ANALYSIS OF PLANAR STRUCTURES LOADED NORMAL TO THE PLANE BY THE PRINCIPLE OF MINIMUM POTENTIAL ENERGY

by

ROY A.J. FRUSTI

Bachelor of Science
Michigan College of Mining and Technology
Houghton, Michigan
1957

Master of Science
Michigan College of Mining and Technology
Houghton, Michigan
1958

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School of the Oklahoma State University
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Thesis Approved:

Thesis Adviser

Dean of the Graduate School

PREFACE

The application of the Principle of Minimum Potential Energy to the analysis of planar structures investigated in this thesis is the culmination of the author's studies at Oklahoma State University. The idea was originally expressed by Professor Jan J. Tuma in the summer of 1964. Professor Tuma suggested that the investigation produce a method that is practical as well as conducive to computer analysis.

The author is greatly indebted and wishes to express his sincere appreciation and gratitude to the following:

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Roy A. J. Frusti

Roy A. J. Frusti

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NOMENCLATURE

A	cross sectional area;
B_1, B_2, B_3	concentrated loads;
C	stiffness coefficient;
CK	carry over stiffness matrix;
E	Young's modulus of elasticity;
f	cantilever flexibility terms;
[F]	flexibility matrix;
FN, FM	fixed end load terms;
G	shear modulus of elasticity;
[H]	force matrix;
I	moment of inertia;
J	torsional constant;
K	stiffness coefficients, or terms of inverse of flexibility matrix;
L	length of member;
M^m_{qix}, M^m_{qiy}	moments at any point q of segment iq in member system;
M_{ijx}, M_{ijy}	moments at end i of member ij in o system;
N_{ijz}	force at end i of member ij in Z direction;
P	external joint force;
Q	external joint moment;

$[r]$ . . . . transmission matrix;

S_{qz} . . . . shear at any arbitrary point q in z direction;

U . . . . strain energy;

V . . . . external potential energy;

w . . . . uniform load;

W . . . . work;

X_{ij} . . . . x distance from end i of member to end j measured in basic system;

Y_{ij} . . . . y distance from end i of member to end j measured in basic system;

Z_i . . . . displacement in z direction of any point i;

α . . . . angle of rotation from basic system to member system;

γ . . . . shape factor;

Δ_{ijz}, Δ_{jiz} . . . . end deflection of member ij in z direction;

δ . . . . end displacements of cantilever member due to loads;

θ_{ijx}, θ_{jix} . . . . end rotation of member ij in x direction;

θ_{ijy}, θ_{jiy} . . . . end rotation of member ij in y direction;

. . . . combined stiffness coefficient;

λ . . . . elemental flexibility;

$[\Psi]$ . . . . displacement matrix;

Π . . . . total potential energy;

ω_{om}, ω_{mo} . . . . rotational matrices;

Superscript T indicates transpose, superscript m indicates the member system, and omission of the superscript refers the term to the basic o system.

CHAPTER I

INTRODUCTION

1.1 General

The analysis of space structures is divided into two major classes: flexibility methods and stiffness methods. The flexibility methods deal with elastic constants known as linear and angular flexibilities; the stiffness methods deal with elastic constants called linear and angular stiffnesses. These elastic constants can be calculated from the principles of elasto-geometry or by means of energy.

The analysis of linearly elastic Order II structures by minimum potential energy is developed in this thesis. These structures are defined as in-plane with loads applied normal to the plane. The Principle of Minimum Potential Energy is used in the analysis and briefly states that for stable equilibrium compatible with constraints, the potential energy of a deformable body will be a minimum. The minimization process is achieved by differentiating the total static potential energy with respect to each end displacement. This produces a set of linear equations with displacements as redundants; thus the method of minimum potential energy leads to a stiffness method.

1.2 Statement of the Problem

Planar frame structures loaded normal to the plane with general type supports and shape of members are investigated by Minimum Potential Energy. The cross section may be variable but symmetric with the vertical axis.

1.3 Assumptions

Material properties are defined by the following assumptions:

1. Material is homogeneous, isotropic, and continuous.
2. Material obeys Hooke's Law.
3. All deformations are small and elastic.

The stereo-geometry of each member is idealized by the following assumptions:

1. Depth to length ratio is small.
2. Cross section is symmetric with respect to vertical axis.
3. Curvature is large and all conventional beam formulas apply.
4. Force and moment vectors are assumed to act at the centroid of the cross section.

Additionally, deformations of each member are calculated from the assumption that all deformations are small and elastic and that deformation of the structure does not alter the point of application of loads.

All forces, moments, and displacement vectors are positive when acting in direction of the respective system axes. The specific application of this rule with respect to elements of cross section are explained where they first appear. All symbols are defined where they first appear and are arranged alphabetically in the table of symbols.

1.4 Historical Comments

The association of minimum potential energy with geometric deflection curves and equilibrium dates primarily to the seventeen-hundreds at the time of Leonard Euler (1) and the Bernoulli family (1). This era provided the basis for significant advances in energy methods and theorems which subsequently emerged, such as Castigliano's Theorems (1) and Engesser's (1) notion of complementary energy.

The energy principles are discussed by Hoff (2), Charlton (3), Argyris and Kelsey (4), Westergaard (5), Williams (6), Langhaar (7), Southwell (8) and others and are well founded and demonstrate methods for analysis of several types of problems. Further, these principles have been rigorously applied to the theory of elasticity and buckling investigations by Timoshenko and Goodier (9), Love (10), Timoshenko and Gere (11), Westergaard (12), and others. Specific applications to civil engineering type structures are given by Li (13), Timoshenko and Young (14), Norris and Wilbur (15), Langhaar (16), Pippard (17), and others.

This thesis employs the most fundamental energy theorem, that of Minimum Potential Energy which produces well-conditioned linear equations as stated by Charlton (18) and Brown (19). Use of nodal displacements as redundants automatically satisfies the compatibility condition and the minimization process assures equilibrium at the nodes (20). The proof of this theorem is given by Sokolnikoff (21).

The strain energy function is developed in terms of end displacements for straight members by Bateman (22), and Langhaar (16) utilizes potential energy to formulate stiffness equations for analysis of planar structures with straight members. His formulation requires the deflection curve for calculation of the potential energy function. In the general case the deflection curve is usually unavailable for this computation. The potential energy expression can, however, be computed as a function of end displacements as presented in the body of this thesis. General matrix operations are demonstrated by McMinn (23).

In the Minimum Potential Energy method of analysis fixed end values are required and must be calculated for each specific curve. For circular shapes the values are readily available from Michalos (24) or Spyropoulos (25). Values for other shaped members must be calculated as necessary.

Eiseman, Namyet, and Woo (26) have demonstrated that the stiffness method with a very large number of redundants can be solved by an electronic computer and can give satisfactory results. The work of this thesis is not primarily concerned with the number of redundants but rather the application of minimum potential energy to analysis of general shaped planar structures.

CHAPTER II

ANALYSIS OF A SEGMENT

2.1 General

The segment is considered to have any general curved shape, to have known elastic properties, and to displace in a restricted manner. End restraint conditions are arbitrary, either hinged or rigidly connected. Static loads are normal to the plane of the segment.

2.2 Coordinate Systems

Two coordinate systems related by the geometry of the segment are utilized. One is an arbitrarily positioned fixed system or basic system, and the other is a member system. Transformation from one system to the other is accomplished by rotational matrices designated $[\omega_{om}]$ and $[\omega_{mo}]$.

$$[\omega_{om}] = \left[\begin{array}{c|c|c} \cos\alpha & -\sin\alpha & 0 \\ \hline \sin\alpha & \cos\alpha & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \qquad [\omega_{mo}] = \left[\begin{array}{c|c|c} \cos\alpha & \sin\alpha & 0 \\ \hline -\sin\alpha & \cos\alpha & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

2.3 Statics

For a general member ij, with forces and moments at i known, the cross sectional elements at any arbitrary section q can be calculated by statics. If, in addition to end forces and moments, loads are acting on segment iq, their effect is superimposed in a similar manner. See Fig. 2.1.

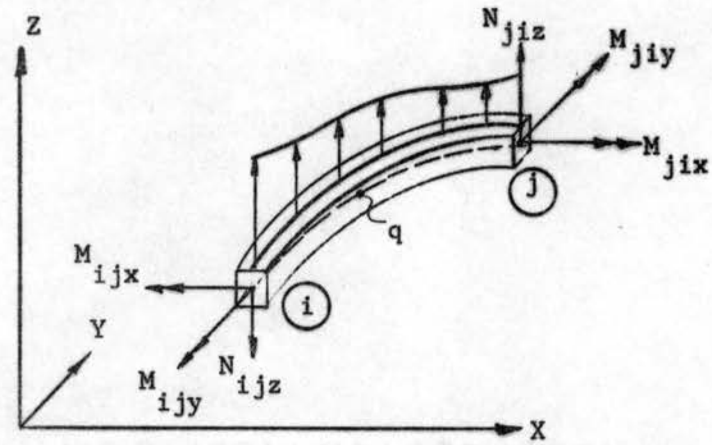

Fig. 2.1 Equilibrium Forces and Moments on Member ij.

2.4 Deformations

Since the principle of statics and geometry are interchangeable, the analogous conjugate member is used to obtain relationships between causes and displacements. Elemental elastic weights are considered as loads on the conjugate structure with cross-sectional elements as slope and displacements.

Let elemental elastic weights be premultiplied and post-multiplied by rotational matrices so that from the conjugate structure the displacements between end i to j due to unit forces and moments on end i can be designated as a flexibility matrix. From elasto-static equilibrium the displacements at the near end can be written as a function of the forces and moments of the near end plus the displacements of the far end or,

$$\begin{bmatrix} \Delta_{ijz} \\ \hline \theta_{ijx} \\ \hline \theta_{ijy} \end{bmatrix} = \left[\begin{array}{c|c|c} f'_{iizz} & f_{iizx} & f_{iizy} \\ \hline f_{iixz} & f_{iixx} & f_{iixy} \\ \hline f_{iiyz} & f_{iiyx} & f_{iiyy} \end{array}\right] \begin{bmatrix} N_{ijz} \\ \hline M_{ijx} \\ \hline M_{ijy} \end{bmatrix}$$

$$+ \left[\begin{array}{c|c|c} 1 & -Y_{ij} & X_{ij} \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \begin{bmatrix} \Delta_{jiz} \\ \hline \theta_{jix} \\ \hline \theta_{jiy} \end{bmatrix} + \left[\begin{array}{c|c|c} f''_{iizz} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \begin{bmatrix} N_{ijz} \\ \hline M_{ijx} \\ \hline M_{ijy} \end{bmatrix} \qquad (2.1)$$

In symbolic form,

$$[\ \Psi_i\ ] = [\ F'_i\ ][\ H_i\ ] + [\ r_{ij}\ ][\ \Psi_j\ ] + [\ F''_i\ ][\ H_i\ ]$$

The coefficients f are flexibility coefficients for a cantilevered basic structure and are defined as follows:

f'_{iizz} = Displacements of near end due to unit cause at near end neglecting the effect of shear. Displacement is at end i in direction z due to unit cause at i in direction z. For this specific value the unit cause must be a force as no moments act in the z direction.

f''_{iizz} = Displacement at near end due to unit cause at near end due to shear effect alone. It is equal to $\int_L \frac{\gamma ds}{GA}$.

All remaining flexibility terms are displacements due to unit causes and can be defined similar to f'_{iizz} as above. From the reciprocal theorem the flexibility matrix is symmetric and all terms can be calculated by applying unit causes to the basic cantilever structure.

2.5 End Forces and Moments

Eq. (2.1) expresses the end forces and moments at i as functions of displacements at both ends. This equation can be solved for the end forces and moments by inverting the flexibility matrix. Defining the inverted matrix as the stiffness matrix,

$$\begin{bmatrix} f'_{iizz} & f_{iizx} & f_{iizy} \\ f_{iixz} & f_{iixx} & f_{iixy} \\ f_{iiyz} & f_{iiyx} & f_{iiyy} \end{bmatrix}^{-1} = \begin{bmatrix} K_{zz} & K_{zx} & K_{zy} \\ K_{xz} & K_{xx} & K_{xy} \\ K_{yz} & K_{yx} & K_{yy} \end{bmatrix} \qquad (2.2)$$

and premultiplying Eq. (2.1) by this matrix gives the end forces and moments at i as functions of end displacements:

$$\begin{bmatrix} N_{ijz} \\ M_{ijx} \\ M_{ijy} \end{bmatrix} = \begin{bmatrix} K_{zz} & K_{zx} & K_{zy} \\ K_{xz} & K_{xx} & K_{xy} \\ K_{yz} & K_{yx} & K_{yy} \end{bmatrix} \begin{bmatrix} \Delta_{ijz} \\ \theta_{ijx} \\ \theta_{ijy} \end{bmatrix}$$

$$+ \begin{bmatrix} -K_{zz} & (Y_{ij}K_{zz} - K_{zx}) & (-X_{ij}K_{zz} - K_{zy}) \\ -K_{xz} & (Y_{ij}K_{xz} - K_{xx}) & (-X_{ij}K_{xz} - K_{xy}) \\ -K_{yz} & (Y_{ij}K_{yz} - K_{yx}) & (-X_{ij}K_{yz} - K_{yy}) \end{bmatrix} \begin{bmatrix} \Delta_{jiz} \\ \theta_{jix} \\ \theta_{jiy} \end{bmatrix}$$

$$+ \begin{bmatrix} -K_{zz}f''_{iizz} & 0 & 0 \\ -K_{xz}f''_{iizz} & 0 & 0 \\ -K_{yz}f''_{iizz} & 0 & 0 \end{bmatrix} \begin{bmatrix} N_{ijz} \\ M_{ijx} \\ M_{ijy} \end{bmatrix} \qquad (2.3)$$

or symbolically,

$$[\,H_i\,] = [\,K_i\,][\,\Psi_i\,] + [\,CK_j\,][\,\Psi_j\,] + [\,K'_i\,][\,H_i\,]$$

All terms are related to the basic reference system.

CHAPTER III

DERIVATION OF POTENTIAL ENERGY EXPRESSIONS

3.1 General

Once the end forces and moments are known in terms of stiffnesses and displacements, the potential energy can be formulated as a function of these same quantities. A brief restatement of the Principle of Potential Energy is introduced and the substitution of Eq. (2.3) into this energy function is shown.

The Principle of Minimum Potential Energy states that of all possible displacements a body can assume which satisfy kinematic boundary conditions, those conditions corresponding to forces and stresses satisfying equilibrium make the total potential energy a stationary value (27). Potential energy Π can be defined as the total static energy possessed by the system, and represented as the sum of U and V, where U is the strain energy stored in the structure and V is the potential energy of external loads. Normally V is taken as equal to minus the work done by the external loads in moving from the initial to the deformed position of the structure. Again, the potential energy is

$$\Pi = U + V$$

For equilibrium, the potential energy assumes a stationary value so that

$$\delta(\Pi) = \delta(U + V) = 0$$

$$\delta(U) = -\delta(V) = \delta(W) \tag{3.1}$$

where W is the external work done by the loads.

3.2 Strain Energy

The strain energy U can be expressed as a function of end displacements, and differentiated with respect to each of these displacements. There will be as many equations derived by this differentiation as there are displacements, forming a set of simultaneous linear equations.

Strain energy of the member ij due to bending, torsion, and shear is given by

$$U = \int_L \frac{(M_{qx}^{m})^2}{2GJ} ds + \int_L \frac{(M_{qy}^{m})^2}{2EI} ds + \int_L \frac{\gamma(S_{qz}^{m})^2}{2GA} ds \tag{3.2a}$$

in which

(M_{qx}^{m}) = torsional moment at arbitrary point q

(M_{qy}^{m}) = flexural moment at arbitrary point q

(S_{qz}^{m}) = shear force in z direction at arbitrary point q

Superscript m identifies the member system and γ is the shape factor of the cross section.

Since the moments and forces are related to the m system and are initially given with respect to the o system, a transformation must be performed:

$$\begin{bmatrix} M_{qix}^{m} \\ M_{qiy}^{m} \\ N_{qiz}^{m} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_{qix} \\ M_{qiy} \\ N_{qiz} \end{bmatrix}$$

or,

$$[\, H_{qi}^{m} \,] = [\, \omega_{mo} \,][\, H_{qi} \,]$$

With the substitution of these terms, the matrix form of Eq. (3.2a) becomes:

$$U = \tfrac{1}{2}\int_L \begin{bmatrix} M_{qx}^{m} & M_{qy}^{m} & S_{qz}^{m} \end{bmatrix} \begin{bmatrix} \lambda_{qx}^{m} & 0 & 0 \\ 0 & \lambda_{qy}^{m} & 0 \\ 0 & 0 & \lambda_{qz}^{m} \end{bmatrix} \begin{bmatrix} M_{qx}^{m} \\ M_{qy}^{m} \\ S_{qz}^{m} \end{bmatrix} \qquad (3.2b)$$

or symbolically,

$$U = \tfrac{1}{2}\int_L [\, H_q^{m} \,]^T [\, \lambda_q^{m} \,][\, H_q^{m} \,]$$

and in the o system

$$U = \tfrac{1}{2}\int_L [\, H_q \,]^T [\, \omega_{om} \,][\, \lambda_q^{m} \,][\, \omega_{mo} \,][\, H_q \,]$$

or,

$$U = \tfrac{1}{2}\int_L [\, H_q \,]^T [\, \lambda_q \,][\, H_q \,]$$

where

λ_{qx}^{m} = angular deformation of differential element ds in x direction or, $\frac{ds}{GJ}$

λ_{qy}^{m} = angular deformation of differential element ds in y direction or, $\frac{ds}{EI}$

λ_{qz}^{m} = linear deformation of differential element ds in z direction or, $\frac{\gamma ds}{GA}$.

Each deformation is due to a unit cause in the corresponding direction of the member system.

The evaluation of the integral in Eq. (3.2b) leads to the general strain energy function of which derivatives with respect to each displacement are recorded in the Appendix and summarized in simple form in Table 3.1. The first matrix of Table 3.1 yields the influence of bending and torsion, and the second matrix represents the contribution of shear effect on the displacements where

$$CF = \frac{-f''_{iizz}}{(1 + K_{zz}f''_{iizz})}$$

The later expression for shear effect is valid for joint loads only. If shear effect of in-span loads is to be included, the shear variation must be substituted into Eq. (3.2a) before differentiation.

The stiffness coefficient matrices can be written directly by performing the required simple calculation with terms obtained from Eq. (2.2).

3.3 External Potential Energy

Obviously for this method to be of any significant value the external potential energy must be expressed as a function of the end displacements. This energy for in-span loads may be considered as consisting of two parts: first, the "fixed end" portion or that portion of work which is done to deflect the member to the equilibrium position while both ends are restrained and second, the work done as a function

TABLE 3.1 FORCE, STIFFNESS-DISPLACEMENT EQUATIONS

$$
\begin{bmatrix}
\frac{\partial U}{\partial \Delta_{ijz}} \\[6pt]
\frac{\partial U}{\partial \theta_{ijx}} \\[6pt]
\frac{\partial U}{\partial \theta_{ijy}} \\[6pt]
\frac{\partial U}{\partial \Delta_{jiz}} \\[6pt]
\frac{\partial U}{\partial \theta_{jix}} \\[6pt]
\frac{\partial U}{\partial \theta_{jiy}}
\end{bmatrix}
=
\begin{bmatrix}
K_{zz} & K_{zx} & K_{zy} & -K_{zz} & (Y_{ij}K_{zz} - K_{zx}) & (-X_{ij}K_{zz} - K_{zy}) \\
K_{zx} & K_{xx} & K_{xy} & -K_{zx} & (Y_{ij}K_{zx} - K_{xx}) & (-X_{ij}K_{zx} - K_{xy}) \\
K_{zy} & K_{xy} & K_{yy} & -K_{zy} & (Y_{ij}K_{zy} - K_{yx}) & (-X_{ij}K_{zy} - K_{yy}) \\
-K_{zz} & -K_{zx} & -K_{zy} & K_{zz} & -(Y_{ij}K_{zz} - K_{zx}) & -(-X_{ij}K_{zz} - K_{zy}) \\
(Y_{ij}K_{zz} - K_{zx}) & (Y_{ij}K_{zx} - K_{xx}) & (Y_{ij}K_{zy} - K_{yx}) & -(Y_{ij}K_{zz} - K_{xz}) & Y_{ij}(Y_{ij}K_{zz} - K_{zx}) - (Y_{ij}K_{zx} - K_{xx}) & Y_{ij}(-X_{ij}K_{zz} - K_{zy}) - (-X_{ij}K_{zx} - K_{xy}) \\
(-X_{ij}K_{zz} - K_{zy}) & (-X_{ij}K_{zx} - K_{xy}) & (-X_{ij}K_{zy} - K_{yy}) & -(-X_{ij}K_{zz} - K_{zy}) & Y_{ij}(-X_{ij}K_{zz} - K_{zy}) - (-X_{ij}K_{zx} - K_{xy}) & -X_{ij}(-X_{ij}K_{zz} - K_{zy}) - (-X_{ij}K_{zy} - K_{yy})
\end{bmatrix}
\begin{bmatrix}
\Delta_{ijz} \\ \theta_{ijx} \\ \theta_{ijy} \\ \Delta_{jiz} \\ \theta_{jix} \\ \theta_{jiy}
\end{bmatrix}
$$

$$
+ \; CF
\begin{bmatrix}
K_{zz}^2 & K_{zx}K_{zz} & K_{zy}K_{zz} & -K_{zz}^2 & K_{zz}(Y_{ij}K_{zz} - K_{zx}) & K_{zz}(-X_{ij}K_{zz} - K_{zy}) \\
K_{zz}K_{zx} & K_{zx}^2 & K_{zy}K_{zx} & -K_{zz}K_{zx} & K_{zx}(Y_{ij}K_{zz} - K_{zx}) & K_{zx}(-X_{ij}K_{zz} - K_{zy}) \\
K_{zz}K_{zy} & K_{zx}K_{zy} & K_{zy}^2 & -K_{zz}K_{zy} & K_{zy}(Y_{ij}K_{zz} - K_{zx}) & K_{zy}(-X_{ij}K_{zz} - K_{zy}) \\
-K_{zz}^2 & -K_{zx}K_{zz} & -K_{zy}K_{zz} & K_{zz}^2 & -K_{zz}(Y_{ij}K_{zz} - K_{zx}) & -K_{zz}(-X_{ij}K_{zz} - K_{zy}) \\
K_{zz}(Y_{ij}K_{zz} - K_{zx}) & K_{zx}(Y_{ij}K_{zz} - K_{zx}) & K_{zy}(Y_{ij}K_{zz} - K_{zx}) & -K_{zz}(Y_{ij}K_{zz} - K_{zx}) & (Y_{ij}K_{zz} - K_{zx})^2 & (Y_{ij}K_{zz} - K_{zx}) \cdot (-X_{ij}K_{zz} - K_{zy}) \\
K_{zz}(-X_{ij}K_{zz} - K_{zy}) & K_{zx}(-X_{ij}K_{zz} - K_{zy}) & K_{zy}(-X_{ij}K_{zz} - K_{zy}) & -K_{zz}(-X_{ij}K_{zz} - K_{zy}) & (Y_{ij}K_{zz} - K_{zx}) \cdot (-X_{ij}K_{zz} - K_{zy}) & (-X_{ij}K_{zz} - K_{zy})^2
\end{bmatrix}
\begin{bmatrix}
\Delta_{ijz} \\ \theta_{ijx} \\ \theta_{ijy} \\ \Delta_{jiz} \\ \theta_{jix} \\ \theta_{jiy}
\end{bmatrix}
$$

of the end displacements. The "fixed end" portion is constant and is not a function of end displacements. External potential energy for joint loads is the product of the joint cause and its displacement.

The external potential energy function is developed in the following derivation. First, the structure is given an infinitesimal virtual displacement toward its initial undeformed position. After these small displacements the end forces and moments can be written as follows:

$$N_{ijz} + dN_{ijz} = N_{ijz} - \frac{\partial N_{ijz}}{\partial \theta_{ijx}} d\theta_{ijx} - \frac{\partial N_{ijz}}{\partial \theta_{ijy}} d\theta_{ijy} - \frac{\partial N_{ijz}}{\partial \Delta_{jiz}} d\Delta_{jiz}$$

$$- \frac{\partial N_{ijz}}{\partial \theta_{jix}} d\theta_{jix} - \frac{\partial N_{ijz}}{\partial \theta_{jiy}} d\theta_{jiy} - K_{zz} d\Delta_{ijz}$$

$$M_{ijx} + dM_{ijx} = M_{ijx} - \frac{\partial M_{ijx}}{\partial \theta_{ijy}} d\theta_{ijy} - \frac{\partial M_{ijx}}{\partial \Delta_{ijz}} d\Delta_{ijz} - \frac{\partial M_{ijx}}{\partial \theta_{jix}} d\theta_{jix}$$

$$- \frac{\partial M_{ijx}}{\partial \theta_{jiy}} d\theta_{jiy} - \frac{\partial M_{ijx}}{\partial \Delta_{jiz}} d\Delta_{jiz} - K_{zz} d\theta_{ijx}$$

$$M_{ijy} + dM_{ijy} = M_{ijy} - \frac{\partial M_{ijy}}{\partial \theta_{ijx}} d\theta_{ijx} - \frac{\partial M_{ijy}}{\partial \Delta_{ijz}} d\Delta_{ijz} - \frac{\partial M_{ijy}}{\partial \theta_{jix}} d\theta_{jix}$$

$$- \frac{\partial M_{ijy}}{\partial \theta_{jiy}} d\theta_{jiy} - \frac{\partial M_{ijy}}{\partial \Delta_{jiz}} d\Delta_{jiz} - K_{yy} d\theta_{ijy}$$

$$N_{jiz} + dN_{jiz} = N_{jiz} - \frac{\partial N_{jiz}}{\partial \theta_{ijx}} d\theta_{ijx} - \frac{\partial N_{jiz}}{\partial \theta_{ijy}} d\theta_{ijy} - \frac{\partial N_{jiz}}{\partial \Delta_{ijz}} d\Delta_{ijz}$$

$$- \frac{\partial N_{jiz}}{\partial \theta_{jix}} d\theta_{jix} - \frac{\partial N_{jiz}}{\partial \theta_{jiy}} d\theta_{jiy} - (-K_{zz})\ d\Delta_{jiz}$$

$$M_{jix} + dM_{jix} = M_{jix} - \frac{\partial M_{jix}}{\partial \theta_{ijx}} d\theta_{ijx} - \frac{\partial M_{jix}}{\partial \theta_{ijy}} d\theta_{ijy} - \frac{\partial M_{jix}}{\partial \Delta_{ijz}} d\Delta_{ijz}$$

$$- \frac{\partial M_{jix}}{\partial \theta_{jiy}} d\theta_{jiy} - \frac{\partial M_{jix}}{\partial \Delta_{jiz}} d\Delta_{jiz}$$

$$- \left[\ Y_{ij}(Y_{ij}K_{zz} - K_{zx}) - (Y_{ij}K_{zx} - K_{xx})\ \right] d\theta_{jix}$$

$$M_{jiy} + dM_{jiy} = M_{jiy} - \frac{\partial M_{jiy}}{\partial \theta_{ijx}} d\theta_{ijx} - \frac{\partial M_{jiy}}{\partial \theta_{ijy}} d\theta_{ijy} - \frac{\partial M_{jiy}}{\partial \Delta_{ijz}} d\Delta_{ijz}$$

$$- \frac{\partial M_{jiy}}{\partial \theta_{jix}} d\theta_{jix} - \frac{\partial M_{jiy}}{\partial \Delta_{jiz}} d\Delta_{jiz}$$

$$- \left[-\ X_{ij}(-X_{ij}K_{zz} - K_{zy}) - (-\ X_{ij}K_{zy} - K_{yy})\ \right] d\theta_{jiy}$$

Since the member is linearly elastic and categorized as a conservative system the Principle of Conservation of Energy applies. Thus, the above expressions are integrated over the real displacements producing the external potential energy as a function of end displacements:

$$[-N_{ijz} + K_{zx}\theta_{ijx} + K_{zy}\theta_{ijy} + (Y_{ijy}K_{zz} - K_{zx})\theta_{jix}$$

$$+ (-X_{ij}K_{zz} - K_{zy})\theta_{jiy} + (-K_{zz})\Delta_{jiz}]\Delta_{ijz} + \frac{K_{zz}}{2}\Delta_{ijz}^2$$

$$+ [-M_{ijx} + K_{xy}\theta_{ijy} + K_{zx}\Delta_{ijz} + (Y_{ij}K_{zx} - K_{xx})\theta_{jix}$$

$$+ (-X_{ij}K_{zx} - K_{xy})\theta_{jiy} + (-K_{zx})\Delta_{jiz}]\theta_{ijx} + \frac{K_{xx}}{2}\theta_{ijx}^2$$

$$+ [-M_{ijy} + K_{xy}\theta_{ijx} + K_{zy}\Delta_{ijz} + (Y_{ij}K_{zy} - K_{yx})\theta_{jix}$$

$$+ (-X_{ij}K_{zy} - K_{yy})\theta_{jiy} + (-K_{zy})\Delta_{jiz}]\theta_{ijy} + \frac{K_{yy}}{2}\theta_{ijy}^2$$

$$+ [-N_{jiz} + (-K_{zx})\theta_{ijx} + (-K_{zy})\theta_{ijy} - (Y_{ij}K_{zz} - K_{zx})\theta_{jix}$$

$$- (-X_{ij}K_{zz} - K_{zy})\theta_{jiy} + (-K_{zz})\Delta_{ijz}]\Delta_{jiz} + \frac{K_{zz}}{2}\Delta_{jiz}^2$$

$$+ \Big\{-M_{jix} + (Y_{ij}K_{zx} - K_{xx})\theta_{ijx} + (Y_{ij}K_{zy} - K_{yx})\theta_{ijy}$$

$$+ (Y_{ij}K_{zz} - K_{zx})\Delta_{ijz} + [Y_{ij}(-X_{ij}K_{zz} - K_{zy}) - (-X_{ij}K_{zx} - K_{xy})]\theta_{jiy}$$

$$- (Y_{ij}K_{zz} - K_{xz})\Delta_{jiz}\Big\}\theta_{jix}$$

$$+ \frac{[Y_{ij}(Y_{ij}K_{zz} - K_{zx}) - (Y_{ij}K_{zx} - K_{xx})]}{2}\theta_{jix}^2$$

$$+ \Big\{-M_{jiy} + (-X_{ij}K_{zx} - K_{xy})\theta_{ijx} + (-X_{ij}K_{zy} - K_{yy})\theta_{ijy}$$

$$+ (-X_{ij}K_{zz} - K_{zy})\Delta_{ijz} + [Y_{ij}(-X_{ij}K_{zz} - K_{zy}) - (-X_{ij}K_{zx} - K_{xy})]\theta_{jix}$$

$$- (-X_{ij}K_{zz} - K_{zy})\Delta_{jiz}\Big\}\theta_{jiy}$$

$$+ \frac{[-X_{ij}(-X_{ij}K_{zz} - K_{zy}) - (-X_{ij}K_{zy} - K_{yy})]}{2}\theta_{jiy}^2$$

$$+ [\text{constant}] + \sum_i B_i Z_i = 0$$

where the value [constant] is independent of end displacements and will vanish in the process of differentiation, and $\sum_i B_i Z_i$ is the potential energy of applied loads.

Since potential energy of loads and corresponding reactions are equivalent and opposite in sense, the equivalent expression above is thus prefixed with a minus sign before differentiating. The equivalent values of the end forces and moments from Eq. (2.3), including the effect of loads, are substituted into this equation, reduced in form, and the partial differentiation of $\sum_i B_i Z_i$ with respect to each end displacement produces the respective fixed end load terms.

With a continuous loading rather than a number of concentrated loads the external potential energy expression changes to an integral form $-\int_L w_s Z_s ds$

where

w_s = distributed load w at any point s along the member

Z_s = vertical deflection of distributed load of any point

ds = differential length of member

and by the same procedure as above the fixed end force and moment values are determined.

In a case in which fixed end forces and moments are unknown they can be determined by the following method. This method utilizes the cantilever beam basic structure from which deflections of the free end due to loads are calculated and the member restored to the fixed end condition. Thus,

$$\begin{bmatrix} FN_{ijz} \\ FM_{ijx} \\ FM_{ijy} \\ FN_{jiz} \\ FM_{jix} \\ FM_{jiy} \end{bmatrix} = \begin{bmatrix} -K_{zz} & -K_{zx} & -K_{zy} \\ -K_{zx} & -K_{xx} & -K_{xy} \\ -K_{zy} & -K_{xy} & -K_{yy} \\ +K_{zz} & +K_{zx} & +K_{zy} \\ -(Y_{ij}K_{zz} - K_{zx}) & -(Y_{ij}K_{zx} - K_{xx}) & -(Y_{ij}K_{zy} - K_{yx}) \\ -(-X_{ij}K_{zz} - K_{zy}) & -(-X_{ij}K_{zx} - K_{xy}) & -(-X_{ij}K_{zy} - K_{yy}) \end{bmatrix} \begin{bmatrix} \delta_{ijz} \\ \delta_{ijx} \\ \delta_{ijy} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ CN_{jiz} \\ CM_{jix} \\ CM_{jiy} \end{bmatrix} \qquad (3.3)$$

where

δ_{ijz} = deflection of free end i of cantilever member ij in z direction due to loads

δ_{ijx} = rotation of free end i of cantilever member ij in x direction due to loads

δ_{ijy} = rotation of free end i of cantilever member ij in y direction due to loads

CN_{jiz} = shear force at cantilevered end j of member ij in z direction due to loads

CM_{jix} = moment at cantilevered end j of member ij in x direction due to loads

CM_{jiy} = moment at cantilevered end j of member ij in y direction due to loads.

External potential energy for loads applied at the joints can be expressed as $-\sum_i P_i\Delta_i$ for concentrated external loads or $-\sum_i Q_i\theta_i$ for externally applied moments. Thus the potential energy is

$$V = -\sum_i P_i\Delta_i - \sum_i Q_i\theta_i$$

The differentiation gives:

$$\begin{bmatrix} \dfrac{\partial V}{\partial \Delta_{ijz}} \\ \dfrac{\partial V}{\partial \theta_{ijx}} \\ \dfrac{\partial V}{\partial \theta_{ijy}} \\ \dfrac{\partial V}{\partial \Delta_{jiz}} \\ \dfrac{\partial V}{\partial \theta_{jix}} \\ \dfrac{\partial V}{\partial \theta_{jiy}} \end{bmatrix} = (-) \begin{bmatrix} P_{ijz} \\ Q_{ijx} \\ Q_{ijy} \\ P_{jiz} \\ Q_{jix} \\ Q_{jiy} \end{bmatrix} \qquad (3.4)$$

3.4 Modification for Hinged Support Conditions

Potential energy expressions for a hinged end condition are functions of the shear force of the hinged end and the applied loads. These expressions are formulated as functions of end displacements in a manner described in sections 3.2 and 3.3.

Utilizing conjugate beam equilibrium for a hinged end member yields

$$\begin{bmatrix} \Delta_{ijz} \\ \hline \theta_{ijx} \\ \hline \theta_{ijy} \end{bmatrix} = \begin{bmatrix} f'_{iizz} \\ \hline f_{iixz} \\ \hline f_{iiyz} \end{bmatrix} \begin{bmatrix} N_{ijz} \end{bmatrix} + \left[\begin{array}{c|c|c} 1 & -Y_{ij} & X_{ij} \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \begin{bmatrix} \Delta_{jiz} \\ \hline \theta_{jix} \\ \hline \theta_{jiy} \end{bmatrix}$$

$$+ \begin{bmatrix} f''_{iizz} \\ \hline 0 \\ \hline 0 \end{bmatrix} \begin{bmatrix} N_{ijz} \end{bmatrix} \qquad (3.5)$$

from which,

$$[\, N_{ijz} \,] = \left[\begin{array}{c|c|c|c} \frac{1}{f'_{iizz}} & \frac{-1}{f'_{iizz}} & \frac{Y_{ij}}{f'_{iizz}} & \frac{-X_{ij}}{f'_{iizz}} \end{array}\right] \begin{bmatrix} \Delta_{ijz} \\ \hline \Delta_{jiz} \\ \hline \theta_{jix} \\ \hline \theta_{jiy} \end{bmatrix}$$

$$+ \left(\frac{-f''_{iizz}}{f'_{iizz} + f''_{iizz}} \right) \left[\begin{array}{c|c|c|c} \frac{1}{f'_{iizz}} & \frac{-1}{f'_{iizz}} & \frac{Y_{ij}}{f'_{iizz}} & \frac{-X_{ij}}{f'_{iizz}} \end{array}\right] \begin{bmatrix} \Delta_{ijz} \\ \hline \Delta_{jiz} \\ \hline \theta_{jix} \\ \hline \theta_{jiy} \end{bmatrix}$$

Then, expressing cross sectional moments as functions of end displacements, rotating values into m system, substituting into Eq. (3.2b), and differentiating the strain energy with respect to each of the four end displacements produces

$$\begin{bmatrix} \dfrac{\partial U}{\partial \Delta_{ijz}} \\ \dfrac{\partial U}{\partial \Delta_{jiz}} \\ \dfrac{\partial U}{\partial \theta_{jix}} \\ \dfrac{\partial U}{\partial \theta_{jiy}} \end{bmatrix} = \frac{1}{f'_{iizz}} \begin{bmatrix} 1 & -1 & Y_{ij} & -X_{ij} \\ -1 & 1 & -Y_{ij} & X_{ij} \\ Y_{ij} & -Y_{ij} & Y_{ij}^2 & -Y_{ij}X_{ij} \\ -X_{ij} & X_{ij} & -Y_{ij}X_{ij} & X_{ij}^2 \end{bmatrix} \begin{bmatrix} \Delta_{ijz} \\ \Delta_{jiz} \\ \theta_{jix} \\ \theta_{jiy} \end{bmatrix}$$

$$+ \frac{CF'}{f'_{iizz}} \begin{bmatrix} 1 & -1 & Y_{ij} & -X_{ij} \\ -1 & 1 & -Y_{ij} & X_{ij} \\ Y_{ij} & -Y_{ij} & Y_{ij}^2 & -Y_{ij}X_{ij} \\ -X_{ij} & X_{ij} & -Y_{ij}X_{ij} & X_{ij}^2 \end{bmatrix} \begin{bmatrix} \Delta_{ijz} \\ \Delta_{jiz} \\ \theta_{jix} \\ \theta_{jiy} \end{bmatrix} \tag{3.6}$$

Again, the first matrix is the influence of bending and torsion, and the second matrix represents the contribution of shear effect on the displacements where

$$CF' = \frac{-f''_{iizz}}{(f'_{iizz} + f''_{iizz})}$$

As in section 3.2 the second matrix is valid for joint loads only.

External potential energy is also expressed as a function of end displacements for a hinged end member. For in-span loads the structure is given an infinitesimal virtual displacement toward its initial position, equilibrating forces and moments are evaluated, and the force-displacement function integrated over the real displacement yields

$$\left[-N_{ijz} + \frac{Y_{ij}}{f'_{iizz}}\,\theta_{jix} + \frac{(-X_{ij})}{f'_{iizz}}\,\theta_{jiy} + \frac{(-1)}{f'_{iizz}}\,\Delta_{jiz}\right]\,\Delta_{ijz}$$

$$+\ \frac{1}{f'_{iizz}}\,\frac{\Delta_{ijz}^{2}}{2}$$

$$+\ \left[-N_{jiz} + \frac{(-Y_{ij})}{f'_{iizz}}\,\theta_{jix} + \frac{X_{ij}}{f'_{iizz}}\,\theta_{jiy} + \frac{(-1)}{f'_{iizz}}\,\Delta_{ijz}\right]\,\Delta_{jiz}$$

$$+\ \frac{1}{f'_{iizz}}\,\frac{\Delta_{jiz}^{2}}{2}$$

$$+\ \left[-M_{jix} + \frac{(-Y_{ij}X_{ij})}{f'_{iizz}}\,\theta_{jiy} + \frac{(-Y_{ij})}{f'_{iizz}}\,\Delta_{jiz} + \frac{Y_{ij}}{f'_{iizz}}\,\Delta_{ijz}\right]\,\theta_{jix}$$

$$+\ \frac{Y_{ij}^{2}}{f'_{iizz}}\,\frac{\theta_{jix}^{2}}{2}$$

$$+\ \left[-M_{jiy} + \frac{(-Y_{ij}X_{ij})}{f'_{iizz}}\,\theta_{jix} + \frac{X_{ij}}{f'_{iizz}}\,\Delta_{jiz} + \frac{(-X_{ij})}{f'_{iizz}}\,\Delta_{ijz}\right]\,\theta_{jiy}$$

$$+\ \frac{X_{ij}^{2}}{f'_{iizz}}\,\frac{\theta_{jiy}^{2}}{2}$$

$$+\ [\text{constant}]' + \sum_i B_i Z_i \;=\; 0$$

The value $[\text{constant}]'$ is independent of end displacements and vanishes under differentiation. To complete the process, the external potential energy function $\sum_i Z_i B_i$ is changed in sign and differentiated with respect to each end displacement, producing the respective fixed end load terms.

The fixed end load terms are then obtained from the following expression:

$$\begin{bmatrix} FN_{ijz} \\ \hline FN_{jiz} \\ \hline FM_{jix} \\ \hline FM_{jiy} \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{f'_{iizz}} \\ \hline \dfrac{1}{f'_{iizz}} \\ \hline \dfrac{-Y_{ij}}{f'_{iizz}} \\ \hline \dfrac{X_{ij}}{f'_{iizz}} \end{bmatrix} \begin{bmatrix} \delta_{ijz} \end{bmatrix} + \begin{bmatrix} 0 \\ \hline CN_{jiz} \\ \hline CM_{jix} \\ \hline CM_{jiy} \end{bmatrix} \tag{3.7}$$

The differentiation of external potential energy due to loads applied at the ends of the member yields

$$\begin{bmatrix} \dfrac{\partial V}{\partial \Delta_{ijz}} \\ \hline \dfrac{\partial V}{\partial \Delta_{jiz}} \\ \hline \dfrac{\partial V}{\partial \theta_{jix}} \\ \hline \dfrac{\partial V}{\partial \theta_{jiy}} \end{bmatrix} = (-) \begin{bmatrix} P_{ijz} \\ \hline P_{jiz} \\ \hline Q_{jix} \\ \hline Q_{jiy} \end{bmatrix} \tag{3.8}$$

3.5 Final Matrix Equations

Once the strain energy and external potential energy of a member are differentiated with respect to each end displacement so as to satisfy Eq. (3.1), the results are superimposed member by member for the complete structure. The following matrix equations show the results of differentiation for a member and synthesis for the complete structure.

The equations for a member with loads applied at the joints are given in Table 3.2 and for a hinged end member

$$
\frac{1}{f'_{iizz}}
\begin{bmatrix}
1 & -1 & Y_{ij} & -X_{ij} \\
-1 & 1 & -Y_{ij} & X_{ij} \\
Y_{ij} & -Y_{ij} & Y_{ij}^2 & -Y_{ij}X_{ij} \\
-X_{ij} & X_{ij} & -Y_{ij}X_{ij} & X_{ij}^2
\end{bmatrix}
\begin{bmatrix}
\Delta_{ijz} \\ \Delta_{jiz} \\ \theta_{jix} \\ \theta_{jiy}
\end{bmatrix}
$$

$$
+ \frac{CF'}{f'_{iizz}}
\begin{bmatrix}
1 & -1 & Y_{ij} & -X_{ij} \\
-1 & 1 & -Y_{ij} & X_{ij} \\
Y_{ij} & -Y_{ij} & Y_{ij}^2 & -Y_{ij}X_{ij} \\
-X_{ij} & X_{ij} & -Y_{ij}X_{ij} & X_{ij}^2
\end{bmatrix}
\begin{bmatrix}
\Delta_{ijz} \\ \Delta_{jiz} \\ \theta_{jix} \\ \theta_{jiy}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
P_{ijz} \\ P_{jiz} \\ Q_{jix} \\ Q_{jiy}
\end{bmatrix}
\qquad (3.9)
$$

TABLE 3.2 GENERAL MEMBER EQUILIBRIUM EQUATIONS

$$
\begin{bmatrix}
K_{zz} & K_{zx} & K_{zy} & -K_{zz} & (Y_{ij}K_{zz} - K_{zx}) & (-X_{ij}K_{zz} - K_{zy}) \\
K_{zx} & K_{xx} & K_{xy} & -K_{zx} & (Y_{ij}K_{zx} - K_{xx}) & (-X_{ij}K_{zx} - K_{xy}) \\
K_{zy} & K_{xy} & K_{yy} & -K_{zy} & (Y_{ij}K_{zy} - K_{yx}) & (-X_{ij}K_{zy} - K_{yy}) \\
-K_{zz} & -K_{zx} & -K_{zy} & K_{zz} & -(Y_{ij}K_{zz} - K_{zx}) & -(-X_{ij}K_{zz} - K_{zy}) \\
(Y_{ij}K_{zz} - K_{zx}) & (Y_{ij}K_{zx} - K_{xx}) & (Y_{ij}K_{zy} - K_{yx}) & -(Y_{ij}K_{zz} - K_{xz}) & Y_{ij}(Y_{ij}K_{zz} - K_{zx}) - (Y_{ij}K_{zx} - K_{xx}) & Y_{ij}(-X_{ij}K_{zz} - K_{zy}) - (-X_{ij}K_{zx} - K_{xy}) \\
(-X_{ij}K_{zz} - K_{zy}) & (-X_{ij}K_{zx} - K_{xy}) & (-X_{ij}K_{zy} - K_{yy}) & -(-X_{ij}K_{zz} - K_{zy}) & Y_{ij}(-X_{ij}K_{zz} - K_{zy}) - (-X_{ij}K_{zx} - K_{xy}) & -X_{ij}(-X_{ij}K_{zz} - K_{zy}) - (-X_{ij}K_{zy} - K_{yy})
\end{bmatrix}
\begin{bmatrix}
\Delta_{ijz} \\ \theta_{ijx} \\ \theta_{ijy} \\ \Delta_{jiz} \\ \theta_{jix} \\ \theta_{jiy}
\end{bmatrix}
$$

$$
+ \ CF
\begin{bmatrix}
K_{zz}^2 & K_{zx}K_{zz} & K_{zy}K_{zz} & -K_{zz}^2 & K_{zz}(Y_{ij}K_{zz} - K_{zx}) & K_{zz}(-X_{ij}K_{zz} - K_{zy}) \\
K_{zz}K_{zx} & K_{zx}^2 & K_{zy}K_{zx} & -K_{zz}K_{zx} & K_{zx}(Y_{ij}K_{zz} - K_{zx}) & K_{zx}(-X_{ij}K_{zz} - K_{zy}) \\
K_{zz}K_{zy} & K_{zx}K_{zy} & K_{zy}^2 & -K_{zz}K_{zy} & K_{zy}(Y_{ij}K_{zz} - K_{zx}) & K_{zy}(-X_{ij}K_{zz} - K_{zy}) \\
-K_{zz}^2 & -K_{zx}K_{zz} & -K_{zy}K_{zz} & K_{zz}^2 & -K_{zz}(Y_{ij}K_{zz} - K_{zx}) & -K_{zz}(-X_{ij}K_{zz} - K_{zy}) \\
K_{zz}(Y_{ij}K_{zz} - K_{zx}) & K_{zx}(Y_{ij}K_{zz} - K_{zx}) & K_{zy}(Y_{ij}K_{zz} - K_{zx}) & -K_{zz}(Y_{ij}K_{zz} - K_{zx}) & (Y_{ij}K_{zz} - K_{zx})^2 & (Y_{ij}K_{zz} - K_{zx}) \cdot (-X_{ij}K_{zz} - K_{zy}) \\
K_{zz}(-X_{ij}K_{zz} - K_{zy}) & K_{zx}(-X_{ij}K_{zz} - K_{zy}) & K_{zy}(-X_{ij}K_{zz} - K_{zy}) & -K_{zz}(-X_{ij}K_{zz} - K_{zy}) & (Y_{ij}K_{zz} - K_{zx}) \cdot (-X_{ij}K_{zz} - K_{zy}) & (-X_{ij}K_{zz} - K_{zy})^2
\end{bmatrix}
\begin{bmatrix}
\Delta_{ijz} \\ \theta_{ijx} \\ \theta_{ijy} \\ \Delta_{jiz} \\ \theta_{jix} \\ \theta_{jiy}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
P_{ijz} \\ Q_{ijx} \\ Q_{ijy} \\ P_{jiz} \\ Q_{jix} \\ Q_{jiy}
\end{bmatrix}
$$

For in-span load conditions the load vector must include the fixed end effects from Eq. (3.3) or (3.7) and the shear expression must be modified.

These equations are then superimposed member by member for the complete structure to produce a matrix of the form

$$\begin{bmatrix} \chi_{iizz} & \chi_{iizx} & \chi_{iizy} & \cdots & \chi_{mizy} \\ \chi_{iixz} & \chi_{iixx} & \chi_{iixy} & \cdots & \chi_{mixy} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \chi_{imyz} & \chi_{imyx} & \chi_{imyy} & \cdots & \chi_{mmyy} \end{bmatrix} \begin{bmatrix} \Delta_{iz} \\ \theta_{ix} \\ \cdot \\ \cdot \\ \cdot \\ \theta_{my} \end{bmatrix} = \begin{bmatrix} \Sigma(P_{iz} - FN_{iz}) \\ \Sigma(Q_{ix} - FM_{ix}) \\ \cdot \\ \cdot \\ \cdot \\ \Sigma(Q_{my} - FM_{my}) \end{bmatrix} \tag{3.10}$$

where

χ_{iizz} = the sum of member stiffnesses at joint i in the z direction. All other stiffness terms are similarly defined.

$\Sigma(P_{iz} - FN_{iz})$ = the sum of all member end forces and fixed end values at end i. All other terms are similarly defined.

This final equation is then solved for the unknown displacements, and subsequently the individual member forces and moments are readily found.

CHAPTER IV

NUMERICAL EXAMPLES

4.1 General

Analysis of six selected frames by the method developed in the preceeding chapters is presented. Each example was completely analyzed by the IBM 1410 Digital Computer. After the loads and geometric properties of the members and structure were designated, the following procedure of analysis was used:

1. The six flexibility terms for each member were determined.
2. Each 3 x 3 flexibility matrix was inverted to obtain the stiffness matrix Eq. (2.2).
3. Stiffness expressions for each member were evaluated by substitution into Table 3.1 or Eq. (3.6), the shear effects were omitted. Stiffnesses of each joint were combined and substituted into Eq. (3.10).
4. Load vectors were determined by evaluation of Eq. (3.3), (3.4), (3.7), or (3.8); terms at each joint were combined and substituted into Eq. (3.10).
5. Eq. (3.10) was solved for unknown displacements.
6. Member shears and moments were calculated utilizing member stiffnesses, end displacements, and fixed end values.

The basic logic of the computer program was as listed above. Imput data provided geometric properties of members and structure configuration as well as loads. Output was limited to final member shears and moments; all other processes were internal to the computer.

4.2 Examples 1 and 2

A planar frame consisting of two circular members with concentrated loads at the center acting normal to the plane and supported by three straight members rigidly restrained at their bases is analyzed in Example 1. Example 2 solves the same structure and loading condition except that the center member is hinged at the support rather than fixed. All members have a constant cross section and EI/GJ is equal to unity. The structure is illustrated in Fig. 4.1 with final results tabulated in Table 4.1 and 4.2.

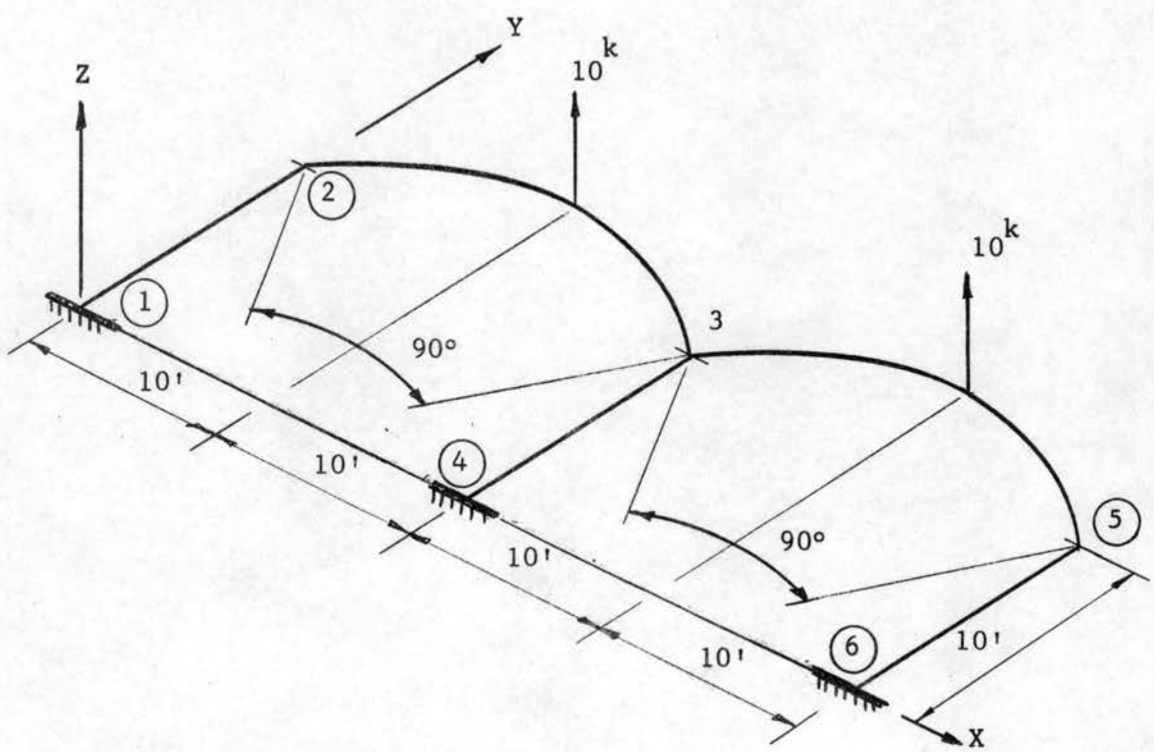

Fig. 4.1 Geometry and Loads Example 1

TABLE 4.1 FINAL SHEARS AND MOMENTS EXAMPLE 1

Member	Shear	X-Moment	Y-Moment
1 2	-5.1	-80.4	17.1
2 1	5.1	29.3	-17.1
2 3	-5.1	-29.3	17.1
3 2	-4.9	-12.2	-14.8
3 4	9.8	24.3	0.0
4 3	-9.8	-122.0	0.0
3 5	-4.9	-12.2	14.8
5 3	-5.1	-29.3	-17.1
5 6	5.1	29.3	17.1
6 5	-5.1	-80.4	-17.1

Final results of Example 2 are tabulated in Table 4.2.

TABLE 4.2 FINAL SHEARS AND MOMENTS EXAMPLE 2

Member	Shear	X-Moment	Y-Moment
1 2	-9.4	-141.4	46.8
2 1	9.4	47.2	-46.8
2 3	-9.4	-47.2	46.8
3 2	-0.6	5.8	41.6
3 4	1.2	-11.6	0.0
4 3	-1.2	0.0	0.0
3 5	-0.6	5.8	-41.6
5 3	-9.4	-47.2	-46.8
5 6	9.4	47.2	46.8
6 5	-9.4	-141.4	-46.8

4.3 Examples 3 and 4

The same planar frame of Example 1, loaded with a uniform load of 0.5 kips per foot on the circular members, is analyzed in Example 3. The supports are rigidly restrained and EI/GJ is unity. Example 4 solves the same structure and loading as Example 3 except the center support is hinged rather than fixed. Fig. 4.2 illustrates the structure with results tabulated in Tables 4.3 and 4.4.

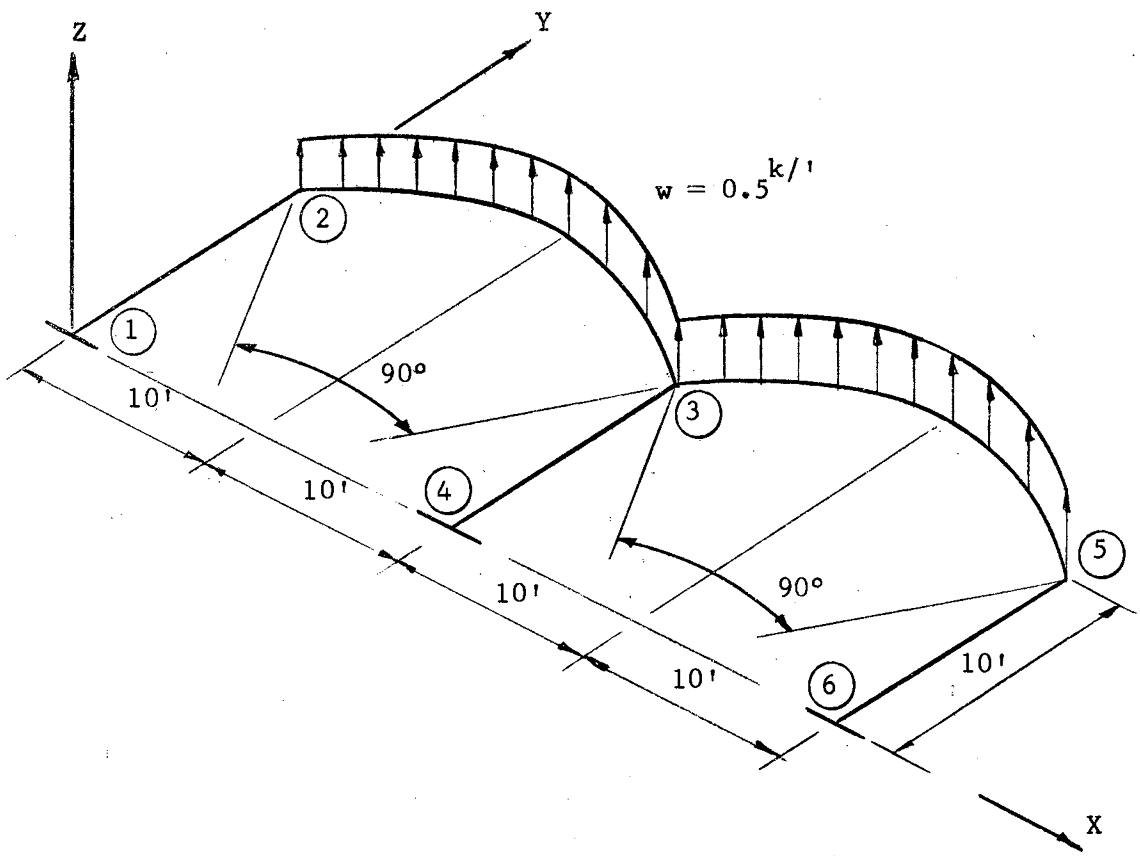

Fig. 4.2 Geometry and Loads Example 3

TABLE 4.3 FINAL SHEARS AND MOMENTS EXAMPLE 3

Member	Shear	X-Moment	Y-Moment
1 2	-5.8	-81.2	13.6
2 1	5.8	22.9	-13.6
2 3	-5.8	-22.9	13.6
3 2	-5.3	-7.4	-8.0
3 4	10.6	14.8	0.0
4 3	-10.6	-120.2	0.0
3 5	-5.3	-7.4	8.0
5 3	-5.8	-22.9	-13.6
5 6	5.8	22.9	13.6
6 5	-5.8	-81.2	-13.6

Final results of Example 4 are tabulated in Table 4.4.

TABLE 4.4 FINAL SHEARS AND MOMENTS EXAMPLE 4

Member	Shear	X-Moment	Y-Moment
1 2	-10.1	-141.3	42.9
2 1	10.1	40.6	-42.9
2 3	-10.1	-40.6	42.9
3 2	-1.0	10.3	47.6
3 4	2.0	-20.6	0.0
4 3	-2.0	0.0	0.0
3 5	-1.0	10.3	-47.6
5 3	-10.1	-40.6	-42.9
5 6	10.1	40.6	42.9
6 5	-10.1	-141.3	42.9

4.4 Examples 5 and 6

Example 5 is analysis of a planar structure consisting of four circular curved members, one straight member, and three concentrated loads applied at the structure joints. All joints are rigidly connected and fixity is assumed at the support positions. All members have constant cross section and EI/GJ is assumed equal to unity. Example 6 analyzes the same structure and loading as Example 5 except that the support at joint number 1 is hinged rather than fixed. Fig. 4.3 illustrates the structure and loads with final results given in the Tables 4.5 and 4.6.

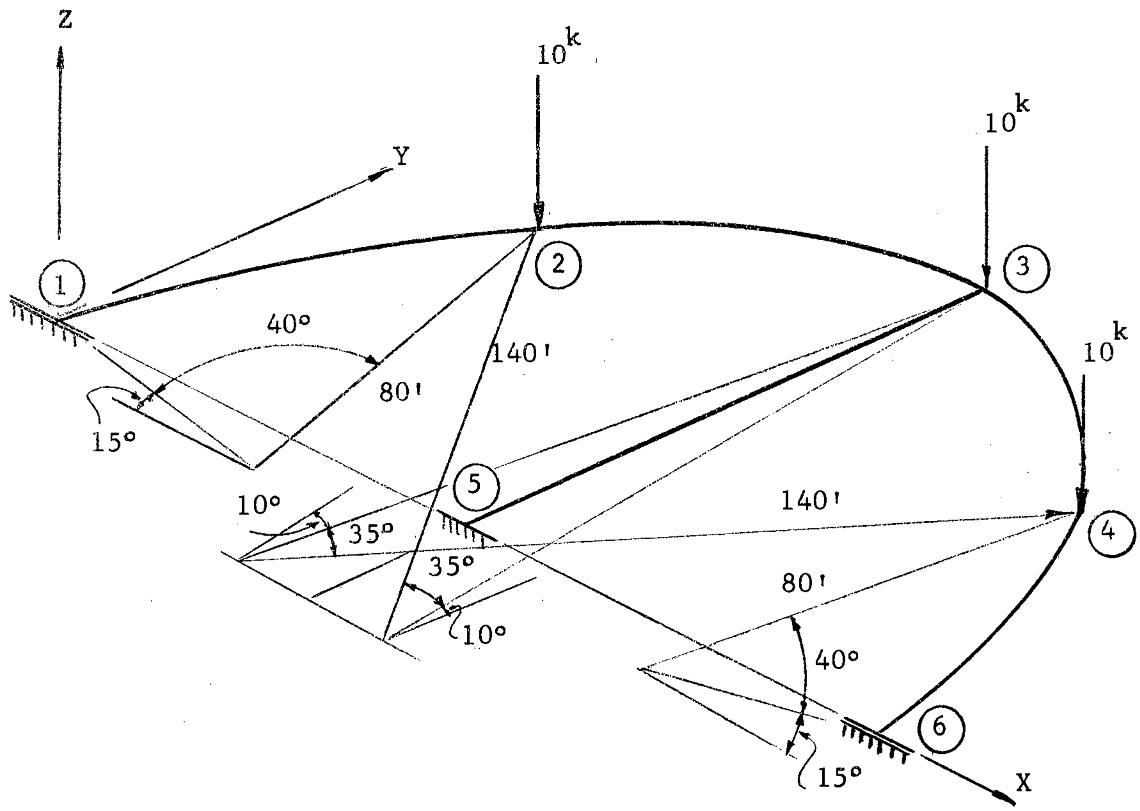

Fig. 4.3 Geometry and Loads Example 5

TABLE 4.5 FINAL SHEARS AND MOMENTS EXAMPLE 5

Member	Shear	X-Moment	Y-Moment
1 2	9.7	511.6	-228.8
2 1	-9.7	-75.9	-76.3
2 3	-0.3	75.9	76.3
3 2	0.3	-86.8	-55.6
3 5	-10.6	173.5	0.0
5 3	10.6	710.0	0.0
3 4	0.3	-86.8	55.6
4 3	-0.3	75.9	-76.3
4 6	-9.7	-75.9	76.3
6 4	9.7	511.6	228.8

Final results of Example 6 are tabulated in Table 4.6.

TABLE 4.6 FINAL SHEARS AND MOMENTS EXAMPLE 6

Member	Shear	X-Moment	Y-Moment
1 2	4.8	0.0	0.0
2 1	-4.8	217.0	-151.9
2 3	-5.2	-217.0	151.9
3 2	5.2	16.4	233.3
3 5	-16.6	225.7	-138.4
5 3	16.6	1161.1	138.4
3 4	1.4	242.2	-94.9
4 3	-1.4	187.3	-10.4
4 6	-8.6	-187.3	10.4
6 4	8.6	572.1	259.2

CHAPTER V

SUMMARY AND CONCLUSIONS

5.1 Summary and Conclusions

The analysis by the Principle of Minimum Potential Energy of planar structures loaded normal to the plane is presented in this work.

By utilization of the conjugate beam analogy the total potential energy consisting of both strain and external energy is written as a function of end displacements for a general shaped planar member. The total potential energy function for an entire structure is the sum of the energy of each member. This function is of a quadratic form in the displacements, thus derivatives with respect to each end displacement leads to stiffness type equations. Coefficients of like terms are combined resulting in a final stiffness equation with displacements as redundant quantities. This equation is solved for the displacements and, subsequently, the member shears and moments.

The procedure of analysis requires calculation of the flexibility terms for each member, inversion of the flexibility matrix, superposition of member stiffnesses at each joint, calculation of fixed end values and joint loads, formulation of a final stiffness matrix equation by summing appropriate member stiffnesses, determination of all joint displacements by any direct method, and then evaluation of member forces and moments utilizing member stiffnesses and solved displacements.

The expressions developed in this work are limited to linear structural materials; however, applications of the Principle of Minimum Potential Energy is not limited to linear systems. Application of the principle automatically satisfies compatibility while the minimization process satisfies equilibrium. Solution of the final stiffness equation provides all displacements necessary to calculate member shears and moments. No additional conditions or shear equations are necessary for complete analysis of a structure. The method is direct and has a physical significance. In comparison to other methods of analysis, it does have in many instances a larger number of redundants; however, because of the systematic approach, and the completeness in analysis the disadvantages are outweighed.

The method is unique in that it utilizes the potential energy function which is the sum of the strain energy of each member plus the work of external loads. The minimization process reduces the expression to the final matrix equations shown in Eq. (3.10).

It can be seen that this method has a great advantage in application to planar frames with in-plane load conditions. The potential energy function is minimized with respect to the permissible displacements and the problem is solved. Other methods of analysis generally require a shear equation as an added condition necessary to solve the problem.

In conclusion, the Principle of Minimum Potential Energy leads to a stiffness method of analysis applicable to any shaped planar type structure with loads normal to the plane. Analysis by this method is systematic, is adaptable to computer solution, and includes any variable support conditions.

5.2 Extension of the Method

The method should be extended to planar structures with in-plane loading. The effect of shear and normal forces should be included and the ease and simplicity compared with other methods.

The method should be applied to three dimensional structures with any general type loading. The shear and normal force effects should be included and the method compared with other techniques of analysis.

The expressions for flexibilities and fixed end effects should be evaluated for various shaped structural members with general loadings. An investigation of the error should be determined when these values are computed by a numerical procedure.

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APPENDIX

Once the strain energy function is expressed in terms of displacements it is differentiated with respect to these displacements and integrated over the length of the member. Differentiation is accomplished before integration; flexibility values are substituted for their equivalent integral expressions and final results expressed in terms of stiffnesses. The first term only is developed to illustrate the procedure; additional terms are obtained by a repetitive process.

As in section 3.2 the strain energy can be written as

$$U = [\ H_q\ ]^T\ [\ \omega_{om}\ ]\ [\ \lambda^m\ ]\ [\ \omega_{mo}\ ]\ [\ H_q\ ] \tag{A.1}$$

where the cross sectional values $[\ H_q\ ]$ are evaluated by statics and their equivalent expression substituted from Eq. (2.3) to produce the strain energy as a function of end displacements alone. For simplicity

$$\begin{aligned}
a &= (K_{zz} - Y_{iq}K_{zz}) \\
b &= (K_{xx} - Y_{iq}K_{zx}) \\
c &= (K_{xy} - Y_{iq}K_{zy}) \\
d &= (-K_{xz} + Y_{iq}K_{zz}) \\
e &= [(Y_{ij}K_{xz} - K_{xx}) - Y_{iq}(Y_{ij}K_{zz} - K_{zx})] \\
f &= [(-X_{ij}K_{xz} - K_{xy}) - Y_{iq}(-X_{ij}K_{zz} - K_{zy})] \\
g &= (K_{yz} + X_{iq}K_{zz})
\end{aligned}$$

$$h = (K_{yx} + X_{iq}K_{zx})$$

$$i = (K_{yy} + X_{iq}K_{zy})$$

$$j = (-K_{yz} - X_{iq}K_{zz})$$

$$k = [(Y_{ij}K_{yz} - K_{yx}) + X_{iq}(Y_{ij}K_{zz} - K_{zx})]$$

$$m = [(-X_{ij}K_{yz} - K_{yy}) + X_{iq}(-X_{ij}K_{zz} - K_{zy})]$$

so that after substitution into Eq. (A.1) differentiation with respect to Δ_{ijz} yields

$$\begin{aligned}
\frac{\partial U}{\partial \Delta_{ijz}} &= \left[\int_L (a \cos\alpha + g \sin\alpha)^2 \frac{ds}{GJ} + \int_L (-a \sin\alpha + g \cos\alpha)^2 \frac{ds}{EI}\right] \Delta_{ijz} \\
&+ \left[\int_L (a \cos\alpha + g \sin\alpha)(b \cos\alpha + h \sin\alpha) \frac{ds}{GJ}\right. \\
&\qquad \left. + \int_L (-a \sin\alpha + g \cos\alpha)(-b \sin\alpha + h \cos\alpha) \frac{ds}{EI}\right] \theta_{ijx} \\
&+ \left[\int_L (a \cos\alpha + g \sin\alpha)(c \cos\alpha + i \sin\alpha) \frac{ds}{GJ}\right. \\
&\qquad \left. + \int_L (-a \sin\alpha + g \cos\alpha)(-c \sin\alpha + i \cos\alpha) \frac{ds}{EI}\right] \theta_{jiy} \\
&+ \left[\int_L (a \cos\alpha + g \sin\alpha)(d \cos\alpha + j \sin\alpha) \frac{ds}{GJ}\right. \\
&\qquad \left. + \int_L (-a \sin\alpha + g \cos\alpha)(-d \sin\alpha + j \cos\alpha) \frac{ds}{EI}\right] \Delta_{jiz} \\
&+ \left[\int_L (a \cos\alpha + g \sin\alpha)(e \cos\alpha + k \sin\alpha) \frac{ds}{GJ}\right. \\
&\qquad \left. + \int_L (-a \sin\alpha + g \cos\alpha)(-e \sin\alpha + k \cos\alpha) \frac{ds}{EI}\right] \theta_{jix} \\
&+ \left[\int_L (a \cos\alpha + g \sin\alpha)(f \cos\alpha + m \sin\alpha) \frac{ds}{GJ}\right. \\
&\qquad \left. + \int_L (-a \sin\alpha + g \cos\alpha)(-f \sin\alpha + m \cos\alpha) \frac{ds}{EI}\right] \theta_{jiy}
\end{aligned}$$

Differentiation with respect to the other displacements are similarly expressed. The above equation does not include shear nor the altered moment due to shear; these effects are differentiated in a similar manner and added as given in the final result. Expansion of the coefficient term C_{11} of Δ_{ijz} yields

$$\int_L (a \cos \alpha + g \sin \alpha)^2 \frac{ds}{GJ} + \int_L (-a \sin \alpha + g \cos \alpha)^2 \frac{ds}{EI}$$

$$= \int_L [(K_{xz}^2 + Y_{iq}^2 K_{zz}^2 - 2Y_{iq}K_{xz}K_{zz}) \cos^2 \alpha$$

$$+ (K_{yz}^2 + X_{iq}^2 K_{zz}^2 + 2X_{iq}K_{yz}K_{zz}) \sin^2 \alpha$$

$$+ 2 \cos \alpha \sin \alpha (K_{xz}K_{yz} - Y_{iq}K_{yz}K_{zz} + X_{iq}K_{xz}K_{zz} - X_{iq}Y_{iq}K_{zz}^2)] \frac{ds}{GJ}$$

$$+ \int_L [(K_{xz}^2 + Y_{iq}^2 K_{zz}^2 - 2Y_{iq}K_{xz}K_{zz}) \sin^2 \alpha$$

$$+ (K_{yz}^2 + X_{iq}^2 K_{zz}^2 + 2X_{iq}K_{yz}K_{zz}) \cos^2 \alpha$$

$$- 2 \cos \alpha \sin \alpha (K_{xz}K_{yz} - Y_{iq}K_{yz}K_{zz} + X_{iq}K_{xz}K_{zz} - X_{iq}Y_{iq}K_{zz}^2)] \frac{ds}{EI}$$

Extraction of the constant terms and substitution of the following flexibility terms

$$f'_{iizz} = [\int_L (Y_{iq}^2 \cos^2 \alpha - 2X_{iq}Y_{iq} \sin \alpha \cos \alpha + X_{iq}^2 \sin^2 \alpha) \frac{ds}{GJ}$$

$$+ \int_L (Y_{iq}^2 \sin^2 \alpha + 2X_{iq}Y_{iq} \sin \alpha \cos \alpha + X_{iq}^2 \cos^2 \alpha) \frac{ds}{EI}]$$

$$f_{iizx} = [\int_L (-Y_{iq} \cos^2 \alpha + X_{iq} \sin \alpha \cos \alpha) \frac{ds}{GJ}$$

$$+ \int_L (-Y_{iq} \sin^2 \alpha - X_{iq} \sin \alpha \cos \alpha) \frac{ds}{EI}]$$

$$f_{iizy} = [\int_L (-Y_{iq} \sin \alpha \cos \alpha + X_{iq} \sin^2 \alpha) \frac{ds}{GJ}$$

$$+ \int_L (Y_{iq} \sin \alpha \cos \alpha + X_{iq} \cos^2 \alpha) \frac{ds}{EI}]$$

$$f_{iixx} = \left[\int_L \cos^2 \alpha \frac{ds}{GJ} + \int_L \sin^2 \alpha \frac{ds}{EI}\right]$$

$$f_{iixy} = \left[\int_L \sin \alpha \cos \alpha \frac{ds}{GJ} + \int_L -\sin \alpha \cos \alpha \frac{ds}{EI}\right]$$

$$f_{iiyy} = \left[\int_L \sin^2 \alpha \frac{ds}{GJ} + \int_L \cos^2 \alpha \frac{ds}{EI}\right]$$

for the equivalent integral expressions reduce the Δ_{ijz} coefficient to

$$C_{11} = K_{zz}^2 f'_{iizz} + K_{xz}^2 f_{iixx} + K_{yz}^2 f_{iiyy}$$

$$+ \; 2K_{xz}K_{yz}f_{iixy} + 2K_{xz}K_{zz}f_{iixz} + 2K_{yz}K_{zz}f_{iiyz}$$

Each coefficient term is thus expressed as a function of stiffness and flexibility terms, which can be further expressed in matrix form as

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} K_{zz} & K_{zx} & K_{zy} \\ K_{zx} & K_{xx} & K_{xy} \\ K_{zy} & K_{xy} & K_{yy} \end{bmatrix} \begin{bmatrix} f'_{iizz} & f_{iizx} & f_{iizy} \\ f_{iizx} & f_{iixx} & f_{iixy} \\ f_{iizy} & f_{iixy} & f_{iiyy} \end{bmatrix} \begin{bmatrix} K_{zz} & K_{zx} & K_{zy} \\ K_{zx} & K_{xx} & K_{xy} \\ K_{zy} & K_{xy} & K_{yy} \end{bmatrix}$$

$$= \begin{bmatrix} K_{zz} & K_{zx} & K_{zy} \\ K_{zx} & K_{xx} & K_{xy} \\ K_{zy} & K_{xy} & K_{yy} \end{bmatrix}$$

The remainder of the coefficient terms are expressed in a similar manner and reduced in form to that of Table 3.1.

For the case of a hinged support the same procedure applies. For simplicity,

$$a' = \frac{-Y_{iq}}{f'_{iizz}} \qquad\qquad d' = \frac{Y_{iq}}{f'_{iizz}}$$

$$e' = \frac{-Y_{iq}Y_{ij}}{f'_{iizz}} \qquad f' = \frac{Y_{iq}X_{ij}}{f'_{iizz}}$$

$$g' = \frac{X_{iq}}{f'_{iizz}} \qquad j' = \frac{-X_{iq}}{f'_{iizz}}$$

$$k' = \frac{X_{iq}Y_{ij}}{f'_{iizz}} \qquad m' = \frac{-X_{iq}X_{ij}}{f'_{iizz}}$$

so that substitution into the strain energy expression and differentiation with respect to Δ_{ijz} yields

$$\begin{aligned}
\frac{\partial U}{\partial \Delta_{ijz}} = {} & \left[\int_L (a' \cos\alpha + g' \sin\alpha)^2 \frac{ds}{GJ} + \int_L (-a' \sin\alpha + g' \cos\alpha)^2 \frac{ds}{EI}\right] \Delta_{ijz} \\
& + \left[\int_L (a' \cos\alpha + g' \sin\alpha)(d' \cos\alpha + j' \sin\alpha) \frac{ds}{GJ}\right. \\
& \qquad \left. + \int_L (-a' \sin\alpha + g' \cos\alpha)(-d' \sin\alpha + j' \cos\alpha) \frac{ds}{EI}\right] \Delta_{jiz} \\
& + \left[\int_L (a' \cos\alpha + g' \sin\alpha)(e' \cos\alpha + k' \sin\alpha) \frac{ds}{GJ}\right. \\
& \qquad \left. + \int_L (-a' \sin\alpha + g' \cos\alpha)(-e' \sin\alpha + k' \cos\alpha) \frac{ds}{EI}\right] \theta_{jix} \\
& + \left[\int_L (a' \cos\alpha + g' \sin\alpha)(f' \cos\alpha + m' \sin\alpha) \frac{ds}{GJ}\right. \\
& \qquad \left. + \int_L (-a' \sin\alpha + g' \cos\alpha)(-f' \sin\alpha + m' \cos\alpha) \frac{ds}{EI}\right] \theta_{jiy}
\end{aligned}$$

Determination of the stiffness coefficients in a manner similar to a fixed end member and including the shear effect for joint loads gives

$$\begin{bmatrix} \dfrac{\partial U}{\partial \Delta_{ijz}} \\[2ex] \dfrac{\partial U}{\partial \Delta_{jiz}} \\[2ex] \dfrac{\partial U}{\partial \theta_{jix}} \\[2ex] \dfrac{\partial U}{\partial \theta_{jiy}} \end{bmatrix} = \frac{1}{f'_{iizz}} \begin{bmatrix} 1 & -1 & Y_{ij} & -X_{ij} \\ -1 & 1 & -Y_{ij} & X_{ij} \\ Y_{ij} & -Y_{ij} & Y_{ij}^2 & -Y_{ij}X_{ij} \\ -X_{ij} & X_{ij} & -Y_{ij}X_{ij} & X_{ij}^2 \end{bmatrix} \begin{bmatrix} \Delta_{ijz} \\ \Delta_{jiz} \\ \theta_{jix} \\ \theta_{jiy} \end{bmatrix}$$

$$+ \frac{CF'}{f'_{iizz}} \begin{bmatrix} 1 & -1 & Y_{ij} & -X_{ij} \\ -1 & 1 & -Y_{ij} & X_{ij} \\ Y_{ij} & -Y_{ij} & Y_{ij}^2 & -Y_{ij}X_{ij} \\ -X_{ij} & X_{ij} & -Y_{ij}X_{ij} & X_{ij}^2 \end{bmatrix} \begin{bmatrix} \Delta_{ijz} \\ \Delta_{jiz} \\ \theta_{jix} \\ \theta_{jiy} \end{bmatrix}$$

VITA

Roy Arthur James Frusti

Candidate for the Degree of

Doctor of Philosophy

Thesis: ANALYSIS OF PLANAR STRUCTURES LOADED NORMAL TO THE PLANE BY THE PRINCIPLE OF MIMIMUM POTENTIAL ENERGY

Major Field: Civil Engineering

Biographical:

Personal Data: Born September 12, 1935, in Calumet, Michigan, the son of Edwin O. and Elsie Frusti.

Education: Graduated from Lake Linden-Hubbell High School, Lake Linden, Michigan in June, 1953. Received the degree of Bachelor of Science in June, 1957, and the Master of Science in Civil Engineering in June, 1958, from the Michigan College of Mining and Technology, Houghton, Michigan. Elected to membership in Chi Epsilon. Completed requirements for the degree of Doctor of Philosophy at Oklahoma State University in August, 1965.

Professional Experience: Engineer Trainee, Michigan State Highway Department, 1957; Commissioned Officer United States Air Force; Civil Engineer at Elmendorf Air Force Base, Anchorage, Alaska, 1958-62; Part time instructor, University of Alaska, 1959-1961; Engineer-in-training with State of Michigan; Member of Society of American Military Engineers; Associate Member of the American Society of Civil Engineers.