STABILITY OF RIGID-JOINTED

SPACE FRAMES

By

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NOMENCLATURE

ds	Length of the cross-section of the wall element
$k_{i,j}$	An element of the beam-column stiffness matrix
k_x, k_y, k_z	Dimensionless coefficients
m	Path of integration
n_{y1}, n_{z1}	End shears of a wall element of the beam-column
r_s	Polar radius of gyration of the cross-section of the beam-column
t	Wall thickness of the beam-column cross-section
$[t]$, $[t_1]$, $[t_2]$, $[t_3]$ . .	Transformation matrices
v, v_s, v_p	Displacements of the centroid, shear center and a point of the beam-column cross-section in Y direction respectively
w, w_s, w_p	Displacements of the centroid, shear center and a point of the beam-column cross-section in Z direction respectively
x	Distance of a beam-column cross-section to the origin
x_i, x_j	X coordinates of the ends of the space member ij
x_m	Length of the projection of a space member in X direction.

y_i, y_j. . . . . Y coordinates of the ends of the space member ij

y_m . . . . Length of the projection of a space member in Y′ direction

y_s. . . . . Y coordinate of the shear center with respect to centroid

z_i, z_j. . . . . Z coordinates of the ends of the space member ij

z_m. . . . . Length of the projection of a space member in Z′ direction

z_s. . . . . Z coordinate of the shear center with respect to origin

A . . . . Cross-sectional area of the beam-column

A_x, A_y, A_z . . . . . . Constants of integration

B_x, B_y, B_z . . . . . . Constants of integration

C . . . . Pure torsional stiffness

C_1 . . . . Warping stiffness

C_x, C_y, C_z . . . . . . Constants of integration

$[D]$ . . . . Column matrix of end deformations transformed to structure oriented axes

$[D_o]$ . . . . Column matrix of end deformations with respect to member oriented axes

$[D_s]$. . . . . Column matrix of joint deformations of the space frame

D_i, D_j . . . . Submatrices of end deformations

E . . . . Modulus of elasticity

$[F_x]$, $[F_y]$, $[F_z]$ . . Column matrices of end actions

G . . . . Modulus of rigidity

I_x, I_y	Moments of inertias of the beam-column cross-section
$[K]$	Stiffness matrix of the beam-column transformed to structure oriented axes
$[K_o]$	Stiffness matrix of the beam-column with respect to member oriented axes
$[K_s]$	Stiffness matrix of the space frame
K_{ii}, K_{ij}	Submatrices of the stiffness matrix
L	Length of the beam-column
L_o	Projected length of a space member in X', Y' plane
M_{ijx}, M_{ijy}, M_{ijz}, M_{jix}, M_{jiy}, M_{jiz} . .	End moments of the space beam-column
M_x, M_y, M_z	Moments acting on the beam-column cross-section
M_{t1}, M_{t2}	Torsional moments acting at the end of beam-column element
$[N]$	Column matrix of end actions transformed to structure oriented axes
$[N_o]$	Column matrix of end actions with respect to member oriented axes
N_{ijx}, N_{ijy}, N_{ijz}, N_{jix}, N_{jiy}, N_{jiz} . .	End forces of the space beam-column
N_{ij}, N_{ji}	Submatrices of end actions
N_y, N_z	Shears acting on the beam-column cross-section
N_{y1}, N_{z1}	Shears at the end of the hinged beam-column element
N_{y2}, N_{z2}	Shears at the end of the beam-column element due to moments

P	Axial force
P_{cr}	Critical load
$[T]$	Transformation matrix
$[W_i]$	Joint-load matrix at joint i
$[W_s]$	Column matrix of joint loads of the space frame
X, Y, Z	Member oriented axes
X', Y', Z'	Structure oriented axes
$\vec{\alpha}$	Rotation vector
α_i	$\Phi_i + \psi_i$
β_i	$\Phi_i - \psi_i$
$\delta_{ix}, \delta_{iy}, \delta_{iz}, \delta_{jx}, \delta_{jy}, \delta_{jz}$	End rotations of the space beam-column
δ_x	Length of the beam-column element
$\delta M_{t1}, \delta M_{t2}$	Incremental torsional moments
$\delta n_{y1}, \delta n_{z1}$	Incremental shears of the wall element of the beam-column
$\delta N_{y1}, \delta N_{z1}, \delta N_{y2}, \delta N_{z2}$	Incremental shears of the beam-column element
θ	Rotation of the beam-column cross-section
$\left.\begin{matrix}\theta_{ix}, \theta_{iy}, \theta_{iz} \\ \theta_{jx}, \theta_{jy}, \theta_{jz}\end{matrix}\right\}$ . . .	End rotations of the space beam-column
θ_X	Rotation of the joint about X' axis
σ	Axial stress in the beam-column
Φ_i	Exponential functions of axial force
ψ_i	Exponential functions of axial force
$\Delta_{DX'}$	Displacement of joint D in the X' direction

CHAPTER I

INTRODUCTION

1-1 The Phenomenon of Buckling

A linearly elastic structure under static loads is stable if it returns to its original deformation configuration after a small disturbance; it is unstable if it tends to move to a different configuration. The necessary condition for stability is the uniqueness of the deformation configuration after small disturbances. The uniqueness of the deformation configuration of a stable structure is characterized by the fact that there is no increase in deformations of the structure without an increase in loads.

The critical load or the buckling load is the level of the given loading pattern at which the structure loses its stability. A loading pattern applied to a structure may be conveniently represented by a reference set of loads or reference load vector. Any level of loading is then obtained by multiplying all of the loads of the reference load vector by a load parameter. The level of loading at which the structure buckles is described by a critical load parameter. It should be noted that for a given structure the critical load parameter differs with the loading pattern and magnitude of reference loads under consideration. In other words, for each loading pattern there exists a critical load parameter.

In the conventional analysis of linearly elastic frames under static loads in the range of small deflections, the equilibrium and compatibility conditions are satisfied for each infinitesimal element of the structure by neglecting the effect of axial forces on the stiffness of the members. Neglecting the effect of the axial forces on the stiffness of the members leads to linear load-deformation relationships. This is a good approximation for frames loaded in such a manner that the axial forces induced in the members of the frame are proportionally small compared to the bending moments induced by the loading. In practice, however, there are certain loading patterns, such as the application of forces to the joints, which induce large axial forces but small moments. For these cases the methods of conventional analysis lead to erroneous results because the change in the stiffness of the structure due to axial forces is neglected. Due to the effect of axial forces at a certain level of loading the frame loses its stiffness, becomes unstable, and fails by elastic buckling.

A non-linear load-deformation relationship is obtained when the effect of axial forces is taken into account in the anlysis of a structure. The load-deformation relationship of a structure affords an excellent means of observing the overall behavior of a structure under increasing loads and determining the type of buckling which may occur. At the critical load, the load deformation curve for a rigid-jointed, elastic frame may show a bifurcation, an infinite type of discontinuity, or a maximum, depending on the geometry of frame and the loading pattern.

A load-deformation curve which exhibits a bifurcation at the buckling load is the characteristic of elastic frames which do not have prebuckling moments (Fig. 1-1a). Bifurcation of the load deformation curve is also obtained for symmetrically loaded symmetrical frames (Fig. 1-2a) which have prebuckling moments[57].

A portal frame and a knee frame are shown in Fig. 1-1a and Fig. 1-1b, respectively. The portal frame does not have any prebuckling moments due to the symmetry of the structure and the loading. The prebuckling moments of the knee frame will be zero if the area of its column is assumed to be infinite. The axial forces in the columns are equal to the joint loads prior to the buckling in both frames. Therefore, the determination of the stiffness of the members does not require an elastic analysis of the frame.

An elastic analysis is not necessary to determine

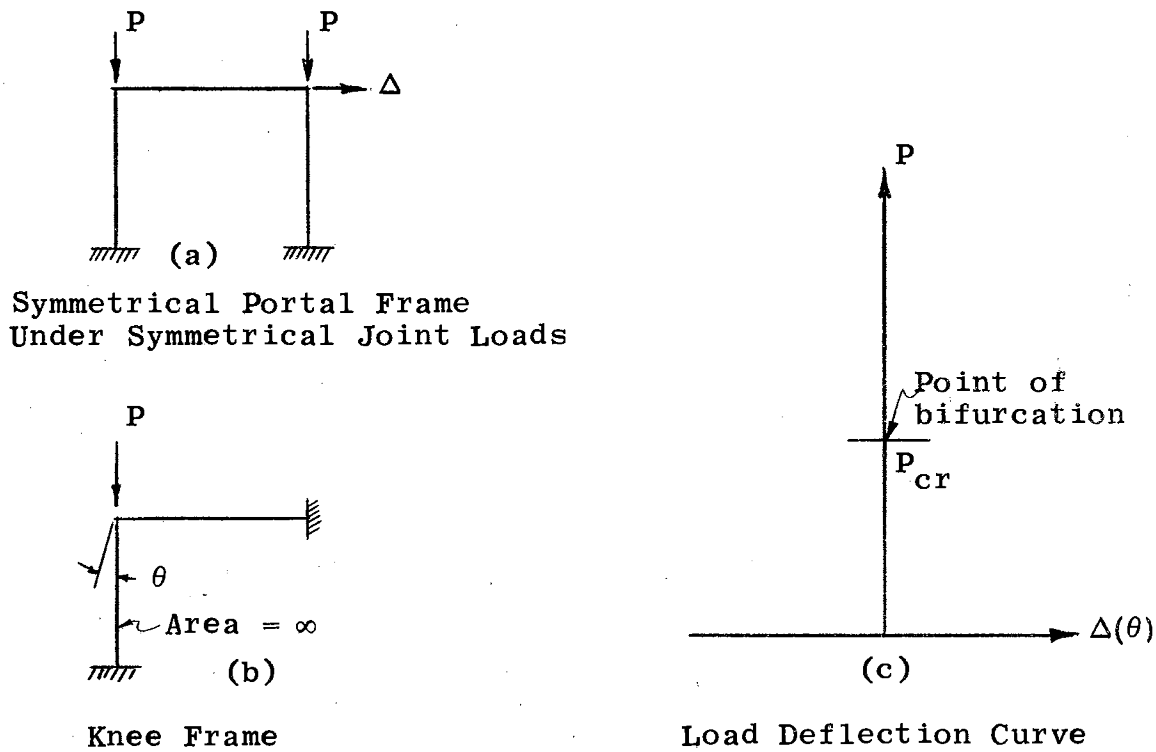
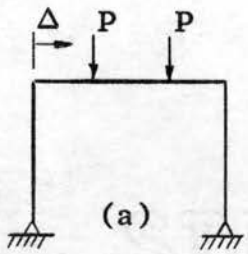


Fig. 1-1

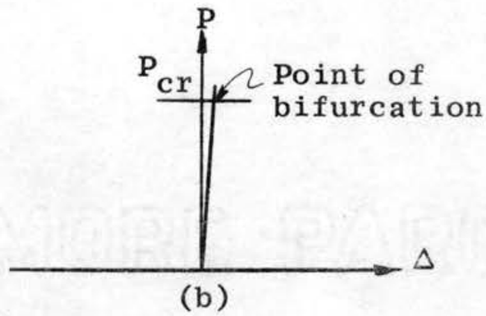
Load-Deformation Relationship of the
Frames Which Do Not Have Prebuckling Moments

the axial forces in the columns of the frame shown in Fig. 1-2a; however, due to the prebuckling moments induced in this frame owing to the positions of the loads, the buckling loading is not the same as that of a similar frame loaded as shown in Fig. 1-1a.

The horizontal displacement Δ for the portal frames and the joint rotation θ for the knee frame are taken as the representative deformation in the load deformation curves which exhibit a bifurcation at the critical load as shown in Figures 1-1c and 1-2b. The portal frames have no resistance to horizontal disturbances while the joint of



Symmetrically Loaded Symmetrical Portal Frame Which Has Prebuckling Moments



Load-Horizontal Displacement Curve

Fig. 1-2

Symmetrically Loaded Symmetrical Frame

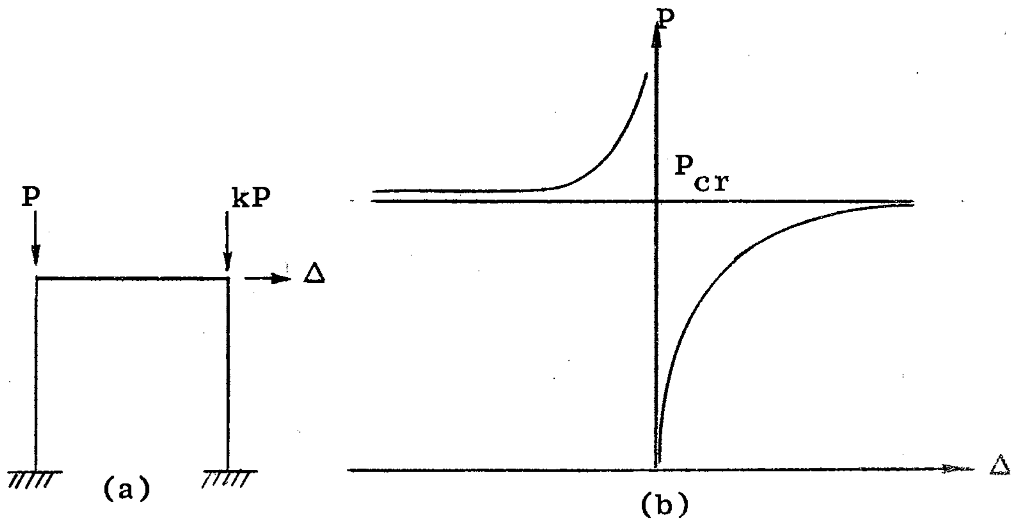
the knee frame has no resistance to rotation at the buckling load. The buckling phenomenon of frames which do not have prebuckling moments is analogous to that of the buckling of initially straight perfect columns.

In general, prebuckling moments cause deformations of the frame as well as the shears which induce the changes in axial forces of the members. Deformations due to prebuckling moments tend to increase without bound for loads near the buckling load because of the decreasing stiffness of the structure. Therefore, for frames which exhibit a sidesway type of buckling, the load-deflection curve shows an infinite type of discontinuity at the buckling load. An eccentrically loaded portal frame is shown in Fig. 1-3a. Fig. 1-3b shows the load-deflection curve of the frame obtained by taking the effect of the prebuckling moments into account. The horizontal displacement Δ of the frame is taken as the representative deformation in the plot[57]. At the buckling load, the load deformation curve is asymptotic to a horizontal line as shown in Fig. 1-3b. The instability of the structure is characterized by unbounded deformation. The plot is not extended to the higher modes of buckling. The phenomenon of buckling in this case is analogous to that of the buckling of eccentrically loaded columns.

It should be noted that the bifurcation type of load-deformation curve is a special case of the general load-deformation curve which has an infinite type of discontinuity at the buckling load. Namely, the deformations of the frame shown in Fig. 1-3a decrease when k approaches to unity. Finally for equal forces, the load deformation curve exhibits a bifurcation as shown in Fig. 1-1c.

A knee frame loaded at the joint is shown in Fig. 1-4a. The load-joint rotation curve obtained by taking the effect of prebuckling moments into account exhibits a maximum at the buckling load as shown in Fig. 1-4b[67]. An interesting feature of this curve is that there is no solution of the frame for the joint loads between P_{cr} and P_1.

In general, the stability analysis made by neglecting the effect of prebuckling moments gives the upper bound of the elastic buckling load of the frames[30,61]. The difference between the upper bound and the actual elastic buckling load was found to be smaller than 3% for the type

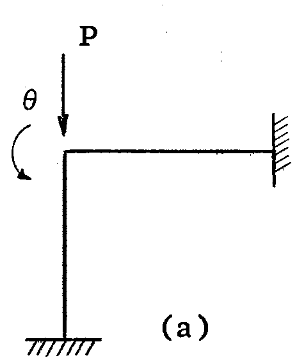


Eccentrically Loaded Portal Frame

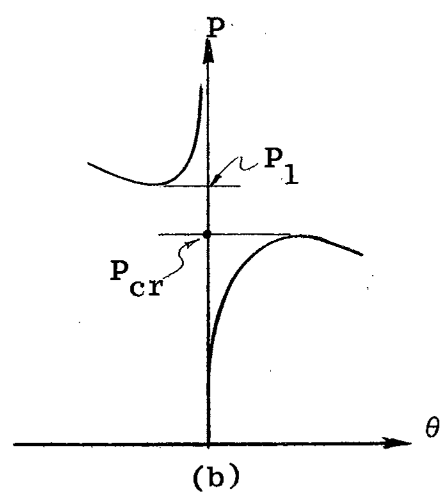
Load Deflection Curve

Fig. 1-3

Load-Deformation Relationships of the Eccentrically Loaded Frames



A Knee Frame Loaded at the Joint



Load Deformation Curve

Fig. 1-4

Load-Deformation Relationship of a Knee Frame of frames shown in Fig. 1-2 (13,42,57) and 7% for the frame shown in Fig. 1-4 (67).

The phenomena discussed above are called the classical buckling of elastic frames. Classical buckling can only be valid if the deflections of the structure are small for the loads close to the buckling load. In some cases there might be another geometric configuration of the frame that it can take by large deformations to be in stable equilibrium. Then, at the buckling load the frame moves from an unstable geometric configuration to a stable geometric configuration as a result of large deflections (59). This phenomenon is called the snap through buckling, and it is characterized by a jump discontinuity instead of an asymptote in the load-deformation curve at the buckling load. The magnitude of the prebuckling moments plays a big role in changing the classical buckling phenomenon to the snap through buckling. When a method of stability analysis which neglects the effect of prebuckling moments is utilized, the magnitude of deflections should be checked for the loads close to buckling load to make sure that the classical buckling will occur. From the above discussion, it can be concluded that the elastic buckling load of frames having prebuckling moments which exhibit the type of load deflection relationships as illustrated in Figures 1-3b and 1-4b can be determined from the load-deformation curve.

The same characteristics as discussed for planar frames are observed in the load-deformation curves of the space frames. They are illustrated in Chapter V.

1-2 Statement of the Problem

Many approximate methods have been introduced for the stability analysis of planar frames. Due to the involved nature of the analysis most of these methods have the tendency to simplify the problem by some assumptions (Chapter II), but still they are good enough to solve the elastic stability problems of planar frames faced in engineering practice. Other, more exact methods involve tedious calculations or the use of an electronic computer[42,57].

In the area of the stability analysis of space frames very little work has been done until recently. Renton[51] introduced a method of stability analysis based on the vanishing determinant of the stiffness matrix. In his derivation, however, he neglected the effect of warping stiffness of the cross sections and the effect of pre-buckling moments. The use of an electronic computer is required for the application of his method.

Unfortunately, due to the effect of the torsional stiffness of the members of structure and the possibility of the torsional buckling of columns, the stability analysis of space frames is not a simple generalization of the methods developed for planar frames. In addition, because of the complexity of the calculations which must be performed, the use of an electronic digital computer is essential for the solution of all but the most trivial structures.

In this dissertation, a method is presented for the elastic stability analysis of rigid-jointed space frames which buckle in a manner analogous to that shown in Figures 1-1, 1-3 and 1-4. The loading considered is limited to those loads which may be applied to the joints of the frame. The effect of prebuckling moments and the warping stiffness of the members is taken into account. Thus, the effect of torsion on the phenomenon of buckling is considered in the method.

1-3 Assumptions

The following assumptions are made in the method presented:

(a) The material is homogeneous, isotropic, and elastic.

(b) Deformations are small and do not change the geometry of the frame.

(c) Deformations due to shears are small and can be neglected.

(d) The frame is made of the members of thin-walled open cross-sections.

(e) Navier's hypothesis remains valid for each of the flat plates of which the member is composed.

(f) The warping of the cross-sections of the members at the joints of the structure is zero.

(g) The directions of the joint loads remain unchanged by the deformations of the structure.

1-4 Procedure of Investigation

In the development of the method of stability analysis presented herein, the following steps of investigation are followed:

(a) The methods of stability analysis of frames are reviewed.

(b) The stiffness matrix for a beam-column is derived from the governing differential equations.

(c) Transformation matrices are established to transform the stiffness matrices from the member oriented axis to the structure oriented axis.

(d) The stiffness matrix for the frame is obtained from the equilibrium of joints.

(e) A determinantal criterion of stability is used for bifurcation type of buckling of space frames.

(f) The second order elastic analysis of the frame is made by an iteration process for each level of loading.

(g) The convergence of this process is postulated as the criterion of stability for space frames loaded at the joints and exhibiting an infinite type of discontinuity or a maximum in the load-deformation curve. For these cases, the determinantal criterion offers only an approximate value of buckling load.

(h) The deformations of the joints, the determinant of the stiffness matrix, and the axial forces in the members are obtained from part "f" in order to verify that the deformations are in the range of small deflection theory.

(i) Necessary computer programs are developed, and the method is illustrated by several examples.

CHAPTER II

SUMMARY OF METHODS OF STABILITY ANALYSIS OF ELASTIC FRAMES

The historical development of the stability analysis of elastic frames is briefly reviewed, and the currently available methods for the elastic stability analysis of rigid-jointed frames are presented in this chapter. A list of selected references is given in the bibliography; however, an extensive survey of available literature may be found in references (21) and (48).

A brief history of the development of methods for the stability analysis of elastic frames is given in Article 2.1 while the methods themselves are briefly described in Articles 2.2 and 2.3. These methods are classified in two groups in these Articles. The first group consists of those methods which neglect the effect of prebuckling moments. The second group consists of those methods which take the effect of prebuckling moments into account.

2-1 Historical Notes

At the beginning of this century, the stability problem of frames which do not have prebuckling moments began to attract the attention of many investigators. In 1919, Bleich[1,2] presented a systematic analysis of the stability of rigid-jointed planar frames by using "the four moment equations." The stability analysis of a plane truss as a unit was made by Mises[3]. Mises and Ratzersdorfer[4] gave a detailed presentation of the stability analysis of frames and extended the investigation by taking the effect of axial shortening in the members into account. The energy method was applied by Kasarnowsky and Zetterholm[5] to the stability analysis of long columns elastically supported at equidistant intermediate points. In 1928, Bleich[6] published a paper dealing with a generalization of the theory of stability of assemblies of bars valid for rigid-jointed space frames whose members buckle spatially. The calculation of the stiffness-coefficient and carry-over factors of bars subjected to axial loads was made by James[7] and established the foundations of a stability analysis by the moment distribution. Prager[9] developed a method for the stability analysis of frames utilizing the analytical stability condition of a column with elastic end supports. Lundquist[10,11] established the "series" and "stiffness" criteria for stability of frames based on the principle of moment distribution. In 1938, Chwalla[13] introduced an approach for the determination of the buckling load of a portal frame by taking the effect of pre-buckling moments into account. Puwein[15] presented an approximate method to solve the same problem. In 1941, the slope deflection method of stability analysis was developed for multi-story frames by Chwalla and Jokisch[17]. At the same time, Hoff[16] gave the rigorous proof of the convergence of the moment distribution and of the uniqueness of the results in the case of stable equilibrium by means of energy considerations. Southwell's relaxation method was applied to the stability analysis of planar frames by Boley[18] in 1947. Winter, et al[19], Perri[20], and Masur[24] proposed modifications of the moment distribution method for the stability analysis of planar frames with sidesway. In 1952, Bleich[22] published a book on buckling strength of structures which contains two chapters on the stability analysis of planar frames. In a series of papers Merchant and his associates[25,28,31,34] presented approximate methods of stability analysis of tall building frames. They defined a single bay and single story portal frame equivalent to the multi-bay and multi-story frame to find the approximate buckling load. In 1956, Livesley[27] introduced a method of stability analysis of rigid-jointed planar frames which was suitable for use on a digital computer and checked the results of the examples solved by Merchant and his associates. Masur and Cukurs[33] applied the determinantal criterion and series criterion of stability to the out of plane buckling of trusses. Historical notes about the research conducted on the stability analysis of frames in Russia was given by Rabinovich[36]. Horne[37] gave a good discussion on the effect of elastic stability to the load carrying capacity of the frames. A solution of the lateral instability of building frames by the energy method was presented in 1960 by Johnson[39]. Lu[40] introduced a method of stability analysis of elastic-plastic frames. In 1961, Masur, et al[42], modified the slope deflection and the moment distribution method to include the effect of prebuckling moments in the stability analysis of frames. A review on the stability of elastic-plastic structures was made by Horne[44] in 1961. Stability analysis of planar frames by the moment distribution has been thoroughly treated by Lightfoot[46] and recently by Gere[58]. Horne[50] and then Lu[55] discussed the effect of finite deformations in the elastic stability of frames. In 1962, Renton[51] presented a method of stability analysis for space frames by an electronic computer. Post buckling deformations of elastic planar frames was investigated by Britvec and Chilver[54] and by Saafan[56]. In 1963, Carter[57] introduced a matrix method of stability analysis of planar frames having prebuckling moments.

2-2 Stability Analysis by Neglecting the Effect of Prebuckling Moments

There are a number of methods available for the stability analysis of frames by neglecting the effect of prebuckling moments. All of these methods are developed by modifying the conventional methods of frame analysis to include the effect of axial forces in the members.

The moments due to loads in the prebuckled state are assumed to be zero. Consequently, the loads which are not applied at the joints of the frame are replaced by the statically equivalent joint loads. The axial forces in the members are computed directly from the equations of statics. The critical load parameter is determined from the loading level for which the deformation configuration of the frame becomes unstable. According to the analytical theory used in the analysis, the methods of stability analysis of frames neglecting the effect of prebuckling moments may be considered in four groups: equilibrium methods, convergence methods, matrix methods, and energy methods.

2-2.1 Equilibrium Methods

The equilibrium methods are based on the bifurcation of the load-deformation relationship. All the deformations in the prebuckled state are zero due to the assumptions made. The equations of equilibrium for the buckled state of the frame are linear and homogenous. The unknown quantities in the equations are the actions and/or deformations induced by buckling. The coefficients of the unknown quantities are the transcendental functions of member properties, dimensions, and the axial forces. For the buckling load, these linear homogenous equations have a non-trivial solution. Therefore, the determinant of the coefficient matrix of the equations should vanish at the buckling load. This is called the determinantal criterion of buckling. The procedure used in establishing the system of linear homogenous equations distinguishes the various methods of this group. But the determination of the buckling load from the determinantal criterion is the same in all the methods of this group. In the case of some simple frames the determinantal criterion may lead to a less complicated transcendental equation from which an algebraic expression for the buckling load can be obtained. Usually, for complex frames, a trial and error approach is used to obtain the buckling load numerically due to the complexity of the transcendental equation obtained from the vanishing determinant. It should be noted that the non-trivial solution of the equations gives the relative values of the unknowns. Therefore, the magnitudes of the deformations for buckling load remain indeterminate, and only the mode of of buckling may be obtained. Detailed information on the methods of this group may be found in references 1, 4, 6, 8, 9, 17, 22, 25, 43, and 57.

2-2.2 Convergence Methods

The methods of this group are developed by modifying the moment distribution and the relaxation methods of conventional frame analysis to take the effect of axial forces into account. In the prebuckled state, a frame offers positive resistance to any externally applied action or deformation. At the buckling load, the frame has no resistance to any external disturbance. Therefore, for the buckling modes which do not have joint translation, the moment distribution and relaxation processes do not converge if the frame is analyzed under a disturbing joint moment M_o[16]. In addition to this criterion of buckling, stiffness and series criteria[10,11] are also available.

For the translational modes of buckling, an arbitrary translation of the frame is imposed in the presence of the loading system. If the frame is in the prebuckled state, a positive force is necessary to hold the frame in the displaced position. For the buckled and post buckled states, a zero force and a negative force are required respectively. Another way is to apply a unit force in the direction of the translation of the frame and compute the translation of the frame due to this force in the presence of the loading system. If the translation is in the direction of the force, the frame is in the prebuckled state. For the critical level of loading the translation is infinite. In the post buckled state translation is in the opposite direction of the applied force[28,31].

A very good discussion of the moment distribution method of stability analysis may be found in references 46, 58.

The relaxation method[18,56] is fundamentally the same as the moment distribution method.

2-2.3 Matrix Methods

Matrix methods became very powerful in the stability analysis of frames with the development of large electronic computers. The displacement method of conventional frame analysis is modified to take the effect of axial forces into account. Joint equilibrium equations of the displacement method are written in matrix form for a number of load parameters. The frame is analyzed by solving these equations for each load parameter. Determinant of the coefficient matrix of the equations may also be computed. At the buckling load this determinant is equal to zero and the frame has no resistance to a disturbance which excites the mode of buckling. In the methods of this group, the buckling load parameter is found from either the load-deflection curve[27] or from the load-determinant curve[51,57].

An excellent presentation of the matrix methods of stability analysis applied to planar frames is given in reference 57.

Matrix methods, in general, are not suitable for hand computations. They are probably the most efficient methods for the systems approach to stability analysis by the computer.

2-2.4 Energy Methods

The energy methods are based on the fact that a frame returns to the unique deformation configuration after any small disturbance if it is in a prebuckled state. In terms of energy, the work done by the external loads through a change of deformation configuration due to a disturbance, is smaller than the change in strain energy. The difference between the two quantities is the energy which returns the frame to the deformation configuration before the disturbance. If the frame is in the buckled state, the work done by the external loads through a change in deformation configuration due to a disturbance is equal to the change in strain energy so that the frame has no tendency to return to the deformation configuration which it held before the disturbance. This criterion leads to a set of linear homogeneous equations analogous to the equations obtained in the equilibrium methods. Then the buckling load is determined from the condition of the vanishing determinant of coefficients of these equations. More information about the methods of this group may be found in references 3, 5, 8, 39 and 43.

2-3 Stability Analysis of Frames Having Prebuckling Moments

In the methods of this group the effects of the moments induced by the loads which are not applied to the joints and by the change in lengths of the members due to axial forces are taken into account. In this case, it is not possible to determine the axial forces by using the equations of statics directly because of the unknown prebuckling moments. Axial forces are the functions of the deformations of the frame, and it is necessary to know the axial forces to determine these deformations. Besides, none of the criteria of stability established by the methods which neglect the effect of prebuckling moments are applicable to this case due to the fact that the equations of equilibrium or energy are linear but not homogeneous.

In the prebuckled state of the frame an increment in the loads induces an increment in the deformations. At the buckling load, deformations can increase without an increment in loads. This fact is used as a criterion of stability in the methods of this group.

In 1938, Chawalla(13) investigated the buckling of a symmetrical portal frame under two symmetrically located concentrated loads by taking the effect of prebuckling moments into account. He obtained a transcendental equation using the differential equations of the members of the frame and solved for the buckling load parameter from this equation. Principally, his method is the extension of the equilibrium method to include the effect of prebuckling moments.

Puwein(15) proposed an approximate method to determine the buckling load of the uniformly loaded portal frame and extended his approach to the gable frames and to the frames with partial base fixity. His approach is primarily the extension of the energy method of stability analysis. The approximation in his solution is the assumption of a buckled deformation configuration for the frame.

Masur, et al[42], introduced two methods for calculating the elastic buckling load of a symmetrical frame under symmetrical loads and prebuckling moments. One method is based on an equilibrium analysis of the buckled frame using the slope-deflection equations, and the other is based on a moment distribution procedure. Both of these methods are the extensions of the methods which do not take the effect of prebuckling moments into account. The relation between the incremental actions and deformations is established. The buckling load is obtained from the fact that the incremental deformations occur without the incremental actions at the buckling load. The results obtained from both of these methods showed a good agreement with those of Chwalla.

Recently Carter[57] developed a matrix method of elastic stability analysis for the planar frames having prebuckling moments. He succeeded in establishing a matrix relation between the incremental deformations and the incremental loads of the frame based on the deformation method. At the buckling load this matrix relation reduces to a set of linear homogeneous equations due to the fact that the incremental deformations can exist even if the incremental loads vanish at the buckling load. This leads to a new determinantal criterion of stability. Because of the involved character of the problem the use of a computer is necessary in the application of this method. For this purpose, a computer program is also developed by the same investigator.

2-4 Summary

From the review of the available methods of stability analysis of elastic frames and their historical development, it is observed that the methods developed before the use of the computers in this field have the tendency to simplify the problem by certain assumptions due to the involved character of the numerical computations. The most common assumption made is that of neglecting the effect of prebuckling moments. In fact, due to this assumption most of the available literature deals with the bifurcation type of buckling. The applications of these methods are also restricted to certain type frames such as portals, single bay, multi-story, and gable frames because of the inconvenience in the hand computations.

In the last decade the use of computers in this field made the development of the general matrix methods of stability analysis possible, but still there has been very little work done in the area of the elastic buckling of space frames, the subject of this dissertation.

CHAPTER III

STIFFNESS MATRIX FOR A SPACE BEAM-COLUMN

The geometry, the positive end deformations, and end actions of a space beam-column* are defined. The differential equations of a space beam-column are derived by using Chilver's[64] concept of corrected discontinuities. The differential equations are integrated for the beam-columns of the cross-section whose shear center coincides with the centroid. The constants of integration are eliminated by assuming a set of end deformations, their corresponding end actions, and zero warping of the cross-section at the ends. Thus, the end actions are obtained in terms of the end deformations in matrix form. The coefficient matrix of the column vector of the end deformations is called the stiffness matrix of the space beam-column. The elements of the stiffness matrix are transcendental functions of the axial force and the length and the section properties of the beam-column.

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*For the purpose of this dissertation a space beam-column is defined as a thin walled structural member which is subjected to end moments and end shears as well as axial force.

3-1 Geometry of A Space Bar

A beam-column ij of constant, thin-walled open cross-section is considered. The centroidal axis of the bar is chosen as the X axis, while the Y and Z axes are the principal axes of the beam cross-section. The positive end actions and end deformations are represented by vectors acting in the positive direction of the coordinate axes (Fig. 3-1,2). Force vectors and displacement vectors are represented by a line with a single arrow designating the sense; moment vectors and the vectors indicating the rotations are represented by a line with a double arrow assigning the sense (Fig. 3-1,2). There are no loads applied to the beam-column between the ends.

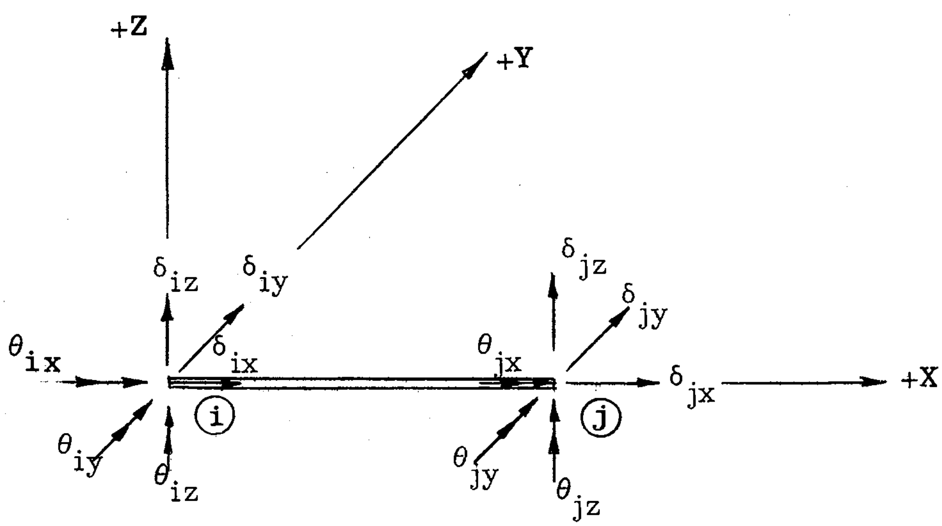


Fig. 3-1

Positive End Deformations

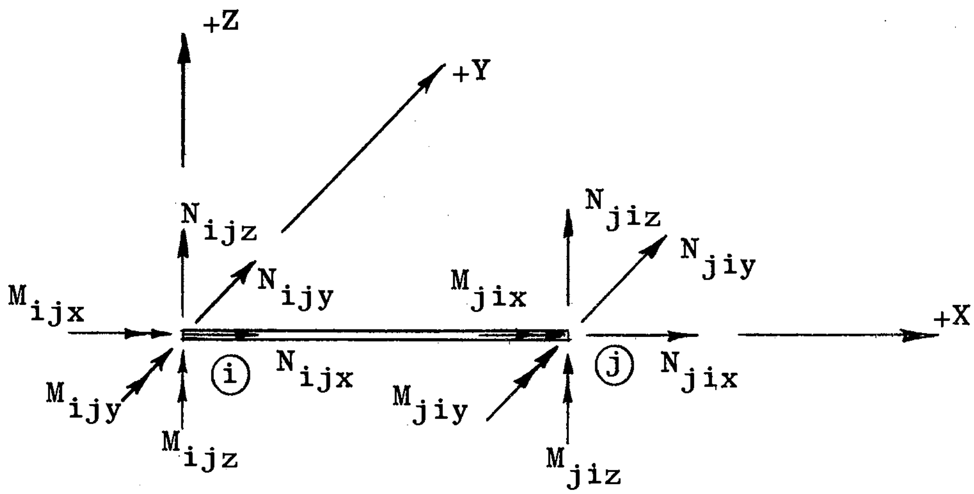


Fig. 3-2

Positive End Actions

The cross-section of the beam is thin-walled, open, and arbitrary. The principal axes Y and Z and the location of the shear center S, with respect to the centroid c, are shown in Fig. 3-3.

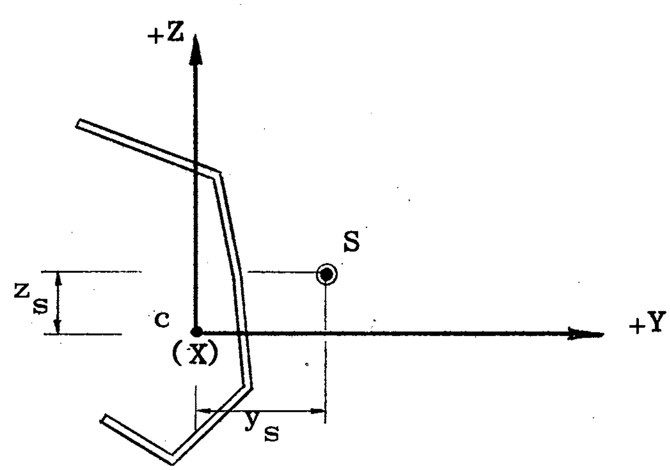


Fig. 3-3

Cross-Section of the Beam-Column

3-2 Differential Equations of A Space Beam-Column

The differential equations of a space beam-column are obtained by using the equilibrium and compatibility conditions for the beam-column element of length δx It is assumed that there is no change in the geometric shape of a plane cross-section due to torsion, but warping of the cross-section is considered in the derivations. This is the conventional assumption made for the torsion of the beams of thin-walled cross-section[22,63].

The modulus of elasticity and the modulus of rigidity of the beam is denoted by E and G respectively. Due to the end actions, a cross-section of the beam column will displace in the Y and Z directions and rotate about an axis parallel to X. The axis of rotation is taken at the shear center for convenience. The lateral displacements of the shear center and of an arbitrary point of the cross-section in the direction of Y and Z axes are denoted by v_s, w_s, and v_p, w_p respectively (Fig. 3-4).

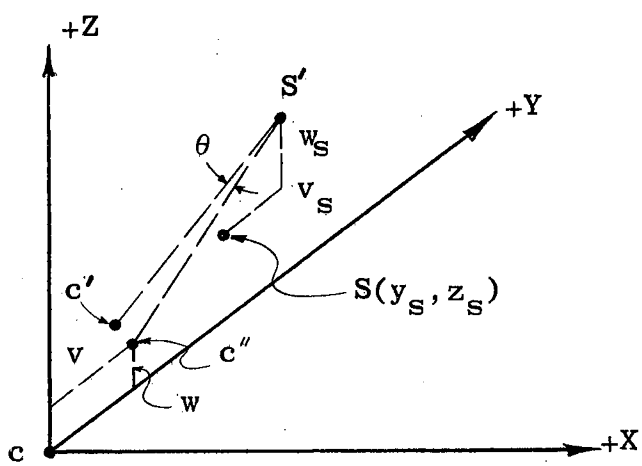


Fig. 3-4

Lateral Displacements of the Shear Center and the Centroid of a Cross-Section

For a small rotation θ the lateral displacements, v and w, of the centroid are

$$v = v_s + \theta z_s$$

$$w = w_s - \theta y_s \qquad . \qquad (3\text{-}1)$$

The lateral displacements of an arbitrary point of the cross-section, having the coordinates y and z, are

$$v_p = v_s + \theta (z_s - z)$$

$$w_p = w_s - \theta (y_s - y) \qquad , \qquad (3\text{-}2)$$

Two sets of actions will be considered at the ends of a beam-column element of length δx. The first set consists of the end shears transmitted from the beam-column to the element if it were connected by hinges at the ends. The second set of actions are the necessary end moments and end shears applied at the shear center to establish the angular compatibility of the element with the beam-column at the assumed hinges. Due to the equilibrium condition of the element, both cases require the application of some transverse loads to the element. Since there is no load applied to the beam column between the ends, the summation of the required loads of the two cases should be zero. This condition gives the differential equations.

A wall strip of a beam column element of length δx width ds, and the thickness t is shown in Fig. 3-5.

The location of the strip is defined by s which is measured from a convenient origin around the center line of the cross-section.

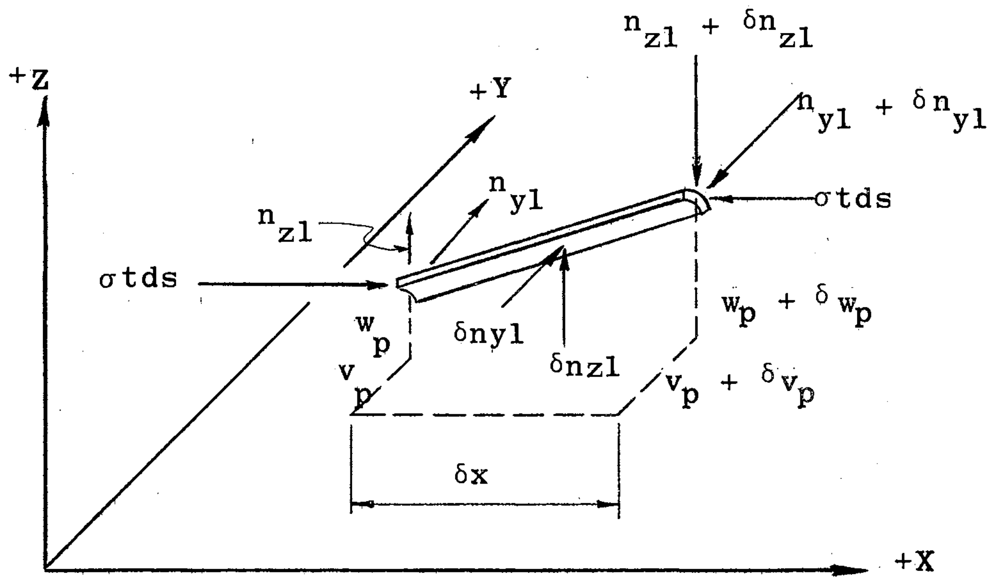


Fig. 3-5

A Wall Element of A Beam Column
Under First Set of Actions

The moment equilibrium of the element about the Y and Z axes, neglecting the second order terms, is

$$n_{z1} = \sigma t \; ds \; w_p' \tag{3-3a}$$

$$n_{y1} = \sigma t \; ds \; v_p' \tag{3-3b}$$

where the primes indicate the derivatives with respect to x. The change in shears is obtained by taking the variation of the Equations 3-3a and 3-3b.

$$\delta n_{z1} = \sigma t \; ds \; w_p'' \; \delta x \tag{3-4a}$$

$$\delta n_{y1} = \sigma t \; ds \; v_p'' \; \delta x \tag{3-4b}$$

The necessary external lateral loads for the equilibrium of the beam-column element in this case can be determined by integrating the Equations 3-4a and 3-4b over the entire cross-section. Thus

$$\delta N_{z1} = \int_m \sigma t \, w_p'' \, \delta x \, ds \tag{3-5a}$$

$$\delta N_{y1} = \int_m \sigma t \, v_p'' \, \delta x \, ds \tag{3-5b}$$

Where δN_{z1} and δN_{y1} are the loads to be applied in the z and y directions to the element for the equilibrium. The integration will be performed around the center line of the cross-section. This path is indicated by m. Taking the derivatives of Equations 3-2, substituting into Equations 3-5a, 3-5b and then performing the integration gives

$$\delta N_{y1} = -P \, \delta x (v_s'' + \theta'' z_s) \tag{3-6a}$$

$$\delta N_{z1} = -P \, \delta x (w_s'' - \theta'' y_s) \tag{3-6b}$$

where P is the axial force in the beam-column. P is assumed to be positive for tension. The transmission of δN_{y1} and δN_{z1} to the shear center results in a torsional moment δM_{t1} which is computed by integrating the torque about the shear center due to δN_{z1} and δN_{y1} as follows.

$$\delta M_{t1} = \int_m \sigma t\ \delta x\ w_p'' \ (y-y_s)\ ds + \int_m \sigma t\ \delta x\ v_p''\ (z_s-z)\ ds$$

Substituting the values for w_p'' and v_p'' from the Equation 3-2 and integrating

$$\delta M_{t1} = -P\ (z_s v_s'' - y_s w_s'' + r_s^2\ \theta'')\ \delta x \qquad (3\text{-}7)$$

where r_s is the polar radius of the gyration of the cross-section about the shear center. The polar radius of gyration r_s can be computed from the Equation 3-8.

$$A\ r_s^2 = I_y + I_z + A\ (y_s^2 + z_s^2) \qquad (3\text{-}8)$$

I_y and I_z are the moments of inertia of the cross-section about Y and Z axes respectively, and A is the cross-sectional area.

If the beam column element of length δx were connected by hinges at the ends, the generalized loads necessary for the equilibrium would be δN_{y1}, δN_{z1}, and δM_{t1} along the shear center axis. The angular continuity at the hinges may be provided by the application of the second set of actions at the hinges as shown in Fig. 3-6.

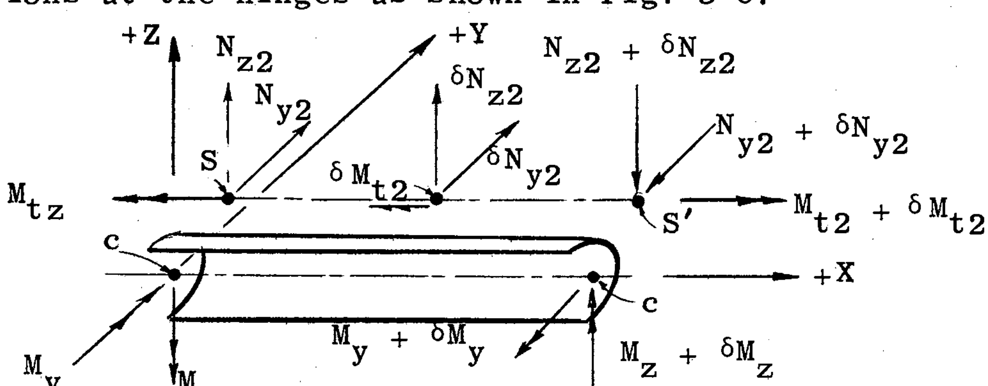


Fig. 3-6

Beam-Column Element Under the Second Set of Actions

The bending and the torsional moments at the left end of the element can be expressed in terms of the derivatives of the deformations at the shear center as follows:

$$M_y = EI_y \, w_s'' \tag{3-9a}$$

$$M_z = EI_z \, v_s'' \tag{3-9b}$$

$$M_{t2} = C\theta' - C_1\theta''' \tag{3-9c}$$

Where C is the pure torsional stiffness and C_1 is the warping stiffness of the cross-section defined in reference 63. The shears due to these moments are

$$N_{y2} = \frac{dM_z}{dx} = EI_z \, v_s''' \tag{3-10a}$$

$$N_{z2} = \frac{dM_y}{dx} = EI_y \, w_s''' \quad . \tag{3-10b}$$

The necessary generalized loads for the equilibrium in this case are obtained by taking the variation of the Equations 3-9c, 3-10a, and 3-10b as given in Equations 3-11a, 3-11b and 3-11c.

$$\delta N_{y2} = EI_z \, v_s^{IV} \, \delta x \tag{3-11a}$$

$$\delta N_{z2} = EI_y \, w_s^{IV} \, \delta x \tag{3-11b}$$

$$\delta M_{t2} = (C\theta'' - C_1\theta^{IV}) \, \delta x \tag{3-11c}$$

The compatibility and the equilibrium conditions for the beam column element can be satisfied by applying the two sets of end actions simultaneously. The generalized loads obtained from the superposition of the two cases should vanish due to the fact that there is no load applied to the element. This is expressed mathematically by the Equation 3-12.

$$\begin{aligned} \delta N_{y1} + \delta N_{y2} &= 0 \\ \delta N_{z1} + \delta N_{z2} &= 0 \\ \delta M_{t1} - \delta M_{t2} &= 0 \end{aligned} \tag{3-12}$$

The differential equations of a space beam-column are then obtained by substituting the values of the necessary generalized forces from Equations 3-6, 3-7, 3-11 into Equation 3-12 as

$$EI_z \, v_s^{IV} - P\,(v_s'' + \theta'' z_s) = 0 \tag{3-13a}$$

$$EI_y \, w_s^{IV} - P\,(w_s'' - \theta'' y_s) = 0 \tag{3-13b}$$

$$C\theta'' - C_1\,\theta^{IV} + P\,(z_s\, v_s'' - y_s\, w_s'' + r_s^2 \theta'') = 0 \tag{3-13c}$$

These differential equations are the same as those obtained by Bleich[22].

3-3 Internal Actions of a Beam-Column

Internal actions can be expressed in terms of the derivatives of the deformation functions. The two sets of actions used in the derivation of the differential equations are superimposed for this purpose. From Equations 3-3a and 3-3b, the shears of the first set of actions acting on the beam-column element are

$$N_{z1} = \int_m \sigma t \; w_p' \; ds$$

$$N_{y1} = \int_m \sigma t \; v_p' \; ds$$

where N_{z1} and N_{y1} are the shears in the Z and Y directions respectively. Substituting the values of W_p' and V_p' from Equations 3-2 and performing the integration around the cross-section

$$N_{z1} = -P \; (w_s' - \theta' y_s) \tag{3-14a}$$

$$N_{y1} = -P \; (y_s' + \theta' z_s) \quad , \tag{3-14b}$$

The torque m_{t1} due to the transmission of the shears to the shear center is obtained by taking the moments of N_{z1} and N_{y1} about the shear center and integrating around the cross-section.

$$M_{t1} = -P \; (z_s \; v_s' - y_s \; w_s' + r_s^{\,2} \, \theta') \tag{3-15}$$

The second set of actions were expressed by Equations 3-9 and 3-10. The superposition of these two sets of actions gives the internal actions of the beam-column in terms of the deformation functions as follows:

$$N_z = -EI_y w_s''' + P (w_s' - \theta' y_s) \tag{3-16a}$$

$$N_y = -EI_z v_s''' + P (v_s' + \theta' z_s) \tag{3-16b}$$

$$M_x = C\theta' - C_1\theta''' + P (z_s v_s' - y_s w_s' + r_s^2 \theta') \tag{3-16c}$$

$$M_y = -EI_y w_s'' \tag{3-16d}$$

$$M_z = EI_z v_s'' \tag{3-16e}$$

where $M_x = M_{t2} - M_{t1}$

The corresponding vectors to the positive actions obtained by Equations 3-16 are in the positive direction of the coordinate axes of the far end and are in the negative direction at the near end of the element. The signs of the actions of point i should be reversed to adopt the sign convention shown in Fig. 3-2.

3-4 The Stiffness Matrix

The stiffness matrix for a space beam-column having the thin-walled open cross-section of two-fold symmetry is obtained from the differential equations of a general space beam-column. For open cross-sections of two-fold symmetry the shear center coincides with the centroid. Consequently, the deformations of the shear center become identical to the deformations of the centroid. Thus, the differential equations 3-13 reduce to

$$v^{IV} - k_y^{\,2}\, v'' = 0 \tag{3-17a}$$

$$w^{IV} - k_z^{\,2}\, w'' = 0 \tag{3-17b}$$

$$\theta^{IV} - k_x^{\,2}\, \theta'' = 0 \tag{3-17c}$$

where $k_y^{\,2} = \dfrac{P}{EI_z}$, $k_z^{\,2} = \dfrac{P}{EI_y}$, $k_x^{\,2} = \dfrac{C + r_s^{\,2} P}{C_1}$.

The internal actions in this case are

$$M_x = (C + P\, r_s^{\,2})\, \theta' - C_1 \theta''' \tag{3-18a}$$

$$M_y = -EI_y\, w'' \tag{3-18b}$$

$$M_z = EI_z\, v'' \tag{3-18c}$$

$$N_y = P\, v' - EI_z\, w''' \tag{3-18d}$$

$$N_z = P\, v' - EI_y\, w''' \tag{3-18e}$$

The solution of the differential equations 3-17 give the following expressions for the deformations as the functions of x.

$$\theta = A_x\, e^{k_x x} + B_x\, e^{-k_x x} + C_x x + D_x \tag{3-19a}$$

$$v = A_y\, e^{k_y x} + B_y\, e^{-k_y x} + C_y x + D_y \tag{3-19b}$$

$$w = A_z\, e^{k_z x} + B_z\, e^{-k_z x} + C_z x + D_z \tag{3-19c}$$

A_x, A_y, A_z, B_x, B_y, B_z, C_x, C_y, C_z and D_x, D_y, D_z are the constants of integration, which will be determined from the end conditions of the beam-column.

The end deformations are expressed in terms of the integration constants in matrix form by taking the appropriate derivatives of the deformation functions and by making the necessary adjustments in signs.

$$\begin{bmatrix} \theta'_{ix} \\ \theta_{ix} \\ \hline \theta'_{jx} \\ \theta_{jx} \end{bmatrix} = \left[\begin{array}{cc|cc} k_x & -k_x & 1 & 0 \\ 1 & 1 & 0 & 1 \\ \hline k_x\Phi_1 & -k_x\psi_1 & 1 & 0 \\ \Phi_1 & \psi_1 & L & 1 \end{array}\right] \begin{bmatrix} A_x \\ B_x \\ \hline C_x \\ D_x \end{bmatrix}$$

or

$$\left[D_x\right] = \left[E_x\right]\left[C_{ox}\right] \tag{3-20}$$

where $\Phi_1 = e^{k_x L}$, $\psi_1 = e^{-k_x L}$

and L is the length of the beam column ij.

$$\begin{bmatrix} \theta_{iy} \\ \delta_{iz} \\ \hline \theta_{jy} \\ \delta_{jz} \end{bmatrix} = \left[\begin{array}{cc|cc} -k_z & k_z & -1 & 0 \\ 1 & 1 & 0 & 1 \\ \hline -k_z\Phi_3 & k_z\psi_3 & -1 & 0 \\ \Phi_3 & \psi_3 & L & 1 \end{array}\right] \begin{bmatrix} A_z \\ B_z \\ \hline C_z \\ D_z \end{bmatrix}$$

or

$$\left[D_z\right] = \left[E_z\right]\left[C_{oz}\right] \tag{3-21}$$

where $\Phi_3 = e^{k_z L}$, and $\psi_3 = e^{-k_z L}$.

$$\begin{bmatrix} \theta_{iz} \\ \delta_{iy} \\ \theta_{jz} \\ \delta_{jy} \end{bmatrix} = \begin{bmatrix} k_y & -k_y & 1 & 0 \\ 1 & 1 & 0 & 1 \\ k_y \Phi_2 & -k_y \psi_2 & 1 & 0 \\ \Phi_2 & \psi_2 & L & 1 \end{bmatrix} \begin{bmatrix} A_y \\ B_y \\ C_y \\ D_y \end{bmatrix}$$

or

$$\left[D_y\right] = \left[E_y\right]\left[C_{oy}\right] \qquad (3-22)$$

where $\Phi_2 = e^{k_y L}$ and $\psi_2 = e^{-k_y L}$.

The end actions are then expressed in terms of the integration constants in matrix form by substituting the appropriate derivatives of the deformation functions into the Equations 3-18 and making the necessary adjustments in signs to adopt the sign convention shown in Figures 3-1 and 3-2.

$$\begin{bmatrix} M_{ijx} \\ M_{jix} \end{bmatrix} = C_1 \begin{bmatrix} 0 & 0 & -k_x^2 & 0 \\ 0 & 0 & k_x^2 & 0 \end{bmatrix} \begin{bmatrix} A_x \\ B_x \\ C_x \\ D_x \end{bmatrix}$$

or

$$\left[F_x\right] = \left[G_x\right]\left[C_{ox}\right] \tag{3-23}$$

and

$$\begin{bmatrix} M_{ijy} \\ N_{ijz} \\ M_{jiy} \\ N_{jiz} \end{bmatrix} = EI_y \begin{bmatrix} k_z^{\;2} & k_z^{\;2} & 0 & 0 \\ 0 & 0 & -k_z^{\;2} & 0 \\ -k_z^{\;2}\Phi_3 & -k_z^{\;2}\Psi_3 & 0 & 0 \\ 0 & 0 & k_z^{\;2} & 0 \end{bmatrix} \begin{bmatrix} A_z \\ B_z \\ C_z \\ D_z \end{bmatrix}$$

or

$$\left[F_z\right] = \left[G_z\right]\left[C_{oz}\right] \tag{3-24}$$

and

$$\begin{bmatrix} M_{ijz} \\ N_{ijy} \\ M_{jiz} \\ N_{jiy} \end{bmatrix} = EI_z \begin{bmatrix} -k_y^{\;2} & -k_y^{\;2} & 0 & 0 \\ 0 & 0 & -k_y^{\;2} & 0 \\ k_y^{\;2}\Phi_2 & k_y^{\;2}\Psi_2 & 0 & 0 \\ 0 & 0 & k_y^{\;2} & {} \end{bmatrix} \begin{bmatrix} A_y \\ B_y \\ C_y \\ D_y \end{bmatrix}$$

or

$$\left[F_y\right] = \left[G_y\right]\left[C_{oy}\right] \tag{3-25}$$

The end actions are obtained in terms of the end deformations in matrix form by solving the column matrix of constants from the matrix equations 3-20, 3-21, 3-22 and substituting into the matrix equations 3-23, 3-24, and 3-25.

$$\left[F_x\right] = \left[G_x\right] \left[E_x\right]^{-1} \left[D_x\right]$$

$$\left[F_z\right] = \left[G_z\right] \left[E_z\right]^{-1} \left[D_z\right] \qquad (3\text{-}26)$$

$$\left[F_y\right] = \left[G_y\right] \left[E_y\right]^{-1} \left[D_y\right]$$

At each end the cross section of the beam-column is assumed to have zero warping. For this reason

$$\theta'_{ix} = \theta'_{jx} = 0 \quad . \qquad (3\text{-}27)$$

The end actions are obtained in one matrix equation (Equation 3-28) by rearranging the Equation 3-26 and including the axial force stiffnesses and considering the Equation 3-27.

$$
\begin{bmatrix} M_{ijx} \\ M_{ijy} \\ M_{ijz} \\ N_{ijx} \\ N_{ijy} \\ N_{ijz} \\ M_{jix} \\ M_{jiy} \\ M_{jiz} \\ N_{jix} \\ N_{jiy} \\ N_{jiz} \end{bmatrix}
=
\begin{bmatrix}
k_{1,1} & & & & & & k_{1,7} & & & & & \\
 & k_{2,2} & & & & k_{2,6} & & k_{2,8} & & & & k_{2,12} \\
 & & k_{3,3} & & k_{3,5} & & & & k_{3,9} & & k_{3,11} & \\
 & & & k_{4,4} & & & & & & k_{4,10} & & \\
 & & & & k_{5,5} & & & & k_{5,9} & & k_{5,11} & \\
 & & & & & k_{6,6} & & k_{6,8} & & & & k_{6,12} \\
 & & & & & & k_{7,7} & & & & & \\
 & & & & & & & k_{8,8} & & & & k_{8,12} \\
 & & & & & & & & k_{9,9} & & k_{9,11} & \\
 & \text{Symmetric} & & & & & & & & k_{10,10} & & \\
 & & & & & & & & & & k_{11,11} & \\
 & & & & & & & & & & & k_{12,12}
\end{bmatrix}
\begin{bmatrix} \theta_{ix} \\ \theta_{iy} \\ \theta_{iz} \\ \delta_{ix} \\ \delta_{iy} \\ \delta_{iz} \\ \theta_{jx} \\ \theta_{jy} \\ \theta_{jz} \\ \delta_{jx} \\ \delta_{jy} \\ \delta_{jz} \end{bmatrix}
$$

(3-28)

or in symbolic form

$$\left[N_o\right] = \left[K_o\right]\left[D_o\right]$$ (3-29)

where $\left[K_o\right]$ is defined as the stiffness matrix of the beam-column. The algebraic expressions for the elements of the stiffness matrix are given by the following equations.

$$k_{1,1} = k_{7,7} = -k_{1,7} = \frac{C_1}{L^3} \cdot \frac{\beta_1 \, (k_x L)^3}{k_x L \beta_1 - 2\alpha_1 + 4} \tag{3-30}$$

$$k_{2,2} = k_{8,8} = \frac{EI_y}{L} \cdot \frac{(k_z L)^2 \, \alpha_3 - k_z L \beta_3}{k_z L \beta_3 - 2\alpha_3 + 4} \tag{3-31}$$

$$k_{2,6} = -k_{2,12} = k_{6,7} = -k_{8,12} = \frac{EI_y}{L^2} \cdot \frac{(k_z L)^2 \, (2 - \alpha_3)}{k_z L \beta_3 - 2\alpha_3 + 4} \tag{3-32}$$

$$k_{2,8} = \frac{EI_y}{L} \cdot \frac{k_z L \beta_3 - 2k_z^2 L^2}{k_z L \beta_3 - 2\alpha_3 + 4} \tag{3-33}$$

$$k_{6,6} = -k_{6,12} = k_{12,12} = \frac{EI_y}{L^3} \cdot \frac{(k_z L)^3 \, \beta_3}{k_z L \beta_3 - 2\alpha_3 + 4} \tag{3-34}$$

$$k_{3,3} = k_{9,9} = \frac{EI_z}{L} \cdot \frac{(k_y L)^2 \, \alpha_2 - k_y L \beta_2}{k_y L \beta_2 - 2\alpha_2 + 4} \tag{3-35}$$

$$k_{3,11} = -k_{3,5} = -k_{5,9} = k_{9,11} = \frac{EI_z}{L^2} \cdot \frac{(k_y L)^2 \, (2 - \alpha_2)}{k_y L \beta_2 - 2\alpha_2 + 4} \tag{3-36}$$

$$k_{3,9} = \frac{EI_z}{L} \cdot \frac{k_y L \beta_2 - 2k_y^2 L^2}{k_y L \beta_2 - 2\alpha_2 + 4} \tag{3-37}$$

$$k_{5,5} = -k_{5,11} = k_{11,11} = \frac{EI_z}{L^3} \cdot \frac{(k_y L)^3 \, \beta_2}{k_y L \beta_2 - 2\alpha_2 + 4} \tag{3-38}$$

$$k_{4,4} = -k_{4,10} = k_{10,10} = \frac{EA}{L} \tag{3-39}$$

Where

$$\alpha_i = \Phi_i + \psi_i \qquad i = 1,2,3$$
$$\beta_i = \Phi_i - \psi_i \qquad i = 1,2,3 \qquad (3\text{-}40)$$

The dimensionless coefficients of the elements of stiffness matrix are the exponential functions of the axial force. For zero axial force these coefficients take an indeterminate form of $\frac{0}{0}$, and the limit by L'Hôpital's rule gives the well-known coefficients of the conventional stiffness matrix. In case of compression the coefficients become imaginary. It is convenient to expand these coefficients into MacLaurin's series for the computer calculations. Appendix A consists of the series expansion and the graphs of their values.

CHAPTER IV

STIFFNESS MATRIX FOR A SPACE FRAME

In this chapter the joint loads of a space frame are expressed in terms of the joint deformations in matrix form. The coefficient matrix of the column vector of joint deformations in this matrix equation is called the stiffness matrix of the space frame.

An orthogonal system of coordinate axes which is called the structure oriented axes is chosen. The matrix relation between the end actions and the end deformations of a space member which is obtained in the previous chapter is transformed from the member oriented axes to the structure oriented axes with the aid of the transformation matrices. The transformed matrix equations of the members are then assembled to form the joint equilibrium equations of the space frame.

4-1 Transformation of Axes

A space member ij where i and j are assumed to be the near and far ends respectively is shown in Fig. 4-1. The member oriented axes are X, Y, Z while X$'$, Y$'$, Z$'$ are the axes parallel to the structure oriented axes at i (Fig. 4-1). The end points i and j of the member are given by the (x_i, y_i, z_i) and (x_j, y_j, z_j) coordinates respectively with respect to the structure oriented axes.

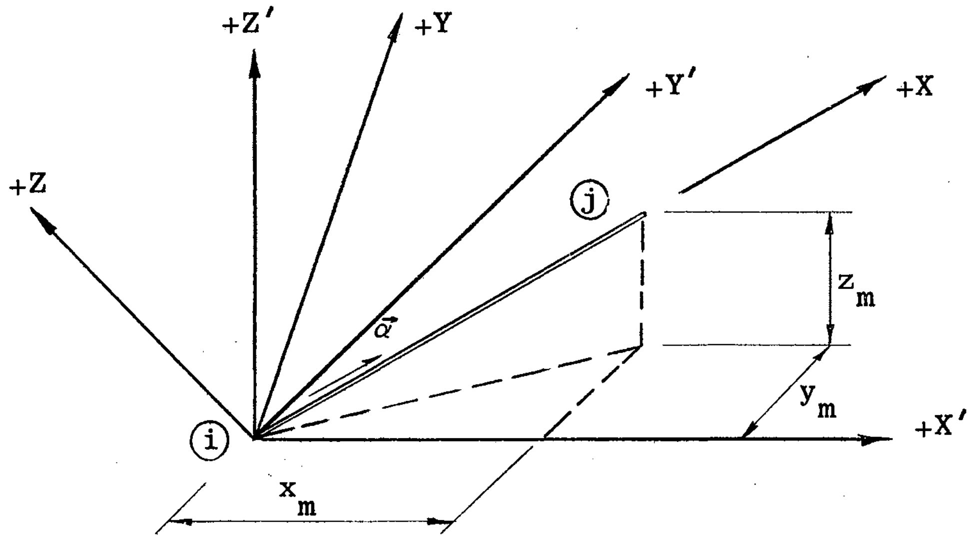


Fig. 4-1

Member and Structure Oriented Axes

The angle between the Y axis and the X'Y' plane is represented by a vector α parallel to X axis. The sign of α is determined by the right hand rule. From the Fig. 4-1

$$L = \sqrt{x_m^2 + y_m^2 + z_m^2} \tag{4-1}$$

$$L_o = \sqrt{x_m^2 + y_m^2} \tag{4-2}$$

and

$$x_m = x_j - x_i$$

$$y_m = y_j - y_i$$

$$z_m = z_j - z_i$$

where L and L_o are the actual and the projected length of the member ij respectively.

A column vector given with respect to the member oriented axes is transformed to the structure oriented axes by premultiplying it with the transformation matrix $[t]$.

The transformation matrix $[t]$ is obtained by the multiplication of the three matrices each of which corresponds to a transformation by rotating the member oriented system about one of the axes. The matrices $[t_1]$, $[t_2]$, and $[t_3]$ correspond to the rotational transformation of the member oriented system about X, Y, Z axes respectively.

$$[t_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix} \tag{4-3a}$$

$$[t_2] = \begin{bmatrix} L_o/L & 0 & -z_m/L \\ 0 & 1 & 0 \\ \frac{z_m}{L} & 0 & \frac{L_o}{L} \end{bmatrix} \tag{4-3b}$$

$$[t_3] = \begin{bmatrix} \frac{x_m}{L_o} & \frac{-y_m}{L_o} & 0 \\ \frac{y_m}{L_o} & \frac{x_m}{L_o} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4-3c}$$

Then the transformation matrix

$$\left[t\right] = \left[t_3\right]\left[t_2\right]\left[t_1\right] \quad . \tag{4-4}$$

The column vector of end actions and end deformations can be transformed from the member oriented axes to the structure oriented axes by the transformation matrix $\left[T\right]$. Since the column vector of end actions and the end deformations consists of four vectors, the transformation matrix

$$\left[T\right] = \left[\begin{array}{c|c|c|c} [t] & & & \\ \hline & [t] & & \\ \hline & & [t] & \\ \hline & & & [t] \end{array}\right] \tag{4-5}$$

where $\left[t\right]$ is given by the Equation 4-4 and the blank areas are the null submatrices. The matrices given by the Equations 4-3 and 4-4 are orthogonal due to the character of the transformations. The matrix $\left[T\right]$ is also orthogonal because it is formed by the orthogonal submatrices at the diagonal (Eq. 4-5). Then

$$\left[T\right]^T = \left[T\right]^{-1} \quad . \tag{4-6}$$

Denoting the column vector of end actions and end deformations with respect to the structure oriented axes by $\left[N\right]$ and $\left[D\right]$ respectively

$$\left[N\right] = \left[T\right]\left[N_o\right] \tag{4-7}$$

and

$$\left[D_o\right] = \left[T\right]^T \left[D\right] \quad , \tag{4-8}$$

By premultiplying both the sides of the matrix equation (3-29) by $\left[T\right]$ and substituting the matrix equations 4-7 and 4-8

$$\left[N\right] = \left[T\right]\left[K_o\right]\left[T\right]^T \left[D\right] \quad , \tag{4-9}$$

Equation 4-9 gives the end actions in terms of the end deformations with respect to structure oriented axes. Equations 4-9 can be written as

$$\left[N\right] = \left[K\right]\left[D\right] \tag{4-10}$$

where $\left[K\right]$ is called the transformed member stiffness matrix and

$$\left[K\right] = \left[T\right]\left[K_o\right]\left[T\right]^T \quad , \tag{4-11}$$

The transformed member stiffness matrix is symmetrical. This can be proved by transposing both the sides of Equation 4-11.

It should be noted that the transformation matrix $\left[T\right]$, in the form presented here, becomes indefinite for the member parallel to Z′ axis, due to the vanishing value of L_o. Nevertheless, the coding system used in the computer programming to form the stiffness matrix does not require any transformation for the members parallel to the structure oriented axes.

4-2 Joint Equilibrium Equations

The end actions transformed to the structure oriented axes are transmitted from the ends of the members to the joints (Fig. 4-2). Then, the equilibrium of the joint is established by equating the summation of the column vector of end actions of the members meeting at the joint to the column vector of the joint loads provided that the joint loads acting in the positive direction of the structure oriented axes are assumed to be positive.

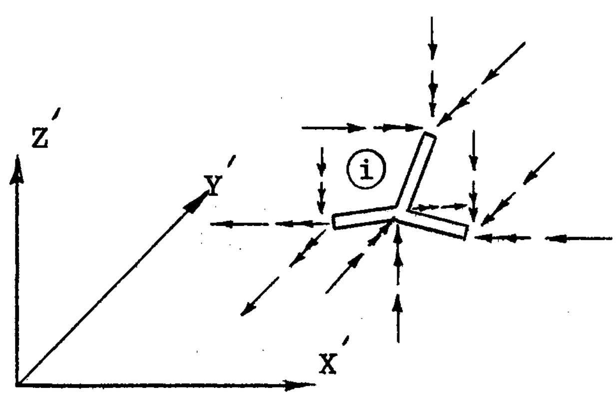


Fig. 4-2

Free Body of a Typical Joint

A free body of a joint of a space structure is shown in Fig. 4-2. The double arrows designate the moment vectors while the forces are represented by the single arrows. It should be noted that the positive end actions acting at the joint are in the opposite direction of the structure oriented axis due to the sign convention for the end actions acting on the members.

The matrix Equation 4-10 can be written for a member ij in the following form:

$$\begin{bmatrix} N_{ij} \\ \hline N_{ji} \end{bmatrix} = \left[\begin{array}{c|c} K_{ii}^{J} & K_{ij} \\ \hline K_{ji} & K_{jj}^{i} \end{array}\right] \begin{bmatrix} D_i \\ \hline D_j \end{bmatrix} \tag{4-12}$$

N_{ij} and N_{ji} are six by one submatrices of the end actions at the ends i and j respectively. D_i and D_j are six by one submatrices of the joint deformations of joints i and j. The six by six submatrices K_{ii}^{J}, K_{ij}, K_{ji}, K_{jj}^{i} are obtained from the transformed stiffness matrix by partitioning. Due to the symmetry of the stiffness matrix and also due to Maxwell's reciprocal theorem

$$\begin{bmatrix} K_{ij} \end{bmatrix} = \begin{bmatrix} K_{ji} \end{bmatrix}^{T} \tag{4-13}$$

The joint equilibrium equations of joint i can be expressed in symbolic form by using Equation 4-12 as

$$\begin{bmatrix} W_i \end{bmatrix} = \sum_{j=1}^{n} \begin{bmatrix} K_{ii}^{J} \end{bmatrix} \begin{bmatrix} D_i \end{bmatrix} + \sum_{j=1}^{n} \begin{bmatrix} K_{ij} \end{bmatrix} \begin{bmatrix} D_j \end{bmatrix} \tag{4-14}$$

where W_i is the column vector of joint loads at joint i, and j is the number of the far end joint of each member meeting at joint i, and n is the number of members coming to joint i. The joint equilibrium equations for the whole frame can be obtained by forming Equation 4-14 for every joint of the structure and by assembling those in one matrix equation

$$\left[W_s\right] = \left[K_s\right]\left[D_s\right] \qquad (4\text{-}15)$$

where W_s and D_s are the column vector of joint loads and joint deformations of the frame respectively and K_s is the stiffness matrix of the structure. The order of the stiffness matrix of a structure is equal to the total number of the degree of freedom of the joints. The matrix K_s is symmetrical due to Maxwell's reciprocal theorem[45, 53].

CHAPTER V

STABILITY ANALYSIS OF SPACE FRAMES

Two criteria of stability are introduced to find the elastic buckling load parameter for a space frame loaded at the joints. The determinantal criterion of buckling is used to obtain the elastic buckling load parameter for frames without prebuckling moments. In the case of frames having prebuckling moments, the convergence of the iteration process to obtain the axial forces in the members by taking into account the prebuckling moments is postulated as a criterion of buckling. The determinantal criterion of buckling is also used to find an approximate buckling load parameter for structures having prebuckling moments. This is very important in the case of the stability analysis of symmetrically loaded, symmetrical space frames having prebuckling moments for which the convergence criterion fails.

5-1 Determinantal Criterion of Buckling

The critical load parameter for frames which do not have prebuckling moments may be determined from the vanishing determinant of the stiffness matrix. The vanishing determinant of the stiffness matrix as a condition of instability is called the determinantal criterion of buckling. This well known and well established criterion has been discussed by several investigators[22,27,44,51,57], among them is Horne[44] who pointed out that the critical load parameter for frames having prebuckling moments obtained from the determinantal criterion is approximate. In this dissertation the determinantal criterion of buckling is used for the stability analysis of space frames which do not have prebuckling moments. For the frames having prebuckling moments the convergence criterion which is discussed in Article 5-2 is postulated.

5-2 Convergence Criterion

The typical load-deformation curves for frames having prebuckling moments are discussed in the introduction. Exact elastic analysis of the structure is necessary to obtain these curves. For a given load parameter, the axial forces in the members of the frame are unknown due to indeterminate prebuckling moments. Since the elements of the stiffness matrix of the structure are the functions of axial forces, Equation 4-15 may be used in an iteration process to compute the axial forces in the members. At first all the axial forces in the members are assumed to be zero. The joint deformations are solved from Equation 4-15, and the axial forces in the members are computed from the deformations of joints. Assuming then the computed axial forces, the elements of stiffness matrix are recalculated and the analysis is repeated. This process is continued until the assumed axial forces approach the computed axial forces to a desired accuracy. For the final axial forces the determinant of the stiffness matrix as well as the joint deformations are calculated.

For the types of load deformation curves introduced in Fig. 1-1c and Fig. 1-3b the iteration to find the axial forces in the members converges rapidly for the loads smaller than the buckling load. At the buckling load this process does not converge due to the fact that there is not a unique set of axial forces to satisfy the equilibrium and the compatibility conditions of the structure. Therefore convergence of iteration process is postulated as a criterion of buckling of frames having prebuckling moments.

This convergence criterion fails for symmetrically loaded, symmetrical frames having prebuckling moments. However, the determinantal criterion of buckling may be used to obtain an approximate solution for the buckling load of this type of frames.

5-3 Procedure of Analysis

In the application of determinantal and for convergence criteria of buckling for the stability analysis of space frames, the use of a high speed electronic computer is necessary. Two computer programs have been developed for this purpose in FORTRAN[68] language. The first program is to compute the critical load parameter of an orthogonal space frame loaded at the joints by using the determinantal criterion of buckling. The second program performs the iteration to find the axial forces in the members for a given load parameter. The flow charts and the input/output of the programs are given in Appendix B. The following is the general procedure of the stability analysis of space frames by using the determinantal and convergence criteria:

(a) A starting value and an increment is chosen for the load parameter. The stiffness matrix of the frame is formed by neglecting the affect of prebuckling moments to the axial forces in the members (axial forces are determined by statics and axial deformations of the members are neglected) and the determinant of the stiffness matrix is computed for each load parameter.

(b) For the load parameters smaller than the critical load parameter, the determinant of the stiffness matrix is positive. When a negative value for the determinant is obtained, the value of load parameter for which the determinant vanishes is determined by successive reductions in the interval at which the determinant changes the sign. This load parameter is the critical load parameter if the frame does not have any prebuckling moments and the computation is finished.

(c) The load parameter obtained in part (b) is an approximate critical load parameter for the frames having prebuckling moments. In this case, the critical load parameter can be obtained by performing an iteration process to obtain the axial forces in the members by taking the prebuckling moments into account. The convergence of the iteration should be checked for the load parameters close to one obtained from the determinantal criterion of buckling. The greatest load parameter for which the iteration converges is the critical load parameter.

5-4 Examples

Example 1

A space frame subject to a joint load analogous to planar knee frame is shown in Fig. 5-1a. The X′, Y′, Z′ axes are the structure oriented axes. All the members of the frame having the same cross-section as is shown in Fig. 5-1b. X, Y, Z are the member oriented axes.

The properties of the cross-section are as follow:

Modulus of Elasticity $E = 29000$ ksi

Modulus of Rigidity $G = 0.40\ E$

Moment of Inertia about Y $I_y = 161.466\ in^4$

Moment of Inertia about Z $I_z = 34.183\ in^4$

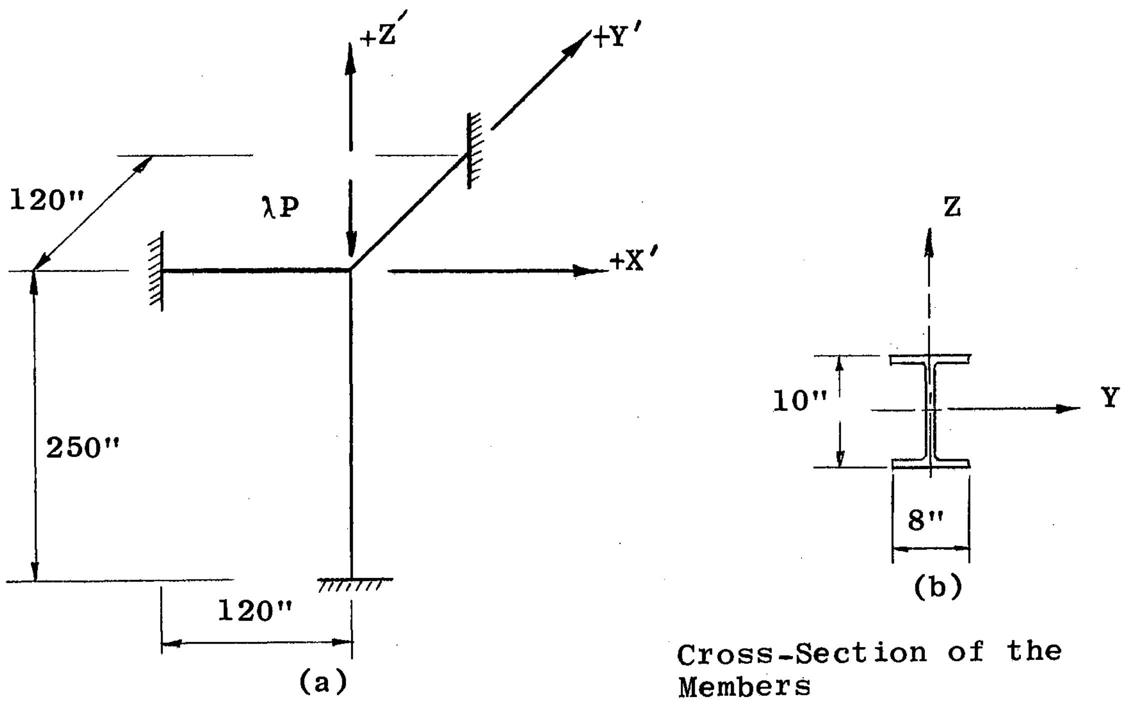
Area of the cross-section $A = 10.80\ in^2$

Pure Torsional Stiffness $C = 6235\ k\text{-}in^2$

Warping Stiffness $C_1 = 20946120\ k\text{-}in^4$

Polar Radius of Gyration $r_s = 4.256$ in.

For the beams, Z axis of the cross-section is parallel to the Z′ axis while for the column, Z axis of the column cross-section is parallel to the X′ axis. The reference load is taken as P = 100 kips.



Space Frame Analogous to Planar Knee Frame

Cross-Section of the Members

Fig. 5-1

Space Frame of Example 1

The approximate critical load parameter is obtained as 5.958 from the determinantal criterion of buckling. When the convergence criterion is used, the greatest value of λ for which the iteration process for the axial forces in the members converged is found to be 5.66. The maximum difference between the corresponding axial forces in the last two cycles of iteration is taken as 0.100 kips.

The buckling occurs by the vanishing of the rotational stiffness of the joint of frame about the X′ axis.

The relation between the load parameter and $\theta_{x'}$, rotation of the joint about the X' axis is shown in Fig. 5-2.

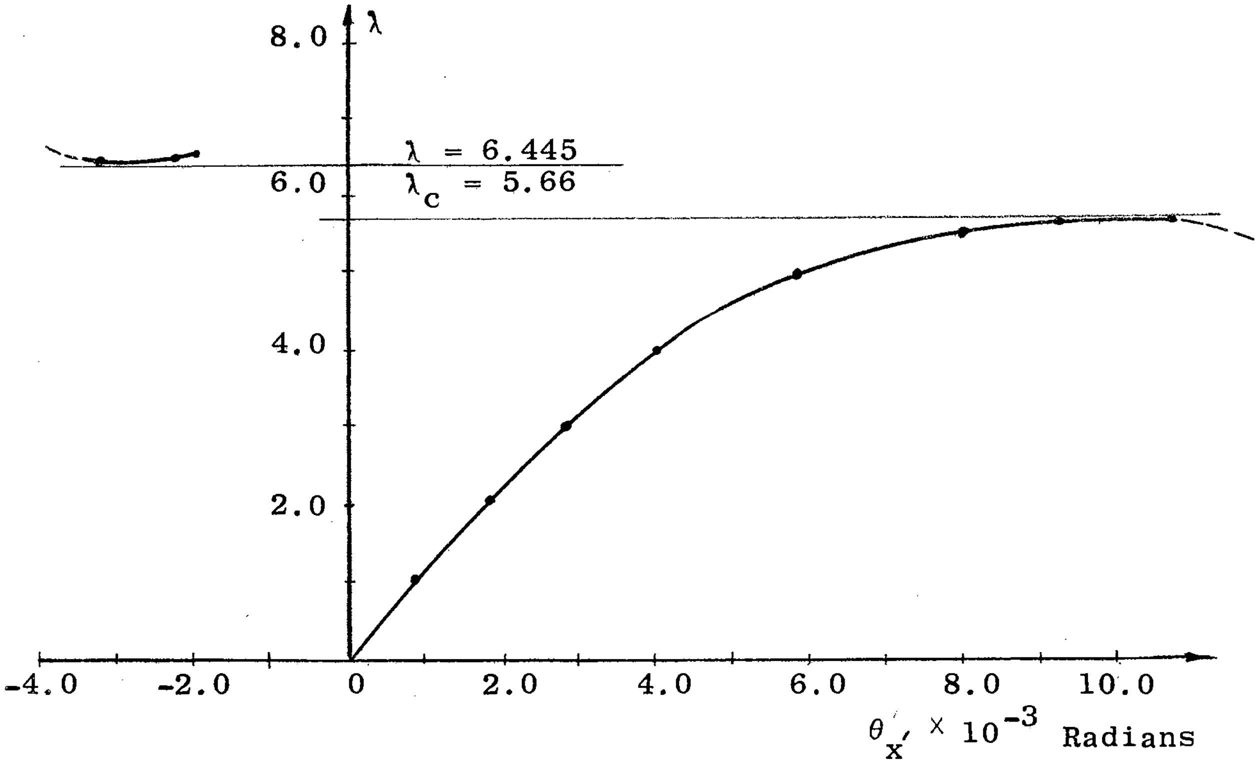


Fig. 5-2

Load-Rotation Relationship of the Example 1

The behavior of the frame in buckling is similar to the behavior of the planar knee frame discussed in Chapter I. It is interesting to note that there is no second order solution of the frame for the load parameters between 5.66 and 6.445 and the determinant of the stiffness matrix does not vanish when the effect of prebuckling moments is considered. The critical load parameter obtained by the convergence criterion is 5.3% smaller than the one obtained by the determinantal criterion. All the deformations of the joint are in the range of the small deflection theory.

Example 2

A space portal frame under vertical joint loads analogous to planar portal frame is shown in Fig. 5-3a. The X′, Y′, Z′ axes are the structure oriented axes. All the members of the frame have the same cross-section which is shown in Fig. 5-3b. X, Y, Z are the member oriented axes. The cross-section of the members of this example is slightly different from that of the first example, with the exception of E and G.

The properties of the cross-section:

Moment of Inertia about Y $I_y = 173.498$ in^4

Moment of Inertia about Z $I_z = 34.182$ in^4

Area of the cross section $A = 10.08$ in^2

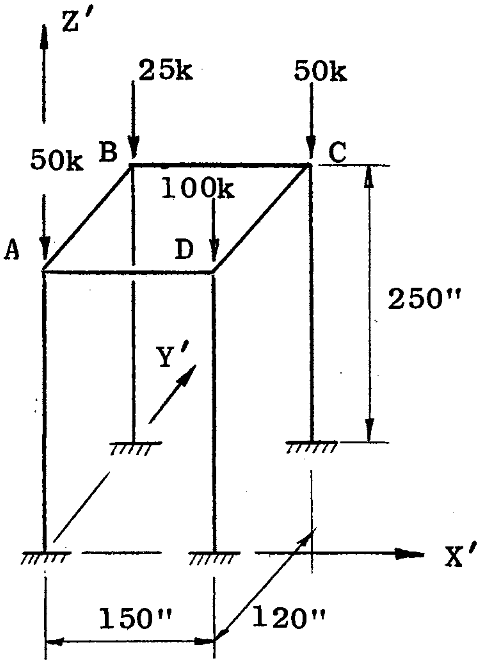
Pure Torsional Stiffness $C = 6235$ k-in^2

Warping Stiffness $C_1 = 22806470$ k-in^4

Polar Radius of Gyration $r_s = 4.539$ in ,

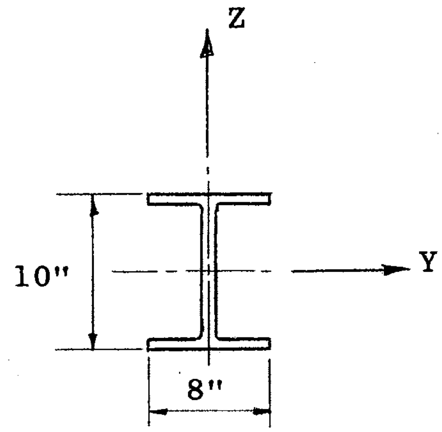
For the beams, Z axis of the cross-section is parallel to the Z′ axis while Z axes of the columns are parallel to Y′ axis.

The critical load parameter for the frame by neglecting the effect of prebuckling moments is obtained as 2.581 from the determinantal criterion. When the convergence criterion is used the critical load parameter is found to be 2.571. The load parameter versus horizontal displacement of joint D in the X′ direction ($\Delta_{Dx'}$) relation is illustrated in Fig. 5-4.



(a)

Space Portal Frame Under Reference Loading



(b)

Cross-Section of the Members

Fig. 5-3

Space Frame of Example 2

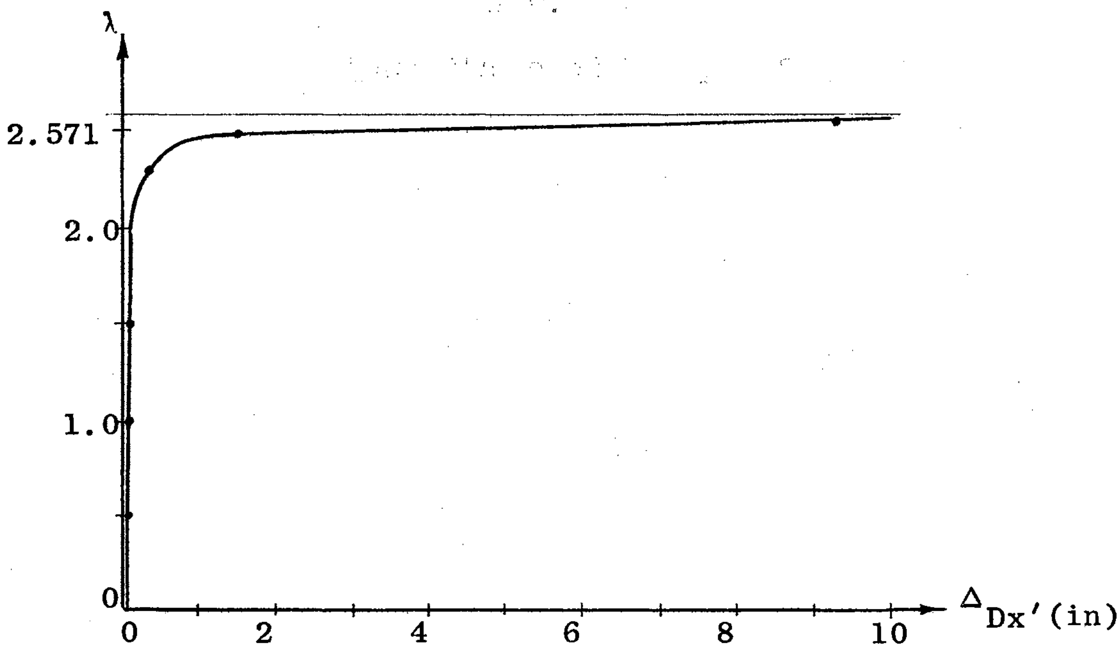


Fig. 5-4

Load Deflection Relationship for the Frame of Example 2

CHAPTER VI

SUMMARY AND CONCLUSIONS

6-1 Summary

A method of stability analysis for rigid-jointed, elastic space frames loaded at the joints is presented. A brief review of the methods of stability analysis of elastic frames, together with the historical development of the subject, is given. The stiffness matrix for a space beam-column of symmetrical, thin-walled open cross-section is derived from the differential equations of bending and torsion. Warping stiffness of the cross-section is taken into account in the derivation. Member stiffness matrices are transformed to a structure oriented coordinate system and then assembled to obtain the joint equilibrium equations of the frame in terms of joint deformations in matrix form, thus obtaining the stiffness matrix of the frame. The vanishing determinant of the stiffness matrix of the frame is used as the criterion of buckling for frames without prebuckling moments. In the case of frames having prebuckling moments, the convergence of the iteration process to obtain the axial forces in the members is postulated as the criterion of buckling.

Two computer programs are developed for the application of the determinantal and the convergence criterion of buckling. These programs are explained in the appendix, and several examples of their use are included.

6-2 Conclusions

A method of elastic stability analysis for the space frames loaded at the joints is developed by taking the warping stiffness of the members into account. Two criteria of buckling have been utilized. The determinantal criterion of buckling is used for the stability analysis of the frames without prebuckling moments. For the frames having prebuckling moments, convergence criterion is postulated. The use of a digital computer is essential for the stability analysis of space frames due to the involved character of the numerical computations.

If the prebuckling moments of the frame are small, determinantal criterion may be used to obtain the buckling load approximately. It is found that the use of determinantal criterion of buckling requires less computer time and core storage compared to convergence criterion. The prebuckling moments caused less than 10% reduction in the buckling load in the example problems solved. The phenomenon of buckling of the space frames is the same as the phenomenon of buckling of the analogous planar frames.

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APPENDIX A

THE STABILITY COEFFICIENTS

The algebraic expressions for the elements of member stiffness matrix are given by the Equations 3-10 through 3-40 in Chapter III. The dimensionless parts of these expressions are called the stability coefficients. It is possible to evaluate all of the stability coefficients from the following basic infinite series which are the reduced MacLaurin's expansions of the algebraic expressions.

$$U1(y) = \sum_{n=1}^{\infty} \frac{2n}{(2n+1)!} y^{n-1}$$

$$U2(y) = \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} y^{n-1}$$

$$U3(y) = \sum_{n=1}^{\infty} \frac{(2n+1)}{(2n+1)!} y^{n-1}$$

$$U4(y) = \sum_{n=1}^{\infty} \frac{2n(2n+1)}{(2n+1)!} y^{n-1}$$

$$B(y) = \sum_{n=1}^{\infty} \frac{n}{(n+1)(2n+1)!} y^{n-1}$$

where y is the variable to be defined according to the stability coefficient to be evaluated. Defining

$$T(y) = \frac{U4(y)}{B(y)} \qquad \text{(A-1)}$$

$$AB(y) = \frac{U1(y)}{B(y)} \tag{A-2}$$

$$CAB(y) = \frac{U2(y)}{B(y)} \tag{A-3}$$

$$DAB(y) = \frac{U3(y)}{B(y)} \tag{A-4}$$

The following are typical elements of the member stiffness matrix in terms of the expressions given by the Equations A-1 through A-4.

$$k_{1,1} = \frac{C_1}{L^3} \; T(y_1)$$

$$k_{2,2} = \frac{EI_y}{L} \; AB(y_3)$$

$$k_{2,6} = - \frac{EI_y}{L^2} \; DAB(y_3)$$

$$k_{2,8} = \frac{EI_y}{L} \; CAB(y_3)$$

$$k_{6,6} = \frac{EI_y}{L^3} \; T(y_3)$$

$$k_{3,3} = \frac{EI_z}{L} \; AB(y_2)$$

$$k_{3,11} = - \frac{EI_z}{L^2} \; DAB(y_2)$$

$$k_{3,9} = \frac{EI_z}{L} \; CAB(y_2)$$

$$k_{5,5} = \frac{EI_z}{L^3} \; T(y_2)$$

where

$$y_1 = k_x^2 \, L^2 = \frac{r_s^2 \, P + C}{C_1} \, . \, L^2$$

$$y_2 = k_y^2 \, L^2 = \frac{PL^2}{EI_z}$$

$$y_3 = k_z^2 \, L^2 = \frac{PL^2}{EI_y}$$

It is interesting to note that the expression for the stability coefficient of torsion obtained by taking the warping stiffness of the cross-section into account is the same as the expression for the stability coefficient of the shear force. This confirms the analogy between the shear force and the torsional moment in structural analysis.

The expressions A-1 through A-4 are evaluated by a high speed computer; once using their exponential equivalents and then by series for the values of y between -50 and +50. It is found that taking only sixteen terms of series is sufficient to obtain at least six digit accuracy in all cases. Since the interval considered for y is much greater than the values of y in practical problems, the elements of stiffness matrix of members are calculated by series in computer analysis. Figure A-1 illustrates the variation of T, AB, CAB and DAB with values of y.

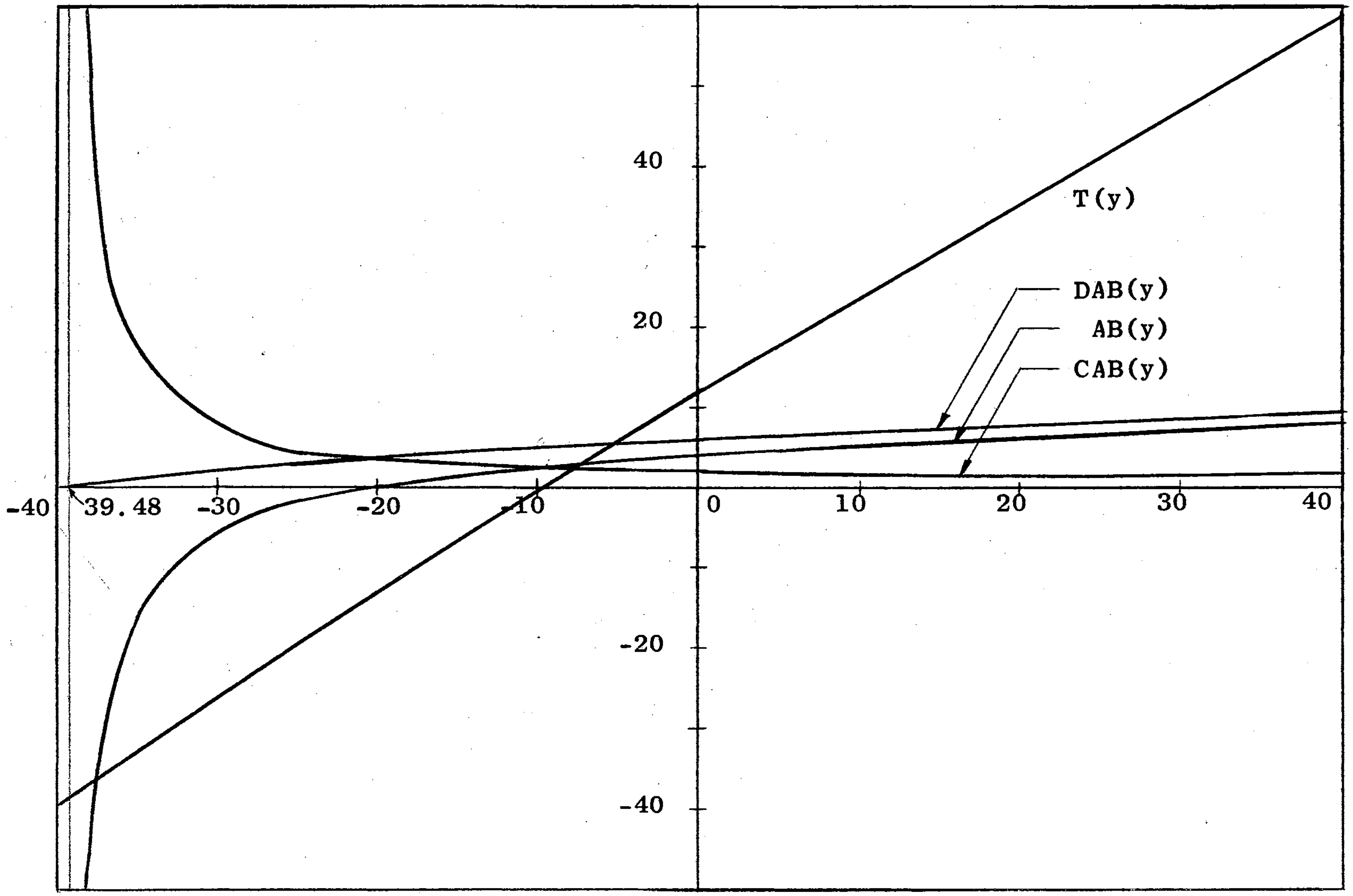

Fig. A-1 Variation of the Typical Stability Coefficients

APPENDIX B

COMPUTER ANALYSIS

The coding used for the deformations of joints to generate the stiffness matrix, the macro flow diagrams and the input-output of the computer programs based on determinantal and convergence criteria for the stability analysis of orthogonal space frames are given.

Generation of the Stiffness Matrix

The components of the deformation vectors of the joints in the direction of structure oriented axis are numbered in any arbitrary order starting with the number "1." All zero deformations are given the same dummy code number that is one greater than the largest number used for possible deformation vector components of the structure. For the orthogonal frames if the axial deformations of the members are neglected, the displacement components of the end joints of the members in the direction of the members are given the same code numbers due to the fact that there will be no relative displacement between the end joints in the direction of the member.

After numbering the deformations of the joints of an orthogonal space frame as explained, code numbers of the near end and far end deformations of each member can be determined from the deformation code numbers of the end joints. The subscripts used in the computer programs indicating the code numbers of the end deformations of a space bar ij, are shown in Figures B-1a and B-1b, where i and j are the near end and far end joints respectively. Fig. B-1a illustrates the subscripts for the code numbers of the end deformations of the case where the axial deformations of the bar is neglected. Fig. B-1b shows the subscripts of a general space bar.

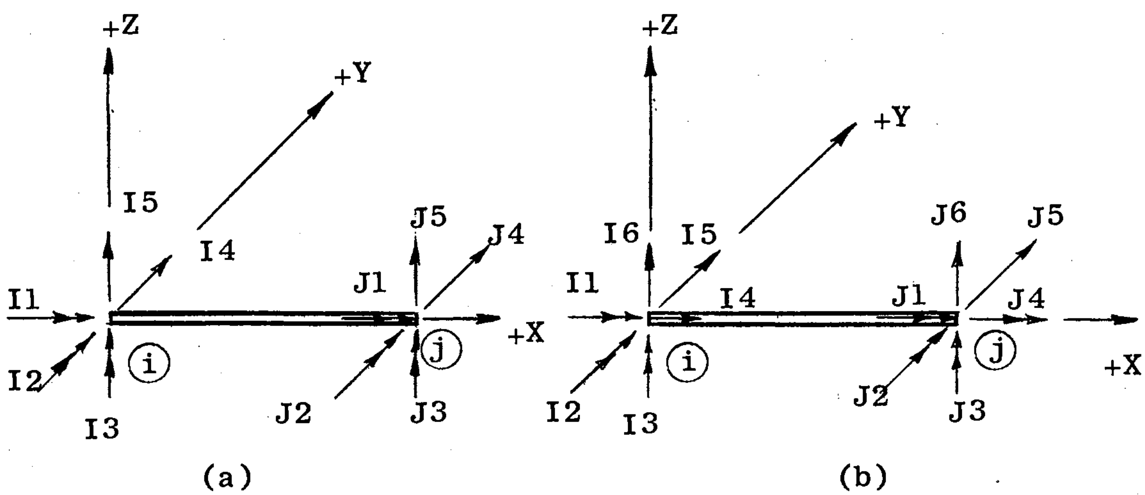

Fig. B-1

Subscripts For the Code Numbers of End Deformations of a Space Bar

The sign convention of the end deformations is the same as the one used in derivations in Chapter III.

If the rows and columns of the beam-column stiffness matrix given by Equation 3-28 are labeled by the subscripts I1, I2, ..., I6, J1, J2, ..., J6, it is possible to set up a general stiffness matrix that is applicable to all members of the frame. The stiffness matrix of the structure is then formed by accumulating the corresponding elements of all member stiffness matrices in one matrix. Due to the dummy deformation code number used for zero deformations, the stiffness matrix of the structure generated by this method contains an extra row and column. The joint loads are subscripted with the code number of the joint deformation in their direction, and they are stored in the extra column of the stiffness matrix to form the joint equilibrium equations. Thus, the waste of computer memory is reduced to a row of the stiffness matrix.

Computer Program Using the Determinantal Criterion of Buckling

A macro flow diagram, Fig. B-2, illustrates the basic logic used in the computer program for the stability analysis of orthogonal space frames by utilizing the determinantal criterion of buckling which is discussed in Chapter V. Since the axial deformations of members are assumed to be zero, the general member stiffness matrix in this case is obtained by deletion of the

4th and 10th rows and columns of the stiffness matrix given by Equations 3-28. The coding scheme for the end deformations of the members is shown in Fig. B-1a. Required input data are indicated below. The output is the values of load parameters and the corresponding determinant of the stiffness matrix. The last load parameter is the critical load parameter. Since it was not the purpose of this dissertation to develop an efficient program, the programming effort terminated when results were obtained. For this reason, the details of the program are omitted. The program was written for a computer having a total storage capacity of only 40,000 decimal digits, therefore the structure was restricted in size to 8 joints, 32 degrees of freedom.

Input Data

Control Cards

AL, DAL, EPS

AL - Starting value of the load parameter

DAL - Equal increments of load parameter

EPS - The allowed error in critical load parameter

E

E - Modulus of elasticity

ND, NM, NK

ND - Number of joint deformations

NM - Number of members

NK - Number of terms to be taken in the computation of stiffness by series.

Member Deformation Coding

M, I1, I2,..., I5, J1, J2,..., J5

M - Code number of the member

I's and J's indicate the code numbers of the end deformations of the member determined according to Fig. B-1a.

Member Properties

ES, ZIY, ZIZ, C1, C, RS, P

ES - Length of the member

ZIY, ZIZ - Moment of inertias of the cross-section about Y and Z axis respectively

C1 - Warping stiffness

C - Pure torsional stiffness

RS - Square of the polar radius of gyration of the cross-section

P - Axial force in the member

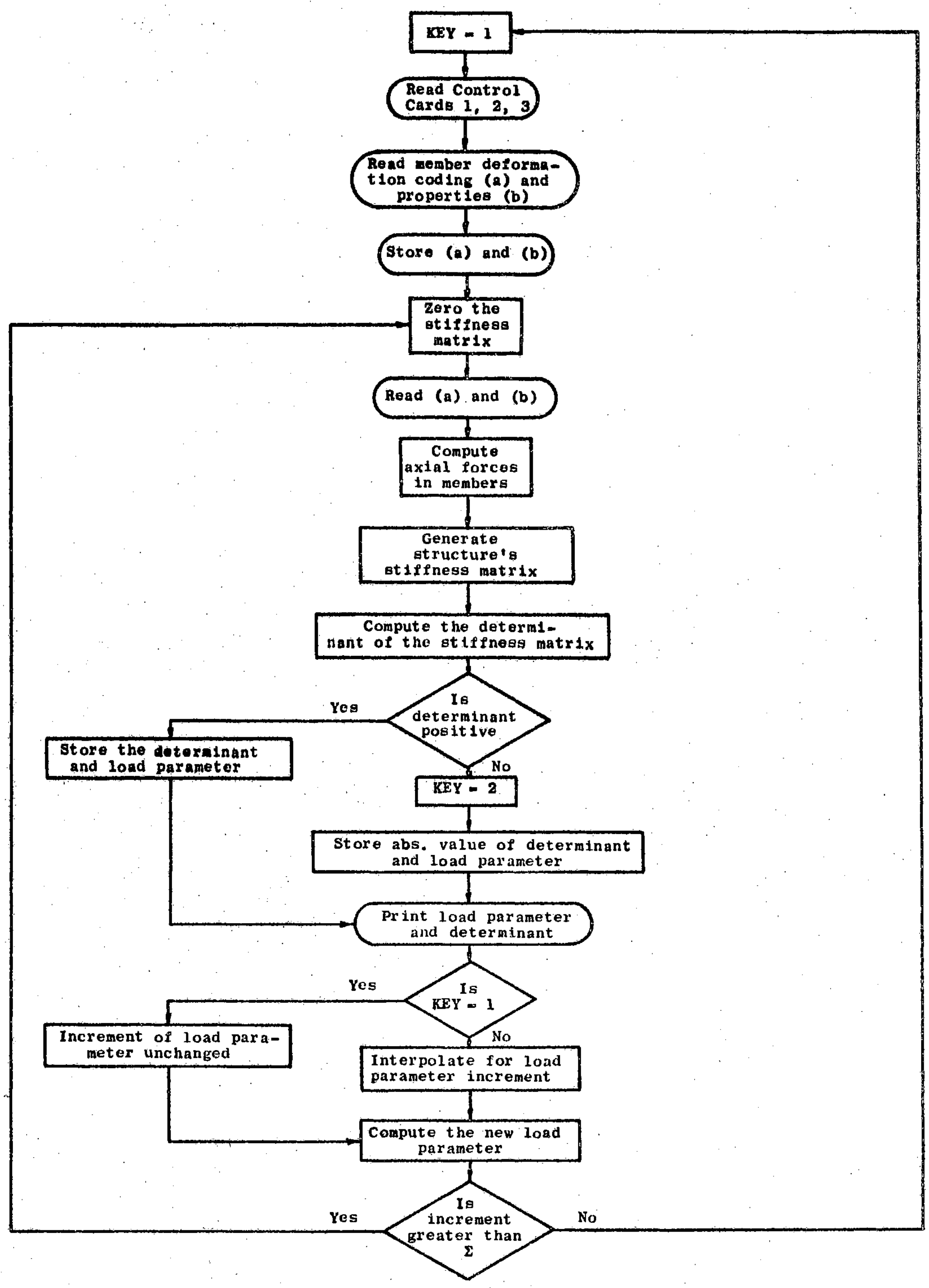

FIG. B.2 MACRO FLOW DIAGRAM OF COMPUTER PROGRAM USING THE DETERMINANTAL CRITERION OF BUCKLING

Computer Program Using the Convergence Criterion of Buckling

A macro flow diagram given in Fig. B-3 illustrates the basic logic used in the computer program for the second order analysis of an orthogonal space frame to apply the convergence criterion of buckling. The program was written for a computer having a total storage capacity of only 40,000 decimal digits; therefore, the structure to be analyzed was restricted in size to 4 joints, 24 degrees of freedom. The details of the program are omitted.

Input Data

Control Cards

E, EPS

E - Modulus of elasticity

EPS - The maximum difference between the corresponding axial forces in last two cycles of iteration

ND, NM, NK, ICL

ND - Number of joint deformations

NM - Number of members

NK - Number of terms to be taken in the computation of stiffnesses by series

ICL - Number of iteration cycles before deciding for non-convergence of the process

Joint Reference Loads

I, F(I)

I - Code number of the joint load

F(I) - Value of the joint load

Member Deformation Coding

M, I1, I2,..., I6, J1, J2,..., J6

M - Code number of the member

I's and J's indicate the code numbers of the end deformations of the member determined according to Fig. B-1b.

Member Properties

ES, ZIY, ZIZ, C1, C, RS, AR

ES - Length of the member

ZIY, ZIZ - Moment of inertias of the cross-section about Y and Z axes respectively

C1 - Warping stiffness

C - Pure torsional stiffness

RS - Square of the polar radius of gyration of the cross-section

AR - Area of the cross-section

Axial Force Information

M, AR, ES, I4, J4

M - Code number of the member

AR - Area of the cross-section

ES - Length of the members

I4 - Displacement code number of the near end joint, in the direction of the member

J4 - Displacement code number of the far end joint, in the direction of the member

Output Data

Output data consists of the axial forces in the members in each cycle and deformations of the joints, load factor and the determinant of the stiffness matrix if the iteration converges.

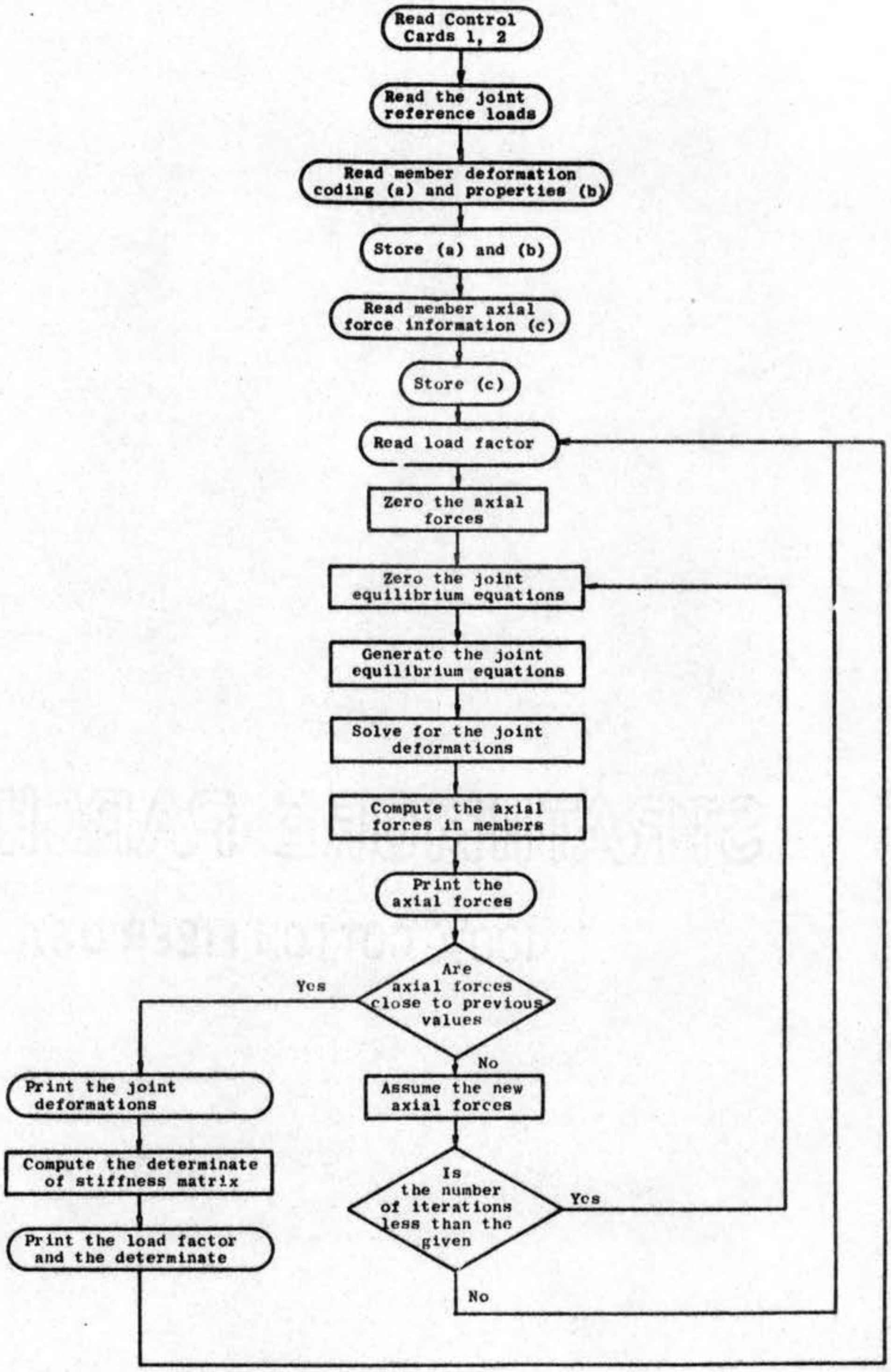

FIG. B.3 MACRO FLOW DIAGRAM OF THE COMPUTER PROGRAM
USING THE CONVERGENCE CRITERION OF BUCKLING

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