ON THE FORMULATION OF STATE MODELS FOR SYSTEMS OF MULTITERMINAL COMPONENTS

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CHAPTER I

INTRODUCTION

1.1 Motivation. There are two major phases in the analysis of a large physical system. In the context employed here, the term physical system is intended to imply a collection of physical components that are interconnected in some meaningful manner. The two phases are often referred to as the "modeling" phase and the "solution" phase. The so-called modeling phase is concerned with the problem of establishing a suitable set of mathematical relationships which are presumed to completely specify the pertinent performance characteristics of the physical system. The solution phase is concerned with the study and analysis of the performance and response characteristics of the physical system through a study of a solution or a partial solution of the previously mentioned mathematical relationships. This thesis is concerned with the modeling phase of system analysis.

In the most primitive form there are two items of information available to the systems engineer when that individual starts to formulate the mathematical model for the system to be studied. These two items of information are often referred to as the primary mathematical model. The two items are the terminal characteristics of the constituent components of the system in some suitable mathematical form and the interconnection scheme.

Trent (1), Koenig and Reed (2), and later Koenig and Blackwell (3) have demonstrated that the mathematical discipline of linear graph theory and the associated generalized Kirchhoff current and voltage laws are tools well suited for the mathematical representation of the interconnection scheme. This tool is largely independent of the form in which the terminal characteristics of the system components are represented.

Now the manner in which one represents the terminal characteristics of the constituent component is dependent first of all upon the characteristics of the devices themselves. If the devices are linear, then one can often represent the terminal characteristics in the form of linear differential or difference equations or possibly as strictly algebraic equations which relate the terminal variables in some prescribed manner. If the devices are nonlinear, then, of course, one must resort to another sort of terminal representation. The manner in which one represents the terminal characteristics of the constituent components is also dependent upon the type of information that the engineer desires, i.e., it is dependent upon the wishes of the analyst also. Koenig and Blackwell (3) in their study of linear systems chose to use linear differential equations, as expressed in the complex-frequency domain, as their primary means of terminal representation.

Another prerogative of the systems engineer is the desired form of the resulting model that must be solved. Much of the time, it is necessary that one solve some set of simultaneous equations. These equations may be differential, algebraic, or both. In recent years the model used most often consisted of a set of simultaneous algebraic equations that result from using the Laplace transform theory to transform derivative and integral operations in the time domain into algebraic operations in the complex-frequency domain. Koenig and Blackwell (3) have presented a great deal of material about this type of mathematical model.

Until very recently, this seemed, from a practical standpoint, to be the only feasible way of handling a system of any extent. However, with the coming of age of large high-speed digital computers, numerical methods for solving sets of ordinary differential equations in the time domain became possible. When this occurred, the systems engineer started to look for more general mathematical models that might include cases where the characteristics of the constituent components are either time varying or possibly depend upon some other parametric variable. The engineer was also hopeful that the new model would be relatively easy to extend to the nonlinear case. It is this type of thinking that brought about the introduction of time-domain modeling and in particular the so-called state model. A state model is one which if it contains differential equations at all, then they are first order and are expressed in normal form.

It is the purpose of this work to present a means whereby this state model for a system containing multiterminal components can be formulated. The linear graph theory techniques mentioned above will be used to represent the interconnection scheme, and a special form of differential and algebraic equations will be introduced to represent the terminal characteristics of the constituent components.

1.2 Scope of the Study. The components considered will be limited to those whose terminal characteristics can be represented by linear ordinary differential equations and/or linear algebraic equations.

Time-varying coefficients in these equations will be allowed. No restriction is placed on the order of the differential equations and no restriction is placed on the number of terminals where a particular constituent component can be interconnected to the remainder of the system. Energy in all forms is admissible.

A set of necessary algebraic and topological conditions are stated and proved in Chapter III. These necessary conditions are then made a part of a set of sufficient conditions to insure that one can formulate the state model for a system of multiterminal components.

Two rather large examples are given to demonstrate the practical application of the theory.

CHAPTER II

FUNDAMENTAL DEFINITIONS AND CONCEPTS

2.1 Introduction. In this chapter some fundamental definitions relating to the "modeling" phase of system analysis are presented. In particular the definitions of the terms primary and secondary mathematical model are given. In the introductory chapter the term "state model" of a physical system was used. In this chapter that term is defined and discussed. The desired form of the time-domain terminal equations for a physical device are given. In the course of presenting these fundamental definitions, references to the literature are given in an effort to establish the present "state of the art" insofar as time-domain system modeling is concerned.

In any modeling scheme some means of representing the interconnection pattern of the components is necessary. Ordinarily, the result of this representation is a set of linear algebraic equations. In the sequel these algebraic equations are derived from an application of the theory and properties of oriented linear graphs as presented by Seshu and Reed (4) and Koenig and Blackwell (3). It is assumed that the reader is familiar with the works of these authors, particularly those which apply to systems of multiterminal components as envisioned by the latter pair of authors. As an aid to the reader who is not familiar with these works, an effort is made to provide a set of detailed references to the previously mentioned books in the form of footnotes. Definitions of the terms used and the properties exploited will be handled in this manner.

2.2 Some Fundamental Concepts of Modeling Theory. The theory of system modeling, as envisioned in this thesis, is concerned with the problem of deriving a set of mathematical relations which characterize a system. This set of relations is referred to as a mathematical model of the physical system. Since this thesis deals with mathematical models only, the term "mathematical model of the system" is often shortened to "model of the system." However, the classification as a mathematical model is always implied. It is normally derived from a knowledge of the terminal characteristics of the components which make up the system and the scheme by which these components are interconnected. As such, the model for a particular system may occur in several different forms. That is, models for systems are not unique. These models may be classified in a number of ways. In this thesis mathematical models will be referred to as primary or secondary in accordance with the following definitions:

Definition 2.2.1. Primary System Mathematical Model. A primary mathematical model of a physical system is one formed by adjoining a set of algebraic equations which describe the interconnection scheme to a set of algebraic and differential or difference equations which describe the terminal characteristics of the components involved. The act of adjoining the two sets of equations is intended to imply that all equations are to be considered in primitive form and that at this point no attempt has been made to reduce the number of equations which must be solved simultaneously.

In the case of a continuous-time linear electric circuit, one primary model consists of the set of first-order differential equations associated with the capacitors and inductors, the set of algebraic equations associated with the resistors, the set of specified functions associated with the drivers, and the set of algebraic equations derived from the application of Kirchhoff's current and voltage laws. Note that it is possible to prescribe the component equations associated with the capacitors, inductors, and drivers in terms of the Laplace transform variable s and thus derive another primary system model.

Definition 2.2.2. Secondary System Mathematical Model. A secondary mathematical model of a physical system is one that is derived from a primary model by means of linear mathematical operations. These linear operations are normally designed to reduce the number of simultaneous equations that must be solved in order to extract a unique solution for all system variables.

The two most classic examples of secondary models are those that electrical engineers most often use to solve linear electric circuits. These models are often referred to as the "mesh" current equation and "node pair" voltage equation models. Each is derived from a primary model for the circuit by means of linear operations. Both secondary models mentioned here can be derived either in the time domain in the form of second-order differential equations or in the complex-frequency domain, and each results in a smaller number of simultaneous equations to solve than the primary model exhibits. In recent years another time-domain secondary model for electric circuits has received considerable attention in the literature (5, 6, 7). It is referred to as the state model for the circuit and is characterized by the form of the differential equations involved. A more specific definition for the state model will be given in the next section.

In this thesis attention will be focused on the formulation of a particular secondary model for physical systems constructed of multiterminal components. The term multiterminal component refers to physical devices which satisfy the following definition:

Definition 2.2.3. Multiterminal Component. A multiterminal component is a physical device which has n points (called terminals) where it may be connected to other physical devices ($n \geq 2$). The phrase "n-terminal component" is sometimes used when it is desired to place in evidence the actual number of terminals present.

Definition 2.2.3 represents a formal statement of the concepts discussed by Koenig and Reed (2), Koenig and Blackwell (3), and more recently Koenig and Tokad (8).

Resistors, inductors, and capacitors are examples of two-terminal components. Vacuum tubes, transistors, transformers, motors, generators, amplifiers, and other similar devices are examples of electrical multiterminal devices with more than two terminals. Combinations of two or more devices such as those mentioned above may also be considered as multiterminal components. For example, filter networks that are constructed from resistors, inductors, and capacitors are multiterminal components. The concept of a multiterminal component is not restricted to the study of electrical phenomena. The ideas presented here are also applicable to the study of systems constructed from mechanical, hydraulic, and pneumatic components.

Every multiterminal component has a number of time variables associated with it. These variables can be classified several different ways. One of these classifications is dependent upon whether or not that particular variable is available for measurement.

Definition 2.2.4. Terminal Variables. The terminal variables of a multiterminal component are those oriented, time-varying quantities which are available for measurement at the terminals of the component. If a variable associated with a multiterminal component is not available for measurement, then it is referred to as a non-terminal variable. Terminal variables are always associated with pairs of terminals.

In the case of two-terminal electrical components (a capacitor, for example), the terminal variables are the voltage developed across the component and the current passing through the component. In the case of an n-terminal filter network, the terminal variables are the voltages that are developed between the various pairs of terminals and the currents which can be measured at each terminal. In the case of a two-terminal hydraulic device, one might consider the terminal pressure and flow rate as the terminal variables.

Terminal variables can be subdivided into two classes in accordance with the manner in which the variables combine when two or more multiterminal components are connected together.

Definition 2.2.5. Across Variable. A terminal variable is classified as an across variable if the sum of such variables about a closed loop or circuit is zero. The symbol x_i will be used to denote the across variable associated with the i-th pair of terminals.

Thus the voltage developed between two terminals of an electrical multiterminal component is an across variable. The displacement or relative velocity between two terminals of a mechanical device and the difference in pressure between two terminals of a hydraulic device are also across variables.

Definition 2.2.6. Through Variable. A terminal variable is classified as a through variable if the sum of such variables is zero at a junction or interface between two or more components. The symbol y_i will be used to denote the through variable associated with the i-th pair of terminals.

The current associated with any pair of terminals of an electrical multiterminal device qualifies as a through variable. The forces and torques of a mechanical device and the flow rates of a hydraulic system are through variables.

Definition 2.2.7. Terminal Graph. A terminal graph for an n-terminal component is an oriented linear graph containing n vertices (one associated with each terminal of the component) and (n - 1) directed line segments connecting the n vertices in such a way that no circuits are formed. Such a graph is sometimes referred to as a tree graph.

It is customary to associate one terminal across variable and one terminal through variable with each element of the terminal graph. These terminal variables are then written in vector notation and referred to as the across and through terminal vectors.

Definition 2.2.8. Across Terminal Vector. If x_i denotes the across variable associated with the i-th element of the terminal graph of the j-th multiterminal component, then the column vector

$$\underline{x}_j (t) = [x_1 (t), x_2 (t), \ldots, x_{(n - 1)} (t)]^T \qquad (2.2.1)$$

is referred to as the across terminal vector for that component. The through terminal vector is similarly defined and denoted as

$$\underline{y}_j (t) = [y_1 (t), y_2 (t), \ldots, y_{(n - 1)} (t)]^T . \qquad (2.2.2)$$

A few words about notation are in order at this point. Throughout this thesis, underscored lower-case Greek and English letters denote vectors. Matrices are denoted by underscored upper-case Greek and English letters. Unless it is stated to the contrary, the elements of these vectors and matrices are continuous functions of the independent variable t. The elements of the vectors and matrices are denoted by lower-case letters (not underscored) that are subscripted in the usual manner to indicate the position of the element in the array. If it is desired to emphasize the value of the elements at a particular time, a notation such as $\underline{u}$ (t) and $\underline{A}$ (t) is used, otherwise symbols such as $\underline{u}$ and $\underline{A}$ are used. Vectors are regarded as special cases of column matrices. A superscript T is used to denote the transpose of a vector or matrix. The term "k-vector" is used to denote a column vector having k elements.

In general, the terminal characteristics of an n-terminal component are governed by the component itself and not by the manner in which it is connected to other multiterminal components. In fact, Koenig and Reed (2) and later Koenig and Blackwell (3) have demonstrated that the terminal characteristics of an n-terminal component are completely specified by a terminal graph containing (n - 1) elements and an (n - 1)-ordered vector function relating the across and through terminal vectors of the component. Hence, the primary terminal model for a multiterminal component is defined as follows:

Definition 2.2.9. Primary Terminal Model. The primary terminal model for the j-th n-terminal component consists of an (n - 1) element terminal graph and an (n - 1)-ordered vector function

$$\underline{g}_j \left(\underline{x}_j (t), \underline{y}_j (t), t\right) = 0 \qquad (2.2.3)$$

relating the terminal across and through variables. Equation 2.2.3 is often referred to as the terminal relations or terminal equations. These equations can take on various forms with the form of the equation being dependent upon the class of device considered. In this thesis, it is assumed that the only physical devices considered are those for which Equation 2.2.3 takes the form of a set of linear ordinary differential and/or algebraic equations in the variables $\underline{x}_j (t)$ and $\underline{y}_j (t)$ with t being the independent variable. Multiterminal components satisfying this last restriction are said to be linear.

There are a few n-terminal components whose terminal characteristics are such that it is possible to completely specify those characteristics by means of terminal graphs which have fewer than (n - 1) elements. These cases arise when electromagnetic coupling or when energy in two forms (electrical and mechanical, for example) is involved. Electric motors, generators, and transformers are examples of such devices. It is customary to represent such devices by means of a two-part terminal graph in which each part is a tree. The coupling effect is accounted for by the inclusion of terms in the terminal equations which cause the terminal variables associated with one part of the terminal graph to depend upon the terminal variables of the other part. Such an n-terminal component will be referred to as a coupled n-terminal component.

Definition 2.2.10. Coupled n-Terminal Component. A coupled n-terminal component is one whose terminal graph contains more than one part (each part is a tree) and whose terminal equations contain terms in which the terminal variables associated with one part of the terminal graph depend upon the terminal variables associated with another part of the graph.

2.3 State Model for a System of Multiterminal Components. This thesis is concerned with the derivation of a particular secondary mathematical model for a system of multiterminal components. The desired model is defined as follows:

Definition 2.3.1. System State Model. The state model for a system of linear multiterminal components whose system graph[1] contains e elements and n vertices is a set of q first-order linear differential equations and 2e linear algebraic equations of the form

$$\frac{d}{dt}\underline{\lambda} = \underline{P}_o\,\underline{\lambda} + \underline{R}_o\,\underline{f} \qquad (2.3.1)$$

$$\underline{Z} = \underline{C}_o\,\underline{\lambda} + \underline{E}_o\,\underline{f} \qquad (2.3.2)$$

where

[1] Herman E. Koenig and William A. Blackwell, Electromechanical System Theory (New York, 1961), p. 42.

1. The q-vector $\underline{\lambda}$ appearing in Equations 2.3.1 and 2.3.2 is referred to as the state vector of the system. The elements of $\underline{\lambda}$ are continuous functions of the independent variable t and are called the state variables of the system and may or may not be terminal variables of one of the system components.
2. The 2e-vector $\underline{Z}$ contains all of the terminal variables of the components which make up the system. $\underline{Z}$ includes both across and through class terminal variables.
3. The k-vector $\underline{f}$ contains the k specified functions of the independent variable t which account for the driver elements in the system. The elements of $\underline{f}$ are assumed to be defined and continuous over the set $T = \{t : t_1 < t < t_2\}$ for some fixed t_1 and t_2.
4. The matrices $\underline{P}_\rho$, $\underline{R}_\rho$, $\underline{C}_\rho$, and $\underline{E}_\rho$ are assumed to have conformable dimensions. It is also assumed that the elements of these matrices are defined and continuous functions of t over the set $T = \{t : t_1 < t < t_2\}$.

There appears to be four reasons for working in the time domain as compared to working in the traditional complex-frequency domain.

If only linear cases are considered, then time-domain models such as the one given in Definition 2.3.1 have the advantage of being applicable to time-varying systems as well as constant-parameter systems. Mathematical models which make use of the Laplace transform complex-frequency variable are also applicable to such systems, but they do not yield simultaneous linear algebraic equations in the variable s which are easily solved. In fact, any advantage offered by the Laplace transform techniques vanishes when the coefficients which describe the system are allowed to vary with time. This is not the case with time-domain models such as the one given in Definition 2.3.1.

In the case of nonlinear systems the complex-frequency models are not applicable while the transition from the linear case to the nonlinear case in the time domain appears to be relatively easy to accomplish.

Another reason for working in the time domain is somewhat philosophical in nature. There appears to be some conceptual value in working in the time domain. This is true since it is in the time domain where one must compare the actual response of the physical system to the response or solution of the model. The intermediate step of transforming into the complex-frequency domain is sometimes confusing and may, in some instances, be the source of errors and misconceptions that are not encountered in a strictly time-domain analysis of a system.

A fourth reason for wishing to work in the time domain has to do with the present control system optimization techniques. Considerable emphasis has been given to this problem in the literature (9, 10, 11), and all of the present techniques are dependent upon the availability of a time-domain model of the system dynamics in a form analogous to that given in Definition 2.3.1.

If it is assumed that a time-domain model is desirable, then one is naturally led to select a form that contains no differential equations of higher order than one. This is the result of a number of items. For example, a considerable amount of information concerning the existence and uniqueness of solutions for a set of simultaneous first-order differential equations is available. The necessary and sufficient conditions to insure a unique solution are well known for equations of that class (12, 13, 14). Still another reason for using a time-domain model that includes first-order differential equations has to do with the present emphasis on the use of high-speed digital computers in the analysis of large systems. Mathematical models of the form given in Definition 2.3.1 are well suited for solution on a computer.

There appears to be two fundamentally different approaches to the problem of formulating the state model for a system if one is given some primary model. One of these involves an intermediate secondary model (not a state model) which can then be reduced to a state model. The intermediate secondary model will normally contain differential equations whose order is greater than one. For example, in the case of linear electric circuit theory one might first formulate a "mesh" current model. Such a model is made up of a set of simultaneous second-order differential equations and a set of linear algebraic equations. The set of second-order differential equations can then be reduced to a set of first-order differential equations by defining a suitable set of auxiliary variables. The resulting equations can be placed in the form of Equations 2.3.1 and 2.3.2. Another example of this approach, as applied to linear control systems, ignores output loading and initial condition effects. In this case the transfer functions of the various components of the control system are manipulated until a system transfer function (the intermediate secondary model) in the complex-frequency domain is derived. This transfer function is then transformed into a suitable time-domain state model by means of some analog computer programming techniques. Tou (11) gives several examples illustrating this technique.

The second fundamental approach does not involve an intermediate secondary model. The state model is formulated directly from a knowledge of some primary model for the system. It is this approach to the problem that will be explored in this thesis.

A number of authors (5, 6, 7) have investigated this approach to the problem for systems of two-terminal electrical components (resistors, conductors, capacitors, and ideal voltage and current sources). However, only Koenig and Tokad (8) have studied the problem for systems of multiterminal components. These authors considered a rather restricted class of multiterminal components and attempted to specify only the necessary topological conditions to insure that a state model of the form of Definition 2.3.1 could be formulated. In this thesis the necessary topological and algebraic restrictions for a larger class of multiterminal components are stated and proved. Sufficient conditions are also considered, and a formulation procedure is developed.

In order to formulate the state model for a system of multiterminal components, it is desirable that the terminal equations for the multiterminal components be in a special form. Attention will now be devoted to defining this special form. Several procedures for deriving this special form are demonstrated in a later chapter.

2.4 State Equations for Multiterminal Components. All of the results of this thesis are based upon the following definition:

Definition 2.4.1. Let the j-th linear multiterminal component of a system have n_j terminals or points where external connections and measurements can be made. It is assumed that the terminal characteristics of the component are entirely specified by a set of p_j first-order linear differential equations and m_j linear algebraic equations in the form given in Equations 2.4.1 and 2.4.2. For coupled multiterminal components in the sense of Definition 2.2.10, m_j is equal to $n_j - n_p$ for an n_p part terminal graph, otherwise m_j is equal to $n_j - 1$.

$$\frac{d}{dt}\underline{\psi}_j = \underline{P}_j\,\underline{\psi}_j + \underline{Q}_j\,\underline{u}_j + \underline{R}_j\,\underline{f}_j \qquad (2.4.1)$$

$$\underline{w}_j = \underline{C}_j\,\underline{\psi}_j + \underline{D}_j\,\underline{u}_j + \underline{E}_j\,\underline{f}_j \qquad (2.4.2)$$

The symbols in Equations 2.4.1 and 2.4.2 are defined as follows:

1. $\underline{\psi}_j$ is a p_j-vector referred to as the <u>state vector</u> for the component. The elements of $\underline{\psi}_j$ may or may not be terminal variables. The component is said to have order p_j.
2. $\underline{u}_j$ is an m_j-vector of terminal variables. It may contain both across and through class terminal variables. In the sequel $\underline{u}_j$ will sometimes be referred to as the <u>generalized input vector</u>.
3. $\underline{w}_j$ is an m_j-vector of terminal variables. It is the complement of $\underline{u}_j$ in the sense that if the across variable associated with a particular element of the terminal graph belongs to $\underline{u}_j$, then the through variable associated with that element belongs to $\underline{w}_j$. A similar statement can be made with the terms across and through interchanged. The two vectors are sometimes said to be complementary vectors. The vector $\underline{w}_j$ is sometimes referred to as the <u>generalized output vector</u> for the component.
4. $\underline{f}_j$ is the k_j-vector of <u>specified functions</u> which account for internal sources within the multiterminal component. The elements of $\underline{f}_j$ are all assumed to be continuous functions of t

over the set $T = \{t : t_1 < t < t_2\}$ for some fixed t_1 and t_2.

5. The matrices $\underline{P}_j$, $\underline{Q}_j$, $\underline{R}_j$, $\underline{C}_j$, $\underline{D}_j$, and $\underline{E}_j$ are assumed to have conformable dimensions. The elements of these matrices are also assumed to be continuous functions of t over the set $T = \{t : t_1 < t < t_2\}$ for some fixed t_1 and t_2.

Equations 2.4.1 and 2.4.2 are referred to as the state equations for the j-th linear multiterminal component of a system.

State equations of the form defined above were first proposed by Koenig and Tokad (8) in their paper concerning the necessary topology of a system of linear multiterminal components. However, those authors placed further restrictions on the rank of $\underline{C}_j$ and the number of rows of identically zero elements in $\underline{D}_j$. In this thesis it is shown that these last restrictions are not necessary and, in fact, severely restrict the class of multiterminal components which can be handled.

There are a number of reasons for choosing the form given in Equations 2.4.1 and 2.4.2 for the terminal relations when one is trying to formulate a state model for a system of multiterminal components. For example, the form appears to be compatible for use with the fundamental interconnection equations.

A second reason for choosing such a form has to do with the idea of giving the term "state of the system or component" some physical significance of its own and not just some secondary significance that is derived from another basic viewpoint. In this author's search of the modern literature on the subject, only Zadeh (15) and Zadeh and Desoer (16) have attempted to define the term "state of the system" for continuous time systems. All other authors are content to derive, by some suitable means, a set of first-order differential equations in normal form[2] and then refer to the vector of variables that appear in differential form as the "state of the system" (7, 11, 17). It is a simple matter to show that the vector $\underline{\psi}_j$ appearing in Equations 2.4.1 and 2.4.2 satisfies the definition of state given by Zadeh and Desoer when one considers $\underline{w}_j$ as a generalized output vector and $\underline{u}_j$ as a generalized input vector. Furthermore, Equation 2.4.1 is a modification of the normal form mentioned above. The modification involves leaving $\underline{u}_j$, the generalized input vector, in an unspecified form.

It might be noted at this point that Equations 2.4.1 and 2.4.2 and the associated terminal graph can be considered as generalized time-domain Thevenin or Norton equivalent representations if $\underline{w}_j$ and $\underline{u}_j$ were suitably restricted. Suppose $\underline{w}_j$ contains only across-type variables, then $\underline{u}_j$ will contain only through class terminal variables. In this case the state equations are time-domain analogies of the Thevenin equivalent terminal equations often considered in complex-frequency domain analysis. If the classification of variables contained in the output and input vectors is reversed, then the Norton equivalent terminal relations result. A hybrid version of these equivalent circuits results when the variables in each vector are mixed. This idea of Thevenin and Norton equivalence is quite useful in the analysis of large-scale systems that are constructed of two-terminal components. This usefulness lies in the concept of dividing the larger system into several subsystems and then finding the generalized equivalent (Thevenin, Norton,

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[2] L. S. Pontryagin, Ordinary Differential Equations, tr. L. Kacinskas (Reading, Massachusetts, 1962), p. 19.

or hybrid as needed) model for each subsystem. The larger system is then studied by treating the generalized equivalent models as the multiterminal components of the system.

A number of techniques that are useful in obtaining state equations for multiterminal components in the form of Equations 2.4.1 and 2.4.2 will be demonstrated in a later chapter.

CHAPTER III

SOME ALGEBRAIC AND TOPOLOGICAL CONSIDERATIONS

3.1 Introduction. This chapter is devoted to the study of some of the algebraic properties of the primary mathematical model of a system of multiterminal components. Certain topological properties of the system graph are also considered. As a preliminary step, a particular primary mathematical model is derived and established as the starting point for further work. The primary model selected as the starting point is one that is derived from a knowledge of the component terminal equations in the time domain (in the form of the equations of Definition 2.4.1) and the interconnection scheme. A set of algebraic conditions that are necessary for a unique solution for all variables to exist is stated and proved in the form of a theorem. This set of necessary algebraic conditions is shown to imply that the system graph must have certain topological properties. A set of rather general sufficient algebraic conditions are also stated and proved in the form of another theorem.

3.2 Necessary Algebraic and Topological Conditions. A system containing k multiterminal components will now be considered. The terminal equations for each component are assumed to have the form of Equations 2.4.1 and 2.4.2. Suppose that $\underline{P}$ represents the direct sum[1] of all the matrices $\underline{P}_j$ for j = 1, 2, ..., k. Let $\underline{Q}$, $\underline{R}$, $\underline{C}$, $\underline{D}$, and $\underline{E}$ be similarly defined with respect to the matrices $\underline{Q}_j$, $\underline{R}_j$, $\underline{C}_j$, $\underline{D}_j$, and $\underline{E}_j$ for j = 1, 2, ..., k. Also let the vector $\underline{\psi}$ be defined as

$$\underline{\psi}^T = [\underline{\psi}_1{}^T \; \underline{\psi}_2{}^T \; \ldots \; \underline{\psi}_k{}^T] \; . \tag{3.2.1}$$

Let $\underline{u}$, $\underline{w}$, and $\underline{f}$ be similarly defined with respect to $\underline{u}_j$, $\underline{w}_j$, and $\underline{f}_j$ for j = 1, 2, ..., k. Now, the terminal equations for all k components can be written in matrix notation as

$$\frac{d}{dt}\underline{\psi} = \underline{P}\,\underline{\psi} + \underline{Q}\,\underline{u} + \underline{R}\,\underline{f} \tag{3.2.2}$$

$$\underline{w} = \underline{C}\,\underline{\psi} + \underline{D}\,\underline{u} + \underline{E}\,\underline{f} \; . \tag{3.2.3}$$

Note that $\underline{w}$ and $\underline{u}$ are each of order e where $e = \sum_{j=1}^{k} m_j$ is the number of elements in the system graph. Also note that if $p = \sum_{j=1}^{k} p_j$, then the vector $\underline{\psi}$ is of order p.

The vectors $\underline{u}$ and $\underline{w}$ may each contain both across and through terminal variables in mixed order. Suppose $\underline{u}$ contains $r \leq e$ through variables, then $\underline{w}$ contains the r corresponding across variables. This is due to the construction of $\underline{u}$ and $\underline{w}$. Let $\underline{u}^{(1)}$ be a simple row rearrangement of $\underline{u}$ such that the r through variables belonging to $\underline{u}^{(1)}$ occur in the first r positions. Let $\underline{w}^{(1)}$ be the complement of $\underline{u}^{(1)}$ in the sense that if the i-th element of $\underline{u}^{(1)}$ is the through variable associated with the j-th element of the system graph, then the i-th element of $\underline{w}^{(1)}$ is the across variable associated with the j-th element

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[1] Edward T. Browne, Introduction to the Theory of Determinants and Matrices (Chapel Hill, North Carolina, 1958), p. 183.

of the system graph. A similar statement must hold if the i-th element of $\underline{u}^{(1)}$ is the across variable associated with the j-th element of the system graph. Thus

$$\underline{u} = \underline{\Gamma}_u^{(1)} \underline{u}^{(1)} \tag{3.2.4}$$

where

$$\underline{u}^{(1)} = \begin{bmatrix} \underline{y}_u^{(1)} \\ \\ \underline{x}_u^{(1)} \end{bmatrix} \tag{3.2.5}$$

and $\underline{\Gamma}_u^{(1)}$ is an e x e nonsingular transformation that results in the desired row rearrangement. The symbol $\underline{y}_u^{(1)}$ denotes the vector of r through variables belonging to $\underline{u}^{(1)}$, and $\underline{x}_u^{(1)}$ denotes the vector of e - r across variables belonging to $\underline{u}^{(1)}$. Similarly, if $\underline{\Gamma}_w^{(1)}$ is an e x e nonsingular transformation, then

$$\underline{w} = \underline{\Gamma}_w^{(1)} \underline{w}^{(1)} \tag{3.2.6}$$

where

$$\underline{w}^{(1)} = \begin{bmatrix} \underline{x}_w^{(1)} \\ \\ \underline{y}_w^{(1)} \end{bmatrix} \tag{3.2.7}$$

and where $\underline{x}_w^{(1)}$ denotes the vector of r across variables belonging to $\underline{w}^{(1)}$, and $\underline{y}_w^{(1)}$ denotes the e - r through variables belonging to $\underline{w}^{(1)}$. Note that $\underline{x}_w^{(1)}$ is the complement of $\underline{y}_u^{(1)}$, and $\underline{y}_w^{(1)}$ is the complement of $\underline{x}_u^{(1)}$.

Substitution of Equations 3.2.4 and 3.2.6 into Equations 3.2.2 and 3.2.3 yields

$$\frac{d}{dt}\underline{\psi} = \underline{P}\,\underline{\psi} + \underline{Q}^{(1)}\,\underline{u}^{(1)} + \underline{R}\,\underline{f} \qquad (3.2.8)$$

$$\underline{w}^{(1)} = \underline{C}^{(1)}\,\underline{\psi} + \underline{D}^{(1)}\,\underline{u}^{(1)} + \underline{E}^{(1)}\,\underline{f} \qquad (3.2.9)$$

where

$$\underline{Q}^{(1)} = \underline{Q}\,\underline{\Gamma}_{u}^{(1)} \, , \qquad (3.2.10)$$

$$\underline{C}^{(1)} = \left[\underline{\Gamma}_{w}^{(1)}\right]^{-1} \underline{C} \, , \qquad (3.2.11)$$

$$\underline{D}^{(1)} = \left[\underline{\Gamma}_{w}^{(1)}\right]^{-1} \underline{D}\,\underline{\Gamma}_{u}^{(1)} \, , \qquad (3.2.12)$$

and

$$\underline{E}^{(1)} = \left[\underline{\Gamma}_{w}^{(1)}\right]^{-1} \underline{E} \, . \qquad (3.2.13)$$

The linear algebraic equations which describe the interconnection scheme are derived from the system graph by applying the concepts of Seshu and Reed (4). Two sets of algebraic equations result. These are the fundamental cutset[2] and circuit[3] equations. The fundamental cutset equations consist of $v - n_p$ linearly independent algebraic equations in the through variables of the system. The symbol v denotes the number of vertices (points where two or more components are interconnected) of the system graph, and the symbol n_p denotes the number of separate parts[4] in the system graph. These cutset equations can be written in

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[2] Sundaram Seshu and Myril B. Reed, Linear Graphs and Electrical Networks (Reading, Massachusetts, 1961), p. 97.

[3] Ibid., p. 91.

[4] H. E. Koenig and W. A. Blackwell, Electromechanical System Theory (New York, 1961), p. 51.

matrix notation as

$$\underline{A}\,\underline{y} = [\underline{A}_w\ \underline{A}_u] \begin{bmatrix} \underline{y}^{(1)}_w \\ \underline{y}^{(1)}_u \end{bmatrix} = 0\ . \qquad (3.2.14)$$

The fundamental circuit equations consist of e - (v - n_p) linearly independent algebraic equations in the across variables of the system. These equations can be written in matrix notation as

$$\underline{B}\,\underline{x} = [\underline{B}_w\ \underline{B}_u] \begin{bmatrix} \underline{x}^{(1)}_w \\ \underline{x}^{(1)}_u \end{bmatrix} = 0\ . \qquad (3.2.15)$$

By properly partioning Equations 3.2.8 and 3.2.9 and adjoining Equations 3.2.14 and 3.2.15 to the result of that partioning operation, the primary mathematical model of the system is derived. It can be written

$$p\ \{ \qquad \frac{d}{dt}\underline{\psi} = \underline{P}\,\underline{\psi} + [\underline{Q}^{(1)}_{11}\ \underline{Q}^{(1)}_{12}] \begin{bmatrix} \underline{y}^{(1)}_u \\ \underline{x}^{(1)}_u \end{bmatrix} + \underline{R}\,\underline{f} \qquad (3.2.16)$$

$$\begin{array}{r} v - n_p\ \{ \\ e - (v - n_p)\ \{ \\ r\ \{ \\ e - r\ \{ \end{array} \begin{bmatrix} \underline{0} & \underline{A}_u & \underline{0} & \underline{0} & \underline{A}_w \\ \underline{0} & \underline{0} & \underline{B}_u & \underline{B}_w & \underline{0} \\ -\underline{C}^{(1)}_{11} & -\underline{D}^{(1)}_{11} & -\underline{D}^{(1)}_{12} & \underline{U}_r & \underline{0} \\ -\underline{C}^{(1)}_{21} & -\underline{D}^{(1)}_{21} & -\underline{D}^{(1)}_{22} & \underline{0} & \underline{U}_{e-r} \end{bmatrix} \begin{bmatrix} \underline{\psi} \\ \underline{y}^{(1)}_u \\ \underline{x}^{(1)}_u \\ \underline{x}^{(1)}_w \\ \underline{y}^{(1)}_w \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{E}^{(1)}_{11} \\ \underline{E}^{(1)}_{21} \end{bmatrix} \underline{f}\ . \qquad (3.2.17)$$

The symbol $\underline{U}_\ell$ denotes a $\ell \times \ell$ identity matrix. An examination of these equations will show that this primary model consists of p first-order linear ordinary differential equations in e + p variables and 2e linear algebraic equations in 2e + p variables. The two sets of equations have e + p variables in common and, therefore, must be solved simultaneously.

The question that is now considered is as follows: Under what conditions does the primary mathematical model given by Equations 3.2.16 and 3.2.17 possess a unique solution? That question is partially answered, insofar as necessary algebraic conditions are concerned, by the following theorem:

Theorem 3.2.1. The primary mathematical model given by Equations 3.2.16 and 3.2.17 has a unique solution for all variables involved (both state and terminal variables) over some subset T_1 of T only if the coefficient matrix on the left of Equation 3.2.17 has rank 2e for every t belonging to the set T_1. The set $T = \{t : t_1 < t < t_2\}$ is an interval over which all of the elements of the coefficient matrices and the specified functions are defined and continuous.

Proof: Suppose that a unique solution for every variable exists. That is, for every set of initial conditions that can be imposed on the independent state variables, there exists a unique representation of all variables (both terminal and state variables) in terms of the specified function vector $\underline{f}$ and those initial conditions. Furthermore, suppose that the rank of the coefficient matrix on the left of Equation 3.2.17 is not 2e. Clearly, the rank cannot be greater than 2e. Therefore, let the rank be 2e - ℓ where ℓ is an integer such that $0 < \ell \leq 2e$. Two cases must be examined. The set of algebraic equations may be either consistent or inconsistent.[5] If the algebraic equations are inconsistent, then a complete[6] solution of those equations does not exist. This implies that a unique solution of the entire set of algebraic and differential equations does not exist. Thus, either a unique solution does not exist or the rank of the coefficient matrix is not less than 2e. If the equations are consistent, then the 2e algebraic equations can be solved for the $2e - \ell$ variables associated with the $2e - \ell$ linearly independent columns of the coefficient matrix in terms of the remaining $p + \ell$ variables and the specified vector function $\underline{f}$. This result, when considered in conjunction with the assumption that a unique solution for all variables exists, implies that the $p + \ell$ unspecified variables appearing in the solution of the algebraic equations can be determined in terms of the specified vector function $\underline{f}$ and the previously mentioned initial conditions by simply solving the p linear first-order differential equations. This is clearly a contradiction, since such a set of ordinary differential equations can be uniquely solved for at most p variables. Since both cases lead to results which contradict the assumption that the rank is not 2e, it is concluded that if a unique solution for all variables exists, then the rank of the coefficient matrix on the left of Equation 3.2.17 must be 2e.

The results of Theorem 3.2.1 can be used to derive some necessary topological properties that the system graph must possess. Suppose that the entries in the vector of variables appearing in Equation 3.2.17

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[5] Fran. E. Hohn, Elementary Matrix Algebra (New York, 1958), p. 111.

[6] Ibid., p. 112.

are rearranged in a manner such that the equation becomes

$$\begin{bmatrix} \underline{0} & \underline{A} & \underline{0} \\ \underline{0} & \underline{0} & \underline{B} \\ \underline{C}^{(1)} & \underline{Z}^{(1)} & \underline{W}^{(1)} \end{bmatrix} \begin{bmatrix} \underline{\psi} \\ \underline{y} \\ \underline{x} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{E}^{(1)} \end{bmatrix} \underline{f} \qquad (3.2.18)$$

where

$$\underline{C}^{(1)} = \begin{bmatrix} -\underline{C}_{11}^{(1)} \\ -\underline{C}_{21}^{(1)} \end{bmatrix}, \qquad (3.2.19)$$

$$\underline{Z}^{(1)} = \begin{bmatrix} -\underline{D}_{11}^{(1)} & \underline{0} \\ -\underline{D}_{21}^{(1)} & \underline{U}_{e-r} \end{bmatrix}, \qquad (3.2.20)$$

$$\underline{W}^{(1)} = \begin{bmatrix} -\underline{D}_{12}^{(1)} & \underline{U}_{r} \\ -\underline{D}_{22}^{(1)} & \underline{0} \end{bmatrix}, \qquad (3.2.21)$$

and $\underline{A}$, $\underline{B}$, $\underline{x}$, and $\underline{y}$ are defined by Equations 3.2.14 and 3.2.15.

Theorem 3.2.2. The primary mathematical model of a system of multiterminal components has a unique solution for all variables involved (both state and terminal variables) only if there exists at least one set of e linearly independent columns in the e x (2e + p) matrix

$$\underline{\Phi} = [\underline{C}^{(1)}\ \underline{Z}^{(1)}\ \underline{W}^{(1)}] \qquad (3.2.22)$$

such that the linearly independent columns taken from $\underline{Z}^{(1)}$ correspond to a subset of the chord set of some tree T_a of the system graph and those taken from $\underline{W}^{(1)}$ correspond to a subset of the branches of some tree T_d of the system graph. The trees T_a and T_d are not necessarily identical. It is understood that the properties stated in this theorem must hold for all t belonging to the set $T = \{t : t_1 < t < t_2\}$ where t_1 and t_2 are the end points where <u>all</u> the terms appearing in the coefficient matrices of the component equations and the specified function vectors are defined and continuous.

Proof: The proof is given for a system whose graph contains one part. The extension to cases where more than one separate part is involved is simply a matter of notation. Theorem 3.2.1 states that it is necessary (for a unique solution for all variables to exist) that there exist at least one 2e x 2e nonsingular submatrix in the coefficient matrix on the left-hand side of Equation 3.2.18. Let $\underline{N}_k$ denote the k-th such submatrix. Consider the Laplace expansion for the determinant of $\underline{N}_k$ about the first e rows. This expansion takes the form

$$\det \underline{N}_k = \sum_{j=1}^{m} (\pm 1) (\det \underline{G}_{kj}) (\det \underline{H}_{kj}) \qquad (3.2.23)$$

where $\underline{G}_{kj}$ is the j-th square minor matrix[7] that can be formed in the first e rows of $\underline{N}_k$. There are

$$m = \frac{2e!}{e!\ e!} \qquad (3.2.24)$$

[7] Edward T. Browne, <u>Introduction to the Theory of Determinants and Matrices</u> (Chapel Hill, North Carolina, 1958), p. 21.

such minor matrices. $\underline{H}_{kj}$ is the e x e complementary minor matrix[8] of $\underline{G}_{kj}$ in $\underline{N}_k$. The sign of each term in the sum is dependent upon the particular columns that are selected for inclusion in $\underline{G}_{kj}$ and is not important to this argument. By hypothesis, $\underline{N}_k$ is nonsingular. This implies that for at least one value of j in Equation 3.2.23

$$\det \underline{G}_{kj} \cdot \det \underline{H}_{kj} \neq 0 . \tag{3.2.25}$$

Equation 3.2.25 leads to the conclusion that for at least one value of j

$$\det \underline{G}_{kj} \neq 0 \tag{3.2.26}$$

and

$$\det \underline{H}_{kj} \neq 0 \tag{3.2.27}$$

The implications of Equation 3.2.26 will now be studied. Clearly, if $\underline{G}_{kj}$ contains any columns that do not belong to either $[\underline{A}^T \; 0]^T$ or $[0 \; \underline{B}^T]^T$, then $\underline{G}_{kj}$ is singular and need not be considered. Therefore, every $\underline{G}_{kj}$ that satisfies Equation 3.2.26 has the form

$$\underline{G}_{kj} = \begin{bmatrix} \underline{A}_{kj,(c)} & \underline{0} \\ \underline{0} & \underline{B}_{kj,[e-c]} \end{bmatrix} \tag{3.2.28}$$

where $\underline{A}_{kj,(c)}$ is a (v - 1) x c submatrix of $\underline{A}$, and $\underline{B}_{kj,[e-c]}$ is an [e - (v - 1)] x [e - c] submatrix of $\underline{B}$. Consider the assertion that c

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[8] Ibid.

is exactly equal to v - 1 where v is the number of vertices in the system graph. Also consider the companion assertion that the columns of $\underline{A}_{kj,(c)}$ correspond to the branches of some tree T_a of the system graph and that the columns of $\underline{B}_{kj,[e-c]}$ correspond to the elements of the chord set of some tree T_d of the system graph. Suppose c is less than v - 1, then e - c is greater than e - (v - 1). This result implies that $\underline{G}_{kj}$ contains more than e - (v - 1) distinct columns taken from $[\underline{0}\ \underline{B}^T]^T$. But since $\underline{B}$ is a fundamental circuit matrix, it has rank[9] e - (v - 1); and it follows that $\underline{G}_{kj}$ is singular and need not be considered if c > v - 1. Similarly, suppose c < (v - 1), then $\underline{G}_{kj}$ contains more than v - 1 columns taken from $[\underline{A}^T\ \underline{0}]^T$. But $\underline{A}$ is a fundamental cutset matrix and, therefore, has rank[10] v - 1. It follows that $\underline{G}_{kj}$ is singular if c < v - 1. Hence, if $\underline{G}_{kj}$ is nonsingular as required by Equation 3.2.26, then c is exactly equal to v - 1. It is now concluded that for every $\underline{G}_{kj}$ which satisfies Equation 3.2.26

$$\det \underline{G}_{kj} = \det \underline{A}_{kj,(v-1)} \cdot \det \underline{B}_{kj,[e-(v-1)]} \neq 0 . \tag{3.2.29}$$

This results in the conclusion that

$$\det \underline{A}_{kj,(v-1)} \neq 0 \tag{3.2.30}$$

and

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[9] S. Seshu and M. B. Reed, Linear Graphs and Electrical Networks (Reading, Massachusetts, 1961), p. 68.

[10] Ibid., p. 74.

$$\det \underline{B}_{kj,[e-(v-1)]} \neq 0 . \qquad (3.2.30)$$

A square submatrix of a fundamental cutset matrix is nonsingular if, and only if, the columns of the submatrix correspond to the elements of some tree T_a of the system graph.[11] Similarly, a square submatrix of a fundamental circuit matrix is nonsingular if, and only if, the columns of the submatrix correspond to the elements of the chord set of some tree T_d of the system graph.[12] Hence, it is concluded that $\underline{G}_{kj}$ is nonsingular (as required by Equation 3.2.26 for some j) only if the columns of $\underline{G}_{kj}$ taken from $[\underline{A}^T \; 0]^T$ correspond to the elements of some tree T_a, and the columns of $\underline{G}_{kj}$ taken from $[\underline{0} \; \underline{B}^T]^T$ correspond to a chord set of some tree T_d belonging to the system graph.

The matrix $\underline{H}_{kj}$ will now be examined in the light of the above results. First, it is noted that $\underline{H}_{kj}$ is a square e x e submatrix of $\underline{\Phi}$. It is also noted that by definition $\underline{H}_{kj}$ is the complementary minor matrix of $\underline{G}_{kj}$ in $\underline{N}_k$. Thus, if Equation 3.2.26 and Equation 3.2.27 are both satisfied, then $\underline{H}_{kj}$ must contain e linearly independent columns, and any of the columns of $\underline{H}_{kj}$ which are taken from $\underline{Z}^{(1)}$ must correspond to a subset of the chord set of the tree T_a, and any columns of $\underline{H}_{kj}$ taken from $\underline{W}^{(1)}$ must correspond to a subset of the branches of some tree T_d. This completes the proof of Theorem 3.2.2.

Koenig and Tokad (8) have stated and proved a theorem similar to the one appearing above. However, they considered only those

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[11] Ibid., p. 69.

[12] Ibid.

multiterminal components which possess the property that the number of rows of zeros in $\underline{D}_j$ is greater than or equal to the rank of $\underline{C}_j$. The theorem as stated and proved here includes a larger and more general class of multiterminal components.

A more useful form of Theorem 3.2.2 would result if one could state the necessary conditions in terms of the component equations of the system. Such a statement is possible as shown below. Let p_1, p_2, n_{b1}, and n_{c1} be four integers which satisfy the following equations;

$$p_1 \leq p \tag{3.2.31}$$

$$p_1 + p_2 = p \tag{3.2.32}$$

$$n_{b1} \leq v - 1 \tag{3.2.33}$$

$$n_{c1} \leq e - (v - 1) \tag{3.2.34}$$

$$p_1 + n_{b1} + n_{c1} = 2e \tag{3.2.35}$$

The symbols p, v, and e have the same significance as in the proofs of Theorems 3.2.1 and 3.2.2. Then Theorem 3.2.2 can be restated as follows:

Theorem 3.2.3. (Alternate form of Theorem 3.2.2). The primary mathematical model of a system of multiterminal components has a unique solution for all variables involved only if there exists two trees T_a and T_d (not necessarily distinct trees) belonging to the system graph such that the algebraic component equations can be written as follows for all t belonging to $T = \{t : t_1 < t < t_2\}$.

$$\begin{bmatrix} \underline{\psi}_1 \\ \underline{x}_{1,bd} \\ \underline{y}_{1,ca} \end{bmatrix} = \begin{bmatrix} \underline{H}_{11} & \underline{H}_{12} & \underline{H}_{13} & \underline{H}_{14} & \underline{H}_{15} \\ \underline{H}_{21} & \underline{H}_{22} & \underline{H}_{23} & \underline{H}_{24} & \underline{H}_{25} \\ \underline{H}_{31} & \underline{H}_{32} & \underline{H}_{33} & \underline{H}_{34} & \underline{H}_{35} \end{bmatrix} \begin{bmatrix} \underline{\psi}_2 \\ \underline{x}_{2,bd} \\ \underline{x}_{cd} \\ \underline{y}_{2,ca} \\ \underline{y}_{ba} \end{bmatrix} + \begin{bmatrix} \underline{E}_\psi \\ \underline{E}_d \\ \underline{E}_a \end{bmatrix} \underline{f} \qquad (3.2.36)$$

The various elements of Equation 3.2.36 are defined as follows:

1. The vector $\underline{\psi}_1$ is a p_1-vector containing a subset of the state variables of the system. $\underline{\psi}_2$ is the p_2 ordered complement of $\underline{\psi}_1$ in $\underline{\psi}$.
2. $\underline{x}_{1,bd}$ is an n_{b1}-vector of the across variables associated with the branches of the tree T_d. $\underline{x}_{2,bd}$ is a $(v - 1 - n_{b1})$-vector containing the remaining across variables associated with the tree T_d.
3. $\underline{y}_{1,ca}$ is an n_{c1}-vector of the through variables associated with a subset of the chord set of the tree T_a. $\underline{y}_{2,ca}$ is the $(e - v + 1 - n_{c1})$-vector containing the remaining through variables associated with the chord set of T_a.
4. $\underline{x}_{cd}$ is the $[e - (v - 1)]$-vector of across variables associated with the chord set of the tree T_d.
5. $\underline{y}_{ba}$ is the $(v - 1)$-vector of through variables associated with the branches of the tree T_a.
6. The elements of the coefficient matrices in Equation 3.2.36 are defined and continuous functions of the independent variable t over the set $T = \{t : t_1 < t < t_2\}$ where t_1 and t_2 are the end

points of an interval over which all of the elements in the component coefficient matrices and specified functions are defined and continuous functions over the set $T = \{t : t_1 < t < t_2\}$.

3.3 Sufficient Algebraic Conditions. For the sake of completeness, the following theorem concerning sufficient algebraic conditions is stated and proved:

Theorem 3.3.1. The primary mathematical model of a system containing k multiterminal components has a unique solution for all variables involved if the algebraic equations belonging to the primary model can be solved for the system terminal vectors $\underline{x}$ and $\underline{y}$ in terms of the specified function vector $\underline{f}$ and the system state vector $\underline{\psi}$. That is, if Equation 3.2.17 can be written

$$\begin{bmatrix} \underline{y}_u^{(1)} \\ \underline{x}_u^{(1)} \\ \underline{x}_w^{(1)} \\ \underline{y}_w^{(1)} \end{bmatrix} = \begin{bmatrix} \underline{N}_{11} \\ \underline{N}_{21} \\ \underline{N}_{31} \\ \underline{N}_{41} \end{bmatrix} \underline{\psi} + \begin{bmatrix} \underline{M}_{11} \\ \underline{M}_{21} \\ \underline{M}_{31} \\ \underline{M}_{41} \end{bmatrix} \underline{f} , \qquad (3.3.1)$$

then the primary mathematical model has a unique solution for all variables. It is noted that the elements of the coefficient matrices in Equation 3.3.1 are defined and continuous functions of the independent variable t over the set $T = \{t : t_1 < t < t_2\}$ where t_1 and t_2 have the same significance as in the statement of Theorem 3.2.1. The elements of $\underline{f}$ are also defined and continuous over the set T.

Proof: Let the first two row blocks of Equation 3.3.1 be substituted into Equation 3.2.16. This results in a set of p first-order differential equations of the form

$$\frac{d}{dt}\underline{\psi} = \underline{P}\,\underline{\psi} + [\underline{Q}_{11}^{(1)}\,\underline{N}_{11} + \underline{Q}_{12}^{(1)}\,\underline{N}_{21}]\,\underline{\psi} + [\underline{Q}_{11}^{(1)}\,\underline{M}_{11} + \underline{Q}_{12}^{(1)}\,\underline{M}_{21}]\,\underline{f} + \underline{R}\,\underline{f}\,. \tag{3.3.2}$$

It is noted that Equations 3.3.1 and 3.3.2 form the system state model as prescribed by Definition 2.3.1. There is a well-known theorem[13] in the theory of ordinary differential equations which states that a set of equations of the form of Equation 3.3.2 possesses a unique solution for the vector $\underline{\psi}$ for every set of initial conditions $\underline{\psi}(t_o)$ and for every vector of specified functions $\underline{f}$ which satisfy the hypothesis of this theorem. On substituting this unique solution for $\underline{\psi}$ into Equation 3.3.1, one obtains a unique solution for every terminal variable of the system. This completes the proof of Theorem 3.3.1.

Insofar as a formulation scheme is concerned, Theorem 3.3.1 contributes very little, in a practical sense, in that it requires that one be able to invert a 2e x 2e matrix. In any but the simplest cases the calculation of such a matrix inverse is no easy task. Also, it is a simple matter to present a counter example showing that the hypotheses of this theorem are not necessary. For example, consider any electrical network containing at least one circuit of capacitors. Such a network possesses a unique solution for certain initial conditions, yet its

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[13] W. Kaplan, Ordinary Differential Equations (Reading, Massachusetts, 1958), p. 494.

component and interconnection equations will not satisfy the hypotheses of Theorem 3.3.1.

CHAPTER IV

STATE MODEL FORMULATION PROCEDURES

4.1 Introduction. The necessary and sufficient algebraic and topological conditions developed and discussed in the preceding chapter do not provide a practical means for formulating the state model for a system of multiterminal components. It is the purpose of this chapter to present a formulation procedure that can, in a practical sense, be applied to large physical systems. A number of special cases will be considered. Before proceeding to the development of this procedure, it will be necessary to consider several special forms for the component terminal equations.

4.2 Some Special Forms for the Component State Equations. The first special form of the terminal equations to be considered is that associated with the class of physical devices for which one of the terminal variables (across or through) is a specified function of the independent variable t. The specified terminal variable is assumed to be independent of its complementary terminal variable. Such a device is referred to as an across or through ideal driver, depending upon which terminal variable is the specified function of t. Only 2-terminal ideal drivers or ideal drivers which can be represented as an interconnection of 2-terminal ideal drivers will be considered in this thesis. The state equations (see Equations 2.4.1 and 2.4.2) for the j-th such ideal driver reduce to the form

$$w_j = f_j \qquad (4.2.1)$$

where the variable w_j is a single terminal variable (either through or across), and f_j is a single specified function of the independent variable t. It is noted that the state vector associated with an ideal driver is an identically zero vector and, therefore, the first-order differential equations which are associated with the general form of the state equations for a multiterminal component vanish. In the sequel, a system containing k_d 2-terminal ideal drivers will be considered. The direct sum of the k_d terminal relations will be written

$$\underline{w}_o = \underline{f}_o \qquad (4.2.2)$$

where $\underline{w}_o$ is a k_d-vector of terminal variables, and $\underline{f}_o$ is a k_d-vector of specified functions. Equation 4.2.2 will be referred to as the "ideal-driver terminal equations" in the developments that follow in this chapter. The symbol n_x will be used to denote the number of across variables appearing in $\underline{w}_o$, and the symbol n_y will be used in a similar manner to denote the number of through variables belonging to $\underline{w}_o$.

The second special form for the component equations can be derived from Equations 2.4.1 and 2.4.2. For easy reference, these equations are

$$\frac{d}{dt} \underline{\psi}_j = \underline{P}_j \, \underline{\psi}_j + \underline{Q}_j \, \underline{u}_j + \underline{R}_j \, \underline{f}_j \qquad (2.4.1)$$

and

$$\underline{w}_j = \underline{C}_j \, \underline{\psi}_j + \underline{D}_j \, \underline{u}_j + \underline{E}_j \, \underline{f}_j \quad . \qquad (2.4.2)$$

The form to be derived is applicable to all classes of multiterminal components except the ideal drivers. This special form is derived by taking advantage of the fact that some of the state variables of an n-terminal component may also be generalized output variables (or linear combinations of the generalized output variables). In that case, some of the algebraic equations belonging to the combined set of algebraic and differential equations may be eliminated. This elimination procedure is described below. In the process of developing this elimination procedure, it will be necessary to perform several sets of elementary operations on the matrices and vectors appearing in Equation 2.4.2. Each time that a set of operations is performed, the resulting matrix or vector will be denoted by the same algebraic symbol as the matrix or vector on which the operations were performed, except that a superscript will be added to denote how many sets of elementary operations have been performed at that point in the derivation. For example, the symbol $\underline{P}_1^{(i)}$ represents the result of i sets of elementary operations on the matrix $\underline{P}_1$.

Consider the matrix $\underline{D}_j$ that appears in Equation 2.4.2. $\underline{D}_j$ will, in general, contain d_j ($d_j \geq 0$) rows whose elements are identically zero. One can, by means of elementary row interchanges, rearrange the terminal algebraic equations so as to place those zero rows in the first d_j positions of $\underline{D}_j$. This operation constitutes the first elementary transformation to be performed on the matrices and vectors of Equation 2.4.2. Since it is desirable that the generalized input vector, $\underline{u}_j^{(k)}$, be the complement of the generalized output vector, $\underline{w}_j^{(k)}$, at every point during the derivation, it is assumed that every operation also includes a rearrangement of elements of $\underline{u}_j^{(k-1)}$ and the columns of

$\underline{D}_j^{(k-1)}$ so as to insure that $\underline{u}_i^{(k)}$ and $\underline{w}_i^{(k)}$ are complementary vectors for every possible i and k. The term complementary, when used in this context, implies that if the m-th entry in $\underline{u}_i^{(k)}$ is the across variable associated with the ℓ-th element in the component terminal graph, then the m-th entry of $\underline{w}_i^{(k)}$ is the through variable associated with the ℓ-th element of the terminal graph and conversely. At any rate, the j-th multiterminal component's algebraic terminal equations can be written, after this first set of row and column interchanges, as shown in Equation 4.2.3.

$$\begin{bmatrix} \underline{w}_1^{(1)} \\ \underline{w}_2^{(1)} \end{bmatrix}_j = \begin{bmatrix} \underline{C}_{11}^{(1)} \\ \underline{C}_{21}^{(1)} \end{bmatrix}_j \underline{\psi}_j + \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{D}_{21}^{(1)} & \underline{D}_{22}^{(1)} \end{bmatrix}_j \begin{bmatrix} \underline{u}_1^{(1)} \\ \underline{u}_2^{(1)} \end{bmatrix}_j + \begin{bmatrix} \underline{E}_{11}^{(1)} \\ \underline{E}_{21}^{(1)} \end{bmatrix}_j [\underline{f}]_j \; . \tag{4.2.3}$$

Note that in the above equation $\underline{w}_1^{(1)}$ is a d_j-vector of terminal variables (both through and across variables are included) and that $\underline{w}_2^{(1)}$ is an $(m_j - d_j)$-vector of terminal variables. As before, m_j is the number of elements belonging to the terminal graph of the j-th multiterminal component.

Suppose that the $d_j \times p_j$ matrix $\underline{C}_{11}^{(1)}$ has rank r_j where

$$r_j \leq \min (d_j, p_j) \; . \tag{4.2.4}$$

If this is the case, then one can, without any loss in generality, rearrange the rows of the first row block of Equation 4.2.3 so as to place r_j linearly independent rows in the first r_j positions of that matrix. Let $[\underline{\Gamma}_\psi^{(1)}]_j$ denote the $m_j \times m_j$ nonsingular linear transformation matrix which accomplishes this row interchange for the j-th multiterminal component. Then, there exists a $p_j \times p_j$ nonsingular transformation $[\underline{\Lambda}_{\psi}^{(1)}]_j$ such that if

$$\underline{C}_j^{(1)} = \begin{bmatrix} \underline{C}_{11}^{(1)} \\ \underline{C}_{21}^{(1)} \end{bmatrix}_j , \qquad (4.2.5)$$

then

$$[\underline{\Gamma}_{\psi}^{(1)}]_j \, \underline{C}_j^{(1)} \, [\underline{\Lambda}_{\psi}^{(1)}]_j = \underline{C}_j^{(2)} = \begin{bmatrix} \underline{U}_{r_j} & \underline{0} \\ \underline{C}_{21}^{(2)} & \underline{C}_{22}^{(2)} \end{bmatrix}_j . \qquad (4.2.6)$$

Let

$$[\underline{\psi}]_j = [\underline{\Lambda}_{\psi}^{(1)}]_j \, [\underline{\psi}^{(1)}]_j . \qquad (4.2.7)$$

Now, suppose that Equation 4.2.3 is premultiplied by $[\underline{\Gamma}_{\psi}^{(1)}]_j$ and Equation 4.2.7 is substituted into the result. These operations plus any operations which are necessary to insure that the input and output vectors remain complementary constitute the second set of linear operations to be performed on the algebraic terminal equations of the j-th multiterminal component. The resulting equation is

$$\begin{bmatrix} \underline{w}_1^{(2)} \\ \underline{w}_2^{(2)} \end{bmatrix}_j = \begin{bmatrix} \underline{U}_{r_j} & \underline{0} \\ \underline{C}_{21}^{(2)} & \underline{C}_{22}^{(2)} \end{bmatrix}_j \begin{bmatrix} \underline{\psi}_1^{(1)} \\ \underline{\psi}_2^{(1)} \end{bmatrix}_j + \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{D}_{21}^{(2)} & \underline{D}_{22}^{(2)} \end{bmatrix}_j \begin{bmatrix} \underline{u}_1^{(2)} \\ \underline{u}_2^{(2)} \end{bmatrix}_j + \begin{bmatrix} \underline{E}_{11}^{(2)} \\ \underline{E}_{21}^{(2)} \end{bmatrix}_j [\underline{f}]_j .$$

(4.2.8)

The first row block of Equation 4.2.8 can now be solved for $\underline{\psi}_1^{(1)}$. The resulting equation yields an expression for r_j state variables in terms of r_j terminal variables and the specified function vector $[\underline{f}]_j$.

$$[\underline{\psi}_1^{(1)}]_j = [\underline{w}_1^{(2)}]_j - [\underline{E}_{11}^{(2)}]_j\ [\underline{f}]_j \ . \qquad (4.2.9)$$

It is assumed that all the elements of the matrices and vectors in Equation 4.2.9 possess continuous first derivatives with respect to t over the set $T = \{t : t_1 < t < t_2\}$ where t_1 and t_2 are the fixed end points of an interval over which all of the elements of the vectors and matrices appearing in Equations 2.4.1 and 2.4.2 are defined and continuous.

Now let Equation 4.2.7 be substituted into a partitioned form of Equation 2.4.1. The result of this substitution can be written in partioned form as shown in Equation 4.2.10.

$$\frac{d}{dt}\begin{bmatrix} \underline{\psi}_1^{(1)} \\ \underline{\psi}_2^{(1)} \end{bmatrix}_j = \begin{bmatrix} \underline{P}_{11}^{(1)} & \underline{P}_{12}^{(1)} \\ \underline{P}_{21}^{(1)} & \underline{P}_{22}^{(1)} \end{bmatrix}_j \begin{bmatrix} \underline{\psi}_1^{(1)} \\ \underline{\psi}_2^{(1)} \end{bmatrix}_j$$

$$+ \begin{bmatrix} \underline{Q}_{11}^{(1)} & \underline{Q}_{12}^{(1)} \\ \underline{Q}_{21}^{(1)} & \underline{Q}_{22}^{(1)} \end{bmatrix}_j \begin{bmatrix} \underline{u}_1^{(2)} \\ \underline{u}_1^{(2)} \end{bmatrix}_j + \begin{bmatrix} \underline{R}_{11}^{(1)} \\ \underline{R}_{12}^{(1)} \end{bmatrix}_j [\underline{f}]_j \qquad (4.2.10)$$

On substituting Equation 4.2.9 into Equation 4.2.10, one obtains

$$\frac{d}{dt}\begin{bmatrix} \underline{w}_1^{(2)} \\ \underline{\psi}_2^{(1)} \end{bmatrix}_j = \begin{bmatrix} \underline{P}_{11}^{(1)} & \underline{P}_{12}^{(1)} \\ \underline{P}_{21}^{(1)} & \underline{P}_{22}^{(1)} \end{bmatrix}_j \begin{bmatrix} \underline{w}_1^{(2)} \\ \underline{\psi}_2^{(1)} \end{bmatrix}_j$$

$$+ \begin{bmatrix} \underline{Q}_{11}^{(1)} & \underline{Q}_{12}^{(1)} \\ \underline{Q}_{21}^{(1)} & \underline{Q}_{22}^{(1)} \end{bmatrix}_j \begin{bmatrix} \underline{u}_1^{(2)} \\ \underline{u}_2^{(2)} \end{bmatrix}_j + \begin{bmatrix} \underline{T}_{11} \\ \underline{T}_{21} \end{bmatrix}_j [\underline{g}]_j \qquad (4.2.11)$$

where $[\underline{T}_{11}]_j$, $[\underline{T}_{21}]_j$, and $\underline{g}_j$ account for the effects of the $[-\underline{E}_{11}^{(2)}]_j\ [\underline{f}]_j$ term in Equation 4.2.9. At any rate, $\underline{T}_{11}$ and $\underline{T}_{21}$ are matrices of defined and continuous functions of t over the set T and $[\underline{g}]_j$ is a vector of defined and continuous functions over the set T. The elements of the other coefficient matrices in Equation 4.2.11 are similarly defined.

One can now write the component terminal equations for the j-th multiterminal component by considering the lower row block of Equation 4.2.8 in conjunction with Equation 4.2.11. Since it is this form of the terminal equations that will be employed in the sequel, the superscript notation will be dropped, and hereafter it is assumed that all component terminal equations (except those associated with ideal drivers) are in the reduced form derived above. These equations take the form of

$$\frac{d}{dt}\begin{bmatrix} \underline{w}_1 \\ \underline{\psi}_2 \end{bmatrix}_j = \begin{bmatrix} \underline{P}_{11} & \underline{P}_{12} \\ \underline{P}_{21} & \underline{P}_{22} \end{bmatrix}_j \begin{bmatrix} \underline{w}_1 \\ \underline{\psi}_2 \end{bmatrix}_j + \begin{bmatrix} \underline{Q}_{11} & \underline{Q}_{12} \\ \underline{Q}_{21} & \underline{Q}_{22} \end{bmatrix}_j \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix}_j + \begin{bmatrix} \underline{T}_{11} \\ \underline{T}_{12} \end{bmatrix}_j [\underline{g}]_j \qquad (4.2.12)$$

and

$$[\underline{w}_2]_j = [\underline{C}_{21} \quad \underline{C}_{22}]_j \begin{bmatrix} \underline{w}_1 \\ \underline{\psi}_2 \end{bmatrix}_j + [\underline{D}_{21} \quad \underline{D}_{22}]_j \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix}_j + [E_{21}]_j [\underline{f}]_j \tag{4.2.13}$$

In the next section, a system containing k_d ideal drivers and k components whose terminal equations are of the form of Equations 4.2.12 and 4.2.13 will be studied. Each of the k components whose terminal equations take the form of these latter equations is assumed to have n_j terminals and m_j elements in its terminal graph. Thus for the j-th such component, $[\underline{w}_2]_j$ contains $(m_j - r_j)$ entries. For such a system, the complete set of system component equations can be written

$$\underline{w}_o = \underline{f}_o \quad , \tag{4.2.14}$$

$$\frac{d}{dt}\begin{bmatrix} \underline{w}_1 \\ \underline{\psi}_2 \end{bmatrix} = \begin{bmatrix} \underline{P}_{11} & \underline{P}_{12} \\ \underline{P}_{21} & \underline{P}_{22} \end{bmatrix} \begin{bmatrix} \underline{w}_1 \\ \underline{\psi}_2 \end{bmatrix} + \begin{bmatrix} \underline{Q}_{11} & \underline{Q}_{12} \\ \underline{Q}_{21} & \underline{Q}_{22} \end{bmatrix} \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix} + \begin{bmatrix} \underline{T}_{11} \\ \underline{T}_{21} \end{bmatrix} \underline{g} \quad , \tag{4.2.15}$$

and

$$\underline{w}_2 = [\underline{C}_{21} \quad \underline{C}_{22}] \begin{bmatrix} \underline{w}_1 \\ \underline{\psi}_2 \end{bmatrix} + [\underline{D}_{21} \quad \underline{D}_{22}] \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix} + \underline{E}_{21} \, \underline{f} \; . \tag{4.2.16}$$

The matrices and vectors in Equations 4.2.15 and 4.2.16 are the direct sums, over j = 1, 2, ..., k, of the similarly denoted matrices in

Equations 4.2.12 and 4.2.13. The entire set of equations, i.e., Equations 4.2.14, 4.2.15, and 4.2.16 will often be referred to as the "direct sum of the component state equations in reduced form" in the remainder of this thesis. Note that this set of equations contains

$$p = \sum_{j=1}^{k} p_j \tag{4.2.17}$$

linear first-order differential equations and

$$k_a = \sum_{j=1}^{k} (m_j - r_j) \tag{4.2.18}$$

linear algebraic equations. It is also noted that the system graph contains

$$e = k_d + \sum_{j=1}^{k} m_j \tag{4.2.19}$$

elements.

<u>4.3 A Formulation Procedure.</u> This section will be devoted to the presentation of a formulation procedure that can be used to derive the state model of a system of multiterminal components. The formulation procedure represents an adaptation to the problem at hand of Wirth's work (18) with nonlinear devices. It will be necessary to place several additional restrictions on the form of the system component equations. These restrictions will be introduced in the form of formal assumptions as needed in the development. The results of this derivation will be summarized in the form of a list of itemized steps near the end of this section.

Throughout this section a system containing k_o multiterminal components will be considered. The system graph, G, will be assumed to have e elements and v vertices. The component equations for the system will be assumed to take the general form of Equations 4.2.14, 4.2.15, and 4.2.16, subject to the exceptions given below.

Before listing the restrictions on the component equations, the technique to be used to select the formulation tree will be presented. Consider the following subgraphs of the system graph G.

1. Let S_1 denote the subgraph containing all the elements of G which correspond to across variables belonging to $\underline{w}_o$.
2. Let S_2 denote the subgraph containing all the elements of G which correspond to across variables belonging to $\underline{w}_1$.
3. Let S_3 denote the subgraph containing all the elements of G which correspond to entries in $\underline{w}_2$.
4. Let S_4 denote the subgraph containing all the elements of G corresponding to through variables that belong to $\underline{w}_1$.
5. Let S_5 denote the subgraph containing all the elements of G corresponding to the through variables that belong to $\underline{w}_o$.

The sets S_1, S_2, ..., S_5 merely classify the elements of the system graph in accordance with the form of the associated terminal equations. In the case of simple electrical networks, one accomplishes an analogous classification when one considers the elements of the system graph that are associated with the capacitors, ideal voltage drivers, resistors, inductors, and ideal current drivers as disjoint subgraphs of the system graph. Now consider the subgraphs G_1, G_2, ..., G_5 of the system graph G that are defined by Equations 4.3.1 through 4.3.5. The symbol $\cup$ denotes set-theoretic union.

$$G_1 = S_1 \qquad (4.3.1)$$

$$G_2 = S_1 \cup S_2 \qquad (4.3.2)$$

$$G_3 = S_1 \cup S_2 \cup S_3 \qquad (4.3.3)$$

$$G_4 = S_1 \cup S_2 \cup S_3 \cup S_4 \qquad (4.3.4)$$

$$G_5 = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 = G \qquad (4.3.5)$$

Define T_i to be a tree of G_i such that T_i is contained in T_{i+1}. It follows that $T_G = T_5$ is a tree of G. T_G is the desired formulation tree. Clearly, one can always select the formulation tree in this manner if the following assumption prevails:

Assumption 4.3.1. It is assumed that there are no complete circuits of G in S_1 and no complete cutsets of G in S_5.

Assumption 4.3.2. The algebraic component equations are assumed to be independent of $\underline{u}_1$. That is

$$\underline{D}_{21} \equiv \underline{0} \qquad (4.3.6)$$

in Equation 4.2.16.

Assumption 4.3.3. It is assumed that if the i-th element of G corresponds to a linear algebraic equation of the set identified as Equation 4.2.16 (as modified to incorporate Assumption 4.3.2) and if that element belongs to the tree T_G, then Equation 4.2.16 is explicitly solvable for the across variable associated with the i-th element of T_G. Similarly, if the i-th element belongs to the complement of T_G in G, then Equation 4.2.16 must be solvable for the through variable associated with the i-th element of G.

It follows from Theorem 3.2.2 and Theorem 3.2.3 that Assumptions 4.3.1 and 4.3.3 are, in a practical sense, necessary conditions. Clearly, if one wrote the direct sum over j = 1, 2, ..., k of all equations of the form of Equation 4.2.9 and considered the result in conjunction with Equation 4.2.14 and Equation 4.2.16 as modified to satisfy Assumptions 4.3.2 and 4.3.3, then one would have a set of equations whose form is the same as Equation 3.2.36 as required by Theorem 3.2.3.

Assumption 4.3.4. It is assumed that there is no ideal branch-to-chord or chord-to-branch coupling in the $\underline{D}_{22}$ matrix appearing on the right of Equation 4.2.16 as modified to satisfy Assumptions 4.3.2 and 4.3.3.

Assumption 4.3.5. It is also assumed that ideal branch-to-chord or chord-to-branch coupling does not exist among the elements whose terminal characteristics are described by the first-order differential equations (Equation 4.2.15). Branch-to-chord and chord-to-branch coupling of an algebraic element to a differential element may appear in the differential equations only.

In light of Assumptions 4.3.2 through 4.3.5, the system component equations can now be written as follows:

$$\begin{bmatrix} \underline{x}_{1b} \\ \underline{y}_{6c} \end{bmatrix} = \begin{bmatrix} \underline{x}_1\ (t) \\ \underline{y}_5\ (t) \end{bmatrix} , \qquad (4.3.7)$$

$$\frac{d}{dt}\begin{bmatrix} \underline{x}_{2b} \\ \underline{x}_{2c} \\ \underline{y}_{4b} \\ \underline{y}_{4c} \\ \underline{\psi}_2 \end{bmatrix} = \begin{bmatrix} \underline{P}_{11} & \underline{P}_{12} & \underline{P}_{13} & \underline{P}_{14} & \underline{P}_{15} \\ \underline{P}_{21} & \underline{P}_{22} & \underline{P}_{23} & \underline{P}_{24} & \underline{P}_{25} \\ \underline{P}_{31} & \underline{P}_{32} & \underline{P}_{33} & \underline{P}_{34} & \underline{P}_{35} \\ \underline{P}_{41} & \underline{P}_{42} & \underline{P}_{43} & \underline{P}_{44} & \underline{P}_{45} \\ \underline{P}_{51} & \underline{P}_{52} & \underline{P}_{53} & \underline{P}_{54} & \underline{P}_{55} \end{bmatrix} \begin{bmatrix} \underline{x}_{2b} \\ \underline{x}_{2c} \\ \underline{y}_{4b} \\ \underline{y}_{4c} \\ \underline{\psi}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} \underline{Q}_{11} & \underline{0} & \underline{Q}_{13} & \underline{Q}_{14} & \underline{0} & \underline{0} \\ \underline{0} & \underline{Q}_{22} & \underline{Q}_{23} & \underline{Q}_{24} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{Q}_{33} & \underline{Q}_{34} & \underline{Q}_{35} & \underline{0} \\ \underline{0} & \underline{0} & \underline{Q}_{43} & \underline{Q}_{44} & \underline{0} & \underline{Q}_{46} \\ \underline{Q}_{51} & \underline{Q}_{52} & \underline{Q}_{53} & \underline{Q}_{54} & \underline{Q}_{55} & \underline{Q}_{56} \end{bmatrix} \begin{bmatrix} \underline{y}_{2b} \\ \underline{y}_{2c} \\ \underline{y}_{3b} \\ \underline{x}_{3c} \\ \underline{x}_{4b} \\ \underline{x}_{4c} \end{bmatrix} + \begin{bmatrix} \underline{g}_{2b} \\ \underline{g}_{2c} \\ \underline{g}_{4b} \\ \underline{g}_{4c} \\ \underline{g}_{2} \end{bmatrix}, \tag{4.3.8}$$

and

$$\begin{bmatrix} \underline{x}_{3b} \\ \underline{y}_{3c} \end{bmatrix} = \begin{bmatrix} \underline{C}_{11} & \underline{C}_{12} & \underline{C}_{13} & \underline{C}_{14} & \underline{C}_{15} \\ \underline{C}_{21} & \underline{C}_{22} & \underline{C}_{23} & \underline{C}_{24} & \underline{C}_{25} \end{bmatrix} \begin{bmatrix} \underline{x}_{2b} \\ \underline{x}_{2c} \\ \underline{y}_{4b} \\ \underline{y}_{4c} \\ \underline{\psi}_2 \end{bmatrix} + \begin{bmatrix} \underline{D}_{11} & \underline{0} \\ \underline{0} & \underline{D}_{22} \end{bmatrix} \begin{bmatrix} \underline{y}_{3b} \\ \underline{x}_{3c} \end{bmatrix} + \begin{bmatrix} \underline{f}_{3b} \\ \underline{f}_{3c} \end{bmatrix}. \tag{4.3.9}$$

It should be noted that the coefficient matrices in Equations 4.3.8 and 4.3.9 are, in general, rearrangements of the coefficient matrices in Equations 4.2.15 and 4.2.16. Also it should be noted that a finer partioning of the coefficient matrices is displayed in Equations 4.3.8

and 4.3.9. The numerical subscripts appearing in the vectors of variables in Equations 4.3.7 through 4.3.9 correspond to the subscripts assigned to the subgraphs S_1, S_2, S_3, S_4, and S_5. The "b" and "c" subscripts denote branch and chord elements respectively.

The symbol, n_{2b}, will be used to denote the number of graph elements belonging to S_2 which are also branches of the tree T_G. Similarly, the symbol, n_{2c}, will be used to denote the number of graph elements belonging to S_2 which are chords of the tree T_G. The symbols, n_{4b}, n_{4c}, n_{3b}, and n_{3c}, are similarly defined. The symbol, n_2, will be used to denote the number of entries in $\underline{\psi}_2$.

Assumption 4.3.6. It is assumed that the matrices $\underline{Q}_{11}$, $\underline{Q}_{22}$, $\underline{Q}_{35}$, and $\underline{Q}_{46}$ appearing in Equation 4.3.8 and the matrices $\underline{D}_{11}$ and $\underline{D}_{22}$ appearing in Equation 4.3.9 are positive definite for all t belonging to the set $T = \{t : t_1 < t < t_2\}$ for some fixed t_1 and t_2. The reader is referred to Definition A.1 for a definition of the term positive definite as used here.

Assumption 4.3.7. The elements of the vectors $\underline{x}_1$ (t) and $\underline{y}_5$ (t) that appear in Equation 4.3.7 are assumed to have continuous and defined first derivatives with respect to t over the set T.

The fundamental circuit equation associated with the tree T_G can be written in partitioned form

$$
\begin{bmatrix}
\underline{B}_{11} & \underline{B}_{12} & \underline{0} & \underline{0} & \underline{U}_{n_{2c}} & \underline{0} & \underline{0} & \underline{0} \\
\underline{B}_{21} & \underline{B}_{22} & \underline{B}_{23} & \underline{0} & \underline{0} & \underline{U}_{n_{3c}} & \underline{0} & \underline{0} \\
\underline{B}_{31} & \underline{B}_{32} & \underline{B}_{33} & \underline{B}_{34} & \underline{0} & \underline{0} & \underline{U}_{n_{4c}} & \underline{0} \\
\underline{B}_{41} & \underline{B}_{42} & \underline{B}_{43} & \underline{B}_{44} & \underline{0} & \underline{0} & \underline{0} & \underline{U}_{n_y}
\end{bmatrix}
\begin{bmatrix}
\underline{x}_{1b} \\ \underline{x}_{2b} \\ \underline{x}_{3b} \\ \underline{x}_{4b} \\ \underline{x}_{2c} \\ \underline{x}_{3c} \\ \underline{x}_{4c} \\ \underline{x}_{5c}
\end{bmatrix}
= \underline{0}
\qquad (4.3.10)
$$

Similarly, the fundamental cutset equations can be written in partitioned form

$$
\begin{bmatrix}
\underline{U}_{n_x} & \underline{0} & \underline{0} & \underline{0} & \underline{A}_{11} & \underline{A}_{12} & \underline{A}_{13} & \underline{A}_{14} \\
\underline{0} & \underline{U}_{n_{2b}} & \underline{0} & \underline{0} & \underline{A}_{21} & \underline{A}_{22} & \underline{A}_{23} & \underline{A}_{24} \\
\underline{0} & \underline{0} & \underline{U}_{n_{3b}} & \underline{0} & \underline{0} & \underline{A}_{32} & \underline{A}_{33} & \underline{A}_{34} \\
\underline{0} & \underline{0} & \underline{0} & \underline{U}_{n_{4b}} & \underline{0} & \underline{0} & \underline{A}_{43} & \underline{A}_{44}
\end{bmatrix}
\begin{bmatrix}
\underline{y}_{1b} \\ \underline{y}_{2b} \\ \underline{y}_{3b} \\ \underline{y}_{4b} \\ \underline{y}_{2c} \\ \underline{y}_{3c} \\ \underline{y}_{4c} \\ \underline{y}_{5c}
\end{bmatrix}
= \underline{0}
\qquad (4.3.11)
$$

The zeros appearing in the first four columns of Equation 4.3.10 and in the last four columns of Equation 4.3.11 are due to the manner in which the formulation tree T_G was selected. For example, consider the first row block of Equations 4.3.10. The fundamental circuit equations appearing in this row block are defined by elements of the subgraph, S_2. Since the tree T_G was selected so as to maximize the number of elements belonging to the set-theoretic intersection of T_G and S_2, any fundamental circuits defined by an element of S_2 can contain only elements which belong to the subgraph, G_2. Similarly, fundamental circuits defined by elements belonging to S_3 can contain only elements belonging to G_3. One may construct a similar argument concerning the coefficient matrix in Equation 4.3.11. However, the desired result can also be obtained by noting that if Equation 4.3.10 and 4.3.11 are written as follows

$$[\underline{B}_1 \ \underline{U}_{e-(v-1)}] \begin{bmatrix} \underline{x}_b \\ \underline{x}_c \end{bmatrix} = 0 \tag{4.3.12}$$

and

$$[\underline{U}_{v-1} \ \underline{A}_1] \begin{bmatrix} \underline{y}_b \\ \underline{y}_c \end{bmatrix} = 0 \tag{4.3.13}$$

then[1]

$$\underline{B}_1 = - \underline{A}_1^T . \tag{4.3.14}$$

This result implies that the submatrices of Equations 4.3.10 and 4.3.11 satisfy

$$\underline{B}_{ij} = - \underline{A}_{ji} \tag{4.3.15}$$

---

[1] H. E. Koenig and W. A. Blackwell, Electromechanical System Theory (New York, 1961), p. 59.

and in light of the argument given that $\underline{B}_{13}$, $\underline{B}_{14}$, and $\underline{B}_{24}$ are identically zero, then $\underline{A}_{31}$, $\underline{A}_{41}$, and $\underline{A}_{42}$ are also identically zero.

The second row block of Equation 4.3.10 and the third row block of Equation 4.3.11 can be used to form

$$\begin{bmatrix} \underline{B}_{22} & \underline{0} \\ \underline{0} & \underline{A}_{33} \end{bmatrix} \begin{bmatrix} \underline{x}_{2b} \\ \underline{y}_{4c} \end{bmatrix} + \begin{bmatrix} \underline{B}_{21} & \underline{0} \\ \underline{0} & \underline{A}_{34} \end{bmatrix} \begin{bmatrix} \underline{x}_{1b} \\ \underline{y}_{5c} \end{bmatrix} + \begin{bmatrix} \underline{0} & \underline{U}_{n_{3c}} \\ \underline{U}_{n_{3b}} & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{y}_{3b} \\ \underline{x}_{3c} \end{bmatrix}$$

$$+ \begin{bmatrix} \underline{B}_{23} & \underline{0} \\ \underline{0} & \underline{A}_{32} \end{bmatrix} \begin{bmatrix} \underline{x}_{3b} \\ \underline{y}_{3c} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{0} \end{bmatrix} \qquad (4.3.16)$$

Substitute Equation 4.3.9 into Equation 4.3.16 to obtain

$$\begin{bmatrix} \underline{B}_{22} + \underline{B}_{23}\underline{C}_{11} & \underline{B}_{23}\underline{C}_{12} & \underline{B}_{23}\underline{C}_{13} & \underline{B}_{23}\underline{C}_{14} & \underline{B}_{23}\underline{C}_{15} \\ \underline{A}_{32}\underline{C}_{21} & \underline{A}_{32}\underline{C}_{22} & \underline{A}_{32}\underline{C}_{23} & \underline{A}_{33} + \underline{A}_{32}\underline{C}_{24} & \underline{A}_{32}\underline{C}_{25} \end{bmatrix} \begin{bmatrix} \underline{x}_{2b} \\ \underline{x}_{2c} \\ \underline{y}_{4b} \\ \underline{y}_{4c} \\ \underline{\psi}_{2} \end{bmatrix}$$

$$+ \begin{bmatrix} \underline{B}_{23}\underline{D}_{11} & \underline{U}_{n_{3b}} \\ \underline{U}_{n_{3b}} & \underline{A}_{32}\underline{D}_{22} \end{bmatrix} \begin{bmatrix} \underline{y}_{3b} \\ \underline{x}_{3c} \end{bmatrix} = - \begin{bmatrix} \underline{B}_{21} & \underline{0} \\ \underline{0} & \underline{A}_{34} \end{bmatrix} \begin{bmatrix} \underline{x}_{1b} \\ \underline{y}_{5c} \end{bmatrix} - \begin{bmatrix} \underline{B}_{23} & \underline{0} \\ \underline{0} & \underline{A}_{32} \end{bmatrix} \begin{bmatrix} \underline{f}_{3b} \\ \underline{f}_{3c} \end{bmatrix}.$$

(4.3.17)

Let n_{3c} be less than n_{3b} and let Equation 4.3.17 be premultiplied by the nonsingular matrix

$$\underline{\Phi}_{1} = \begin{bmatrix} \underline{U}_{n_{3c}} & -\underline{B}_{23}\underline{D}_{11} \\ \underline{0} & \underline{U}_{n_{3b}} \end{bmatrix}. \qquad (4.3.18)$$

The coefficient matrix of the vector $[\underline{y}_{3b}^T \; \underline{x}_{3c}^T]^T$ in the resulting equation is (after one application of Equation 4.3.15)

$$\underline{\Phi}_2 = \begin{bmatrix} \underline{0} & \underline{U}_{n_{3c}} + \underline{A}_{32}^T \underline{D}_{11} \underline{A}_{32} \underline{D}_{22} \\ \underline{U}_{n_{3b}} & \underline{A}_{32} \underline{D}_{22} \end{bmatrix} . \qquad (4.3.19)$$

The upper row block of Equation 4.3.17, after multiplication by $\underline{\Phi}_1$, is independent of $\underline{y}_{3b}$ and can be solved directly for $\underline{x}_{3c}$ if the inverse of $\underline{U}_{n_{3c}} + \underline{A}_{32}^T \underline{D}_{11} \underline{A}_{32} \underline{D}_{22}$ exists. The following identity holds.

$$\underline{U}_{n_{3c}} + \underline{A}_{32}^T \underline{D}_{11} \underline{A}_{32} \underline{D}_{22} = (\underline{D}_{22}^{-1} + \underline{A}_{32}^T \underline{D}_{11} \underline{A}_{32}) \underline{D}_{22} \qquad (4.3.20)$$

Assumption 4.3.6 implies $\underline{D}_{22}^{-1}$ exists, and Theorems A.4 and A.5 show that $(\underline{D}_{22}^{-1} + \underline{A}_{32}^T \underline{D}_{11} \underline{A}_{32})^{-1}$ exists. Since the product of two nonsingular matrices is nonsingular, it is concluded that the desired inverse exists. Therefore, Equation 4.3.17 can be solved for

$$\begin{bmatrix} \underline{y}_{3b} \\ \underline{x}_{3c} \end{bmatrix} = \begin{bmatrix} \underline{K}_{31} & \underline{K}_{32} & \underline{K}_{33} & \underline{K}_{34} & \underline{K}_{35} \\ \underline{K}_{41} & \underline{K}_{42} & \underline{K}_{43} & \underline{K}_{44} & \underline{K}_{45} \end{bmatrix} \begin{bmatrix} \underline{x}_{2b} \\ \underline{x}_{2c} \\ \underline{y}_{4b} \\ \underline{y}_{4c} \\ \underline{\psi}_2 \end{bmatrix} + \begin{bmatrix} \underline{L}_{31} & \underline{L}_{32} \\ \underline{L}_{41} & \underline{L}_{42} \end{bmatrix} \begin{bmatrix} \underline{x}_{1b} \\ \underline{y}_{5c} \end{bmatrix}$$

$$+ \begin{bmatrix} \underline{M}_{31} & \underline{M}_{32} \\ \underline{M}_{41} & \underline{M}_{42} \end{bmatrix} \begin{bmatrix} \underline{f}_{3b} \\ \underline{f}_{3c} \end{bmatrix} . \qquad (4.3.21)$$

where the elements of the coefficient matrices are, in general, defined and continuous functions of the independent variable t over some set $T = \{t : t_1 < t < t_2\}$ for the fixed t_1 and t_2 of Assumption 4.3.6. The elements of the vectors $\underline{f}_{3b}$ and $\underline{f}_{3c}$ are also defined and continuous functions of the variable t over the same open set T. Note that if n_{3b} is less than n_{3c}, then one could suitably redefine $\underline{\Phi}_1$ in such a way that one could derive Equation 4.3.21 by finding the inverse of an $n_{3b} \times n_{3b}$ matrix rather than an $n_{3c} \times n_{3c}$ matrix as given in this development.

Substitute Equation 4.3.21 into 4.3.9 to form

$$\begin{bmatrix} \underline{x}_{3b} \\ \underline{y}_{3c} \end{bmatrix} = \begin{bmatrix} \underline{G}_{11} & \underline{G}_{12} & \underline{G}_{13} & \underline{G}_{14} & \underline{G}_{15} \\ \underline{G}_{21} & \underline{G}_{22} & \underline{G}_{23} & \underline{G}_{24} & \underline{G}_{25} \end{bmatrix} \begin{bmatrix} \underline{x}_{2b} \\ \underline{x}_{2c} \\ \underline{y}_{4b} \\ \underline{y}_{4c} \\ \underline{\psi}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} \underline{H}_{11} & \underline{H}_{12} \\ \underline{H}_{21} & \underline{H}_{22} \end{bmatrix} \begin{bmatrix} \underline{x}_{1b} \\ \underline{y}_{5c} \end{bmatrix} + \begin{bmatrix} \underline{J}_{11} & \underline{J}_{12} \\ \underline{J}_{21} & \underline{J}_{23} \end{bmatrix} \begin{bmatrix} \underline{f}_{3b} \\ \underline{f}_{3c} \end{bmatrix} . \qquad (4.3.22)$$

The coefficient matrices of this equation have the same restrictions as those in Equation 4.3.21.

The third row block of Equation 4.3.10 and the second row block of Equation 4.3.11 can be used in conjunction with Equation 4.3.21 to form

$$
\begin{bmatrix} \underline{y}_{2b} \\ \underline{y}_{2c} \\ \underline{y}_{3b} \\ \underline{x}_{3c} \\ \underline{x}_{4b} \\ \underline{x}_{4c} \end{bmatrix}
=
\begin{bmatrix}
\underline{0} & \underline{0} & \underline{0} & -\underline{A}_{23} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{K}_{31} & \underline{K}_{32} & \underline{K}_{33} & \underline{K}_{34} & \underline{K}_{35} \\
\underline{K}_{41} & \underline{K}_{42} & \underline{K}_{43} & \underline{K}_{44} & \underline{K}_{45} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
-\underline{B}_{32} & \underline{0} & \underline{0} & \underline{0} & \underline{0}
\end{bmatrix}
\begin{bmatrix} \underline{x}_{2b} \\ \underline{x}_{2c} \\ \underline{y}_{4b} \\ \underline{y}_{4c} \\ \underline{\psi}_{2} \end{bmatrix}
+
\begin{bmatrix}
\underline{0} & -\underline{A}_{24} \\
\underline{0} & \underline{0} \\
\underline{L}_{31} & \underline{L}_{32} \\
\underline{L}_{41} & \underline{L}_{42} \\
\underline{0} & \underline{0} \\
-\underline{B}_{31} & \underline{0}
\end{bmatrix}
\begin{bmatrix} \underline{x}_{1b} \\ \underline{y}_{5c} \end{bmatrix}
$$

$$
+
\begin{bmatrix}
-\underline{A}_{21} & \underline{0} \\
\underline{U}_{n_{2c}} & \underline{0} \\
\underline{0} & \underline{0} \\
\underline{0} & \underline{0} \\
\underline{0} & \underline{U}_{n_{4b}} \\
\underline{0} & -\underline{B}_{34}
\end{bmatrix}
\begin{bmatrix} \underline{y}_{2c} \\ \underline{x}_{4b} \end{bmatrix}
+
\begin{bmatrix}
\underline{0} & -\underline{A}_{22} \\
\underline{0} & \underline{0} \\
\underline{0} & \underline{0} \\
\underline{0} & \underline{0} \\
\underline{0} & \underline{0} \\
-\underline{B}_{33} & \underline{0}
\end{bmatrix}
\begin{bmatrix} \underline{x}_{3b} \\ \underline{y}_{3c} \end{bmatrix}
+
\begin{bmatrix}
\underline{0} & \underline{0} \\
\underline{0} & \underline{0} \\
\underline{M}_{31} & \underline{M}_{32} \\
\underline{M}_{41} & \underline{M}_{42} \\
\underline{0} & \underline{0} \\
\underline{0} & \underline{0}
\end{bmatrix}
\begin{bmatrix} \underline{f}_{3b} \\ \underline{f}_{3c} \end{bmatrix}
\; . \qquad (4.3.23)
$$

Substitute Equation 4.3.22 into Equation 4.3.23 to obtain an equation of the form

$$
\begin{bmatrix} \underline{y}_{2b} \\ \underline{y}_{2c} \\ \underline{y}_{3b} \\ \underline{x}_{3c} \\ \underline{x}_{4b} \\ \underline{x}_{4c} \end{bmatrix}
=
\begin{bmatrix}
\underline{K}_{11} & \underline{K}_{12} & \underline{K}_{13} & \underline{K}_{14} & \underline{K}_{15} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{K}_{31} & \underline{K}_{32} & \underline{K}_{33} & \underline{K}_{34} & \underline{K}_{35} \\
\underline{K}_{41} & \underline{K}_{42} & \underline{K}_{43} & \underline{K}_{44} & \underline{K}_{45} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{K}_{61} & \underline{K}_{62} & \underline{K}_{63} & \underline{K}_{64} & \underline{K}_{65}
\end{bmatrix}
\begin{bmatrix} \underline{x}_{2b} \\ \underline{x}_{2c} \\ \underline{y}_{4b} \\ \underline{y}_{4c} \\ \underline{\psi}_{2} \end{bmatrix}
+
\begin{bmatrix}
\underline{L}_{11} & \underline{L}_{12} \\
\underline{0} & \underline{0} \\
\underline{L}_{31} & \underline{L}_{32} \\
\underline{L}_{41} & \underline{L}_{42} \\
\underline{0} & \underline{0} \\
\underline{L}_{61} & \underline{L}_{62}
\end{bmatrix}
\begin{bmatrix} \underline{x}_{1b} \\ \underline{y}_{5c} \end{bmatrix}
$$

$$
+
\begin{bmatrix}
\underline{M}_{11} & \underline{M}_{12} \\
\underline{0} & \underline{0} \\
\underline{M}_{31} & \underline{M}_{32} \\
\underline{M}_{41} & \underline{M}_{42} \\
\underline{0} & \underline{0} \\
\underline{M}_{61} & \underline{M}_{62}
\end{bmatrix}
\begin{bmatrix} \underline{f}_{3b} \\ \underline{f}_{3c} \end{bmatrix}
+
\begin{bmatrix}
-\underline{A}_{21} & \underline{0} \\
\underline{U}_{n_{2c}} & \underline{0} \\
\underline{0} & \underline{0} \\
\underline{0} & \underline{0} \\
\underline{0} & \underline{U}_{n_{4b}} \\
\underline{0} & -\underline{B}_{34}
\end{bmatrix}
\begin{bmatrix} \underline{y}_{2c} \\ \underline{x}_{4b} \end{bmatrix}
\; . \qquad (4.3.24)
$$

Now, substitute Equation 4.3.24 into Equation 4.3.8 and collect terms so as to form

$$\frac{d}{dt}\begin{bmatrix} \underline{x}_{2b} \\ \underline{x}_{2c} \\ \underline{y}_{4b} \\ \underline{y}_{4c} \\ \underline{\psi}_{2} \end{bmatrix} = \begin{bmatrix} \underline{P}'_{11} & \underline{P}'_{12} & \underline{P}'_{13} & \underline{P}'_{14} & \underline{P}'_{15} \\ \underline{P}'_{21} & \underline{P}'_{22} & \underline{P}'_{23} & \underline{P}'_{24} & \underline{P}'_{25} \\ \underline{P}'_{31} & \underline{P}'_{32} & \underline{P}'_{33} & \underline{P}'_{34} & \underline{P}'_{35} \\ \underline{P}'_{41} & \underline{P}'_{42} & \underline{P}'_{43} & \underline{P}'_{44} & \underline{P}'_{45} \\ \underline{P}'_{51} & \underline{P}'_{52} & \underline{P}'_{53} & \underline{P}'_{54} & \underline{P}'_{55} \end{bmatrix} \begin{bmatrix} \underline{x}_{2b} \\ \underline{x}_{2c} \\ \underline{y}_{4b} \\ \underline{y}_{4c} \\ \underline{\psi}_{2} \end{bmatrix}$$

$$+ \begin{bmatrix} \underline{L}'_{11} & \underline{L}'_{12} \\ \underline{L}'_{21} & \underline{L}'_{22} \\ \underline{L}'_{31} & \underline{L}'_{32} \\ \underline{L}'_{41} & \underline{L}'_{42} \\ \underline{L}'_{51} & \underline{L}'_{52} \end{bmatrix} \begin{bmatrix} \underline{x}_{1b} \\ \underline{y}_{5c} \end{bmatrix} + \begin{bmatrix} \underline{M}'_{11} & \underline{M}'_{12} \\ \underline{M}'_{21} & \underline{M}'_{22} \\ \underline{M}'_{31} & \underline{M}'_{32} \\ \underline{M}'_{41} & \underline{M}'_{42} \\ \underline{M}'_{51} & \underline{M}'_{52} \end{bmatrix} \begin{bmatrix} \underline{f}_{3b} \\ \underline{f}_{3c} \end{bmatrix}$$

$$+ \begin{bmatrix} -\underline{Q}_{11}\,\underline{A}_{21} & \underline{0} \\ \underline{Q}_{22} & \underline{0} \\ \underline{0} & \underline{Q}_{35} \\ \underline{0} & -\underline{Q}_{46}\,\underline{B}_{34} \\ \underline{0} & \underline{Q}_{55} - \underline{Q}_{56}\,\underline{B}_{34} \end{bmatrix} \begin{bmatrix} \underline{y}_{2c} \\ \underline{x}_{4b} \end{bmatrix} + \begin{bmatrix} \underline{g}_{2b} \\ \underline{g}_{2c} \\ \underline{g}_{4b} \\ \underline{g}_{4c} \\ \underline{g}_{2} \end{bmatrix} . \qquad (4.3.25)$$

The primes appearing in Equation 4.3.25 indicate that the terms resulting from the use of Equation 4.3.24 have been collected. As it now stands, Equation 4.3.25 represents a system of p linear first-order differential equations in $p + n_{2c} + n_{4b}$ variables. Such a system does not possess a unique solution. The only way to insure that the system has a unique solution is to eliminate $n_{2c} + n_{4b}$ of the variables from the equation by means of the remaining interconnection equations. This elimination process is given next.

The first row block of Equation 4.3.10 and the last row block of Equation 4.3.11 can be used to form

$$\begin{bmatrix} \underline{x}_{2b} \\ \underline{x}_{2c} \\ \underline{y}_{4b} \\ \underline{y}_{4c} \\ \underline{\psi}_{2} \end{bmatrix} = \begin{bmatrix} \underline{U}_{n2b} & \underline{0} & \underline{0} \\ -\underline{B}_{12} & \underline{0} & \underline{0} \\ \underline{0} & -\underline{A}_{43} & \underline{0} \\ \underline{0} & \underline{U}_{n4c} & \underline{0} \\ \underline{0} & \underline{0} & \underline{U}_{n2} \end{bmatrix} \begin{bmatrix} \underline{x}_{2b} \\ \underline{y}_{4c} \\ \underline{\psi}_{2} \end{bmatrix} + \begin{bmatrix} \underline{0} & \underline{0} \\ -\underline{B}_{11} & \underline{0} \\ \underline{0} & -\underline{A}_{44} \\ \underline{0} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{x}_{1b} \\ \underline{y}_{5c} \end{bmatrix} . \qquad (4.3.26)$$

Substitute Equation 4.3.26 into Equation 4.3.25. The submatrices $\underline{N}_{ij}$ have been introduced to denote that the $\underline{P}'_{ij}$ and $\underline{L}'_{ij}$ matrices have been modified to incorporate the effects of Equation 4.3.26.

$$\frac{d}{dt} \begin{bmatrix} \underline{U}_{n2b} & \underline{0} & \underline{0} \\ -\underline{B}_{12} & \underline{0} & \underline{0} \\ \underline{0} & -\underline{A}_{43} & \underline{0} \\ \underline{0} & \underline{U}_{n4c} & \underline{0} \\ \underline{0} & \underline{0} & \underline{U}_{n2} \end{bmatrix} \begin{bmatrix} \underline{x}_{2b} \\ \underline{y}_{4c} \\ \underline{\psi}_{2} \end{bmatrix} = \begin{bmatrix} \underline{N}_{11} & \underline{N}_{12} & \underline{N}_{13} \\ \underline{N}_{21} & \underline{N}_{22} & \underline{N}_{23} \\ \underline{N}_{31} & \underline{N}_{32} & \underline{N}_{33} \\ \underline{N}_{41} & \underline{N}_{42} & \underline{N}_{43} \\ \underline{N}_{51} & \underline{N}_{52} & \underline{N}_{53} \end{bmatrix} \begin{bmatrix} \underline{x}_{2b} \\ \underline{y}_{4c} \\ \underline{\psi}_{2} \end{bmatrix} + \begin{bmatrix} \underline{N}_{14} & \underline{N}_{14} \\ \underline{N}_{24} & \underline{N}_{25} \\ \underline{N}_{34} & \underline{N}_{35} \\ \underline{N}_{44} & \underline{N}_{45} \\ \underline{N}_{54} & \underline{N}_{55} \end{bmatrix} \begin{bmatrix} \underline{x}_{1b} \\ \underline{y}_{5c} \end{bmatrix}$$

$$+ \begin{bmatrix} \underline{M}'_{11} & \underline{M}'_{12} \\ \underline{M}'_{21} & \underline{M}'_{22} \\ \underline{M}'_{31} & \underline{M}'_{32} \\ \underline{M}'_{41} & \underline{M}'_{42} \\ \underline{M}'_{51} & \underline{M}'_{52} \end{bmatrix} \begin{bmatrix} \underline{f}_{3b} \\ \underline{f}_{3c} \end{bmatrix} + \begin{bmatrix} -\underline{Q}_{11}\underline{A}_{21} & \underline{0} \\ \underline{Q}_{22} & \underline{0} \\ \underline{0} & \underline{Q}_{35} \\ \underline{0} & -\underline{Q}_{46}\underline{B}_{34} \\ \underline{0} & \underline{Q}_{55} - \underline{Q}_{56}\underline{B}_{34} \end{bmatrix} \begin{bmatrix} \underline{y}_{2c} \\ \underline{x}_{4b} \end{bmatrix} + \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{B}_{11} & \underline{0} \\ \underline{0} & \underline{A}_{44} \\ \underline{0} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \underline{x}_{1b} \\ \underline{y}_{5b} \end{bmatrix} + \begin{bmatrix} \underline{g}_{2b} \\ \underline{g}_{2c} \\ \underline{g}_{4b} \\ \underline{g}_{4c} \\ \underline{g}_{2} \end{bmatrix} \qquad (4.3.27)$$

Now, premultiply Equation 4.3.27 by the nonsingular matrix

$$\underline{\Phi}_3 = \begin{bmatrix} \underline{U}_{n_{2b}} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{U}_{n_{4c}} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{U}_{n_2} \\ \underline{B}_{12} & \underline{U}_{n_{2c}} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{U}_{n_{4b}} & \underline{A}_{43} & \underline{0} \end{bmatrix} \qquad (4.3.28)$$

The results of this multiplication are

$$\frac{d}{dt}\begin{bmatrix} \underline{U}_{n_{2b}} & \underline{0} & \underline{0} \\ \underline{0} & \underline{U}_{n_{4c}} & \underline{0} \\ \underline{0} & \underline{0} & \underline{U}_{n_2} \\ \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{x}_{2b} \\ \underline{y}_{4c} \\ \underline{\psi}_2 \end{bmatrix} = \begin{bmatrix} \underline{N}_{14} & \underline{N}_{15} \\ \underline{N}_{44} & \underline{N}_{45} \\ \underline{N}_{54} & \underline{N}_{55} \\ \underline{B}_{12}\,\underline{N}_{14} + \underline{N}_{24} & \underline{B}_{12}\,\underline{N}_{15} + \underline{N}_{25} \\ \underline{N}_{34} + \underline{A}_{43}\,\underline{N}_{44} & \underline{N}_{35} + \underline{A}_{43}\,\underline{N}_{45} \end{bmatrix} \begin{bmatrix} \underline{x}_{1b} \\ \underline{y}_{5c} \end{bmatrix}$$

$$+ \begin{bmatrix} \underline{N}_{11} & \underline{N}_{12} & \underline{N}_{13} \\ \underline{N}_{41} & \underline{N}_{42} & \underline{N}_{42} \\ \underline{N}_{51} & \underline{N}_{52} & \underline{N}_{53} \\ \underline{B}_{12}\,\underline{N}_{11} + \underline{N}_{21} & \underline{B}_{12}\,\underline{N}_{12} + \underline{N}_{23} & \underline{B}_{12}\,\underline{N}_{13} + \underline{N}_{23} \\ \underline{N}_{31} + \underline{A}_{43}\,\underline{N}_{41} & \underline{N}_{32} + \underline{A}_{43}\,\underline{N}_{42} & \underline{N}_{33} + \underline{A}_{43}\,\underline{N}_{43} \end{bmatrix} \begin{bmatrix} \underline{x}_{2b} \\ \underline{y}_{4c} \\ \underline{\psi}_2 \end{bmatrix} + \begin{bmatrix} \underline{g}_{2b} \\ \underline{g}_{4c} \\ \underline{g}_2 \\ \underline{B}_{12}\,\underline{g}_{2b} + \underline{g}_{2c} \\ \underline{g}_{4b} + \underline{A}_{43}\,\underline{g}_{4c} \end{bmatrix}$$

$$+ \begin{bmatrix} \underline{M}'_{11} & \underline{M}'_{12} \\ \underline{M}'_{41} & \underline{M}'_{42} \\ \underline{M}'_{51} & \underline{M}'_{52} \\ \underline{B}_{12}\underline{M}'_{11} + \underline{M}'_{21} & \underline{B}'_{12}\,\underline{M}'_{12} + \underline{M}'_{22} \\ \underline{M}'_{31} + \underline{A}_{43}\,\underline{M}'_{41} & \underline{M}'_{32} + \underline{A}_{43}\,\underline{M}'_{42} \end{bmatrix} \begin{bmatrix} \underline{f}_{3b} \\ \underline{f}_{3c} \end{bmatrix} + \begin{bmatrix} -\underline{Q}_{11}\,\underline{A}_{21} & \underline{0} \\ \underline{0} & -\underline{Q}_{46}\,\underline{B}_{34} \\ \underline{0} & \underline{Q}_{55} - \underline{Q}_{56}\,\underline{B}_{34} \\ -\underline{B}_{12}\,\underline{Q}_{11}\,\underline{A}_{21} + \underline{Q}_{22} & \underline{0} \\ \underline{0} & \underline{Q}_{35} - \underline{A}_{43}\,\underline{Q}_{46}\,\underline{B}_{34} \end{bmatrix} \begin{bmatrix} \underline{y}_{2c} \\ \underline{x}_{4b} \end{bmatrix}$$

$$+ \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{0} & \underline{0} \\ \underline{0} & \underline{0} \\ \underline{B}_{11} & \underline{0} \\ \underline{0} & \underline{A}_{44} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \underline{x}_{1b} \\ \underline{y}_{5c} \end{bmatrix} \qquad (4.3.29)$$

It is seen that $n_{2c} + n_{4b}$ of the p equations have been reduced to simple linear algebraic equations. These linear algebraic equations are in the last two row blocks of Equation 4.3.29. Consider the matrix coefficient of the vector $[\underline{y}_{2c}^T \ \underline{x}_{4b}^T]^T$ in these algebraic equations. Let that matrix be denoted $\underline{\Phi}_4$. $\underline{\Phi}_4$ can be written (with the application of Equation 4.3.15) as follows:

$$\underline{\Phi}_4 = \begin{bmatrix} \underline{A}_{21}^T \underline{Q}_{11} \underline{A}_{21} + \underline{Q}_{22} & \underline{0} \\ \underline{0} & \underline{Q}_{35} + \underline{B}_{34}^T \underline{Q}_{46} \underline{B}_{34} \end{bmatrix} \tag{4.3.30}$$

Theorems A.4 and A.5 and Assumption 4.3.6 imply that $\underline{\Phi}_4^{-1}$ exists. Hence, the linear algebraic equations belonging to Equations 4.3.29 can be solved for the following relationship:

$$\begin{bmatrix} \underline{y}_{2c} \\ \underline{x}_{4b} \end{bmatrix} = \begin{bmatrix} \underline{K}_{21} & \underline{K}_{22} & \underline{K}_{23} \\ \underline{K}_{51} & \underline{K}_{52} & \underline{K}_{53} \end{bmatrix} \begin{bmatrix} \underline{x}_{2b} \\ \underline{y}_{4c} \\ \underline{\psi}_2 \end{bmatrix} + \begin{bmatrix} \underline{K}_{24} & \underline{K}_{25} \\ \underline{K}_{54} & \underline{K}_{55} \end{bmatrix} \begin{bmatrix} \underline{x}_{1b} \\ \underline{y}_{5c} \end{bmatrix} + \begin{bmatrix} \underline{K}_{2,13} & \underline{K}_{2,14} \\ \underline{K}_{5,13} & \underline{K}_{5,14} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \underline{x}_{1b} \\ \underline{y}_{5c} \end{bmatrix}$$

$$+ \begin{bmatrix} \underline{K}_{26} & \underline{K}_{27} & \underline{K}_{28} & \underline{K}_{29} & \underline{K}_{2,10} & \underline{K}_{2,11} & \underline{K}_{2,12} \\ \underline{K}_{56} & \underline{K}_{57} & \underline{K}_{58} & \underline{K}_{59} & \underline{K}_{5,10} & \underline{K}_{5,11} & \underline{K}_{5,12} \end{bmatrix} \begin{bmatrix} \underline{f}_{3b} \\ \underline{f}_{3c} \\ \underline{g}_{2b} \\ \underline{g}_{4b} \\ \underline{g}_{4c} \\ \underline{g}_{2c} \\ \underline{g}_2 \end{bmatrix} \tag{4.3.31}$$

Substituting these results into the linear first-order differential equations belonging to Equation 4.3.29 and substituting Equation 4.3.7 into the result yields a set of p_o differential equations in p_o variables. These equations are

$$\frac{d}{dt}\begin{bmatrix}\underline{x}_{2b}\\ \underline{y}_{4c}\\ \underline{\psi}_{2}\end{bmatrix} = \begin{bmatrix} \underline{N}'_{14} & \underline{N}'_{15} & \underline{N}'_{16} & \underline{N}'_{17} & \underline{N}'_{18} & \underline{N}'_{19} & \underline{N}'_{1,10} & \underline{N}'_{1,11} & \underline{N}'_{1,12} & \underline{N}'_{1,13} & \underline{N}'_{1,14}\\ \underline{N}'_{24} & \underline{N}'_{25} & \underline{N}'_{26} & \underline{N}'_{27} & \underline{N}'_{28} & \underline{N}'_{29} & \underline{N}'_{2,10} & \underline{N}'_{2,11} & \underline{N}'_{2,12} & \underline{N}'_{2,13} & \underline{N}'_{2,14}\\ \underline{N}'_{34} & \underline{N}'_{35} & \underline{N}'_{36} & \underline{N}'_{37} & \underline{N}'_{38} & \underline{N}'_{39} & \underline{N}'_{3,10} & \underline{N}'_{3,11} & \underline{N}'_{3,12} & \underline{N}'_{3,13} & \underline{N}'_{3,14} \end{bmatrix} \begin{bmatrix}\underline{x}_{1}\\ \underline{y}_{5}\\ \underline{f}_{3b}\\ \underline{f}_{3c}\\ \underline{g}_{2b}\\ \underline{g}_{4b}\\ \underline{g}_{4c}\\ \underline{g}_{2c}\\ \underline{g}_{2}\\ \dot{\underline{x}}_{1}\\ \dot{\underline{y}}_{5}\end{bmatrix}$$

$$+\begin{bmatrix} \underline{N}'_{11} & \underline{N}'_{12} & \underline{N}'_{13}\\ \underline{N}'_{21} & \underline{N}'_{22} & \underline{N}'_{23}\\ \underline{N}'_{31} & \underline{N}'_{32} & \underline{N}'_{33}\end{bmatrix}\begin{bmatrix}\underline{x}_{2b}\\ \underline{y}_{4c}\\ \underline{\psi}_{2}\end{bmatrix} \qquad (4.3.32)$$

The primes in Equation 4.3.32 indicate that all the coefficients of like terms have been collected. An inspection of Equation 4.3.32 shows that

$$p_o = n_{2b} + n_{4c} + n_2 \ . \qquad (4.3.33)$$

Substitution of Equation 4.3.26 and 4.3.31 into Equation 4.3.24 and then substituting Equation 4.3.7 into the result yields an equation of the form

$$
\begin{bmatrix} \underline{y}_{2b} \\ \underline{y}_{2c} \\ \underline{y}_{3b} \\ \underline{x}_{3c} \\ \underline{x}_{4b} \\ \underline{x}_{4c} \end{bmatrix}
=
\begin{bmatrix}
\underline{K}'_{14} & \underline{K}'_{15} & \underline{K}'_{16} & \underline{K}'_{17} & \underline{K}'_{18} & \underline{K}'_{19} & \underline{K}'_{1,10} & \underline{K}'_{1,11} & \underline{K}'_{1,12} & \underline{K}'_{1,13} & \underline{K}'_{1,14} \\
\underline{K}'_{24} & \underline{K}'_{25} & \underline{K}'_{26} & \underline{K}'_{27} & \underline{K}'_{28} & \underline{K}'_{29} & \underline{K}'_{2,10} & \underline{K}'_{2,11} & \underline{K}'_{2,12} & \underline{K}'_{2,13} & \underline{K}'_{2,14} \\
\underline{K}'_{34} & \underline{K}'_{35} & \underline{K}'_{36} & \underline{K}'_{37} & \underline{K}'_{38} & \underline{K}'_{39} & \underline{K}'_{3,10} & \underline{K}'_{3,11} & \underline{K}'_{3,12} & \underline{K}'_{3,13} & \underline{K}'_{3,14} \\
\underline{K}'_{44} & \underline{K}'_{45} & \underline{K}'_{46} & \underline{K}'_{47} & \underline{K}'_{48} & \underline{K}'_{49} & \underline{K}'_{4,10} & \underline{K}'_{4,11} & \underline{K}'_{4,12} & \underline{K}'_{4,13} & \underline{K}'_{4,14} \\
\underline{K}'_{54} & \underline{K}'_{55} & \underline{K}'_{56} & \underline{K}'_{57} & \underline{K}'_{58} & \underline{K}'_{59} & \underline{K}'_{5,10} & \underline{K}'_{5,11} & \underline{K}'_{5,12} & \underline{K}'_{5,13} & \underline{K}'_{5,14} \\
\underline{K}'_{64} & \underline{K}'_{65} & \underline{K}'_{66} & \underline{K}'_{67} & \underline{K}'_{68} & \underline{K}'_{69} & \underline{K}'_{6,10} & \underline{K}'_{6,11} & \underline{K}'_{6,12} & \underline{K}'_{6,13} & \underline{K}'_{6,14}
\end{bmatrix}
\begin{bmatrix} \underline{x}_1 \\ \underline{y}_5 \\ \underline{f}_{3b} \\ \underline{f}_{3c} \\ \underline{g}_{2b} \\ \underline{g}_{4b} \\ \underline{g}_{4c} \\ \underline{g}_{2c} \\ \underline{g}_2 \\ \mathring{\underline{x}}_1 \\ \mathring{\underline{y}}_5 \end{bmatrix}
$$

$$
+ \begin{bmatrix}
\underline{K}'_{11} & \underline{K}'_{12} & \underline{K}'_{13} \\
\underline{K}'_{21} & \underline{K}'_{22} & \underline{K}'_{23} \\
\underline{K}'_{31} & \underline{K}'_{32} & \underline{K}'_{33} \\
\underline{K}'_{41} & \underline{K}'_{42} & \underline{K}'_{43} \\
\underline{K}'_{51} & \underline{K}'_{52} & \underline{K}'_{53} \\
\underline{K}'_{61} & \underline{K}'_{62} & \underline{K}'_{63}
\end{bmatrix}
\begin{bmatrix} \underline{x}_{2b} \\ \underline{y}_{4c} \\ \underline{\psi}_2 \end{bmatrix}
\qquad (4.3.34)
$$

As before, the prime notation indicates that, after the required substitution has been performed, all coefficients of like terms are grouped together.

Substitute Equation 4.3.7 into the first four row blocks of Equation 4.3.26 to form

$$\begin{bmatrix} \underline{x}_{2b} \\ \underline{x}_{2c} \\ \underline{y}_{4b} \\ \underline{y}_{4c} \end{bmatrix} = \begin{bmatrix} \underline{U}_{n_{2b}} & \underline{0} & \underline{0} \\ -\underline{B}_{12} & \underline{0} & \underline{0} \\ \underline{0} & -\underline{A}_{43} & \underline{0} \\ \underline{0} & \underline{U}_{n_{4c}} & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{x}_{2b} \\ \underline{y}_{4c} \\ \underline{\psi}_2 \end{bmatrix} + \begin{bmatrix} \underline{0} & \underline{0} \\ -\underline{B}_{11} & \underline{0} \\ \underline{0} & -\underline{A}_{44} \\ \underline{0} & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{y}_5 \end{bmatrix} . \quad (4.3.35)$$

Substitute Equation 4.3.26 into Equation 4.3.22 and then substitute Equation 4.3.7 into the result to form

$$\begin{bmatrix} \underline{x}_{3b} \\ \underline{y}_{3c} \end{bmatrix} = \begin{bmatrix} \underline{K}'_{11,1} & \underline{K}'_{11,2} & \underline{K}'_{11,3} \\ \underline{K}'_{12,1} & \underline{K}'_{12,2} & \underline{K}'_{12,3} \end{bmatrix} \begin{bmatrix} \underline{x}_{2b} \\ \underline{y}_{4c} \\ \underline{\psi}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} \underline{K}'_{11,4} & \underline{K}'_{11,5} & \underline{K}'_{11,6} & \underline{K}'_{11,7} \\ \underline{K}'_{12,4} & \underline{K}'_{12,5} & \underline{K}'_{12,6} & \underline{K}'_{12,7} \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{y}_5 \\ \underline{f}_{3b} \\ \underline{f}_{3c} \end{bmatrix} \quad (4.3.36)$$

Substitute Equations 4.3.34, 4.3.35, 4.3.36, and 4.3.7 into the matrix equations that can be formed by taking the direct sum of the last row block of Equation 4.3.10 and the first row block of Equation 4.3.11 and solving it for $[\underline{x}_{5c}^T \ \underline{y}_{1b}^T]^T$. This operation results in the equation

$$\begin{bmatrix} \underline{x}_{5c} \\ \underline{y}_{1b} \end{bmatrix} = \begin{bmatrix} \underline{K}'_{13,1} & \underline{K}'_{13,2} & \underline{K}'_{13,3} \\ \underline{K}'_{14,1} & \underline{K}'_{14,2} & \underline{K}'_{14,3} \end{bmatrix} \begin{bmatrix} \underline{x}_{2b} \\ \underline{y}_{4c} \\ \underline{\psi}_{2} \end{bmatrix}$$

$$+ \begin{bmatrix} \underline{K}'_{13,4} & \underline{K}'_{13,5} & \underline{K}'_{13,6} & \underline{K}'_{13,7} & \underline{K}'_{13,8} & \underline{K}'_{13,9} & \underline{K}'_{13,10} & \underline{K}'_{13,11} & \underline{K}'_{13,12} & \underline{K}'_{13,13} & \underline{K}'_{13,14} \\ \underline{K}'_{14,4} & \underline{K}'_{14,5} & \underline{K}'_{14,6} & \underline{K}'_{14,7} & \underline{K}'_{14,8} & \underline{K}'_{14,9} & \underline{K}'_{14,10} & \underline{K}'_{14,11} & \underline{K}'_{14,12} & \underline{K}'_{14,13} & \underline{K}'_{14,14} \end{bmatrix} \begin{bmatrix} \underline{x}_{1} \\ \underline{y}_{5} \\ \underline{f}_{3b} \\ \underline{f}_{3c} \\ \underline{g}_{2b} \\ \underline{g}_{4b} \\ \underline{g}_{4c} \\ \underline{g}_{2c} \\ \underline{g}_{2} \\ \mathring{\underline{x}}_{1} \\ \mathring{\underline{y}}_{5} \end{bmatrix} \tag{4.3.37}$$

If $\underline{\lambda}$ and $\underline{f}$ are defined in the following manner

$$\underline{\lambda} = [\underline{x}_{2b}^T \ \underline{y}_{4c}^T \ \underline{\psi}_{2}^T]^T \ , \tag{4.3.38}$$

$$\underline{f} = [\underline{x}_{1}^T \ \underline{y}_{5}^T \ \underline{f}_{3b}^T \ \underline{f}_{3c}^T \ \underline{g}_{2b}^T \ \underline{g}_{4b}^T \ \underline{g}_{4c}^T \ \underline{g}_{2c}^T \ \underline{g}_{2}^T \ \mathring{\underline{x}}_{1}^T \ \mathring{\underline{y}}_{5}^T]^T \ , \tag{4.3.39}$$

then the coefficient matrices appearing in Equation 4.3.32 can be identified as $\underline{P}_o$ and $\underline{R}_o$ that appear in Equation 2.3.1 of Definition 2.3.1. If, in addition, one considers Equations 4.3.7, 4.3.34, 4.3.35, 4.3.36, and 4.3.37 and defines the vector $\underline{Z}$ to be the direct sum of the vectors appearing on the left of these equations, then one can easily identify the matrices $\underline{C}_o$ and $\underline{E}_o$ appearing in Equation 2.3.2 of Definition 2.3.1. Therefore, Equations 4.3.32, 4.3.34, 4.3.35, 4.3.36, and 4.3.37 form the state model for the system of k_o multiterminal components. It is noted that all of the entries in the coefficient matrices and in the vector $\underline{f}$ are defined and continuous functions of the independent variable t over the set $T = \{t : t_1 < t < t_2\}$ for some fixed t_1 and t_2.

Let t_o be a fixed value of t belonging to T. Then, there is a well-known theorem[2] in the theory of ordinary differential equations which states that there exists a unique solution for the equation

$$\frac{d}{dt}\underline{\lambda} = \underline{P}_o\,\underline{\lambda} + \underline{R}_o\,\underline{f} \qquad (4.3.40)$$

for every possible value of the vector $\underline{\lambda}\,(t_o)$. Then, one can substitute this solution for $\underline{\lambda}\,(t)$ into Equations 4.3.34, 4.3.35, 4.3.36, and 4.3.37 and obtain a unique solution for every terminal variable of the system. Thus, a complete solution is assured.

In summary, it is seen that there are eight major steps involved in the formulation of the state model for a system of multiterminal components whose terminal equations satisfy Assumptions 4.3.1 through 4.3.7. The major steps in this procedure are as follows:

1. Select a formulation tree in accordance with the procedure discussed in the paragraph that immediately follows Equation 4.3.5. Write the fundamental cutset and circuit equations

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[2] W. Kaplan, Ordinary Differential Equations (Reading, Massachusetts, 1958), p. 494.

defined by that formulation tree.

2. Use the n_{3c} fundamental cutset and the n_{3b} fundamental circuit equations that are defined by algebraic elements (that is, elements whose component equations are of the form of Equation 4.3.9) in conjunction with the algebraic component equations (Equation 4.3.9) to solve for the vector $[\underline{y}_{3b}^T\ \underline{x}_{3c}^T]^T$. The inverse of an s x s matrix is required here. The symbol s denotes the smaller of the two numbers n_{3b} and n_{3c}.
3. Use the results of step 2 to solve for the vector $[\underline{x}_{3b}^T\ \underline{y}_{3c}^T]^T$. No matrix inverse is required here.
4. Use the fundamental circuit equations that are defined by the graph elements belonging to S_4 that are also chords of T_G (there are n_{4c} such equations) and the fundamental cutset equations that are defined by the graph elements belonging to S_2 that are also branches of T_G (there are n_{2b} such equations) in conjunction with the results of step 2 to form an explicit equation for the vector

$$[\underline{y}_{2b}^T\ \underline{y}_{2c}^T\ \underline{y}_{3b}^T\ \underline{x}_{3c}^T\ \underline{x}_{4b}^T\ \underline{x}_{4c}^T]^T \quad .$$

Substitute these results into the differential component equations (Equation 4.3.8). No matrix inverse is required here.
5. Use the fundamental circuit equations that are defined by the graph elements which belong to S_2 and are chords of T_G (there are n_{2c} such equations) and the fundamental cutset equations that are defined by graph elements which belong to S_4 and are

branches of T_G (there are n_{4b} such equations) to reduce the $n_{2c} + n_{4b}$ linearly dependent first-order differential equations, that are the result of step 4, to linear algebraic equations. No matrix inverse is required here.

6. Solve the $(n_{2c} + n_{4b})$ linear algebraic equations that are derived in step 5 for the vector $[\underline{y}_{2c}^T, \underline{x}_{4b}^T]^T$. The inverse of an $(n_{2c} + n_{4b})$-ordered square matrix is required here. This inverse can be obtained by inverting two smaller inverses separately. These smaller inverses are of order $n_{2c} \times n_{2c}$ and $n_{4b} \times n_{4b}$, respectively.
7. Use the $(n_{2b} + n_{4c} + n_2)$ first-order differential equations that remain after step 5 is completed to form the differential equation portion of the state model of the system (see Definition 2.3.1).
8. Collect the results of steps 2, 3, 4, and 6 and use these equations in conjunction with the fundamental cutset and circuit equations that are defined by elements belonging to S_1 and S_5 and the component equations which describe the terminal characteristics of the ideal drivers to form the 2e algebraic equations which belong to the system state model.

Two different matrix inverses are required in the formulation procedure. However, these inverses are assured for those systems in which the components satisfy Assumptions 4.3.1 through 4.3.7. This concludes the development of the state model formulation procedure; however, there are a number of special cases that merit some discussion.

One such case involves the elements which belong to S_2 and S_4. Suppose that every element which belongs to S_2 can be placed in T_G and that every element that belongs to S_4 can be placed in the complement of T_G in G. Then, if Assumptions 4.3.1 through 4.3.7 remain valid, all of the first-order differential equations mentioned in step 4 in the above formulation procedure will be independent. This implies that the $(n_{2c} + n_{4b}) \times (n_{2c} + n_{4b})$ matrix inverse mentioned in step 6 is not required. Since this inverse is not required, the restriction in Assumption 4.3.6 requiring that $\underline{Q}_{11}$, $\underline{Q}_{22}$, $\underline{Q}_{35}$, and $\underline{Q}_{36}$ in Equation 4.3.8 be positive definite can be removed. All other steps in the formulation procedure are required. This special case is analogous, in the case of simple electric circuits, to requiring that all capacitors be placed in the formulation tree and all inductors be placed in the chord set of the formulation tree.

A second special case arises when each of the fundamental circuits and cutsets associated with T_G contains no more than one algebraic element (that is, an element whose terminal equation has the form of Equation 4.3.9). In that case

$$\underline{B}_{23} \equiv \underline{A}_{32} \equiv \underline{0} . \tag{4.3.41}$$

Then, the vector $[\underline{x}_{3b}^T \; \underline{y}_{3c}^T]^T$ does not appear in Equation 4.3.16. In that case the matrix inverse mentioned in step 2 of the formulation procedure reduces to the inverse of an identity matrix. As in the first special case, the restriction in Assumption 4.3.6 requiring that $\underline{D}_{11}$ and $\underline{D}_{22}$ in Equation 4.3.9 be positive definite can now be removed. This special case is analogous, in the case of electric networks, to requiring that for some formulation tree every fundamental cutset and every fundamental circuit contain at most one resistor.

If one combines both of the special cases mentioned above, then no inverses are required in the formulation procedure. In that case all of the restrictions given in Assumption 4.3.6 can be removed.

CHAPTER V

EXAMPLES AND APPLICATIONS

5.1 Introduction. In this chapter, a number of applications of the formulation procedure given in Chapter IV are demonstrated by means of examples. Two major examples are considered.

The first one deals with the formulation of the state model for a relatively large system containing two-terminal components. In that case it is demonstrated that by considering the system to be an interconnection of several subsystems (each of which is a multiterminal component), one can formulate the system state model in a straightforward manner. As a preliminary step in this formulation procedure, a technique whereby the state equations for a multiterminal component constructed of two-terminal devices is demonstrated. This technique, as applied to the formulation of state equations, is treated in the work of Koenig and Tokad (8). The procedure is similar to the technique given by Koenig and Blackwell (3) for the derivation of terminal equations for multiterminal components in the complex-frequency domain.

The second major example deals with the formulation of the state equations for an interconnection of electromechanical and mechanical-hydraulic coupled n-terminal components (see Definition 2.2.10) that are typical of components often found in automatic control systems. In this case it is necessary that one be able to derive the state equations for each of the constituent non-reducible coupled n-terminal components. A number of techniques that are useful in deriving these state equations are presented and discussed.

5.2 An Example of State Model Formulation. Consider the electrical network shown in Figure 5.2.1. Suppose that one is interested in formulating a state model for the network that has the characteristic that the currents and voltages associated with the elements labeled R_{10} and C_{11} are given as explicit functions of the state of the network and the known driver E_6. One way of solving this problem is to use the formulation techniques of Bryant (6) or Brown (4). However, one can also derive a suitable state model by considering the network to be an interconnection of four subsystems each of which is a multi-terminal component. The subsystems to be considered in this example are indicated by the dotted lines in Figure 5.2.1.

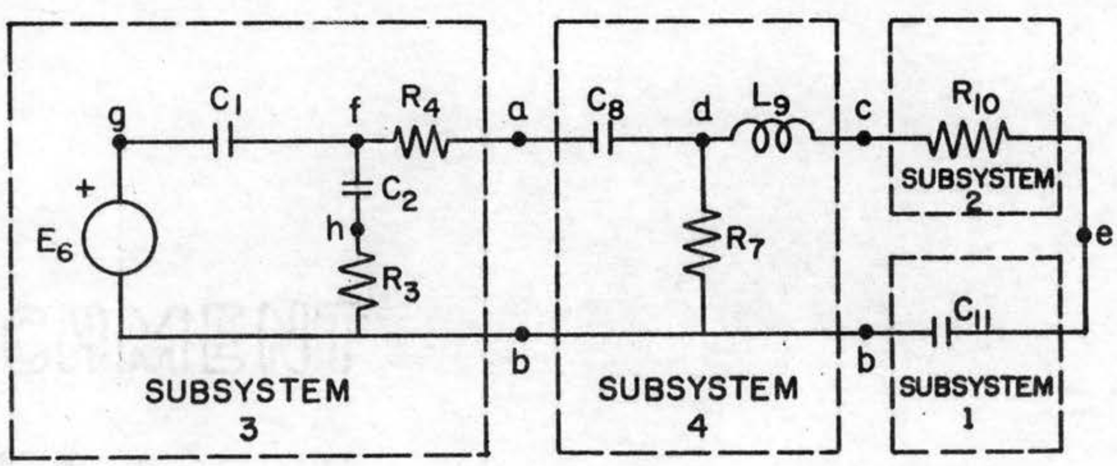


Figure 5.2.1. The Network to be Studied

Prior to actual formulation of the required state model, one must first derive a set of state equations in the form given in Definition 2.4.1 for each of the subsystems of the network.

For those subsystems which contain only one element, the state equations can be determined by inspection of the terminal equations of the device. For example, the terminal equation for a single capacitor can be written

$$\frac{dv}{dt} = \frac{1}{C} i \ . \tag{5.2.1}$$

If one denotes the state of the capacitor by ψ, then

$$\psi = v \tag{5.2.2}$$

and one can form

$$\frac{d\psi}{dt} = 0 \ \psi + \frac{1}{C} i \ , \tag{5.2.3}$$

and Equations 5.2.2 and 5.2.3 form the state equations for the capacitor. However, due to the simplicity of Equation 5.2.2, Equation 5.2.1 represents the state equation of a capacitor in reduced form (see Section 4.2 for a discussion of the idea of the reduced form of the state equations). It is the reduced form that will be used in the sequel. Similarly, the reduced form of the state equations for an inductor take the form

$$\frac{di}{dt} = \frac{1}{L} v \ . \tag{5.2.4}$$

Since the terminal equations for a resistor do not involve a time-derivative term, the state equations for a resistor reduce to one of the following algebraic equations:

$$i = Gv \tag{5.2.5}$$

$$v = Ri \tag{5.2.6}$$

Note that, for each of the simple two-terminal components discussed above, the terminal graph consists of a single-line segment and two vertices.

Now consider Subsystem 3 of Figure 5.2.1. It is seen that this subsystem is connected to the remainder of the network at only 2 points, i.e., at points "a" and "b". Such a subsystem can be treated as a two-terminal device. The corresponding state equations can be derived by assuming that an appropriate ideal-driver element is applied to the terminals "a" and "b" and solving for the resulting complementary variable. This technique will now be demonstrated for Subsystem 3. The technique is an adaptation of the one presented by Koenig and Blackwell (3) in their work in the complex-frequency domain. Figure 5.2.2 shows the system graph and the desired terminal graph. The element labeled A in Figure 5.2.2 represents the ideal driver that will be used to determine the terminal characteristics of Subsystem 3. For the purposes of this derivation, A will be assumed to be an ideal-voltage driver. This choice is arbitrary so long as Assumptions 4.3.1 through 4.3.7 are satisfied.

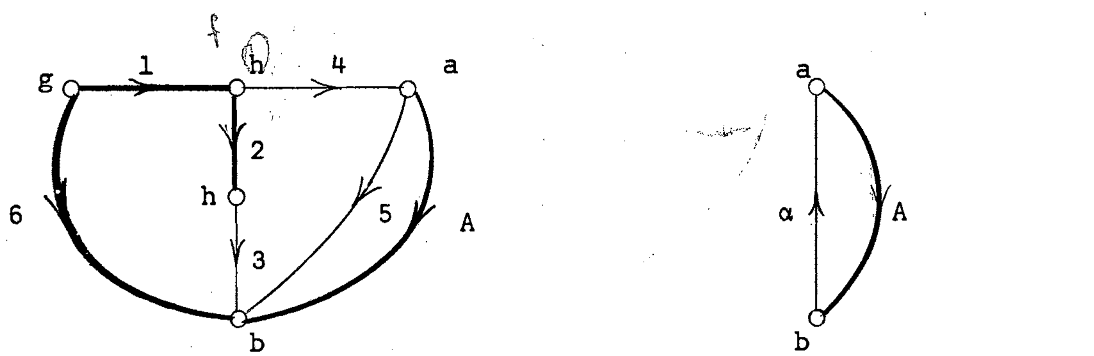
In light of Equations 5.2.1 through 5.2.6 and the accompanying discussion, the direct sum of the reduced form state equations of the constituent components of Subsystem 3 can be written

$$\begin{bmatrix} v_6 \\ v_A \end{bmatrix} = \begin{bmatrix} E_6 \\ E_A \end{bmatrix} \tag{5.2.7}$$

$$\frac{d}{dt}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{C_2} & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}, \tag{5.2.8}$$

and

$$\begin{bmatrix} i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} G_3 & 0 & 0 \\ 0 & G_4 & 0 \\ 0 & 0 & G_5 \end{bmatrix}\begin{bmatrix} v_3 \\ v_4 \\ v_5 \end{bmatrix}. \tag{5.2.9}$$



(a) System Graph with the Arbitrary Driver

(b) Desired Terminal Graph with the Arbitrary Driver

Figure 5.2.2. Linear Graphs for Subsystem 3

The set S_1 contains elements 6 and A, while the set S_2 contains elements 1 and 2. The set S_3 contains elements 3, 4, and 5. The sets S_4 and S_5 are empty in this case. The formulation tree to be used is shown by the heavier line segments in Figure 5.2.2. The corresponding fundamental cutset and circuit equations can be written in the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_6 \\ i_A \\ i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = 0 \qquad (5.2.10)$$

and

$$\begin{bmatrix} -1 & 0 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_6 \\ v_A \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = 0 , \qquad (5.2.11)$$

respectively.

Now, if one substitutes the circuit equations into the algebraic component equations (Equation 5.2.9), then an explicit expression for $[i_3, i_4, i_5]^T$ in terms of $[v_1, v_2]^T$ and $[v_6, v_A]^T$ results. If this explicit expression is substituted into the last two cutset equations and the result of that operation is substituted into the differential component equations (Equation 5.2.8), one obtains the equation

$$\frac{d}{dt}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{-(G_3 + G_4)}{C_1} & -\frac{G_3}{C_1} \\ -\frac{G_3}{C_2} & -\frac{G_3}{C_2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} \frac{G_3 + G_4}{C_1} \\ \frac{G_3}{C_2} \end{bmatrix} v_6 + \begin{bmatrix} -\frac{G_4}{C_1} \\ 0 \end{bmatrix} v_A \quad . \tag{5.2.12}$$

Also, if one substitutes the explicit expression for $[i_3, i_4, i_5]^T$ into the second cutset equation, then one obtains

$$i_A = [-G_4 \quad 0]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + [G_4]\, v_6 + [-(G_4 + G_5)]\, v_A \quad . \tag{5.2.13}$$

The interconnection equations for the desired terminal graph are

$$i_A = i_\alpha \tag{5.2.14}$$

and

$$v_A = -v_\alpha \tag{5.2.15}$$

The state equations for Subsystem 3 are obtained by substituting Equations 5.2.14 and 5.2.15 into Equations 5.2.12 and 5.2.13. Also note that the variables v_1 and v_2 are not terminal variables of the subsystem. Therefore, in order to avoid confusion later, let

$$v_1 = \psi_1 \tag{5.2.16}$$

and

$$v_2 = \psi_2 \quad . \tag{5.2.17}$$

The state equations for Subsystem 3 are

$$\frac{d}{dt}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}_3 = \begin{bmatrix} \frac{-(G_3 + G_4)}{C_1} & -\frac{G_3}{C_1} \\ -\frac{G_3}{C_2} & -\frac{G_3}{C_2} \end{bmatrix}_3 \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}_3 + \begin{bmatrix} \frac{G_4}{C_1} \\ 0 \end{bmatrix}_3 \begin{bmatrix} v_\alpha \end{bmatrix}_3 + \begin{bmatrix} \frac{1}{C_1}(G_3+G_4) \\ \frac{G_3}{C_2} \end{bmatrix}_3 E_6 \tag{5.2.18}$$

$$[i_\alpha]_3 = [-G_4 \quad 0]_3 \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}_3 + [G_4 + G_5]_3 \; [v_\alpha]_3 + [G_4]_3 \; E_6 \tag{5.2.19}$$

Subsystem 3 is an example of a device in which none of the state variables are also terminal variables, hence the above equations are also the reduced state equations. Subsystem 3 is also an example of a device that has an internal energy source. The terminal representation derived here can be considered as a time-domain Norton equivalent representation for the original subsystem. The terminal characteristics of the device are completely specified by Equations 5.2.18 and 5.2.19 and the terminal graph consisting of the single-oriented line segment α and the two vertices "a" and "b".

Subsystem 4 is an example of a device where one of the state variables is also a terminal variable. If the procedure illustrated above for Subsystem 3 is applied to Subsystem 4 for the desired terminal graph shown in Figure 5.2.3, then the following state equations result:

$$\frac{d}{dt}\begin{bmatrix} v_8 \\ i_9 \end{bmatrix}_4 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{R_7}{L_9} \end{bmatrix}_4 \begin{bmatrix} v_8 \\ i_9 \end{bmatrix}_4 + \begin{bmatrix} -\frac{1}{C_8} & 0 \\ \frac{R_7}{L_9} & \frac{1}{L_9} \end{bmatrix}_4 \begin{bmatrix} i_\delta \\ v_\sigma \end{bmatrix}_4 \quad (5.2.20)$$

$$\begin{bmatrix} v_\delta \\ i_\sigma \end{bmatrix}_4 = \begin{bmatrix} -1 & -R_7 \\ 0 & 1 \end{bmatrix}_4 \begin{bmatrix} v_8 \\ i_9 \end{bmatrix}_4 + \begin{bmatrix} R_7 & 0 \\ 0 & 0 \end{bmatrix}_4 \begin{bmatrix} i_\delta \\ v_\sigma \end{bmatrix}_4 \quad (5.2.21)$$

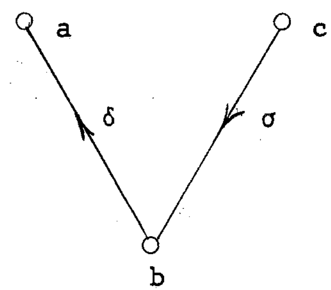


Figure 5.2.3. Desired Terminal Graph for Subsystem 4

The state variable i_9 is seen to be expressible as a single-terminal variable as follows

$$i_9 = i_\sigma . \quad (5.2.22)$$

Also, in order to avoid future confusion, let

$$\psi_3 = v_8 . \quad (5.2.23)$$

Then the state equations for Subsystem 4, in reduced form, become

$$\frac{d}{dt}\begin{bmatrix} i_\sigma \\ \psi_3 \end{bmatrix}_4 = \begin{bmatrix} -\frac{R_7}{L_9} & 0 \\ 0 & 0 \end{bmatrix}_4 \begin{bmatrix} i_\sigma \\ \psi_3 \end{bmatrix}_4 + \begin{bmatrix} \frac{1}{L_9} & \frac{R_{11}}{L_9} \\ 0 & -\frac{1}{C_8} \end{bmatrix}_4 \begin{bmatrix} v_\sigma \\ i_\delta \end{bmatrix}_4 \quad (5.2.24)$$

and

$$[v_\delta]_4 = [-\,R_7 \quad -1]_4 \begin{bmatrix} i_\sigma \\ \psi_3 \end{bmatrix}_4 + [0 \quad R_7]_4 \begin{bmatrix} v_\sigma \\ i_\delta \end{bmatrix}_4 . \qquad (5.2.25)$$

Attention will now be focused on the problem of formulating a state model for the network shown in Figure 5.2.1. The system graph for that network takes the form of Figure 5.2.4, when one treats the network as an interconnection of the multiterminal components whose state equations are derived above.

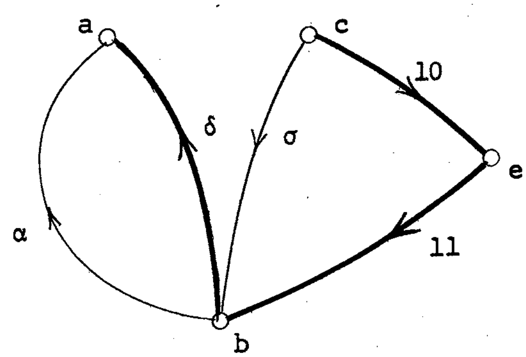


Figure 5.2.4. The System Graph

An examination of Equation 5.2.1 as applied to C'_{11} and Equation 5.2.6 as applied to R_{10} along with Equations 5.2.18, 5.2.19, 5.2.24, and 5.2.25 reveals that

1. the set S_1 as defined in Chapter IV is empty,
2. the set S_2 contains element 11,
3. the set S_3 contains elements α, δ, and 10,
4. the set S_4 contains element σ, and

5. the set S_5 is empty.

A formulation tree T_G can be selected in accordance with the procedure described in Chapter IV. One such tree contains elements δ, 10, and 11. It is this tree that will be used to formulate the state model for the system. The fundamental cutset and circuit equations associated with that tree are

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_{11} \\ i_\delta \\ i_{10} \\ i_\alpha \\ i_\sigma \end{bmatrix} = 0 \qquad (5.2.26)$$

and

$$\begin{bmatrix} 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_\delta \\ v_{10} \\ v_\alpha \\ v_\sigma \end{bmatrix} = 0 \;, \qquad (5.2.27)$$

respectively.

The direct sum of all of the component state equations as arranged in the form of Equations 4.3.8 and 4.3.9 can be written as follows:

$$\frac{d}{dt}\begin{bmatrix} v_{11} \\ i_\sigma \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{R_7}{L_9} & 0 & 0 & 0 \\ 0 & 0 & -\frac{(G_3+G_4)}{C_1} & -\frac{G_3}{C_1} & 0 \\ 0 & 0 & -\frac{G_3}{C_2} & -\frac{G_3}{C_2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_{11} \\ i_\sigma \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$$

$$+\begin{bmatrix} \frac{1}{C_{11}} & 0 & 0 & 0 & 0 \\ 0 & \frac{R_{11}}{L_9} & 0 & 0 & \frac{1}{L_9} \\ 0 & 0 & 0 & \frac{G_4}{C_1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{C_8} & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} i_{11} \\ i_\delta \\ i_{10} \\ v_\alpha \\ v_\sigma \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{G_3+G_4}{C_1} \\ \frac{G_3}{C_2} \\ 0 \end{bmatrix} E_6 \qquad (5.2.28)$$

$$\begin{bmatrix} v_\delta \\ v_{10} \\ i_\alpha \end{bmatrix} = \begin{bmatrix} 0 & R_7 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -G_4 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_{11} \\ i_\sigma \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$$

$$+\begin{bmatrix} R_7 & 0 & 0 \\ 0 & R_{10} & 0 \\ 0 & 0 & G_4+G_5 \end{bmatrix}\begin{bmatrix} i_\delta \\ i_{10} \\ v_\alpha \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ G_4 \end{bmatrix} E_6 \quad . \qquad (5.2.29)$$

An examination of the above equations will show that the conditions of

Assumptions 4.3.2 through 4.3.7 prevail, hence the formulating procedure developed in Chapter IV is applicable. It is also noted that

$$n_{3b} = 2 \tag{5.2.30}$$

and

$$n_{3c} = 1 \, , \tag{5.2.31}$$

hence the inverse of a single equation is required.

When the first equation of Equation 5.2.27 and the second and third equations of Equation 5.2.26 are substituted into Equation 5.2.29, a matrix equation of the form

$$\begin{bmatrix} 1 & 0 & R_7 \\ 0 & 1 & 0 \\ -(G_4 + G_5) & 0 & 1 \end{bmatrix} \begin{bmatrix} v_\delta \\ v_{10} \\ i_\alpha \end{bmatrix} = \begin{bmatrix} 0 & R_7 & 0 & 0 & -1 \\ 0 & -R_{10} & 0 & 0 & 0 \\ 0 & 0 & -G_4 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{11} \\ i_\sigma \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ G_4 \end{bmatrix} E_6 \tag{5.2.32}$$

results.

Let $\underline{\Phi}_1$ be defined as shown in Equation 5.2.33. The reader is referred to Equation 4.3.18 and the paragraph preceding Equation 4.3.22 for the significance of the symbol $\underline{\Phi}_1$.

$$\underline{\Phi}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ G_4 + G_5 & 0 & 1 \end{bmatrix} \tag{5.2.33}$$

Now, when Equation 5.2.32 is premultiplied by $\underline{\Phi}_1$, the resulting equation can be solved for

$$\begin{bmatrix} v_\delta \\ v_{10} \\ i_\alpha \end{bmatrix} = \begin{bmatrix} 0 & \frac{R_7}{1 + K_o} & \frac{R_7\ G_4}{1 + K_o} & 0 & \frac{1}{1 + K_o} \\ 0 & -\frac{R}{1 + K_o} & 0 & 0 & 0 \\ 0 & \frac{K_o}{1 + K_o} & -\frac{G_4}{1 + K_o} & 0 & -\frac{K_o}{R_7(1 + K_o)} \end{bmatrix} \begin{bmatrix} v_{11} \\ i_\sigma \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$$

$$+ \begin{bmatrix} -\frac{R_7\ G_4}{1 + K_o} \\ 0 \\ \frac{G_4}{1 + K_o} \end{bmatrix} E_6 \qquad (5.2.34)$$

by inverting a single equation. The symbol K_o is defined to be

$$K_o = R_7\ (G_4 + G_5) \qquad (5.2.35)$$

in Equation 5.2.34 and in the equations that follow. Equation 5.2.34 can now be substituted into the first circuit equation and the second and third cutset equations to form

$$\begin{bmatrix} i_\delta \\ i_{10} \\ v_\alpha \end{bmatrix} = \begin{bmatrix} 0 & \frac{-K_o}{1 + K_o} & \frac{G_4}{1 + K_o} & 0 & \frac{K_o}{R\ (1 + K_o)} \\ 0 & -1 & 0 & 0 & 0 \\ 0 & \frac{R_7}{1 + K_o} & \frac{R_7\ G_4}{1 + K_o} & 0 & \frac{1}{1 + K_o} \end{bmatrix} \begin{bmatrix} v_{11} \\ i_\sigma \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} + \begin{bmatrix} -\frac{G_4}{1 + K_o} \\ 0 \\ \frac{-R_7\ G_4}{1 + K_o} \end{bmatrix} E_6 \qquad (5.2.36)$$

The first cutset equation and the last circuit equation can be used in conjunction with Equation 5.2.36 to form an explicit expression for $[i_{11}, i_\delta, i_{10}, v_\alpha, v_\sigma]^T$ in terms of the state variables and the known function E_6. One can then substitute this explicit expression into Equation 5.2.28. The result of this operation is

$$\frac{d}{dt}\begin{bmatrix} v_{11} \\ i_\sigma \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{1+K_o}\,\frac{R_{11}\,G_4}{L_9} \\ \frac{1}{C_1}\left(\frac{G_4{}^2\,R_7}{1+K_o} + G_3 + G_4\right) \\ \frac{G_3}{C_2} \\ \frac{1}{1+K}\left(\frac{G_4}{C_8}\right) \end{bmatrix} E_6$$

$$+ \begin{bmatrix} 0 & -\frac{1}{C_{11}} & 0 & 0 & 0 \\ \frac{1}{L_9} & \frac{1}{L_9}\left(\frac{R_{11}K_o}{1+K_o} - R_{10} - R_7\right) & \frac{1}{1+K_o}\left(-\frac{R_{11}\,G_4}{L_9}\right) & 0 & \frac{K_o}{1+K_o}\left(-\frac{R_{11}}{R_7 L_9}\right) \\ 0 & \frac{1}{1+K_o}\cdot\frac{G_4}{C_1\,R_7} & \frac{1}{C_1}\left(\frac{1}{1+K_o}R_7 G_4^2 - G_3 - G_4\right) & -\frac{G_3}{C_1} & \frac{1}{1+K_o}\left(\frac{G_4}{C_1}\right) \\ 0 & 0 & -\frac{G_3}{C_2} & -\frac{G_3}{C_2} & 0 \\ 0 & \frac{K_o}{1+K_o}\cdot\frac{1}{C_8} & \frac{1}{1+K_o}\left(-\frac{G_4}{C_8}\right) & 0 & \frac{1}{1+K_o}\left(-\frac{1}{C_8}\right) \end{bmatrix} \begin{bmatrix} v_{11} \\ i_\sigma \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}.$$

(5.2.37)

Equations 5.2.34 and 5.2.36 together with the explicit expression for $[i_{11}, v_\alpha]^T$ mentioned above can be used to form

$$
\begin{bmatrix} v_\delta \\ v_{10} \\ i_\alpha \\ i_\delta \\ i_{10} \\ v_\alpha \\ i_{11} \\ v_\sigma \\ v_{11} \\ i_\sigma \end{bmatrix}
=
\begin{bmatrix}
0 & \frac{R_7}{1+K_o} & \frac{R_7\ G_4}{1+K_o} & 0 & \frac{1}{1+K_o} \\
0 & -\frac{R_{10}}{1+K_o} & 0 & 0 & 0 \\
0 & \frac{K_o}{1+K_o} & -\frac{G}{1+K_o} & 0 & -\frac{K_o}{R_7(1+K_o)} \\
0 & -\frac{K_o}{1+K_o} & \frac{G_4}{1+K_o} & 0 & \frac{K_o}{R_7(1+K_o)} \\
0 & -1 & 0 & 0 & 0 \\
0 & \frac{R_7}{1+K_o} & \frac{R_7\ G_4}{1+K_o} & 0 & \frac{1}{1+K_o} \\
0 & -1 & 0 & 0 & 0 \\
1 & -R_{10} & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} v_{11} \\ i_\sigma \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}
$$

$$
+ \begin{bmatrix} \frac{-G_7\ G_4}{1+K_o} \\ 0 \\ \frac{G_4}{1+K_o} \\ -\frac{G_4}{1+K_o} \\ 0 \\ -\frac{R_7\ G_4}{1+K_o} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} E_6 \qquad (5.2.38)
$$

Equations 5.2.37 and 5.2.38 form a state model for the system given in Figure 5.2.1. One can easily identify the various matrices and vectors given in Equations 2.3.1 and 2.3.2 of Definition 2.3.1.

5.3 The State Equations of an Electromechanical-Hydraulic Device. Consider the electromechanical-hydraulic device[1] shown in Figure 5.3.1. Suppose that this device is a component in a larger system for which a state model is desired. If it is assumed that the design of the device under consideration is fixed in the sense that one cannot alter its characteristics, then there is no point in considering all of the constituent parts (i.e., the push-pull amplifier, the solenoid, the power piston, etc.) individually when one formulates the state model for the larger system. For this reason, one might wish to consider the complete electromechanical device as a single coupled 4-terminal device (see Definition 2.2.10). If this is the case, then one must be able to derive the state equations for the 4-terminal representation of the device from a knowledge of the terminal characteristics of each of the constituent components and their interconnection relations. It is the purpose of this example to demonstrate how these state equations can be derived for the device shown in Figure 5.3.1. The basic approach will be to first find a set of state equations for each of the individual constituent components and then use a modified version of the formulating procedure developed in Chapter IV to derive the desired state equations for the larger device. Blackwell (19) has

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[1] J. L. Bower and P. M. Schultheiss, Introduction to the Design of Servomechanisms (New York, 1958), p. 94.

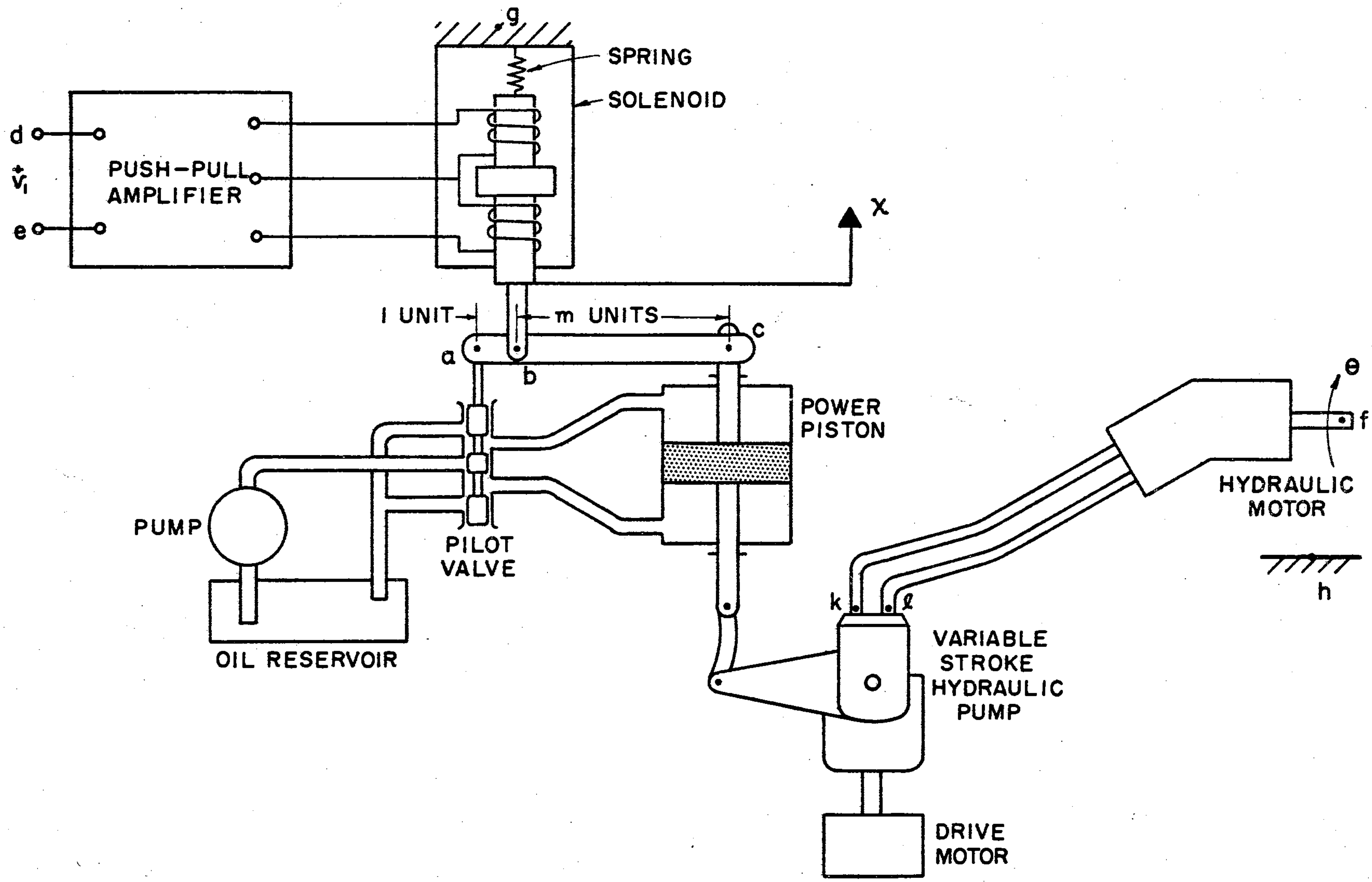
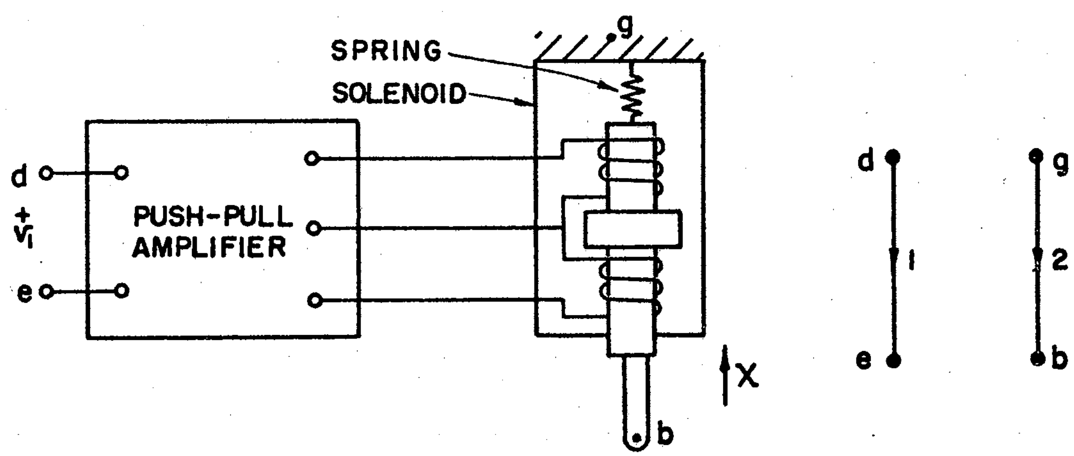


Figure 5.3.1. A Hydraulic Transmission

presented a similar analysis for this device in the complex-frequency domain.

Consider the push-pull amplifier and solenoid as a single multi-terminal component. It is assumed that the input current to the amplifier is negligible. Furthermore, it is assumed that the force that the solenoid armature exerts on the lever is proportional to the input voltage of the amplifier and the displacement and velocity of the solenoid armature. The mass of the armature is assumed to be either negligible or to have been lumped with the mass of the pilot valve of the hydraulic power amplifier. Figure 5.3.2 presents the schematic and the terminal graph for the amplifier-solenoid combination.



(a) Schematic (b) Terminal Graph

Figure 5.3.2. Component No. 1 - Amplifier-Solenoid Combination

The terminal equations for this coupled 4-terminal device can be written

$$i_1 = 0 \qquad (5.3.1)$$

$$f_2 = -K_{12}\, v_1 + \left(K_2 + B_2 \frac{d}{dt}\right) x_2 \qquad (5.3.2)$$

where K_{12}, K_2, and B_2 are known constants. The reduced form of the state equations for this component can be derived by solving Equation 5.3.2 for

$$\frac{d}{dt} x_2 = [-\frac{K_2}{B_2}]\, x_2 + [\frac{K_{12}}{B_2} \quad \frac{1}{B_2}] \begin{bmatrix} v_1 \\ f_2 \end{bmatrix}. \qquad (5.3.3)$$

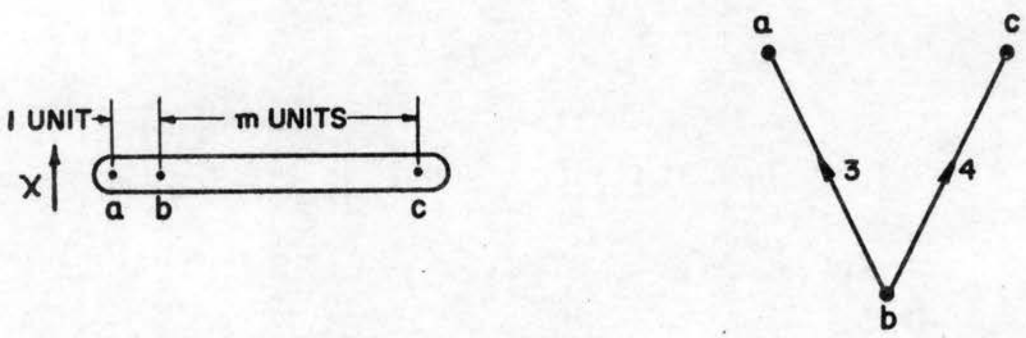
Equations 5.3.1 and 5.3.3 are the desired state equations.

A few words about notation are in order at this point. Throughout this example, the symbols x_i, p_j, and $\dot{\theta}_k$ will be used to denote the translational displacement of the i-th element, the hydraulic pressure associated with the j-th element, and the rotational angular velocity of the k-th element, respectively. All are across variables. The symbols f_i, $\dot{g}_j$, and T_k denote the corresponding force, volume flow rate, and torque. The latter variables are all through variables. The symbol ψ_i will be used to denote the i-th state variable. It may also be a terminal variable in which case this fact will be apparent from an inspection of the algebraic component equations.

Now consider the lever in Figure 5.3.1. It is assumed that the lever is ideal, i.e., the lever is rigid and has negligible mass. If that is the case, then all displacements, forces, and small rotations are transformed ideally with the ratio of transformation being dependent upon the relative distances of the end points "a" and "c" from the fulcrum point "b". Let m denote the ratio of the distance from "c" to "b" to the distance from "a" to "b". Figure 5.3.3 presents the schematic and terminal graphs for the ideal lever. The terminal equations for the component are given in matrix form by Equation 5.3.4.

$$\begin{bmatrix} f_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & m \\ -m & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ f_4 \end{bmatrix} \qquad (5.3.4)$$

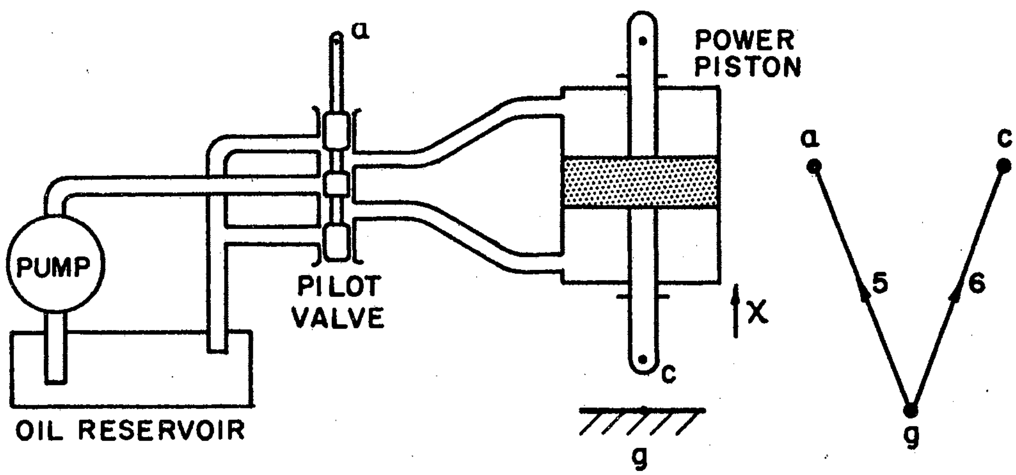
Since no time derivations are involved, these equations are also the desired state equations.



(a) Schematic (b) Terminal Graph

Figure 5.3.3. Component No. 3 - Lever

The hydraulic power amplifier presents a slightly different problem. For the purposes of this analysis this component is assumed to include the pump and oil reservoir, the pilot valve, and the power piston. The pilot valve piston is assumed to possess both viscous damping and mass. In fact the mass of the driving solenoid, if appreciable, is assumed to be a part of the total mass of the pilot valve piston mass. The force developed by the power piston is assumed to be proportional to the displacement of the pilot valve piston and the displacement and velocity of the power piston itself. As with the solenoid, the output mass effects of the power piston will be lumped together with the input mass effects of the variable stroke hydraulic pump unit. A schematic and terminal graph for the hydraulic power amplifier are shown in Figure 5.3.4.



(a) Schematic (b) Terminal Graph

Figure 5.3.4. Component No. 3 - Hydraulic Power Amplifier

The terminal equations for the component can be written

$$f_5 = B_5 \frac{d}{dt} x_5 + M_5 \frac{d^2}{dt^2} x_5 + K_5 x_5 \qquad (5.3.5)$$

and

$$f_6 = - K_{65} x_5 + B_6 \frac{d}{dt} x_6 + K_6 x_6 \quad . \qquad (5.3.6)$$

Now, let

$$\psi_2 = x_5 \quad , \qquad (5.3.7)$$

$$\psi_3 = \frac{d}{dt}\psi_2 = \frac{d}{dt}x_5 \quad , \tag{5.3.8}$$

and

$$\psi_4 = x_6 \quad . \tag{5.3.9}$$

Equation 5.3.5 can be solved for $\frac{d}{dt}\psi_3$, and Equation 5.3.6 can be solved for $\frac{d}{dt}\psi_4$ as follows

$$\frac{d}{dt}\psi_3 = -\frac{B_5}{M_5}\psi_3 - \frac{K_5}{M_5}\psi_2 + \frac{1}{M_5}f_5 \tag{5.3.10}$$

and

$$\frac{d}{dt}\psi_4 = \frac{K_{65}}{B_6}\psi_2 - \frac{K_6}{B_6}\psi_4 \quad . \tag{5.3.11}$$

Equations 5.3.8, 5.3.10, and 5.3.11 are the first-order differential equation portion of the state equations for the hydraulic power amplifier, while Equations 5.3.7 and 5.3.9 form the algebraic equation portion. However, due to the simplicity of these algebraic equations, one can write the matrix state equations in the reduced form as follows:

$$\frac{d}{dt}\begin{bmatrix} x_5 \\ x_6 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{K_6}{B_6} & \frac{K_{65}}{B_6} \\ -\frac{K_5}{M_5} & 0 & -\frac{B_5}{M_5} \end{bmatrix}\begin{bmatrix} x_5 \\ x_6 \\ \psi_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{B_6} \\ \frac{1}{M_5} & 0 \end{bmatrix}\begin{bmatrix} f_5 \\ f_6 \end{bmatrix} \quad . \tag{5.3.12}$$

The schematic and terminal graph for the variable-stroke hydraulic pump are shown in Figure 5.3.5. Using a technique similar to that illustrated above for the hydraulic power amplifier, one can easily show that if the terminal equations for the pump are

$$f_7 = K_7\, x + B_7 \frac{d}{dt} x_7 + M_7 \frac{d^2}{dt^2} x_7 + K_{79}\, p_9 \tag{5.3.13}$$

and

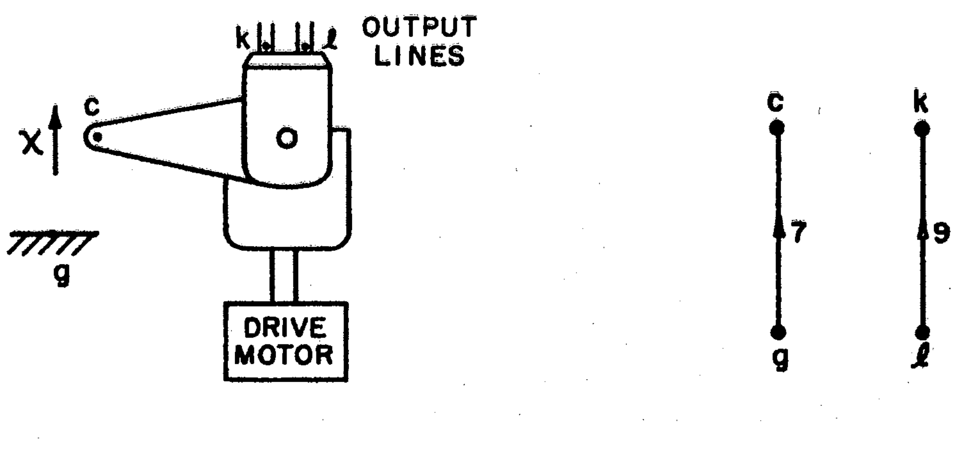
$$\dot{g}_9 = - K_{97}\, x_7 \quad , \tag{5.3.14}$$

then the reduced state equations for this device take the form

$$\frac{d}{dt}\begin{bmatrix} x_7 \\ \psi_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K_7}{M_7} & -\frac{B_7}{M_7} \end{bmatrix}\begin{bmatrix} x_7 \\ \psi_6 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{M_7} & -\frac{K_{79}}{M_7} \end{bmatrix}\begin{bmatrix} f_7 \\ p_9 \end{bmatrix} \tag{5.3.15}$$

$$\dot{g}_9 = - K_{97}\, x_7 \quad . \tag{5.3.16}$$

Note that the leakage and line expansion effects will be accounted for at the input side of the hydraulic motor.

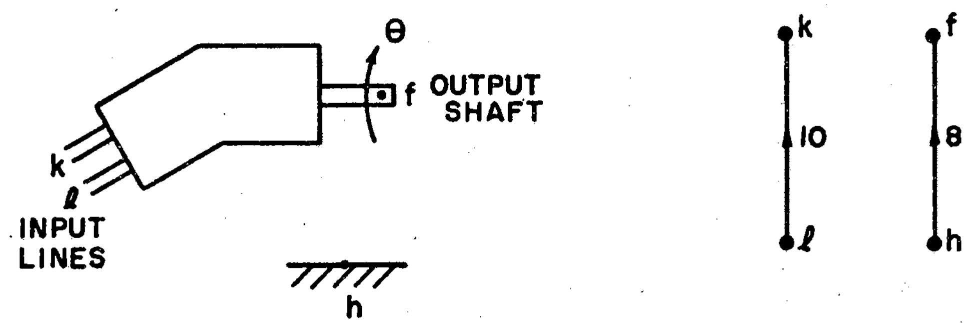


(a) Schematic

(b) Terminal Graph

Figure 5.3.5. Component No. 4 - Variable-Stroke Hydraulic Pump

The schematic and terminal graph for the hydraulic motor are presented in Figure 5.3.6. The terminal equations for this device are assumed to take the form of Equations 5.3.17 and 5.3.18.



(a) Schematic (b) Terminal Graph

Figure 5.3.6. Component No. 5 - Hydraulic Motor

$$\mathring{g}_{10} = K_{10}\, p_{10} + A_{10} \frac{d}{dt} p_{10} + K_{10,8}\, \mathring{\theta}_8 \tag{5.3.17}$$

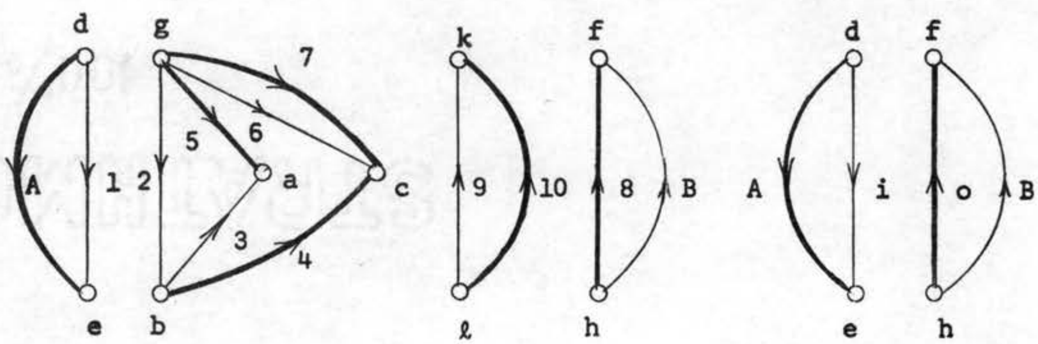
$$T_8 = - K_{8,10}\, p_{10} + B_8\, \mathring{\theta}_8 + J_8 \frac{d\, \mathring{\theta}_8}{dt} \tag{5.3.18}$$

It can be shown that the reduced state equations for this device can be written

$$\frac{d}{dt} \begin{bmatrix} p_{10} \\ \mathring{\theta}_8 \end{bmatrix} = \begin{bmatrix} -\frac{K_{10}}{A_{10}} & -\frac{K_{10,8}}{A_{10}} \\ \frac{K_{8,10}}{J_8} & -\frac{B_8}{J_8} \end{bmatrix} \begin{bmatrix} p_{10} \\ \mathring{\theta}_8 \end{bmatrix} + \begin{bmatrix} \frac{1}{A_{10}} & 0 \\ 0 & \frac{1}{J_8} \end{bmatrix} \begin{bmatrix} \mathring{g}_{10} \\ T_8 \end{bmatrix}. \tag{5.3.19}$$

A slight digression is in order at this point. In all of the cases encountered thus far, it has been possible to derive the state equations for the multiterminal components either by considering the device to be an interconnection of simpler (2-terminal components) or by inspection of the device's terminal equations. Not all components fall into one of these categories. It may be that the device's terminal equations involve time derivatives on both sides of the terminal equations, i.e., it may be that one cannot solve the terminal relations for n - 1 of the terminal variables explicitly. If this is the case, then one must resort to some other technique to derive the state equations for such a device. Zadeh and Desoer (16) have studied this problem extensively and present several procedures which seem to cover the 2-terminal case very well. These authors also present some material which might be adapted to fit cases where more than 2 terminals are involved.

Attention will now be focused on the problems of determining the state equations for the electromechanical-hydraulic device. The system graph including the assumed driving elements A and B is shown in Figure 5.3.7. The desired terminal graph along with the assumed driver elements is also shown in Figure 5.3.7.



(a) System Graph (b) Desired Terminal Graph

Figure 5.3.7. Linear Graph for an Electromechanical-Hydraulic Device

In accordance with the tree-selecting procedure given in Chapter IV, the formulation tree is selected so as to include elements A, 4, 5, 7, 10, and 8. This means element A must be considered as an across driver, while element B must be considered as a through driver. The interconnection equations defined by this tree are

$$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_A \\ x_5 \\ x_7 \\ p_{10} \\ \mathring{\theta}_8 \\ x_4 \\ x_3 \\ p_9 \\ x_2 \\ x_6 \\ v_1 \\ \mathring{\theta}_B \end{bmatrix} = 0 \tag{5.3.20}$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_A \\ f_5 \\ f_7 \\ \dot{g}_{10} \\ T_8 \\ f_4 \\ f_3 \\ \dot{g}_9 \\ f_2 \\ f_6 \\ i_1 \\ T_B \end{bmatrix} = 0 \; . \tag{5.3.21}$$

The direct sum of all component state equations in the form of Equations 4.3.7, 4.3.8, and 4.3.9 can be written as follows:

$$\begin{bmatrix} v_A \\ T_B \\ i_1 \end{bmatrix} = \begin{bmatrix} V_A\ (t) \\ T_B\ (t) \\ 0 \end{bmatrix} \tag{5.3.22}$$

$$\frac{d}{dt}\begin{bmatrix} x_5 \\ x_7 \\ p_{10} \\ \dot{\theta}_8 \\ x_2 \\ x_6 \\ \psi_3 \\ \psi_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{K_{10}}{A_{10}} & -\frac{K_{10,8}}{A_{10}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{K_{8,10}}{J_8} & -\frac{B_8}{J_8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{K_2}{B_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{K_6}{B_6} & \frac{K_{65}}{B_6} & 0 \\ -\frac{K_5}{M_5} & 0 & 0 & 0 & 0 & 0 & -\frac{B_5}{M_5} & 0 \\ 0 & -\frac{K_7}{M_7} & 0 & 0 & 0 & 0 & 0 & -\frac{B_7}{M_7} \end{bmatrix} \begin{bmatrix} x_5 \\ x_7 \\ p_{10} \\ \dot{\theta}_8 \\ x_2 \\ x_6 \\ \psi_3 \\ \psi_6 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{A_{10}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{J_8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{B_2} & 0 & \frac{K_{12}}{B_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{B_6} & 0 & 0 \\ \frac{1}{M_5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{M_7} & 0 & 0 & 0 & 0 & 0 & -\frac{K_{79}}{M_7} \end{bmatrix} \begin{bmatrix} f_5 \\ f_7 \\ \mathring{g}_{10} \\ T_8 \\ f_2 \\ f_6 \\ v_1 \\ p_9 \end{bmatrix} \qquad (5.3.23)$$

$$\begin{bmatrix} x_4 \\ f_3 \\ \dot{g}_9 \end{bmatrix} = \begin{bmatrix} 0 & -m & 0 \\ m & 0 & 0 \\ 0 & 0 & -K_{97} \end{bmatrix} \begin{bmatrix} f_4 \\ x_3 \\ x_7 \end{bmatrix} \qquad (5.3.24)$$

Suppose that the first two equations of Equation 5.3.24 are substituted into the first circuit equation (Equation 5.3.20) and the last cutset equations (Equation 5.3.21). The resulting equation can be solved for

$$\begin{bmatrix} f_4 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{m_o} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_2 \\ f_6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{m_o} & -\frac{1}{m_o} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_5 \\ x_7 \\ p_{10} \\ \dot{\theta}_8 \end{bmatrix} \qquad (5.3.25)$$

where $m_o = m + 1$. One can then write

$$\begin{bmatrix} x_4 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{m}{m_o} & 0 \end{bmatrix} \begin{bmatrix} f_2 \\ f_6 \end{bmatrix} + \begin{bmatrix} -\frac{m}{m_o} & \frac{m}{m_o} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_5 \\ x_7 \\ p_{10} \\ \dot{\theta}_8 \end{bmatrix}. \qquad (5.3.26)$$

Now, suppose that the last equation of the algebraic component equations (Equation 5.3.24) and Equation 5.3.26 are substituted into the second and fifth circuit equations and into the second through the fifth cutset equations. This operation will yield an explicit expression for the generalized input vector $[f_5, f_7, \dot{g}_{10}, T_8, f_2, f_6, v_1, p_9]^T$ that appears on the right of Equation 5.3.23. If this explicit expression is then substituted into Equation 5.3.23, the following set of first-order differential equations results.

$$
\frac{d}{dt}\begin{bmatrix} x_5 \\ x_7 \\ p_{10} \\ \mathring{\theta}_8 \\ x_2 \\ x_6 \\ \psi_3 \\ \psi_6 \end{bmatrix}
=
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\frac{1}{B_2} & 0 \\
0 & \frac{1}{B_6} \\
-\frac{1}{M_5}\frac{m}{m_o} & 0 \\
\frac{1}{M}\left(\frac{m}{m_o} - 1\right) & -\frac{1}{M_7}
\end{bmatrix}
\begin{bmatrix} f_2 \\ f_6 \end{bmatrix}
+
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & -\frac{1}{J_8} \\
\frac{K_{12}}{B_2} & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix} v_A \\ T_B \end{bmatrix}
$$

$$
+
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & \frac{K_{97}}{A_{10}} & -\frac{K_{10}}{A_{10}} & -\frac{K_{10,8}}{A_{10}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{K_{8,10}}{J_9} & -\frac{B_8}{J_8} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{K_2}{B_2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{K_6}{B_6} & \frac{K_{65}}{B_6} & 0 \\
-\frac{K_5}{M_5} & 0 & 0 & 0 & 0 & 0 & -\frac{B_5}{M_5} & 0 \\
0 & -\frac{K_7}{M_7} & -\frac{K_{79}}{M_7} & 0 & 0 & 0 & 0 & -\frac{B_7}{M_7}
\end{bmatrix}
\begin{bmatrix} x_5 \\ x_7 \\ p_{10} \\ \mathring{\theta}_8 \\ x_2 \\ x_6 \\ \psi_3 \\ \psi_6 \end{bmatrix}
\tag{5.3.27}
$$

The situation here is identical to that discussed in conjunction with Equation 4.3.25. There are 8 first-order differential equations in 10 variables. The excessive variables can be eliminated by noting that the third and fourth circuit equations (middle row block of Equation 5.3.20) provide a means whereby the vector of variables appearing on the left of Equation 5.3.27 can be written as an explicit function of $[x_5, x_7, p_{10}, \theta_8, \psi_3, \psi_6]^T$ and the variable x_4. Equation 5.3.26 can be used to eliminate the dependence on x_4. Let the resulting expression be substituted into Equation 5.3.27 and let the result be premultiplied by the matrix

$$\underline{\Phi}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{m}{m_o} & -(1-\frac{m}{m_o}) & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} . \qquad (5.3.28)$$

The resulting set of equations contains 6 first-order differential equations in 8 variables two of which are f_2 and f_6. The operations described above also result in two linear algebraic equations which can be solved for an explicit expression for f_2 and f_6 in terms of the vectors $[x_5, x_7, p_{10}, \dot{\theta}_8, \psi_3, \psi_6]^T$ and $[v_A, T_B]^T$. Substitution of this explicit expression for f_2 and f_6 into the six differential equations yields the desired set of first-order differential equations in normal form. The sixth circuit equation and the first cutset equation give the associated algebraic terminal equations for the assumed driver elements A and B.

The six differential equations and the two algebraic equations mentioned in the previous paragraph contain terms in v_A, i_A, $\dot{\theta}_B$, and T_B. These variables must be replaced by their equivalent expressions in terms of the terminal variables associated with the elements of the desired terminal graph. These equivalent expressions can be obtained from the interconnection equations for the linear graph labeled desired terminal graph in Figure 5.3.7. At any rate, after elimination of v_A, i_A, $\dot{\theta}_B$, and T_B in favor of v_i, i_i, $\dot{\theta}_o$, and T_o, the six differential and two algebraic equations become

$$\frac{d}{dt}\begin{bmatrix} x_5 \\ x_7 \\ p_{10} \\ \dot{\theta}_8 \\ \psi_3 \\ \psi_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a_{32} & a_{33} & a_{34} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & 0 & a_{65} & a_{66} \end{bmatrix}\begin{bmatrix} x_5 \\ x_7 \\ p_{10} \\ \dot{\theta}_8 \\ \psi_3 \\ \psi_6 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & a_{48} \\ a_{57} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_i \\ T_o \end{bmatrix} \qquad (5.3.29)$$

and

$$\begin{bmatrix} i_i \\ \mathring{\theta}_o \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_5 \\ x_7 \\ p_{10} \\ \mathring{\theta}_8 \\ \psi_3 \\ \psi_6 \end{bmatrix} \qquad (5.3.30)$$

where the elements of the coefficient matrices in Equation 5.3.29 are defined as follows:

$$a_{32} = \frac{K_{97}}{A_{10}} \qquad a_{55} = -\frac{1}{M_5}\left(B_5 + \frac{m^2}{m_o{}^2} B\right)$$

$$a_{33} = -\frac{K_{10}}{A_{10}} \qquad a_{56} = B_2\left(1 - \frac{m}{m_o}\right)$$

$$a_{34} = -\frac{K_{10,8}}{A_{10}} \qquad a_{57} = \frac{m}{m_o}\ \frac{K_{12}}{M_5}$$

$$a_{43} = \frac{K_{8,10}}{J_8} \qquad a_{61} = \frac{m}{m_o}\ \frac{K}{M_7}\left(\frac{m}{m_o} - 1\right)$$

$$a_{44} = -\frac{B_8}{J_8} \qquad a_{62} = \frac{1}{M_7}\left[K_2\left(\frac{m}{m_o} - 1\right) - K_7 - K_6\right]$$

$$a_{48} = \frac{1}{J_8} \qquad a_{63} = -\frac{K_{79}}{M_7}$$

$$a_{51} = \frac{1}{M_5}\left(K_5 - \frac{m^2}{m_o} K_2\right) \qquad a_{65} = \frac{1}{M_7}\left[K_{65} + \frac{m}{m_o} B_2\left(\frac{m}{m_o} - 1\right)\right]$$

$$a_{52} = \frac{K_2}{M_5}\ \frac{m}{m_o}\left(1 - \frac{m}{m_o}\right) \qquad a_{66} = -\frac{1}{M_7}\left[B_7 + B_6 + B_2\left(\frac{m}{m_o} - 1\right)^2\right]$$

Equations 5.3.29 and 5.3.30 are the state equations for the electro-mechanical-hydraulic device shown in Figure 5.3.1 when that device is considered as a coupled 4-terminal component with external connections at points d, e, f, and h only. Those equations and the terminal graph consisting of elements i and o in part b of Figure 5.3.7 completely specify the terminal characteristics of the device.

An examination of Equations 5.3.22, 5.3.23, and 5.3.24 will show that Assumptions 4.3.2, 4.3.4, and 4.3.6 are not satisfied by the component state equations for this system. However, formulation of the state model is possible as demonstrated here. This merely illustrates that the assumptions presented in Chapter IV are only sufficient conditions and are not necessary.

This representation can now be used in a number of ways. The most obvious application is that one is now in a position to treat the entire device as simply a component in a larger system. An examination of Equations 5.3.29 and 5.3.30 will show that these equations meet the requirements of the formulation procedure given in Chapter IV and, therefore, that procedure might be applied to a system in which the device is a simple component.

Another observation is worthy of mention at this point. Several authors (11, 16, 20) have extensively studied various analog programming techniques for generating state equations of linear devices from a knowledge of the transfer function of the device. Since these techniques start with the transfer function, one in inherently forced to assume that there are no output loading effects. The set of state equations derived above for this rather complex electromechanical-hydraulic device do not suffer from this limitation. Another interesting characteristic of this procedure for deriving the state equations for a multiterminal component is that it appears that, with some slight modifications, it can be used to handle some nonlinear cases; whereas any technique which makes use of a transfer function is not applicable to any nonlinear cases.

CHAPTER VI

SUMMARY

6.1 Principal Results and Conclusions. Probably the most important result of this work is the establishment of the formulation procedure given in Chapter IV. Clearly, the state model for any system of linear continuous-time multiterminal components whose terminal state equations and whose interconnection topology satisfy the list of formal assumptions given in that chapter can be formulated in a straightforward manner.

The formulation procedure is applicable to two fundamental problems. These problems are (1) the formulation of the system state model as demonstrated by the first example in Chapter V and (2) the derivation of the component state equations for multiterminal components that can be considered as an interconnection of simpler multiterminal components. This latter application is demonstrated in both of the major examples given in Chapter V.

The procedure is applicable to any system whose component state equations and topology satisfy the assumptions of Chapter IV regardless of the form of the energy involved. That is, the procedure is applicable to electrical, mechanical, hydraulic, or any other form of system. It is applicable to systems in various forms. The second example in Chapter V is a graphic example of that fact.

In the opinion of the author, the state model or state equations for a multiterminal component or system that are derived using this formulation procedure enjoy one fundamental advantage when compared to the state model or state equations that are derived by other methods now in general use. This fundamental advantage is that the knowledge of an intermediate secondary mathematical model is not required if one applies the formulation procedure given in this thesis. All other methods now in use require that one have available some other secondary mathematical model such as the "mesh current" equations or possibly the transfer function. In the latter case, any state model which is derived from the transfer function suffers from the basic deficiency that all loading at the output terminals is ignored. State models and state equations derived in the manner demonstrated in this thesis do not make the assumption that one of the output variables is zero as is common practice in the derivation of transfer functions.

Another result of this thesis that is of some importance has to do with the theorem concerning the topological conditions that are necessary if a unique solution of the primary mathematical model is to exist. This theorem is an extension of the work of Wirth (18) with systems containing components whose terminal equations are strictly algebraic. The theorem as presented in this thesis admits to the possibility that some of the components may have terminal equations involving derivatives of the terminal variables. Thus, the theorem as presented here contains the work of Wirth as a special case.

Koenig and Tokad (8) presented a similar theorem recently. However, those authors placed some rather severe restrictions upon the form of the component terminal equations. In fact their work excluded all multiterminal components for which none of the state variables are also terminal variables. Subsystem Number 3 of the first example in Chapter V is an example of a component which would fall into the excluded category of the last mentioned authors. By considering a more general form for the state equations, as in this thesis, one can include such components into a theorem on necessary conditions.

Practically speaking, this theorem can be used to say that after one has eliminated, in favor of state variables, as many terminal variables in the primary mathematical model as possible, then the remaining algebraic component equations must be solvable for a subset of across variables corresponding to a subset of the branches of some formulation tree and a subset of through variables which correspond to a subset of the chords of some formulation tree. The two formulation trees mentioned here may or may not be distinct, although from the standpoint of practical formulation procedure it is desirable that they be identical.

6.2 Recommendations for Future Study. The fact that the formal assumptions given in Chapter IV as sufficient conditions to insure that one can formulate the state model of a system of multiterminal components are not also necessary is demonstrated in the second example of Chapter V. In particular the conditions of Assumption 4.3.2, 4.3.4, and 4.3.6 do not hold for all of the components considered in that example, yet it was possible to formulate the state model for the system using the major steps given the formulation procedure of Chapter IV. The problem here is that the assumptions mentioned above were included as a part of the general formulating scheme so as to insure that the two required matrix inverses would exist. It is felt that the restrictions given here are overly severe and that one might be able to relax these conditions, at least partially. This certainly appears to be possible in cases where one can restrict the class of components to be studied.

Along the same lines, one might be able to clarify to a certain extent the relationship between the necessary algebraic and topological conditions given in Chapter III and the other sufficient formulating conditions given in Chapter IV. In particular, one might be able to show that some subset of the sufficient conditions given in Chapter IV are also necessary.

Another problem that merits some consideration is that of classification of the various component state equations. It may be that one could classify the forms of the state equations (in reduced form or not) into a few classes and as a result be able to establish a formulating procedure accordingly. For example, one might be able to classify the form of the state equations according to whether or not the differential equations are explicit in an across variable, a through variable, or a state variable that is not a terminal variable. One might also consider classification according to the form of the algebraic equations that are an integral part of the state equations. At any rate, a table showing the various forms that the state equations might take for a representative sample of the classes of multiterminal components that a system engineer encounters would be useful.

Zadeh and Desoer (16) in their recent book attempted to solve the problem of finding the state model for an interconnection of components through the use of a "signal-state-graph" and some conventions with respect to the classification of certain terminal variables. It appears that the methods discussed in this thesis are applicable to the problem as described by those authors; and, in fact, the procedures presented here appear to be superior to those presented in the referenced work. However, this latter statement has not been fully researched by the author and surely warrants some attention.

Last, but not least, someone should consider the possibility of using a digital computer to perform the complete formulation and solution of the state model for a system of multiterminal components. Much work has been done for systems of two-terminal ideal linear elements such as resistors, capacitors, and inductors; but, insofar as the author knows, very little has been done with systems containing more complicated devices.

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APPENDIX

SOME DEFINITIONS AND THEOREMS

In the process of formulating the state model for a system of multiterminal components in Chapter IV, it was necessary to calculate the inverses of two matrices. A number of conditions were assumed to prevail in order to insure that the required inverse existed. The following definitions and theorems taken from matrix algebra are designed to show that these prescribed conditions are indeed sufficient to insure that the required matrix inverses exist.

Definition A.1. Positive Definite Matrix. An n x n real matrix $\underline{A}$ is said to be positive definite if it is symmetric and if for every non-zero vector $\underline{x}$, the quadratic form $\underline{x}^T \underline{A}\, \underline{x}$ satisfies

$$\underline{x}^T \underline{A}\, \underline{x} > 0 \,. \tag{A.1}$$

Definition A.2. Semi-Definite Matrix. An n x n real matrix $\underline{A}$ is said to be positive semi-definite if it is symmetric and if for every non-zero vector $\underline{x}$, the quadratic form $\underline{x}^T \underline{A}\, \underline{x}$ satisfies

$$\underline{x}^T \underline{A}\, \underline{x} \geq 0 \,. \tag{A.2}$$

Theorem A.1. If $\underline{A}$ is a real positive definite (semi-definite) n x n matrix and $\underline{B}$ is an n x n nonsingular matrix, then the matrix

$\underline{B}\ \underline{A}\ \underline{B}^T$ is positive definite (semi-definite).

Proof. See Hohn[1] and allow equality to include the semi-definite case.

Theorem A.2. A necessary and sufficient condition that a real n x n matrix $\underline{A}$ be positive definite (semi-definite) is that every principal minor determinant of the matrix $\underline{A}$ be greater than (greater than or equal to) zero.

Proof. This is merely a restatement of a theorem given by Browne.[2]

Theorem A.3. Let $\underline{A}$ be an n x n positive definite matrix and let $\underline{B}$ be an n x n nonsingular matrix. Then the square matrix

$$\underline{\Phi}_1 = \begin{bmatrix} \underline{B}\ \underline{A}\ \underline{B}^T & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} \tag{A.3}$$

is positive semi-definite.

Proof. The matrix $\underline{B}\ \underline{A}\ \underline{B}^T$ is positive definite by Theorem A.1; and, therefore, every principal minor determinant of $\underline{\Phi}_1$ is greater than or equal to zero. The conclusion that $\underline{\Phi}_1$ is positive semi-definite follows directly from Theorem A.2.

Theorem A.4. Let $\underline{A}$ be a real positive definite n x n matrix and

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[1] F. E. Hohn, Elementary Matrix Algebra (New York, 1958), p. 257.

[2] E. T. Browne, The Theory of Determinants and Matrices (Chapel Hill, North Carolina, 1958), pp. 120-121.

let $\underline{B}$ be a real b x n matrix with rank $r < b \leq n$. Then $\underline{B}\ \underline{A}\ \underline{B}^T$ is positive semi-definite.

Proof. This proof is taken from Koenig, Tokad, and Kesavan (24). There exist a b x b nonsingular matrix $\underline{L}$ and an n x n nonsingular matrix $\underline{K}$ such that

$$\underline{L}\ \underline{B}\ \underline{K} = \begin{bmatrix} \underline{B}_{11} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} \qquad (A.4)$$

where $\underline{B}_{11}$ is r x r and nonsingular. Then

$$(\underline{L}\ \underline{B}\ \underline{K})^T = \underline{K}^T\ \underline{B}^T\ \underline{L}^T = \begin{bmatrix} \underline{B}_{11}^T & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix}. \qquad (A.5)$$

Also

$$\underline{B} = \underline{L}^{-1} \begin{bmatrix} \underline{B}_{11} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} \underline{K}^{-1} \qquad (A.6)$$

and

$$\underline{B}^T = (\underline{K}^T)^{-1} \begin{bmatrix} \underline{B}_{11}^T & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} (\underline{L}^T)^{-1} . \qquad (A.7)$$

Thus one can form

$$\underline{B}\ \underline{A}\ \underline{B}^T = \underline{L}^{-1} \begin{bmatrix} \underline{B}_{11} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} \underline{K}^{-1}\ \underline{A}\ (\underline{K}^T)^{-1} \begin{bmatrix} \underline{B}_{11}^T & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} (\underline{L}^T)^{-1} . \qquad (A.8)$$

The three matrices appearing in the center of the right-hand side of

Equation A.8 can be written

$$\underline{C} = \underline{K}^{-1} \, \underline{A} \, (\underline{K}^{T})^{-1} = (\underline{K}^{-1}) \, \underline{A} \, (\underline{K}^{-1})^{T} \tag{A.9}$$

which by Theorem A.1 is positive definite. $\underline{C}$ can be written in the partioned form

$$\underline{C} = \begin{bmatrix} \underline{C}_{11} & \underline{C}_{12} \\ \underline{C}_{21} & \underline{C}_{22} \end{bmatrix} \tag{A.10}$$

where $\underline{C}_{11}$ is r x r and $\underline{C}_{22}$ is (n - r) x (n - r). Theorem A.2 implies that $\underline{C}_{11}$ is positive definite. Substitution of Equation A.9 into Equation A.8 and Equation A.10 into the result yields

$$\underline{B} \, \underline{A} \, \underline{B}^{T} = \underline{L}^{-1} \begin{bmatrix} \underline{B}_{11} \, \underline{C}_{11} \, \underline{B}_{11}^{T} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} (\underline{L}^{-1})^{T} = \underline{L}^{-1} \, \underline{D} \, (\underline{L}^{-1})^{T} \tag{A.11}$$

where

$$\underline{D} = \begin{bmatrix} \underline{B}_{11} \, \underline{C}_{11} \, \underline{B}_{11}^{T} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} \tag{A.12}$$

is positive semi-definite by Theorem A.3. Thus Theorem A.1 implies that $\underline{B} \, \underline{A} \, \underline{B}^{T}$ is positive semi-definite.

Theorem A.5. Let $\underline{A}_{11}$ be an n x n positive definite matrix and let $\underline{A}_{22}$ be an n x n positive semi-definite matrix. Then the sum $\underline{A}_{11}$ + $\underline{A}_{22}$ is nonsingular.

Proof. The quadratic form

$$\underline{x}^{T} \, (\underline{A}_{11} + \underline{A}_{22}) \, \underline{x} = \underline{x}^{T} \, \underline{A}_{11} \, \underline{x} + \underline{x}^{T} \, \underline{A}_{22} \, \underline{x} > 0$$

for every non-zero vector $\underline{x}$ since $\underline{x}^T \underline{A}_{11} \underline{x}$ is strictly greater than zero and $\underline{x}^T \underline{A}_{22} \underline{x}$ is never less than zero by hypothesis. Hence $\underline{A}_{11} + \underline{A}_{22}$ is positive definite and by Theorem A.2 nonsingular.

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