

5.1 Summary. This thesis has considered the problem of analyzing a linear system which is governed by a time-domain matrix differential equation. This type of equation is called a state-variable equation. The analysis of the system consists of operations performed on the state equation. The operations bring out certain properties concerning the system. These properties include the stability of the system, the type of the system, and others.

The first step in the analysis is to find the Laplace transfer function of the system. This step proves useful for two reasons. First, a canonical form for the state equations can be written directly from inspection of the transfer function. Second, the determination of the transfer function allows a great deal of information from conventional control theory to be used directly. This includes such techniques as root locus, Bode plots, and Nyquist criteria. This is useful for engineers who are familiar with such methods.

The second step in the analysis is to write the canonical form of the state equation by inspection of the transfer function. This form of the state equation is convenient for the various manipulations corresponding to the interconnection of systems.

Where systems are interconnected, the next step is to adjoin the canonical state equations yielding the state equation for the entire system. This final state equation is then examined. Various parameters within individual systems may then be selected to cause the over-all system to have the desired characteristics. The state equation for the interconnected system shows the effect the various subsystems have upon the performance of the entire system.

The reduction of the given state equation to canonical form is motivated by the simplicity of the canonical form. This simplicity makes possible the recognition of effects in the over-all system caused by component systems. For a single n-th order system there are $n^2 + 2n$ constants in the initial state equation describing the system. The reduction of the state equation to the $\underline{F}$ canonical form has a corresponding reduction in the number of constants from $n^2 + 2n$ to $2n$.

The last part of this thesis shows how the state equations are treated in cases involving common interconnections of systems and furnishes several examples illustrating the methods discussed.

5.2 Conclusions. A method has been presented which simplifies the analysis of a linear system governed by a state equation. The method makes possible a reduction of the given state equation to a canonical form which is preferable from the viewpoint of an engineer.

A simple relationship has been shown to exist between the Laplace transfer function of a system and the $\underline{F}$ canonical form of the state equation. It is felt that this relationship will have an application in control theory, network synthesis, and analog computation.

A general observation is that there seem to be more similarities between the methods in control system theory and the methods in state-variable theory than there are differences.

5.3 Recommendations for Further Study. The implications for network synthesis of the simple relationship between the Laplace transfer function and the $\underset{\sim}{F}$ canonical state equation should be investigated.

The procedure of reducing general state equations to canonical form should be programmed for solution on a digital computer.

The area of compensation of systems described by canonical state equations should be investigated at length.

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A P P E N D I X   A

APPENDIX A

TABLES FOR OBTAINING TRANSFER FUNCTIONS

FROM TRANSFORMED STATE EQUATIONS

A.1 Explanation of Tables. Appendix A supplies a list of tables which assist in determining the transfer function from the transformed state equations for systems of order six and below.

The transformed state equation has the form

$$\frac{d}{dt}\underline{Y} = \underline{F}\,\underline{Y} + \underline{E}\,r(t)$$

$$c(t) = \underline{D}\,\underline{Y} \qquad (A.1.1)$$

where

$$\underline{F} = \begin{bmatrix} 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -a_n \\ 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -a_{n-1} \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & -a_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -a_2 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & -a_1 \end{bmatrix} \qquad (A.1.2)$$

$$\underline{E} = \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_n \end{bmatrix} \qquad (A.1.3)$$

and

$$\underline{D} = [d_1 \; d_2 \; \ldots \; d_n] \qquad (A.1.4)$$

The numbers in $\underline{F}$, $\underline{E}$, and $\underline{D}$ can be substituted into the appropriate expression in Appendix A for the evaluation of

$$G(s) = \frac{C(s)}{R(s)} \qquad (A.1.5)$$

For each type system, the coefficients of two polynomials, M (s) and A (s), are listed, where

$$G(s) = \frac{M(s)}{A(s)} = \frac{m_1 s^{n-1} + m_2 s^{n-2} + \ldots + m_{n-1} s + m_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} \qquad (A.1.6)$$

Thus, m_i is the coefficient of s^{n-i} in M (s), and a_i is the coefficient of s^{n-i} in A (s).

A.2 First-Order Systems.

$$A(s) = s + a_1$$

$$M(s) = m_1 \qquad (A.2.1)$$

$$m_1 = e_1 d_1$$

A.3 Second-Order Systems.

$$A(s) = s^2 + a_1 s + a_2$$

$$M(s) = m_1 s + m_2$$

$$m_1 = e_1 d_1 + e_2 d_2 \qquad (A.3.1)$$

$$m_2 = e_1 d_2 + a_1 e_1 d_1 - a_2 e_2 d_1$$

A.4 Third-Order Systems.

$$A(s) = s^3 + a_1 s^2 + a_2 s + a_3$$

$$M(s) = m_1 s^2 + m_2 s + m_3$$

m_1	m_2	m_3
$+ e_1 d_1$	$+ e_1 d_2$	$+ e_1 d_3$
$+ e_2 d_2$	$+ e_2 d_3$	$+ a_1 e_1 d_2$
$+ e_3 d_3$	$+ a_1 e_1 d_1$	$+ a_2 e_1 d_1$
	$+ a_1 e_2 d_2$	$- a_3 e_3 d_2$
	$- a_3 e_3 d_1$	$- a_3 e_2 d_1$
	$- a_2 e_3 d_2$	

(A.4.1)

A.5 Fourth-Order Systems.

$$A(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$$

$$M(s) = m_1 s^3 + m_2 s^2 + m_3 s + m_4$$

m_1	m_2	m_3	m_4
$+ e_1 d_1$	$+ e_1 d_2$	$+ e_1 d_3$	$+ e_1 d_4$
$+ e_2 d_2$	$+ e_2 d_3$	$+ e_2 d_4$	$+ a_1 e_1 d_3$
$+ e_3 d_3$	$+ e_3 d_4$	$+ a_1 e_1 d_2$	$+ a_2 e_1 d_2$
$+ e_4 d_4$	$+ a_1 e_1 d_1$	$+ a_1 e_2 d_3$	$+ a_3 e_1 d_1$
	$+ a_1 e_2 d_2$	$+ a_2 e_1 d_1$	$- a_4 e_4 d_3$
	$+ a_1 e_3 d_3$	$+ a_2 e_2 d_2$	$- a_4 e_3 d_2$
	$- a_4 e_4 d_1$	$- a_4 e_4 d_2$	$- a_4 e_2 d_1$
	$- a_3 e_4 d_2$	$- a_3 e_4 d_3$	
	$- a_2 e_4 d_3$	$- a_4 e_3 d_1$	
		$- a_3 e_3 d_2$	

(A.5.1)

A.6 Fifth-Order Systems.

$$A(s) = s^5 + a_1 s^4 + a_2 s^3 + a_3 s^2 + a_4 s + a_5$$

$$M(s) = m_1 s^4 + m_2 s^3 + m_3 s^2 + m_4 s + m_5$$

m_1	m_2	m_3	m_4
$+ e_1 d_1$	$+ e_1 d_2$	$+ e_1 d_3$	$+ e_1 d_4$
$+ e_2 d_2$	$+ e_2 d_3$	$+ e_2 d_4$	$+ e_2 d_5$
$+ e_3 d_3$	$+ e_3 d_4$	$+ e_3 d_5$	$+ a_1 e_1 d_3$
$+ e_4 d_4$	$+ e_4 d_5$	$+ a_1 e_1 d_2$	$+ a_1 e_2 d_4$
$+ e_5 d_5$	$+ a_1 e_1 d_1$	$+ a_1 e_2 d_3$	$+ a_2 e_1 d_2$
	$+ a_1 e_2 d_2$	$+ a_1 e_3 d_4$	$+ a_2 e_2 d_3$
	$+ a_1 e_3 d_3$	$+ a_2 e_1 d_1$	$+ a_3 e_1 d_1$
m_5	$+ a_1 e_4 d_4$	$+ a_2 e_2 d_2$	$+ a_3 e_2 d_2$
$+ e_1 d_5$	$- a_5 e_5 d_1$	$+ a_2 e_3 d_3$	$- a_5 e_5 d_3$
$+ a_1 e_1 d_4$	$- a_4 e_5 d_2$	$- a_5 e_5 d_2$	$- a_4 e_5 d_4$
$+ a_2 e_1 d_3$	$- a_3 e_5 d_3$	$- a_4 e_5 d_3$	$- a_5 e_4 d_2$
$+ a_3 e_1 d_2$	$- a_2 e_5 d_4$	$- a_3 e_5 d_4$	$- a_4 e_4 d_3$
$+ a_4 e_1 d_1$		$- a_5 e_4 d_1$	$- a_5 e_3 d_1$
$- a_5 e_5 d_4$		$- a_4 e_4 d_2$	$- a_4 e_3 d_2$
$- a_5 e_4 d_3$		$- a_3 e_4 d_3$	
$- a_5 e_3 d_2$			
$- a_5 e_2 d_1$			

(A.6.1)

A.7 Sixth-Order Systems.

$$A(s) = s^6 + a_1 s^5 + a_2 s^4 + a_3 s^3 + a_4 s^2 + a_5 s + a_6$$

$$M(s) = m_1 s^5 + m_2 s^4 + m_3 s^3 + m_4 s^2 + m_5 s + m_6$$

m_1	m_2	m_3	m_4	m_5
$+ e_1 d_1$	$+ e_1 d_2$	$+ e_1 d_3$	$+ e_1 d_4$	$+ e_1 d_5$
$+ e_2 d_2$	$+ e_2 d_3$	$+ e_2 d_4$	$+ e_2 d_5$	$+ e_2 d_6$
$+ e_3 d_3$	$+ e_3 d_4$	$+ e_3 d_5$	$+ e_3 d_6$	$+ a_1 e_1 d_4$
$+ e_4 d_4$	$+ e_4 d_5$	$+ e_4 d_6$	$+ a_1 e_1 d_3$	$+ a_1 e_2 d_5$
$+ e_5 d_5$	$+ e_5 d_6$	$+ a_1 e_1 d_2$	$+ a_1 e_2 d_4$	$+ a_2 e_1 d_3$
$+ e_6 d_6$	$+ a_1 e_1 d_1$	$+ a_1 e_2 d_3$	$+ a_1 e_3 d_5$	$+ a_2 e_2 d_4$
	$+ a_1 e_2 d_2$	$+ a_1 e_3 d_4$	$+ a_2 e_1 d_2$	$+ a_3 e_1 d_2$
m_6	$+ a_1 e_3 d_3$	$+ a_1 e_4 d_5$	$+ a_2 e_2 d_3$	$+ a_3 e_2 d_3$
$+ e_1 d_6$	$+ a_1 e_4 d_4$	$+ a_2 e_1 d_1$	$+ a_2 e_3 d_4$	$+ a_4 e_1 d_1$
$+ a_1 e_1 d_5$	$+ a_1 e_5 d_5$	$+ a_2 e_2 d_2$	$+ a_3 e_1 d_1$	$+ a_4 e_2 d_2$
$+ a_2 e_1 d_4$	$- a_6 e_6 d_1$	$+ a_2 e_3 d_3$	$+ a_3 e_2 d_2$	$- a_6 e_6 d_4$
$+ a_3 e_1 d_3$	$- a_5 e_6 d_2$	$+ a_2 e_4 d_4$	$+ a_3 e_3 d_3$	$- a_5 e_6 d_5$
$+ a_4 e_1 d_2$	$- a_4 e_6 d_3$	$- a_6 e_6 d_2$	$- a_6 e_6 d_3$	$- a_6 e_5 d_3$
$+ a_5 e_1 d_1$	$- a_3 e_6 d_4$	$- a_5 e_6 d_3$	$- a_5 e_6 d_4$	$- a_5 e_5 d_4$
$- a_6 e_6 d_5$	$- a_2 e_6 d_5$	$- a_4 e_6 d_4$	$- a_4 e_6 d_5$	$- a_6 e_4 d_2$
$- a_6 e_5 d_4$		$- a_3 e_6 d_5$	$- a_6 e_5 d_2$	$- a_5 e_4 d_3$
$- a_6 e_4 d_3$		$- a_6 e_5 d_1$	$- a_5 e_5 d_3$	$- a_6 e_3 d_1$
$- a_6 e_3 d_2$		$- a_5 e_5 d_2$	$- a_4 e_5 d_4$	$- a_5 e_3 d_2$
$- a_6 e_2 d_1$		$- a_4 e_5 d_3$	$- a_6 e_4 d_1$	
		$- a_3 e_5 d_4$	$- a_5 e_4 d_2$	
			$- a_4 e_4 d_3$	

(A.7.1)

A P P E N D I X   B

APPENDIX B

B.1 Description of System. In Section 1.1 the class of systems under consideration was limited to those having the state model description

$$\frac{d}{dt}\underline{X}(t) = \underline{A}\,\underline{X}(t) + \underline{B}\,r(t)$$
$$c(t) = \underline{X}(t), \qquad X(0) = \underline{X}_o \tag{B.1.1}$$

where the terms are defined in Equation 1.1.3. Appendix B provides an extension of the method to include systems having the state model description

$$\frac{d}{dt}\underline{X}(t) = \underline{A}\,\underline{X}(t) + \underline{B}\,r(t)$$
$$c(t) = \underline{T}\,\underline{X}(t) + k\,r(t), \quad \underline{X}(0) = \underline{X}_o \tag{B.1.2}$$

B.2 Procedure. In Equation B.1.2 make the definitions that

$$c_1(t) = \underline{T}\,\underline{X}(t) \tag{B.2.1}$$

$$c_2(t) = k\,r(t) \tag{B.2.2}$$

where k is a constant. Then it follows that

$$c(t) = c_1(t) + c_2(t) \tag{B.2.3}$$

At this point the system

$$\frac{d}{dt}\underline{X}(t) = \underline{A}\,\underline{X} + \underline{B}\,r(t)$$

$$c_1(t) = \underline{T}\,\underline{X}(t) \qquad (B.2.4)$$

can be analyzed by the general method or by the secondary method to obtain

$$G_1(s) = \frac{C_1(s)}{R(s)} \qquad (B.2.5)$$

where $C_1(s)$ is the Laplace transform of $c_1(t)$. Taking the Laplace transform of both sides of Equation B.2.2 and Equation B.2.3, it is seen that

$$G_2(s) = \frac{C_2(s)}{R(s)} = k \qquad (B.2.6)$$

and

$$C(s) = C_1(s) + C_2(s) \qquad (B.2.7)$$

Dividing Equation B.2.7 by R (s) and substituting Equations B.2.5 and B.2.6 into this equation, it is seen that

$$G(s) = \frac{C(s)}{R(s)} = G_1(s) + k \qquad (B.2.8)$$

which is the transfer function for the system described by Equation B.1.2.

In general $G_1(s) + k$ will be a ratio of polynomials in s where the degree of numerator is the same as the degree of the denominator.

Next, it is desired that the method be extended to include a canonical form for the state equations when the numerator of G (s) is of degree n, the same degree as the denominator. In this case perform a partial division of the numerator by the denominator

$$G(s) = k + \frac{M(s)}{A(s)} \tag{B.2.9}$$

where k is the coefficient of s^n in the numerator divided by the coefficient of s^n in the denominator, A (s) is the denominator polynomial of G (s), and M (s) is the resultant polynomial of degree n - 1 or less.

At this point a canonical state equation of

$$G_1(s) = \frac{M(s)}{A(s)} \tag{B.2.10}$$

can be formed in the usual way

$$\begin{aligned} \frac{d}{dt}\underline{Z}(t) &= \underline{F}\,\underline{Z}(t) + \underline{M}\,r(t) \\ c_1(t) &= z_n(t) \end{aligned} \tag{B.2.11}$$

Then it can be seen that the canonical form of the state equation is

$$\begin{aligned} \frac{d}{dt}\underline{Z}(t) &= \underline{F}\,\underline{Z}(t) + \underline{M}\,r(t) \\ c(t) &= z_n(t) + k\,r(t) \end{aligned} \tag{B.2.12}$$

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