STRICTLY CONVEX METRICS AND NATURAL DELTA FUNCTIONS

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PREFACE

This paper will be concerned with two special types of metrics, namely those which are convex and those which are strictly convex. In addition, a particular function is investigated, the natural delta function. Chapter I is an introductory chapter giving the definitions of the above mentioned metrics. In Chapter II the relationship between the convex metric and the strictly convex metric is considered. Chapter III combines a general result on the structure of continua which are strictly convex metrizable with a complete characterization of such continua in the plane. The material of Chapter IV is devoted to the study of similarities and contrasts between the two types of metrics. In Chapter V the natural delta function is defined and its relationships to other concepts, including that of a strictly convex metric, are considered. Chapter VI contains a summary of the results. The results in this paper rely heavily upon the material in Whyburn [13] and in Moore [12]. (Numbers in brackets refer to the bibliography at the end of the paper).

I should like to express my sincere appreciation to Olan H. Hamilton for his wise counsel during the preparation of this thesis; to the other members of my advisory committee, R. B. Deal, Eugene K. McLachlan, John E. Hoffman, and J. D. Parker; to L. Wayne Johnson for his innumerable efforts in my behalf; to the National Science Foundation for the two fellowships; and, most of all, to my wife and children.

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CHAPTER I

INTRODUCTION

This paper will be devoted to the development of certain properties of convex and strictly convex metrics, the nature of spaces which allow such metrics, and the relationships between these spaces. In addition, the natural delta function will be defined and some of its properties investigated.

A topological space is metrizable if there is a distance function $D(x,y)$ such that if x,y, and z are points, then

(1) $D(x,y) \geq 0$, the equality holding only if $x = y$,

(2) $D(x,y) = D(y,x)$ (symmetry),

(3) $D(x,y) \leq D(x,z) + D(z,y)$ (triangle condition),

(4) $D(x,y)$ preserves limit points.

Menger, in [9], defined the metric $D(x,y)$ to be convex if it has the additional property that

(5) for each pair of points x,y there is a point u such that $D(x,u) = D(u,y) = D(x,y)/2$.

A subset M of a space S is said to have a convex metric (even though S may have no metric) if the subspace M of S has a convex metric.

Menger proved in [9] that a compact continuum is locally connected if it has a convex metric; showed that M is convexifiable if it possesses a metric D such that for each point p of M and each positive number ϵ there is an open subset R of M containing p such that each point of R can be joined in M to p by a rectifiable arc of length (under D) less than ϵ; and raised the question as to whether or not a compact, locally connected continuum M can be assigned a convex metric. This question was answered in the affirmative by Bing and Moise in [4] and [10], respectively, but not until after twenty years had elapsed and a number of attempts had been made by others. In [8], Kuratowski and Whyburn proved that M has a convex metric if each of its cyclic elements does. Beer considered in [1] the case where M is one-dimensional. Harrold, in [7], found M to be convexifiable if it has the additional property of being a plane continuum with only a finite number of complementary domains.

It was the above mentioned results which led this author to consider a metric which, in addition to being a convex metric, has the property that the point u of (5) is unique.

Definition 1.1. The metric D(x,y) is said to be a strictly convex metric if it has the additional property that

(5') for each pair of points x,y there is a unique point u such that $D(x,u) = D(u,y) = D(x,y)/2$.

Definition 1.2. Let S be a topological space with topology T. Then S is said to be c-metrizable if and only if it is possible to define a convex metric which will induce the topology T. Furthermore, S is said to be sc-metrizable if and only if it is possible to define a strictly convex metric which will induce the topology T.

In Chapter II the basic relationship between the concepts of convex metric and strictly convex metric are discussed. It is shown that the requirement (5') is equivalent to the requirement that each pair of points x,y determines a unique arc whose length under D is equal to D(x,y). Using this fact a strictly convex metric is characterized as a convex metric such that each pair of points determines a unique arc of length equal to the distance between the points. In this discussion the notion of arc length in an arbitrary metric space is required and is obtained in a manner analogous to that used in E_n, by using a partition of the arc and defining for the partition $P = \{x_o, x_1, \cdots, x_n\}$ the number $\pi(P) = \Sigma\, D(x_{i-1}, x_i)$; $i = 1,2,\cdots,n$, where D is the metric. The number $\pi(P)$ then plays the role of the length of the inscribed polygon in E_n. The notions of norm, refinement, etc., all follow easily and the length of the arc A, from x_o to x_n, is defined by sup $\{\pi(P): P \in \mathcal{P}(A)\}$ where $\mathcal{P}(A)$ is the collection of all partitions of A. If this number exists and is finite, A is rectifiable and the length of A is denoted by $\ell(A)$.

Chapter III is concerned with the study of plane continua which are sc-metrizable. It is first shown that no compact continuum which is sc-metrizable can separate the plane. The primary result of the chapter is that any compact and locally connected continuum which can be represented as the countable union of sc-metrizable continua satisfying certain conditions has a strictly convex metric which preserves the metrics on each of the countable collection of continua. It is shown that any locally connected and point-like plane continuum can be so expressed and is therefore sc-metrizable. The sc-metrizable continua in the plane are then characterized. Some of the results can easily be seen to be adaptable to a more general setting.

Chapter IV is devoted to a comparison of some of the properties of c-metrizable and sc-metrizable continua. The properties of being c-metrizable and sc-metrizable are seen to be topological properties. Every sc-metrizable continuum is unicoherent. An example is given of a convex metric on a closed 2-cell which is not a strictly convex metric. The Cartesian product of two c-metrizable continua is shown to be c-metrizable. The notion of D-convex hull is defined in a manner which is analogous to the concept in a linear space.

In Chapter V the natural delta function is defined; is a non-negative, non-decreasing and left-continuous function which is bounded and therefore, Riemann integrable. The natural delta function is a mapping of C(K) X R^+ into R, where C(K) is the collection of continuous functions on a compact, metric continuum K. The integral of the delta function is a uniformly continuous function from C(K) into R, where C(K) has the topology of uniform convergence. A necessary and sufficient condition that the delta function be continuous is stated. The right-hand derivative of the delta function at zero is shown to be a determining factor relative to Lipschitz conditions, complex functions and periodic points. Finally, the natural delta function is determined to be super-additive when the set K has a strictly convex metric.

CHAPTER II

RELATIONSHIP BETWEEN CONVEX METRICS AND STRICTLY CONVEX METRICS

The purpose of this chapter is to relate the concept of a convex metric with that of the strictly convex metric. It will be shown that a strictly convex metric is a convex metric for which each pair of points determines a unique arc whose length is given by the distance between the points under that metric.

Lemma 2.1. Let M be a compact continuum with a convex metric D and having the property that if x and y are any two points of M, there exists a unique arc [x,y] between x and y such that $\ell[x,y] = D(x,y)$. Then if z is any point of [x,y], $D(x,z) + D(z,y) = D(x,y)$.

Proof. Let z be a point of [x,y] distinct from x and y. By the triangle inequality, $D(x,z) + D(z,y) \geq D(x,y)$. Since [x,y] is an arc, $[x,y] - z = H_x \cup K_y$ where $\overline{H}_x$ is an arc from x to z and $\overline{K}_y$ is an arc from z to y. Then $\overline{H}_x$ and $\overline{K}_y$ are rectifiable with the metric D and $\ell(\overline{H}_x) + \ell(\overline{K}_y) = \ell[x,y]$. By hypothesis, there exist unique arcs [x,z] and [z,y] such that $\ell[x,z] = D(x,z)$ and $\ell[z,y] = D(z,y)$. Thus $D(x,z) + D(z,y) \leq D(x,y)$ and the equality is established.

Theorem 2.1. Let M be a compact continuum satisfying the hypothesis of Lemma 2.1. Then the metric D of Lemma 2.1 is a strictly convex metric for M.

Proof. Assume D is not a strictly convex metric. There exist points x,y,z and v, all distinct, such that

$$D(x,z) = D(z,y) = D(x,v) = D(v,y) = D(x,y)/2.$$

Without loss of generality, z can be taken in [x,y], since there is a point z in [x,y] such that $D(x,z) = D(x,y)/2$, and from Lemma 2.1,

$$D(z,y) = D(x,y) - D(x,z) = D(x,y) - D(x,y)/2 = D(x,y)/2.$$

Now v does not lie on [x,y], for if it does, one of the points z, v must precede the other in the order from x to y. Suppose z precedes v. Then $D(x,v) \leq D(x,z) + D(z,v)$ and since $z \neq v$ and $D(x,v) = D(x,z)$, $D(x,v) < D(x,z) + D(z,v)$. There exists an arc [x,v] such that $\ell[x,v]$ is less than the distance from x to v along [x,y]. If $\overline{K}_y$ is the arc from y to v along [x,y], then $\ell([x,v] \cup \overline{K}_y) < \ell[x,y]$. If this were the case, however, there would exist an arc lying in $[x,v] \cup \overline{K}_y$ and having length less than that of [x,y].

Since v is not in [x,y] and $D(x,v) = D(v,y) = D(x,y)/2$, there exists a pair of arcs [x,v] and [v,y] such that $\ell[x,v] = D(x,v)$ and $\ell[v,y] = D(v,y)$. Then $[x,v] \cup [v,y]$ contains an arc A from x to y and $\ell(A)$ cannot exceed the combined lengths of [x,v] and [v,y]. This implies, however, that $\ell(A) \leq D(x,y)$, contradicting the uniqueness of [x,y]. Hence, there must exist a unique point u such that $D(x,u) = D(u,y) = D(x,y)/2$ and that point must lie on [x,y].

The following lemma is presented with no pretense of originality. It is, instead, an elementary result of the theory of rectifiable curves included at this point for the purpose of completeness.

Lemma 2.2. Let M be a metric space with metric D and let A be an arc in M. Let $\{P_i(A)\}$ be a sequence of partitions of A such that $\cup P_i(A)$; $i \in I$, is dense in A. Then if $\sup \{\pi(P_i)\}$ is finite, A is rectifiable and $\ell(A) = \sup \{\pi(P_i)\}$.

Proof. The above result will be established by showing that for any partition $P = \{x_o, x_1, \ldots, x_m\}$ of A, $\pi(P)$ is bounded by $\sup \{\pi(P_i)\}$. It should be understood that by the term "sequence of partitions", as used above, is meant a nested sequence such that for each integer n, $P_n(A)$ is a refinement of $P_{n-1}(A)$. With this understanding and the hypothesis that $\cup P_i(A)$, $i \in I$, is dense in A, it is clear that for any positive real number ϵ there exists an integer $N > 0$ such that for $n > N$, and any point x_i of P there exists a point y_i of $P_n(A)$ such that $D(x_i, y_i) < \epsilon/2m$.

Now let x_{i-1} and x_i denote any pair of adjacent points of P and let y_{i-1} and y_i be the associated points of $P_n(A)$. Application of the triangle property then gives

$$\begin{aligned} D(x_{i-1}, x_i) &\leq D(x_{i-1}, y_{i-1}) + D(y_{i-1}, y_i) + D(y_i, x_i) \\ &< D(y_{i-1}, y_i) + \epsilon/m, \end{aligned}$$

from which it follows that

$$\begin{aligned} \pi(P) &= \Sigma\, D(x_{i-1}, x_i);\ i = 1,2,\ldots,m \\ &< \Sigma\, D(y_{i-1}, y_i) + \epsilon;\ i = 1,2,\ldots,m \\ &\leq \pi(P_n) + \epsilon;\ n > N \\ &\leq \sup \{\pi(P_i)\} + \epsilon;\ i = 1,2,\ldots, \end{aligned}$$

Since the above statement is true for every $\epsilon > 0$, $\pi(P) \leq \sup \{\pi(P_i)\}$ and the desired result is obtained.

<u>Theorem 2.2.</u> Let M be a compact continuum with a strictly convex metric D. If x and y are any two distinct points of M, there exists a unique arc, [x,y], from x to y such that $\ell[x,y] = D(x,y)$.

<u>Proof.</u> By definition there exists a unique point, call it U(1/2) in M such that

$$D(x,U(1/2)) = D(U(1/2),y) = D(x,y)/2.$$

Similarly, there exist unique points U(1/4) and U(3/4) in M such that

$$D(x,U(1/4)) = D(U(1/4),U(1/2)) = D(x,U(1/2))/2 = D(x,y)/4$$

and

$$D(U(1/2),U(3/4)) = D(U(3/4),y) = D(U(1/2),y)/2 = D(x,y)/4.$$

Applying the triangle property,

$$D(U(1/4),y) \leq D(U(1/4),U(1/2)) + D(U(1/2),y) = 3D(x,y)/4.$$

Assume $D(U(1/4),y) < 3D(x,y)/4$, then

$$\begin{aligned} D(x,y) &\leq D(x,U(1/4)) + D(U(1/4),y) \\ &< D(x,y)/4 + 3D(x,y)/4, \end{aligned}$$

which is a contradiction. Thus, $D(U(1/4),y) = 3D(x,y)/4$ and, in a similar manner, $D(x,U(3/4)) = 3D(x,y)/4$.

For each integer n, let $P_n(x,y) = \{U(m/2^n) \mid m = 0,1,\ldots,2^n\}$. Assume that for the positive integer n, $P_n(x,y)$ has been defined and has the property that for any two elements, $U(i/2^n)$ and $U(j/2^n)$,

$$D(U(i/2^n),U(j/2^n)) = |i - j| \cdot D(x,y)/2^n.$$

Now for each integer i; $i = 0,1,\ldots,2^n-1$, there exists a unique point q in M such that

$$\begin{aligned} D(U(i/2^n),q) = D(q,U((i+1)/2^n)) &= D(U(i/2^n),U((i+1)/2^n))/2 \\ &= D(x,y)/2^{n+1}. \end{aligned}$$

By repeating the argument of the above paragraph,

$$D(x,q) = |2i+1| \cdot D(x,y)/2^{n+1}$$

and

$$D(q,y) = |1 - (2i+1)/2^{n}| \cdot D(x,y)$$

Thus, $q = U((2i+1)/2^{n+1})$ is an element of $P_{n+1}(x,y)$ and $P_{n+1}(x,y)$ also has the above property.

It is easily seen from the above discussion that for each positive integer n, $P_n(x,y) \subset P_{n+1}(x,y)$, so that $\{P_i(x,y): i = 1,2,\ldots,\}$ is an expanding collection of compact sets. Let U represent the union of this collection. Then $\overline{U}$ is a compact set. Assume $\overline{U}$ is not connected. There exists a pair of disjoint closed, and therefore compact, sets, A and B, such that $\overline{U} = A \cup B$. If $k > 0$ represents the distance from A to B, there exists an integer N such that $1/2^N < k$. Suppose the point $U(1/2^N)$ is in A, then for each integer $n < N$, the point $U(1/2^n)$ is contained in A since $D(u(1/2^N),U(1/2^n)) < 1/2^N$. Also, for each $n \geq N$, if $U(i/2^n)$ is in A, then $U((i+1)/2^n)$ is in A for the same reason. Hence, $\overline{\bigcup_n P_n(x,y)}$; $n \geq N$, is contained in A.

Let $p = U(m/2^n)$ be a point of U, where $n < N$ and $0 < m < 2^n$. Then $p = \lim_{k \to \infty} U[(2^k m-1)/2^{n+k}]$ and since all but a finite number of points of U are in A, U is in A and $\overline{U} = A$. Thus, the assumption that $\overline{U}$ is not connected is false and $\overline{U}$ is a compact continuum.

Let z be a point of $\overline{U}$ and $D(x,z) = \alpha \cdot D(x,y)$. There exists a sequence of points $\{U(m_n/2^n)\}$; $n \in I$, such that $p = \lim_{n \to \infty} U(m_n/2^n)$, from which it follows that $\alpha = \lim_{n \to \infty} (m_n/2^n)$. On the other hand, let $\alpha \in (0,1)$ be a real number, there exists a sequence $\{m_n/2^n\}$ of real numbers such that $\alpha = \lim_{n \to \infty} (m_n/2^n)$ and a point $z = \lim_{n \to \infty} U(m_n/2^n)$ which has the property that $D(x,z) = \alpha \cdot D(x,y)$, by the continuity of D. Similarly, $D(z,y) = (1-\alpha) \cdot D(x,y)$.

Suppose there exists a pair of distinct points, z and w, in $\overline{U}$ such that $D(x,z) = D(x,w)$. Then $D(z,w) = c > 0$, and there exist sequences, $\{U(m_j/2^{n_j})\}$ and $\{U(m_i/2^{n_i})\}$ converging to z and w, respectively. There exists an integer $N > 0$ such that $1/2^N < c/4$, an integer $n_i > N$ such that $D(z,U(m_i/2^{n_i})) < 1/2^N$ and an integer $n_j > N$ such that $D(w,U(m_j/2^{n_j})) < 1/2^N$. Now, both $U(m_i/2^{n_i})$ and $U(m_j/2^{n_j})$ are points of $P_{n_k}(x,y)$, where $k = \max\{i; j\}$. Hence,

$$D(U(m_i/2^{n_i}),U(m_j/2^{n_j})) < 1/2^N,$$

which is a contradiction. Thus, $z = w$.

It follows now that if z is a point of $\overline{U} - \{x \cup y\}$, $D(x,z) = \alpha > 0$, and

$$\overline{U} - z = \{w: D(x,w) < \alpha\} \cup \{w: D(x,w) > \alpha\},$$

separated. $\overline{U}$ is a compact continuum with at most two non-cut points and is, therefore, an arc from x to y.

Since, as established above, for any positive integer n and any positive integer $i < 2^n$,

$$D(U((i-1)/2^n),U(i/2^n)) = D(x,y)/2^n,$$

the collections $P_n(x,y)$ may be considered as regular partitions of the arc $\overline{U}$, and for each integer n, $\pi(P_n) = \Sigma\ D(x,y)/2^n;\ i = 1,2,\ldots,2^n$, and $\pi(P_n) = D(x,y)$. It is obvious then, from Lemma 2.2, that $\ell(\overline{U}) = D(x,y)$.

It remains only to show that $\overline{U}$ is unique. Assume the contrary, that there exists an arc $A \neq \overline{U}$ from x to y such that $\ell(\overline{U}) = D(x,y) = \ell(A)$. Then there exists a point z of $\overline{U} - A$. Let $D(x,z) = \alpha \cdot D(x,y)$, there exists a point w in A, $w \neq z$, such that $D(x,w) = D(x,z)$. Now $D(z,y) \neq D(w,y)$, thus $D(w,y) > D(z,y)$. Consider the partition $P = \{x; w; y\}$

of A. Then

$$\pi(P) = D(x,w) + D(x,y) > D(x,y)$$

and $\ell(A) \geq \pi(P) > D(x,y)$, and the assumption is clearly false.

The following theorem, the result of Theorem 2.1 and Theorem 2.2, gives a characterization of the strictly convex metric in relation to the convex metric which will prove quite useful in later discussions.

Theorem 2.3. Let M be a compact continuum with a convex metric D. A necessary and sufficient condition that D be a strictly convex metric for M is that if x and y are any two points of M, there exists a unique arc, $[x,y]$, from x to y such that $\ell[x,y] = D[x,y]$.

CHAPTER III

STRICTLY CONVEX METRICS ON PLANE CONTINUA

This chapter is devoted to the study of continua in the plane which are sc-metrizable. It will be shown in this chapter that the collection of plane continua which are sc-metrizable is precisely the collection of all locally connected and point-like continua. This result will be obtained by establishing first that no compact continuum which is sc-metrizable can separate the plane. The converse will then be established by showing that every locally connected and point-like continuum in the plane can be represented as a particular combination of sc-metrizable sets and that, in general, such a structure is sc-metrizable.

<u>Lemma 3.1</u>. Let J be a simple closed curve in the plane, x and y two distinct points of J, and M an arc from x to y which contains no point of the bounded complementary domain of J. Let A_1 and A_2 be the arcs of J such that $A_1 \cup A_2 = J$ and $A_1 \cap A_2 = \{x \cup y\}$. If U is the unbounded complementary domain of $M \cup J$, then some point p of J - M is accessible from U. Also, if p is a point of A_1, then no point of $A_2 - M$ is accessible from U. (Figure 1)

<u>Proof</u>. Let q be a point of U and assume that no point of J - M is accessible from U. Now, M and J are locally connected and $M \cup J$ is locally connected. Hence, $M \cup J$ is a compact, locally connected

continuum separating the plane. The boundary of U, F(U), is a locally connected continuum by the Torhorst theorem [13-p 106], and F(U) $\subset$ (M $\cup$ J). Thus, F(U) is a bounded, locally connected continuum separating the plane, and every point of F(U) is accessible from U [13-p 112]. Since, by assumption, no point of J - M is accessible from U, F(U) must be a subset of M. Then F(U) contains a simple closed curve [13-p 107]. However, this is impossible since M is an arc. Therefore, the assumption is false and there must exist a point p of J - M which is accessible from U.

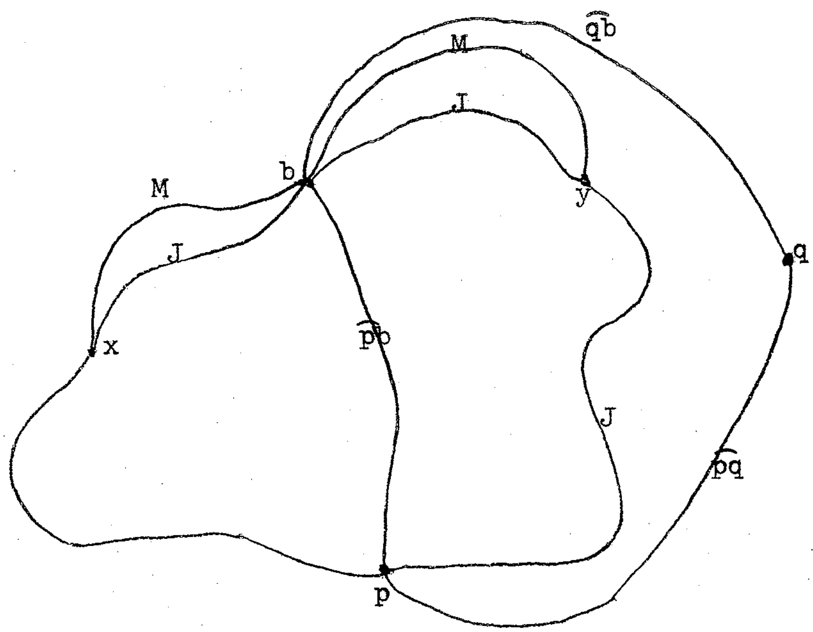

Figure 1.

Suppose p is a point of A_1 and assume there exists a point b of A_2 - M which is accessible from U. Let $\widehat{pq}$ be an arc from p to q such that p = $\widehat{pq} \cap$ F(U), there exists an arc $\widehat{qb}$ such that $\widehat{qb} \cap$ F(U) = b and $\widehat{pq} \cap \widehat{qb}$ = q. Then $p\widehat{q}b$ is an arc from p to b such that $p\widehat{q}b \cap$ F(U) = {p $\cup$ b}. Since p and b are points of J, there exists an arc $\widehat{pb}$ from b to p which lies, with the exception of its end points, entirely in the bounded complementary domain of J. Thus, $\widehat{pb} \cap \widehat{pqb} = \{p \cup b\}$ and $\widehat{pb} \cup \widehat{pqb} = J_1$ is a simple closed curve.

Now, J_1 is a simple closed curve containing p and b and must therefore, separate x from y. However, J_1 contains no point of the arc M, which is a contradiction. It follows that there can exist no point b in A_2 - M which is accessible from U.

Lemma 3.2. Let M be a compact plane continuum with a strictly convex metric D. Let J be a simple closed curve in M, and let x_o be a point of J such that for every point y of J - $\{x_o\}$ there exists a unique arc, $[x_o,y]$, containing no points of the bounded complementary domain I of J and having length equal to $D(x_o,y)$. For each point y of J - $\{x_o\}$ let E(y) be the unbounded complementary domain of $J \cup [x_o,y]$. Let $A_+(y)$ and $A_-(y)$ represent, respectively, the positively and negatively oriented arcs of J such that $A_+(y) \cup A_-(y) = J$ and $A_+(y) \cap A_-(y) = \{x_o \cup y\}$ (See Figure 2). Let P = {y: y $\in$ J - $\{x_o\}$, $A_-(y)$ contains a point of J - $[x_o,y]$ which is accessible from E(y)} and let N = {y: y $\in$ J - $\{x_o\}$, $A_+(y)$ contains a point of J - $[x_o,y]$ which is accessible from E(y)}. Then P and N are nonempty separated sets.

Proof. From Lemma 3.1, J - $\{x_o\}$ = N $\cup$ P. One of the sets, say P, must contain an uncountable number of points. Let y_o be a point of J which is a limit point of P distinct from x_o, there exists a sequence $\{y_i\}$ of points of P which converges to y_o. Let $\{[x_o,y_i]\}$ be the sequence of arcs such that $\ell[x_o,y_i] = D(x_o,y_i)$. By the theorem of Janiszewski [12-p 23], the limiting set L of $\{[x_o,y_i]\}$ is a compact continuum and

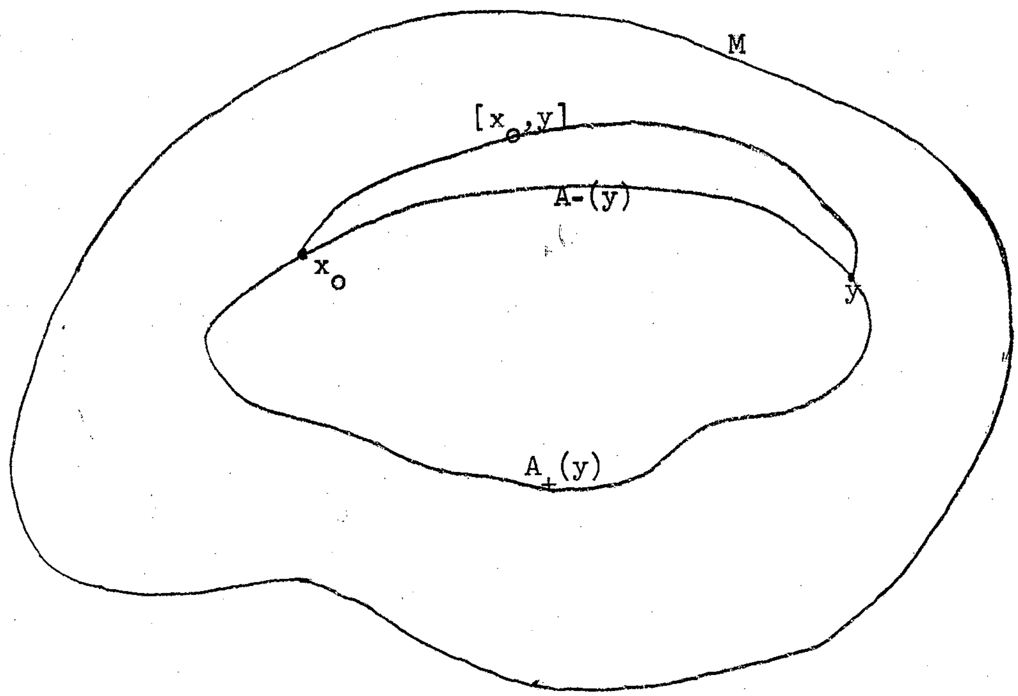

Figure 2.

there exists a subsequence $\{[x_o, y_j]\}$ which has L as a sequential limiting set [12-p 24]. For each real number α, $0 < \alpha < 1$, there exists a unique point $U(\alpha)$ of M such that $D(x_o, U(\alpha)) = \alpha \cdot D(x_o, y_o)$ and $D(U(\alpha), y_o) = (1 - \alpha) \cdot D(x_o, y_o)$. Also, for each j, there exists a unique point $U_j(\alpha)$ of M such that $D(x_o, U_j(\alpha)) = \alpha \cdot D(x_o, y_j)$ and $D(U_j(\alpha), y_j) = (1-\alpha) \cdot D(x_o, y_j)$. If z is a limit point of the set $\{U_j(\alpha)\}$, there exists a subsequence $\{U_k(\alpha)\}$ converging to z and

$$\begin{aligned} D(x_o, z) &= \lim_{k \to \infty} D(x_o, U_k(\alpha)) = \alpha \cdot \lim_{k \to \infty} D(x_o, y_k) \\ &= \alpha \cdot D(x_o, y_o) \end{aligned}$$

Also,

$$\begin{aligned} D(z, y_o) &= \lim_{k \to \infty} D(U_k(\alpha), y_k) = (1-\alpha) \cdot \lim_{k \to \infty} D(x_o, y_k) \\ &= (1-\alpha) \cdot D(x_o, y_o), \end{aligned}$$

from which it follows that $U(\alpha) = z$ is contained in L.

Thus, $[x_o, y_o]$ is in L and $\ell[x_o, y_o] = \lim_{k \to \infty} \ell[x_o, y_k]$.

Now, for each integer k, $F(E(y_k))$ is a simple closed curve [13-p 108]. Let $I(y_k)$ represent the bounded complementary domain of $F(E(y_k))$ and let q be a point of $A_+(y_o) - \{x_o \cup y_o\}$. There exists an integer $N > 0$ such that for every integer $k > N$, q is in $\overline{I(y_k)}$. Then q is either a point of $F(E(y_o))$ or is in $I(y_o)$. In either case q cannot be point of $A_+(y_o) - [x_o, y_o]$ which is accessible from $E(y_o)$. Hence y_o is an element of P, and by repetition of the above argument, N and P are mutually separated.

Finally, it must be established that P and N are non-empty. Assume N is empty and let $\{y_i\}$ be a sequence of points of J such that y_{i+1} is in $A_-(y_i)$ for each i, and the sequence $\{y_i\}$ converges to x_o. Let $\{y_k\}$ be the subsequence, as above, such that $\{[x_o, y_k]\}$ has a sequential limiting set, L. Let p be a point of I and let b be a point of the unbounded complementary domain of M. Then for each integer k, $\{[x_o, y_k] \cup A_-(y_k)\}$ is a simple closed curve separating p from b. The sequential limiting set for $\{[x_o, y_k] \cup A_-(y_k)\}$ is $L \cup \bigcap_{k=1}^{\infty} A_-(y_k)$, which also separates p from b. Also, $\{[x_o, y_k]\}$ must converge to the point x_o, since $\lim_{k \to \infty} D(x_o, y_k) = 0$. But this implies that $L - \{x_o\}$ separates p from b, which is a contradiction. Therefore, N is non-empty and the theorem follows.

<u>Theorem 3.1.</u> Let M be a compact plane continuum with a strictly convex metric D. Then M does not separate the plane.

<u>Proof.</u> Since M has a strictly convex metric, and therefore a convex metric, M is locally connected. Assume M separates the plane, then there exist points p and b which lie in disjoint complementary

domains of M. One of the points, say p, must lie in a bounded complementary domain, K, of M. Applying the Torhorst theorem [13-p 106], $F(K) \cap M$ is a compact, locally connected continuum separating p from b. Then there exists a simple closed curve, J, lying in F(K) and separating p from b.

Now let x and y be any two distinct points of J, and let D be the bounded complementary domain of $E_2 - J$. Suppose there exists an arc A in M from x to y such that $A \cap D$ is non-empty. If z is a point of $A \cap D$, then there must exist a last point q in $A \cap J$ such that q precedes z in the order from x to y, and a first point v of $A \cap J$ such that z precedes v in the order from x to y. The sub-arc $q\widehat{z}v$ of A lies in D, with the exception of its end points, q and v. The curve J is the union of two arcs, B and C, such that $B \cap C = \{q \cup v\}$. If $J_1 = B \cup q\widehat{z}v$ and $J_2 = C \cup q\widehat{z}v$, then J_1 and J_2 are simple closed curves having exactly the arc $q\widehat{z}v$ in common. The set $E_2 - (J_1 \cup J_2)$ has exactly two mutually exclusive bounded complementary domains, U_1 and U_2, which are bounded by J_1 and J_2, respectively [12-p 180]. Since $(J_1 \cup J_2) \subset M$, $K \subset (E_2 - M)$ is connected and must lie either in U_1 or in U_2. If K is in U_1, some point of C must fail to lie in F(K), and if K is in U_2, some point of B must fail to lie in F(K). In either case, a contradiction is obtained to the fact that J is a subset of F(K). Hence, every arc in M connecting two points of J must lie entirely in M - D.

Choose x_o to be a fixed base point in J and for every point y of $J - \{x_o\}$ let $[x_o,y]$ denote the unique arc in M such that $\ell[x_o,y] = D(x_o,y)$. By the preceding paragraph, $[x_o,y]$ must lie in M - D. In addition, each point y determines, with x_o, a pair of arcs, $A_+(y)$ and

$A_-(y)$, such that $A_+(y) \cup A_-(y) = J$ and $A_+(y)$ intersects $A_-(y)$ in $\{x_o \cup y\}$. Let $A_+(y)$ indicate the arc from x_o to y obtained by proceeding along J in a counter-clockwise manner, and let $A_-(y)$ be the remaining arc. By Lemma 3.1, points of one and only one of the sets, $A_+(y) - [x_o, y_o]$ and $A_-(y) - [x_o, y_o]$, are accessible from the unbounded complementary domain of $J \cup [x_o, y]$. Let P represent the set of points y of $J - \{x_o\}$ for which the set $A_-(y) - [x_o, y]$ contains such a point, and let N designate the set of points y of $J - \{x_o\}$ for which $A_+(y) - [x_o, y]$ contains such a point. Then $J - \{x_o\} = N \cup P$. By Lemma 3.2, however, P and N are mutually separated and J is separated by x_o, contradicting the assumption that J is a simple closed curve. Hence, the assumption that M separates the plane is false.

The remainder of the chapter will be devoted to the converse of the previous theorem. It will be established that every compact, locally connected plane continuum which does not separate the plane is sc-metrizable. This will be accomplished by showing that every such continuum has a particular composition and that, in general, any set so composed is sc-metrizable.

Lemma 3.3. Let $M_1, M_2, \ldots, M_n$ be a finite collection of nondegenerate compact continua with strictly convex metrics $d_1, d_2, \ldots, d_n$, respectively, and having the property that for each integer $k > 1$, $M_k \cap (\bigcup_{i=1}^{k-1} M_i)$ consists of a single point, b_{k-1}. Then there exists a strictly convex metric D_n on $\bigcup_{i=1}^{n} M_i$ which preserves the metric d_i on M_i for each integer i.

Proof: The above result can easily be established by induction.

Thus, it is only necessary, in this instance, to prove the statement for N = 2. Let M_1 and M_2 be two compact continua with strictly convex metrics d_1 and d_2, respectively, and let $b = M_1 \cap M_2$. Let $M = M_1 \cup M_2$ and define the function D_2 of M X M into R by:

$$D_2(x,y) = d_1(x,y);\ x,y \text{ in } M_1$$
$$= d_2(x,y);\ x,y \text{ in } M_2$$
$$= d_1(x,b) + d_2(b,y);\ x \text{ in } M_1,\ y \text{ in } M_2$$

Then D_2 is a metric on M by the following:

(i) $D_2(x,x) = d_1(x,x) = 0$ if x is in M_1

$= d_2(x,x) = 0$ if x is in M_2

(ii) Obviously, $D_2(x,y) \geq 0$. If $D_2(x,y) = 0$ and x and y both lie in M_1, then $D_2(x,y) = d_1(x,y) = 0$ implies $x = y$. If x and y are both elements of M_2, then $D_2(x,y) = d_2(x,y) = 0$ implies $x = y$. If x is in M_1 and y in M_2, then $D_2(x,y) = d_1(x,b) + d_2(b,y) = 0$ implies $x = b = y$.

(iii) Let x,y and z be any three points of M. If all three lie in M_1, or in M_2, the triangle property for D_2 is obtained from the original metric, d_1 or d_2, whichever the case. Assume then that one of the points, say z, is an element of M_2 while x and y are in M_1. Then $D_2(x,y) = d_1(x,y)$, $D_2(x,z) = d_1(x,b) + d_2(b,z)$, and $D_2(z,y) = d_1(y,b) + d_2(b,y)$. Hence,

$$D_2(x,y) = d_1(x,y)$$
$$\leq d_1(x,b) + d_2(b,y)$$
$$\leq D_2(x,z) + D_2(z,y)$$

Also,
$$D_2(x,z) = d_1(x,b) + d_2(b,z)$$
$$\leq d_1(x,y) + d_1(y,b) + d_2(b,z)$$
$$= D_2(x,y) + D_2(y,z)$$

(iv) Let x be a point of M and $\{x_i\}$ a sequence of distinct points of M which converges to x. If x is a point of $M_1 - \{b\}$, there exists an integer $N > 0$ such that $i > N$ implies x_i is a point of $M_1 - \{b\}$, since $M_1 - \{b\}$ is open in M. Thus,

$$\lim_{i \to \infty} D_2(x_i, x) = \lim_{i \to \infty} d_1(x_i, x) = 0$$

Similarly, if x is a point of $M_2 - \{b\}$, then

$$\lim_{i \to \infty} D_2(x_i, x) = \lim_{i \to \infty} d_2(x_i, x) = 0$$

If $x = b$, there exists a subsequence $\{x_j\}$ of $\{x_i\}$ which lies in only one of the sets. In either case, $\lim_{i \to \infty} D_2(x_i, b) = 0$. Thus D_2 preserves limit points.

It remains to be shown that the metric D_2 is a strictly convex metric. This can be accomplished by showing that there exists a unique arc $[x,y]$ between the arbitrary points x and y such that $\ell[x,y] = D_2(x,y)$. If both points lie in the same subset, M_1 or M_2, this fact is obvious from the original metrics. If x is a point of M_1 and y is a point of M_2, any arc from x to y in M must contain the point b. Then the arc $[x,y]$ determined by $[x,b]_1 + [b,y]_2$ where $\ell[x,b] = d_1(x,b)$ and $\ell[b,y] = d_2(b,y)$ has the property that $\ell[x,y] = D_2(x,y)$ and takes its uniqueness from the uniqueness of $[x,b]$ and $[b,y]$.

<u>Theorem 3.2</u>. Let $\{M_i\}$ be a countable collection of nondegenerate compact continua satisfying the following conditions:

(i) $M = \bigcup M_i$, where $\bar{M}$ is compact and locally connected and $\bar{M} - M$ has no nondegenerate component,

(ii) For each integer $n > 1$, $M_n \cap \bigcup_{i=1}^{n-1} M_i$ consists of exactly one point, b_{n-1}, which separates $M_n - \{b_{n-1}\}$ from $\bigcup_{i=1}^{n-1} M_i - \{b_{n-1}\}$ in $\bar{M}$,

(iii) Each M_i has a strictly convex metric d_i such that M_i has diameter W_i under d_i,

(iv) The series ΣW_i converges.

Then $\overline{M}$ has a strictly convex metric D which preserves d_i on M_i for each i.

Proof: By Lemma 3.3, there exists for each positive integer n, a strictly convex metric D_n on $\bigcup_{i=1}^{n} M_i$ which preserves d_i on M_i for $i = 1,2,\ldots,n$. Then for $k > n$, D_k preserves D_n on $\bigcup_{i=1}^{n} M_i$. Define a function $D : \overline{M} \times \overline{M} \to R$ as follows:

$$D(x,y) = \lim_{n \to \infty} D_n(x,y);\ x,y \in M$$
$$= \lim_{i \to \infty} D(x_i,y_i);\ x,y \in \overline{M};\ x_i,y_i \in M,\ x_i \to x,\ y_i \to y.$$

(1) $D(x,y)$ is well defined on M.

By Lemma 3.3, there exists, for each pair of points, x and y, of M an integer N such that $D_k(x,y) = D_n(x,y)$ for $k > N$. Then $D(x,y) = \lim_{n \to \infty} D_n(x,y) = D_N(x,y)$.

(2) In M, $D(x,y) \geq 0$, $D(x,y) = D(y,x)$ and $D(x,y) = 0$ if and only if $x = y$.

These properties all follow from the associated properties on D_N by the discussion in (1).

(3) Let x,y and z be any three points of M. Then $D(x,y) \leq D(x,z) + D(z,y)$.

As above, there exists an integer N such that $\{x \cup y \cup z\} \subset \bigcup_{i=1}^{N} M_i$. Thus,

$$D(x,y) = D_N(x,y) \leq D_N(x,z) + D_N(z,y) = D(x,z) + D(z,y).$$

In the following discussion let $B = \bigcup_{i=1}^{\infty} b_i$ where $\{b_i\}$ is the collection of points mentioned in (ii) of the hypothesis.

(4) If x is a point of $\overline{M} - M$, then $x \in B'$.

Let $\{x_j\}$ be a sequence of points of M which converges to x and has the property that no two points of the sequence lie in the same element of the collection $\{M_i\}$. For each j, let M_j be the continuum of $\{M_i\}$ which contains x_j and let b_{j-1} be the point of M_j satisfying (ii). Assume $\{b_{j-1}\}$ does not converge to x. Then there exists a subsequence $\{b_{k-1}\}$ which converges to a point $y \neq x$. The sequence $\{M_k\}$ has a limiting set L which is a compact continuum [12-p. 23]. Also, L is nondegenerate, since $\{x \cup y\} \subset L$. By (i), L contains a point p of $M - \overline{B}$. Since $\overline{M}$ is locally connected, $\overline{M} - \overline{B}$ is locally connected. However, $\overline{M} - \overline{B}$ cannot be locally connected at p, since every region of p contains points of infinitely many of the $\{M_i\}$. Thus the assumption that $x \notin B'$ is false.

(5) If x is a point of M_k and x is a limit point of $\overline{M} - M_k$, then x is a limit point of B'.

This result can be established by applying the same argument as that used in the preceeding statement.

(6) If x is a point of $\overline{M}$ and $\{x_i\}$ and $\{a_i\}$ are any two sequences in M converging to x, then $\lim_{i \to \infty} D(x_i, a_i) = 0$.

If x is a point of $\overline{M} - M$, there exists a collection of compact continua $\{R_k\}$ which closes down on x and has the property that R_k intersects no element of the collection $\{M_i\}$ which has a lower subscript than k. For each R_k, let a_k and x_k be, respectively, the first elements of the sequences $\{a_i\}$ and $\{x_i\}$ which lie in R_k. Then $D(a_k, x_k) \leq \sum_{i=k}^{\infty} W_i$

and $\lim_{k \to \infty} D(a_k,x_k) = 0$, since $\sum_{i=1}^{\infty} W_i$ converges. Thus $\lim_{i \to \infty} D(x_i,a_i) = 0$.

If x is a point of M, let M_k be the set of lowest subscript which contains x. If x is not a limit point of $\overline{M} - M_k$, then all but a finite number of points of the sequences, $\{x_i\}$ and $\{a_i\}$, must lie in M_k. Then since D_k is a metric on $\bigcup_{i=1}^{k} M_i$, $\lim_{i \to \infty} D(x_i,a_i) = \lim_{i \to \infty} D_k(x_i,a_i) = 0$. If x is a limit point of $\overline{M} - M$, then each of the sequences, $\{x_i\}$ and $\{a_i\}$, must contain points of infinitely many of the elements of the collection $\{M_i\}$. Now, by repetition of the argument of the preceeding paragraph, with the exception that the R_j intersects no point of the collection $\{M_i\}$ of lower subscript than j, other than M_k, it can be established that $\lim_{i \to \infty} D(x_i,a_i) = 0$.

(7) D(x,y) is well defined on $\overline{M}$.

Let x and y be any two points of $\overline{M}$, and let $\{x_i\}$, $\{a_i\}$ and $\{y_i\}$ be sequences in M such that $\{x_i\}$ and $\{a_i\}$ converge to x, while $\{y_i\}$ converges to y. For each integer i, $D(x_i,y_i) \leq D(x_i,a_i) + D(a_i,y_i)$ by the triangle property for M. Then

$$\lim_{i \to \infty} D(x_i,y_i) \leq \lim_{i \to \infty} D(x_i,a_i) + \lim_{i \to \infty} D(a_i,y_i),$$

and by (6) $\lim_{i \to \infty} D(x_i,a_i) = 0$. Hence $\lim_{i \to \infty} D(x_i,y_i) \leq \lim_{i \to \infty} D(a_i,y_i)$. By reversing the roles of a_i and x_i, $\lim_{i \to \infty} D(a_i,y_i) \leq \lim_{i \to \infty} D(x_i,y_i)$. Therefore $\lim_{i \to \infty} D(x_i,y_i) = \lim_{i \to \infty} D(a_i,y_i)$ and the value D(x,y) is shown to be independent of the choice of sequence.

(8) In $\overline{M}$, $D(x,y) \geq 0$ and $D(x,y) = D(y,x)$

This is an obvious result of the preceding discussion.

(9) For any three points x, y and z of $\overline{M}$, $D(x,y) \leq D(x,z) + D(z,y)$.

Let x, y and z be any three points of $\bar{M}$ and $\{x_i\}$, $\{y_i\}$ and $\{z_i\}$ be sequences of points of M converging, respectively, to x, y and z. For each integer i, the triangle property for M gives $D(x_i,y_i) \leq D(x_i,z_i) + D(z_i,y_i)$. Then,

$$\lim_{i \to \infty} D(x_i,y_i) \leq \lim_{i \to \infty} D(x_i,z_i) + \lim_{i \to \infty} D(z_i,y_i)$$

or $D(x,y) \leq D(x,z + D(z,y)$.

(10) If $\{x_i\}$ is a sequence in M converging to a point x of $\bar{M}$, then $\lim_{i \to \infty} D(x_i,x) = 0$. Also, for x in $\bar{M}$, $D(x,x) = 0$.

Let x and y be points of $\bar{M}$ and $\{x_i\}$ and $\{y_i\}$ sequences in M converging to x and y, respectively. If x = y, then $\{x_i\}$ and $\{y_i\}$ are both sequences converging to x. By (6), $D(x,y) = \lim_{i \to \infty} D(x_i,y_i) = 0$. Thus, if $\{x_i\}$ is a sequence in M converging to x, $\lim_{i \to \infty} D(x_i,x) = D(x,x) = 0$.

(11) If $\{x_i\}$ is a sequence in M converging to a point y and x is a point of $\bar{M}$ such that $\lim_{i \to \infty} D(x_i,x) = 0$, then $D(x,y) = 0$.

For each integer i, $D(x,y) \leq D(x,x_i) + D(x_i,y)$. By (10) $\lim_{i \to \infty} D(x_i,y) = 0$, and, by hypothesis, $\lim_{i \to \infty} D(x,x_i) = 0$. Thus, $D(x,y) = 0$.

(12) If $\{x_i\}$ is a sequence in $\bar{M}$ converging to the point x, then $\lim_{i \to \infty} D(x_i,x) = 0$.

For each integer i, there exists a sequence $\{z_{i_j}\}$ in M such that $D(x_i,z_{i_j}) < 1/ij$. Since $\{x_i\}$ converges to x, there exists a subsequence $\{z_{k\ell}\}$ which converges to x and has the property that $k_i\ell_i < k_j\ell_j$ when $i < j$. Designate this sequence by $\{z_k\}$ and the associated point of

$\{x_i\}$ by x_k. Then $\{z_k\}$ is a sequence in M converging to x. For each integer k, $D(x_k,x) \leq D(x_k,z_k) + D(z_k,x)$. By construction, $\lim_{k \to \infty} D(x_k,z_k) = 0$ and $\lim_{k \to \infty} D(z_k,x) = 0$ by (10). Thus $\lim_{k \to \infty} D(x_k,x) = 0$.

(13) In $\overline{M}$, if $D(x,y) = 0$, then $x = y$.

Suppose there exists in $\overline{M}$ a pair of distinct points x and y for which $D(x,y) = 0$. There exist distinct sequences $\{x_i\}$ and $\{y_i\}$ in M converging to x and y, respectively. For each integer i there exists an integer $N = N(i)$ such that $\{x_i \cup y_i\} \subset \bigcup_{i=1}^{N} M_i$, and an arc A_i from x_i to y_i in $\bigcup_{i=1}^{N} M_i$ such that $D(x_i,z) + D(z,y_i) = D(x_i,y_i)$ for every point z in A_i. Since $\overline{M}$ is compact, the collection $\{A_i\}$ is seen to satisfy [12-2] and must have a sequential limiting set L which contains both x and y, and is a compact continuum.

Let z be any point of L, there exists a sequence $\{z_i\}$ such that z_i is a point of A_i and $\{z_i\}$ converges to z. For each integer i, $D(x_i,z_i) + D(z_i,y_i) = D(x_i,y_i)$. Thus, $D(x,z) + D(z,y) = D(x,y) = 0$ and $D(x,z) = 0$.

If z and w are any two points of L, then $D(z,w) \leq (D(z,x) + D(x,w) = 0$. By hypothesis, however, L cannot lie entirely in $\overline{M} - M$ and there must exist points z and w in $L \cap M$. If z and w are distinct points in M, then $D(z,w) > 0$. Hence, a contradiction has been reached and the assumption that x and y are distinct is false.

(14) If $\{x_i\}$ is a sequence in $\overline{M}$, and x is a point of $\overline{M}$ for which $\lim_{i \to \infty} D(x_i,x) = 0$, then $\{x_i\}$ converges to x.

Let $\{x_i\}$ be a sequence in $\overline{M}$ such that $\lim_{i \to \infty} D(x_i,x) = 0$. Assume $\{x_i\}$ does not converge to x. Without loss of generality, $\{x_i\}$

can be taken as a sequence converging to $y \neq x$. By (12), $\lim_{i \to \infty} D(x_i,y) = 0$. For each integer i, $D(x,y) \leq D(x,x_i) + D(x_i,y)$. Then $D(x,y) \leq \lim_{i \to \infty} D(x,x_i) + \lim_{i \to \infty} D(x_i,y) = 0$. Therefore, $D(x,y) = 0$ and, by (13), $x = y$.

The above discussion establishes the fact that D is a metric on $\overline{M}$. It remains to be shown that D is strictly convex on $\overline{M}$. In the following discussion, it will be shown that each pair of points of $\overline{M}$ determines a unique arc in $\overline{M}$ whose length under D is equal to the distance, under D, between the points.

(15) Let A be an arc in $\overline{M}$ containing a point **x** of $\overline{M} - M$. Then x is an end point of A.

Assume the contrary, that $A - x = H \cup K$ disjoint, where $\overline{H} = H \cup x$ and $\overline{K} = K \cup x$ are arcs each having x as an end point. Then $\overline{H}$ and $\overline{K}$ must contain points of infinitely many of the sets $\{M_i\}$. Let j be the least integer for which $\overline{H} \cap M_j$ contains a point a and let k be the least integer for which $\overline{K} \cap M_k$ contains a point b. There exists an arc $N \subset A$ having a and b as end points and containing x. Let n be an integer for which $n > \max\{j,k\}$ and $(M_n \cap N) - B$ is non-empty. Let c be a point of $(M_n \cap N) - B$. The point b_{n-1} must lie in N and separate c from $\{a \cup b\}$ in N. However, this is impossible since b_{n-1} must either lie in the subarc determined by a and c or the subarc determined by c and b. Thus, x does not separate A.

(16) If x and y are points of M and A is an arc in $\overline{M}$ from x to y, there exists an integer N for which $A \subset \bigcup M_i; i=1,2,\ldots,N$.

There exists an integer N for which $\{x \cup y\} \subset C_N = \bigcup M_i; i=1,2,\ldots N$.

Suppose A is not contained in C_N. Let j be the least integer, $j > N$, and p a point of A such that $p \in M_j$. The point b_{j-1} must lie in $A \cap C_N$ and separate p from $\{x \cup y\}$ in A. The point p, however, determines in A two arcs, A_1 and A_2, from x to p and p to y, respectively such that $A_1 \cup A_2 = A$. Then the point b_{j-1} can lie in only one of the arcs A_1, A_2 and must fail to separate both x and y from p. Hence A must lie in C_N.

(17) If x and y are two points of M, there exists a unique arc $[x,y]$ in $\overline{M}$ such that $\ell[x,y] = D(x,y)$.

There must exist an integer N for which $C_N = \cup M_i; i=1,2,\cdots,N$, contains $\{x \cup y\}$. By (16), all arcs in $\overline{M}$ from x to y must lie in C_N. By (1), D preserves D_N on C_N and by Lemma 3.3, D_N is a strictly convex metric on C_N. Thus the arc $[x,y]$ in C_N such that $\ell[x,y] = D_N(x,y) = D(x,y)$ is unique in $\overline{M}$.

(18) If A is an arc in $\overline{M}$, then for each integer n such that $A \cap C_n$ is non-empty, $A \cap C_n$ is connected.

Let x and y be any two points of $A \cap C_n$. Then there exists an arc in A from x to y. By (16), every arc in $\overline{M}$ from x to y must lie in C_n. Therefore, $A \cap C_n$ is connected.

(19) Let x be a point of $\overline{M} - M$ and p a point of M. Let $\{A_\alpha\}$ be the collection of all arcs in $\overline{M}$ from p to x. Then there exists a sequence $\{b_k\}$ in $B \cap (\bigcap_\alpha A_\alpha)$ which converges to x.

Let A_1 and A_2 designate any two arcs of the collection $\{A_\alpha\}$ and let N be the first integer for which $p \in C_N$. For each integer $n > N$, let $j(n) > n$ and $k(n) > n$ denote, respectively, the least integers for which $A_1 \cap M_j$ and $A_2 \cap M_k$ are non-empty.

Assume there exists an integer $n > N$ such that $j(n) < k(n)$. Then there is a point q in $A_1 \cap M_j$, $q \neq b_{j-1}$. By (18), $A_1 \cap C_n$ is connected and contains p. Thus, there is an arc from q to x in $A_1 - b_{j-1}$. The point b_{j-1} cannot lie in A_2, since $A_1 \cap M_j = \emptyset$. But this implies b_{j-1} does not separate q from p in $\overline{M}$, contradicting the nature of b_{j-1}. Thus, for each integer n, $j(n) = k(n)$ and the point b_{k-1} must lie in $A_1 \cap A_2$.

Let $\{b_k\}$ denote the sequence so determined. Then $\{b_k\}$ is easily seen to converge to x by (18). Also, since $\{b_k\}$ is common to each pair of arcs from p to x, $\{b_k\}$ is contained in $\bigcap_{\alpha} A_\alpha$.

(20) Let x be a point of $\overline{M} - M$ and y a point of M. There exists in $\overline{M}$ a unique arc $[x,y]$ from y to x such that $\ell[x,y] = D(x,y)$.

Let $\{b_k\}$ be the sequence of (19). For each k, let $[b_k,y]$ be the unique arc in $\overline{M}$ for which $\ell[b_k,y] = D(b_k,y)$. If A is an arc in M from y to x, and b_j, b_ℓ, $j < \ell$ are any two points of the sequence $\{b_k\}$, then there exists an arc S in A from b_ℓ to x. From (16) and (18), S does not contain the point b_j. Assume b_j does not lie on $[y,b_k]$. Then $S \cup [y,b_k]$ contains an arc from y to x which does not contain b_j, contradicting (19). Thus, b_j lies on $[y,b_k]$ and $[y,b_j] \subset [y,b_k]$.

For each integer k, let $A_k = [y,b_k]$. The collection $\{A_k\}$ is a nested collection of sets, $A_j \subset A_{j+1}$ for each j. Let $\{x_i\}$ be any sequence in $\bigcup_k A_k$ which converges to a point p not in $\bigcup_k A_k$. Then there exists a subsequence $\{x_j\}$ such that for each integer n, x_j does not lie in A_n for $j > n$; and there exists a subsequence $\{b_j\}$ of $\{b_k\}$ such that for each integer j, x_j is in $[b_{j-1}, b_j]$. It follows that $D(x_j, b_j) \leq D(b_{j-1}, b_j)$ and $0 \leq \lim_{j \to \infty} D(x_j, b_j) \leq \lim_{j \to \infty} D(b_{j-1}, b_j) = 0$. Thus, $\{x_j\}$, and consequently $\{x_i\}$, must converge to x.

It is easily seen that neither x nor y separates $\overline{\bigcup_k A_k}$, and every other point p of $\overline{\bigcup_k A_k}$ does separate $\overline{\bigcup_k A_k}$. Therefore, $\overline{\bigcup_k A_k} = [y,x]$ is an arc in $\overline{M}$ from y to x and $\ell[y,x] = \lim_{k \to \infty} \ell[y,b_k] = \lim_{k \to \infty} D(y,b_k) = D(y,x)$.

Assuming $[y,x]$ is not unique, there exists an arc A in $\overline{M}$, from x to y, such that $\ell(A) = D(y,x)$ and $A \neq [y,x]$. No point of $\overline{M} - M$ other than x can lie on A or $[y,x]$ and the sequence $\{b_k\}$ must lie in both. Thus, A and $[y,x]$ must differ between y and b_k for some integer k. Let j be such an integer and let S designate the subarc of A from y to b_j. However, $\ell[y,b_j] < \ell(S)$ by the uniqueness of $[y,b_j]$ and $\ell[y,x] < \ell(A)$.

(21) If x and y are two points of $\overline{M} - M$, there exists a unique arc $[x,y]$ from x to y in $\overline{M}$ such that $\ell[x,y] = D(x,y)$.

If p is any point of M, there exists a sequence $\{b_k\}$ in B such that $\{b_k\}$ converges to y and lies in every arc of $\overline{M}$ from p to y. By (20), for each integer k, there exists a unique arc in $\overline{M}$ from x to b_k, $[x,b_k]$. The collection $\{[x,b_k]\}$ has a sequential limiting set L which is a compact continuum containing x and y.

If z is any point of L, there exists a sequence $\{z_k\}$ such that z_k is a point of $[x,b_k]$ and $\{z_k\}$ converges to z. For each integer k, $D(x,z_k) + D(z_k,b_k) = D(x,b_k)$ and

$$D(x,z) + D(z,y) = \lim_{k \to \infty} D(x,z_k) + \lim_{k \to \infty} D(z_k,b_k)$$

$$= \lim_{k \to \infty} D(x,b_k) = D(x,y).$$

For each α, $0 \leq \alpha \leq 1$, and each integer k, let $z(k,\alpha)$ be the unique point of $[x,b_k]$ for which $D(x,z(k,\alpha)) = \alpha D(x,b_k)$. There exists a point $z(\alpha)$ in L such that $D(x,z(\alpha)) = \alpha D(x,y)$. By repeating part of the argument of Theorem 2.2 it can be shown that L contains an arc, $[x,y]$, from x to y such that $D(x,z) + D(z,y) = D(x,y)$ for every point z of $[x,y]$. Obviously, $\ell[x,y] = D(x,y)$.

Let A be any other arc in $\overline{M}$ from x to y, neither A nor $[x,y]$ contains any other point of $\overline{M} - M$. The arcs A and $[x,y]$ must intersect in infinitely many points, since by (15), no arc can lie in $A \cup [x,y]$ and have x as a cut point. Let p and q be any two points in $A \cap [x,y]$, and let N be the least integer that $\{p \cup q\}$ is contained in C_N. By (16), both subarcs of A and $[x,y]$ determined by p and q must lie in C_N. The subarc of $[x,y]$ determined by p and q is actually $[p,q]$. Thus, if A is assumed to have length equal to $D(x,y)$, A must coincide with $[x,y]$ between p and q. Then A must coincide with $[x,y]$ on every subarc of $[x,y]$ and must, in fact, coincide with $[x,y]$. Hence $[x,y]$ is unique.

Corollary 3.2.1. Let M be a dendrite and $k > 0$ a real number. Then there exists a strictly convex metric D on M such that the diameter of M, under D, does not exceed k.

Proof: Let $\{e_i\}$ be the collection of all end points of $\overline{M}$. Then $\{e_i\}$ is countable. For each integer $i > 1$, let $[e_i,e_1]$ be the unique arc in $\overline{M}$ containing e_i and e_1. Let $A_1 = [e_2, e_1]$ and let b_1 be the first point of A_1 on $[e_3, e_1]$ in the order from e_3 to e_1. Let $A_2 = [e_3, b_1]$. In general, for each integer n, let b_n be the first point of $\bigcup_{i=1}^{n} A_i$ on $[e_{n+2}, e_1]$ in the order from e_{n+2} to e_1. Let $A_{n+1} = [e_{n+2}, b_n]$. For each arc A_n let h_n be the homeomorphism of A_n onto the interval $[0, k/2^n]$ such that $h_1(e_1) = h_n(b_{n-1}) = 0$ and $h_n(e_{n+1}) = k/2^n$.

It is easily seen that the above construction satisfies the hypothesis of Theorem 3.2, since $\{A_i\}$ is a countable collection of nondegenerate compact continua (arcs in this case) and:

(i) $A = \bigcup_{i=1}^{\infty} A_i$, where $M = \overline{A}$ is compact and locally connected and $\overline{A} - A$ has no nondegenerate component.

(ii) For each integer n, $A_n \cap \bigcup_{i=1}^{n-1} A_i = b_{n-1}$, which separates $A_n - \{b_{n-1}\}$ from $\bigcup_{i=1}^{n-1} A_i - \{b_{n-1}\}$ in M.

(iii) Each A_i has a strictly convex metric d_i, such that $d_i(x,y) = |h_i(x) - h_i(y)|$ and A_i has diameter $w_i = k/2^i$ under d_i.

(iv) $\sum_{i=1}^{\infty} w_i = \sum_{i=1}^{\infty} k/2^i = k$.

Thus, M has a strictly convex metric which preserves d_i on A_i for each integer i.

<u>Lemma 3.4</u>. Let M be a compact, locally connected plane continuum not separating the plane. Then $M = A^* \cup B^* \cup C$, where A is a countable collection of disjoint dendrites, B is a countable number of nondegenerate closed 2-cells, and C is a set having no nondegenerate component, such that (1) no two elements of A + B intersect in more than one point, (2) any point common to two elements of A + B is a cut point of M,

(3) $C = M - [A^* \cup B^*]$.

Proof: The proof of the above statement follows from several results in Whyburn [13].

The continuum M is a semi-locally connected continuum [13-p. 20], each true cyclic element (simple link) of M is a closed 2-cell [13-p. 172], and the true cyclic elements form a null sequence at most [13-p. 71]. Hence, M contains a countable number of closed 2-cells, $B = \{B_j\}$. Also, any two simple links in M can intersect in at most one point and any point common to two simple links is a cut point of M [13-p. 65].

Every simple link, B_j, of M contains at most a countable number of cut points of M [13-p. 65]. For each integer j, let B_j^o be the set of points of B_j which are not cut points of M. For each point x of $M - \bigcup_j B_j^o$, let H_x be the component of $M - \bigcup_j B_j^o$, containing x. Let C be the collection of degenerate components. For each nondegenerate H_x and each integer j, $H_x \cap B_j$ consists of at most one point, since H_x is contained in the closure of a component R of $M - B_j$ and $R \cap B_j$ is at most one point [13-p. 66]. Also, the common point, if it exists, is a cut point [13-p. 66].

Let y be a point of $M - \overline{\bigcup B_j}$. Then y is a point of some nondegenerate H_x and there exists a connected open region containing y whose closure intersects no point of $M - H_x$. Thus, each H_x is locally connected. Each H_x is a compact, locally connected continuum every cyclic element of which reduces to a single point and is by definition, a dendrite. Then the collection $A = \{H_x\}$ of nondegenerate components must be a countable collection of disjoint dendrites.

Proposition 3.1. Let M be a closed topological n-cell and $k > 0$ a real number. Then there exists a strictly convex metric D on M such that the diameter of M under D does not exceed k.

Proof: Let C be the closed ball in E_n with center at the origin and radius $k/2$, there exists a homeomorphism h of M onto C. For each pair of points of M let $D(x,y) = \rho[h(x),h(y)]$, where ρ is the standard Euclidean metric. The metric D is the desired metric.

Theorem 3.3. Let M be a compact, locally connected plane continuum not separating the plane. Then M has a strictly convex metric.

Proof: By Lemma 3.4, $M = A^* \cup B^* \cup C$ where $A = \{A_i\}$ is a countable collection of disjoint dendrites, $B = \{B_j\}$ is a countable collection of closed 2-cells, $C = M - (A^* \cup B^*)$ contains no nondegenerate component, for each i, j, $A_i \cap B_j$ is at most one point, and for $i \neq j$, $B_i \cap B_j$ is at most one point.

Case I: $M - C$ is connected.

Let $\{W_i\}$ be any sequence of positive real numbers such that $\sum_i W_i$ converges and let $H = A + B$ be the combined collection of dendrites and closed 2-cells. It will be shown that H can be so ordered that the collection $\{H_i\}$ will have the property that for each integer n, $H_n \cap \bigcup_{i=1}^{n-1} H_i$ consists of exactly one point of M and that point is a cut point of M. Proposition 3.1 and Corollary 3.2.1 will then make it possible, for each integer i, to give the element H_i a strictly convex metric d_i such that H_i has diameter W_i under d_i.

Choose any element of H and call it H_1. Let H_2 be any element of $H - \{H_1\}$ which intersects H_1. Such an element exists since M is connected. By Lemma 3.4, $H_1 \cap H_2$ consists of exactly one point, call it b_1. The element H_3 will then be chosen from the previously unchosen elements of H which intersect $H_1 \cup H_2$. It is obvious that the collection H can be ordered in this way, but it remains to be shown that for each integer n, $H_n \cap \bigcup_{i=1}^{n-1} H_i$ is a single point.

Assume that each of the first n - 1 elements has been shown to have the desired relationship with the union of those which preceed it. Suppose $H_n \cap \bigcup_{i=1}^{n-1} H_i$ contains more than one point. Let x and y be any pair of points in $H_n \cap \bigcup_{i=1}^{n-1} H_i$, they determine in $\bigcup_{i=1}^{n-1} H_i$ an arc K and in H_n an arc L such that $K \cap L = \{x \cup y\}$. Thus $K \cup L$ is a simple closed curve in M. This is impossible, however, since the only true cyclic elements of M are closed 2-cells and $K \cup L$ must lie in some element of B, meaning $K \cup L$ must lie in a single element of H. Thus, $H_n \cap \bigcup_{i=1}^{n-1} H_i$ is a single point, b_{n-1}, for each integer n. By Lemma 3.4, each point of the collection $\{b_{n-1}\}$ is a cut point of M. Repeating the argument used in this paragraph will show that, for each n, b_{n-1} separates $H_n - \{b_{n-1}\}$ from $\bigcup_{i=1}^{n-1} H_i - \{b_{n-1}\}$ in M.

It may now be observed that $M = H^* = \bigcup_{i=1}^{\infty} H_i$ and $M - H^*$ contains no nondegenerate component. Thus, M has been so described that it satisfies the hypothesis of Theorem 3.2 and must have a strictly convex metric.

Case II: M - C is not connected.

Let $K = \{K_n\}$ be the collection of components of M - C. Then $M = K^* \cup C$, K is countable and for each n, $\overline{K}_n$ is a compact and locally connected continuum satisfying the conditions of Case I. For each n, let $\overline{K}_n$ have a strictly convex metric d_n such that $\overline{K}_n$ has diameter W_n under d_n, where ΣW_n converges. The space $M' = K \cup C$ is an upper semi-continuous decomposition of M satisfying [13-p. 129], where the associated transformation $f : M \longrightarrow M'$ is monotonic and continuous. Hence, M' is a dendrite and has, by Corollary 3.2.1, a strictly convex metric d.

Let x and y be any pair of points of M and let $A[x,y]$ be the collection of all arcs in M from x to y. If A' and A'' are any two elements of $A[x,y]$, then $f(A') = f(A'')$. Thus $\overline{A' \cap C} = \overline{A'' \cap C}$ and A' can differ from A'' only in elements of K. Also if K_n is any element of K not containing x and if S_x is the component of $M - \overline{K}_n$ containing x, then $\overline{S}_x \cap \overline{K}_n$ contains exactly one point, by [12-p. 23] and the local connectedness of M.

Now, if x and y are any pair of points of M, let A be any arc in M from x to y. For each integer n, let a_n and b_n designate, respectively, the first and last points of $\overline{K}_n$ on A in the order from x to y and let

$$D(x,y) = d(f(x),f(y)) + \sum_{n=1}^{\infty} d_n(a_n,b_n)$$

where $d_n(a_n,b_n) = 0$ if $\overline{K}_n \cap A = \emptyset$. Then $D(x,y)$ is a metric and the arc $[x,y] = \overline{A \cap C} \cup (\bigcup_n [a_n,b_n]_n)$ where $[a_n,b_n]_n$ is the arc in $\overline{K}_n$ such that $\ell(a_n,b_n) = d_n(a_n,b_n)$, is the unique arc in M such that $\ell[x,y] = D(x,y)$.

<u>Theorem 3.4</u>. Let M be a compact and locally connected plane continuum. Then M is sc-metrizable if and only if M does not separate the plane.

<u>Proof</u>: The necessity is established in Theorem 3.1. Sufficiency is exhibited in Theorem 3.3.

CHAPTER IV

OTHER PROPERTIES OF CONVEX AND STRICTLY CONVEX METRICS

In this chapter additional properties of the convex and strictly convex metrics will be examined and some similarities and differences noted.

Proposition 4.1. Both c-metrizability and sc-metrizability are topological properties.

Proof: The result is obvious, for if G and H are any spaces such that G has a metric ρ and h is a homeomorphism from G onto H, the function $D(x,y) = \rho(h^{-1}(x), h^{-1}(y))$, where x and y are points of H, is a metric on H. Then h is an isometry from (G,ρ) to (H,D).

Bing showed [2] that if K is a closed subset of the metrizable space S and D_1 is a metric on K, then there is a metric D_2 on S that preserves D_1 on K. He later [3] used this result to show that if M_1 and M_2 are two intersecting compact continua with convex metrics D_1 and D_2, respectively, there is a convex metric D_3 on $M_1 \cup M_2$ that preserves D_1 on M_1. That the latter result is not valid when the word convex is replaced by strictly convex may be seen in the following example.

Example 1: Let M_1 and M_2 be the compact plane continua (Figure 3) obtained by taking the simple closed curves, J_1 and J_2, and their respective bounded complementary domains. In other words, M_1 is the closed 2-cell having J_1 as its boundary and M_2 is the closed 2-cell having J_2 as its boundary.

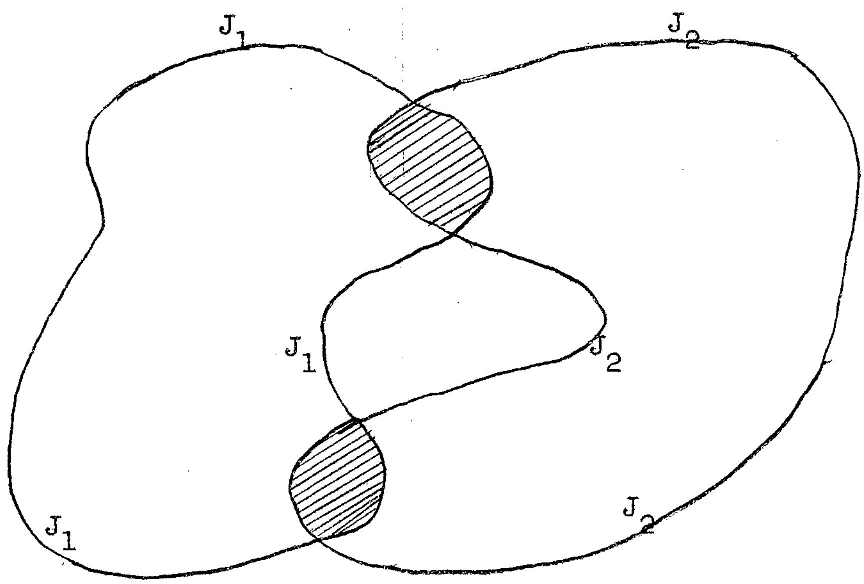

Figure 3.

Each of the two cells has a strictly convex metric induced by its homeomorphic relationship with the unit disk. However, $M_1 \cup M_2$ separates the plane and, as previously shown, can have no strictly convex metric.

It can be shown by using Lemma 3.4 that if $\overline{M_2 - M_1}$ consists of a finite number of components each intersecting M_1 in a single point, then there is a strictly convex metric D_3 on $M_1 \cup M_2$ which preserves D_1 on M_1. It remains an open question, however, as to whether this can be done in general since before the construction given by Bing in [3] can be a strictly convex metric there must exist for each pair of points,

x in M_1 and y in $M_2 - M_1$, a unique point z in $F(M_1) \cap M_2$ such that $D_3(x,y) = D_3(x,z) + D_3(z,y)$.

It will be shown, in the following theorem, that in general, a necessary condition for sc-metrizability is unicoherence.

Theorem 4.1. Let M be a compact continuum with a strictly convex metric, D. Then M is unicoherent.

Proof. Assume the contrary, that M is not unicoherent. Then M can be considered as the union of two continua, M_1 and M_2, whose intersection is not connected. Thus, $M_1 \cap M_2 = H \cup K$, where H and K are disjoint closed sets. Let p be a point of $(M_1 \cup M_2) - (M_1 \cap M_2)$. Suppose p is a point of M_1. Then for every point x of M_2, there exists a unique arc $[x,p]$ from x to p such that $\ell[x,p] = D(x,p)$. Designate by $f(x)$ the first point of M_1 on $[x,p]$ in the order from x to p, and let H_x and K_x represent, respectively, the points of M_2 for which $f(x)$ lies in H and those for which $f(x)$ is in K. It should now be obvious that f is a mapping of M_2 onto $H \cup K$, since for each x in M_2, $f(x)$ is uniquely determined and $f(x) = x$ for x in $M_1 \cap M_2$. Hence, if it is shown that f is continuous, the desired result will be obtained in the form of a contradiction.

Let $\{x_i\}$ be a sequence of points of M_2 such that the associated sequence $\{f(x_i)\}$ converges to a point q. The sequence $\{x_i\}$ can be taken, without loss of generality, to be convergent to a point x_o. For each integer i, the arcs $[x_i,f(x_i)]$ and $[f(x_i),p]$ satisfy the following relationships:

$$(1) \quad \ell[x_i,p] = \ell[x_i,f(x_i)] + \ell[f(x_i),p]$$

$$(2)\quad D(x_o,p) = \underset{i \to \infty}{\text{limit}}\, D(x_i,p)$$
$$= \underset{i \to \infty}{\text{limit}}\, D(x_i,f(x_i)) + \underset{i \to \infty}{\text{limit}}\, D(f(x_i),p)$$
$$= D(x,q) + D(q,p)$$

Statement (1) is obtained by construction and (2) is the result of the continuity of D. It follows then that $q = f(x_o)$ and f is continuous.

Bing, in addition to the above mentioned result on two intersecting continua with convex metrics, has also established [4], along with Moise [10] that every locally connected and compact continuum is c-metrizable. These two results will now be combined to show the existence of a convex metric on a closed 2-cell which is not a strictly convex metric.

<u>Example 2</u>: Let M_1 be the unit circle and M_2 the closed unit disk. If x and y are any two points of M_1, let $D_1(x,y)$ be the length of the shortest arc in M_1 determined by the points. Then D_1 is a convex metric for M_1, which is obviously not a strictly convex metric. Now, M_2 has a convex metric, D_2 which can even be a strictly convex metric. Then $M_1 = M_1 \cap M_2$ and $M_2 = M_1 \cup M_2$. By application of Bing's theorem, there exists a convex metric D_3 on $M_1 \cup M_2$ which preserves D_1 on M_1. The metric D_3 is obviously not a strictly convex metric, since for a pair of diametrically opposite points, x and y, of M_1 there exist two arcs, in M_1, whose length is $D_1(x,y)$.

If M_1 and M_2 are metric spaces with metrics D_1 and D_2 respectively, it is known that $M_1 \times M_2$ has a metric D given by

$$(A)\quad D[(x_1,y_1),(x_2,y_2)] = [D_1^{\,2}(x_1,x_2) + D_2^{\,2}(y_1,y_2)]^{1/2},$$

where (x_1,y_1) and (x_2,y_2) are elements of $M_1 \times M_2$.

It has not been determined whether D will be a strictly convex metric when D_1 and D_2 are, but it will now be shown that D is a convex metric when D_1 and D_2 are convex metrics.

Theorem 4.2. Let M_1 and M_2 be compact continua with convex metrics D_1 and D_2, respectively. Then $M_1 \times M_2$ is c-metrizable and has the metric D of (A) as a convex metric.

Proof: Let (x_1,y_1) and (x_2,y_2) be any pair of elements of $M_1 \times M_2$. There exist points x_o in M_1 and y_o in M_2, such that $D_1(x_1,x_o) = D_1(x_o,x_2) = D_1(x_1,x_2)/2$ and $D_2(y_1,y_o) = D_2(y_o,y_2) = D_2(y_1,y_2)/2$. Then,

$$\begin{aligned} D[(x_1,y_1), (x_o,y_o)] &= [D_1^2(x_1,x_o) + D_2^2(y_1,y_o)]^{1/2} \\ &= [D_1^2(x_1,x_2) + D_2^2(y_1,y_2)]^{1/2}/2 \\ &= D[(x_1,y_1), (x_2,y_2)]/2. \end{aligned}$$

Similarily, $D[(x_o,y_o), (x_2,y_2)] = D[(x_1,y_1), (x_2,y_2)]/2$, and (x_o,y_o) is point satisfying (5) in the definition of convex metric.

The remaining portion of this chapter is devoted to defining the notions of D-convex subset and D-convex hull and showing that the latter is analogous to the linear concept for strictly convex metrics but not for convex metrics.

Definition 4.1. Let S be a topological space with a convex metric D. A subset M of S is D-convex if D is a convex metric for M.

Proposition 4.2. If S is a topological space with a strictly convex metric D and M is D-convex subset of S, then D is a strictly convex metric for M.

Proof: Let x and y be any two points of M, there exists a point z in M such that $D(x,z) = D(z,y) = D(x,y)/2$. The point z is unique in S and therefore in M.

Proposition 4.3. Let S be a topological space with a strictly convex metric D and let $\{M_\alpha\}$ be an arbitrary collection of D-convex subsets of S having a non-void intersection. Then $\bigcap_\alpha M_\alpha$ is a D-convex subset of S.

Proof: If $\bigcap_\alpha M_\alpha$ is a single point, the result is obvious. Otherwise, if x and y are any two points of $\bigcap_\alpha M_\alpha$, the unique point z such that $D(x,z) = D(z,y) = D(x,y)/2$ must also lie in $\bigcap_\alpha M_\alpha$.

That the result of Proposition 4.3 does not hold for convex metrics is seen in the following example.

Example 3: Let S be the unit circle. Let S have the convex metric D_1 of Example 2 and let x and y be the diametrically opposite points. If M_1 and M_2 are the two arcs of S determined by x and y, D_1 is a strictly convex metric for each of the arcs, but $M_1 \cap M_2$ is $\{x \cup y\}$.

Definition 4.2. Let S be a topological space with a strictly convex metric D and let X be a subset of S. The D-convex hull of X is defined to be the intersection of all D-convex subsets of S which contain X.

Proposition 4.4. If S is a topological space with a strictly convex metric D and X is a subset of S, then the D-convex hull of X is a

D-convex subset of S.

Proof: The argument is essentially the same as that of Proposition 4.3.

Proposition 4.5. If S is a topological space with strictly convex metric D, the D-convex hull of any pair of distinct points of S, x and y, is [x,y].

Proof: Let z and w be any pair of points of [x,y]. Assume z precedes w in the order from x to y. By Lemma 2.1, $D(z,w) = D(x,y) - D(x,z) - D(z,y)$ and the arc [z,w] must coincide with the subarc of [x,y] determined by z and w. Hence, [x,y] is a D-convex subset of S.

If M is any D-convex subset of S containing x and y, then M contains [x,y]. Thus, if $\{M_\alpha\}$ is the collection of all D-convex subsets of S containing x and y, $[x,y] \subset \bigcap_\alpha M_\alpha$. Now, since $\bigcap_\alpha M_\alpha \subset [x,y]$, $[x,y] = \bigcap_\alpha M_\alpha$.

CHAPTER V

NATURAL DELTA FUNCTIONS

Let S and T be metric spaces, T complete, with metrics ρ_1 and ρ_2 respectively, and let K be a nondegenerate compact subset of S. If C(K) denotes the collection of all continuous functions of K into T, then C(K) with the topology U of uniform convergence is a complete metric space with the metric $d(g,h) = \sup \{\rho_2(g(x),h(x))\}$; $x \in K$, where g and h represent elements of C(K). If S = T, let I(K) denote the collection of all continuous functions of K into K. Then I(K) is a closed subset of C(K) and is also a complete metric space with the topology U. In the following discussion, C(K) will be treated as a collection of mappings of S into S for the simplicity of notation, with the understanding that the results obtained are applicable when S and T are distinct.

The following concept, obtained directly from the notion of uniform continuity, is the subject of investigation in this chapter.

<u>Definition 5.1</u>. Let K be a non-degenerate compact subset of S, and let b represent the diameter of K. For each element g of C(K) and each real number t, let the real valued function $\delta(g,t)$ be defined as follows:

$$\delta(g,t) = \sup \{\delta \mid x,y \in K,\ \rho(x,y) < \delta \leq b \text{ implies } \rho(g(x),g(y)) < t\}, \text{ for } t > 0$$

$$\delta(g,0) = \lim_{t \to 0+} \delta(g,t)$$

The above defined function relates with each element of $C(K)$ a function of R into R which will be shown to be monotonic, non-negative and integrable. The Riemann integral of this function will be used to define a uniformly continuous function of $C(K)$ into R which takes on certain values for specific types of functions. The number $\delta_+'(g,0)$ when defined will also be strongly influenced by certain structual properties of g. The structure of K will be seen to exert a certain influence on $\delta(g,t)$ also, for it will be shown that $\delta(g,t)$ is superadditive when ρ is a strictly convex metric on K.

<u>Theorem 5.1.</u> For each element g of $C(K)$, $\delta(g,t)$ is a non-negative, non-decreasing function of t, defined in $[0,\infty]$ and bounded above by b.

<u>Proof:</u> Obviously, $\delta(g,t)$ is non-negative and bounded above by b, from Definition 5.1 and the fact that g is uniformly continuous, since K is compact and g is continuous. Thus, it need only be shown that $\delta(g,t)$ is well defined and non-decreasing. Since g is uniformly continous, the set $\{\delta \mid x,y \in K,\ 0 < \rho(x,y) < \delta \text{ implies } \rho(g(x),g(y)) < t\}$ is non-empty for $t > 0$ and is bounded above by b. Thus, $\delta(g,t)$ is positive and takes its unique existence from that of the supremum, for each $t > 0$.

Now let t' and t'' be any pair of real numbers such that $0 < t' < t''$ and let r be any real number between 0 and $\delta(g,t')$. For each pair of points, x and y, in K, such that $\rho(x,y) < r < \delta(g,t')$, $\rho(g(x),g(y)) < t' < t''$. Then $\delta(g,t'')$ is an upper bound for the set $\{r \mid r < \delta(g,t')\}$ and $\delta(g,t') \leq \delta(g,t'')$.

The function $\delta(g,t)$ is now seen to be uniquely determined at 0, since $\lim_{x \to 0+} \delta(g,t)$ is defined.

Proposition 5.1. For each $t > 0$, if $\rho(x,y) < \delta(g,t)$, then $\rho(g(x),g(y)) < t$.

Proof: Let x and y be any two points of K such that $\rho(x,y) < \delta(g,t)$ and let r be a real number such that $\rho(x,y) < r < \delta(g,t)$. Assume $\rho(g(x),g(y)) \geq t$, then r is an upper bound for the set $\{\delta \mid x,y \in K, \rho(x,y) < \delta \text{ implies } \rho(g(x),g(y)) < t\}$ and, since $\delta(g,t)$ is the least upper bound of the set, $\delta(g,t) \leq r$, contradicting the choice of r.

Theorem 5.2. Let K be a compact continuum with diameter b and let g be an element of C(K). Then $\delta(g,0) = b$ if and only if g is constant. $\delta(g,0) = 0$ otherwise.

Proof: Assume first that g is constant. Then for every $t > 0$, $\delta(g,t) = b$ and, as a consequence, $\delta(g,0) = \lim_{t \to 0+} \delta(g,t) = b$. On the other hand, assume $\delta(g,0)$ is positive and let $r = \delta(g,0)/2$. Then $0 < r < \delta(g,0)/2$ and for any pair of points, x and y, in K such that $\rho(x,y) < r$, $\rho(g(x),g(y)) < t$ for every $t > 0$. Thus, $\rho(g(x),g(y)) = 0$ and $g(x) = g(y)$.

Now, since K is a compact continuum in a metric space, K is totally bounded and there exists a finite set of points $\{x_1,x_2,\cdots,x_n\}$ in K such that $\bigcup_{i=1}^{n} S(x_i,r)$ covers K. Then the range of g is the finite set, $\{g(x_1);g(x_2);\cdots g(x_n)\}$. However, since K is connected and g is continuous, this is impossible unless $g(x_1) = g(x_2) = \cdots = g(x_n)$. Hence, g is constant.

<u>Proposition 5.2.</u> Let K be a compact set and let g be an element of C(K). Then $\delta(g,0) = 0$ if and only if g(K) is infinite.

<u>Proof:</u> Since K is compact and g is continuous, g(K) is compact. If g(K) is infinite, there exists a convergent sequence $\{x_i\}$ in K converging to a point x and having the property that $\{g(x_i)\}$ is a sequence of distinct points converging to g(x). Then,

$$\delta(g,0) = \underset{t \to 0+}{\text{limit}}\, \delta(g,t) = \underset{x_i \to x}{\text{limit}}\, \rho(g(x_i),g(x)) = 0.$$

Assume g(K) is finite. There exists a real number $k > 0$ such that if y' and y'' are any two points of g(K) and $S(y',k)$ and $S(y'',k)$ are spherical neighborhoods of radius k about y' and y'' respectively, $S(y',k) \cap S(y'',k) = \emptyset$. By the uniform continuity of g there exists a number $\delta > 0$ such that if x and x' are any two points of K and $\rho(x,x') < \delta$, then $\rho(g(x),g(x')) < k$. However, this implies that $g(x) = g(x')$ and $\rho(g(x),g(x')) = 0$. Thus, $0 < \delta \leq \delta(g,0)$.

<u>Theorem 5.3.</u> Let K be a compact continuum and let g be an element of C(K). Then $\delta(g,t)$ is left-continuous.

<u>Proof:</u> Let b denote the diameter of K. For every positive number t', $\underset{t \to t'-}{\text{limit}}\, \delta(g,t)$ exists, since $\delta(g,t)$ is non-decreasing and bounded above by b. Also, $\underset{t \to t'-}{\text{limit}}\, \delta(g,t) \leq \delta(g,t')$. Assume the inequality holds and let r be a real number such that $\underset{t \to t'-}{\text{limit}}\, \delta(g,t) < r < \delta(g,t')$. Let $\{t_n\}$ be a non-decreasing sequence of real numbers converging to t'. For each n, there exists a pair of points, x_n and y_n, in K such that $\rho(x_n,y_n) < r$ and $\rho(g(x_n),g(y_n)) \geq t_n$. By the compactness of K, there exists a subsequence $\{n_i\}$ of integers and points x and y in K such that

$\{x_{n_i}\}$ converges to x and $\{y_{n_i}\}$ converges to y. Now, from continuity, $\rho(x,y) \leq r < \delta(g,t')$ and

$$\rho(g(x),g(y)) = \lim_{n_i \to \infty} \rho(g(x_{n_i}),g(y_{n_i})) \geq \lim_{n \to \infty} t_n = t'$$

contradicting Proposition 5.1.

It should be pointed out at this time that if x and y are any two points of K and $\rho(g(x),g(y))= t$, then $\rho(x,y) \geq \delta(g,t)$. Theorems 5.4 and 5.5 and Proposition 5.3, which follow, yield more specific information about $\delta(g,t)$ and the values taken on for certain values of t.

Theorem 5.4. Let K be a compact continuum and let g be an element of C(K). If $t_o \in (0,b)$, where b is the diameter of K, and if $\delta(g,t) < b$, there exists a pair of points, x_o and y_o, of K for which $\rho(x_o,y_o) = \delta(g,t_o)$ and $\rho(g(x_o),g(y_o)) = t_o$.

Proof: Let b' denote the diameter of g(K). It is easily seen that $t_o \leq b'$, for otherwise, $\rho(g(x),g(y)) < t_o$ for every pair of points in g(K) and $\delta(g,t_o) = b$, contrary to hypothesis.

For every $t_o < b'$ there exists at least one pair of points, x and y, in K for which $\rho(g(x),g(y)) = t_o$, by the connectedness of g(K). Also, by the continuity of g, $\rho(x,y) = \delta(g,t_o)$ implies $\rho(g(x),g(y)) \leq t_o$. For each real number r between $\delta(g,t_o)$ and b, there exists a pair of points, x and y, in K for which $\rho(x,y) = r$ and $\rho(g(x),g(y)) \geq t_o$. For each integer n, let $r_n = \delta(g,t_o) + 1/n$, and let x_n and y_n be points in K such that $\rho(x_n,y_n) = r_n$ and $\rho(g(x_n),g(y_n)) \geq t_o$. There must exist points, x_o and y_o, in K and subsequences $\{x_{n_i}\}$ and $\{y_{n_i}\}$ converging to x_o and y_o, respectively, from which it follows that:

$$\rho(x_o,y_o) = \lim_{n \to \infty} r_n = \delta(g,t_o)$$

and, $\rho(g(x_o),g(y_o)) = \lim_{n_i \to \infty} \rho(g(x_{n_i}),g(y_{n_i})) \geq t_o$.

Theorem 5.5. Let K be a compact continuum of diameter b and let g be an element of C(K). If b' is the diameter of g(K), then b' = inf $\{t \mid \delta(g,t) = b\}$.

Proof: Let $t_o = \inf \{t \mid \delta(g,t) = b\}$. Assume there exists a number t' such that $b' < t' < t_o$. Then $\delta(g,t') < b$ and, by Theorem 5.4, there must exist a pair of points x and y, in K for which $\rho(x,y) = \delta(g,t')$ and $\rho(g(x),g(y)) = t'$. However, this is impossible, since b' is the diameter of g(K).

Similarly, assume there exists a number t' such that $t_o < t' < b'$. By Theorem 5.1, $\delta(g,t') = b$. Then for any pair of points of K, $\rho(x,y) \leq \delta(g,t')$ and, by the continuty of g, $\rho(g(x),g(y)) \leq t'$. Thus, $b' \leq t'$, contradicting the choice of t.

Proposition 5.3. Let K be a compact continuum and let g be an element of C(K). If t is a positive number less than the diameter of g(K), then $\delta(g,t) = \inf \{\rho(x,y) \mid x,y \in K \text{ and } \rho(g(x),g(y)) = t\}$.

Proof: Let $C = \inf \{\rho(x,y) \mid x,y \in K \text{ and } \rho(g(x),g(y)) = t\}$. Then $C \leq \delta(g,t)$, from Theorem 5.4. Assume $C < \delta(g,t)$. There exists a pair of points, x and y, in K such that $\rho(x,y) = (c + \delta(g,t))/2 < \delta(g,t)$ and $\rho(g(x),g(y)) \geq t$, contradicting $\delta(g,t)$. Hence, $C = \delta(g,t)$.

It was shown in Theorem 5.3 that $\delta(g,t)$ is left-continuous. The following example will illustrate a case in which the function g is continuous, monotone and differentiable almost everywhere, but $\delta(g,t)$ is not continuous. Theorem 5.6, which follows the example, gives a necessary and sufficient condition that $\delta(g,t)$ be continuous.

Example 4. Let S be the real line and $K = [0,1]$. Let the function $g(x)$ be an element of $I(K)$ defined as follows:

$$\begin{aligned} g(x) &= x \quad &; 0 \leq x \leq 1/4 \\ &= 1/4 \quad &; 1/4 \leq x \leq 3/4 \\ &= x - 1/2; \quad & 3/4 \leq x \leq 1 \end{aligned}$$

The function g is obviously continuous and non-decreasing. g' exists except at $x = 1/4$ and $x = 3/4$. (See Figure 4-A).

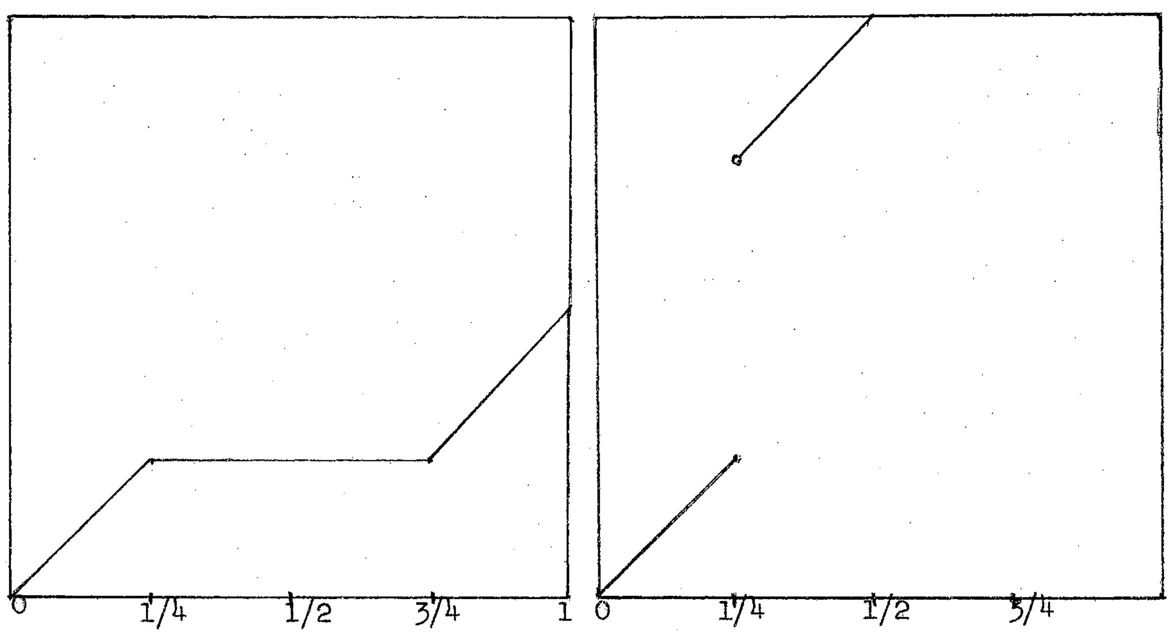

Figure 4-A. Figure 4-B.

For $t \leq 1/4$, $\delta(g,t) = t$, since from Proposition 5.3, $\delta(g,t)$ is $\inf \{\rho(x,y) \mid x,y \in K \text{ and } \rho(g(x),g(y)) = t\}$ and from Theorem 5.4, there exists a pair of points in K for which $\rho(x,y) = \delta(g,t)$ and $\rho(g(x),g(y)) = t$. The number 0 and the number t will suffice for $t \leq 1/4$. Thus

$\delta(g,t) = t$ for $t \leq 1/4$. If x and y are points of K for which $\rho(g(x),g(y)) > 1/4$, then x must lie in $[0,1/4]$ and y in $[3/4,1]$. Thus, for $t \in [1/4,1/2]$, $\delta(g,t) = \delta(g,t - 1/4) + 3/4$ and $\delta(g,t) = t + 1/2$ for $t \in [1/4,1/2]$. The graph of $\delta(g,t)$ is illustrated in Figure 4-B. Obviously since $g(K) = [0,1/2]$, $\delta(g,t) = 1$ for $t > 1/2$.

It might be pointed out that the function g has a "flat" spot, namely $[1/4,3/4]$. The following theorem shows that the existence of such "flat spots" can prevent the function $\delta(g,t)$ being continuous.

<u>Theorem 5.6</u>. Let K be a compact continuum and let g be an element of C(K) such that g(K) has diameter b. If t' is any number for which $0 < \delta(g,t') < b$, a necessary and sufficient condition that $\delta(g,t)$ be continuous at t' is that for every $\epsilon > 0$ there exist points x and y in K and a $t > t'$ such that $\rho(x,y) < \delta(g,t') + \epsilon$ and $\rho(g(x),g(y)) = t$.

Proof: The necessity is established first, assuming $\delta(g,t)$ is continuous at $t = t'$. Let $\{t_n\}$ be a decreasing sequence in the interval (t',b'), where b' is the diameter of g(K), such that $\{t_n\}$ converges to t'. By Theorem 5.5, $\delta(g,t_n) < b$ for each n, and by Theorem 5.4, there exists a pair of points x_n and y_n in K such that $\rho(x_n,y_n) = \delta(g,t_n)$ and $\rho(g(x_n),g(y_n)) = t_n$. Then for every $\epsilon > 0$ there exists an integer $N > 0$ such that $\delta(g,t_n) < \delta(g,t') + \epsilon$ whenever n exceeds N.

Assume, on the other hand, that for each $\epsilon > 0$ there exists a $t > t'$ and a pair of points, x and y, in K such that $\rho(x,y) < \delta(g,t') + \epsilon$ and $\rho(g(x),g(y)) = t$. In view of Theorem 5.3, it is necessary only to show that $\delta(g,t)$ is right-continuous at t'. If this is not the case, there exists a real number $k > 0$ such that $k = \lim_{t \to t'+} \delta(g,t) - \delta(g,t')$.

Let $\epsilon = k/2$, there must exist points x and y in K and a $t > t'$ such that $\rho(x,y) < \delta(g,t') + k/2$ and $\rho(g(x),g(y)) = t$. However, this implies that $\rho(x,y) < \delta(g,t)$ and $\rho(g(x),g(y)) = t$, which is the desired contradiction.

The right-hand derivative of $\delta(g,t)$ at 0, $\delta_+'(g,0)$ will now be examined and will be shown to have certain properties similar to those of the derivative. It will also be shown that when $\delta_+'(g,0)$ meets desired conditions the function g will have certain properties.

Theorem 5.7. Let K be a compact continuum and let g be an element of C(K) such that $\delta_+'(g,0)$ exists. Then $\delta_+'(g,0) > 0$ if and only if g satisfies a uniform Lipschitz condition of order 1 on K.

Proof: Assuming $\delta_+'(g,0) > 0$, let k be any positive number less than $\delta_+'(g,0)$. There exists a positive number t_k such that $\delta(g,t)/t > k$ and $\delta(g,t) > kt$ for $0 < t < t_k$. Let x and y be any pair of distinct points of K such that $\rho(x,y) < \delta(g,t_k)$, and let $t = \rho(g(x),g(y))$. Then $t < t_k$ and $\rho(x,y) \geq \delta(g,t) > kt = k \cdot \rho(g(x),g(y))$. Thus, for x and y in K such that $0 < \rho(x,y) < \delta(g,t_k)$, $\rho(g(x),g(y)) < \rho(x,y)/k$.

If g satisfies a uniform Lipschitz condition, there exists a pair of positive numbers δ and M such that for x and y in K, $\rho(g(x),g(y)) < M \cdot \rho(x,y)$ whenever $0 < \rho(x,y) < \delta$. If it is assumed that $\delta_+'(g,0) = 0$, then for each positive number, and in particular for 1/M, there must exists a real number $t_m > 0$ such that $\delta(g,t) < t/M$ for $0 < t < t_m$. Let t' be a real number such that $0 < t' < \min\{t_m;\delta\}$. There exists a pair of points, x and y, in K such that $\rho(x,y) = \delta(g,t') > 0$ and $\rho(g(x),g(y)) = t'$. Now, $t' < t_m$ implies $\rho(x,y) = \delta(g,t') < t'/M =$

$\rho(g(x),g(y))/M$ or, in other words, $\rho(g,(x),g(y)) > M \cdot \rho(x,y)$. However, this contradicts the Lipschitz condition and the assumption that $\delta_+^!(g,0) = 0$ is false.

Theorem 5.8. Let K be the complex plane and let g be a function which is holomorphic in the finite plane but is not constant. Then $g(z) = kz + c$, $k \neq 0$ if and only if $\delta_+^!(g,0)$ exists and is non-zero.

Proof: Assume first that $g(z) = kz + c$, $k \neq 0$. Then for z' and z'' in K, $|g(z') - g(z'')| = |(kz' + c) - (kz'' + c)| = |k| \cdot |z' - z''|$. Thus, $|g(z') - g(z'')| < t$ if and only if $|z' - z''| < t/|k|$, and $\delta(g,t) = t/|k|$. By Proposition 5.2, $\delta(g,0) = 0$ since g(K) is not constant and must be connected. Hence,

$$\delta_+^!(g,0) = \underset{t \to 0+}{\text{limit}}\ \delta(g,t)/t = \underset{t \to 0+}{\text{limit}}\ 1/|k| = 1/|k| \neq 0.$$

In the other direction, assume $\delta_+^!(g,0)$ exists and is non-zero. As above, $\delta_+^!(g,0) = \underset{t \to 0+}{\text{limit}}\ \delta(g,t)/t$ and $\underset{t \to 0+}{\text{limit}}\ t/\delta(g,t) = 1/\delta_+^!(g,0) < \infty$. Let z_o be a point of K such that $g'(z_o) \neq 0$. Such a point exists since g is holomorphic and not constant. There exists a sequence $\{z_n\}$ of points of K such that (i) for each integer n, $|z_n - z_o| < 1/n$ and (ii) $g(z_n) \neq g(z_o)$.

For each n, let $t_n = |g(z_n) - g(z_o)|$. Then $|z_n - z_o| \geq \delta(g,t_n)$ and $|g(z_n) < g(z_o)| / |z_n - z_o| \leq t_n/\delta(g,t_n)$. Also, t_n approaches 0 if and only if n approaches infinity. Now, from the existence of $g'(z_o)$ and $\delta_+^!(g,0)$, the following result is obtained.

$$|g'(z_o)| = \underset{z \to z_o}{\text{limit}}\ \{|g(z) - g(z_o)| / |z - z_o|\}$$

$$= \lim_{z_n \to z_o} \{ |g(z_n) - g(z_o)| / |z_n - z_o| \}$$

$$\leq \lim_{t_n \to 0+} \{t_n/\delta(g,t_n)\}$$

$$= \lim_{t \to 0+} \{t/\delta(g,t)\}$$

$$= 1/\delta_+^!(g,0)$$

Since g is holomorphic in the finite plane, g' is holomorphic in the finite plane and is bounded by $1/\delta_+^!(g,0)$, as established in the preceeding paragraph. Application of Liouville's Theorem shows that g'(z) is constant. Then $g'(z) \neq 0$, since g is not constant and $g(z) = kz + c$, $k \neq 0$.

Proposition 5.4. Let [a,b] be a closed interval and let g be an element of C[a,b] such that

(1) g' is strictly monotone in (a,b)

(2) g^{-1} exists in [a,b]

Then $\delta(g,t) = |g^{-1}(g(a) + t) - a|$ if $|g'(a+)| > |g'(b-)|$
and $\delta(g,t) = |b - g^{-1}(g(b)-t)|$ if $|g'(b-)| > |g'(a+)|$

Proof: By (1) and (2) of the hypothesis g' is either positive throughout (a,b) or negative throughout (a,b) and by (1), g' is strictly monotone in (a,b). Also, g is either increasing in (a,b) or decreasing in (a,b). Thus, four cases are established by considering the alternatives on g'.

Assume g' is positive and decreasing in (a,b). Then g is increasing in (a,b) and g^{-1} is increasing since $D_x g^{-1}(x) = 1/D_x g(x)$. Now, let x and y be numbers in (a,b) and t a real number. It follows that

$g(x) + t < g(y) + t$ and $g^{-1}(g(x) + t) < g^{-1}(g(y) + t)$.

Let x and y be points of $[a,b]$ and t a real number such that $x < y$ and $g(y) - g(x) = t$. Since g is increasing, $g(y) - g(x) = t$ and $g(y) = g(x) + t$. Then $y = g^{-1}(g(x) + t)$ and

$$|y - x| = y - x = g^{-1}(g(x) + t) - x = |g^{-1}(g(x) + t) - x|.$$

By Proposition 5.3,

$$\begin{aligned}\delta(g,t) &= \inf\{|y - x|: x,y \in [a,b],\ |g(x) - g(y)| = t\} \\ &= \inf\{|g^{-1}(g(x) + t) - x| : x \in [x,b]\} \\ &= |g^{-1}(g(a) + t) - a|\end{aligned}$$

Also, since g' is decreasing in (a,b),

$$|g'(a+)| = g'(a+) > g'(x) \geq g'(b-) = |g'(b-)| \text{ for } x \text{ in } (a,b).$$

Example 5. Let $g(x) = x^2$ and consider any interval $[a,b]$, $a \geq 0$. By the above proposition, $g'(x) = 2x$ is positive and increasing in (a,b) and $g^{-1}(x) = \sqrt{x}$ in $(0,\infty)$. Then $|g'(b-)| = 2b$, $|g'(a+)| = 2a$ and $|g'(b-)| > |g'(a+)|$. Thus, $\delta(g,t) = |b - g^{-1}(g(b) - t)| = \sqrt{b^2 - t}$. In particular, for the interval $[0,1]$, $\delta(g,t) = 1 - \sqrt{1 - t}$.

Example 6. If $g(x) = \sqrt{x}$, $g'(x) = 1/2\sqrt{x}$ is defined, positive and decreasing in $[a,b]$, $a \geq 0$. Then $|g'(a+)| > |g'(b-)|$ and $g^{-1}(x) = x^2$. By Proposition 5.4, $\delta(g,t) = |(\sqrt{a} + t)^2 - a|$. Again, for the interval $[0,1]$, $\delta(g,t) = t^2$.

Let (X,ρ) be a metric space and let g be an element of $I(X)$. The mapping g is said to be ϵ-contractive if there exists an $\epsilon > 0$ such that $\rho(g(x),g(y)) < \rho(x,y)$ whenever $0 < \rho(x,y) < \epsilon$. Since this definition is clearly a uniform Lipschitz condition of order 1, the mapping g is uniformly continuous. Thus, $\delta(g,t)$ exists and is bounded on any compact subset of X. If, in addition, it is required that $\delta_+'(g,0)$ exist, the following relationships exist between the notions of ϵ-contractive function and natural delta function.

Theorem 5.9. Let K be a compact subset of the metric space (X,ρ) and let g be a non-constant element of $I(K)$. If g is ϵ-contractive and $\delta_+'(g,0)$ exists, then $\delta_+'(g,0) \geq 1$.

Proof: By Proposition 5.2, $\delta(g,0) = 0$ and since $\delta_+'(g,0)$ exists, $\delta_+'(g,0) = \lim_{t\to 0+} \delta(g,t)/t$. There exists a real number $\epsilon > 0$ such that if x and y are points in K and $0 < \rho(x,y) < \epsilon$, then $\rho(g(x),g(y)) < \rho(x,y)$. Now, $\delta(g,0) = \delta(g,0+)$ implies the existence of a real number $t_o > 0$ such that $\delta(g,t) < \epsilon$ for $0 < t < t_o$. For each such t, there is a pair of points, x and y, in K such that $\rho(x,y) = \delta(g,t)$ and $\rho(g(x),g(y)) = t$. Then $t < \delta(g,t)$ for every $t < t_o$ and $\delta(g,t)/t > 1$. Therefore, $\delta_+'(g,0) \geq 1$.

Theorem 5.10. Let K be a compact continuum in the metric space (X,ρ) and let g be an element of $I(K)$. If $\delta_+'(g,0)$ exists and $\delta_+'(g,0) > 1$, then g is ϵ-contractive.

Proof: If $\delta_+'(g,0) \neq 0$, g is not constant and $\delta(g,0) = 0$. Thus, $\delta_+'(g,0) = \lim_{t\to 0+} \delta(g,t)/t$ and since $\delta_+'(g,0) > 1$, there must exist a real number $k > 1$ and $t_k > 0$ such that $\delta(g,t)/t > k$ whenever $0 < t < t_k$. In other words, $\delta(g,t) > kt$ whenever $0 < t < t_k$. Let x and y be points of K such that $0 < \rho(x,y) < \delta(g,t_k)$ and let $t' = \rho(g(x),g(y))$. Then

$t' < t_k$ and $\rho(x,y) \geq \delta(g,t') > kt'$ which yields $\rho(x,y) > \rho(g(x),g(y))$. Hence, g is ϵ-contractive.

Edelstein [5] established the following results concerning ϵ-contractive mappings:

I. If K is a compact metric space and g is an ϵ-contractive mapping on K, then there exists a periodic point α.

II. If, in addition, K is ϵ-chainable, then α is a unique fixed point and $\alpha = \lim_{n \to \infty} g^n(x)$ for each x in K.

These results may now be stated in terms of the natural delta function.

Corollary 5.9.1. If K is a compact subset of a metric space (X,ρ) and g is an element of I(K) such that $\delta'_+(g,0)$ exists and $\delta'_+(g,0) > 1$, then there exists a periodic point α.

Corollary 5.10.1. If K is a compact metric continuum and g is an element of I(K) such that $\delta'_+(g,0)$ exists and is greater than 1, then g has a unique fixed point α and $\alpha = \lim_{n \to \infty} g^n(x)$ for each x in K.

Proof: Obviously, since K is a compact continuum, K is ϵ-chainable and, by Theorem 5.10, is ϵ-contractive. Hence, the hypothesis of Edelstein's theorem (II) is satisfied and the desired result follows.

In Theorems 5.1 and 5.3 it was shown that if K is a compact continuum and g is an element of C(K), then $\delta(g,t)$ is a positive, non-decreasing and left-continuous function defined on K and bounded by b, the diameter of g(K). If it is observed that $\delta(g,t)$ is Riemann-integrable on [0,b], the integral may be used to define a uniformly continuous function from (I(K),U) to the interval $[0,b^2]$. This will be accomplished in Theorem 5.11, with the values taken on by this function for certain special elements of I(K) observed in Theorems 5.12 and 5.13 and their corollaries.

Theorem 5.11. Let K be a compact continuum in a metric space (X,ρ) with b the diameter of K. For each element g of I(K), let $F(g) = \int_0^b \delta(g,t)dt$. Then F is a uniformly continuous function from (I(K),U) to the interval $[0,b^2]$.

Proof: Let $\epsilon' > 0$ be given and take $\epsilon = \epsilon'/b$. Then for g and h in I(K) such that $d(g,h) < \epsilon/2$, and any two points, x and y, in K,

$$\rho(h(x),h(y)) \leq \rho(h(x),g(x)) + \rho(g(x),g(y)) + \rho(g(y),h(y)).$$

Thus,

$$\rho(h(x),h(y)) < \rho(g(x),g(y)) + \epsilon.$$

Similarly,

$$\rho(g(x),g(y)) < \rho(h(x),h(y)) + \epsilon.$$

Now, if $\rho(x,y) < \delta(g,t)$, then $\rho(g(x),g(y)) < t$ and $\rho(h(x),h(y)) < t + \epsilon$. Therefore, $\delta(g,t) \leq \delta(h,t + \epsilon)$ and $\delta(h,t) \leq \delta(g,t + \epsilon)$.

Since $\delta(g,t)$ and $\delta(h,t)$ are non-negative and non-decreasing in [0,b],

$$\int_0^b \delta(g,t)dt \leq \int_0^b \delta(h,t + \epsilon)dt$$

$$= \int_\epsilon^{b+\epsilon} \delta(h,t)dt$$

$$= \int_0^b \delta(h,t)dt + \int_b^{b+\epsilon} \delta(h,t)dt - \int_0^\epsilon \delta(h,t)dt$$

$$< \int_0^b \delta(h,t)dt + \epsilon \cdot b.$$

Similarly,

$$\int_0^b \delta(h,t) < \int_0^b \delta(g,t) + \epsilon\cdot b.$$

Thus, $F(g) < F(h) + \epsilon'$ and $F(h) < F(g) + \epsilon'$ or, in other words, $-\epsilon' < F(g) - F(h) < \epsilon'$. Hence, $|F(g) - F(h)| < \epsilon'$ whenever $d(g,h) < \epsilon'/2b$ and F is uniformly continuous.

Theorem 5.12. Let K be a compact continuum of diameter b and let g be an element of I(K) which is a homeomorphism. Then g is an isometry if and only if $\delta(g,t) = t$ for every t in the interval [0,b].

Proof: If g is not an isometry, then by a theorem of Montgomery [11], g must increase the distance between some two points of K. That is, there must exist points x and y in K such that $\rho(x,y) < \rho(g(x),g(y))$. Let $t' = \rho(g(x),g(y))$. Then

$$\delta(g,t') \leq \rho(x,y) < \rho(g(x),g(y)) = t'.$$

Conversely, suppose there exists a t' in (0,b) for which $\delta(g,t') \neq t'$. Now, $\delta(g,t') < b$ since, by Theorem 5.5, the diameter of g(K) is given by $\inf \{t \mid \delta(g,t) = b\}$ and g is a homeomorphism. From Theorem 5.4, there exists a pair of points, x and y, in K such that $\rho(x,y) = \delta(g,t')$ and $\rho(g(x),g(y)) = t'$, from which it follows that $\rho(x,y) \neq \rho(g(x),g(y))$ and g is not an isometry.

Corollary 5.12.1. If K is a compact continuum of diameter b and g is an isometry of K into K, then $F(g) = b^2/2$.

Proof: By Theorem 5.12, $\delta(g,t) = t$ and $F(g) = \int_0^b t\,dt = b^2/2$.

Theorem 5.13. Let K be a compact continuum of diameter b and let g be an element of I(K) which is a contraction mapping. Then for every t in $(0,b), \delta(g,t) > t$.

Proof: Since g is a contraction mapping, there exists a real number r, $0 < r < 1$, such that $\rho(g(x),g(y)) \leq r \cdot \rho(x,y)$ for every pair of points in K. Let a be any number in $(0,b)$ and let $m = \min\{b; a/r\}$. Then $a < m$ since $a < a/r$ for $r < 1$. Let a' be a real number such that $a < a' < m$ and let x and y be any pair of points in K such that $\rho(x,y) < a'$. Now,

$$\rho(g(x),g(y)) \leq r \cdot \rho(x,y) < ra' < rm \leq r \cdot a/r = a$$

Therefore, $a' \leq \delta(g,a)$ and $\delta(g,a) > a$.

Corollary 5.13.1. Let K be a compact continuum of diameter b and let g be a contraction mapping of K into K. Then $F(g) > b^2/2$.

Proof: The function $\delta(g,t)$ is integrable and $\delta(g,t) - t > 0$ for every number t in the interval $(0,b)$ by Theorem 5.13. Then,

$$\int_0^b [\delta(g,t) - t]dt = \int_0^b \delta(g,t)dt - \int_0^b tdt > 0$$

and $F(g) > \int_0^b tdt$. Thus, $F(g) > b^2/2$.

The remaining two theorems of this chapter are devoted to studying the effect on $\delta(g,t)$ and $F(g)$ when the domain K has a strictly convex metric.

Theorem 5.14. Let K be a compact continuum in a complete metric space with metric D such that D is a strictly convex metric for K. If g is an element of $C(K)$, then $\delta(g,t)$ is super-additive in $[0,b']$ where b' denotes the diameter of $g(K)$.

Proof: Let t', t'' and t be real numbers in the interval $[0,b']$

such that $t = t' + t''$. Assume $\delta(g,t) < \delta(g,t') + \delta(g,t'')$. There exists a pair of points, x and y, in K such that $D(x,y) = \delta(g,t)$ and $D(g(x),g(y)) = t$. Let $\epsilon = [\delta(g,t') + \delta(g,t'') - \delta(g,t)]/2$. There exists a point z in K such that $D(x,z) + D(z,y) = D(x,y)$ and $D(x,z) = \delta(g,t') - \epsilon$. Now,

$$\begin{aligned} D(z,y) &= \delta(g,t) - D(x,z) \\ &= \delta(g,t) - \delta(g,t') + \epsilon \\ &= \delta(g,t) - \delta(g,t') + [\delta(g,t') + \delta(g,t'') - \delta(g,t)]/2 \\ &= \delta(g,t'') - \epsilon \end{aligned}$$

Then $D(g(x),g(z)) < t'$, $D(g(z),g(y)) < t''$ and,

$$\begin{aligned} D(g(x),g(y)) &\leq D(g(x),g(z)) + D(g(z),g(y)) \\ &< t' + t'' = t \end{aligned}$$

This, however, contradicts the choice of points, x and y, and the original assumption that $\delta(g,t) < \delta(g,t') + \delta(g,t'')$ is false. Thus $\delta(g,t)$ is super-additive in $[0,b']$.

<u>Theorem 5.15</u>. Let K be a compact continuum with a strictly convex metric D such that the diameter of K, under D, is b. Let g be a non-constant element of C(K) such that the diameter of g(K) is b'. Then $F(g) \leq b^2 - (bb')/2$.

<u>Proof</u>: Since D is a strictly convex metric and g is not constant, $\delta(g,t)$ is super-additive in $[0,b']$ by Theorem 5.14 and $\delta(g,t) \leq bt/b'$ for every t in $(0,b)$. Then,

$$\int_0^{b'} \delta(g,t)dt \leq \int_0^{b'} (bt/b')dt = bb'/2.$$

By Theorem 5.5, $\delta(g,t) = b$ for every number t in the interval $[b',b]$.

Hence,

$$\int_{b'}^{b} \delta(g,t)dt = \int_{b'}^{b} bdt = b^2 - bb'.$$

Now,

$$\begin{aligned} F(g) &= \int_{0}^{b} \delta(g,t)dt \\ &= \int_{0}^{b'} \delta(g,t)dt + \int_{b'}^{b} \delta(g,t)dt \\ &\leq (bb')/2 + b^2 -(bb') \end{aligned}$$

and $F(g) \leq b^2 - bb'/2$.

CHAPTER VI

SUMMARY

This paper is concerned with two concepts, namely those of strictly convex metrics and of natural delta functions.

Comparing the strictly convex metric with the convex metric, it is found that a strictly convex metric is a convex metric for which each pair of points determines a unique arc whose length is given by the distance between the points under that metric.

One of the principal results of this paper is contained in Chapter III. If a compact continuum can be expressed as the union of a discrete set and a countable collection of compact sc-metrizable continua $\{M_i\}$ such that for each integer n the continuum M_n intersects the union of those continua of the collection of index less than n in a single point, which separates the space, then M is sc-metrizable. As a corollary to this, every dendrite is sc-metrizable.

In the plane, the sc-metrizable sets are characterized as the collection of all locally connected and point-like continua. In general, every sc-metrizable continuum is unicoherent.

In Chapter IV both c-metrizability and sc-metrizability are found to be topological properties. Examples are given to show that not every convex metric on a sc-metrizable continuum is a strictly convex metric and that Bing's method in [3] of extending a convex metric is not applicable to strictly convex metrics. The Cartesian product of two c-metrizable continua is found to be c-metrizable and the question is raised as to whether the same is true of sc-metrizable continua.

It is observed that the concept of a strictly convex metric is more closely analogous to the original linear concept of convexity than is the concept of a convex metric, since with sc-metrizability the concept of convex hull is definable and consistent with linear results but is not with c-metrizability.

Focusing attention on the collection of all continuous functions between two compact, metric continua, the natural delta function is defined and is shown to be, for each such continuous function, a non-negative, non-decreasing and Riemann-integrable function of one variable, defined in $(0,\infty)$ and bounded above. The value of the natural delta function at the origin is zero if and only if the range of its determining function is infinite. The natural delta function is found to be left-continuous and a necessary and sufficient condition that it be continuous is given.

Paying particular attention to the right-hand derivative of the natural delta function at the origin, a number of results are established. The requirement that this right-hand derivative exist and be non-zero is shown to be equivalent to a uniform Lipshitz condition of order 1, where the Lipshitz condition is defined in a general way. When applied to functions holomorphic in the complex plane, the same requirement yields a translation.

If g is a uniformly continuous function on a compact set such that the right-hand derivative of $\delta(g,t)$ at the origin exists and exceeds one, g is an ϵ-contractive function. Applying this result to Edelstein's theorems on ϵ-contractive functions, the following results are obtained. If K is a compact subset of a metric space and g is a continuous function of K into itself for which the right-hand derivative of $\delta(g,t)$ at zero exists and exceeds one, then g has a periodic point α. If, in addition, K is a continuum, then α is a unique fixed point and $\alpha = \lim_{n \to \infty} g^n(x)$ for each x in K.

If K is a compact metric continuum of diameter b and I(K) is the collection of all continuous mappings of K into itself, the function F from I(K) to $[0,b^2]$ is found to be a uniformly continuous function where I(K) is given the topology of uniform convergence and $F(g) = \int_0^b \delta(g,t)dt$ for each element g of I(K). The value of F(g) is then determined for certain special types of functions.

If the metric on the continuum K of the preceeding paragraph is a strictly convex metric, then $\delta(g,t)$ is a super-additive function on the interval $[0,b']$, where b' denotes the diameter of g(K).

Some questions for further study might include the following. What are the conditions under which a strictly convex metric on a subset M of a space S can be extended to S? Under what conditions will a non-compact continuum have a strictly convex metric? If a space S has a strictly convex metric D, are the spherical neighborhoods D-convex subsets of S? Finally, under what conditions can a continuous function f on a subcontinuum M_1 of the metric continuum M_2 be so extended to M_2 that the natural delta function of the extension and the natural delta function of f coincide on the interval from zero to the diameter of M_1? In other words, when does the Tietze extension theorem yield a smooth extension?

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