CLASSES OF NONCONTINUOUS

TRANSFORMATIONS

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CHAPTER I

INTRODUCTION AND STATEMENT OF THE PROBLEM

INTRODUCTION

Point set topology is one of the newer branches of mathematics, having emerged as a discipline in the early part of the twentieth century. Although topology has not been incorporated in many college undergraduate programs, recent recommendations by the Committee on the Undergraduate Program in Mathematics of the Mathematics Association of America and by other groups indicate that topology may soon become an important part of the undergraduate mathematics program. Also, some of the current experimental programs in secondary and elementary school mathematics indicate that some concepts from topology will be introduced in the secondary schools and even in the elementary schools.

The Committee on the Undergraduate Program in Mathematics was established by the Mathematics Association of America to study the undergraduate program in mathematics and to make recommendations for a sound undergraduate program. The committee was divided into a number of panels in recognition of the fact that the undergraduate program in mathematics must serve a number of student groups with divergent needs. Specifically, the following panels were established:

(1) The Panel on Teacher Training.

(2) The Panel on Mathematics for the Physical Sciences

and Engineering.

(3) The Panel on Mathematics for the Biological, Management and Social Sciences.

(4) The Panel on Pregraduate Training.

Each of these panels issued a report describing a program of studies the members felt would constitute a sound program in mathematics for that panel's area of interest.

The panel on Teacher Training issued a report in December, 1960 [19][1] in which the following minimum requirements for secondary teachers were given.

(A) Three courses in analysis.

(B) Two courses in abstract algebra.

(C) Two courses in geometry beyond analytic geometry.

(D) Two courses in probability and statistics.

(E) Two upper level courses, e.g., introduction to real variables, number theory, topology, history of mathematics, or numerical analysis (including use of high speed computers).

For elementary teachers the following minimum requirements were listed.

(A) A two-course sequence devoted to the structure of the real number system.

(B) A course devoted to the basic concepts of algebra.

(C) A course in informal geometry.

The inclusion of topology in the list of suggested electives for secondary teachers indicates that the committee feels certain concepts from topology will probably emerge in the high school mathematics curriculum.

1. Numbers in brackets refer to references in the bibliography.

The committee also recommended that the course in informal geometry for elementary teachers include the consideration of closed curves and separation properties which are topological concepts.

The Panel on Pregraduate Training issued a report in December, 1964 [26] with recommendations for the undergraduate program. In this report the panel suggested that every college offering a pregraduate program in mathematics should offer a core of basic courses for upper division students. These courses are roughly described as: real analysis, complex analysis, abstract algebra, geometry, topology, and probability or mathematical physics. The panel further suggested that as far as resources will permit, an institution offering pregraduate training should offer courses in algebra, analysis, applied mathematics, foundations and logic, geometry, mathematical statistics, number theory and topology.

From these recommendations, one can conclude that topology shows promise of being important in a sound undergraduate program of the future, and quite possibly will be important in teacher education.

STATEMENT OF THE PROBLEM

In the field of mathematics, as in other academic areas, there is always a gap between the material in current textbooks and material in recent research. This gap often exists because the recent research is usually published in a number of professional journals with little or no unification nor standardization of terminology and symbolism. In point set topology, for example, many of the current textbooks do not discuss noncontinuous transformations extensively; however, much research concerning certain classes of noncontinuous transformations has been published in recent years.

The purpose of this paper is to review and organize the recent research concerning certain classes of noncontinuous transformations in a single paper with standardized notation and symbolism so that this material will be more available and readable for the student of topology.

PROCEDURE

A careful survey and analysis of the literature to locate the published results concerning noncontinuous transformations will be made. The Mathematical Reviews, bibliographies of texts, bibliographies of unpublished theses and the bibliographies of published papers will be used as primary tools for locating source papers. This material will then be presented in a systematic manner relating the results of each source to results of a similar nature in other sources. Most of the proofs given will not be original; however, the proofs given by the various authors will be modified to obtain a standardization of notation and symbolism.

SCOPE AND LIMITATIONS

The published material concerning noncontinuous transformations is quite extensive. Therefore, this paper will be limited to a relatively small number of classes of noncontinuous transformations, so that a more complete discussion can be given for these classes. The classes of non-continuous transformation for which a sufficient volume of material has been published to justify an attempt at correlation will be covered in this paper. Since it is intended that this paper be readable by a student taking a first course in topology, material from algebraic topology and some branches of more advanced mathematics will not be included. Occasional references to and use of more advanced concepts will be made, however, for the purpose of giving a more complete discussion. The use of such materials will be limited to cases in which they will lead to particularly significant results concerning one or more of the classes of transformations covered by this study.

EXPECTED OUTCOMES

It is expected that as a result of reading this paper an individual will become more aware of the current and past research in point set topology. He should also develop an awareness of the continuous changes through which this development has progressed and should anticipate continued change in topology and related disciplines as new materials are developed through research.

It is also expected that the presentation of the published results concerning certain classes of transformations in one source will whet the student's curiosity for future study and help in the identification of areas for such study.

CHAPTER II

BASIC CONCEPTS OF POINT SET TOPOLOGY

INTRODUCTION

Throughout this paper it is assumed that the reader is familiar with the basic notions, notations, definitions, and operations used in point set theory, and has a working knowledge of many of the basic concepts of point set topology. Even if an individual is familiar with many of these basic concepts, however, the differences in definitions and approaches to the theory of topology in various texts may cause him to encounter unnecessary obstacles in reading and interpreting the theorems and the proofs. It is for this reason that some of the basic concepts of point set topology will be introduced in this chapter. In general, the concepts given in this chapter will be concepts which will be used frequently in the remaining chapters. Concepts which will not be used extensively will be presented as needed. Since it is assumed that the reader is familiar with most of the concepts in one form or another, the proofs will not be given for several of the theorems stated in this chapter.

DE MORGAN LAWS

One of the basic theorems from point set theory which will be useful in this paper is DeMorgan's theorem (DeMorgan's laws). This theorem will now be stated and a proof supplied for part (a). This proof is given as an example of a proof by set containment.

Theorem 2.1. (DeMorgan's Theorem). Let β be an index set, S a set, and $\{A_\alpha\}_{\alpha\epsilon\beta}$ a collection of subsets of S indexed by β. Then

(a) $S - \bigcup_{\alpha\epsilon\beta} A_\alpha = \bigcap_{\alpha\epsilon\beta}(S - A_\alpha)$, and

(b) $S - \bigcap_{\alpha\epsilon\beta} A_\alpha = \bigcup_{\alpha\epsilon\beta}(S - A_\alpha)$.

Proof of (a). Let $p \in (S - \bigcup_{\alpha\epsilon\beta} A_\alpha)$. Then $p \notin \bigcup_{\alpha\epsilon\beta} A_\alpha$, hence $p \notin A_\alpha$ for any α. This means $p \in (S - A_\alpha)$ for every α, so that $p \in \bigcap_{\alpha\epsilon\beta}(S - A_\alpha)$, and $(S - \bigcup_{\alpha\epsilon\beta} A_\alpha) \subset \bigcap_{\alpha\epsilon\beta}(S - A_\alpha)$.

On the other hand, if $p \in \bigcap_{\alpha\epsilon\beta}(S - A_\alpha)$, $p \in S$ and $p \notin A_\alpha$ for any α. Thus $p \in S$ and $p \notin \bigcup_{\alpha\epsilon\beta} A_\alpha$. It then follows that $p \in (S - \bigcup_{\alpha\epsilon\beta} A_\alpha)$ so that $\bigcap_{\alpha\epsilon\beta}(S - A_\alpha) \subset (S - \bigcup_{\alpha\epsilon\beta} A_\alpha)$. Since $(S - \bigcup_{\alpha\epsilon\beta} A_\alpha) \subset \bigcap_{\alpha\epsilon\beta}(S - A_\alpha)$, and since $\bigcap_{\alpha\epsilon\beta}(S - A_\alpha) \subset (S - \bigcup_{\alpha\epsilon\beta} A_\alpha)$, we have $(S - \bigcup_{\alpha\epsilon\beta} A_\alpha) = \bigcap_{\alpha\epsilon\beta}(S - A_\alpha)$.

The proof of (b) follows in an analogous manner.

BASIC TERMS OF POINT SET TOPOLOGY

The definition of a topological space varies somewhat from text to text. The following definition of a topological space will be used in this paper.

Definition 2.1. A set S, together with a collection of subsets called open sets, is called a <u>topological space</u> if and only if the collection of open sets satisfy the following three properties:

(1) S and $\emptyset$ are open sets,

(2) the union of any collection of open sets is an open set,

(3) the intersection of any finite collection of open sets is open.

The collection of open sets is called the topology of S.

Example 2.1. Let S = {a,b,c} and let the open sets of S be the following: ∅, {a}, {a} ∪ {b}, {a} ∪ {c}, S. One can easily verify that S, with the collection of open sets listed, is a topological space.

Example 2.2. Let S be the set of real numbers, and let a and b be any two elements of S with a < b. Define the open interval (a,b) by (a,b) = {x | a < x < b}. Let a subset U of S be an open set if and only if U is the union of a collection of open intervals in S. Set S with the open sets just defined is a topological space.

The topology of S defined in Example 2.2 is called the usual topology for S.

Given a set S, several topologies can be defined for S. The next example gives another topology that can be defined for the set of real numbers.

Example 2.3. Let S be the set of real numbers, and let a subset U of S be an open set if and only if the complement of U in S is finite. Set S with the collection of open sets thus defined is a topological space.

In Example 2.1 all of the open subsets of space S were listed. We can frequently avoid listing all of the open sets by defining a subcollection of the open subset of S which "generates" the entire collection of open sets of S. Such a subcollection of the open sets is called a basis of S. Let us now give a formal definition of this concept.

Definition 2.2. Let S be a set and let σ and ω be two collections of subsets of S. Then the collection σ is said to generate the collection ω if and only if, for every $K \in \omega$, K is the union of a collection of elements of σ. Collection σ is said to form a basis for ω.

In Example 2.2 the topology of S was defined by first defining the collection of open intervals in S as a basis for the open sets.

For a given topological space, it may be possible to define more than one basis for that topological space. The following discussion will illustrate this fact.

Example 2.4. Let S be the Euclidean plane, let x be an element of S, and let $\epsilon > 0$ be a real number. Define a spherical neighborhood of radius ϵ about x, $(S_\epsilon(x))$, to be the set of all $y \in S$ such that $d(x,y) < \epsilon$ where $d(x,y)$ means the distance from x to y. The real number ϵ is called the radius of $S_\epsilon(x)$. The collection of all spherical neighborhoods about all points of S can be used as a basis for a topology of S. The collection of spherical neighborhoods about all points of S with rational radii generates the same topology for S.

Closely associated with the open sets of a topological space is the collection of subsets called closed sets.

Definition 2.3. A subset K of a topological S is said to be closed if and only if S - K is open.

A subset H of a space S may be open, closed or both open and closed.

It is also possible for a subset H of a space S to be neither open nor closed.

Example 2.5. Let $S = \{a,b,c\}$ with every subset of S open. Then every subset of S is also closed.

Example 2.6. Let S be the set of all real numbers with the usual topology and let a, $b \in S$ such that $a < b$. The set $[a,b) = \{x \mid a \leq x < b\}$ is neither open nor closed.

Definition 2.4. If S is a topological space and if $x \in S$ then U is said to be a <u>neighborhood of x</u> if and only if U is an open set containing x.

When working with open and/or closed sets, it is frequently useful to work with characterizations of these sets other than the definitions. The next two theorems give characterizations of these sets which are often used.

Theorem 2.2. A subset G of a space S is open if and only if, for each point $p \in G$, there exists a neighborhood U_p of p contained in G.

Proof: If G is open, then for each point $p \in G$, G is a neighborhood of p such that G is contained in G.

On the other hand, if for each $p \in G$ there exists a neighborhood U_p of p contained in G, then $G = \bigcup_{p \in G} U_p$. Hence G is open as the union of open sets.

Definition 2.5. Let H be a subset of space S. Then p is a <u>limit point of H</u> if and only if every neighborhood U of p contains at least one point q of H such that $q \neq p$.

Theorem 2.3. A subset H of a space S is closed if and only if H contains all of its limit points.

Proof: Let H be a closed set and let x be a limit point of H. Assume $x \notin H$. Then $x \in (S - H)$ which is an open set. By Theorem 2.2 there exists a neighborhood U of x such that U is contained in (S - H). This contradicts the fact that x is a limit point of H, hence we must conclude $x \in H$.

Now suppose H is a subset of S such that H contains all of its limit points. Consider (S - H). For any $x \in (S - H)$, x is not a limit point of H, since $x \notin H$. Thus, there exists some neighborhood U of x such that U is contained in (S - H). By Theorem 2.2, (S - H) is open. Hence $H = S - (S - H)$ is closed.

The first portion of the proof of Theorem 2.3 is an example of a proof by contradiction. This technique will be used frequently throughout this paper.

Definition 2.6. If K is a subset of S, the <u>closure of K</u> is the union of set K with all of its limit point. The closure of K is denoted by $\overline{K}$.

A concept closely associated with the concept of a limit point is the concept of a boundary point.

Definition 2.7. A point p is a <u>boundary point</u> of a subset H of a space S if and only if every neighborhood U of p contains at least one point of H and at least one point not in H.

Given a topological space S and a subset K of S, we can form a new topological space using K as the set of points for the new space and the open sets of S intersected with K as the open sets in K. The following definition will give a formal characterization of this space.

Definition 2.8. Let S be a topological space, let ω be the collection of open sets in S, and let K be a subset of S. Then the set K with collection $\{\omega \cap K\}$, where $\{\omega \cap K\}$ denotes the collection of sets of the form $H \cap K$, $H \in \omega$, is a topological space. Such a space is said to be a <u>subspace of S</u>.

One can easily verify that the collection $\{\omega \cap K\}$ described in Definition 2.8 satisfies the three conditions for a topology. One can also verify that a basis for S will generate a basis for space K.

Example 2.7. Let S be the space of real numbers with the usual topology and let $T = [0,1] = \{x \mid 0 \leq x \leq 1\}$. Let the topology of T be the topology of S intersected with set T. It is interesting to note that sets of the form $[0,y) = \{x \mid 0 \leq x < y,\ y \leq 1\}$ are open in T although they are neither open nor closed in S.

Given two topological spaces, one can describe a new topological space using the cartesian product. Let us now give a formal definition for this space which will be used frequently in this paper.

Definition 2.9. Let S and T be sets. The set $S \times T = \{(x,y) \mid x \in S,\ y \in T\}$ is called the <u>cartesian product</u> of sets S and T.

Definition 2.10. Let S and T be topological spaces. The set $S \times T$ with $\{U \times V\}$, where $\{U \times V\} = \{(U \times V) \mid U \text{ open in } S, V \text{ open in } T\}$, as the topology, is called the topological product of S and T.

One can easily verify that $\{U \times V\}$ satisfies the three conditions for a topology.

SEQUENCES

When a topology is placed on a set S, certain subsets of S take on significant properties. For example, certain subsets become open or closed sets. A subset A of S can also take on significant properties if it is indexed by the set of positive integers. Sets indexed by the positive integers will be used frequently in this paper. Therefore, a formal definition of this concept will be stated and certain basic theorems concerning this concept will be given.

Definition 2.11. A sequence is a set A indexed by the set I of all positive integers. The nth element of the sequence is the element a which is indexed by the integer n. The nth element is denoted by a_n and the sequence is symbolized by $\{a_n\}$, where $A = \bigcup_{n \in I} a_n$.

It is important to note that a sequence is not just a set of points, but is a set of points indexed by the positive integers. The significance of this fact is that the same set indexed in two different ways gives rise to two different sequences.

Definition 2.12. Let S be a topological space and let $\{a_n\}$ be a sequence of points in S. Then $\{a_n\}$ is said to converge to the point p of S if and only if, given any neighborhood U of p, there exists a positive integer N such that $a_n \in U$ for all $n > N$. If there exists a point $p \in S$ such that $\{a_n\}$ converges to the point p then $\{a_n\}$ is said to be a convergent sequence. If $\{a_n\}$ converges to a point $p \in S$ then we say limit $a_n = p$. Point p is called the sequential limit point of $\{a_n\}$.

Definition 2.13. The sequence $\{n_i\}$ is a subsequence of the sequence of positive integers if and only if the following conditions hold:

(1) Each n_i is a positive integer, and

(2) For each positive integer i, $n_i < n_{i+1}$.

From condition (2) it follows at once that $n_i \geq i$ for every $i \in I$.

Definition 2.14. The sequence $\{b_n\}$ is a subsequence of $\{a_n\}$ if and only if there exists a subsequence $\{n_i\}$ of positive integers such that $b_i = a_{n_i}$ for every i.

Theorem 2.4. If point p is a sequential limit point of the sequence $\{a_n\}$, and if $\{b_n\}$ is a subsequence of $\{a_n\}$, then p is a sequential limit point of $\{b_n\}$.

Proof: Let p be a sequential limit point of $\{a_n\}$ and let U be any neighborhood of p. Then there exists a positive integer N such that $a_n \in U$ for any $n > N$. Consider b_i where $i > N$. Since $b_i = a_{n_i}$, and since $n_i \geq i > N$, $b_i \in U$. Thus, p is a limit point of $\{b_n\}$.

CLASSES OF TOPOLOGICAL SPACES

In the remaining chapters of this paper, the topological spaces used will often have properties not common to all topological spaces. In the following discussion some of the important classes of topological spaces will be defined.

Definition 2.15. A topological space S is said to be a <u>Hausdorff space</u> if and only if, given any two distinct points p, q of S, there exists disjoint open subsets U and V of S such that $p \in U$, $q \in V$.

The next theorem is an example of a theorem which is true for a topological space with a particular property, but is not true for topological spaces in general. The particular property in this case is that the space be Hausdorff. This theorem also gives the second characteristic of a sequence which will be used extensively in the remainder of this paper.

Theorem 2.5. Let $\{a_n\}$ be a sequence of points of a Hausdorff space S. If this sequence converges to a point $p \in S$ and also to a point $q \in S$, then $p = q$.

Proof: Suppose $p \neq q$. Since S is a Hausdorff space, there exists open sets U and V containing p and q, respectively, such that $U \cap V = \emptyset$. Since $\{a_n\}$ converges to both p and q, there exists integers N_1 and N_2 such that $n > N_1$ implies $a_n \in U$ and $n > N_2$ implies $a_n \in V$. For $n >$ maximum $\{N_1, N_2\}$ we have $a_n \in U \cap V$. This is a contradiction, hence we must conclude $p = q$.

The next example shows that this theorem is not true for topological spaces in general.

Example 2.8. Let $S = [0,1]$ and let U be an open set in $[0,1]$ if and only if $U = [0,1]$, $U = \emptyset$ or $[0,1] - U$ is finite. Let $\{a_n\} = \{1/n\}$ for $n = 1, 2, 3, \cdots$. Sequence $\{a_n\}$ converges to every point $p \in S$ with this topology, for any open set U that contains $p \in S$ will contain all except possibly a finite number of points of $\{a_n\}$. Thus if all points of $\{a_n\}$ belong to U, let $N = 1$, and if all but a finite number of points of $\{a_n\}$ belongs to U, let N be the maximum index of the elements of $\{a_n\}$ not contained in U.

Definition 2.16. A space S is said to be <u>regular</u> if and only if, given any closed subset F of S and any point p of S not in F, there exists disjoint open setsU and V in S containing F and p, respectively.

Definition 2.17. A space S is said to be <u>normal</u> if and only if, given any two disjoint closed subsets F_1 and F_2 of S there exists disjoint open sets U and V containing F_1 and F_2, respectively.

Many important topological spaces such as the real numbers with the usual topology and Euclidean n-spaces are Hausdorff, regular and normal. The following example illustrates the fact that an arbitrary topological space need not posses any of these properties.

Example 2.9. Let $S = [0,1] = \{x \mid x \text{ is a real number and } 0 \leq x \leq 1\}$, and let U be an open set if and only if $U = \emptyset$, or the complement of U is finite. One can easily verify that set S with this topology is a topological space. Space S is neither Hausdorff, regular, nor normal.

Before defining the next class of topological spaces it will be necessary to define mutually separated sets. This concept will be important in the remainder of this paper as well as being useful in defining completely normal spaces.

Definition 2.18. Two subsets A and B of space T are said to be <u>mutually separated</u> if and only if $A \neq \emptyset$, $B \neq \emptyset$, $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.

Definition 2.19. A space S is said to be <u>completely normal</u> if and only if, given any two mutually separated subsets A and B of S, there exists disjoint open sets U and V containing A and B, respectively.

For an example of a space that is normal but not completely normal see [8,191].

When working with a space such as the real numbers with the usual topology, one often uses the fact that a set consisting of a single point is closed. A space having this property is said to be a T_1 space.

Definition 2.20. A space S is said to be a <u>T_1 space</u> if and only if every point p in S is a closed subset of S.

The spaces described in Examples 2.8 and 2.9 are T_1. The space defined in Example 2.1 is not T_1 since $\{a\}$ is not closed.

Some preliminary definitions will now be given in preparation for the definition of first and second countable spaces.

Definition 2.21. Two sets X and Y are said to be in a <u>one-to-one correspondence</u> if the elements of X and Y can be paired in such a way that distinct elements in X are paired with distinct elements in Y, and distinct elements in Y are paired with distinct elements in X.

Definition 2.22. A set A is said to be countable if A can be placed into a one-to-one correspondence with set $I = \{1,2,3,4,\ldots\}$ or with any subset of I.

Definition 2.23. A collection σ of neighborhoods of a point p in a space S is said to be a basis at p if and only if, given any neighborhood U of p in S, there exists a $V \in \sigma$ such that $V \subset U$.

Definition 2.24. A space S is said to be first countable if and only if, for any point p in S, there exists a countable basis at p.

Definition 2.25. A space S is said to be second countable if and only if there exists a countable basis for S.

From the definitions of first and second countable spaces, it follows that a second countable space is first countable. The next example shows that the converse is not true.

Example 2.10. Let S be the set of real numbers with the discrete topology. That is, each point of S is an open set. For each point x in S the open set $\{x\}$ constitutes a countable basis at x. Since set S is not countable, space S is not second countable.

One of the most important classes of topological spacesis the class of metric spaces. This class of spaces will be defined in Definition 2.29. Some preliminary definitions will be presented first.

Definition 2.26. A rule f is called a Transformation of a set S into a set T if and only if f associates with each element x in S a unique element y of T. This association is usually symbolized by

$f(x) = y$. Set S is called the <u>domain</u> of f and T is called the <u>range</u> of f.

The words mapping and function will be used as synonyms for transformation.

Definition 2.27. A set S is said to be a <u>metric set</u> if and only if there is associated with S a mapping ρ from $S \times S$ into R, where R is the space of all real numbers, having the following properties for every triple x, y, z of elements in S.

(1) $\rho(x,y) \geq 0$, and $\rho(x,y) = 0$ if and only if $x = y$,

(2) $\rho(x,y) = \rho(y,x)$, and

(3) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$.

This mapping is called the <u>metric for S</u>.

Example 2.11. For the set R of all real numbers, the usual metric function is $\rho(x,y) = |x - y|$. For the Euclidean plane the usual metric function is $\rho(x,y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, where $x = (x_1,y_1)$, $y = (x_2,y_2)$.

Intimately associated with a metric set S are the subsets of S known as spherical neighborhoods.

Definition 2.28. Let K be a metric set. Then with each point $p \in S$ and each real number r, we associate a subset $S_r(p)$ called a <u>spherical neighborhood of radius r about p</u>. A point q of K is an element of $S_r(p)$ if and only if $\rho(p,q) < r$.

Definition 2.29. A <u>metric space</u> is a metric set with the collection ω of all spherical neighborhoods in S as a basis for its topology.

For a proof that a metric set S with the collection ω as its basis is a topological space, see [8,60].

Many relationships exist between and among the various classes of topological spaces that have been defined above. The following collection of theorems are examples of such relationships which will be used in this paper. Proofs will not be given, since these theorems are stated and proven in most elementary texts on point set topology.

Theorem 2.6. Every Hausdorff space is a T_1 space [8,64].

Theorem 2.7. Every regular T_1 space is Hausdorff [8,111].

Theorem 2.8. Every regular second countable Hausdorff space is completely normal [8,111].

Theorem 2.9. Every completely normal space is normal [8,110].

Theorem 2.10. Every second countable space is first countable [8,107].

Theorem 2.11. Every metric space is Hausdorff [8,61].

Other theorems stating relationships that exist between and among the spaces in the various classes defined above will be stated and references given as needed in the remaining chapters.

It should be noted that the space R of real numbers with the usual topology is a metric space, a Hausdorff space, a T_1 space, a first countable space, a second countable space, a normal space, a completely normal space and a regular space. As a consequence many of the theorems presented in this paper can be stated as theorems for function defined on the space of real numbers.

Two other significant properties that a topological space and/or a subset of a topological space may have are compactness and connectedness.

Definition 2.30. A collection of sets $\{D_\alpha\}_{\alpha \in \omega}$ is said to be a covering of a set A if and only if $A \subset \bigcup_{\alpha \in \omega} \{D_\alpha\}$. If, in addition, each of the sets D_α is an open set, then the collection $\{D_\alpha\}$ is said to be an open covering of A. Any subcollection of $\{D_\alpha\}$ covering A is said to be a subcovering of A.

Definition 2.31. Let A be a subset of a space S. Then A is said to be compact if and only if every open covering of A contains a finite subcovering of A.

Closely associated with compact subsets of a space S is a class of subsets called the countably compact subsets of S. The definition for a countably compact subset of a topological space will now be presented. The relationships between compact and countably compact subsets will be given in Chapter 4.

Definition 2.32. A subset A of a space S is said to be countably compact if and only if every infinite subset of A has at least one limit point in A.

Definition 2.33. Let A be a topological space or a subset of a topological space. Then A is said to be connected if and only if A can not be expressed as the union of two mutually separated sets.

TRANSFORMATIONS

The definition of a transformation f from a set S into a set T was given in Definition 2.26. When working with a particular problem one usually requires that a mapping from S into T satisfies additional conditions. One might, for instance, require that the mapping f be one-to-one, onto, or continuous. Let us now define these concepts and other concepts associated with functions which will be used frequently in this paper.

Definition 2.34. Let S and T be sets and let f from S into T be a mapping. Then for any subset A of S, we define $f(A) = \bigcup_{x \in A} f(x)$. The subset f(A) of T is called the *image of A* under f.

Definition 2.35. Let S and T be sets, let f from S into T be a mapping, and let B be a subset of T. We define $f^{-1}(B) = \bigcup_{x \in S, f(x) \in B} x$. The subset $f^{-1}(B)$ of S is called the *source* or *inverse of B*.

Example 2.12. Let S = T be the set of real numbers and let A = {1,2}, B = {1,4}, where A is a subset of S and B is a subset of T. Define f from S into T by $f(x) = x^2$. Then f(A) = B and $f^{-1}(B) = \{+1, -1, +2, -2\}$. Note that $f^{-1}f(A) \neq A$.

Definition 2.36. Let S and T be sets. A mapping f from S into T is *one-to-one* if and only if, for every $y \in f(S)$, $f^{-1}(y)$ is a single point.

Definition 2.37. Let S and T be sets. A mapping f from S into T is an *onto* mapping if and only if f(S) = T.

A mapping from a space S into a space T can be modified in several ways. We may change f by changing the rule of association, or we may change the set S on which f is defined. A situation that will frequently arise in this paper is that we will want to consider f defined on a subset A of S. This modification of f is known as the restriction of f to A.

Definition 2.38. Let S and T be sets, let f from S into T be a mapping, and let A be a subset of S. A mapping g from A into T is said to be the <u>restriction of f to A</u> if and only if $f(x) = g(x)$ for every $x \in A$. The restriction of f to A is frequently denoted by $f \mid A$.

Example 2.13. Let S, T and f be defined as in Example 2.12, and let $A = \{x \mid x \in S \text{ and } x \geq 0\}$. $f \mid A$ is a one-to-one mapping, but f defined on S is not. One frequently uses a restriction of a mapping to obtain some such desirable property.

The properties of functions defined thus far have been properties of functions from a point set into a point set. If in addition S and T are topological spaces, certain other properties for functions can be defined. One of the most fundamental of the properties for functions is continuity.

Definition 2.39. A mapping f from a space S into a space T is said to be <u>continuous at a point p</u> in S if and only if, for every open set U in T containing $f(p)$, there exists an open set V in S containing p such that $f(V) \subset U$.

Definition 2.40. A mapping f from a space S into a space T is said to be continuous if and only if f is continuous at every point p in S.

Other properties of functions from one topological space into another will be defined in the following chapters as needed.

CHAPTER III

OPEN AND CLOSED TRANSFORMATIONS

INTRODUCTION

Given two topological spaces S and T, one is often concerned with whether or not S and T have similar structures with similarity of structures defined in terms of mappings. In Chapter II, continuous, one-to-one, and onto functions were defined. Each of these conditions is a strong condition to place on a function; however, even if a function which is continuous, one-to-one, and onto can be defined from S onto T, spaces S and T may have very dissimilar properties as is shown by the following example.

Example 3.1. Let S be the unit interval $0 \leq x \leq 1$ with the discrete topology. That is, let every point of S be an open set. Let T be the unit interval $0 \leq x \leq 1$ with the usual topology and define a mapping f from S onto T by:

$$f(x) = x \text{ for every } x \in S.$$

This function is easily seen to be continuous, one-to-one and onto. However, it is quite obvious that spaces S and T are dissimilar, since the respectively topologies are of a different nature. In fact, many of the open sets in T are not open in S.

If one requires, in addition to the three conditions previously mentioned, that a function map open sets of S onto open sets of T, one will find that S and T have many properties in common. In particular, it will be implied in Theorem 3.14 that the open sets of S and T will be placed into a one-to-one correspondence. Formal definitions for these concepts will now be presented.

Definition 3.1. Let f be a mapping from a space S into a space T. The mapping f is said to be <u>open</u> if and only if the image of every open set in S is open in T.

Definition 3.2. A mapping f from a space S onto a space T is said to be a <u>homeomorphism</u> if and only if f is one-to-one, onto, open, and continuous. The spaces S and T are said to be <u>homeomorphic</u>.

The word homeomorphism is derived from Greek and means of a similar form or structure. The following facts suggest that the term homeomorphism is appropriate. If spaces S and T are homeomorphic and if S is a metric space, a Hausdorff space, a normal space, a completely normal space, a first countable space, a second countable space, or a T_1 space, then T is of the same type. The open subsets of S and T will be in a one-to-one correspondence as will the closed subsets, the connected subsets and the compact subsets. This is by no means a complete listing of the properties S and T will have in common, however, a complete discussion of the properties of homeomorphisms is not the purpose of this paper. Proofs of the above statements and further properties of homeomorphisms can be found in [8].

Closely associated with the class of open mappings is the class of mappings which map the closed subsets of the domain space onto closed subsets of the range space.

Definition 3.3. A mapping f from a space S into a space T is said to be <u>closed</u> if and only if the image of every closed subset of S is a closed subset of T.

It will be shown in Theorem 3.10 that a homeomorphism could also be defined as a one-to-one, onto, closed, continuous mapping from a space S onto a space T.

Open and closed functions have been introduced here as functions possessing one of the properties of a homeomorphism. The purpose of this chapter is to give a systematic presentation of other interesting properties these functions are known to possess. The following discusion will show that a function may be open, closed or continuous without possessing either of the other two properties.

The function f defined in Example 3.1 is continuous, however, it is neither open nor closed. To verify that f is not open, consider a point p in S. The set $\{p\}$ is open in S, but $\{f(p)\} = \{p\}$ is not open in T, hence f is not open. Any subset of S is closed, including sets of the form $a < x < b$, where $0 \leq a < b \leq 1$. The images of such subsets are not closed in T, hence f is not a closed mapping. The following examples show that it is possible to define functions which are open but not closed or continuous, and functions which are closed but not open or continuous.

Example 3.2. Let S and T be the closed intervals $[0,2]$ and $[0,1]$, respectively, each with the usual topology. Define a function f from S into T by:

$$f(x) = x, \text{ if } 0 \leq x < 1, \text{ and}$$

$$f(x) = x - 1, \text{ if } 1 \leq x \leq 2.$$

The function f is open, but is neither closed nor continuous.

Example 3.3. Let S = T be the open interval (0,1) with the usual topology. Define f from S into T as follows.

$$f(x) = 1/4, \text{ if } x \text{ is irrational},$$

$$f(x) = 3/4, \text{ if } x \text{ is rational}.$$

This function maps all subsets of S onto one of the closed subsets $\{1/4\}$, $\{3/4\}$ or $\{1/4, 3/4\}$ of T, and is thus closed. The function f is neither open nor continuous.

One can also construct examples of functions possessing any two of these three properties but not the third.

CHARACTERIZATIONS

The following discussion gives characterizations of open functions and closed functions. Necessary definitions and preliminary theorems will be stated as needed in the development of these characterizations.

Theorem 3.1. A function f from a space S into a space T is closed if and only if $f(\overline{R}) \supset \overline{f(R)}$ where R is any subset of S.

Proof. Assume f is closed and let R be any subset of S. Since $\overline{R}$ is closed, and since f is a closed mapping, $f(\overline{R})$ is a closed subset of T. Thus $f(\overline{R}) = \overline{f(\overline{R})}$. Now $R \subset \overline{R}$, so that $f(R) \subset f(\overline{R})$. It now follows that $\overline{f(R)} \subset \overline{f(\overline{R})} = f(\overline{R})$.

Now assume $\overline{f(R)} \subset f(\overline{R})$ for any subset R of S, and let H be any closed subset of S. Since H is closed, $H = \overline{H}$ so that $f(H) = f(\overline{H})$. By hypothesis $\overline{f(H)} \subset f(\overline{H}) = f(H)$, which implies f(H) is closed. Thus f is a closed mapping.

Definition 3.4. The <u>interior of a set A</u> (Int A) is the union of all open sets contained in A.

The following theorem follows immediately from the definition of Int A.

Theorem 3.2. Let A be any subset of a space S, then,

(i) Int $A \subset A$,

(ii) if $A \subset$ Int A, then A is open, and

(iii) the set A is open if and only if Int A = A.

Theorem 3.3. A function f from a space S into a space T is open if and only if f(Int A) $\subset$ Int f(A) for any subset A of S.

Proof. Assume f is open and let A be any subset of S. Since Int A is open by Theorem 3.2, and since f is open, f(Int A) is open. Furthermore, f(Int A) $\subset$ f(A), since Int $A \subset A$. Thus f(Int A) is an open subset of f(A), so that f(Int A) $\subset$ Int f(A) by definition.

Now assume f(Int A) $\subset$ Int f(A) for any subset A of S, and let G be any open subset of S. By Theorem 3.2, G = Int G, so that f(G) = f(Int G). From the hypothesis, f(Int G) $\subset$ Int f(G), which implies f(G) $\subset$ Int f(G). It now follows from Theorem 3.2 (ii) that f(G) is open, so that f is an open function.

A second characterization of open transformations can be expressed in terms of the limit inferior of a sequence of subsets of a space S.

Definition 3.5. If $\{X_n\}$ is a sequence of subsets of a space S, then the <u>limit inferior</u> (lim inf) $\{X_n\}$ is the set of all x such that, for each neighborhood U of x, U contains points from all but a finite number of the sets in $\{X_n\}$.

The following theorem will be used in the proof of the next characterization theorem.

Theorem 3.4. If S is a first countable metric space and if x is a limit point of a subset A of S, then there exists a sequence $\{X_n\}$ of distinct points of A that converges to x. [8,102].

Theorem 3.5. Let S and T be topological spaces and let $f(S) = T$ be an open transformation. Then for every convergent sequence $\{y_n\}$ in T, the relationship $f^{-1}(y) \subset \lim\inf \{f^{-1}(y_n)\}$ holds, where $y = \lim y_n$. If in addition, T is a first countable metric space, the converse is true.

Proof. Let f be an open transformation $f(S) = T$ and let $\{y_n\}$ be a convergent sequence of points in T with $y = \lim y_n$. Suppose $x \in f^{-1}(y)$ and that U is an open set containing x. Since f is an open transformation and $x \in f^{-1}(y)$, it follows that $f(U)$ is an open set in T containing y. Now $y = \lim y_n$ implies there exists some positive integer N such that $y_n \in f(U)$ for all $n \geq N$. Point y_n in $f(U)$ for all $n \geq N$ implies there exists an x_n in U such that $f(x_n) = y_n$ for all $n \geq N$. Therefore, for all $n \geq N$, $U \cap f^{-1}(y_n) \neq \emptyset$ so that x is an element of lim inf $\{f^{-1}(y_n)\}$ and hence $f^{-1}(y) \subset \lim\inf \{f^{-1}(y_n)\}$.

Now let T be a first countable metric space with the property that for every convergent sequence $\{y_n\}$ in T with $y = \lim y_n$, $f^{-1}(y) \subset \lim\inf f^{-1}(y_n)$. Assume f is not an open mapping. Then there must exist some open set U in S such that f(U) is not open in T. Now f(U) not open in T implies there exists some y in f(U) such that y is a limit point of T - f(U). Since T is a first countable metric space, there must exist, by Theorem 3.4, a sequence of distinct points $\{y_n\}$ in T - f(U) such that $y = \lim y_n$. Point y is an element of f(U), so that there must exist some x in U such that f(x) = y. Now y_n is not an element of f(U) for any n, so that $f^{-1}(y_n) \cap U = \emptyset$ for every n. By hypothesis, however, x is an element of $\lim\inf \{f^{-1}(y_n)\}$ which implies U must contain points from all but a finite number of the sets $\{f^{-1}(y_n)\}$. This is a contradiction, hence f must be an open mapping.

Theorems 3.3 and 3.5 give characterizations of open transformations which are stated in terms of the interior of a set and the limit inferior of a sequence of subsets of the domain space. A third characterization theorem for open transformations and a second characterization theorem for closed transformations can be stated in terms of inverse sets.

Definition 3.6. Given space S and T and a mapping f from S into T, a subset Q of S said to be an <u>inverse set</u> if and only if $f^{-1}(f(Q)) = Q$.

Definition 3.7. If A is a subset of a space S, then a subset H of A is said to be <u>closed with respect to A</u> if and only if H contains all of its limit points which belong to A.

Theorem 3.6. [17] A transformation f from a space S into a space T is closed if and only if f is closed on every inverse set Q of S.

Proof. Let f be a closed transformation from S into T, and let Q be an inverse set in S. Let H be a subset of Q that is closed with respect to Q. Then $H = \bar{\bar{H}} \cap Q$. By hypothesis, $f(\bar{\bar{H}})$ is closed in T. Now $f(H) = f(\bar{\bar{H}} \cap Q) = f(\bar{\bar{H}}) \cap f(Q)$ [27,146]. Since $f(\bar{\bar{H}})$ is closed, $f(\bar{\bar{H}}) = \overline{f(\bar{\bar{H}})}$, so that $f(H) = \overline{f(\bar{\bar{H}})} \cap f(Q)$. By Theorem 3.1, $\overline{f(H)} \subset f(\bar{\bar{H}}) = \overline{f(\bar{\bar{H}})}$ so that $(\overline{f(H)} \cap f(Q)) \subset (\overline{f(\bar{\bar{H}})} \cap f(Q)) = f(H)$. On the other hand, $f(H) \subset (\overline{f(H)} \cap f(Q))$. For let y be an element of f(H). Set $f^{-1}(y) \subset Q$, since Q is an inverse set, hence y is in $(f(H) \cap f(Q)) \subset (\overline{f(H)} \cap f(Q))$. Thus $f(H) = \overline{f(H)} \cap f(Q)$ which implies f(H) is closed with respect to f(Q). Therefore, f is closed with respect to Q.

On the other hand, if f is closed with respect to every inverse set in S, then, in particular, f is closed with respect to $S = f^{-1}(f(S))$. Thus f is a closed mapping.

Definition 3.8. If A is a subset of a space S, and H is a subset of A, then H is said to be <u>open with respect to A</u> if and only if $H = U \cap A$ for some open set U in S.

Theorem 3.7. [17] A transformation f from a space S into a space T is open if and only if f is open with respect to every inverse set in S.

Proof. Let Q be an inverse set in S and let f be an open transformation on S. For a set H which is open with respect to Q, $H = U \cap Q$ for some open set U in S. Thus $f(H) = f(U \cap Q) = (f(U) \cap f(Q))$ [27,146].

Since f is an open mapping, f(U) is open in T and hence f(H) is open with respect to f(Q).

Conversely, if a mapping f from S into T is open with respect to every inverse set in S, then f is open on $S = f^{-1}(f(S))$.

SOME GENERAL PROPERTIES OF CLOSED MAPPINGS

Let us turn now to a consideration of some general properties of closed mappings. The first two theorems in this section make use of a class of subsets called conditionally compact subsets.

Definition 3.9. A subset A of a topological space S is said to be <u>conditionally</u> <u>compact</u> if and only if every infinite subset of A has a limit point which belongs to S.

Theorem 3.8. [17] If the transformation f from a space S onto a space T is closed, and if F is any conditionally compact subset of T, then there exists a conditionally compact subset H of S such that f(H) = F.

Proof. Let H be any subset of $f^{-1}(F)$ such that f is one-to-one from H onto F. Assume H is not conditionally compact. Then there exists an infinite subset A of H such that A has no limit point in S. Now set A and all subsets of A are vacuously closed since A has no limit points. Set f(A) has infinitely many points in F, and since F is conditionally compact, f(A) must have a limit point t in F. Furthermore, f(A) is a closed subset of T as the image of a closed subset under a closed mapping, so that t must be an element of f(A). Let $s = f^{-1}(t) \cap A$ and consider set $A - \{s\}$. Set $A - \{s\}$ is closed in S, hence

$f(A - \{s\}) = f(A) - \{t\}$ must be closed in T. But the set $f(A) - \{t\}$ has the limit point t. This is a contradiction, hence H must be conditionally compact.

Theorem 3.9. [17] If function $f(S) = T$ is a closed mapping from a space S onto a countably compact space T, and if $f^{-1}(y)$ is conditionally compact for each y in T, then S is countably compact.

Proof. Assume S is not countably compact. Then there exists an infinite sequence $\{a_n\}$ of points in S such that no point of S is a limit point of $\{a_n\}$. Now $\{a_n\}$ must contain point from only a finite number $f^{-1}(y)$, y in T. Otherwise, $\{a_n\}$ would contain an infinite number of points from at least one $f^{-1}(y)$, and hence, by hypothesis, $\{a_n\}$ would have a limit point in S. Therefore, the image set $\{f(a_n)\}$ must be an infinite subset of T.

Since T is countably compact, $\{f(a_n)\}$ must have a limit point t in T which belongs to $\{f(a_n)\}$. Now consider the set $\left\{\{a_n\} - (f^{-1}(t) \cap \{a_n\}\right\}$ which is vacuously closed in S. The set $f(\{a_n\} - (f^{-1}(t) \cap \{a_n\}))$ must be a closed subset of T. But this set does not contain the limit point t, which gives a contradiction. Thus S must be countably compact.

TRANSFORMATIONS THAT ARE BOTH OPEN AND CLOSED

In examples 3.1, 3.2, and 3.3 it was shown that a function might be open without being closed or continuous, a function might be closed without being open or continuous, and that a function could be continuous without being open or closed. These facts naturally lead one to inquire what conditions must be placed on a function having one of these properties to insure that function will have one or both of the other properties. The following discussion is concerned with conditions that imply a function is both open and closed.

Theorem 3.10. If F(S) = T is one-to-one then f is open if and only if f is closed.

Proof. Suppose f(S) = T is one-to-one and open, and let H be any closed subset of S. Set (S - H) is an open set in S, and since f is an open mapping, f(S - H) is open in T. Since f is one-to-one and onto, f(H) must equal T - f(S - H), which is closed as the complement of an open subset of T. Thus the image of a closed subset of S is closed in T, and f is a closed mapping.

The proof that a one-to-one closed mapping is open follows in an analogous manner.

Theorem 3.10 implies that a homeomorphism could be defined as a one-to-one, onto, continuous closed mapping from one space into another.

For a sequence $\{X_n\}$ of subsets of a space S, the definition of the limit inferior of $\{X_n\}$ was given in Definition 3.5. A related subset associated with the sequence $\{X_n\}$ is the limit superior $\{X_n\}$. This concept will be defined and some preliminary results will be demonstrated in preparation for the next theorem concerning open and closed mappings.

Definition 3.10. If $\{X_n\}$ is a sequence of subsetsof a space S, the <u>limit superior</u> (lim sup) of $\{X_n\}$ is the set of all x such that for each neighborhood U of x, U contains points from infinitely many of the sets in $\{X_n\}$.

Example 3.4. For the sequence $\{X_n\}$ where each X_n is the set $\{(-1)^n + 1/n\}$, both ± 1 are elements of the lim sup $\{X_n\}$, but neither is an element of lim inf $\{X_n\}$. Thus lim sup $\{X_n\}$ does not necessarily equal lim inf $\{X_n\}$.

Lemma 3.1. For any sequence $\{X_n\}$ of subsets of a topological space S, lim inf $\{X_n\} \subset$ lim sup $\{X_n\}$.

Proof. The proof follows immediately from the definitions.

Theorem 3.11. For any sequence $\{X_n\}$ of subsets of a topological space, both lim inf $\{X_n\}$ and lim sup $\{X_n\}$ are closed.

Proof. Suppose x is a limit point of lim inf $\{X_n\}$. Then, any neighborhood U of x contains a point y of lim inf $\{X_n\}$. Since U is an open set containing y, and since y is an element of lim inf $\{X\ \}$, U must contain points from all but a finite number of the sets in $\{X_n\}$. This implies x is an element of lim inf $\{X_n\}$, so that lim inf $\{X_n\}$ is closed.

The proof for lim sup $\{X_n\}$ follows in a similar manner.

Definition 3.11. Let $\{X_n\}$ be a sequence of subsets of a topological space. If lim inf $\{X_n\}$ = lim sup $\{X_n\}$ then sequence $\{X_n\}$ is said to <u>converge to limit</u> $\{X_n\}$ = lim inf $\{X_n\}$ = lim sup $\{X_n\}$.

Theorem 3.12. [17] Let S and T be first countable metric spaces. The closed transformation f(S) = T is open if and only if for each sequence $\{y_n\}$ in T converging to a point y in T,

$$\lim \{f^{-1}(y_n)\} = \overline{f^{-1}(y)}.$$

Proof. Let f be a closed and open mapping from S onto T, and let $\{y_n\}$ be a sequence of points in T converging to a point y in T. Since f is open, Theorem 3.5 implies $f^{-1}(y) \subset \lim\inf \{f^{-1}(y_n)\}$, and Lemma 3.1 implies that $f^{-1}(y) \subset \lim\inf \{f^{-1}(y_n)\} \subset \lim\sup \{f^{-1}(y_n)\}$. Let us now show that $\lim\sup \{f^{-1}(y_n)\} \subset \overline{f^{-1}(y)}$. Suppose there exists a point x in $\lim\sup \{f^{-1}(y_n)\} - f^{-1}(y)$. Let U be a neighborhood of x. Set U contains points from infinitely many of the sets $f^{-1}(y_n)$, and hence f(U) contains infinitely many points of $\{y_n\}$. The set $f(\overline{U})$ is closed by hypothesis, and hence must contain the limit point y of $\{y_n\}$. Therefore, $(\overline{U} \cap f^{-1}(y)) \neq \emptyset$. If $U \cap f^{-1}(y) = \emptyset$, then one could choose a neighborhood V of x such that $\overline{V} \subset U$ [8, 70], and such that $\overline{V} \cap f^{-1}(y) = \emptyset$. Since U was chosen arbitrarily, however, the argument given for U must also hold for V so that $\overline{V} \cap f^{-1}(y) \neq \emptyset$. This gives a contradiction, hence $U \cap f^{-1}(y) \neq \emptyset$, and x is a limit point of $f^{-1}(y)$. Thus we have,

$$f^{-1}(y) \subset \lim\inf \{f^{-1}(y_n)\} \subset \lim\sup \{f^{-1}(y_n)\} \subset \overline{f^{-1}(y)}.$$

But $\lim\inf \{f^{-1}(y_n)\}$ and $\lim\sup \{f^{-1}(y_n)\}$ are always closed sets, so that

$$\lim\inf \{f^{-1}(y_n)\} = \lim\sup \{f^{-1}(y_n)\} = \lim \{f^{-1}(y_n)\} = \overline{f^{-1}(y)}.$$

Assume now that $\lim \{f^{-1}(y_n)\} = \overline{f^{-1}(y)}$ and that f is a closed mapping. It remains to be shown that f is open. By hypothesis $f^{-1}(y) \subset \lim\sup \{f^{-1}(y_n)\}$ for any sequence $\{y_n\}$ in T that converges to a point y in T. Let U be any open set in S and assume f(U) is not open in T. Then there exists a point y in f(U) such that y is a limit point of T - f(U). Since T is a first countable metric space there exists a sequence $\{y_n\}$ of distinct points in $T - f(U)$ that converges to y. Since $f^{-1}(y) \subset \lim\sup \{f^{-1}(y_n)\}$ and since y is an element of $f(U)$, it follows that U must contain an element x of $f^{-1}(y)$, and that U, as a neighborhood of x, must contain points from infinitely many of the sets $f^{-1}(y_n)$. This implies $f(U)$ contains infinitely many of the points of $\{y_n\}$. This is a contradiction, hence $f(U)$ must be open in T and f must be an open mapping.

The next theorem gives a property possessed by functions which are both open and closed.

Definition 3.12. A space S is said to be locally connected at a point p if and only if, given any neighborhood U of p, there exists a neighborhood V of p such that $V \subset U$ and V is connected.

Definition 3.13. A space S is said to be locally connected if and only if S is locally connected at each of its points.

Definition 3.14. A subset Q of a topological space S is said to be a component of S if and only if Q satisfies the following conditions:

(1) Q is non-empty,

(2) Q is connected, and

(3) if C is any connected subset of S satisfying $C \cap Q \neq \emptyset$, then $C \subset Q$.

Theorem 3.13. [17] Let S be locally connected and let $f(S) = T$ be an open and closed mapping from S onto T. Then if T is connected, and if Q is any component of S, $f(Q) = T$.

Proof. It will first be shown that Q is both open and closed in S. Since Q is a component of S, Q is a connected subset of S, and Q is not contained in any other connected subset of S. Let p be an element of Q, and let U be a neighborhood of p. Since S is locally connected there exists a connected neighborhood V_p of p such that $V_p \subseteq U$. Now V_p and Q are both connected and $V_p \cap Q \neq \emptyset$. Thus $V_p \cup Q$ is a connected set. But Q is a component of S, so that $V_p \cup Q \subset Q$, and hence $V_p \subseteq Q$. It now follows that $Q = \bigcup_{p \in Q} V_p$ so that Q is an open set.

Now let p be a limit point of Q. Since the union of a connected set with one or all of the limit points of that set is a connected set, $Q \cup \{P\}$ is a connected set. If p is not an element of Q then $Q \cup P$ is a connected set such that $Q \subset (Q \cup P)$ and $(Q \cup P) \not\subset Q$. This contradicts the hypothesis that Q is a component of S. Therefore, p is an element of Q, and Q is a closed set.

Since Q is both open and closed, and since f is both open and closed, f(Q) is both open and closed in T. Assume now that $f(Q) \neq T$. Then f(Q) and $(T - f(Q))$ are each open and closed in T and are disjoint. This implies that f(Q) and $(T - f(Q))$ are mutually separated. But $f(Q) \cup (T - f(Q)) = T$, which contradicts the hypothesis that T is connected. Thus $f(Q) = T$ as claimed.

CONTINUITY OF OPEN MAPPINGS AND CLOSED MAPPINGS

The fact that a mapping can be open or closed without being continuous raises the following question. Under what condition will an open mapping be continuous, and under what conditions will a closed mapping be continuous? In the following discussion, this question will be considered.

Before proving the first theorem for closed continuous mappings it will be necessary to prove the following theorems concerning continuous functions.

Theorem 3.14. A function f from a space S onto a space T is continuous if and only if for every open subset G in T, $f^{-1}(G)$ is an open subset of S.

Proof. Let f be a continuous function and let G be an open subset of T. For any y in G, and for any x in $f^{-1}(y)$, there exists a neighborhood V_x of x such that $f(V_x) \subset G$ by the continuity of f. For each x in $f^{-1}(G)$, let V_x be a neighborhood of x such that $f(V_x) \subset G$. The union of all such V_x is an open set and furthermore, $\bigcup_{x \in f^{-1}(G)} V_x = f^{-1}(G)$.

Now suppose $f^{-1}(G)$ is an open set in S whenever G is open in T. Let y be an element of T and let U be any neighborhood of y. For any x in $f^{-1}(y)$, $f^{-1}(U)$ is an open set about x such that $f(f^{-1}(U)) \subset U$. Thus f is continuous.

Corollary. If f is a one-to-one continuous mapping from a space S onto a space T, then the mapping f^{-1} from T onto S is open.

Theorem 3.15. A function f from a space S onto a space T is continuous if and only if $f^{-1}(H)$ is closed in S whenever H is closed in T.

Proof. Suppose f is continuous and H is a closed subset of T. Then $G = (T - H)$ is an open subset of T, and by Theorem 3.12, $f^{-1}(G)$ is open. Therefore $f^{-1}(H) = (S - f^{-1}(G))$ is closed in S.

If on the other hand $f^{-1}(H)$ is closed in S whenever H is closed in

T, then for any open set G in T, $f^{-1}(G) = (S - f^{-1}(T - S))$ is open in S. Thus f is continuous by Theorem 3.14.

Theorem 3.16. [17] A function f from a space S onto a space T is closed and continuous if and only if $f(\overline{R}) = \overline{f(R)}$, for any subset R of S.

Proof: Assume f is closed and continuous on S, and let R be any subset of S. By Theorem 3.1, $\overline{f(R)} \subset f(\overline{R})$, so it remains to be shown that $f(\overline{R}) \subset \overline{f(R)}$. Let y be an element of $f(\overline{R})$. Since y is an element of $f(\overline{R})$, $f^{-1}(y)$ contains an element x of $\overline{R}$. Given any neighborhood U of y, there exists a neighborhood V os x such that $f(V) \subset U$ because of the continuity of f. Now since x is an element of $\overline{R}$, x is an element of R or x is a limit point of R. In either case, V must contain a point of R, so that U must contain a point of f(R). Therefore, y is a point of f(R) or a limit point of f(R). In any case y is any element of $\overline{f(R)}$ so that $f(\overline{R}) \subset \overline{f(R)}$. Since $\overline{f(R)} \subset f(\overline{R})$ and $f(\overline{R}) \subset \overline{f(R)}$, $f(\overline{R}) = \overline{f(R)}$.

If $f(\overline{R}) = \overline{f(R)}$ for any subset of R of S, then f is closed. This follows since for any closed subset H of S, $f(H) = f(\overline{H}) = \overline{f(H)}$ which is closed in T. Let us now show that f is continuous by showing that $f^{-1}(K)$ is closed in S whenever K is closed in T, and applying Theorem 3.15.

Let K be a closed subset of T and let x be a limit point of $f^{-1}(K)$. Since x is a limit point of $f^{-1}(K)$, f(x) is an element of $\overline{f(f^{-1}(K))}$ by hypothesis. But $\overline{f(f^{-1}(K))} = K$ since K is closed which implies f(x) is an element of K. Thus x is an element of $f^{-1}(K)$ and $f^{-1}(K)$ is closed. Now by Theorem 3.15, f is continuous. This completes the proof.

The following theorem concerning continuous functions will be useful in the remainder of this chapter.

Theorem 3.17. Let S and T be first countable metric spaces and let $f(S) = T$ be a transformation of S onto T. The mapping f is continuous if and only if for every sequence of points $\{x_n\}$ in S converging to a point x in S, sequence $\{f(x_n)\}$ converges to $f(x)$ in T.

Proof: Let the transformation $f(S) = T$ be continuous at the point x in S, and let $\{x_n\}$ be a sequence of points converging to x. Consider $f(x)$ in T and let U be an open set containing $f(x)$. By the continuity of f there exists an open set V in S, containing x, such that $f(V) \subset U$. Since $x = \lim x_n$, there exists a positive integer N such that for all $n \geq N$, x_n is in V. This implies for all $n \geq N$, $f(x_n)$ is in U. Therefore, $f(x) = \lim f(x_n)$.

Now let x be a point of S such that for every sequence $\{x_n\}$ converging to x, $\{f(x_n)\}$ converges to $f(x)$; and assume f is not continuous at x. Then there must exist some neighborhood U of $f(x)$ such that for any neighborhood V of x, $f(V) \not\subset U$. Let V_1 be a spherical neighborhood of radius 1 about x and pick x_1 in V_1 such that $f(x_1)$ is not an element of U. Let $r_2 = \rho(x,x_1)$. Since S is first countable, it is possible to choose a neighborhood V_2 of x such that V_2 is contained in the spherical neighborhood of radius r_2 about x. Pick x_2 in V_2 such that $f(x_2)$ is not an element of U. If points $x_1, x_2, \ldots, x_n$ have been chosen, let $r_{n+1} = \rho(x,x_n)$ and let V_{n+1} be an open set about x such that V_{n+1} is contained in the spherical neighborhood of radius r_{n+1} about x. Choose x_{n+1} in V_{n+1} such that $f(x_{n+1})$ is not an element of U.

Continuing in this manner, one can inductively choose a sequence $\{x_n\}$ of points in S such that $x = \lim x_n$, but such that $\{f(x_n)\}$ does not converge to x. This contradicts the hypothesis, hence f must be continuous at x.

The next two theorems as well as the last two theorems in this chapter are consequences of Theorem 3.17.

Theorem 3.18. [17] Let S and T be separable metric spaces and let f from S onto T be a closed transformation. If for each $y \in T$, $f^{-1}(y)$ is countable compact, and if for each convergent sequence $\{x_n\}$ in S, $\{f(x_n)\}$ has a limit point in T or is finite, then f is continuous.

Proof: Let $\{x_n\}$ be a convergent sequence in S with limit point x and assume $\{f(x_n)\}$ is infinite. Assume $\{f(x_n)\}$ has a limit point z in T such that $z \neq f(x)$. Since the only limit point of a sequence in a metric space is the sequential limit point, this is equivalent to assuming f is not continuous by Theorem 3.17. Since $x \cap f^{-1}(z) = \emptyset$, and since a metric space is completely normal [8,110], we can find disjoint open sets U and V containing x and $f^{-1}(z)$ respectively. Furthermore, since S is a metric space, there exists a neighborhood U_x of x such that $\overline{U}_x \subset U$. Now $\overline{U}_x \cap V = \emptyset$ since $U \cap V = \emptyset$. But $f(\overline{U}_x)$ is closed since f is a closed mapping, and furthermore $f(\overline{U}_x)$ must contain all but a finite number of the points of $\{f(x_n)\}$. Thus $f(\overline{U}_x)$ must contain z, which leads to a contradiction. Hence it must be true that $z = f(x)$, so that f is continuous at x. Now by Theorem 3.17 f is continuous on S.

If the sequence $\{f(x_n)\}$ is finite, then $\{f(x_n)\} = b_1, b_2, \ldots, b_k$, where k is finite. Now there must exist some i, $1 \leq i \leq k$ such that

$f^{-1}(b_i)$ is an infinite subsequence of $\{x_n\}$. By hypothesis, $f^{-1}(b_i)$ is countably compact, hence must contain a limit point in S. But $f^{-1}(b_i)$ as a subsequence of $\{x_n\}$ can have only the point x as a limit point, so that $x \in f^{-1}(b_i)$ and $f(x) = b_i$. Therefore, f(x) is a sequential limit point of $f(x_n)$, and f is continuous at x. Now by Theorem 3.17, f must be continuous at every point of S.

If we require that spaces S and T in Theorem 3.18 be countably compact, then the requirement that for each convergent sequence $\{x_n\}$ in S, $\{f(x_n)\}$ have a limit point in T or be finite can be dropped. Furthermore, this theorem can be generalized to the following if and only if theorem.

Theorem 3.19. Let S and T be first countable metric spaces and let f be a transformation from S onto T. Then S is countably compact and f is continuous on S if and only if f is closed, T is countably compact, and for each y in T, $f^{-1}(y)$ is countably compact.

Proof. Let f be closed, T be countably compact, and, for each y in T, let $f^{-1}(y)$ be countably compact. By Theorem 3.9, S is countably compact. Since T is countably compact for any sequence $\{x_n\}$ in S, the set $\{f(x_n)\}$ in T must have a limit point or be finite. Hence by Theorem 3.18, mapping f is continuous.

Now assume S is countably compact and that f is continuous. Let H be a closed subset of S and consider f(H). If y is a limit point of f(H) there exists a convergent sequence $\{y_n\}$ of distinct points of H such that lim $y_n = y$, since T is a first countable metric space. Pick a sequence K from the indexed collection of sets $\{f^{-1}(y_n)\}$ such that

$K \subset H$ and f is one-to-one from K onto $\{y_n\}$. Since S is countably compact, K has a limit point x in S such that x in an element of K. Furthermore, x is a sequential limit point of K since S is a first countable metric space. By the continuity of f, $f(K) = \{y_n\}$ has limit point $f(x) = y$. Thus y is an element of f(H), f(H) is closed, and f is a closed mapping on S.

To show that T is countably compact when S is countably compact, let A be an infinite subset of T. Let $\{y_n\}$ be an infinite sequence of points in A. Now by the argument used for sequence $\{y_n\}$ in the preceeding paragraph, one can show that $\{y_n\}$ must have a sequential limit point y which belongs to $\{y_n\}$. Thus A has a limit point y which belongs to A, and T is countably compact.

For $y \in T$, $f^{-1}(y)$ is either finite or infinite. If $f^{-1}(y)$ is infinite, let B be an infinite subset of $f^{-1}(y)$. Since S is countably compact, there exists some x in B such that x is a limit point of B. Thus $f^{-1}(y)$ is countably compact. This completes the proof.

The next theorem concerning continuity of closed mappings will be useful in Chapter IV.

Theorem 3.20. [6] If S is a regular space, T is a compact space, and if f is a closed mapping from S onto T such that $f^{-1}(y)$ is closed for each $y \in T$, then f is continuous.

Proof. Suppose f is not continuous at a point x in S. Then there exists a neighborhood V of f(x) such that for any neighborhood U of x, $f(U) \cap (T - V) \neq \emptyset$. Since T is compact and since (T - V) and $f(\overline{U})$ are compact, it follows that $(T - V) \cap f(\overline{U})$ is closed and compact. For any finite collection $U_1, U_2, \ldots, U_n$ of neighborhoods of x, $(\bigcap_{i=1}^{n} \overline{U}_i) \cap (T - V) \neq \emptyset$. Otherwise, $U = \bigcap_{i=1}^{n} U_i$ is an open set containing x such that $f(U) \subset V$, and f is continuous. This implies that the intersection of all sets of the form $f(\overline{U}) \cap (T - U)$, where U is an open set containing x, is non-empty [14,136]. Let y be an element of the intersection of all sets of the form $f(\overline{U}) \cap (T - V)$, where U is an open set containing x. Since $y \neq f(x)$, $x \notin f^{-1}(y)$ and since $f^{-1}(y)$ is closed and S is regular, there exists disjoint open sets U_1 and U_2 containing $f^{-1}(y)$ and x, respectively. Since $f^{-1}(y) \subset U_1$ and since $U_1 \cap U_2 = \emptyset$ it follows that $y \notin f(\overline{U}_2) \cap (T - V)$. This is a contradiction, hence f must be continuous.

Let us turn now to a consideration of theorems concerning continuity of transformation which are both open and closed.

Theorem 3.21. Let S and T be first countable metric spaces with T countably compact and let f be a transformation from S onto T which is both open and closed. The transformation f is continuous if and only if $f^{-1}(y)$ is closed for each y in T.

Proof. Let f be an open and closed mapping and let $\{x_n\}$ be a sequence in S with sequential limit point x. If we can show that f(x) is a sequential limit point of $\{f(x_n)\}$, then f will be continuous by Theorem 3.17. Now $\{f(x_n)\}$ is either an infinite subset of T or $\{f(x_n)\}$ can be expressed as a set $\{y_1, y_2, \ldots, y_k\}$ of point in T with k finite. If $\{f(x_n)\}$ is infinite, then $\{f(x_n)\}$ must have a limit point y in T, and since y is a first countable metric space, y is a sequential limit point of $\{f(x_n)\}$. Since $f^{-1}(y) = \overline{f^{-1}(y)}$ for each $y \in T$, and since $\{f(x_n)\}$ converges to y, $\lim\inf \{f^{-1}(f(x_n))\} \subset f^{-1}(y)$ by Theorem 3.5. Furthermore, x is an element of $\lim\inf \{f^{-1}(f(x_n))\}$, so that $f(x) = y$ and f is continuous by Theorem 3.17.

If $\{f(x_n)\} = \{y_1, y_2, \ldots, y_k\}$ with k finite, then there exists at least one i, $1 \leq i \leq k$, such that $f^{-1}(y_i)$ is infinite. Thus $f^{-1}(y_i)$ is a subsequence of $\{x_n\}$ and x is a sequential limit point of $\{f^{-1}(y_i)\}$. But $\{f^{-1}(y_i)\}$ is closed by hypothesis so that $x \in f^{-1}(y_i)$ and $f(x) = y_i$. Now suppose there exists some j, $1 \leq j \leq k$, $j \neq i$, such that $f^{-1}(y_j)$ is infinite. Then $f(x) = y_j$ so that $y_i = y_j$ and $i = j$. This is a contradiction, so that $\{f^{-1}(y_j)\}$ is finite if $i \neq j$. This implies that all but a finite number of points of $\{x_n\}$ map onto $y_i = f(x)$ so that $f(x)$ is a sequential limit point of $\{f(x_n)\}$ and f is continuous by Theorem 3.17.

Now assume f is continuous and let y be an element of T. If x is a limit point of $f^{-1}(y)$, then there exists a sequence $\{x_n\}$ of points in $f^{-1}(y)$ which converges to x. By the continuity of f, $\{f(x_n)\}$ must converge to $f(x)$. But $\{f(x_n)\} = \{y\}$ so that $f(x) = y$. Therefore, x is an element of $f^{-1}(y)$ and $f^{-1}(y)$ is closed as claimed.

Corollary. If f is a one-to-one function from a first countable metric space S onto a first countable metric space T which is both open and closed, then f is continuous.

Theorem 3.22. Let S and T be first countable metric spaces and let f from S onto T be an open mapping. If T is countably compact and if $f^{-1}(y)$ is a single point for all but a finite number of limit points in T, then f is a homeomorphism.

Proof. Let us first show f is one-to-one. Suppose there exists a limit point y in T such that $f^{-1}(y)$ contains two or more points. Now y a limit point of T implies there exists a sequence of distinct points $\{y_n\}$ converging to y. Since f is open, $f^{-1}(y) \subset \liminf \{f^{-1}(y_n)\}$ by Theorem 3.5. Let x_1 and x_2 be elements of $f^{-1}(y)$ and let U and V be disjoint open sets in S containing x_1 and x_2, respectively. It now follows that both U and V must contain points from $f^{-1}(y_n)$ for all but a finite number of n. This contradicts the hypothesis, hence $f^{-1}(y)$ must be a single point and f is one-to-one. The one-to-one, open mapping f is onto by hypothesis and is continuous by the corollary to Theorem 3.21. Thus f is a homeomorphism.

CHAPTER IV

COMPACT PRESERVING MAPPINGS AND CONNECTED MAPPINGS

INTRODUCTION

The following theorems give two characteristics of a continuous function f from a space S into a space T.

Theorem 4.1. Let S and T be spaces and let f be a continuous mapping from S into T. If C is a connected subset of S then f(C) is a connected subset of T [8,78].

Theorem 4.2. Let S and T be spaces and let f be a continuous mapping from S into T. If C is a compact subset of S then f(C) is a compact subset of T.

Proof. Let C be a compact subset of S and consider f(C). Let K be any open covering of f(C). Since f is continuous, for any open set U in K, $f^{-1}(U)$ is open in S. Let $H = \{f^{-1}(U) \mid U \in K\}$. Now H is an open covering of C, and since C is compact, a finite collection $f^{-1}(U_1)$, $f^{-1}(U_2), \ldots, f^{-1}(U_k)$ of open set in H will cover C. Thus the finite collection $U_1, U_2, \ldots, U_k$ of open sets in K will cover f(C), and f(C) is compact.

These two fundamental properties of continuous functions naturally lead to two lines of research. The requirements for a subset of a space to be either connected or compact are rather strong. Thus one might expect that a function that preserves either connected subsets or compact subsets would have interesting properties. One is also led to inquire what conditions, other than preserving connected sets or compact sets, a function must possess to be continuous. The purpose of this chapter is to investigate these lines of inquiry. Formal definitions will now be presented for connected and compact preserving mappings.

Definition 4.1. A function f from a space S into a space T is said to be connected if and only if for every connected subset C of S, f(C) is a connected subset of T.

Definition 4.2. A function f from a space S into a space T is said to be compact preserving if and only if for every compact subset C of S, f(C) is a compact subset of T.

PROPERTIES OF COMPACT PRESERVING MAPPINGS AND CONTINUITY OF COMPACT PRESERVING MAPPINGS

Compact sets and countably compact sets were defined in Chapter II. The following theorem relating these concepts will be useful in the development of the properties of compact preserving mappings.

Theorem 4.3. Every compact subset H of a space S is countably compact. If in addition, S is a metric space, the converse is also true [8,108].

In the following, properties of compact preserving mappings, and the relationship of compact preserving mappings to other mappings will be discussed in one section. This organization has been chosen since the theorems giving properties of compact preserving mappings lead naturally into theorems of the other type.

Theorem 4.4. [15] Let S and T be metric spaces and let f be a compact preserving mapping from S into T, such that f is discontinuous at a point p in S. Then there exists a point q in T and a sequence $\{p_i\}$ of points in S converging to point p such that, $f(p) \neq q$ and $f(p_i) = q$ for each i.

Proof. Since f is discontinuous at p and since S is a metric space, we can find a sequence of points $\{x_i\}$ in S with $\lim x_i = p$ and an open set V in T with $f(p) \in V$ and $V \cap \{f(x_i)\} = \emptyset$. If an infinite number of points from $\{x_i\}$ map onto a single point q in T we are finished since the subsequence $f^{-1}(q) \cap \{x_i\}$ of sequence $\{x_i\}$ can be taken as the sequence $\{p_i\}$. Thus assume that each point q in the image of $\{x_i\}$ is the image of only a finite number of points of $\{x_i\}$. Hence a subsequence $Y = \{y_i\}$ of the sequence $\{x_i\}$ must map one-to-one onto the set $Q = \{q_1, q_2, q_3, \ldots\}$ of image points of $\{x_i\}$ under f. The set $Y \cup \{p\}$ is a countably compact subset of a metric space, hence is compact. Therefore, the image set $f(Y \cup \{p\}) = Q \cup f(p)$ is compact and countably compact, since f is a compact preserving. Since $f(Y) \cap V = \emptyset$, p cannot be a limit point of $(f(y) \cup \{p\})$, and hence set f(Y) must be countably compact. This implies that for some j, $f(y_j)$ is a limit point f(Y). Now set $\{(Y - \{y_j\}\}) \cup \{p\}$ is compact and, as above, $f(Y - \{y_j\})$ is compact. However the set $f(Y - \{y_j\}) = \{f(Y) - \{f(y_j)\}\}$

is not closed. This gives a contradiction since a compact subset of a metric space is always closed. Thus we must assume that an infinite number of the points of $\{x_i\}$ maps onto a single point q in T and $\{p_i\}$ can be chosen as the subsequence $f^{-1}(q) \cap \{x_i\}$ of sequence $\{x_i\}$.

The preceeding theorem states an interesting property of a compact preserving mapping. The disclosure of this characteristic, however, is not the only significance of this theorem, since the next theorem relating compact preserving mappings to continuous mappings is a consequence of this theorem.

Theorem 4.5. [15] Let S and T be metric spaces and let f from S onto T be a compact preserving mapping. If $f^{-1}(q)$ is closed for every q in T, then f is continuous.

Proof. Assume f is not continuous and let p be an element of S such that f is not continuous at p. By Theorem 4.4, we find a point q in T and a sequence $\{p_i\}$ of points in S such that $\lim p_i = p$, $f(p) \neq q$, and $f(p_i) = q$ for all i. Now the set $f^{-1}(q)$ is closed by hypothesis, hence must contain p. This is a contradiction, so that f must be continuous.

Corollary. Every one-to-one compact preserving mapping from a metric space S onto a metric space T is continuous.

Proof. The proof follows immediately from Theorem 4.5 since every point of a metric space is a closed subset of that space.

Using the results of Theorem 4.4, one can easily construct examples of functions that are not continuous but are compact preserving. The following is one such example.

Example 4.1. Let S be the real numbers with the usual topology and let f be defined by:

$$f(x) = 0 \text{ if } x \text{ is rational, and}$$
$$f(x) = 1 \text{ if } x \text{ is irrational.}$$

Function f is discontinuous everywhere, but is compact preserving since every subset of S is mapped onto one of the compact sets $\{0,1\}$, $\{0\}$, or $\{1\}$.

If spaces S and T are not metric spaces, the conclusion of Theorem 4.5 may no longer follow. However, it is sometimes possible to place alternate conditions on the spaces that will insure continuity. The next two theorems give examples of such alternate conditions for certain spaces.

Definition 4.3. A space S is <u>locally compact</u> if and only if, for every point p in S and for every neighborhood U of p, there exists a neighborhood V of p such that $V \subset U$ and $\overline{V}$ is compact.

Theorem 4.6. [7] Let S be a locally compact Hausdorff space and let T be a Hausdorff space. Then if f is compact preserving, and if $f^{-1}(y)$ is closed for each $y \in T$, f is continuous.

Proof. Consider any point x in S. Since S is locally compact, there exists a neighborhood U of x such that $\overline{U}$ is compact. Since continuity is a local property, one need only consider f restricted to $\overline{U}$ with $\overline{U}$ regarded as a subspace of S. Let us now show that the conditions of Theorem 3.20 are satisfied and the conclusion will follow.

To see that $\overline{U}$ is regular, let F be a closed subset of $\overline{U}$ and let x be an element of $\overline{U} - F$. For each y in $\overline{U}$ choose neighborhood V_y and U_y of y and x, respectively, such that $V_y \cap U_y = \emptyset$. This is possible since $\overline{U}$ is Hausdorff. The collection $\{V_y\}$, $y \in F$, is an open covering of F. Now F is a closed subset of a compact space and is, therefore, compact. Hence there exists a finite subcollection $V_{y_1}, V_{y_2}, \ldots, V_{y_k}$ of $\{V_y\}$, $y \in F$, that covers F. The sets $\bigcup_{i=1}^{k} V_{y_i}$ and $\bigcap_{i=1}^{n} U_{y_i}$ are the desired open sets containing F and x, respectively, so that $\overline{U}$ is regular.

Space $F(\overline{U})$ is compact since $\overline{U}$ is compact and f is compact preserving.

To verify that f is closed on $\overline{U}$, let F be any closed subset of $\overline{U}$. Since $\overline{U}$ is compact, any closed subset of $\overline{U}$ is compact. Therefore, f(F) is a compact subset of $f(\overline{U})$. Because f(F) is a compact subset of the Hausdorff space $f(\overline{U})$, f(F) is closed [8,66].

Since all of the conditions of Theorem 3.20 are satisfied, f is continuous at x. Point x was chosen arbitrarily, however, so f is continuous on S.

Definition 4.4. A space S will be said <u>to have property K^* at point p</u> if and only if, for every infinite subset A of S having p as an accumulation point, there exists a compact subset of $A \cup \{p\}$ having p as an accumulation point.

Theorem 4.7. [7] Let S and T be Hausdorff spaces and let f from S onto T be a compact preserving mapping. Then if S has property K^* at x and if $f^{-1}(y)$ is closed for each $y \in T$, f is continuous at x.

Proof: If x is an isolated point the proof is trivial, so it may be assumed x is not isolated. Suppose f is not continuous at x. Then there exists some neighborhood V of $f(x)$ such that for each open set U containing x there exists an x_u in $U \cap f^{-1}(T - V)$. For each neighborhood U of x choose a point x_u and let A be the set of all such x_u. Set A is infinite since x is an accumulation point of A. Hence, there exists some compact subset K of $A \cup \{x\}$ such that x is an accumulation point of K. By Theorem 4.6, f restricted to K is continuous. This is a contradiction since $f(K) \subset (T - V)$ and $f(x)$ is in V. Thus function f is continuous at x.

The next theorems state a relationship between compact subsets and closed subsets of compact Hausdorff spaces which implies a corresponding relationship between closed mappings and compact preserving mappings.

Theorem 4.8. Let S be a compact Hausdorff space. A subset H of S is closed if and only if H is compact.

Proof: Assume H is a closed subset of a compact Hausdorff space S, and let $\{U_\alpha\}$ be an open covering of H. The collection $\{U_\alpha\} \cup (S - H)$ is an open covering of S. Since S is compact a finite number of the sets in collection $\{U_\alpha\} \cup (S - H)$ will cover S. Therefore, a finite number of sets from collection $\{U_\alpha\}$ will cover H and H is compact.

Now assume H is a compact subset of the compact Hausdorff space S. Let us show that H is closed by showing that no point of $(S - H)$ is a limit point of H. Let q be any point of (S - H). For each y in H, choose disjoint neighborhoods U_y and V_y containing y and q, respectively. This is possible since S is a Hausdorff space. The collection $\{U_y\}$, $y \in H$, is an open covering of the compact set H, hence a finite subcollection $U_{y_1}, U_{y_2}, \ldots, U_{y_k}$ will cover H. The open set $\bigcap_{i=1}^{k} V_{y_k}$ is an open set containing x which does not intersect H. Thus q is not a limit point of H, and H is closed.

The significance of Theorem 4.8 is that a mapping f from a compact Hausdorff space S into a compact Hausdorff space T will be closed if and only if f is compact preserving. Thus most of the theorems of Chapter III concerning closed mapping give rise to theorems concerning compact preserving mappings.

PROPERTIES OF CONNECTED MAPPINGS

In the following theorems some properties of connected mapping will be developed. As with compact preserving mappings, these theorems will lead into theorem relating connected mappings to continuous mappings.

Theorem 4.9. [21] Let f be a connected mapping of the Hausdorff space S into the Hausdorff space T. If C is any connected subset of S then $f(\overline{C}) \subset \overline{f(C)}$.

Proof: Let C be a connected subset of S and let q be an element of $f(\overline{C})$. We wish to show that q is an element of $\overline{f(C)}$. Since $q \in f(\overline{C})$, there exists some p in $\overline{C}$ such that f(p) = c. If p is in C, then f(p) = q is in f(C) and hence in $\overline{f(C)}$. If p is not in C, then p is a limit point of C. Now set C U {P} is a connected set since the union of a connected set with a limit point of that set is connected and $f(C \cup \{P\}) = f(C) \cup \{q\}$ is connected. Now assume q is not an element of $\overline{f(C)}$. Since T is Hausdorff, no point of C is a limit point of $\{q\}$. But this implies $f(C) \cup \{q\}$ is not connected which is a contradiction. Therefore, $q \in \overline{f(C)}$ and $f(\overline{C}) \subset \overline{f(C)}$.

The next theorem is a consequence of Theorem 4.9.

Theorem 4.10 [21] Let S_1, S_2 and S_3 be Hausdorff spaces and let f be a connected mapping of $S_1 \times S_2$ into S_3. If f is a connected mapping, then f has the following properties: (i) $f(x,B)$ is connected for any x in S_1 and for any connected subset B in S_2, (ii) $f(A,y)$ is connected for any connected subset A of S_1 and for any y in S_2.

Proof of (i). Assume there exists a point x in S_1 and a connected subset B of S_2 such that $(x,B) = \{(x,y) \mid y \in B\}$ is not connected. Then there exists disjoint nonempty subsets H_1 and H_2 in $S_1 \times S_2$ such that $(x,B) \subset H_1 \cup H_2$, $\overline{H}_1 \cap H_2 = \emptyset$ and $H_1 \cap \overline{H}_2 = \emptyset$. Let $T_1 = \{y \mid y \in B$ and $(x,y) \in H_1\}$ and let T_2 $\{y \mid y \in B$ and $(x,y) \in H_2\}$. For any $y \in B$, $y \in T_1$, or $y \in T_2$, since $(x,B) \subset (H_1 \cup H_2)$. Furthermore, T_1 and T_2 are nonempty, for otherwise, H_1 or H_2 is empty. It is also true that $\overline{T}_1 \cap T_2 = \emptyset$ and $T_1 \cap \overline{T}_2 = \emptyset$; otherwise, $\overline{H}_1 \cap H_2 \neq \emptyset$ or $H_1 \cap \overline{H}_2 \neq \emptyset$. This, however, implies B is not connected, which is a contradiction. Therefore, (x,B) is connected and f connected implies $f(x,B)$ is connected.

The proof of (ii) follows in an analogous manner.

The following example shows that conditions (i) and (ii) of Theorem 4.10 are not sufficient for a function to be connected.

Example 4.2. [21] Let f be defined on the Euclidean plane as follows:

$$f(x,y) = \frac{xy}{x^2 + y^2} \quad \text{if } x \neq 0 \text{ or } y \neq 0$$

$$f(0,0) = 0.$$

Function f is continuous in each variable separately, and is therefore connected in each variable. This means that f satisfies conditions (i) and (ii) of Theorem 4.10. However, along the line $x = y$, which is a connected subset of the plane, $f(x,x) = 1/2$ if $x \neq 0$ while $f(0,0) = 0$. Hence, mapping f is not connected.

A partial converse does exist for Theorem 4.10.

Theorem 4.11. [21] Let S_1, S_2 and S_3 be Hausdorff spaces and let f be a transformation from $S_1 \times S_2$ into S_3. If f has properties (i) and (ii) of Theorem 4.10, then $f(A,B)$ is connected whenever A is connected in S_1 and B is connected in S_2.

Proof. Let f satisfy conditions (i) and (ii) and let A and B be connected subsets of S_1 and S_2 respectively. Assume $f(A,B) = \{f(x,y) \mid x \in A, y \in B\}$ is not connected. Then $f(A,B)$ can be expressed as the union of two nonempty disjoint sets H and K such that $H \cap \overline{K} = \emptyset$ and $\overline{H} \cap K = \emptyset$. Now for a fixed point x_1 of A, $f(x_1,B) = \{f(x_1,y) \mid y \in B\}$ is connected by condition (i), hence must be entirely contained in either H or K, say H. Similarly for a fixed point y_1 in B, $f(A,y_1)$ must be contained in either H or K by condition (ii). However, $f(x_1,y_1)$ is an element of H, so $f(A,y_1)$ must be a subset of H. Since the same argument is true for every $y \in B$, $f(A,B)$ is contained in H.

This implies $K = \emptyset$ which is a contradiction. Therefore, $f(A,B)$ must be connected as claimed.

A set will now be defined which will lead to a theorem giving a property of connected mappings as well as a theorem giving necessary and sufficient conditions for a connected mapping to be continuous.

Definition 4.5. Let f be a mapping from a space S into a space T. For every point p in S let the <u>set of limit points of f at p</u>, denoted by $L(f,p)$, be the set of all points q in T for which there exists a sequence of $\{p_n\}$ of points in S such that limit $p_n = p$ and limit $f(p_n) = q$.

The following property of set $L(f,p)$ will be used in the proof of the next two theorems.

Lemma 4.1. [21] Suppose f is a mapping from a first countable Hausdorff space S into a first countable space T. For every point p in S, $L(f,p)$ is closed.

Proof: Let p be any point in S and let q be a limit point of $L(f,p)$. Since S is first countable, there exists a sequence $\{U_n\}$ of open sets containing p such that for any open set U containing p there exists a positive integer N_1 such that $n > N_1$ implies $U_n \subset U$. We may assume that $\{U_n\}$ is monotone decreasing. Similarly, a monotone sequence $\{V_n\}$ of open sets can be chosen in T with the property that $q \in V_n$ for each n, and such that for each open set V containing q there exists a positive integer N_2 such that $n > N_2$ implies $V_n \subset V$.

Now consider V_j for some fixed positive integer j. Since q is a limit point of $L(f,p)$ there exists some point q' of $L(f,p)$ in $V_j \cap (L(f,p) - \{q\})$. Because $q' \in L(f,p)$, there exists a sequence of points $\{p_n\}$ in S such that limit $p_n = p$ and limit $f(p_n) = q'$. Now since limit $f(p_n) = q'$, and since V_j is an open set containing q', there exists a positive integer N_3 such that $n > N_3$ implies $f(p_n) \in V_j$. Furthermore, since limit $p_n = p$ there exists a positive integer N_4 such that $n > N_4$ implies p_n is in U_j. Let $N = \max\{N_3, N_4\}$ and choose a point p_n where $n > N$. Then p_n is in U_j and $f(p_n)$ is in V_j. Relabel the point p_n as x_j. By the above construction we can pick a point x_j in U_j for each j such that $f(x_j)$ will be an element of V_j. The sequence $\{x_j\}$, so selected, will have the property that limit $x_j = p$ and limit $f(x_j) = q$. Thus q is an element of $L(f,p)$ and $L(f,p)$ is closed.

Theorem 4.12. [21] If f is a connected mapping from a locally connected first countably Hausdorff space S into the compact first countable Hausdorff space T, then $L(f,p)$ is a connected subset of T for every p in S.

Proof. Let us first note that T as a compact Hausdorff space is normal. Now assume $L(f,p)$ is not connected for some p in S. Then $L(f,p)$ can be expressed as the union of two sets A and B where $A \neq \emptyset$, $B \neq \emptyset$, $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$. By Lemma 4.1, $L(f,p)$ is closed. This implies that both A and B are closed. Since space T is normal, open sets U and V can be found such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$, so that $L(f,p) = (A \cup B) \subset (U \cup V)$. We shall now obtain a contradiction by showing that $L(f,p)$ is a subset of either U or V.

It will first be shown that there exists at least one open set M containing p such that $f(M) \subset (U \cup V)$. Assume that no open set containing p is mapped into $(U \cup V)$. Space S is first countable, hence there exists a monotone decreasing sequence $\{M_i\}$ of open sets each containing p and such that for any open set M containing p there exists an integer N such that $M_i \subset M$ for all $i > N$. For each i, pick an element p_i in M_i such that $f(p_i) \in T - (U \cup V)$. Since $T - (U \cup V)$ is a closed subset of a compact metric space, it is compact, hence countably compact. Thus sequence $\{f(p_n)\}$ must have a limit point q in $T - (U \cup V)$, and some subsequence of $\{f(p_n)\}$ will have q as a sequential limit point. This is a contradiction since limit $p_n = p$ which implies $q \in L(f,p)$. Thus we must conclude that some open set M containing p maps into $(U \cup V)$.

Now consider an open set M about p such that $f(M) \subset (U \cup V)$. Since S is locally connected, there exists a connected open set C containing p such that $f(C) \subset (U \cup V)$. Transformation f is connected, so that $f(C)$ must be connected. Therefore, $f(C) \subset U$ or $f(C) \subset V$. This implies $L(f,p)$ must be contained in either U or V which contradicts the assumptions $L(f,p) \cap U \neq \emptyset$ and $L(f,p) \cap V \neq \emptyset$. Hence $L(f,p)$ must be connected.

One can note the $L(f,p)$ is never empty, since $f(p)$ is always an element of $L(f,p)$. This follows from the fact $f(p)$ is the limit of the sequence $\{f(p_i)\}$, where $p_i = p$ for each i.

CONTINUITY OF CONNECTED MAPPINGS

Theorem 4.12 leads to the following theorem which states a necessary and sufficient condition for a connected function to be continuous.

Theorem 4.13. [21] If f is a connected mapping from the locally connected first countable Hausdorff space S into the compact first countable Hausdorff space T, then f is continuous at a point p in S if and only if set $L(f,p)$ is finite or denumerable.

Proof. Since a continuous mapping is connected and has $L(f,p) = f(p)$, we need only prove that the condition $L(f,p)$ is finite or denumerable is sufficient. By Theorem 4.12, $L(f,p)$ is connected and as a closed subset of a compact space is compact. Let us now show that $L(f,p)$ is either a single point or is non-denumerable. Assume set $L(f,p)$ is denumerable but does not consist of a single point. If $L(f,p)$ is assumed to be finite we get an immediate contradiction since each point of a finite subset of a Hausdorff space is an isolated point which implies $L(f,p)$ is not connected. If $L(f,p)$ is assumed to be an infinite denumerable set, a contradiction can be obtained as follows. Let $L(f,p)$ be ordered by the positive integers and let M_1 and U_2 be disjoint open set containing x_1 and x_2, respectively. Now consider $\overline{M}_1$. Since $M_1 \cap U_2 = \emptyset$, x_2 is not an element of $\overline{M}_1$. Thus $\overline{M}_1$ can not contain all of $L(f,p)$. Now if $L(f,p) - \overline{M}_1$ is finite we can obtain a contradiction by constructing an open set about x_2 that would contain no other point of $L(f,p)$. This would lead to a contradiction since no point of a connected set can be an isolated point.

Now consider the point x_2 in $L(f,p) - \overline{M}_1$ and let x_{i_3} be the element of least index in $L(f,p) - \overline{M}_1$. Consider the closed set $B_2 = x_2 \cup$ (Boundry M_2). Since this set is a compact subset of a Hausdorff space it is possible to construct disjoint open sets M_2 and U_{i_3} containing B and x_{i_3} respectively. Now x_{i_3} is not a limit point of $\overline{M}_1 \cup \overline{M}_2 = \overline{M_1 \cup M_2}$

and as above, $(L(f,p) - \overline{M_1 \cup M_2})$ must be infinite. Continuing in this manner until the elements of $L(f,p)$ are exhausted one can construct an open covering $\bigcup_{i=1}^{\infty} M_i$ of $L(f,p)$ and a sequence $\{x_i\}$ of points of $L(f,p)$ such that $x_{i_{n+1}}$ is not an element of $\bigcup_{i=1}^{n} M_i$. Thus no finite subcollection of $\bigcup_{i=1}^{\infty} M_1$ can cover $L(f,p)$. But $L(f,p)$ as a closed subset of a compact space is compact. This is a contradiction, hence $L(f,p)$ must be a single point or non-denumerable. By hypothesis, $L(f,p)$ is finite, so $L(f,p)$ must be a single point and f must be continuous.

The following theorems give alternate sufficient conditions for a connected function to be continuous. Since a continuous function is always connected, these conditions will not need to be stated as necessary and sufficient conditions. The first two of these theorems make use of the concept of at worst a removable discontinuity.

Definition 4.6.a. A mapping f from a Hausdorff space S into a Hausdorff space T is said to have <u>at worst a removable discontinuity at a point p</u> in S if and only if for every sequence $\{p_n\}$ of points in S converging to p with each $p_n \neq p$, $\lim f(p_n) = q$ for some q in T.

Under definition 4.6.a., a function f which is continuous at a point p in S has at worst a removable discontinuity at p. The following example gives a function which is not continuous, has at worst a removable discontinuity at each point, and is connected.

Example 4.3. Let S be the set of points on the real number line of the form $1/n$, $n = 1, 2, 3, \ldots$, along with the point 0, and let S have the usual topology. Define f by:

$$f(p) = 1 \text{ if } p \neq 0$$

$$f(0) = 0.$$

The function f is continuous at every point in S except 0. At the point 0, f has at worst a removable discontinuity since for any sequence $\{p_n\}$ of points converging to 0, $\lim f(p_n) = 1$. Furthermore, f is connected on S since the only connected subsets of S consist of single points.

The next theorem gives conditions that will imply a connected function with at worst a removable discontinuity is continuous.

Theorem 4.14. [21] Let f be a connected mapping of the locally connected first countable Hausdorff space S into the Hausdorff space T. Then f is continuous at a point p in S if and only if f has at worst a removable discontinuity at p.

Proof. Suppose f is connected and has at worst a removable discontinuity at p. Assume there exists a sequence $\{p_n\}$ which converges to p and is such that the unique point $q = \lim f(p_n)$ is not equal to f(p). This is equivalent to assuming f is not continuous. Since T is Hausdorff, disjoint open sets U and V can be chosen such that $q \in U$ and $f(p) \in V$. Now there must exist at least one open set M containing p such that for any point x in $\{p_n\} \cap M$, f(x) is an element of U. If this is not true, one can use the fact that S is first countable to construct a sequence of points $\{p_n\}$ converging to p and such that $\lim f(p_n) \neq q$. Since S is locally connected, there exists a connected open set C containing p such that $C \subset M$. Set f(C) must be a connected subset of T since f is connected. But f(p) is contained in V and $f(C - \{p\})$ must be contained in U. This is a contradiction, since this implies f(C) is not connected. Thus

$(f(p))$ must equal q and f is continuous at p.

Since a continuous mapping is connected and has at most a removable discontinuity at a point p, the conditions given are both necessary and sufficient.

The conditions placed on space S in Theorem 4.14 were rather strong conditions. If the definition of at worst a removable discontinuity at a point p is generalized appropriately, the restriction that S be first countable can be removed. The desired generalization is stated in the next definition.

Definition 4.6.b. A function f from a space S into a space T is said to have at worst a removable discontinuity at a point p of S if there exists a point q in T such that for every neighborhood U of q, there exists a neighborhood V of p such that $f(U - \{p\}) \subset V$.

From the definition of a convergent sequence, one can easily show that Definition 4.6.b always implies Definition 4.6.a. In the proof of Theorem 4.14 it was shown that Definition 4.6.a implies definition 4.6.b whenever S is a first countable Hausdorff space. Therefore, Definitions 4.6.a and 4.6.b are equivalent for first countable Hausdorff spaces. Theorem 4.14 can now be restated as follows if Definition 4.6.b is used.

Theorem 4.15. [7] If f is a connected mapping from a locally connected Hausdorff space S into a Hausdorff space T, then f is continuous at a point p in S if and only if f has at worst a removable discontinuity at p.

Proof. The proof is essentially the same as the proof of Theorem

4.14.

In the next example, a function is presented which is connected everywhere, but is discontinuous at one point. All of the hypotheses of Theorem 4.15 are satisfied except the hypothesis that f has at worst a removable continuity at each point.

Example 4.4. [21] Let S be the Euclidean plane and let T be the space of real numbers. Define f from S into T by:

$$f(x,y) = \frac{2xy}{x^2 + y^2} \sin \frac{\pi}{(x^2 + y^2)^{1/2}} \quad \text{if } x \text{ and } y \text{ are not both } 0,$$

$$f(0,0) = 0.$$

This function is continuous at every point except possibly (0,0) and therefore maps any connected subset not containing (0,0) onto a connected subset of T. f is not continuous at (0,0) as one can verify by considering the line x = y. On this line f reduces to the following function:

$$g(x) = \sin \frac{\pi}{\sqrt{2}\,|x|} \text{ if } x \neq 0$$

$$g(0) = 0.$$

Now consider points of the form $(\sqrt{2}/n, \sqrt{2}/n)$. This sequence of points converges to (0,0) but the sequence $f(\sqrt{2}/n, \sqrt{2}/n)$ does not converge to f(0,0) since $f(\sqrt{2}/n, \sqrt{2}/n) = \pm 1$, depending on whether n is even or odd. Thus f is not continuous at (0,0).

Let us now verify that f maps connected subsets of S containing (0,0) onto connected subsets of T. Suppose C is a connected subset of S containing (0,0) and such that f(C) is not connected. Then f(C) can be expressed as the union of two sets A and B such that $A \neq \emptyset$, $B \neq \emptyset$, $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$. Now f(0,0) is an element of either A or B, say B.

Now $f^{-1}(A) \cup f^{-1}(B) = C$ and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Let us show that for any x in $f^{-1}(A)$, x is not a limit point of $f^{-1}(B)$. Suppose x is an element of $f^{-1}(A)$. Then $f(x)$ is an element of A and since $A \cap \bar{B} = \emptyset$, there exists a neighborhood U of $f(x)$ such that $U \cap B = \emptyset$. By the continuity of f at x there exists an open set V about x such that $f(V) \subset U$. Now $V \cap f^{-1}(B)$ must be empty since $f(V) \subset U$ and $U \cap B = \emptyset$. Thus x is not a limit point of $f^{-1}(B)$. Similarly for x in B, if $x \neq (0,0)$, x is not a limit point of $f^{-1}(A)$. Now $(0,0)$ must be a limit point of $f^{-1}(A)$. If not, $f^{-1}(A)$ and $f^{-1}(B)$ are mutually separated and C is not connected. Now $f^{-1}(A)$ cannot consist of a single point; for if $f^{-1}(A)$ is a single point $(0,0)$ is not a limit point of $f^{-1}(A)$ and C is not connected. Thus let x_1 be an element of $f^{-1}(A)$ let $d = \rho((0,0), x_1)$. Pick a positive integer n such that $1/n^2 < d$ and consider the spherical neighborhood N of radius $1/n^2$ about $(0,0)$. Now the boundary of N must contain a point x_2 of $f^{-1}(A)$. If not, C can be expressed as $(f^{-1}(B) \cup \text{Int } N) \cup (f^{-1}(A) - f^{-1}(A) \cap \bar{N}))$. But these sets are mutually separated so that C is not connected. Now for point x_2, $f(x_2) = 0 = f(0,0)$ which implies $f(x_2)$ is in B. This is a contradiction, hence $f(C)$ must be connected, and f is a connected mapping.

A connected function f maps connected subsets of the domain space onto connected subsets of the range space. If in addition f is one-to-one and f^{-1} maps connected subsets of the image space onto connected subsets of the range space several theorems concerning continuity of connected mappings can be proven.

Definition 4.7. A mapping f from a space S onto a space T is said to be biconnected if and only if f is one-to-one, f(C) is connected in T whenever C is connected in S and $f^{-1}(H)$ is connected in S whenever H is connected in T.

Definition 4.8. A space S is said to be semi-locally-connected if and only if, for any point p in S and for any open set U containing p, there exists an open set V containing p such that $V \subset U$ and S - V consists of a finite number of closed connected sets.

Example 4.5. The set R of real numbers with the usual topology is semi-locally-connected. This follows since for any open set U containing p there exists an open interval (a,b) with $p \in (a,b) \subset U$. The complement of (a,b) is two closed rays each of which is connected.

Theorem 4.16. [21] If f is a biconnected mapping of the Hausdorff space S onto the semi-locally-connected Hausdorff space T, then f is continuous.

Proof. The proof will follow if it can be shown that the inverse image of every open set in T is open in S. Let U be an open set in T, and consider $f^{-1}(U)$. Let p be an element of $f^{-1}(U)$. Since T is semi-locally-connected, there exists an open set V such that $f(p) \in V \subset U$ and such that (T - V) consists of a finite number of closed connected sets. Thus (T - V) can be expressed as $H_1 \cup H_2 \cup \ldots \cup H_n$ where H_i is closed and connected for each i = 1, 2, ..., n. Let $C_i = f^{-1}(H_i)$ for each i = 1,2,...n. Each C_i is connected since f is a biconnected mapping. Let us now show that $p \notin \overline{C}_i$ for any i. Suppose p were en element of $\overline{C}_i$

for some i. Then $C_i \cup \{p\}$ is connected, since the union of any connected subset with some or all if its limit points is connected. Mapping f is biconnected, hence $f(C_i \cup \{p\}) = H_i \cup f(p)$ is connected. If $H_i \cup \{f(p)\}$ is connected, however, then f(p) must be a point of H_i or a limit point of H_i. Either assumption contradicts the fact that $H_i \subset T - V$ where V is an open set containing f(p). Thus, the assumption $p \in \bar{C}_i$ for some i leads to contradiction, and we must conclude $p \notin \bar{C}_i$ for any i. Since $p \notin \bar{C}_i$ for any i, then for each i, let $M_i = S - \bar{C}_i$ and let $M = \bigcap_{i=1}^{n} M_i$. Set M is open as the intersection of a finite number of open set. Now $M \cap C_i = \emptyset$ for each i so that $f(M) \cap f(C_i) = f(M) \cap H_i = \emptyset$, for each i, and $f(M) \subset V \subset U$. By the above construction we can find an open set M_p for every $p \in f^{-1}(U)$ such that $f(M_p) \subset U$. One can easily verify that $f^{-1}(U) = \bigcup_{p \in f^{-1}(U)} M_p$ which is open as the union of open sets. Since $f^{-1}(U)$ is open in S whenever U is open in f, f is continuous.

If space S is also required to be semi-locally-connected the following stronger theorem holds.

Theorem 4.17.[2] Let f be a biconnected mapping of the semi-locally-connected Hausdorff space S onto the semi-locally-connected Hausdorff space T. Then f is a homeomorphism.

Proof. Mapping f is one-to-one and onto by definition of biconnected. By Theorem 4.16 f is continuous. Also, by Theorem 4.16 f^{-1} is continuous so that f is open. Hence f is a homeomorphism.

One can replace the requirement that S and T be semi-locally-connected with the requirement S and T be locally connected, provided space T is compact.

Theorem 4.18. [21] Let f be a biconnected mapping of the locally connected compact Hausdorff space S onto the locally connected compact Hausdorff space T. Then f is a homeomorphism.

Proof. As in Theorem 4.16 it will be shown that f is continuous by showing that $f^{-1}(U)$ is open for every U open in T. Let U be an open subset of T and let p be an element of $f^{-1}(U)$. Since T is locally connected, there exists a connected open set C in T such that $f(p) \in C \subset U$. Consider $(T - C)$ which is a closed subset of T. Since T is Hausdorff, for each $q \in (T - C)$ there exists an open set U_q such that $q \in U_q$ and $p \notin \overline{U}_q$. Also, because T is locally connected, one can find a connected subset V_q such that $q \in V_q \subset U_q$ for each $q \in (T - C)$. Now $\bigcup_{q \in (T-C)} V_q$ is an open covering of $(T - C)$ and by the compactness of $(T - C)$, which is a closed subset of T, one can find a finite number of sets $V_1, V_2, \ldots, V_n$ from collection $\{V_q\}_{q \in (T-C)}$ which covers $(T - C)$. Let us note that $f(p) \notin \overline{V}_i$ for any $i = 1, 2, \ldots, n$, since $f(p) \notin \overline{U}_q$ for any q, and for each i, $\overline{V}_i = \overline{V}_q \subset \overline{U}_q$, for some q. Now let $C_i = f^{-1}(V_i)$ for each i. One can now complete the proof that $f^{-1}(U)$ is open by the same constructive argument as in Theorem 4.16.

By the same argument, f^{-1} is continuous, hence f is open. Thus, f is one-to-one, onto, open and continuous, and is, therefore, a homeomorphism.

Several of the theorems from the first of this chapter concerning compact preserving mappings required that the domain space be locally compact and that $f^{-1}(y)$ be closed for each point y in the range space.

Results of a similar nature can be obtained for connected mappings if one requires that the domain space be locally connected and that $f^{-1}(y)$ be closed for each point y in the range space.

Theorem 4.19. [15] Let M be a metric space and let p be an element of M. The following conditions are equivalent: (i) M is locally connected at p; (ii) every connected map f from M onto a metric space f(M) with the property, $f^{-1}\{z \mid z \in f(M) \text{ and } \rho(z,q) = \epsilon\}$ is closed for each $\epsilon > 0$ and for every $q \in f(M)$ is continuous at p, (iii) every real valued connected map with the property that $f^{-1}(q)$ is closed for $q \in f(M)$ is continuous at p.

Proof. For a real valued map the requirement that $f^{-1}(q)$ is closed implies $f^{-1}\{z \mid z \in f(M) \text{ and } \rho(z,q) = \epsilon\}$ is closed for every $\epsilon > 0$ and for every $q \in f(M)$. This follows from the fact that for a given ϵ and a given q, $N = \{z \mid z \in f(M) \text{ and } \rho(z,q) = \epsilon\}$ is either empty, contain one of the points $q - \epsilon$, $q + \epsilon$, or contain both of the points $q - \epsilon$, $q + \epsilon$. If N is $\emptyset$, $f^{-1}(N) = \emptyset$ which is closed. If N contains only one point $f^{-1}(N)$ is closed by hypothesis. If N contains two points, $f^{-1}(N)$ $= f^{-1}(q - \epsilon) \cup f^{-1}(q + \epsilon)$ which is the union of two closed sets, hence closed. Thus condition (ii) implies condition (iii). It remains to be shown that (i) implies (ii) and (iii) implies (i).

Suppose condition (i) is true, and let f be a connected mapping from M onto f(M) such that $f^{-1}\{z \mid z \in f(M) \text{ and } \rho(z,q) = \epsilon\}$ is closed for every $\epsilon > 0$ and for every $q \in f(M)$. Let V be an open set containing point f(p) and let $\eta > 0$ be such that the spherical neighborhood of radius η about f(p) is contained in V.

Let $B_\eta = \{z \mid z \in f(M) \text{ and } \rho(z, f(p)) = \eta\}$. Set $f^{-1}(B_\eta)$ is closed and $p \notin f^{-1}(B_\eta)$. Since M is locally connected at p, there exists a connected set $U \subset (S - f^{-1}(B\ ))$ such that p is interior to U. Now f(U) is connected, contains f(p) and does not intersect B. Hence f(U) is contained in the spherical neighborhood of radius η about f(p) which is contained in V. Hence f is continuous at p. Thus, condition (i) implies condition (ii).

To see that condition (iii) implies (i), assume (iii) is true and that M is not locally connected at p. Since M is not locally connected at p, there is a δ, $0 < \delta < 1$, such that p is not an element of any open connected open set in the spherical neighborhood $N_\delta(p)$ of radius δ about p. Let Q be the component of N_δ which contains p. Note the p cannot be an interior point of Q. Define f by:

$$f(x) = \delta \text{ if } x \in M - N_\delta(p),$$

$$f(x) = \rho(x,p) \text{ if } x \in Q, \text{ and}$$

$$f(x) = \delta - (\delta - \rho(x,p)\ [\rho(x,Q].$$

Now $f \mid Q$ is continuous, hence $f \mid Q$ is connected. Also, f is continuous on $(S - N_q(p)) \cup (N_s p - Q) = (S - Q)$, hence $f \mid (S - Q)$ is connected. Now for any connected subset C of S, $C \subset Q$ or $C \cap Q = \emptyset$ since Q is a component. Thus f(C) is connected and f is a connected mapping. Furthermore, point inverses are closed. To verify this note that $f^{-1}(\delta) = M - N_q(p)$ which is closed. $f^{-1}(0) = \{x\}$ which is closed. For $0 < \epsilon < \delta$, $f^{-1}(\epsilon) = A \cup B$ where $A = \{x \mid x \in Q \text{ and } \rho(x,p) = \epsilon\}$ and $B = \{x \mid x \in (N_\delta(p) - Q) \text{ and } \delta - (\delta - \rho(x,p)) \cdot [\rho(x,Q] = \epsilon\}$. Let us note that Q as a component of $N_\delta(p)$ must be closed with respect to $N_\delta(p)$. Furthermore, $\{x \mid \rho(x,p) = \epsilon\}$ is closed. Therefore, set $A = Q \cap \{x \mid (x,p)' = \epsilon\}$ is closed with respect to $N_\delta(p)$. But A is contained in $Int(N_\delta(p))$, hence A is closed. Now consider B. Let us show that no point of Q can be a limit point of B. Suppose some y in Q is a limit point of B. Then pick z in B such that $\rho(z,y) < \delta - \epsilon$. Then $(\delta - \rho(z,p)) \cdot [\rho(z,Q)] < \delta - \epsilon$ since $(\delta - \rho(z,p)) < 1$ and hence $\delta - (\delta - \rho(z,p)) \cdot [\rho(z,Q)] > \epsilon$ which contradicts the choice of $\delta \in B$. Thus no point of Q is a limit point of B so that B is closed with respect to $(N_\delta(p) - Q)$. Now $(N_\delta(p) - Q)$ is open as the complement of $(M \cup Q)$ so that B is a closed set. Now this implies $f^{-1}(\epsilon) = A \cup B$ is closed so that point inverses are closed.

Function f is discontinuous at p, since p is a limit point $N_\delta(p) - Q$, and in any neighborhood of p there must exist a point x for which $|f(x) - f(p)|$ is arbitrarily closed to δ. Thus under the assumption M is not locally connected we have been able to construct a real valued function f on M which is connected, has closed point inverses, and is discontinuous. This contradicts the hypothesis, hence M must be locally connected and (iii) implies (i). This completes the proof.

In Theorem 4.19 (iii) the requirement that $f^{-1}(q)$ be closed for every q in the range space can be replaced with the requirement that f be monotone. These results will be given in Theorem 4.21 after the following backgroup information is given.

Definition 4.9. If S and T are two Hausdorff spaces and if f is a mapping of S into T, then f is <u>monotone</u> if and only if for every p in T, $f^{-1}(p)$ is a connected subset of p.

Theorem 4.20. [21] If f is a monotone connected mapping of a Hausdorff space S onto a Hausdorff space T then for every q in T, $f^{-1}(q)$

is a closed subset of K.

Proof. Suppose $f^{-1}(q)$ is not closed for some $q \in T$. Let p be a limit point of $f^{-1}(q)$ which does not belong to $f^{-1}(q)$. Because f is monotone, $f^{-1}(q)$ is a connected subset of S and $f(f^{-1}(q)) = q$. Now consider the set $f^{-1}(q) \cup p$ which is also a connected set since the union of a connected set with some or all of its limit points is connected. Now $f(f^{-1}(q) \cup \{p\})$ is a connected subset of T, since f is connected. But $f(f^{-1}(q) \cup \{p\}) = q \cup f(p)$. Since a connected subset of a Hausdorff space cannot consist of two distinct points, f(p) must equal q. This contradicts the statement $p \notin f^{-1}(q)$, which implies $f^{-1}(q)$ is closed.

Since a metric space is Hausdorff, we can now restate a part of Theorem 4.19 as follows.

Theorem 4.21. Let M be a metric space and let f be a monotone connected mapping from M onto a metric space f(M). Mapping f is continuous at a point p in M if and only if M is locally connected at point p.

Corollary. A monotone real valued connected mapping defined on a connected subset of the real numbers is continuous.

MAPPINGS THAT ARE BOTH COMPACT PRESERVING AND CONNECTED

Conditions have now been given for a compact preserving function to be continuous and for connected functions to be continuous. Conditions will now be given which will imply a function which is both compact preserving and connected will be continuous.

Theorem 4.22. [15] Let S and T be metric spaces and let f be a connected, compact preserving mapping from S onto T. If S is locally connected, then f is continuous. If S is not locally connected there exists a compact preserving connected mapping from S into the real numbers which is not continuous.

Proof. To prove the first assertion, suppose S is locally connected but that f is not continuous at a point p in S. By Theorem 4.4 there exists a sequence of points $\{p_i\}$ in S converging to p and a point q in T, $q \neq p$, such that $f(p_i) = q$ for each i. By the local connectedness of S at p and because $\{p_i\}$ converges to p, one can pick a subsequence $\{x_i\}$ of $\{p_i\}$ and a sequence $\{C_i\}$ of connected open sets about p such that $\{p, x_i\} \subset C_i$ for each i, and C_i is contained in the spherical neighborhood of radius 1/i about p. Thus set $\{f(p, f(x_i)\} = \{f(p), q\}$ must be contained in $f(C_i)$ which is connected. Now point q cannot be an isolated point in C_i, for any i, since connected sets cannot contain isolated points. Thus, for each i, there exists a point z_i in $f(C_i)$ such that $0 < \rho(z_i, q) < 1/i$. For each i, let y_i be an element of $C_i \cap f^{-1}(z_i)$. Set $\{p, y_1, y_2, y_3, \ldots\}$ is a countably compact subset of S, and is, therefore compact. But set $\{f(p), z_1, z_2, z_3 \ldots\}$ is not compact since it does not contain the limit point q. This contradicts the hypothesis that f is compact preserving. Therefore, we must conclude f is continuous.

Now suppose M is not locally connected at some point p in S. Then there exists a real number δ, $0 < \delta < 1$, such that no connected open subset of the spherical neighborhood $N_{2\delta}(p)$ about p contains p. Let Q be the component of $N_{2\delta}(p)$ that contain p and note the p cannot be an interior point of Q. Define f from S into the real numbers by:

$$f(x) = 0 \text{ for } x \in S - N_{2\delta}(p),$$

$$f(x) = 2\delta - \rho(x,p) \text{ for } x \in (N_{2\delta}(p) - N_{\delta}(p)),$$

$$f(x) = \rho(x,p) \text{ for } x \in N_{\delta}(p) \cap Q, \text{ and}$$

$$f(x) = \delta \text{ for } x \in N_{\delta}\ p - Q.$$

Let $A = Q \cup (S - N_{\delta}(p))$. One can easily verify f restricted to A is continuous and f restricted to $S - (N_{\delta}(p) \cap Q)$ is continuous. Let us now show that f is compact preserving and connected. Consider an arbitrary compact subset C of S. Now $f(C \cap A)$ is compact since A is closed, hence compact, and f restricted to A is continuous. Furthermore, $f(C - A) = \{\delta\}$, so f(C) must be compact and f is compact preserving. Now consider an arbitrary connected set K in S. Since f restricted to A is continuous and f restricted to $S - N_{\delta}(p) \cap Q$ is continuous, the set f(K) must be connected if K is contained in either of the sets. In the remaining case, K intersects both $(N_{\delta}(p) \cap Q)$ and $N_{\delta}(p) - Q$, and being connected cannot be contained in $N_{2\delta}(p)$. This follows since $K \subset N_{2\delta}(p)$ would imply K would be a subset of the component Q of $N_{2\delta}(p)$ since $K \cap Q \neq \emptyset$. Hence $\{\rho(y,p) \mid y \in K\} \supset [\delta, 2\delta]$ and f(K), from the definition of f, must equal $[0,\delta]$. Thus in any case f is a connected mapping. The function f is discontinuous at p since $f(p) = 0$, but every neighborhood of p must contain a point z of $(N_{\delta} - Q)$ for which $f(z) = \delta$. This completes the proof of the second assertion of the theorem.

The metric spaces S and T in the first assertion of Theorem 4.23 can be replaced with Hausdorff spaces if S is required to have the property K^* defined in Definition 4.4.

Theorem 4.23. [7] If S is a Hausdorff space with property K^* at each point p and if f is a connected and compact preserving mapping from S onto a Hausdorff space T, then f is continuous.

Proof. By Theorem 4.7 it will only be necessary to show point inverses are closed. Let y be an element of T and assume there exists a limit point x of $f^{-1}(y)$ such that x is not an element of $f^{-1}(y)$. Let $\{C_\alpha\}$ be the collection of connected neighborhoods of x and let $\{U_\gamma\}$ be the collection of neighborhoods of y. Since T is Hausdorff, and since $f(x) \neq y$ it is possible to select disjoint open set U and V containing y and x, respectively. For each C_α in $\{C_\alpha\}$ and for each U_γ in $\{U_\gamma\}$ let $y_{\alpha,\gamma}$ be an element of $f(C_\alpha) \cap ((U_\gamma \cap U)$, and let $x_{\alpha,\gamma}$ be an element of $(f^{-1}(y_{\alpha,\gamma}) \cap C_\alpha)$. The set A of all such $x_{\alpha,\gamma}$ is infinite and has x as an accumulation point. By the property K^*, $A \cup \{x\}$ has an infinite compact subset K with x as an accumulation point. Since K is compact, hence closed, x must be an element of K. Let g denote function f restricted to K. Then $M = (g(K) - g(x)) = g(K) \cap (T - V)$ since $g(K) \subset U$ and $U \cap V = \emptyset$. Now $g(K) \cap (T - V)$ is an infinite compact set, hence must have a limit point z in T. If $g^{-1}(z)$ is an isolated point in K, then $K - \{f^{-1}(z)\}$ and $M - \{z\}$ are compact, which is a contradiction. Thus for each accumulation point z of M, $f^{-1}(z)$ must be an accumulation point of K.

Let L be the set of all limit points of K with the exception of x. For each p in L, select disjoint open sets W_p and V_p containing p and x, respectively. Each $(K - W_p)$ is closed hence compact and each B = $g(K - W_p) \cap M$ is a closed non-empty subset of M. Let $f = \{B_p \mid p \in L\}$.

Suppose now that there exists a finite subcollection $B_{p_1}, B_{p_2}, \ldots, B_{p_n}$ such that $\bigcap_{i=1}^{n} B_{p_n} = \emptyset$. Then the corresponding closed subsets $(K - W_{p_i})$ in $(K - \{x\})$ must have the property that $\bigcup_{i=1}^{n}(K - W_{p_i}) = \emptyset$. Now by DeMorgan's law, $\bigcup_{i=1}^{n} W_{p_i}$ must cover $(K - \{x_i\})$. However, for each W_{p_i}, there exists an open set V_{p_i} containing x such that $V_{p_i} \cap W_{p_i} = \emptyset$. The set $\bigcap_{i=1}^{n} V_{p_i}$ is an open set containing x such that $(\bigcap_{i=1}^{n} V_{p_i}) \cap (\bigcup_{i=1}^{n} W_{p_i}) = \emptyset$. This leads to a contradiction since $\bigcap_{i=1}^{n} V_{p_i}$ must contain points of $(K - \{x\})$. Thus for any finite subcollection $B_{p_i}, B_{p_2}, \ldots B_{p_n}$ of F, $\bigcap_{i=1}^{n} B_{p_i} \neq \emptyset$. This implies $\bigcap_{B_p \in F} B_p \neq \emptyset$, [14 ,136]. Now for each point q in $\bigcap_{B_p \in F} B_p$, $g^{-1}(q)$ is an isolated point of K since $g^{-1}(q)$ is not an element of W_p for any p.

Let D denote the collection of all such isolated points in K. Since D is open in K, for each p in L, the set $K - (W_p \cup D)$ is closed hence compact and non-empty. Then $\{g(K - (W_p \cup D)) \mid p \in L\} \cap M$ is a null intersection of non-empty closed subset of the compact space M and there must exist some finite cubcollection of these sets which has an empty intersection, and which cover M. [14,136]. This implies, by DeMorgan's laws, that a finite subcollection $W_{p_1}, W_{p_2}, \ldots$, with $P_i \in L$ for each i, must cover $K - (D \cup \{x\})$. Since x is an accumulation point of K, set D must be infinite and hence $D \cup \{x\}$ must have an infinite subset H such that x is the only accumulation point of H. Then $g(H) \cap M$ is an infinite compact subset of S and must have an accumulation point z which belongs to H. This is a contradiction since $g^{-1}(z)$ is an isolated point of K.

Since that assumption $x \notin (f^{-1}(y))$ leads to a contradiction, x must be an element of $f^{-1}(y)$, $f^{-1}(y)$ must be closed, and by Theorem 4.7, f is continuous.

CHAPTER V

CLIQUISH AND NEIGHBORLY TRANSFORMATIONS

INTRODUCTION

The requirement that a function be continuous is very restrictive. Therefore, one is often tempted to define classes of functions that satisfy weaker conditions than continuity and to investigate these classes of functions to see if problems that are solvable using continuity would also be solvable using a less restrictive condition. Since a considerable body of material has been developed concerning continuous functions, one will naturally inquire what properties the new class of functions will have in common with continuous functions, and what properties of continuous functions are not true for the new class of functions. Another line of inquiry is to ask if the new class of functions will be useful in dealing with topological problems which are not solvable using continuity.

In this chapter, two classes of functions satisfying weaker conditions than continuity will be defined and their properties investigated. Special emphasis will be placed on discovering whether or not certain properties of continuous functions are true for these new classes of functions. At the end of the chapter, several theorems will be presented which show that the new classes of functions can be used to characterize derivative functions of continuous real valued functions.

This is particularly significant since it is well known that the derivative of a continuous function is not, in general, continuous. Formal definitions will now be presented for these new classes of functions.

Definition 5.1. A function f from a space S into a space T is said to be neighborly at a point x of S if and only if for every neighborhood V of f(x) and for every neighborhood U_x of x there exists an open set U such that $U \subset U_x$ and $f(U) \subset V$. Function f is said to be neighborly on S if f is neighborly at every point of S.

One should note that x is required to be an element of U_x, but is not necessarily an element of U.

For T a metric space, with metric ρ, one can restate Definition 5.1 as follows:

Definition 5.1.b. A function f from a space S into a metric space T, with metric ρ, is neighborly at a point x of S if and only if for every $\epsilon > 0$ and for every neighborhood U_x of x there exists an open set $U \subset U_x$ such that $\rho(f(x), f(y)) < \epsilon$ for every $y \in U$.

Definition 5.2. A function f from a space S into a metric space T, with metric ρ, is said to be cliquish at a point x of S if and only if for every $\epsilon > 0$, and for every neighborhood U_x of X, there exists an open set $U \subset U_x$ such that $\rho(f(y), f(z)) < \epsilon$ for every pair y, z of element in U. A function f is said to be cliquish on S if f is cliquish at every point in S.

One can easily verify that every continuous function is neighborly and that continuous functions and neighborly functions are cliquish, provided the range space is a metric space.

Example 5.1. The function f from the real numbers R into R defined by:

$$f(x) = \begin{cases} \sin \frac{1}{x}, \text{ if } x = 0, \\ 0, \text{ if } x = 0, \end{cases}$$

is not continuous at x = 0 but is neighborly and cliquish at that point. To verify that f is not continuous at x = 0, one can note that the sequence $\left\{\frac{2}{n\pi}\right\}_{n=1}^{\infty}$ converges to 0 but the sequence $\left\{f(\frac{2}{n\pi})\right\}_{n=1}^{\infty}$ does not converge to f(0) = 0. This follows from the fact that $f(\frac{2}{n\pi}) = \pm 1$, or 0, depending on the choice of n.

Function f is neighborly, however. To show this, let U_0 be a neighborhood of 0 and choose a positive integer N such that $x = \frac{1}{2N\pi}$ is contained U_0. Now f is continuous at point x since f is a composition of continuous functions at all points except 0. Furthermore, $f(x) = f(\frac{1}{2n\pi}) = \sin(2N\pi) = 0$. Now given $\epsilon > 0$, let V be a neighborhood of x such that $\rho(f(x),f(y)) < \epsilon$ for all $y \in V$ and let $U = V \cap U_0$. For any $y \in U$, $\rho(f(0),f(y)) = \rho(f(x),f(y)) < \epsilon$ since f(0) = f(x) and since $y \in V$. This implies f is neighborly at x = 0. Since a neighborly function whose range is a metric space is cliquish, f is also cliquish.

Example 5.2. The function from R into R defined by:

$$f(x) = \begin{cases} \sin \frac{1}{x}, \text{ if } x = 0, \\ 2, \text{ if } x = 0, \end{cases}$$

is cliquish, but is neither neighborly nor continuous.

PROPERTIES OF CLIQUISH AND NEIGHBORLY FUNCTIONS

Since every continuous function is neighborly and since every neighborly function whose range space is a metric space is cliquish, neighborliness is a weaker condition than continuity and clisuishness is a weaker condition than neighborliness. This suggests that neighborly functions might possess certain properties of continuous functions that the still weaker cliquish functions might not possess. The following discussion will verify that such properties exist.

Definition 5.3. A subset A of a topological space S is said to be <u>everywhere dense in S</u> if $\overline{A} = S$.

Definition 5.4. A function f is said to be <u>pointwise continuous</u> on a space S if the set of point where f is noncontinuous is everywhere dense in S but is not closed relative to S. A function f is said to be <u>pointwise noncontinuous</u> on S if the set of points where S is continuous is everywhere dense in S but is not closed in S.

Definition 5.5. A function f is said to be <u>pointwise neighborly</u> on a space S if the set of points of S where f is non-neighborly is everywhere dense in S but is not closed in S. A function f is said to be <u>pointwise non-neighborly</u> on a space S if the set of points of S where f is neighborly is everywhere dense in S but is not closed in S.

The following example gives a function that is pointwise continuous, pointwise noncontinuous, pointwise neighborly, and pointwise non-neighborly on the open interval (0,1) with the usual topology.

Example 5.3. [25] Let f be defined on (0,1) as follows:

$$f(x) = \begin{cases} 0 \text{ if } x \text{ is irrational, and} \\ \frac{1}{q} \text{ if } x = \frac{p}{q} \text{ where p and q are relatively prime.} \end{cases}$$

Let us now show that f is pointwise continuous and pointwise non-continuous by showing that f is continuous at every irrational point, and discontinuous at each rational point.

Let x be an irrational point and let $\epsilon > 0$ be given. There exists only a finite number of q for which $1/q > \epsilon$. Let $A = \{p/q \mid p$ and q are relatively prime, $1/q > \epsilon$ and $|x - p/q| < \epsilon\}$. Now set A contains at most a finite number of points. For each $p/q \in A$, let $\delta_{p/q} = |x - p/q|$ and let $\delta = 1/2 \min \{\delta_{p/q} \mid p/q \in A\}$. Now for any $y \in N_\delta(x)$, y is irrational or $y = p/q$ where $1/q < \epsilon$. In either case, $|f(x) - f(y)| < \epsilon$ so that f is continuous at x.

If x is rational, $f(x) = 1/q$ for some integer q. Now for $\epsilon < 1/q$ it is impossible to find a δ such that $|f(x) - f(y)| < \epsilon$ for any $y \in N_\delta(x)$. This is true, since every neighborhood of x must contain an irrational point y and $|f(x) - f(y)| = |1/q - 0| = 1/q > \epsilon$. Thus f is discontinuous at every rational point.

Since f is continuous at every irrational point, and discontinuous at every rational point, f is pointwise continuous and pointwise non-continuous on R.

The function defined in Example 5.3 is also neighborly and non-neighborly on (0,1). To verify this one can note that f is neighborly at each irrational point since f is continuous at each irrational point.

To verify that f is non-neighborly at each rational point, let x be a rational point. Since x is a rational point, $f(x) = 1/q$ for some positive integer q. Choose $\epsilon < 1/q$. Now for each neighborhood U of x and for each open set N contain in U, there exists an irrational point y in N. Now $|f(x) - f(y)| = |1/q - 0| = 1/q > \epsilon$, so that f is not neighborly at x.

The following theorem shows that it is impossible to find a function that is both pointwise cliquish and pointwise non-cliquish on any space S.

Theorem 5.1. [24] Let f be a function defined on a space S. If f is cliquish at each point of a set which is everywhere dense in S, then f is cliquish on S.

Proof. Let f be a function defined on S which is cliquish on a set which is everywhere dense in S. Then there exists a set C, everywhere dense in S, such that for every point $c \in C$ the function f is cliquish at c. Let x be an arbitrary point of S and let N_x be an arbitrary neighborhood of x. In N_x there must exist at least one point c of C, since x is either a point of C or a limit point of C. Let a positive number ϵ be given, and let N_c be a neighborhood of c such that N_c is contained in N_x. Since f is cliquish at c, there exists a neighborhood N contained in N_c, and hence in N_x, such that for every pair x_1, x_2 of elements of N, $\rho[f(x_1), f(x_2)] < \epsilon$. Since N is contained in N_x and since N_x was an arbitrary neighborhood of x, f is cliquish at x. But x was an arbitrary point of S, so that f is cliquish at every point of S.

As a consequence of the above theorem, every pointwise noncontinuous function, whose range is a metric space, is cliquish, and every pointwise non-neighborly function whose range is a metric space is cliquish at all points.

Definition 5.6. A subset A of a space S is said to be <u>nowhere</u>

dense in S if and only if for every open subset U of S there exists an open subset $V \subset U$ such that $V \cap A = \emptyset$.

The function defined in Example 5.3 was both pointwise continuous and pointwise neighborly. The points where f was continuous and neighborly was the set of irrational points in (0,1). The set of irrational points in (0,1) is not nowhere dense. Thus it is possible to have pointwise continuous functions whose points of continuity are not nowhere dense, and to have pointwise neighborly functions with an analogous property. In contrast, the points where a pointwise cliquish function is cliquish must be nowhere dense.

Theorem 5.2. [24] The set of points at which a pointwise cliquish function is cliquish is nowhere dense.

Proof. Suppose the set of point C at which a pointwise cliquish function f is cliquish is not nowhere dense in the domain of definition of f. Then there would exist at least one neighborhood N such that C would be everywhere dense in N. By Theorem 5.1, f would be cliquish at every point of N. This contradicts the hypothesis that the set of points where f is non-cliquish is everywhere dense.

Definition 5.7. A subset A of a metric space S is said to be of the first ρ category if A can be expressed as the union of a denumerable number of nowhere dense sets.

We shall now show, for a function f which is the limit of a sequence of neighborly function, that the points of discontinuity of f form a set of the first ρ category. Since a sequence of continuous functions is also a sequence of neighborly functions, we will also obtain the result that for a function f which is the limit of a sequence of continuous functions, the points of discontinuity of f forms a set of the first ρ category. In contrast, in Theorem 5.4 we shall show the set of points of discontinuity of a convergent sequence of cliquish functions need not be of the first ρ category.

Theorem 5.3. [1] If g is a function from a metric space S, with metric ρ, into a metric space T, with metric ρ', and if $\{f_n\}$ is a sequence of neighborly function such that $\lim \rho'(f_n(x),g(x)) = 0$, for every x in S, then the points of discontinuity of g form a set of the first ρ category.

Proof. Let $\omega(x) = \limsup_{y \to x} \rho'(g(x),g(y))$ for x in S. Since the set of points of discontinuity of g is the set of points for which $\omega(x) > 0$, the desired conclusion follows from the following statement:

Statement. If n is a positive integer, if $0 < \epsilon < \infty$ and, if $A_n = \{x \mid \omega(x) \geq \epsilon \text{ and } \rho'(f_m(x),g(x)) \leq \epsilon/16 \text{ for each integer } m \geq n\}$, then A_n is nowhere dense.

Proof. Suppose (1) A_n is everywhere dense in some open sphere α. Let x_1 be an element of $A_n \cap \alpha$ and use the neighborliness of f_n to find an open sphere $\alpha_1 \subset \alpha$ such that (2) $\rho'(f_n(x_1), f_n(z)) \leq \epsilon/16$ whenever $z \in \alpha_1$. Let x be an element of α_1 and choose an integer m such that $m \geq n$ and (3) $\rho'(f_m(x), g(x)) \leq \epsilon/16$. Now use the neighborliness of f_m to secure an open sphere α_2 such that $\alpha_2 \subset \alpha_1$ and (4) $\rho'(f_m(x),f_m(z)) \leq \epsilon/16$ whenever $z \in \alpha_2$. Let x_2 be an element of $A_n \cap \alpha_2$. From statements

3, 4, 2 and from the fact that x_1 and x_2 are elements of A, it follows that

$$\rho'(g(x),g(x_1)) \leq \rho'(g(x),f_n(x)) + \rho'(f_n(x),f_n(x_2))$$
$$+ \rho'(f_n(x_2),g(x_2)) + \rho'(g(x_2),f_m(x_2)) +$$
$$\rho'(f_m(x_2),f_m(x_1)) + \rho'(f_m(x_1),g(x_1)) \leq$$
$$\epsilon/16 + \epsilon/16 + \epsilon/16 + \epsilon/16 + \epsilon/16 + \epsilon/16 = 3\epsilon/8.$$

Thus $\rho'(g(x),g(x_1)) \leq 3\epsilon/8$ whenever $x \in \alpha_1$. Accordingly, $\rho'(g(x),g(y)) \leq 3\epsilon/4$ whenever $x \in \alpha_1$, $y \in \alpha_1$. Thus $\omega(x) \leq 3\epsilon/4$ whenever $x \in \alpha_1$, $A \cap \alpha_1$ is empty, and in contradiction to (1), A_n is nowhere dense in α. Hence set $A = \bigcup_{n=1}^{\infty} A_n$, which is the set of points of discontinuity of g, is of the first ρ category as claimed.

Corollary. The points of discontinuity of a neighborly function f constitute a set of the first ρ category.

Proof. Let sequence $\{f_n\}$ be defined by $f_n = f$ for each n and apply Theorem 5.3.

Theorem 5.4. [24] The limit $f(x)$ of a sequence of cliquish functions can be non-cliquish at every point of its domain of definition.

Proof. Let f be defined on the interval (0,1) of the real numbers with the usual topology. Define f by:

$$f(x) = 0 \text{ if } x \text{ is irrational, and}$$
$$f(x) = 1 \text{ if } x \text{ is rational.}$$

Function f is non-cliquish at every point, and is the limit of the sequence $\{f_n\}$ defined by:

$$f_n(x) = \sum_{q=2}^{n} g_q(x),$$

where $g_q(x) = 1$ if $x = p/q$, $p < q$, and p and q relatively prime integers, while $g_q(x) = 0$ otherwise. Each $f_n(x)$ is cliquish at every point in $(0,1)$.

In Theorem 5.3 it was shown that the points of discontinuity of a function which is the limit of a sequence of neighborly function is of the first ρ category. Theorem 5.4 implies this is not true for a function which is the limit of a sequence of cliquish functions. However, the points of discontinuity of a cliquish function must be of the first ρ category. This property of neighborly function was shown in the corollary to Theorem 5.3.

Theorem 5.5. [24] The points of discontinuity of a cliquish function must be of the first ρ category.

Proof. Let f be a cliquish function defined on a space S. For each $x \in S$ let $\omega(x) = \lim \sup_{y \to x} \rho(f(x), f(y))$. The set of points of discontinuity of f is the set of points of S for which $\omega(x) > 0$.

$$\text{Let } A_1 = \{x \mid x \in S,\ \omega(x) > 1\}$$
$$A_2 = \{x \mid x \in S,\ \omega(x) > 1/2\}$$
$$A_n = \{x \mid x \in S,\ \omega(x) > 1/n\}$$

Since each point of discontinuity of f is an element of A_n for some n, $\bigcup_{n=1}^{\infty} A_n$ is the set of points of discontinuity for f. Let us show each A_n is nowhere dense.

Suppose for some n the set A_n is not nowhere dense. Then there exists some open set α in S such that A_n is everywhere dense in α. Let $x \in A_n \cap \alpha$. By the cliquishness of f, there exists some neighborhood $\alpha_1 \subset \alpha$ such that $(f(y), f(z)) < 1/2n$ for every pair y, z of elements in α_1.

But this implies that no point of A_n is contained in α_1, which contradicts the assumption that A_n is not nowhere dense. Thus A_n is nowhere dense for each n and $\bigcup_{n=1}^{\infty} A_n$ which is the set of points of discontinuity of f is of the first ρ category.

Corollary. Every cliquish function is at most pointwise discontinuous.

Proof. Assume the contrary of the corollary as stated. Then the points of discontinuity of f would be everywhere dense in the space S of definition and closed with respect to S. This implies that the points of discontinuity of f would equal S. This contradicts Theorem 5.5.

The following example shows that a function which is the limit of a sequence of neighborly functions need not be neighborly at all points.

Example 5.4. Let S be the closed interval [0,1] with the usual topology. Define sequence $\{f_n\}$ of functions on S by:

$$f_1(x) = \begin{cases} 1 \text{ if } x = 1 \text{ or } 0 \\ 0 \text{ otherwise,} \end{cases}$$

$$f_2(x) = \begin{cases} 1 \text{ if } x = 0,\ 1 \text{ or } 1/2, \text{ and for each } n = 3,4,5,\ldots, \\ 0 \text{ otherwise,} \end{cases}$$

$$f_n(x) = \begin{cases} 1 \text{ if } x = 0 \text{ or if } x = 1/k \text{ for } K = 1,2,3,\ldots,n. \\ 0 \text{ otherwise.} \end{cases}$$

Let $f = \lim f_n$. Now $f(x) = 1$ if $x = 0$ or if $x = 1/i$, where $i = 1,2,3,\ldots$, and $f(x) = 0$ otherwise. Function f is not neighborly at point $x = 0$.

As was shown in Example 5.4 and Theorem 5.4, the limit of a sequence of neighborly functions need not be neighborly at all points and the limit of a sequence of cliquish functions need not be cliquish at all points. It is also true that a convergent sequence of continuous functions need not be continuous at all points. It is true, however, that the limit of a uniformly convergent sequence of continuous functions from a space S into a metric space M is continuous. Analogous results hold for uniformly convergent sequences of neighborly functions and uniformly convergent sequences of cliquish functions.

Theorem 5.6. A function f from a space S into a metric space M which is the uniform limit of a convergent sequence of neighborly functions is neighborly.

Proof. Suppose f is the limit of a uniformly convergent sequence $\{f_n\}$ of neighborly functions from a space S into a space T. Let $\epsilon > 0$ be given. Pick an N such that $\rho(f_n(x), f(x)) < \epsilon/3$ for all $x \in S$ and for all $n \geq N$. Choose x in the domain of f and let α be a neighborhood of x. By the neighborliness of $f_n(x)$, there exists an open set α_1 contained in α such that for all y in α_1, $(f_n(x), f_n(y)) < \epsilon/3$. Now consider $(f(x), f(y))$ for any $y \in \alpha_1$. $(f(x), f(y)) \leq (f(x), f_n(x)) + (f_n(x), f_n(y)) + (f_n(y), f(y)) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$. Thus f is neighborly at x, and since x was chosen arbitrarily, f is neighborly on S.

Theorem 5.7. A function which is the limit of a uniformly convergent sequence of cliquish functions is cliquish.

Proof. Suppose f is the limit of a uniformly convergent sequence $\{f_n\}$ of cliquish functions from a space S into a metric space M. Let

$\epsilon > 0$ be given. By the uniform convergence of $\{f_n\}$, an integer N can be chosen such that $\rho(f(x), f_n(x)) < \epsilon/3$ for all $n > N$ and for all x in S. Now let x be an arbitrary element of S, let α be a neighborhood of x, and let $n > N$ be given. By the cliquishness of f_n there exists an open set $\alpha_1 \subset \alpha$ such that $\rho(f_n(x_1), f_n(x_2)) < \epsilon/3$ for all pairs of elements x_1 and x_2 in α_1. Now for x_1, x_2 in α_1, $\rho(f(x_1), f(x_2)) \leq \rho(f(x_1), f_n(x_1)) + \rho(f_n(x_1), f_n(x_2)) + \rho(f_n(x_2), f(x_2)) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$. Therefore, f is cliquish at x. Since x was chosen arbitrarily f is cliquish at every point of S as claimed.

CHARACTERIZATIONS OF DERIVATIVE FUNCTIONS

For a continuous real valued function defined on a subset of the real numbers, it is well known that the derivative function may not be continuous. The following example gives a function for which the derivative exists at every point, but the derivative is discontinuous at the point $x = 0$.

Example 5.5. Let a function f be defined by:

$$f(x) = \begin{cases} x^2 \sin 1/x \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

Now $f'(0) = \lim_{x \to 0} \left(\frac{x^2 \sin 1/x - 0}{x - 0}\right) = \lim_{x \to 0} (x \sin 1/x) = 0$, and $f'(x) = \cos 1/x + 2x \sin 1/x$ if $x \neq 0$. For $x \neq 0$, $f'(x)$ is continuous, but $f'(x)$ is not continuous at $x = 0$ as one can see by considering points of the form $1/2n\pi$.

This example can be generalized for the interval $(0,1)$ to give a function whose derivative exists at all points in $(0,1)$, but for which

$f'(x)$ is continuous at all irrational ~~points,~~ and discontinuous at all rational points.

Example 5.6. [22] Let the domain of f be the interval $(0,1)$. Order the rational in $(0,1)$ as sequence $\{x_n\}$.

Let $g_n(x) = (x - x_n)^2 \sin(1/x-x_n)$ for $x \neq x_n$, and for each n, let $g_n(x_n) = 0$.

Let $f(x) = \sum_{n=1}^{\infty} g_n/n^2(x)$.

Now $\{g_n(x)/n^2\}$ is a uniformly convergent sequence since $-1 \leq g_n(x) \leq 1$. Thus $f'(x) = \sum_{n=1}^{\infty} g'_n(x)/n^2$.

Now $f'(x) = \sum_{n=1}^{\infty} - \cos 1/(x-x_n) + (x - x_n)^2 \sin 1/(x-x_n)$ if $x \neq x_n$ for any n, and is continuous at all points where $x \neq x_n$ for any n. However, $f'(x) = \sum_{k=1}^{n-1} g'(x)/K^2 + \sum_{n+1}^{\infty} f'(x)/K^2 + 0$ if $x = x_n$ for some n. At the point where $x = x_n$, $f'(x)$ is discontinuous since the sequence of points $\{1/2n\pi + x_n\}$ converges to x_n, but $f\{(1/2n\pi + x_n)\}$ does not converge to $f'(x_n)$.

From Example 5.5 and 5.6 one can easily see that the derivative function of a continuous real valued function defined on a subset of the real numbers need not be continuous even though it may be defined at all points. Such derivatives of continuous real valued functions defined on certain subsets of the real number can, however, be characterized as being either neighborly or cliquish. The next theorem states conditions under which the derivative function of a continuous real valued function will be neighborly. This theorem was proven by Smith [22]. A similar theorem can be found in [18].

Theorem 5.8. [22] Let f be a continuous real valued function defined on the real numbers, an open interval, or a closed interval of the real numbers. If the function f has a derivative at each point of its domain S of definition, then f' is neighborly.

The proof of this theorem will depend on a theorem due to Baire [12]. It will be necessary to give some preliminary definitions before stating the theorem by Baire and giving the proof of Theorem 5.8.

Definition 5.8. A subset A of the real number is said to be <u>dense-in-itself</u> if every point x in A is a limit point of A.

Definition 5.9. A subset A of the real numbers is said to be <u>perfect</u> if it is closed and dense-in-itself.

One should note that a closed interval of the real numbers is a perfect set and that an open interval when thought of as a subspace of the real numbers is perfect.

Definition 5.10. A function f defined on a subset S of the real numbers is said to have the <u>Darboux property</u> on S if for every pair of points x_1, x_2 in S with $x_1 < x_2$ such that $f(x_1) \neq f(x_2)$ and for every η with $\min \{f(x_1), f(x_2)\} < \eta < \max \{f(x_1), f(x_2)\}$, there exists an x, $x_1 < x < x_2$, such that $f(x) = \eta$.

This property is of interest since continuous real valued functions defined on the real numbers, closed intervals of the real numbers, or open intervals of the real numbers are known to possess this property. The derivatives of such functions also possess this property.

Theorem of Baire. Let E be a subset of the real numbers which is either perfect or open, and let f be a function defined on E. Function f is the limit of a sequence of functions, each of which is continuous on E, if and only if f is at most pointwise discontinuous with respect to every perfect set contained in E.

Definition 5.11. A function f is said to be of Baire's class less than two if f is continuous or is the limit of a sequence of continuous functions.

A lemma and a theorem will now be proven, from which the proof of Theorem 5.8 will follow.

Lemma 5.1. [22] If the real valued function f is defined on an open interval I of the real numbers and if x is a point of I where f is not neighborly, there exists a positive number ϵ and a neighborhood N of x such that for each point y of continuity of f in $N \cap I$, $|f(x) - f(y)| \geq \epsilon$.

Proof. Let x be an element of I at which f is not neighborly. Suppose for every $\epsilon > 0$ and for every N_x, there exists a continuity point y of f in $N_x \cap I$ such that $|f(x) - f(y)| < \epsilon$. Choose a positive number ϵ_1 such that $|f(x) - f(y)| + \epsilon_1 < \epsilon$. Since y is a continuity point of f, there exists a neighborhood $N_y \subset N_x \cap I$ such that for z in N_y, $|f(z) - f(y)| < \epsilon_1$. Thus for z in N_y, $|f(x) - f(z)| \leq |f(x) - f(y)| + |f(y) - f(z)| < |f(x) - f(y)| + \epsilon_1 < \epsilon$. This, however, contradicts the hypothesis f is not neighborly at x.

Theorem 5.9. [22] If the real valued function f, defined on an open interval I, is a Bain's class less than two and has the Darboux property, then f is neighborly on I.

Proof. Suppose f is not neighborly at the point ξ in I. From Baire's Theorem, it follows that the set of points of continuity of f forms a set which is everywhere dense in I. Since f(x) is not neighborly at ξ, Lemma 5.1 implies there exists a positive number ϵ and a neighborhood $N_1(\xi)$ such that for every continuity point x of f in $N_1(\xi) \cap I$, $|f(x) - f(\xi)| \geq \epsilon$. Choose a neighborhood $N(\xi)$ of ξ such that $N(\xi) \subset N_1(\xi)$ and such that the end points of $N(\xi)$ are continuity point of f. Denote by R the set of continuity points of f in $N(\xi)$. Let $A = \{x \mid x \in N(\xi), \text{ and } |f(x) - f(\xi)| < \epsilon\}$. Consider the set B of points of A at which the saltus $S_f(x)$ relative to A satisfies $S_f(x) > \epsilon/2$. Set B is not null since ξ is an element of B. Let $\bar{B}$ denote the closure of B. Now every point of $\bar{B}$ is an interior point of $N(\xi)$ since the end points of $N(\xi)$ are points of continuity of f and $S_f(x) = 0$ at these points. Let us now show that $\bar{B}$ is perfect by showing that every point of $\bar{B}$ is a limit point of $\bar{B}$.

If x in $\bar{B}$ is such that $|f(x) - f(\xi)| \geq \epsilon$, then $x \notin A$ and $x \notin B$, so that x must be a limit point of B and a limit point of $\bar{B}$. If x in $\bar{B}$ is such that $|f(x) - f(\xi)| < \epsilon$, then x is in B since $x \in A$ and the set of points where $S_f(x) > \epsilon/2$ is closed relative to B. Now if x is in B for an arbitrary neighborhood N_x of x which is contained in $N(\xi)$, there exists two points x_1, and x_2 in N_x such that $|f(x_1) - f(x_2)| \geq \epsilon/2$. The following possibilities can hold.

(1) $|f(x_1) - f(\xi)| < \epsilon/2$

(2) $|f(x_2) - f(\xi)| < \epsilon/2$

(3) $f(\xi) - \epsilon < f(x_1) \leq f(\xi) - \epsilon/2$, and

$f(\xi) + \epsilon/2 \leq f(x_2) < f(\xi) + \epsilon$.

Since R is everywhere dense in $N(\xi)$ and since f satisfies the Darboux property, in Case (1) x_1 is an element of $\bar{\bar{B}}$, in case (2) x_2 is an element of $\bar{\bar{B}}$, and in case (3) there exists an x_3 in N_x with $f(x_3) = f(\xi)$ and therefore x_3 is in $\bar{\bar{B}}$. If it should happen that any one of the points x_1, x_2 or x_3 equals x then by the Darboux property there exists an x_4 in N_x with $x_4 \neq x$ and such that $f(x_4) - f(\xi) < \epsilon/2$. In this case x_4 is in $\bar{\bar{B}}$. In any case, the arbitrary neighborhood N_x of x must contain a point of $\bar{\bar{B}}$ and $\bar{\bar{B}}$ is perfect.

The saltus $S_f(x) \geq \epsilon/2$ for each point of $\bar{\bar{B}}$ and, therefore, each point of $\bar{\bar{B}}$ is a discontinuity point of f relative in $\bar{\bar{B}}$. By Baire's theorem, f could not be the limit of a sequence of continuous functions. This contradicts the hypothesis that f is of Baire's class less than two and, therefore, f must be neighborly on I.

Proof of Theorem 5.8. If f has a derivative $f'(x)$ at each point x in S, then f' is of Baire's class less than two, since

$$f(x) = \lim_{n\to\infty} \frac{f(x) - f(x + 1/n)}{1/n}.$$

Therefore, by Theorem 5.9 f' is neighborly on S.

Theorem 5.8 gives a characterization of the derivative functions of continuous real valued function defined on the real numbers, open intervals of the real numbers and closed intervals of the real numbers, provided the derivative exists at all points in the domain. The following theorem shows that a characterization of the derivative function can also be given if the derivative function is defined on all but a nowhere dense set in S.

Theorem 5.10. [22] Let space S be the real numbers, an open interval, or a closed interval. If the real valued function f defined on S has a derivative everywhere on S with the possible exception of a nowhere dense set D in S, then the derivative function f' is cliquish on S.

Proof. Let ξ be an arbitrary point of S and let $\epsilon > 0$ be given. For any neighborhood $N(\xi)$ of ξ, there exists an open set N contained in $N(\xi)$ such that $N \cap D = \emptyset$. By Theorem 5.8, f' is neighborly on N. Thus for any x_1 in N there exists an open set $N_1 \subset N$ such that $|f(x_1) - f(y)| < \epsilon/2$ for any y in N_1. Now let y,x be any two elements in N_1. $|f(x) - f(y)| \leq |f(x) - f(x_1)| + |f(x_1) - f(y)| < \epsilon/2 + \epsilon/2 = \epsilon$. Since N_1 is a subset of $N(\xi)$, f is cliquish at ξ.

Let us now give an example of a continuous function whose derivative is cliquish but not neighborly.

Example 5.7. [22] Let S be the closed interval [0,1] and let f be defined by:

$$f(x) = \frac{(2n+1) - (2n^2+2n+1)\,x}{n(n+1)} \text{ if } \frac{1}{n+1} < x \leq \frac{1}{n}, \text{ where}$$

$$n = 0,1,2,3,\ldots, \text{ and}$$

$$f(0) = 0.$$

Now $f'(0) = 0$, but at all other points where the derivative exists, $f'(x) < -2$. Function f' is, therefore, not neighborly at point 0. However, by Theorem 5.10, f' is cliquish at all points in S.

CHAPTER VI

CONNECTIVITY AND PERIPHERALLY CONTINUOUS MAPPINGS

INTRODUCTION

Much of the recent research in topology has been concerned with determining if a mapping f from a space S into itself leaves a point of S fixed. That is, in determining if there exists some point x in S such that $f(x) = x$. For example, it is well known that a continuous mapping from a closed n-cell I into I will leave a point of I fixed. A closed n-cell I is any homeomorphic image of the subset of Euclidean n-space consisting of points of the form $(x_1, x_2, \ldots, x_n)$, where $0 \leq x_i \leq 1$ for each i, $i = 1, 2, \ldots, n$. Many functions which satisfy conditions other than continuity can also be shown to leave points of an n-cell fixed. John Nash in studying fixed point problems defined a mapping which he called a connectivity map and inquired whether or not this kind of mapping left a point of the n-cell fixed [9]. Professor O. H. Hamilton [9] of Oklahoma State University and Professor J. Stalling [23] of Princeton investigated the problem further and gave an affirmative answer. In Hamilton's investigation he defined and made use of another noncontinuous function which he called the peripherally continuous mapping.

Although connectivity mappings and peripherally continuous mappings were defined in connection with fixed point theorems, a considerable amount of research concerning other properties of these functions has taken place. In this chapter, such results concerning these functions will be given. Definitions will now be stated for connectivity and perifpherally continuous transformations.

Definition 6.1 A mapping f from a space S into a space T is said to be a connectivity mapping if and only if the induced mapping g of S into $S \times T$, defined by $g(p) = p \times f(p)$, transforms connected subsets of S onto connected subsets of $S \times T$.

Using the definitions one can easily show that a connectivity map is a connected map.

Definition 6.2. A mapping f from a space S into a space T is said to be peripherally continuous if and only if for each point p of S and for each pair of open sets U and V containing p and f(p), respectively, there exists an open set $D \subset U$ containing p such that f transforms the boundry F of D into V.

The following examples show that connectivity maps and peripherally continuous mappings need not be continuous.

Example 6.1 Let S be the unit interval $0 \leq x \leq 1$ with the usual topology and define f on S by:

$$f(x) = 1 \text{ if } x \text{ is rational}$$

$$f(x) = 0 \text{ if } x \text{ is irrational.}$$

Function f is peripherally continuous at all points but is discontinuous at all points.

Example 6.2. Let S be the set of rational numbers in [0,1] with the usual topology, let $A = \{x \mid x \in S,\ x = p/q$, where p and q are relatively prime and q is prime$\}$, and let $B = \{x \mid x \in S,\ X = p/q$, where p and q are relatively prime and q is not prime$\}$. Define f on S by:

$$f(x) = 1 \text{ if } x \in A, \text{ and}$$

$$f(x) = 0 \text{ if } x \in B.$$

Mapping f is a connectivity map since the only connected subsets in S are single points. Mapping f is not continuous on S since both A and B are everywhere dense in S.

SOME FIXED POINT PROPERTIES

Since the original work with connectivity maps and peripherally continuous maps was by O. H. Hamilton in connection with fixed point theorems, and since many of the other theorems concerning these mappings followed from his work, it seems appropriate to discuss his results first. After the presentation of Hamilton's work, a systematic presentation of the other theorems concerning these mappings will be given.

Theorem 6.1. [9] If f is a connectivity map from a Hausdorff space S onto a Hausdorff space T, p is a point of S, U and V are open set containing p and f(p), respectively, then every nondegenerate connected subset of S containing p contains a point q of U distinct from p such that f(q) is an element of V.

Proof. Suppose C is a nondegenerate connected subset of S containing p but such that C contains no other point q of U such that $f(q) \in V$. Then g(C) is the union of the two mutually separated sets

$g(p) = \{p \times f(p)\}$ and $g(C - \{p\})$, since $U \times V$ contains $p \times f(p)$ but no point of $g(C - \{p\})$. This contradicts the hypothesis, hence C must contain a point q of U such that $q \neq p$ and $f(q) \in V$.

Theorem 6.2. [9] If f is a connectivity map from a Hausdorff space S into a Hausdorff space T and if C is a closed subset of T, then each component of $f^{-1}(C)$ is a closed subset of S.

Proof. Suppose C is a closed subset of T and that some component E of $f^{-1}(C)$ is not closed. Then there exists a limit point p of E such that p is not an element of E. Thus f(p) is not an element of C. Since C is closed, f(p) is not a limit point of C, and there exists an open set U in T such that f(p) is in U and $U \cap C = \emptyset$. Therefore, $f^{-1}(U) \cap E = \emptyset$. This leads to a contradiction, since the connected set $E \cup \{p\}$ must contain a point q distinct from p such that f(q) is an element of U, by Theorem 6.1. Hence E is closed.

Corollary. If f is a connectivity map from a Hausdorff space S into a Hausdorff space T, p is a point in S, U is an open subset of T containing f(p), and if C is the subset of S consisting of all points q of S such that f(q) is an element of $\overline{U}$, then each component E of C is closed.

The next theorem was stated by Hamilton [9], and a proof of a generalization of this theorem was given by Stalling [23]. The proof of Stalling's theorem will not be given since it involves terms and techniques of algebraic topology which would not be appropriate in this paper. The statement of Hamilton's theorem will be given, however, since his remaining theorems rely on this result.

Theorem 6.3. [9] If f is a connectivity map of a closed n-cell I, $n \geq 2$, onto a subset B of I, then f is peripherally continuous on I. Furthermore, if p is any point of I and U and V are open subsets of I containing p and f(p), respectively, there is a connected set D of I with connected boundary F such that $p \in D$, $D \cup F \subset U$, and $f(F) \subset V$.

It should be noted that the second statement in Theorem 6.3 follows because f is peripherally continuous, and not from Stalling's generalized theorem. A partial converse of this theorem will be presented later.

Theorem 6.4. [9] Let f be a peripherally continuous transformation of a closed n-cell I, $n \geq 2$ into itself. Let it be assumed that I is the closed n-cube consisting of the points $(x_1, x_2, \ldots, x_n)$ given by the inequalities $0 \leq x_i \leq 1$ for each i. Let the faces $x_i = 0$ and $x_i = 1$ be designated by A_1 and B_i respectively. For each point $x = (x_1, x_2, \ldots, x_n)$ in I, let f(x) be designated by $x' = (x_1', x_2', \ldots x_n')$. For each i, $1 \leq i \leq n$, let M_i, L_i, and N_i designate the subsets of I for which $x_1' \leq x_i$, $x_i' = x_i$, and $x_1' \geq x_i$, respectively. Then the components M_i, L_i, and N_i are closed and if $q = (q_1, q_2, \ldots, q_n)$ is a point in the common boundary between a component E of M_i or N_i and a connected subset of I - E, then q is an element of L_i.

Proof. Let $q = (q_1, q_2, \ldots, q_n)$ be a limit point of a component E of M_i and suppose that q does not belong to M_i. Then by the definition of M_i, $q_1' = q_i + d$ for some $d > 0$. Then, since f is peripherally continuous, by Theorem 6.3, there exists a connected open set D of diameter $< d/3$ containing q such that

(1) $E - (\overline{D} \cap E) \neq \emptyset$, and

(2) if x is a point of F, the boundary of D, then $\rho[f(x),f(q)] < d/3$.

The connected set E, since it contains points outside of $\overline{D}$ and within D, must contain a point x of F. This means, $\rho[f(x),f(q)] < d/3$, and $\rho(x,q) < d/3$. Hence, $|x_i' - q_i'| < d/3$ and $|x_i - q_i| < d/3$. With $q_i' = q_i + d$, these inequalities give $x_i' > x_i + d/3$ and this contradicts the fact that x is in M_i. Hence the assumption that q does not belong to M_i is false and E is closed.

A similar argument can be used to show that each component of L_i or N_i is closed.

Now let $q = (q_1,q_2,\ldots,q_n)$ be a point in the common boundary between a component E of M_i and some connected subset R of $I - E$, and suppose q does not belong to L_i. Since q is an element of M_i, $q_i = q_i'$ + d for some $d > 0$. Let δ be a positive real number such that $\delta < d/3$ and such that the spherical neighborhood with center q and diameter δ does not contain all of E. Then since f is peripherally continuous, it follows from Theorem 6.3 that there is a connected domain D with respect to I of diameter $< \delta$ containing q with connected boundary F such that

(1) D contain a point z of R,

(2) $R - (\overline{D} \cap R) = \emptyset$,

(3) if $x = (x_1,x_2,\ldots,x_n)$ is in F then $\rho(f(x), f(q)) < d/3$.

Then $|x_1' - q_i'| < d/3$, $|x_i - q_i| < d/3$, and since $q_i = q_1' + d$, it follows that $x_i > x_i'$. Hence $x \in M_i$, and therefore $F \subset E$. But the connected set R contains a point of F and hence a point of E. This contradicts $R \subset I - E$. Hence the assumption q does not belong to L_i is false. By a similar argument, it can be shown that each point common to the boundaries of a component E of N_i and a connected subset of I - E is in L_i.

The main theorems from Hamilton's paper will now be stated and proofs given.

Theorem 6.5. If f is a peripherally continuous transformation of a closed n-cell I, $n \geq 2$, into itself, then f leaves a point of I fixed.

Proof. Let sets N_i, M_i, L_i and faces A_i and B_i be defined as in Theorem 6.4. Since set A_i must be a subset of N_i for each i and since A_i is a connected subset for each i, let E_i be the component of N_i which contains A_i for each i. By Theorem 6.4, E_i is closed. Let $\{G_\alpha^i\}$ be the collection of all components of $I - E_i$ which contain points of B_i. Let H_i be $[\bigcup_\alpha G_\alpha^i] \cup B_i$. Now H_i is connected since B_i is connected and since each G_α^i is connected and contains a point of B_i. Let K_i be the subset of E_i consisting of all points in the common boundary between H_i and E_i. Then by Theorem 6.4, K_i is a closed subset of L_i and hence $F_i = K_i \cup (B_i \cap L_i)$ is a closed subset of L_i. Now,

(1) No component C of $I - F_i$ contains point of both A_i and B_i. For suppose C contains a point of A_i and a point b of B_i. Then $a \in E_i$ and $b \in H_i$. Hence C contains a point of K_i, the common boundary between H_i and E_i. This contradicts $K_i \subset F_i$ and $C \subset I - F_i$.

Now for each point $x = (x_1, x_2, \ldots, x_n)$ in I let $d_i(x) = \rho(x, F_i)$, and define a mapping W on I as follows. Let $W(x) = W(x_1, x_2, \ldots, x_n)$ be designated by $(x''_1, x''_2, \ldots, x''_n)$. If x belongs to a component of $I - F_i$ which intersects B_i, hence contains no point of A_i, let $x''_i = x_i - 1/2[d_i(x) \cdot x_i]$. Then since $x_i \neq 0$, $x_i \neq x''_i$. If x is an element of a component of $I - F_i$ which does not intersect B_i let $x''_i = x_i + 1/2[d_i(x) \cdot (1 - x_i)]$. Since $(1 - x_i) \neq 0$, $x''_i \neq x_i$. If x is an element of F_i let $x''_i = x_i$.

Now since $d_i(x)/2 < 1$, we have

(2) $0 \leq x''_i \leq 1$, and

(3) $x_i = x''_i$ if and only if $x \in F_i \subset L_i$.

The function W is by its definition a continuous function of I into itself and hence, by the well known Brouwer fixed point theorem for n-cells must leave some point z of I fixed. That is, for each i, $i = 1, 2, \ldots, n$, $x''_i = x_i$. The point z must be an element of $\bigcap_{i=1}^{n} F_i \subset \bigcap_{i=1}^{n} L_i$. But z in $\bigcap_{i=1}^{n} L_i$ implies $x'_i = x_i$ for each i, so that $f(z) = z$, and f leaves a point of I fixed, as required.

Theorem 6.6. If f is a connectivity map of a closed n-cell I into itself, f leaves a point of I fixed.

Proof. If $n = 1$ and if f is a connectivity map of interval I, into itself, then g(I) is connected by the definition of a connectivity map. Furthermore, g(I) contains the points $0 \times f(0)$ and $1 \times f(1)$ in the subset $0 \times I$ and $1 \times I$ respectively. Hence the connected set g(I) must contain a point of the closed connected set $x \times x$ in $I \times I$. This implies $f(x) = x$ for xome x in I.

If $n \geq 2$ then Theorem 6.6 follows directly from Theorems 6.3 and 6.5.

GENERAL PROPERTIES

As previously mentioned, a considerable amount of research concerning the properties of connectivity and peripherally continuous mappings has taken place since the publication of Hamilton's paper concerning fixed point theorems. A systematic discussion of these findings will now be given. General properties of connectivity and peripherally continuous transformations will be given first. The first of these general theorems gives a property of peripherally continuous mappings which is analogous to the property of connectivity mappings given in Theorem 6.2.

Theorem 6.7. [16] If a function f from a Hausdorff space S into a Hausdorff space T is peripherally continuous and if C is a closed subset of T, then each component of $f^{-1}(C)$ is closed in S.

Proof. Suppose some component E of $f^{-1}(C)$ is not closed. Then there exists some limit point p of E such that p is not an element of E. Now f(p) is not an element of C, and since C is closed, there must exist an open set V about f(p) in T such that $V \cap C = \emptyset$.

Since E is non-degenerate, there exists an open subset U of S containing p such that $(S - U) \cap E \neq \emptyset$. There also exists an open subset D of S such that $D \subset U$, $p \in D$ and $f(F(D)) \subset V$, since f is peripherally continuous. Now $D \subset U$ and p a limit point of E implies there exists points of E in D and (S - D). Therefore, F(D) contains at least one point of E, and it follows that f(F(D)) is not a subset of V which is a contradiction. Thus the assumption that E is not closed is false, and the conclusion of the theorem follows.

Corollary 1. If for each closed set C in T $f^{-1}(C)$ consists of a finite number of somponents, $f^{-1}(C)$ is closed for each closed set C in T.

Proof. The conclusion follows from Theorem 6.7 since the union of a finite collection of closed sets is closed.

Corollary 2. If for each closed set C in T $f^{-1}(C)$ consists of a finite number of components, f is continuous.

Proof. By Corollary 1, $f^{-1}(C)$ is closed if C is closed. This implies $f^{-1}(U)$ is open if U is open. Therefore, f is continuous.

The next two theorems are concerned with point set properties which are preserved by peripherally continuous mappings.

Theorem 6.8. [16] If f is a peripherally continuous transformation of a Hausdorff space S into a Hausdorff space T, if N is a connected subset of S, and if $x \in \bar{N}$ then f(x) is an element of $\overline{f(\bar{N})}$.

Proof. Suppose there exists a connected subset N of S and a limit point x of N such that f(x) is not an element of $\overline{f(\bar{N})}$. Then since $\overline{f(N)}$ is closed, there exists some open set V about f(x) such that $V \cap \overline{f(N)} = \emptyset$.

Since N is non-degenerate, there exists an open subset U of S containing x such that $(S - U) \cap N \neq \emptyset$. There also exists an open subset D of S such that $D \subset U$, $x \in D$, and $f(F(D)) \subset U$, since f is peripherally continuous. But $(S - D) \cap N \neq \emptyset$ and $N \cap D \neq \emptyset$. Therefore, F(D) contains at least one point of N since N is connected, and it follows that f(F(D)) is not a subset of V. This is a contradiction, hence the conclusion of the theorem follows.

Theorem 6.9. [16] Let S and T be Hausdorff spaces, let f be a one-to-one peripherally continuous transformation from S into T, and let $M \subset S$ be a non-degenerate connected subset such that S - M has a finite number of components. If x is a boundary point of M, then f(x) is a boundary point of f(M).

Proof. Let x be a boundary point of M. If x is not an element of M, then f(x) is a limit point of f(M) by Theorem 6.8, but f(x) is not an element of f(M) due to the one-to-one property of f. Thus f(x) is a boundary point of f(M).

Now suppose x is a boundary point of M which belongs to M but that f(x) is not a boundary point of f(M). Then f(x) is an interior point of f(M). Since there exists only a finite number of components of S - M, x must be a limit point of some non-degenerate component $E \subset (S - M)$. But $f(E) \subset (T - f(M))$ due to the one-to-one property of f, and therefore $f(x) \notin \overline{F(E)}$ which contradicts Theorem 6.8. Thus f(x) must be a boundary point of f(M).

Another interesting point set property of peripherally continuous mappings is given in the next theorem.

Theorem 6.10. [16] If f from S onto T is a peripherally continuous transformation of a non-degenerate, connected, regular, Hausdorff space S onto a Hausdorff space T, and if y is an interior point of a subset M of T, then every point of $f^{-1}(y)$ is a limit point of $f^{-1}(M)$.

Proof. Let y be an interior point of M and let V be an open subset of T containing y and lying entirely in M. Suppose there exists a point x of $f^{-1}(y)$ that is not a limit point of $f^{-1}(M)$. Then there exists an open subset U of S containing x such that $\bar{\bar{U}}$ contains no point of $f^{-1}(M)$ due to S being regular. Consequently, any open set $D \subset U$ containing x has the property that F(D) contains no point of $f^{-1}(M)$ and, furthermore, F(D) is non-empty since S is connected. Hence F(D) is not a subset of V which contradicts the hypothesis that f is peripherally continuous. The conclusion of the theorem thus follows.

In Theorem 6.2 a point set property of connectivity mappings was given in connection with Hamilton's fixed point theorems. Similar results for peripherally continuous mappings were given in Theorems 6.7, 6.8, 6.9 and 6.10. Let us turn now to a further consideration of point set properties of connectivity maps.

Theorem 6.11. [5] Let f be a connectivity map from the T_1 space S into the T_1 space T. If V is an open subset of T and K is a non-degenerate component of $f^{-1}(V)$, then any point p in the closure of K such that p is not in K has the property that f(p) is in F(V).

Proof. Let p be a limit point of K which is not in K. Since $K \cup \{p\}$ is connected and connectivity maps map connected sets onto connected sets, $f(K \cup \{p\}) = f(K) \cup \{f(p)\}$ is connected. Now f(K) contained in V, f(p) not in V, and $f(K) \cup \{f(p)\}$ connected implies f(p) is a limit point of f(K) by Theorem 6.8. Hence f(p) is a limit point of V which is not in V. Therefore, f(p) is in F(V).

Theorem 6.12. [6] Let f be a connectivity mapping of the locally connected and connected T_1 space S into the T_1 space T. If V is an open subset of T, then $f^{-1}(V)$ is dense-in-itself.

Proof. Suppose $f^{-1}(V)$ is not dense-in-itself. Then there is a point p in $f^{-1}(V)$ and an open set U containing p such that $U - \{p\}$ contains no point of $f^{-1}(V)$. Since S is locally connected there exists a connected open subset C of U containing p. Therefore, $C \times V$ is an open set in $S \times T$ containing only the point $p \times f(p)$ of $g(C)$. This implies $g(C)$ is not connected contradicting the hypothesis that f is a connectivity map. Therefore, every point of $f^{-1}(V)$ is a limit point of $f^{-1}(V)$ and hence $f^{-1}(V)$ is dense-in-itself.

With any class of functions, it is always of interest to determine whether or not a convergent sequence of such function will always converge to a function of the same class. Example 6.3 below proves that the limit function of a sequence of connectivity maps or peripherally continuous maps need not be of the same type. However, if the sequence of functions is required to be uniformly convergent the limit function will be of the same class for certain spaces.

Example 6.3. Let S be the unit unterval $0 \leq x \leq 1$ with the usual topology and define a sequence of function $\{f_n\}$ on S by, $f_n(x) = x^n$, for each $x \in S$. Now each f_n is a connectivity map and is peripherally continuous. Furthermore, sequence $\{f_n\}$ converges to the function f defined by:

$$f(x) = 0 \text{ if } x \neq 1, \text{ and}$$

$$f(1) = 1.$$

Function f is neither a connectivity map nor a peripherally continuous mapping.

Theorem 6.13. [5] Let $\{f_n\}$ be a sequence of peripherally continuous mappings of a space S into a metric space T. If sequence $\{f_n\}$ converges uniformly to a function f on S, then f is peripherally continuous.

Proof. Let p be a point of S and let U and V be open sets containing p and f(p) respectively. Since T is a metric space, there exists an $\epsilon > 0$ such that the spherical neighborhood R of radius ϵ about f(p) is contained in V. Let R' be the spherical neighborhood of radius $\epsilon/4$ about f(p). Since the convergence is uniform there exists a positive integer N such that for every $n > N$, $\rho(f_n(x),f(x)) < \epsilon/4$ for every x in S. Let n_o be a fixed positive integer such that $n_o > N$. Then $f_{n_o}(p)$ is contained in R' and since f_{n_o} is peripherally continuous at p, there exists an open set $D \subset U$ and containing p such that $f(F(D)) \subset R'$. If y is an element of F(D), then $\rho(f(y),f(p)) \leq \rho(f(y),f_{n_o}(y)) + \rho(f_{n_o}(y),f(p))$. Now $\rho(f(y),f_{n_o}(y)) < \epsilon/4$ by the uniform convergence and $\rho(f_{n_o}(y),f(p)) < \epsilon/4$ since $f_{n_o}(y)$ is in R'. Hence $(f(y),f(p)) < \epsilon/2$ and f(y) is in R. Therefore, $f(F(D)) \subset R \subset V$, and D is the required neighborhood which implies f is peripherally continuous.

An analogous theorem holds for connectivity maps; however, the proof for this theorem requires the use of Theorem 6.22 which states conditions under which a peripherally continuous mapping is a connectivity map. This result will be discussed further after the proof of Theorem 6.22.

CHARACTERIZATIONS

Some characterization theorems for peripherally continuous mapping will now be given.

Definition 6.4. A sequence $\{D_i\}$ of open sets is said to <u>close down on a point x</u> if and only if $\{x\} = \bigcap_{i=1}^{\infty} D_i$ and for every open set U containing x there exists a positive integer N such that $D_i \subset U$ for all $i > N$.

Theorem 6.14. [16] Let f from S into T be a transformation where spaces S and T are regular and first countable. Then a necessary and sufficient condition that f be peripherally continuous is that for each $x \in S$ there exists a monotone decreasing sequence of open set $\{D_i\}$, $i = 1,2,3,\ldots,$ closing down of x such that the sequence $\{f(F(D_i))\}$, $i = 1,2,3,\ldots,$ converges to f(x).

Proof. The fact that the condition is necessary follows from the definition of a peripherally continuous mapping.

Now let $\{D_i\}$, $i = 1,2,3,\ldots,$ be a monotone sequence of open sets in S converging to x such that the sequence $f(F(D_i))$, $i = 1,2,3,\ldots$ converges to f(x). If R and V are any two open sets containing x and f(x), respectively, there exists an open set $D_j \in \{D_i\}$, $i = 1,2,3\ldots,$ such that $\bar{D}_j \subset R$. This follows since $\{D_i\}$ closes down on x, and since S is regular. Since the sequence $\{f(F(D_i))\}$ converges to f(x) there exists an open set $D_k \in \{D_i\}$, $i = 1,2,3,\ldots,$ where $k \geq j$ such that $f(F(D_k)) \subset U$. Therefore, by definition, f is peripherally continuous.

Corollary. Let f from S into T be a peripherally continuous transformation of a regular space S into a regular space T such that if x is an element of S there exists a sequence $\{D_i\}$, $i = 1,2,3,\ldots$, of open sets closing down on x such that for each i, $F(D_i) \neq \emptyset$. Then for every point x in S there exists at least one sequence of distinct points converging to x such that their images under f converges to f(x).

Theorem 6.15. [5] If f is a mapping from a space S into a space T, then f is peripherally continuous if and only if g is peripherally continuous.

Proof. Suppose f is peripherally continuous. Let p be a point of S and let U and V be open sets containing p and $p \times f(p)$, respectively, where V is of the form $H \times K$ with H open in S and K open in T. Then $H \cap U$ is an open set containing p and K is an open set containing f(p). Since f is peripherally continuous, there exists an open set $D \subset U \cap H$ containing p such that $f(F(D)) \subset K$. Thus $g(F(D)) \subset V$ and g is peripherally continuous.

Conversely, suppose g is peripherally continuous. Let p be a point of S and let U and V be open set containing p and f(p) respectively. Then $U \times V$ is an open set containing $p \times f(p)$, and hence there exists an open set $D \subset U$ containing p such that $f(F(D)) \subset U \times V$. Therefore, $f(F(D)) \subset V$ and f is peripherally continuous.

MAPPINGS THAT ARE BOTH PERIPHERALLY CONTINUOUS AND CONNECTIVITY MAPS

Hamilton in his original work with connectivity mapping made use of the fact that a connectivity mapping from a closed n-cell, $n \geq 2$, onto a subset of that n-cell was peripherally continuous. Some additional theorems relating connectivity mappings to peripherally continuous mappings will be given next. The first of these theorems is an extension of Hamilton's theorem.

Theorem 6.16. [16] Let I be the closed unit interval, $0 \leq x \leq 1$. If mapping f from I into I is a connectivity map, then f is peripherally continuous.

Proof. Assume that f is not peripherally continuous at some point p in I. Then there exists some subinterval V containing f(p) such that for some open connected subinterval U = (a,b) containing p, no subinterval $D \subset U$ containing p has the property $f(F(D)) \subset V$. There exists, by Theorem 6.1, a point q in U, $q \neq p$, such that f(q) is an element of V. Suppose q is an element of (p,b). It follows from our assumption that no point of (a,p) can be mapped into V under f. Hence the graph of (a,p), $g(a,p) \subset I \times I - U \times V$.

The set $P \times I$ separates $I \times I$ into two mutually separated sets such that the graph g(a,p) is contained in one and g(p,b) is contained in the other. Since g(p) is not a limit point of g(a,p), g(a,b) = $g(a,p) \cup \{g(p) \cup g(p,b)\}$, where g(a,p) and $g(p) \cup g(p,b)$ are mutually separated sets, which is a contradiction of the fact that f is a connectivity map. Thus f is a peripherally continuous mapping.

The following example shows that the converse of Theorem 6.16 is not true.

Example 6.4. Let I be the unit interval, $0 \leq x \leq 1$, and define f on I by:

$$f(x) = \pi/4 \text{ if } x \text{ is rational, and}$$

$$f(x) = 3/4 \text{ if } x \text{ is irrational.}$$

Mapping f is peripherally continuous, but is not a connectivity map.

Closely related to Theorems 6.1 and 6.16 is the following theorem concerning peripherally continuous mappings defined on n-cells. The following definition will be used in the proof of this theorem.

Definition 6.5. Let S be a topological space and let A be a subset of A. Any component of the subspace (S - A) is said to be a component complementary domain of A.

Theorem 6.17. [16] There exists no peripherally continuous transformation f that maps an n-cell I, $n \geq 2$, into itself such that f(I) is the union of two closed disjoint subsets of I.

Proof. Suppose there does exist a peripherally continuous transformation from an n-cell I, $n \geq 2$, into itself such that $f(I) = H \cup K$, where H and K are closed disjoint subsets of I. Then the components of both $f^{-1}(H)$ and $f^{-1}(K)$ are closed by Theorem 6.7, and $f^{-1}(H) \cap f^{-1}(K) = \emptyset$. Since $f^{-1}(H) \cup f^{-1}(K) = I$, and I is not the union of a countable number of disjoint closed sets, one of the set $f^{-1}(H)$ or $f^{-1}(K)$ must have uncountably many components. Consider this to be $f^{-1}(H)$; similar results hold if this is $f^{-1}(K)$.

Let x be a point of I and let $U \subset I$ be any open set containing x. If $D \subset U$ is an open set containing x and D^* is defined as the component of D containing x unioned with its component complementary domains which are bounded in E^n, then $F(D^*)$ is connected [27,106].

We shall now consider the two possible resulting cases under our assumption that f is peripherally continuous and show a contradiction of the hypothesis. First, if $f^{-1}(H)$ has at most a finite number of non-degenerate components, $M_1, M_2, \ldots M_n$, then $\bigcup_{i=1}^{n} M_i$ is closed since each M_i is closed by Theorem 6.7. Let x be a degenerate component of $f^{-1}(H)$, and let $U \subset I$ be an open set containing x such that $\bar{U} \cap [\bigcup_{i=1}^{n}, M_i] = \emptyset$, and let V an open set containing f(x) such that $V \cap K = \emptyset$. If $D \subset U$ is any open set containing x, then $F(D) \cap f^{-1}(K) \neq \emptyset$ for if not, $F(D^*) \subset F(D)$ would be a subset of a non-degenerate component N of $f^{-1}(H)$. But $N \subset U$ is impossible by the definition of U. Hence there exists no set $D \subset U$ containing x such that $f(F(D)) \subset V$ and consequently f is not peripherally continuous at x, contrary to hypothesis.

Alternately, suppose $f^{-1}(H)$ has infinitely many non-degenerate components. Then there exists a non-degenerate component E of $f^{-1}(H)$ such that E union its complementary domains which are bounded in E^n, denoted by E^*, does not equal I. Let x be a point on the boundary of E^* which is a limit point of $I - E^*$ and let U and V be open sets containing x and f(x), respectively, such that $(I - U) \cap E^* \neq \emptyset$ and $V \cap K = \emptyset$. Now if $D \subset U$ is any open set containing x, $F(D^*) \subset F(D)$ must contain a point of $f^{-1}(K)$. For if not, $D^* \cup F(D^*)$ would belong to E^* since $F(D^*)$ is connected and $E^* \cap F(D^*) = \emptyset$; but this contradicts the fact that x is a boundary point of E^*. Thus there exists no open set $D \subset U$ such that $f(F(D)) \subset V$, which again contradicts the fact that f is peripherally continuous.

The following theorem will be useful in proving the next theorem relating peripherally continuous mappings to connectivity maps.

Theorem 6.18. [5] If f is a mapping from the T_1 space S into the T_1 space T and if K is a connected subset of g(S), then $g^{-1}(K)$ is connected.

Proof. Suppose $g^{-1}(K) = M \cup N$ where M and N are mutually separated. Then $K = g(M) \cup g(N)$ and $g(M) \cap g(N) = \emptyset$ since $M \cap N = \emptyset$. Therefore, one of the sets g(M) and g(N) must contain a limit point of the other, say, g(M) contains a limit point $p \times f(p)$ of g(N). Then there is a sequence $\{q_n \times f(q_n)\}$ of points in g(N) converging to $p \times f(p)$. Now q_n is in N, point p is in M and q_n converges to p. This implies that p is a limit point of N belonging to M contradicting the assumption M and N mutually separated. Therefore, $g^{-1}(K)$ is connected.

Theorem 6.19. [5] Let f be a peripherally continuous mapping from a T_1 space into a T_1 space T. If for every connected set K in S, g(K) has a finite number of components, then f is a connectivity map.

Proof. Since f is peripherally continuous, g is peripherally continuous by Theorem 6.15.

Now let K be a connected subset of S and suppose g(K) is not connected. By hypothesis, g(K) has a finite number of components $C_1, C_2, \ldots, C_n$. Thus $g(K) = \bigcup_{i=1}^{n} C_i$, $K = \bigcup_{i=1}^{n} g^{-1}(C_i)$, and $g^{-1}(C_i) \cap g^{-1}(C_j) = \emptyset$ for $i \neq j$, since $C_i \cap C_j$ are mutually separated. Since K is connected, not all of the $g^{-1}(C_i)$ are mutually separated. Let p be a point of C_i for some i, such that $\bar{p}$ is a limit point of $\bigcup_{i=1}^{n} g^{-1}(C_j)$, $j \neq i$. Then p must be a limit point of $g^{-1}(C_j)$ for some $j \neq i$. Now $p \times f(p)$ is in C_i and there is an open set V containing $p \times f(p)$ such that $V \cap C_k = \emptyset$ if $k \neq i$, since sets C_K are mutually separated. Then g peripherally continuous implies, for any open set U containing p, there exists an open set $D \subset U$ and containing p such that $g(F(D)) \subset V$.

By Theorem 6.18, $g^{-1}(C_j)$ is connected since C_j is connected. Since p is a limit point of $g^{-1}(C_j)$, $g^{-1}(C_j)$ is non-degenerate and the open set D can be chosen such that $g^{-1}(C_j)$ has points interior to D and exterior to D. Therefore, $g^{-1}(C_j)$ must have points in common with f(D), since $g^{-1}(C_j)$ is connected. Thus g(F(D)) is not a subset of V. This is a contradiction of the hypothesis that f is peripherally continuous, hence g(K) must be connected. Therefore f is a connectivity map.

Theorem 6.20. [5] Let f be a peripherally continuous mapping of the T_1 space S into the T_1 space T. If for every non-degenerate connected set K in S, g(K) has no degenerate components, then f is a connectivity map.

Suppose f is not a connectivity map. Then there is a non-degenerate connected set K in S such that $g(K) = M \cup N$ where M and N are mutually separated. By hypothesis the components of M and N are non-degenerate. Hence $g^{-1}(M)$ and $g^{-1}(N)$ have non-degenerate components. For suppose the point p is a component of $g^{-1}(M)$. Then $g(p) = p \times f(p)$ lies in some non-degenerate component C of M and $g^{-1}(C)$ is connected. Therefore $g^{-1}(C) = p$ and this contradicts the fact that g is always a one-to-one mapping.

Now $M \cap N = \emptyset$ implies $g^{-1}(M) \cap g^{-1}(N) = \emptyset$, and $K = g^{-1}(M) \cup g^{-1}(N)$ being connected implies $g^{-1}(M)$ and $g^{-1}(N)$ are not mutually separated. Let p be a point of $g^{-1}(M)$ which is a limit point of $g^{-1}(N)$. Then $p \times f(p)$ is in M and there is an open set V containing $p \times f(p)$ such that $V \cap N = \emptyset$ since M and N are mutually separated. Let U be an open set containing p. Then $U \cap g^{-1}(N) \neq \emptyset$ since p is a limit point of

$g^{-1}(N)$. Hence U intersects some non-degenerated component of $f^{-1}(N)$. Since g is peripherally continuous, there is an open set W containing p and contained in U such that $f(F(W)) \subset V$. Now U and W can be chosen such that $C \not\subset W$ but $C \cap W \neq \emptyset$ since C is non-degenerate. Since C is connected, and since C must have point interior to W and exterior to W, $F(W) \cap C \neq \emptyset$. This is a contradiction since $f(F(W)) \subset V$, $g(C) \subset N$ and $V \cap N = \emptyset$. Thus f is a connectivity map.

In Theorems 6.3 and 6.16 it was shown that a connectivity map from an n-cell, $n \geq 1$, into itself is peripherally continuous. The question of whether or not a peripherally continuous mapping from an n-cell into itself is a connectivity map has not yet been answered. Example 6.4 showed that for n = 1, the conclusion need not follow. In Theorem 6.21 a partial solution to this question for $n \geq 1$ will be given. The following lemma will be used in the proof of Theorem 6.21.

Lemma 6.1. [16] Let S and T be Hausdorff spaces, and let f be a mapping from S into T. If M is a subset of S such that g(M) is the union of two mutually separated sets H and K and if p is a point of $g^{-1}(H)$ or $g^{-1}(K)$ which is a limit point of the other, then p is a point of discontinuity of f.

Proof. Let p be a point of $g^{-1}(H)$ which is a limit point of $g^{-1}(K)$. Since H and K are mutually separated sets, there exists open sets U and V containing p and f(p), respectively, such that $U \times V$ contains no points of K. Thus no point of $g^{-1}(K) \cap U$ maps into V under f. There exists a sequence of points $\{p_i\}$ belonging to $g^{-1}(K)$ which converges to p. Hence infinitely many of the points p_i lie in U and the images of these points lie in $(T - V)$. Therefore, the sequence $\{f(p_i)\}$ cannot converge to $f(p)$ and p is a point of discontinuity of f.

A similar argument holds if p is an element of $f^{-1}(K)$ which is a limit point of $f^{-1}(H)$.

Theorem 6.21. [16] If f is a peripherally continuous mapping from a regular Hausdorff space S into a regular Hausdorff space T which has at most a finite number of points of discontinuity, then f is a connectivity map of S into T.

Proof. Suppose that f is not a connectivity map. Then there exists a connected set M in S such that g(M) is the union of two mutually separated set H and K. By Lemma 6.1, every point of $g^{-1}(H) \cap \overline{g^{-1}}(K)$ and $g^{-1}(K) \cap \overline{g^{-1}(H)}$ is a point of discontinuity of f. Since the points of discontinuity of f is a finite set, let $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_r$ denote the points of $g^{-1}(H) \cap \overline{g^{-1}(K)}$ and $g^{-1}(K) \cap \overline{g^{-1}(H)}$, respectively.

Since H and K are mutually separated, there exists open set U_i and W_i containing x_i and y_i, respectively, and open sets L_i and N_i containing $f(x_i)$ and $f(y_i)$, respectively, such that $(U_i \cap L_i)$, $i = 1,2,3,\ldots,K$, contains no point of K and $(W_i \cap N_i)$, $i = 1,2,3,\ldots,r$, contains no point of H. By the peripheral continuity of f and the fact that S is a regular Hausdorff space, there exists open sets $D_i \subset U_i$ and $E_i \subset W_i$ containing x_i and y_i, respectively, having the following properties: (1) $f(F(D_i)) \subset N_i$, $i = 1,2,3,\ldots,r$ and (2) the closure of no two of the sets D_i and E_i have a point in common.

Let $D = \bigcup_{i=1}^{k} D_i$ and let $E = \bigcup_{i=1}^{r} E_i$. Then M may be expressed as the union of two sets

$$M_1 = (D \cap M) \cup [(S - D - E) \cap g^{-1}(H)],$$

and

$$M_2 = (E \cap M) \cup [(S - D - E) \cap g^{-1}(K)],$$

which can be shown to be mutually separated sets. Since $g^{-1}(H) \cap g^{-1}(K) = \emptyset$ and $D \cap E = \emptyset$, $M_1 \cap M_2 = \emptyset$. Now the only points of $g^{-1}(H)$ that are limit points of $g^{-1}(K)$ lie in the open set D and thus $[(S - D - E) \cap g^{-1}(H)]$ contains no limit points of $[(S - D - E) \cap g^{-1}(K)]$; since $F(E) \subset g^{-1}(K)$, and $[(S - D - E) \cap g^{-1}(H)]$ contains no limit point of $E \cap M$. By construction $D \cap M$ contains no limit point of $E \cap M$ and since $F(D) \subset g^{-1}(H)$, $D \cap M$ contains no limit point of $[(S - D - E) \cap g^{-1}(K)]$. Therefore, M_1 contains no limit point of M_2 and in a similar manner M_2 contains no limit point of M_1. Thus $M = M_1 \cup M_2$ is expressed as the union of two mutually separated sets, which contradicts the fact that M is a connected set. Hence f must be a connectivity map.

As previously mentioned, Stalling [23] proved a theorem stating conditions under which a connectivity map will be peripherally continuous. Stalling's theorem and its proof were not presented, since it involved concepts from Algebraic Topology which are not appropriate for this paper. Hagan [5] proved a partial converse for this theorem using a topological space called a Moore space [20]. It should be noted that a Moore space is regular and T_1. Some preliminary definitions and a lemma will be given in preparation for the presentation of Hagan's converse to Stalling's theorem.

Definition 6.3. A space S is said to be <u>locally peripherally connected at the point p</u> if for every open set U containing p there is an open set V containing p and contained in U such that F(V) is connected. A space is locally peripherally connected if it is locally peripherally connected at every point.

Definition 6.6. A space S is said to satisfy property II if for every closed connected subset M of S and for every component C of S - M, the boundary of C is closed and connected [23].

Lemma 6.2. [5] Let W be an open connected subset of the locally peripherally connected, Moore space S such that f(W) is connected. Let W_1 and W_2 be open connected sets such that $W_1 \cap W_2 \neq \emptyset$, $F(W_1)$ and $F(W_2)$ are connected, and $\overline{W_1} \cup \overline{W_2} \subset W$. If $W_3 = (W_1 \cup W_2) \cup (\bigcup_\alpha C_\alpha)$, where $\{C_\alpha\}$ is the collection of all component of $\overline{W} - (W_1 \cup W_2)$ such that $F(C_\alpha) \subset F(W_1) \cup F(W_2)$, and if C is the component of $\overline{W} - (W_1 \cup W_2)$ containing the connected set F(W), then

(1) $F(W_3) \subset F(W_1) \cup F(W_2)$,

(2) $\overline{W} = C \cup W_3$,

(3) W_3 is open and connected, and

(4) if the space S has property II, $F(W_3)$ is connected.

Proof of (1). Suppose there is an x in $F(W_3) - (F(W_1) \cup F(W_2))$. Then since S is regular, T_1 and peripherally connected there exists an open set G such that F(G) is connected, $x \in G$, and $\overline{G} \cap (\overline{W}_1 \cup \overline{W}_2) = \emptyset$. Since $F(C_\alpha) \subset F(W_1) \cup F(W_2)$ for each α, $x \notin C_\alpha$ for any α. Therefore, x is a limit point of $\bigcup C_\alpha$ such that $x \notin C_\alpha$ for any α. This implies G must intersect infinitely many C_α. If $C_\alpha \subset G$ for some α, then $F(C_\alpha) \subset G$ since C_α is closed. This is a contradiction since $F(C_\alpha)$ is contained in $F(W_1) \cup F(W_2)$. Therefore, if $C_\alpha \cap G \neq \emptyset$, then C_α has point interior to

G and points exterior to G. This implies $f(G) \cap C_\alpha \neq \emptyset$ since C_α is connected. Now $F(G) \cap F(C_\alpha) = \emptyset$ since $\bar{G} \cap (\bar{W}_1 \cup \bar{W}_2) = \emptyset$. Hence $F(G) = (F(G) - C_\alpha) \cup (F(G) \cap C_\alpha)$, where $F(G) - C_\alpha$ and $F(G) \cap C_\alpha$ are nonempty and mutually separated. This contradicts $F(G)$ being connected. Hence $F(W_3) \subset F(W_1) \cup F(W_2)$.

Proof of (2). If K is a component of $W - (W_1 \cup W_2)$ such that $K \cap C = \emptyset$, then $F(K) \subset (F(W_1) \cup F(W_2))$. For suppose there is a point x in $F(K) - (F(W_1) \cup F(W_2))$. Since K is closed, $x \in K$. Now $K - (F(W_1) \cup F(W_2))$ is equal to $\cup K_\alpha$ where $\{K_\alpha\}$ is the collection of components of the set $K - (F(W_1) \cup F(W_2))$. Then each K_α is also a component of $W - (C \cup \bar{W}_3)$ since $K \cap C_\alpha = \emptyset$, and $K \cap C = \emptyset$. Set $K \cap C_\alpha = \emptyset$ is implied by the fact that C_α a component and if $K \cap C_\alpha \neq \emptyset$, C_α must contain K so that $F(K) \subset F(C_\alpha) \subset (F(W_1) \cup F(W_2))$ which gives a contradiction. Thus $(K - (F(W_1) \cup F(W_2))) \cap (C \cup \bar{W}_3) = \emptyset$. Since $W - (C \cup \bar{W}_3)$ is open K_α is open for each α, and $F(K_\alpha) \subset (C \cup \bar{W}_3)$. But $\bar{K}_\alpha \cap C = \emptyset$ implies $F(K_\alpha) \subset \bar{W}_3$. Now $\bar{W}_3 = ((\bar{W}_3 - F(W_3))$ and $\bar{W}_3 - F(W_3)$ is an open set disjoint from K_α. Therefore, $F(K_\alpha) \subset F(W_3)$. Now $K = (K - F(W_3)) \cup (K \cap F(W_3))$. Thus, since x is in $(F(K) - F(W_3))$, x is in K_α for some α. But $x \notin$ interior K_α since interior K_α is contained in interior of K. Therefore x is in $F(K_\alpha)$. This is a contradiction since $F(K_\alpha) \subset F(W_3) \subset (F(W_1) \cup F(W_2))$. Hence $F(K) \subset (F(W_1) \cup F(W_2))$. Now suppose there is a point x in $(W - (C \cup W_3))$. Then x is in some component K of $W - (W_1 \cup W_2)$. By the above argument $F(K) \subset (F(W_1) \cup F(W_2))$ and hence $K = C_\alpha$ for some α. But $C_\alpha \subset W_3$. This contradiction implies that $\bar{W} = (C \cup W_3)$.

Proof of (3). Since $C \cap W_3 = \emptyset$ and since C is closed $\overline{W} - C = W - C = W_3$ is open. Also, W_3 is connected since $W_1 \cup W_2$ is connected and each C_α is connected, and $C_\alpha \cap \overline{W_1 \cup W_2} \neq \emptyset$.

Proof of (4). Since W_3 is open, $F(W_3) \cap W_3 = \emptyset$ and hence $F(W_3) \subset C$. Therefore $\overline{W_3} \cap C = F(W_3)$. Since $\overline{W_3}$ is closed and S has property II, every component of $S - \overline{W_3}$ has a connected boundary. The closed connected set C contains $F(W_3)$, and W_3 is connected, hence by Theorem 34 of [20,103], $\overline{W_3} \cap C = F(W_3)$ is connected.

Theorem 6.22. [5] If f is a peripherally continuous mapping of the locally connected, Moore space S having property II into the space T and if $S \times T$ is completely normal, then f is a connectivity map.

Suppose f is not a connectivity map and let A be a connected subset of S such that $g(A) = M \cup N$, where M and N are mutually separated. Let $g^{-1}(M) = H$ and $g^{-1}(N) = K$. Then $A = H \cup K$, where $H \cap K$ is empty. Since A is connected H and K are not separated and hence one must contain a limit point of the other. Let p be a point of H which is a limit point of K. Since $S \times T$ is completely normal, there exists disjoint open set U and V in $S \times T$ containing M and N, respectively.

Let R be an open set containing p such that A is not entirely contained in R. Then f peripherally continuous and S locally connected implies there exists an open connected set W containing p and contained in R such that W and F(W) are both connected and $g(F(W)) \subset U$ [23]. Since p is a limit point of K there is a point q of K in W.

Let Q be the collection of all open connected sets D such that q is in D, $\overline{D} \subset W$, D and F(D) are connected, and $g(F(D)) \subset V$. The collection Q is non-empty since f is peripherally continuous at point q. Let Q^+ denote the union of the collection of sets in Q. Then Q^+ is an open subset of W. Consider the boundary $F(Q^+)$ of Q^+. If $F(Q^+) \cap A = \emptyset$, then $A = (A - Q^+) \cup (A \cap Q^+)$ and $(A - Q^+)$ and $(A \cap Q^+)$ are mutually separated. For $A - Q^+ \neq \emptyset$ since A does not lie entirely in Q^+ and $A \cap Q^+ \neq \emptyset$ since q is in $A \cap Q^+$. Furthermore, $A \cap Q^+$ is open in A and hence cannot contain any limit point of $A - Q^+$, and any limit point of $A \cap Q^+$ which is in $A - Q^+$ is in $F(Q^+)$ which is disjoint from A. Thus, since $A - Q^+$ and $A \cap Q^+$ are disjoint they are mutually separated and this contradicts A being connected. Therefore, $F(Q^+) \cap A \neq \emptyset$.

Since $F(Q^+) \cap A \neq \emptyset$, either $F(Q^+)$ contains a point of H or a point of K. Suppose there is a point h of H in $F(Q^+) \cap H$. Then there is an open set E containing h but not q such that F(E) is connected and $g(F(E)) \subset U$. Since h is a limit point of Q^+, E must intersect some set D belonging to the collection Q. Now $E \not\subset D$ since h lies in $E - D$ and $D \not\subset E$ since q is in $D - E$. Thus E and D both have point interior and exterior to one another and F(D) and F(E) being connected implies $F(D) \cap F(E) \neq \emptyset$. But this contradicts the fact that $g(F(D)) \subset V$, $g(F(E)) \subset U$ and $U \cap V = \emptyset$. Hence $F(Q^+) \cap H = \emptyset$ and therefore $F(Q^+) \cap K \neq \emptyset$.

Let k be a point of $F(Q^+) \cap K$. Now k is not a point of F(W) since $g(F(W)) \subset U$ and $g(k) \in V$. Thus k is in W and there is an open connected set W_1 containing k and contained in W such that $F(W_1)$ is connected, $\overline{W}_1 \subset W$ and $g(F(W_1)) \subset V$. Since k is a limit point of Q^+ there is a set W_2 in the collection Q such that $W_1 \cap W_2 \neq \emptyset$.

Now from the set W_3 referred to in Lemma 6.2. By this lemma, the set W_3 is open, connected, $F(W_3)$ is connected, $\overline{W}_3 \subset W$, and q is in W_3.

Furthermore, $g(F(W_3)) \subset V$ since $F(W_3) \subset F(W_1) \cup F(W_2)$. Therefore W_3 possesses all of the requirements to belong to Q. But W_3 is not in Q since k is in $W_3 \cap F(Q^+)$. Therefore, the assumption that g(A) is not connected leads to a contradiction. Hence f is a connectivity map.

It was shown in Theorem 6.13 that a uniformly convergent sequence of peripherally continuous functions from a space S into a metric space T will converge to a function which is peripherally continuous. If the spaces S and T are required to satisfy the hypothesis of Stalling's theorem and Hagan's theorem, then the same result must hold for connectivity maps since connectivity maps and peripherally continuous mappings are equivalent under these conditions.

Since an n-cell, $n \geq 2$, satisfies the hypothesis for both Stalling's theorem and Hagan's theorem, a mapping f from an n-cell into itself, $n \geq 2$, is peripherally continuous if and only if f is a connectivity map.

CONTINUITY OF PERIPHERALLY CONTINUOUS MAPPINGS AND CONNECTIVITY MAPS

It was shown in Examples 6.1 and 6.2 that peripherally continuous mappings and connectivity maps are not necessarily continuous. The problem of when connectivity maps and peripherally continuous maps will be continuous will now be investigated.

Theorem 6.23. [16] If f is a one-to-one real valued connectivity map defined on a locally connected metric space S, then f is continuous on S.

Proof. Since connectivity maps carry connected subsets of S onto connected subsets of the image space, f is continuous by Theorem 4.19.

Theorem 6.24. [5] If f is a connectivity mapping of the T_1 space S into the T_1 space T and if g(S) is semi-locally connected, then f is continuous.

Proof. Suppose f is not continuous at a point p in S. Then g is not continuous at p and hence there exists a sequence $\{p_n\}$ of points of S converging to p such that $\{p_n \times f(p_n)\}$ does not converge to $p \times f(p)$. Since g(S) is semi-locally connected and since $\{p_n \times f(p_n)\}$ does not converge to f(p), there is an open set U containing $p \times f(p)$ such that $p_n \times f(p_n)$ is not in U for infinitely many n and g(S) - U has only a finite number of components. Thus infinitely many of the points of $\{p_n \times f(p_n)\}$ lie in a single component K of g(S) - U. Now $K \cup \{p \times f(p)\}$ is not connected but $g^{-1}(K \cup \{p \times f(p)\} = g^{-1}(K) \cup \{p\}$ is connected since $g^{-1}(K)$ is connected by Theorem 6.18 and p is a limit point of $f^{-1}(K)$. Point p is a limit point of $f^{-1}(K)$ since infinitely many of the points p_n lie in K and p = limit p_n. Since the set $g^{-1}(K) \cup \{p\}$ is connected and since f is a connectivity map, the set $g(g^{-1}(K) \cup \{p\}) = K \cup \{p \times f(p)\}$ must be connected. This is a contradiction, so that f must be continuous.

The following theorem shows that a similar result holds for peripherally continuous functions.

Theorem 6.25. [16] Let f be a peripherally continuous transformation from a Hausdorff space S into a Hausdorff space T. If x is a point of S such that for any open set R containing f(x) there exists an open set $V \subset R$ containing $f(x)$ having the property that $S - f^{-1}(V)$ has only a finite number of components, then f is continuous at x.

Proof. Suppose f is not continuous at x. Then there exists some sequence $\{x_n\}$ in S converging to x such that $\{f(x_n)\}$ does not converge to $f(x)$. Thus, by the hypothesis, an open set V can be found such that $f(x) \in V$, an infinite number of the points of $f(x_n)$ belong to $T - V$, and such that $S - f^{-1}(V)$ has only a finite number of components. Thus an infinite number of the points of sequence $\{x_n\}$ must lie in some component E of $S - f(V)$ and x must be a limit point of the connected set E, but $f(x)$ is not a limit point of $f(E)$ since $f(x)$ is an element of V. This contradicts Theorem 6.8, so that f must be continuous.

Theorem 6.26. [16] If a mapping f from a Hausdorff space S into a Hausdorff space T is peripherally continuous and is such that for each closed subset N of T and for each $x \in (S - f^{-1}(N))$ there exists an open set U containing x such that U intersects at most a finite number of components of $f^{-1}(N)$, then f is continuous.

Proof. Suppose f is not continuous. Then there must exist some closed set N in T such that $f^{-1}(N)$ is not closed. Let x be a limit point of $f^{-1}(N)$ which does not belong to $f^{-1}(N)$. By the hypothesis, there exists an open set U containing x such that U intersects at most a finite number of components of $f^{-1}(N)$. This implies some component of $f^{-1}(N)$ is not closed which contradicts Theorem 6.7. Thus f must be continuous.

The following theorem gives a necessary and sufficient condition for a monotone peripherally continuous mapping to be open. This result will be used to obtain conditions which imply a peripherally continuous mapping is continuous.

Theorem 6.27. [5] Let f be a monotone peripherally continuous mapping of the compact metric space S onto the regular T_1 space T. Then f is open if and only if every sequence $\{y_n\}$ of points of T with sequential limit point y, $\lim \{f^{-1}(y_n)\} = \{f^{-1}(y)\}$.

Proof. Suppose f is an open, monotone, peripherally continuous mapping and let $\{y_n\}$ be a sequence of points of T with sequential limit point y. Let $G = f^{-1}(y)$ and $G_n = f^{-1}(y_n)$ for each n. Since f is monotone, G and G_n are connected. Furthermore, G and G_n are closed by Theorem 6.7. Now S is a compact space, so that G and G_n are compact. Suppose there exists a point x in G and a neighborhood U of x such that $U \cap G_n = \emptyset$ for all but a finite number of n. Then since f is open f(U) is an open set containing f(x) = y such that f(U) contains only a finite number of points of $\{y_n\}$. This contradicts the fact that y = limit y_n since T is a regular T_1 space. Therefore, $G \subset \lim\inf \{G_n\} \subset \lim\sup \{G_n\}$. Let us now show that $\lim\sup G_n \subset G$.

Suppose there exists a point p in $\lim\sup \{G_n\}$ such that p is not in G. Since $\{p\}$ and G are closed subsets in a metric space and since G is compact, it is possible to find disjoint open sets U and V such that $p \in U$, $G \subset V$, and $U \cap V = \emptyset$. Let N be any open set about f(p). Since f is peripherally continuous, there exists an open set $D \subset U$ containing p such that $f(F(D)) \subset N$. Now $p \in \lim\sup \{G_n\}$ and $G \subset \lim\inf \{G_n\}$ implies an infinite number of G_n must intersect both D and V. Furthermore, $D \subset \overline{U}$ and $\overline{U} \cap V = \emptyset$ so that an infinite number of set G_n must have points interior to D and exterior to D. Since G_n is connected for each n, this implies F(D) contains points from infinitely many G_n. Thus N contains infinitely many of the points of $\{y_n\}$. This is a contradiction, since the only limit point of a sequence in a regular T_1 space is the sequential limit point. Thus lim sup $(G_n) \subset G$.

We now have

$G \subset \liminf \{G_n\} \subset \limsup \{G_n\} \subset G$ so that $F = \lim \{G_n\}$ or $f^{-1}(y) = \lim \{f^{-1}(y_n)\}$ which completes the proof of the first assertion.

Conversely, suppose U is an open set in S such that f(U) is not open. Then there exists a point y in f(U) and a sequence of points $\{y_n\}$ in T - f(U) such that $\lim y_n = y$. By hypothesis, $f^{-1}(y) = \lim \{f^{-1}(y_n)\}$. Now $U \cap \{f^{-1}(y_n)\} = \emptyset$ for every n since $y_n \notin f(U)$. But $U \cap f^{-1}(y) \neq \emptyset$ since y is in f(U), and by hypothesis U must intersect all but a finite number of set $\{f^{-1}(y_n)\}$. This contradiction implies f(U) is open, so that f is an open mapping.

Theorem 6.28. [5] Let f be a peripherally continuous mapping of the compact metric S onto the countable compact, regular T_1 space T. If f is an open monotone mapping, then f is continuous.

Proof. Let $\{x_n\}$ be a sequence of points in S with sequential limit point x. Let $y_n = f(x_n)$ for each n. Since T is countably compact, some subsequence $\{y_{n_i}\}$ of $\{y_n\}$ must have a sequential limit point in T. By Theorem 6.27, $\lim \{f^{-1}(y_{n_i})\} = f^{-1}(y)$. Since $\{x_{n_i}\}$ must converge to x. Therefore x is in $f^{-1}(y)$ and $y = f(x)$. Since every sequence $\{x_n\}$ converging to x has a subsequence converging to f(x), f is continuous at x.

The following example shows that the inverse image of a connected set under a connectivity map or a peripherally containuous mapping need not be connected. In Theorem 4.29 conditions which will imply that the inverse image of a connected set under a peripherally continuous mapping will be connected will be given. The results of Theorem 4.29 will then be used to prove a theorem giving conditions under which a peripherally continuous mapping will be continuous.

Example 6.5. [5] Let S be the union of the intervals $(-1,0) \cup (0,1)$ and let T be the interval $(-1,1)$. Define f from S into T by:

$$f(x) = x - 1 \text{ if } x \in (0,1), \text{ and}$$

$$f(x) = x + 1 \text{ if } x \in (-1,0).$$

The mapping f is a connectivity map and is peripherally continuous. The inverse map f^{-1} is neither a connectivity map nor peripherally continuous as one can verify by considering the point 0 in T.

Theorem 6.29. [5] Let f be an open, monotone, peripherally continuous mapping of the compact metric space S onto the regular T_1 space T. If K is a connected subset of T, then $f^{-1}(K)$ is a connected subset of S.

Proof. Suppose $f^{-1}(K) = M \cup N$ where M and N are mutually separated, Then $K = f(M) \cup f(M)$. Now suppose there is a point y in $f(M) \cap f(M)$. Then there exists points m and n in M and N, respectively, such that $f(m) = f(m) = y$. Hence $f^{-1}(y) \cap M \neq \emptyset$ and $f^{-1}(y) \cap N \neq \emptyset$. This is a contradiction, since $f^{-1}(y)$ is connected. Therefore, $f(M) \cap f(N) = \emptyset$. Since K is connected, one of the sets f(M) or f(N) must contain a limit point of the other, say f(N) contains a limit point p of f(M). Then there exists a sequence of points $\{p_n\}$ in f(M) such that $\lim p_n = p$.

By Theorem 6.27, $\lim \{f^{-1}(y_n)\} = f^{-1}(y)$. This is a contradiction since $f^{-1}(y_n) \subset M$ for every n, $f^{-1}(y) \subset N$, and M and N are mutually separated. Therefore, the assumption $f^{-1}(K)$ is not connected leads to a contradiction so that $f^{-1}(K)$ is connected.

Theorem 6.30. [5] Let f be an open, monotone, peripherally continuous mapping of the compact metric space S onto the semi-locally connected, regular T_1 space T. Then f is continuous.

Proof. Suppose f is not continuous. Then there is a point x in S and a sequence $\{x_n\}$ of points of S converging to x such that $\lim f(x_n) \neq f(x)$. Since T is semi-locally connected there is an open set U containing f(x) such that T - U has a finite number of component, $K_1, \ldots, K_n$, and such that infinitely many of the points of $\{f(x_n)\}$ are in T - U. Hence infinitely many points of $\{f(x_n)\}$ are in some K_i. By Theorem 6.29, $f^{-1}(K_i)$ is connected, and x is a limit point of $f^{-1}(K_i)$ since infinitely many x_n are in $f^{-1}(K_i)$ and $\{x_n\}$ converges to x.

Set $f^{-1}(K_i) \cup \{x\}$ is non-degenerate, so that one can choose an open set V about x such that $f^{-1}(K_i)$ has points interior to V and exterior to V. Since f is peripherally continuous, there exists an open set $D \subset U$ such that $f(F(P)) \subset U$. Now $f^{-1}(K_i)$ must have pointsinterior to D, and exterior to D. Therefore, F(D) must contain a point of the connected set $f^{-1}(K_i)$. This is a contradiction, since $f(F(D)) \subset U$ and $K_i \cap U = \emptyset$. Thus f must be continuous.

Corollary. If the hypothesis that T be semi-locally connected in Theorem 6.30 is replaced with the requirement that T be locally connected and locally compact, f is continuous.

Proof. Every locally connected, locally compact space is semi-locally connected [27,20]. Therefore, f is continuous by Theorem 6.30.

CHAPTER VII

SUMMARY AND EDUCATIONAL IMPLICATION

In this paper the recent research concerning certain classes of noncontinuous transformations in point set topology is organized and summarized with standardization of terminology and symbolisms. This presentation makes the recent research concerning these transformations both more readable and more readily available to the student of topology. Several examples are supplied to help the reader grasp the significance of the various concepts and theorems.

SUMMARY

In Chapter I, the statement of the problem, the scope of the paper, methods and procedures, and expected outcomes are given. In Chapter II a brief introduction to point set topology is given. This is included in the interest of standardizing notation and terminology, since texts on point set topology differ slightly in both. In Chapter III a discussion of open and closed mappings is given. This discussion is by no means complete since open and closed functions are defined in connection with homeomorphisms in elementary texts on topology and many results concerning these functions are included in these texts. Such results are not included in this study since the intent of this study is to present the results of recent research which are not readily available to students of topology. Chapter IV gives a rather complete discussion of compact preserving and connected mappings. These mappings are significant since continuous functions, which are the most fundamental functions in topology, are both compact preserving and connected. In Chapter V a review of the recent research concerning neighborly and cliquish functions is given. Theorems 5.8 and 5.10 from this chapter are particularly significant since they give characterizations of derivative functions of real valued continuous functions defined on the real numbers. In Chapter VI the recent research concerning connectivity mappings and peripherally continuous mappings is reviewed. It is noted in Chapter VI that these functions were originally defined and studied in connection with fixed point properties. Fixed point properties have been studied extensively by topologists in recent years.

Throughout Chapters III, IV, V, and VI relationships between and among the various classes of noncontinuous transformations have been emphasized. Also, the relationships between the classes of noncontinuous transformations and continuous transformations have been considered in detail.

EDUCATIONAL IMPLICATIONS

Since the body of material and ideas are constantly expanding in mathematics, it is increasingly important that such be made available in systematic, readable sources. These sources should enable the student of mathematics to become aware of the research that has been done and the areas that need to be investigated further. The reader of this paper will come abreast of the frontiers of knowledge in the study of some important aspects of noncontinuous transformation. From this vantage point the reader can then proceed in further study of functions and topology by study of the professional journals or by independent research into properties of the functions considered in this study.

For the future mathematician or mathematics teacher, and particularly for the college teacher of mathematics, it is important to realize that curricular changes in the various disciplines will occur as new knowledge is discovered. An acquaintanceship with the ideas presented in this thesis should help one to anticipate changes that may occur as point set topology becomes more involved in the mathematics curriculum.

References to the bibliography are given for most of the theorems in this thesis. By consulting this bibliography, one may gain an awareness of the men who have contributed to topology in recent years. It is likely that these men, many of them contemporaries of the reader, will play a significant role in shaping point set topology and the mathematics curriculum of the future. Awareness of these potential leaders should help interested individuals keep abreast of the developing mathematics curriculum and should contribute to their implementation of curricular changes.

Perhaps the most significant result of the development of this paper has been the extent to which the investigator has developed his own interest and knowledge of point set topology. The skills developed and the research experiences encountered will add to the background needed for effective teaching at the college level.

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APPENDIX

INDEX

VITA

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Biographical:

Personal Data: Born in Tuttle, Oklahoma, January 19, 1935, the son of Jolly Joseph and Lois Elva Burns.

Education: Attended Meridian, Tuttle, and Pioneer grade schools in Grady County, Oklahoma; graduated from Verden High School at Verden, Oklahoma, in 1954; received the Bachelor of Science degree from Oklahoma Agricultural and Mechanical College at Stillwater, Oklahoma, in 1957 with a major in Education (Mathematics); received the degree of Master of Science from Oklahoma State University at Stillwater, Oklahoma, in 1958, with a major in Secondary Education; attended North Texas State College, Denton, Texas, in the summer of 1959, Kansas State University, Manhattan, Kansas, in the summer of 1960, and Kansas University, Lawrence, Kansas, 1960-61; completed requirements for the Doctor of Education degree at Oklahoma State University, Stillwater, Oklahoma, in August of 1965.

Professional experience: Taught mathematics at Temple High School, Temple, Oklahoma, 1958-1959, and at Del City High School, Del City, Oklahoma, 1959-1960; was an instructor of mathematics at Central Missouri State College, Warrensburg, Missouri from 1961-1963; was a graduate assistant at Oklahoma State University in 1963 and a staff assistant from 1963-1965 while completing the requirements for the doctorate.

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