AN INTRODUCTION TO GRAPH THEORY

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CHAPTER I

NATURE AND SIGNIFICANCE OF THE PROBLEM

There are many situations that can be illustrated by a set of points joined by lines. Transportation and communication situations are outlined on paper with maps. A chemist draws molecular diagrams to represent the chemical bonds between the atoms of complex molecules. Engineers use diagrams of electrical circuitry and flow in pipe line networks. A geneologist draws family trees. Sociograms are used by psychologists. The administrative organization of an institution can be represented by points to indicate the staff, with lines representing the line of command. Blue prints are used by architects. Diagrams of some nature are used in some respect by nearly everyone. The question arises as to whether all such patterns have anything in common? Certainly all the diagrams consist of points representing atoms, people, cities, electrical connections, etc. with lines indicating some relationship between the things represented by the points.

The first published systematic attempt to study all such patterns was made by a German mathematician Denes König in 1936. He named such geometric figures or patterns "graphs." It is unfortunate that such a name was picked since it means something quite different from the graphs studied in analytical geometry and function theory. Nevertheless, bowing to tradition, any set of elements and a relationship between the elements will be called a graph.

Need for the Study

The very nature of graphs is so fundamental that nearly any discipline has at some time an occasion to use graph theory. Therefore it is natural that such disciplines have developed theories pertaining to graphs. However, each discipline developed only that part of the theory significant in the discipline. The terminology arising from such a diversified development is unrelated and sometimes misleading. Such a beginning has led to a need for a systematic development of the theory divorced of any particular setting and that can be universally applied.

There are many fields of mathematics now in the mathematics curriculum that overlap into graph theory. Large areas of set theory, pure combinatorics, algebra, geometry and especially topology consider problems of graph theory. However, since graph theory now makes its appearance in so many fields and especially since a large amount of graph theory could be developed and presented at the high school or undergraduate level, it would seem to merit more than just a passing glance in the curriculum. First, however, a great effort is needed to introduce graphs as a logical abstract mathematical system. A sequential rigorous development with preciseness of definitions and sufficiently complete to reveal its basic nature and applications is needed.

There are also pedagogical implications involved in developing a suitable introduction to graph theory. Graph theory can be introduced in several ways, depending upon the nature of the concepts used in the definitions and the analysis used in the development. Some means of developing the subject lend themselves more directly to an intuitive understanding while others may lend themselves better to expedite the subject. Each method has certain advantages and disadvantages. One can first determine the most expedient way to develop the subject and then investigate the amount of maturity and knowledge prerequisite to its understanding; or he may assume a certain level of maturity and background and develop the subject accordingly.

Since the theory of graphs has been developed in so many fields and referred to by so many names, it is easy to overlook important contributions. To this extent, there exists a need to compile an extensive bibliography complete enough to reveal the major contributions to the subject.

Procedure

It is for the above reasons that this study has been written. The general approach was a careful survey of the existing contextual development of graph theory and its applications in order to redevelop the theory in an abstract formal manner with universal application.

Since graph theory is an old subject dating back to the solution of the Köengsberg bridge problem by Euler in 1735, much of the language used in graph theory has historical overtones and is out-dated. For instance, very few of the earlier papers on graph theory used the language of sets which is in common practice today. Great care was used in this paper to formulate explicit and precise definitions in an up-to-date mathematical language. The definitions are complete enough to avoid ambiguous remarks and paradoxical situations.

The theory in this paper is developed and presented in a sequential deductive manner better to relate the theory and to expedite its development. A logical and complete argument is presented for nearly all theorems. It was intended that the rigor used in this paper will merit the use of the material as a basis for further future development.

The simple nature of graph theory lends itself to many applications. Any introduction to graph theory would be sadly incomplete if certain applications were neglected. This paper has numerous examples and applications appearing where it is appropriate. Such applications and examples will not only show the significance of the subject, but will serve as models to help trigger the associations needed to reason with the abstract development.

The maturity and mathematical concepts attained by an undergraduate student of mathematics are sufficient for him to understand the material presented in this paper. It was found that a great amount of graph theory can be developed in a concise manner without utilizing any far-reaching tools of mathematics. There are several occasions in Chapter VII to use some basic concepts of point-set topology, but all such concepts are found in an undergraduate course in point-set topology. However, the material in Chapter VII can be read with a great amount of understanding without knowledge of point-set topology. The most important prerequisite for an understanding of the material presented is the maturity to work with abstractions and knowledge of the nature of "proof."

An extensive search of the literature was made to compile a bibliography revealing work pertinent to the theory and application of graphs. There is no way of knowing that the list is complete, but it does include most of the notable works.

Limitations

This paper is limited to the very basic concepts that make up the subject of graph theory. It is further limited by the mathematical tools used in its development. Certainly more complex and detailed conjectures can be proved provided a broader background is assumed of the reader. The applications presented are limited to only basic and immediate understandable situations.

Expected Outcomes

It is expected that this paper will serve as a readable source in an up-to-date language for an undergraduate to gain an understanding and appreciation of the nature of graphs. The reading of this paper reveals the history of the subject and an acquaintance with the men who contributed to its growth.

It is further intended that the preciseness and rigor used in developing the subject will furnish a suitable foundation for any further detailed development. The large bibliography is intended to supplement the efforts made for such future study.

CHAPTER II

BASIC DEFINITIONS

Any mathematical system consists of a set of elements. In dealing with number systems one may discuss the set of integers, rational numbers, real numbers, complex numbers, Gaussian integers, or quaternions. Algebra is concerned with sets for which binary operations are defined on the set. Geometry ordinarily deals with various sets of points called lines, planes, triangles and so forth. However, in constructing a useful mathematical system more is needed than just a set of elements; a relation between the elements of the set is needed. A set by itself is not very exciting unless some relation is expressed between the elements. In the various number systems one defines relations such as equality and order or as in the integers, the relation a divides b. In geometry one deals with relations such as are illustrated by the terms collinear points, parallel lines, perpendicular lines, congruent triangles and so forth.

Relations and Graphs

Relations are of such importance in mathematics that a general definition and convenient notation is needed:

Definition 1.1. A <u>binary relation</u> R between sets A and B is a subset of $A \times B$.

When A = B a binary relation between elements of A and elements of A is called a relation on A. By subset we are including the empty set and the whole set. The symbol $A \subset B$ denotes that A is a subset of B.

If R is a relation and $(a,b) \in R$ then it is said that, "a is in relation R to b." $(a,b) \in R$ is sometimes denoted by aRb. The notation Rb will be used to mean the set of all x such that xRb. The notation aR will be used in a similar manner to mean the set of all x such that aRx. That is,

$$Rb = \{x \mid xRb\} \text{ and}$$

$$aR = \{x \mid aRx\}.$$

The theory of graphs is a very general theory with many applications. In fact, the theories of graphs and of relations are very closely related as revealed by the definition.

Definition 1.2. A <u>graph</u> consists of a nonempty set S and a relation R on the set, denoted by (S,R).

Note that a graph is not just a set nor a relation on S, but both.

Definition 1.3. An element of S in a graph (S,R) is called a <u>vertex</u>.

Definition 1.4. An element (a,b) of R in a graph (S,R) is called an <u>arc</u> from a to b. The vertex a of arc (a,b) is called the <u>initial vertex</u> and vertex b is called the <u>terminal vertex</u>.

In order to gain an intuitive feeling for graphs a dot is used to represent a vertex and a line segment with arrows from a to b to represent an arc (a,b). For example, if $S = \{a,b,c,d,e\}$ and $R = \{(a,a),(a,b),(b,a),(b,c),(d,d)\}$ then the graph (S,R) can be represented as in Figure 1.

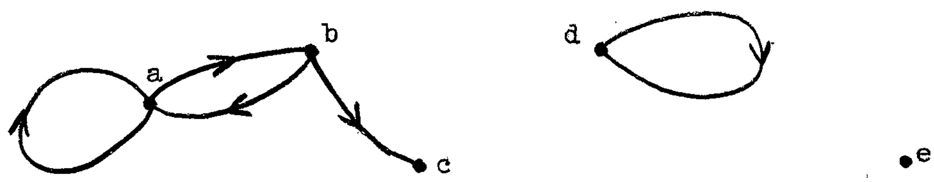

Figure 1.

Since graphs and relations are similar much of the terminology regarding relations will be carried over into graphs. It may happen that in a relation R every element is in relation to itself. A relation R such that aRa for all a is called a <u>reflexive relation</u>.

Definition 1.5. A graph (S,R) where R is a reflexive relation is called a <u>reflexive graph</u>.

Definition 1.6. If $(a,a) \in R$ in a graph (S,R), then the arc (a,a) is called a <u>loop</u>.

In Figure 1. there are two loops, namely, (a,a) and (d,d).

If a relation R is such that aRa does not hold for any element, then R is said to be <u>irreflexive</u>.

Definition 1.7. A graph (S,R) where R is an irreflexive relation is called an <u>irreflexive graph</u>.

For a graph (S,R) to have an irreflexive relation R is equivalent to the property that it has no loops.

Certain relations R have the property that whenever aRb then bRa. Such relations are said to be <u>symmetric</u>.

Definition 1.8. A graph (S,R) where R is a symmetric relation is called a <u>symmetric graph</u>.

It may be that in a relation R, aRb always implies that b is not in relation R to a, denoted by b$\not R$a. Such relations are called <u>asymmetric</u>.

Definition 1.9. A graph (S,R) where R is an asymmetric relation is called an <u>asymmetric graph</u>.

An asymmetric graph has no loops and is therefore irreflexive.

Another important property of relations is that of transitivity. If R is a relation such that aRb and bRc implies aRc, then the relation is said to be <u>transitive</u>.

Definition 1.10. A graph (S,R) in which R is a transitive relation is called a <u>transitive graph</u>.

In a transitive graph if there exists arcs (a,b) and (b,c), then there is an arc (a,c).

One of the most important types of relations is that of equivalence relations. A relation R on a set S is an <u>equivalence relation</u> if and only if:

1. For every a ϵ S, aRa. (Reflexive)
2. For a,b ϵ S whenever aRb, then bRa. (Symmetric)
3. For a,b,c ϵ S whenever aRb and bRc, then aRc. (Transitive)

For example, let $S = \{r_1, r_2, w_1, w_2, w_3, b_1\}$ be a set of colored blocks with r_1 and r_2 red blocks; w_1, w_2 and w_3 white blocks; and b_1 a blue block. Define the relation R on S by the property that xRy if and only if x has the same color as y. The relation R is an equivalence relation. The graph (S,R) is illustrated in Figure 2.

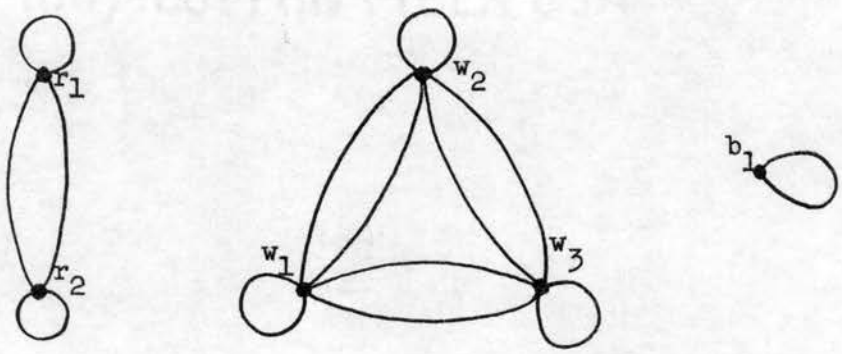

Figure 2.

One of the major reasons why equivalence relations are of interest is because any equivalence relation defined on a set partitions the set into disjoint subsets, called <u>equivalence classes</u>. Conversely, a partition of equivalence classes of a set defines an equivalence relation of the set. A <u>partition</u> of a set is a set of subsets, called <u>cells</u>, such that each cell is nonempty, the intersection of any two cells is empty and the union of all the cells is the set. A subset A of a set S is said to be an <u>equivalence class</u> with respect to an equivalence relation R if and only if:

1. $A \neq \emptyset$.
2. If $x, y \in A$, then xRy.
3. If $x \in A$ and yRx, then $y \in A$.

In the example of the colored blocks the equivalence classes would be the subsets of blocks with the same color. It is easily seen that such equivalence classes partition the set of blocks into disjoint subsets whose union is the whole set of blocks.

If the relation R in a graph (S,R) is the empty set, then the graph has only isolated vertices. Each vertex is isolated for there are no arcs.

Definition 1.11. A graph (S,R) where R is the empty set is called a <u>null graph</u>.

For example, if S is a set of baseball teams and a relation R is defined as aRb if and only if team a wins over team b, the graph (S,R) at the beginning of the season is a null graph.

The other extreme to a null graph is a graph in which $a \not R b$ implies bRa. That is, every pair of vertices a and b has either $(a,b) \in R$ or $(b,a) \in R$ or perhaps both. A relation with this property is said to be <u>determinate</u>.

Definition 1.12. A graph (S,R) in which R is a determinate relation is said to be <u>complete</u>.

Subsystems of a Graph

In nearly any mathematical system it becomes necessary to consider "subsystems" of the system such as subsets of a set, subgroups of a group, linear manifolds of a vector space and so forth. This same situation is true in the theory of graphs. However, there are basically two kinds of "subsystems" of a graph that will be of interest.

Definition 1.13. A <u>subgraph</u> of a graph (S,R) is a graph (T,W) such that $T \subset S$ and W is the relation consisting of all $(a,b) \in R$ such that $a,b \in T$.

It may be that one would want to consider the entire set S of a graph (S,R) and any subset W of R.

Definition 1.14. A <u>partial graph</u> of a graph (S,R) is a graph (S,W) such that $W \subset R$.

In fact, one may want to consider something more or less a combination of a subgraph and partial graph as defined by:

Definition 1.15. A <u>partial subgraph</u> of a graph (S,R) is a graph (T,W) such that $T \subset S$ and $W \subset R$ with $W \subset T \times T$.

To illustrate subgraphs, partial graphs and partial subgraphs consider the following example. Let the graph (S,R) in Figure 3 represent a road map of a small country with S the set of towns in the country and $(x,y) \in R$ any road from town x to town y. Further suppose that some roads are gravel, indicated by dotted lines, and that towns e and f are on islands not connected by roads.

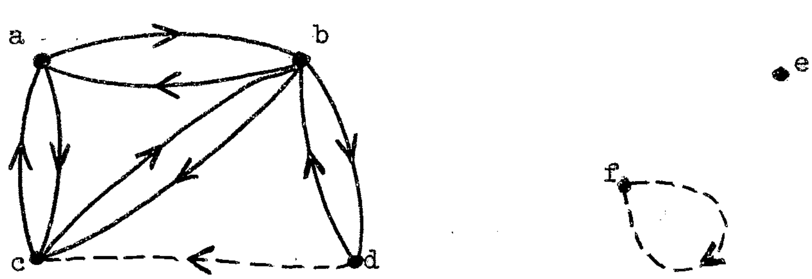

Figure 3.

One may be interested only in the roads directly connecting a set T of particular towns of the country S. Such a map would be a subgraph. If $T = \{a,c,d\}$ and $W = \{(a,c),(c,a),(d,c)\}$ then the graph (T,W) is a subgraph of the graph (S,R) as represented in Figure 4.

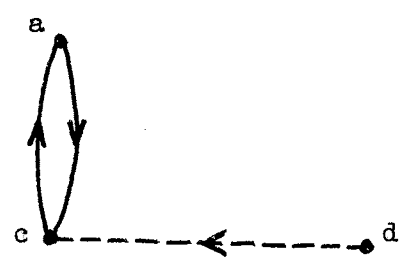

Figure 4.

However, if one is interested only in the gravel roads of the country S, then he would consider the partial graph (S,W) with $W = \{(d,c),(f,f)\} \subset R$ as given in Figure 5.

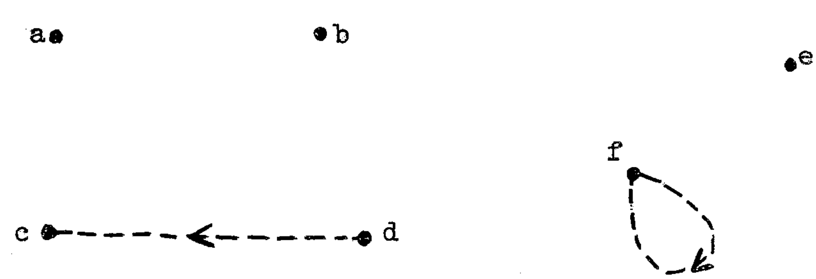

Figure 5.

Moreover, if one is interested only in the gravel roads connecting towns in set T, then this would be a partial subgraph of (S,R) as illustrated in Figure 6.

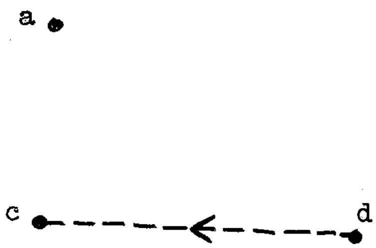

Figure 6.

Directed and Undirected Graphs

The orientation of arcs may or may not be of significance in a particular problem. For instance, if (S,R) is a graph with S a set of people and the relation R defined as aRb if and only if a can "contact" b then the importance of orientation depends on the means of contact. If the contact is by direct personal contact, telephone, etc., then the graph is symmetric. However, if the contact is by a one-way communication system, carrier pigeon, jungle drums, etc., then the orientation becomes an important factor for aRb need not imply bRa. Because the orientation may or may not be important in a particular problem, a dual set of definitions will be used to distinguish the two situations.

Definition 1.16. An <u>edge</u>, denoted by [a,b], is any pair of vertices a and b of a graph (S,R) such that aRb or bRa.

An edge is different from an arc for an edge does not carry a direction whereas an arc does. Notice that the notation [a,b] for an edge is not an ordered pair since [a,b] = [b,a]. The graph illustrated in Figure 3. has six edges and ten arcs. An edge is usually illustrated with a solid line between a and b without the arrows.

Definition 1.17. The order of a vertex is the number of edges containing the vertex. A vertex of finite order will be called even or odd according as it's order is even or odd.

Definition 1.18. A path in a graph (S,R) is any ordered set ((a,b), (b,c),(c,d),$\cdots$) of arcs with the terminal vertex of each arc being the initial vertex of the succeeding arc, if it exists.

A path ((a,b),(b,c),(c,d),(d,e)$\cdots$) is sometimes denoted by abcde$\cdots$.

Definition 1.19. A path is finite or infinite depending on the number of arcs.

Definition 1.20. If each arc in a path is used only once then the path is said to be simple and otherwise composite.

Definition 1.21. If a path goes through each of its vertices only once then it is called an elementary path.

Every elementary path is necessarily simple. In Figure 3. the path acbd is a simple elementary path whereas the path dcbdcab is a composite path.

An important special case of a finite path is that in which it repeats itself.

Definition 1.22. If a finite path has the initial vertex of the first arc the same as the terminal vertex of the last arc it is called a <u>circuit</u>.

In Figure 3. the path abdca is a circuit.

Associated with the terms path and circuit for directed graphs are the terms chain and cycle in undirected graphs.

Definition 1.23. A <u>chain</u> is an ordered set ([a,b],[b,c],[c,d]...) of edges with one vertex of each edge coinciding with a vertex in the succeeding edge and the other vertex coinciding with a vertex in the preceeding edge.

In Figure 3. acdb would be a chain but not a path.

Definition 1.24. If each edge in a chain is used only once, then the chain is said to be <u>simple</u> and otherwise <u>composite</u>.

Definition 1.25. If a chain begins and ends at the same vertex it is called a <u>cycle</u>.

Connectedness

The idea of paths and chains gives rise to other important properties of a graph. Two such properties are (1) that there exists a path or (2) there exists a chain between any two distinct vertices of (S,R).

Definition 1.26. A graph (S,R) with the property of a path existing between any two distinct vertices is said to be <u>strongly connected</u>.

Definition 1.27. A graph (S,R) with the property of a chain existing between any two distinct vertices is said to be <u>connected</u>.

It should be noted that a strongly connected graph is connected; however, the converse need not be true. Figure 7. illustrates a graph that is connected but not strongly connected.

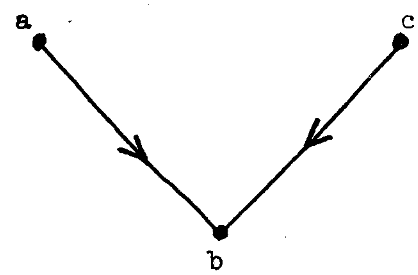

Figure 7.

Definition 1.28. A strongly connected graph (S,R) with the property that for all arcs (a,b), (S,R - (a,b)) is not strongly connected, is said to be <u>minimally strongly connected</u>.

Definition 1.29. A connected graph (S,R) with the property that for all edges [a,b], (S,R) - [a,b] is not connected, is said to be <u>minimally connected</u>.

Many times it is of interest to find all the vertices of a graph that are connected by a chain to a particular vertex.

Definition 1.30. The set of all vertices connected by a chain to a vertex a and including vertex a is called the <u>component</u> of the vertex a and denoted by C_a.

In Figure 3., $C_a = C_b = C_c = C_d = \{a,b,c,d\}$, $C_f = \{f\}$ and $C_e = \{e\}$.

Isomorphic Graphs

It is easily seen that two graphs (S_1,R_1) and (S_2,R_2) such that S_1 and S_2 consist of different elements could have the same geometric representation. For instance, let $S_1 = \{a,b,c\}$, $R_1 = \{(a,b),(a,c)\}$ and $S_2 = \{1,2,3\}$, $R_2 = \{(2,3),(2,1)\}$. Both graphs (S_1,R_1) and (S_2,R_2) are represented in Figure 8. In such a case one may feel that there is no significant difference between the graphs. In a

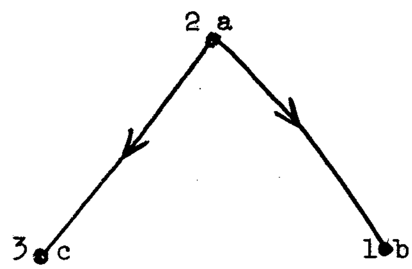

Figure 8.

sense (S_1,R_1) and (S_2,R_2) are the same graph. This situation is described in general by saying the graphs are "isomorphic."

Definition 1.31. Two graphs (S_1,R_1) and (S_2,R_2) are said to be <u>isomorphic</u> if and only if there exists a one-to-one mapping f of S_1 onto S_2 such that if $(a,b) \in R_1$ then $(f(a),f(b)) \in R_2$ and conversely. The mapping f is called an <u>isomorphism</u>.

In the example illustrated in Figure 8., (S_1,R_1) and (S_2,R_2) are isomorphic since the mapping f defined by

$$f(a) = 2$$
$$f(b) = 1$$
$$f(c) = 3$$

is an isomorphism between (S_1, R_1) and $S_2, R_2)$.

CHAPTER III

FUNDAMENTAL THEOREMS

This chapter includes some of the very basic theorems relating the terms defined in Chapter II and that will be needed for discussions to follow. An illustration or application is given for some of the theorems in order to help reveal the nature of the theorem and show its application.

Theorem 3.1. If in a graph (S,R) there exists a path (chain) from a to b and a path (chain) from b to c, then there exists a path (chain) in the graph (S,R) from a to c.

The proof of Theorem 1.1 follows directly by merely combining the two given paths (chains).

Theorem 3.2. If there exists a path (chain) in a finite graph (S,R) from a to $b \neq a$, then there exists an elementary path (chain) from a to b.

If any two vertices coincide in a path (chain) then the circuit (cycle) between them can be deleted from the path (chain) without destroying the defining properties of a path (chain). If this procedure is repeated until no vertex is used twice in the resulting path (chain) then an elementary path (chain) will be obtained. It should be noted that this can be done in a finite number of steps. This proves the theorem.

Corollary 3.3. Every circuit (cycle) of a finite graph (S,R) contains an elementary circuit (cycle).

Theorem 3.4. Every arc in a strongly connected graph (S,R) is contained in some elementary circuit of (S,R).

Let (a,b) be any arc in a strongly connected graph (S,R). Since (S,R) is strongly connected than there exists a path from b to a. By Theorem 3.2 there exists an elementary path from b to a. The elementary path from b to a together with the arc (a,b) constitute an elementary circuit containing arc (a,b).

Theorem 3.5. A subgraph of a complete graph is complete.

Let (T,W) be any subgraph of a complete graph (S,R). If $a,b \in T$ then $a,b \in S$ since $T \subset S$. Since (S,R) is complete the $(a,b) \in R$ or $(b,a) \in R$. By definition 1.13 of a subgraph then $(a,b) \in W$ or $(b,a) \in W$ and (T,W) is complete.

Theorem 3.6. A connected symmetric graph is strongly connected.

Let (S,R) be a connected symmetric graph and $x,y \in S$. Since (S,R) is connected then there exists a chain $([x,a_1],[a_1,a_2],\cdots[a_n,y])$ from x to y. Since (S,R) is symmetric then the paths $((x,a_1),(a_2,a_3)\cdots(a_n,y))$ and $((y,a_n),(a_n,a_{n-1})\cdots(a_1,x))$ exist. Therefore (S,R) is strongly connected.

Theorem 3.7. A strongly connected graph is connected.

The theorem is immediate since there exists a path between any two vertices and hence a chain exists between them.

Theorem 3.8. For any two distinct edges of a connected graph there is an elementary chain with the given edges as ends.

Consider any two distinct edges [a,b] and [c,d] of a connected graph (S,R). If either vertex of [a,b] coincides with a vertex of [c,d] then the other vertices are necessarily distinct and ([a,b],[c,d]) is a required elementary chain. If vertices a,b,c, and d are all distinct then since (S,R) is connected there exists a chain between b and c. By Theorem 3.2 there exists an elementary chain $bx_1x_2x_3\cdots x_nc$ between b and c. If $a \neq x_i$, $d \neq x_i$ for all $i = 1,2,\cdots n$, then $abx_1x_2x_3\cdots x_ncd$ is an elementary chain. If $a = x_i$ for some i, $1 \leq i \leq n$ and $d \neq x_i$ for all $i = 1,2,\cdots n$, then $abx_ix_{i+1}\cdots x_ncd$ is an elementary chain. If $a \neq x_i$ for all $i = 1,2,\cdots n$ and $d = x_i$ for some i, $1 \leq i \leq n$, then $abx_1\cdots x_{i-1}x_icd$ is an elementary chain. If $a = x_i$ for some i, $1 \leq i \leq n$ and $d = x_j$ for some j, $1 \leq j \leq n$, then either $i < j$ or $j < i$. For $i < j$ then $abx_ix_{i+1}\cdots x_{j-1}x_jcd$ is an elementary chain and for $j < i$ $cdx_jx_{j+1}\cdots x_{i-1}x_iab$ is an elementary chain.

Theorem 3.9. Every component C_a of a graph (S,R) determines a connected subgraph (C_a,W) of the graph (S,R).

$C_a \subset S$ by the definition of a component and if W is the relation consisting of all $(x,y) \in R$ such that $x \in C_a$, then (C_a,W) is a subgraph of (S,R). Consider any two distinct vertices $x,y \in C_a$. If $x = a$ or $y = a$ then there exists a chain including x and y since C_a is a component. If $x \neq a$ and $y \neq a$ then there exists a chain including y and a chain including x and a. Hence by Theorem 3.1 there exists a chain including x and y. Therefore the subgraph (C_a, W) is connected.

Theorem 3.10. The set of all components of a graph (S,R) is a partition of the set S.

By a partition of S into components it is understood that each component is nonempty, the intersection of distinct components is empty and the union of all components is the set S. It is obvious that each component C_a of the graph (S,R) is nonempty since $a \in C_a$ by definition 1.30 of a component. Also $\bigcup_{a \in S} C_a = S$.

In order to prove that the intersection of distinct components is empty it will suffice to show that $C_a \cap C_b \neq \emptyset$ implies $C_a = C_b$. $C_a \cap C_b \neq \emptyset$ implies there exists an $x \in C_a \cap C_b$. By Theorem 3.1 and the definition 1.30 of a component then a and b are connected by a chain. Furthermore, if $y \in C_a$ then there exists a chain from y to b going through a and $y \in C_b$ and conversely. Thus $C_a \subset C_b$, $C_b \subset C_a$ and $C_a = C_b$.

A simple illustration of Theorem 3.10 is an organization where person x is in relation to person y if and only if x has contact with y. A component C_x in the organization is the set of all people, including x, that can make contact with x through a sequence of people. One might call component C_x a "grapevine" which includes x. Theorem 3.10 indicates that all grapevines (components) partition the organization. That is, every person belongs to some grapevine, but only one. Any two members either belong to the same grapevine or they cannot make contact with each other through the grapevines.

Theorem 3.11. The set of all components of a graph (S,R) determines a partition of the graph (S,R) into connected subgraphs.

By Theorem 3.9 and 3.10 the set S is partitioned into connected subgraphs. Consider any $(a,b) \in R$. The vertex a is contained in some component C_x since the set of all components partition the set S, and $b \in C_x$ since $(a,b) \in R$. Since $a,b \in C_x$ then the arc (a,b) is contained in the connected subgraph (C_x,W_x). Therefore, each arc of (S,R) is contained in some connected subgraph. Any two such connected subgraphs (T_1,W_1) and (T_2,W_2) are disjoint since $T_1 \cap T_2 = \emptyset$ and the definition of a subgraph implies $W_1 \cap W_2 = \emptyset$.

Theorem 3.12. A graph (S,R) is connected if and only if S is a component.

If the graph (S,R) is not connected then there exists vertices a and b that are not connected by a chain. Hence S is not a component. Conversely, if S is not a component then by Theorem 3.10 there exists at least two disjoint components C_a and C_b. Thus b is not connected to a by a chain and the graph (S,R) is not connected.

Theorem 3.13. A complete graph has only one component.

Let a and b be distinct vertices of a complete graph (S,R). Since (S,R) is complete then $(a,b) \in R$ or $(b,a) \in R$. Thus (S,R) is connected. By Theorem 3.11 (S,R) has only one component.

Theorem 3.14 (7)[1]. A finite graph (S,R) is strongly connected if and only if there are no proper subsets $A \subset S$ such that $\{xR \mid x \in A\} \subset A$.

If a graph (S,R) consists of only one vertex then the theorem follows since it is strongly connected and S has no proper subsets. Let (S,R) be a strongly connected graph with A any proper subset of S. Since A is a proper subset of S then there exists a vertex b such that $b \notin A$. Since (S,R) is strongly connected then there exists a path from any vertex $a \in A$ to $b \notin A$. Hence there exists an arc (c,d) in the path such that $c \in A$ and $d \notin A$. This means that $d \in \{xR \mid x \in A\}$, but $d \notin A$ and therefore $\{xR \mid x \in A\} \not\subset A$. Conversely, let a and b be any two vertices of a graph (S,R) that has no proper subsets $A \subset S$ such that $\{xR \mid x \in A\} \subset A$. Consider the subset $B = \{x \mid \text{there is a path from } a \text{ to } x\}$. If $y \in \{xR \mid x \in B\}$ then there is an arc (x,y) and a path from a to x which together give a path from a to y. This means $y \in B$ and therefore $\{xR \mid x \in B\} \subset B$. Since there are no proper subsets such that $\{xR \mid x \in B\} \subset B$ then $B = S$. Hence $b \in S$ and there is a path from a to b. Therefore (S,R) is strongly connected.

Let a graph be represented by a complex system of pipes with oil flowing in the system. Suppose it is known that oil flows from every proper subset of joints (vertices) directly to at least one joint not in the set. By Theorem 3.14 it is known that oil flows from any joint directly or indirectly to any other joint.

Theorem 3.15. If (S,R) is a finite minimally strongly connected graph

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[1]Arabic numerals in parenthesis indicate a reference to the Bibliography.

with at least two vertices, then in (S,R) there are at least two vertices each contained in exactly two arcs.

Theorem 3.15 will be proved by induction on the number of edges. Let (S,R) be a finite minimally strongly connected graph. First note that (S,R) is irreflexive since if it contained a loop (a,a) then $(S,R - \{(a,a)\})$ would be strongly connected contradicting that (S,R) is minimally strongly connected. If (S,R) contained only one edge [a,b], then to be strongly connected $(a,b) \in R$ and $(b,a) \in R$. Since (S,R) contains no loops then vertices a and b are each contained in exactly two arcs, namely, (a,b) and (b,a). Assume the theorem is true for a minimally strongly connected graph with less than k edges. Let (S,R) be any minimally strongly connected graph with k edges.

It will now be shown that the new graph (T,W) is also minimally strongly connected. Let a and b be any two vertices of (T,W). There exists an elementary path $ax_1x_2\cdots b$ from a to b in (S,R) since it is strongly connected. If the path contains only one or no vertex in the circuit then the path also exists in (T,W). If the elementary path $ax_1x_2\cdots b$ contains more than one vertex in the circuit, then the part of the path between the first and last vertices in the circuit can be deleted from the path $ax_1x_2\cdots b$ still leaving a path from a to b in (T,W). Therefore (T,W) is strongly connected. Now suppose there exists an arc $(a,b) \in W$ such that $(T,W - \{(a,b)\})$ is strongly connected. Thus there exists a path in (T,W) from a to b other than (a,b). This path, with part of the circuit if necessary, provides a path in (S,R) from a to b other than (a,b), contradicting that (S,R) is minimally strongly connected. Therefore (T,W) is minimally strongly connected. Since

(T,W) is minimally strongly connected and has less than k edges, then by assumption (T,W) contains at least two vertices x and y each contained in exactly two arcs.

Lemma 3.16. Any vertex of (S,R) not in the circuit is contained in the same number of arcs in (T,W) as in (S,R).

If the vertex is not connected to the circuit by an edge, then the lemma follows since any arcs in (S,R) that contain the vertex are also in (T,W) and vice versa. Let b be a vertex not in the circuit, but connected to the circuit by an edge. Vertex b cannot be contained in three or more arcs each with the other vertex in the circuit, since at least two of the arcs would be from b to the circuit or from the circuit to b and one would make the other superfluous (unneeded for minimally strongly connected) in (S,R). Similarily, if vertex b is contained in exactly two such arcs, then one is from b to the circuit and the other from the circuit to b. Hence, these two arcs and any other arc in (S,R) containing b are transformed to distinct arcs in (T,W), so b is contained in the same number of arcs in (T,W) as in (S,R). It should also be clear that if b is contained in only one arc with <u>another</u> vertex in the circuit, then such an arc and any other arcs containing b are transformed to distinct arcs in (T,W). This proves the lemma.

Now in (S,R) there are either, (1) three or more distinct vertices not in the circuit, but each contained in an arc with the other vertex in the circuit, (2) exactly two such vertices, (3) exactly one such vertex or (4) none if (S,R) is the circuit.

First consider case (1). If there are three or more vertices not in the circuit but contained in arcs with a vertex in the circuit, then the circuit is transformed into a vertex in (T,W) that is contained in three or more distinct arcs in (T,W). It follows that neither vertex x or y is that vertex. Therefore x or y are vertices of (S,R) not in the circuit. By the lemma, x and y are vertices in (S,R) each contained in exactly two arcs of (S,R).

Case (2) is a little more involved. Let b and c be the only two vertices in (S,R) not in the circuit, but each contained in an arc with a vertex in the circuit. Note that there cannot be two arcs from b or from c to the circuit or two arcs from the circuit to b or to c, since one would be superfluous in (S,R). Hence, if there is a total of three or more arcs each containing b or c with the other vertex in the circuit, then the circuit is transformed into a vertex of (T,W) included in at least three arcs of (T,W). Neither vertex x or vertex y is that vertex. By the lemma, then x and y are vertices in (S,R) each contained in exactly two arcs of (S,R). If there is a total of only two arcs between the circuit and vertices b and c, then one arc is from a vertex, say b, to the circuit and the other from the circuit to c. Now there must exist a path not including a vertex of the circuit from c to b. Therefore, if the circuit contained only two vertices then the path from c to b would make one arc in the circuit superfluous in (S,R). Hence the circuit must have at least three vertices, with at least one contained in exactly two arcs of (S,R). Also either x or y must be a vertex of (S,R) not included in the circuit. By the lemma, that vertex is contained in exactly two arcs in

(S,R). This makes at least two such vertices in (S,R).

Now consider case (3). Let b be the only vertex not in the circuit, but contained in an arc with a vertex in the circuit. There must be two arcs, one from the circuit to b and the other from b to the circuit, since (S,R) is strongly connected. If both arcs include the same vertex in the circuit, then any other vertex in the circuit (there is at least one) is contained in exactly two arcs of (S,R). Also x and y in (T,W) is a vertex in (S,R) not in the circuit and by the lemma each is contained in exactly two arcs of (S,R). The only other possibility is for the two arcs including b to contain different vertices of the circuit. If this happens then the circuit must contain more than two vertices or otherwise one arc in the circuit is superfluous in (S,R). Thus, there is a third vertex in the circuit, belonging to exactly two arcs. Again x or y is not in the circuit and there are at least two of the desired vertices in (S,R).

Theorem 3.17. A finite irreflexive graph has an even number of odd vertices.

Theorem 3.17 is easily proved by mathematical induction. A graph can have any number of edges. An irreflexive graph with only one edge obviously has two odd vertices. Assume the theorem is true for a finite irreflexive graph with k edges and let (S,R) be any graph with k + 1 edges. If any edge of (S,R) is deleted, then the number of odd vertices changes by 0 or 2. By assumption the deleted graph has an even number of odd vertices and hence (S,R) has an even number of odd vertices. This completes the proof.

Corollary 3.18. A finite reflexive graph has an even number of even vertices.

Theorem 3.19. The sum of the orders of all the vertices of a finite irreflexive graph is even.

By Theorem 3.17 the sum of the orders of the odd vertices is even. Since the sum of the orders of all even vertices is even then the sum of the orders of all vertices is even.

Corollary 3.20. The sum of the orders of all the vertices of a finite reflexive graph if even or odd according as the number of vertices is even or odd.

Theorems 3.17 and 3.19 lead to interesting comments that one can make about various situations. For example, consider all the animals in the world with two animals related if and only if one depends on the other for survival. Do not consider that an animal depends on himself for survival, that is, make the graph irreflexive. It is therefore known that an even number of animals exist that give vital assistance to or depend on an odd number of animals. Also if one adds the total number of animals that directly dependent on and necessary to a particular animal, then it is even.

Theorem 3.21. If all vertices a_i, $i = 1, 2, \cdots n$ and b_j, $j = 1, 2 \cdots m$ of an irreflexive graph are such that $|a_i R| = |R b_j| = k_1$ and $|R a_i| = |b_j R| = k_2$, $k_1 \neq k_2$, then $n = m$.

The theorem states that if each vertex is either a terminal vertex to exactly k_1 arcs and an initial vertex to exactly k_2 arcs or a terminal vertex to exactly k_2 arcs and an initial vertex to exactly k_1 arcs, then there are as many vertices of one kind as there are of the other. Such a graph exists as illustrated by Figure 9. with $k_1 = 1$ and $k_2 = 2$.

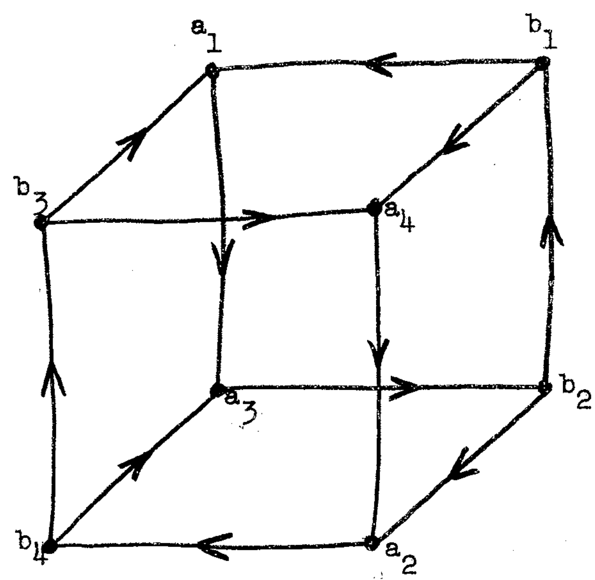


Figure 9.

Theorem 3.21 can be proved by counting the number of arcs. One can count the arcs by associating with each vertex a_i the k_1 arcs with a_i its initial vertex and with each vertex b_j the k_2 arcs with b_j as its initial vertex. Thus $k_1n + k_2m$ is the total number of arcs. But, one can also count the arcs by associating with each vertex a_i the k_2 arcs with a_i as its terminal vertex and with each b_j the k_1 arcs with b_j as its terminal vertex. Thus $k_2n + k_1m$ also is the number of arcs and $k_1n + k_2m = k_2n + k_1m$ which implies $n = m$.

Theorem 3.22. If a graph (S,R) is connected, then the partial graph

$(S,R - \{(a,b)\})$ where $(a,b) \in R$ has one component (connected) or two components C_a and C_b.

Let $p \in S$. If $p = a$ or $p = b$ then $p \in C_a$ or $p \in C_b$. If $p \neq a$ and $p \neq b$ then since (S,R) is connected there exist chains from p to a and p to b. By Theorem 3.12 there exist elementary chains from p to a and from p to b. If at least one of the elementary chains from p to a does not include the edge $[a,b]$ then p is connected to a in the graph $(S,R - \{(a,b)\})$. If every elementary chain from p to a includes the edge $[a,b]$, necessarily as the last edge in the chain, then p is connected to b by an elementary chain not including $[a,b]$ in the graph $(S,R - \{(a,b)\})$. A similar argument holds for a and b interchanged. Therefore each vertex p of $(S,R - \{(a,b)\})$ is connected to a or b, or $p = a$ or $p = b$. This implies for components C_a and C_b that $C_a \cup C_b = S$ in the graph $(S,R - \{(a,b)\})$. By Theorem 3.10 either $C_a = C_b$ or $C_a \cap C_b = \emptyset$ and Theorem 3.22 follows.

It should be noted that Theorem 3.22 could be restated by deleting an edge $[a,b]$ instead of arc (a,b) from the graph (S,R) and still hold. The argument would be identical.

Definition 3.1. If (S,R) is a connected graph and $(S,R) - [a,b]$ has two components then the edge $[a,b]$ is called a <u>separating edge</u> of (S,R). If $(S,R) - [a,b]$ is connected then the edge $[a,b]$ is called a <u>non-separating edge</u>.

Theorem 3.23. An edge of a connected graph is a separating edge if and only if there exist two vertices such that every chain between them contains the edge.

Let [a,b] be a separating edge of a connected graph (S,R). The partial graph (S,R) - [a,b] has two components C_a and C_b by definition of a separating edge and alternate form of Theorem 3.22. Hence there are no chains in (S,R) - [a,b] connecting vertices a and b. Therefore, every chain in (S,R) that connects a and b must contain edge [a,b]. Conversely, suppose a and b are vertices such that every chain between them contains an edge [c,d]. Consider the components C_a and C_b in the partial graph (S,R) - [c,d]. Now $a \notin C_b$ since by hypothesis every chain between a and b includes [c,d] which is deleted in (S,R) - [c,d]. By Theorem 3.10 then $C_a \cap C_b = \emptyset$. By Theorem 3.22, C_a and C_b are the only components and edge [c,d] is therefore a separating edge.

Theorem 3.24. If (S,R) is an asymmetric strongly connected graph, then every edge is a non-separating edge.

Consider any edge [a,b] of a strongly connected graph (S,R). Since (S,R) is asymmetric then either $(a,b) \in R$ and $(b,a) \notin R$ or $(b,a) \in R$ and $(a,b) \notin R$. Suppose without loss of generality that $(a,b) \in R$ and $(b,c) \notin R$. There must be a path from b to a since (S,R) is strongly connected. The path cannot be just (b,a) since $(b,a) \notin R$. Since every path is a chain then there is at least one chain from b to a not including the edge [a,b]. By Theorem 3.23 then edge [a,b] is a non-separating edge.

Theorem 3.25. If (S,R) is a finite connected graph with every edge a non-separating edge, then there exists an asymmetric strongly connected graph (S,T) such that (S,R) and (S,T) contain the same edges.

Let (S,R) be any finite connected graph with every edge a non-separating edge. For an edge to be a non-separating edge, then there must exist a cycle that includes the edge. Since (S,R) is finite and there exists at least one simple cycle, then there exists a finite number of simple cycles. Since (S,R) is connected then every vertex is contained in an edge and consequently included in some cycle. Consider any one of the finite number of simple cycles, call it u_1. Give an orientation to the cycle so that it becomes a simple circuit. Any simple circuit is necessarily strongly connected. Hence if u_1 is the only cycle, then the proof is complete. If u_1 is not the only cycle, then it can be shown there exists another cycle u_2 that has at least one vertex in common with u_1 since (S,R) is connected. Proceed with cycle u_2 as if to give it an orientation to become a circuit except leave any of the edges that are in cycle u_1 with the orientation previously assigned. Together, cycles u_1 and u_2 with the selected orientation is strongly connected, because: u_1 by itself with the given orientation is strongly connected. By the orientation given u_2 any of its vertices has a path to a vertex of strongly connected u_1 and has a path from some vertex of u_1 to this vertex. Hence there is a path from any vertex of u_2 to any vertex of u_1, a path from any vertex of u_1 to any vertex of u_2, and a path from any vertex of u_2 to any vertex of u_2. If all the edges are oriented, then the proof is finished. If not, then there exists a cycle u_3 that contains a vertex in common with the already strongly connected set of vertices with the selected orientation. Continue the process as before. Since there is only a finite number of edges then the process must stop and the sets have an orientation by which it is strongly connected. Since every edge was given only one orientation the graph is asymmetric. Every edge of (S,R) was considered so (S,R) and the new graph contain the same edges. This completes the proof.

As an application of Theorem 3.25 suppose Figure 10 A represents a monorail system between various displays, indicated by letters, at a large fair. Notice that the system is connected and each track is involved in some cycle of tracks. The fair committee decides since several trains will be in use at the same time that every track should be made one-way to prevent head-on collisions. Can it be done and still assure service between any two displays and if so how? Theorem 3.25 assures that it is possible and the proof of Theorem 3.25 indicates how a solution might be obtained. First consider any simple cycle, say cycle aefgibcda. Make it into a simple circuit as illustrated in Figure 10 B by following the single arrows. Next consider a simple cycle that has a vertex in common with that cycle, such as cycle abiehgida. The edges in this cycle that are not already oriented are [a,b],[i,e],[e,h],[h,g] and [i,d]. If one were to proceed as if to give this cycle an orientation, say the reverse of the order named, except leaving edges already oriented with their orientation, then the result would be as indicated in Figure 10 B with the double arrows showing the additional orientation. There is no need to consider any other cycles for all edges are now directed. The graph illustrated in Figure 10 B is one of many solutions.

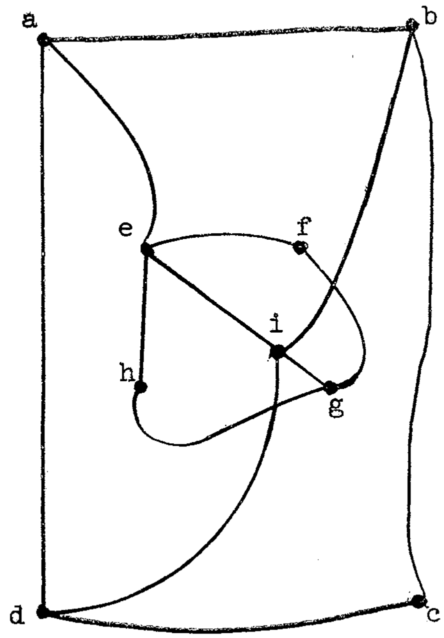


Figure 10 A.

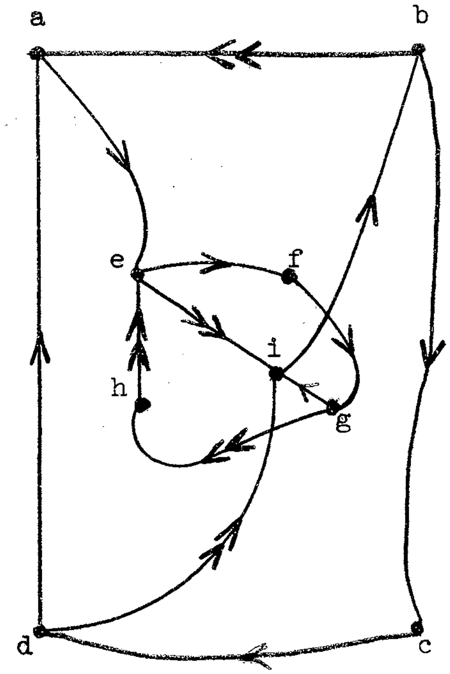


Figure 10 B.

Theorem 3.25 dealt with a finite connected graph without separating edges. A generalization of Theorem 3.25 is given in the following theorem.

Theorem 3.26 (87). If (S,R) is a finite connected graph then there exists a finite strongly connected graph (S,W) with the same edges, but for any non-separating edge [a,b] of (S,R) either $(a,b) \in W$ or $(b,a) \in W$ but not both, and for any separating edge [c,d] of (S,R) both $(c,d) \in W$ and $(d,c) \in W$.

It should be obvious that a connected graph (S,R) can be made into a strongly connected graph (S,W) by including a set of arcs T such that $W = R \cup T$ is a symmetric relation. However, this statement is not nearly so strong as Theorem 3.26. Not only is Theorem 3.26 interesting, but it takes on special importance in dealing with certain traffic problems. In terms of traffic control of a city with linking bridges, dead-end streets, etc. as separating edges, the theorem assures one that proper traffic connections can be made everywhere by making the bridges, dead-end streets, etc. as two-way streets and by making the other streets one-way. This kind of map may be more restrictive than necessary, but the theorem does assure that such a solution is always possible.

The proof of Theorem 3.26 involves giving a proper orientation to all non-separating edges. Let (S,R) be a finite connected graph. If a separating edge [a,b] is deleted from (S,R) then (S,R) - [a,b] has two distinct components C_a and C_b by definition of a separating edge. Furthermore, the subgraphs (C_a,W_a) and (C_b,W_b) of (S,R) have no common edges or otherwise C_a and C_b are not disjoint. Also, if there are any other separating edges in (S,R) then necessarily it is a separating edge of (C_a,W_a) or (C_b,W_b). Now if either subgraph contains a separating edge of (S,R) then delete it from the graph and obtain additional subgraphs. Since there can only be a finite number of separating edges then such a process must stop. The result is a finite set of connected subgraphs of (S,R) which together contain all the non-separating edges of (S,R) and no two subgraphs contain a common edge. Consider any one such subgraph (C_x,W_x). If (C_x,W_x) contains an edge, necessarily a non-separating edge, then there exists at least one cycle of non-separating edges containing the edge or otherwise the edge would have been a separating edge. All the edges of the cycle are contained in (C_x,W_x)

since these edges were not deleted and the subgraphs are disjoint. Hence by Theorem 3.25 there exists an asymmetric strongly connected graph (C_x, T_x) with the same edges as (C_x, W_x). Do this for every one of the finite number of the connected subgraphs for the components. Now the only edges of (S,R) not contained in the subgraphs are the separating edges. Since (S,R) is connected then each separating edge must connect two subgraphs and all subgraphs are connected. Therefore, if all separating edges are oriented both ways then there exists a path from any vertex to any other. It should be noted that if the vertices are in different subgraphs the path between them may go through several other of the strongly connected subgraphs.

A kind of vertex that is similar to a separating edge is that of a separating vertex.

Definition 3.2. If (S,R) is a connected graph and the subgraph $(S - \{x\}, W)$ is not connected, then the vertex x is called a <u>separating vertex</u>. If $(S - \{x\}, W)$ is connected then the vertex x is called a <u>non-separating vertex</u>.

Definition 3.3. A separating vertex x is <u>simple</u> if and only if there is only one edge connecting x to each of the components of the subgraph $(S - \{x\}, W)$.

It should be obvious that separating vertices are of prime concern in dealing with problems where connectivity is important. For example, separating vertices in a graph representing a communication system are invaluable, since without them certain communications would be impossible.

The following theorem characterizes separating vertices in the same manner that Theorem 3.22 characterized separating edges.

Theorem 3.27. A vertex of a connected graph is a separating vertex if and only if there exist two vertices such that every chain between them contains the vertex.

Let (S,R) be a connected graph. If x is a separating vertex then $(S - \{x\},W)$ is not connected. Hence there are at least two components C_a and C_b. There are no chains between a and b in the subgraph since C_a and C_b are components. Hence, any chains between a and b in (S,R) must pass through x. Conversely, if every chain between a and b contains x, then in $(S - \{x\},W)$ there is no chain between a and b. Hence the subgraph is not connected and x is a separating vertex.

CHAPTER IV

FUNCTIONS OF GRAPH THEORY

This chapter contains an introduction to some important integral-valued functions. Some of these functions are defined on the set of all finite graphs, such as to associate with each finite graph the number of arcs it contains. Other functions are defined on parts of a graph, such as to associate with each simple path in a graph the number of arcs it contains. There will be occasions when it is beneficial to define an arbitrary function on parts of a graph. For example, in a graph represented by a road map one may wish to associate with a road some number that may indicate the condition or cost of the road. Functions of this nature lead to applicable relations and many lead to characterizations of terms and properties already discussed.

M-Graphs

One function associates with each arc of a graph a positive integer called the multiplicity of the arc. Because of the definition of a graph, any two vertices of a graph have at most two arcs between them. However, there are many problems which present a need to have numerous arcs from one vertex to another. For example, two towns a and b may have n direct highways from a to b or two offices a and b in an organization may have n different means of direct communication from a to b.

This situation can be described by a function, call it M, that associates with each arc a positive integer. In the example, $M((a,b)) = n$, that is, the multiplicity of arc (a,b) is n.

Definition 4.1. A function that assigns to each arc (a,b) of a graph (S,R) a positive integer $M((a,b))$, called its multiplicity, is an M-function.

Definition 4.2. An M-graph, denoted by (S,R,M) consists of the vertices of a graph (S,R) and the arcs $(a,b)_1,(a,b)_2,\cdots(a,b)_n$ where $n = M((a,b))$ assigned by an M-function.

It should be noted that a graph (S,R) is a special case of an M-graph (S,R,M) where $M((a,b)) = 1$ for all $(a,b) \in R$. Also, the definitions pertaining to graphs will have corresponding definitions for M-graphs. There is no need to list all such corresponding definitions, but a few will be given to reveal the general idea.

Definition 4.3. An edge $[a,b]_j$ in an M-graph (S,R,M) is any pair of vertices a and b and a number j such that arcs $(a,b)_j$ or $(b,a)_j$ exist in (S,R,M).

Notice that the number of edges connecting a pair of vertices a and b in (S,R,M) is the maximum of $M((a,b))$ and $M((b,a))$. An edge $[a,b]_j$ in an M-graph is represented by an undirected line segment with a and b as endpoints. Edges are undirected as was the case for graphs.

Definition 4.4. The order of a vertex x in an M-graph (S,R,M) is the total number of edges $[a,b]_j$ in (S,R,M) containing x.

Definition 4.5 A path in an M-graph (S,R,M) is an ordered set $((a,b)_i,(b,c)_j,(c,d)_k\cdots)$ of arcs in (S,R,M) with one vertex of each arc coinciding with a vertex in the succeeding arc and the other vertex coinciding with a vertex in the preceeding arc.

The definition of a chain in an M-graph is similar to that of a path, but using edges instead of arcs.

Definition 4.6. A chain in an M-graph is simple if all the edges of the chain are distinct and otherwise composite.

Definition 4.7. An M-subgraph of a M-graph (S,R,M) is the M-graph (T,W,M) such that (T,W) is a subgraph of (S,R) with the M-function of (S,R) restricted to (T,W).

Definition 4.8. A partial M-graph of an M-graph (S,R,M) is an M-graph obtained from (S,R,M) by deleting any set of arcs.

It should be noted that a partial M-graph of (S,R,M) is an N-graph (S,T,N) such that $T \subset R$ and $M((a,b)) \geq N((a,b))$ for all (a,b) in (S,T). A partial M-graph is also obtained by deleting any set of edges from (S,R,M) where the deletion of an edge means deleting the arcs that give rise to the edge.

An M-graph is represented geometrically by drawing n lines with arrows from a to b when the multiplicity of arc (a,b) is n. Figure 11-A represents an M-graph with $M((b,c)) = 3$, $M((c,b)) = 1$, $M((c,d)) = 2$, $M((f,f)) = 3$ and $M((e,f)) = 1$. There are three edges between b and c; two edges between c and d; three edges (loops) at f; and one edge between f and e. Figure 11-B represents the edges.

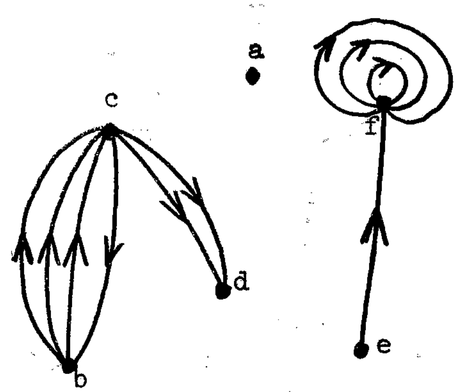


Figure 11 A.

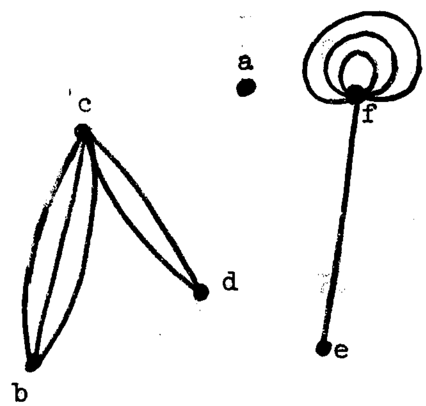


Figure 11 B.

Many of the previous theorems in Chapter III stated for graphs also hold for M-graphs. In most cases the proofs for these theorems stated for M-graphs, instead of for graphs, are similar to the proofs given for graphs. In fact, many of the theorems are easier to prove for M-graphs than for graphs. Since the previous theorems which hold for M-graphs can be proved by arguments similar to the ones given for graphs, then for the sake of brevity we merely state the theorems.

Theorem 4.1. If in an M-graph (S,R,M) there exists a path (chain) from a to b and a path (chain) from b to c, then there exists a path (chain) in the M-graph (S,R,M) from a to c.

Theorem 4.2. If there exists a path (chain) in a finite M-graph (S,R,M) from a to b $\neq$ a, then there exists an elementary path (chain) from a to b.

Corollary 4.3. Every circuit (cycle) of a finite M-graph (S,R,M) contains an elementary circuit (cycle).

Theorem 4.4. Every arc in a strongly connected M-graph (S,R,M) is contained in some elementary circuit of (S,R,M).

Theorem 4.5. An M-subgraph of a complete M-graph is complete.

Theorem 4.6. A strongly connected M-graph is connected.

Theorem 4.7. For any two distinct edges of a connected M-graph there is an elementary chain with the given edges as ends.

Theorem 4.8. Every component C_a of a M-graph (S,R,M) determines a connected M-subgraph (C_a,W,M) of the M-graph (S,R,M).

Theorem 4.9. The set of all components of an M-graph (S,R,M) is a partition of the set S.

Theorem 4.10. The set of all components of an M-graph (S,R,M) determines a partition of (S,R,M) into connected M-subgraphs.

Theorem 4.11. A M-graph (S,R,M) is connected if and only if S is a component.

Theorem 4.12. A complete M-graph has only one component.

Theorem 4.13. If (S,R,M) is a finite minimally strongly connected M-graph with at least two vertices, then in (S,R,M) there are at least two vertices each contained in exactly two arcs.

Theorem 4.14. The sum of the orders of all the vertices of a finite irreflexive M-graph is even.

Theorem 4.15. If an M-graph (S,R,M) is connected, then the partial

M-graph $(S,R,M) - [a,b]_j$ where $[a,b]_j$ is an edge, has one component or two components.

Theorem 4.16. An edge of a connected M-graph is a separating edge if and only if there exists two vertices such that every chain between them contains the edge.

Theorem 4.17. A vertex c of a connected M-graph is a separating vertex if and only if there exist two vertices a and b such that every chain between a and b contains the vertex c.

Fundamental Numbers of Graph Theory

Any finite graph or M-graph has a certain number of vertices, arcs, edges and components. The following non-negative integer-valued functions are defined on the set of all finite graphs or M-graphs:

a function V that associates with each finite graph (S,R) or M-graph (S,R,M) its number of vertices v, that is,

$$v = V((S,R) = V((S,R,M))$$

(V is the cardinal number of set S);

a function A that associates with each finite graph (S,R) or M-graph (S,R,M) its number of arcs r, that is,

$$r = A((S,R))$$

$$r = A((S,R,M)) = \Sigma\ M((a,b))$$

(summed over all arcs (a,b) of (S,R));

a function E that associates with each finite graph (S,R) or M-graph (S,R,M) its number of edges e, that is,

$$c = C((S,R)) = C((S,R,M))$$

Lemma 4.18. Theorems which hold for all M-graphs hold for graphs.

The conversion is done easily by considering an ordinary graph to be an M-graph with each multiplicity 1, so that $\Sigma M((a,b))$ counts the arcs of (S,R).

The following theorems show relationships between the numbers v, r, e, and c.

Theorem 4.19. If (S,R) is a finite graph then:

(1) $e \leq r \leq 2e$ or $\frac{1}{2}r \leq e \leq r$

(2) $c \leq v$ (also holds for M-graphs)

Theorem 4.19 is straightforward since every arc is associated with one and only one edge and every edge can be associated with at most two arcs. Every vertex belongs to one and only one component.

Theorem 4.20. An M-graph is connected if and only if $c = 1$.

Theorem 4.20 follows directly from Theorem 4.11.

Theorem 4.21. A graph is irreflexive and symmetric if and only if $r = 2e$.

In an irreflexive symmetric graph, for every edge $[a,b]$ there are exactly two arcs (a,b) and (b,a). Irreflexitivity guarantees they are different. Also, for every arc (a,b) there is one and only one edge $[a,b]$. Hence $r = 2e$. Conversely, let $r = 2e$. From Theorem 4.19 $r \leq 2e$. To have equality every edge must arise from two arcs. So there cannot be any loops. Hence the graph is irreflexive and symmetric.

Theorem 4.22. Of the three following properties of a finite graph, any two imply the third:

(1) The graph is reflexive

(2) The graph is symmetric

(3) $2e = r + v$

That (1) and (2) implies (3) and that (1) and (3) implies (2) follows easily by using Theorem 4.21. To show (2) and (3) imply (1) let the edges be divided into ℓ loops and e_1 non-loops. To be symmetric, then $r = \ell + 2e_1$. Hence, with property (3) $2(\ell + e_1) = (\ell + 2e_1) + v$ and $\ell = v$. But $\ell = v$ implies reflexitivity.

Theorem 4.23. In a finite connected M-graph $e + 1 \geq v$.

If a finite connected graph has v vertices then the minimum number of edges needed to connect them is $v - 1$. Thus, in any finite connected graph $e + 1 \leq v$. In an M-graph, since each arc is assigned an integer ≥ 1 giving rise to the same number or more edges, this inequality is merely strengthened.

One may wonder by Theorem 4.23 if there are any distinguishing characteristics of a connected graph when $e + 1 > v$ or when $e + 1 = v$. The answer is definitely "yes" as will be seen later by Theorem 6.2.

Theorem 4.24. If (S,R) is a finite irreflexive graph, then $r < v(v - 1)$.

The largest number of arcs in a finite irreflexive graph is the number of arrangements of v vertices taken two at a time, that is,

$v!/(v - 2)! = v(v - 1)$. Thus $r \leq v(v - 1)$.

Theorem 4.25. A finite irreflexive graph is complete if and only if $2e = v(v - 1)$.

In a complete graph any pair of vertices determines a unique edge and all edges arise only from pairs of vertices since it is irreflexive. Hence there are as many edges as there are combinations of v things taken two at a time, that is, $e = v!/2(v - 2)! = v(v - 1)/2$. Conversely, if $e = v(v - 1)/2$ and since irreflexitivity means edges can only be formed between distinct pairs of vertices, then each pair of vertices must determine an edge. Hence the graph is complete.

Corollary 4.25. In a finite complete M-graph $2e \geq v(v - 1)$.

Theorem 4.26. A finite graph is reflexive and complete if and only if $2e = v(v + 1)$.

In a reflexive complete graph any pair of vertices, distinct or not, determines an edge. Hence, the number of edges in a reflexive complete graph with v vertices is the same as the number of edges in an irreflexive graph with v vertices plus the v edges formed by the loops. By Theorem 4.25, the number of edges $e = v(v - 1)/2 + v$ and $2e = v(v + 1)$. Conversely, if $2e = v(v + 1)$ then every edge possible must be formed. Hence the graph is reflexive and complete.

It is easily seen in a graph or M-graph that either $e \geq v$ or $e \leq v$ may be true. This does not outrule the possibility of a simple relationship between the number of vertices and the number of edges in a graph.

Theorem 4.27. A finite M-graph has $e + c \geq v$.

Let (S,R) be a finite graph. Partition (S,R) into connected subgraphs by Theorem 3.11. By Theorem 4.23 each subgraph $e_i + 1 \geq v_i$ where e_i is the number of edges in the subgraph and v_i the number of vertices. Since there are c subgraphs then

$$\sum_{i=1}^{c} (e_i + 1) \geq \sum_{i=1}^{c} v_i,$$

$$\sum_{i=1}^{c} e_i + \sum_{i=1}^{c} 1 \geq \sum_{i=1}^{c} v_i \quad \text{and}$$

$$e + c \geq v.$$

In an M-graph this inequality is merely strengthened.

Theorem 4.28. If (S,R) is a finite strongly connected graph with $v > 1$, then $v \leq r \leq v^2$.

Theorem 4.28 follows since in a strongly connected graph every vertex is an initial vertex of some arc and no two vertices can be the initial vertex of the same arc. The other inequality is true of any graph.

Theorem 4.29. If k subsets of edges of a complete graph, $v < 2$, have the properties that each edge is contained in some subset and that any two edges in the same subset have a common vertex, then $k \geq v - 2$.

The theorem is proved by mathematical induction. The theorem follows easily for $v = 3$ since $k > 0$. Assume the theorem holds for a complete graph with v vertices. Let (S,R) be any complete graph with $v + 1$ vertices. Consider the collection of k subsets of edges of (S,R) with the required properties. Note that a subset cannot contain a cycle of more than three edges and that for a cycle of three edges there could not be any other edge in the subset because of the second property (and because two vertices have at most one edge in common). If three edges do not have a common vertex, then one does not contain the vertex in common to the other two, but by containing their other vertices forms a cycle. Therefore, either (1) there exists a subset with a vertex in common with all of its edges or (2) every subset contains three edges.

If (1) occurs then let x be the common vertex and consider the subgraph $(S - \{x\},W)$. By theorem 3.5, $(S - \{x\},W)$ is complete. Furthermore, since x is in common with each edge in one of the k subsets then the other k - 1 subsets satisfy the conditions for $(S - \{x\},W)$. Hence by assumption it follows that $k - 1 \geq (v - 1) - 2$ or $k \geq v - 2$. Now consider case (2). If every subset contains three edges then $3k \geq e$ in (S,R), where the inequality occurs only if the subsets are not disjoint. By Theorem 4.25, $2e = v(v - 1)$. For v an integer $(v - 3)(v - 4) \geq 0$ gives $v(v - 1) \geq 6(v - 2)$. Hence $6k \geq 2e = v(v - 1) \geq 6(v - 2)$ and $k \geq v - 2$.

Suppose one is to color the edges of the complete graph represented in Figure 11. such that any two edges with the same color have a common vertex. Theorem 4.29 indicates that no matter how one attempts to do this, at least five colors will be needed. Figure 12. shows one such coloring with r meaning red, b for blue, etc..

Theorem 4.30. If every vertex of a connected M-graph is of order k, then $v \leq 2e/k$.

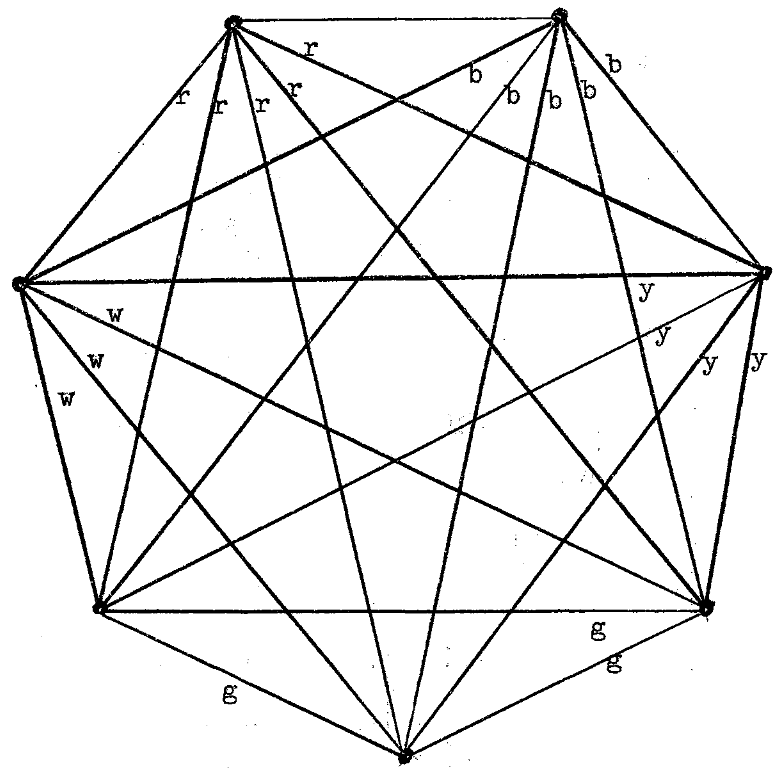


Figure 12.

Let $X = \{(a,b) \mid a$ is an edge and b a vertex of $a\}$. The cardinal number of X, $|X| \leq 2e$, since each edge contains at most two vertices. Also, $kv \leq |X|$, since each vertex is contained in k distinct edges. Therefore, $kv \leq |X| \leq 2e$ and $v \leq 2e/k$.

Theorem 4.31. The sum of the orders of the vertices in an irreflexive M-graph with e edges is $2e$.

Theorem 4.31 is actually a generalization of Theorem 4.14 which states that in an irreflexive graph the sum of the orders of the vertices is even. Theorem 4.31 follows since every edge not being a loop accounts of an order of one in each of two vertices.

Corollary 4.32. The sum of the orders of the vertices in an M-graph with e edges of which ℓ are loops is $2e - \ell$.

Theorem 4.33. An irreflexive finite graph with the order of each vertex $\geq (v - 1)/2$, $v < 1$, is connected.

If the number of vertices v is odd, say $2k + 1$, then the order of each vertex is $\geq k$. If the number of vertices v is even, say $2k$, then the order of each vertex is $\geq (2k - 1)/2 = k - \frac{1}{2}$ and hence the order is $\geq k$. Consider any two vertices a and b. Since the order of each vertex is $\geq k$ then it follows that there exists elementary chains $aa_1a_2\cdots a_k$ and $bb_1b_2\cdots b_k$ each consisting of at least $k + 1$ distinct vertices. The two chains are not vertexwise disjoint for if they were then there would be at least $2k + 2$ vertices which is a contradiction. Hence the two chains have a vertex x in common and the chain $aa_1\cdots x\cdots b_1b$ exists between a and b. Therefore the graph is connected.

The Cyclomatic Function

Another integer-valued function defined on the set of all finite graphs or M-graphs is concerned with eliminating the cycles of the graph. In any finite M-graph there is a set of edges that when deleted gives a partial graph without cycles. Hence there is a set A of edges of minimum cardinality such that when deleted give a partial graph with no cycles.

Definition 4.9. A function that associates with every M-graph the minimum of the set of cardinalities of all sets of edges that when deleted gives a partial graph without cycles, is called a cyclomatic function, denoted by Q.

Definition 4.10. The number $Q((S,R,M)) = \theta$ is called the cyclomatic number of (S,R,M).

Cyclomatic numbers lead to interesting characterizations.

Theorem 4.34. An M-graph (S,R,M) contains no cycles if and only if the cyclomatic number $\theta = 0$.

Theorem 4.34 is immediate.

Theorem 4.35. An M-graph (S,R,M) contains a unique elementary cycle if and only if the cyclomatic number $\theta = 1$.

The "only if" part of Theorem 4.35 is immediate. Suppose that $\theta = 1$ and there are two elementary cycles. The elementary cycles are not edgewise disjoint for then $\theta \geq 2$. If an edge in their intersection is deleted, then the remaining parts of the cycles contain a cycle. Hence $\theta \geq 2$. Therefore, there is only one elementary cycle.

Is there a way of finding the cyclomatic number of a graph other than by exhaustive examination? The answer is "yes", due to the following theorem.

Theorem 4.36. In a finite M-graph (S,R,M) the cyclomatic number $\theta = e + c - v$.

The proof of Theorem 4.36 is by mathematical induction. It is easily seen that in a M-graph with no edges (null graph) that $\theta = e + c - v$ since $c = v$. Next consider any finite M-graph with exactly one edge. If the edge is a loop, then every vertex is a distinct component and the theorem holds. If the edge is not a loop, then $v = 1 + c$ while θ is still zero, so the theorem still holds. Now assume the theorem is true for an M-graph with k edges. Let (S,R,M) be any M-graph with $k + 1$ edges and (S,R,M) - $[a,b]_j$ the partial M-graph by deleting the edge $[a,b]_j$. By assumption the cyclomatic number of the partial M-graph is $\theta_1 = k + c_1 - v_1$. If the edge $[a,b]_j$ is not contained in a cycle of (S,R,M) then the cyclomatic number of (S,R,M) $\theta = \theta_1$. However, since $[a,b]_j$ is not contained in a cycle of (S,R,M) then it is a separating edge and necessarily connects two components of (S,R,M) - $[a,b]_j$. Therefore $c_1 = c + 1$. By substituting $\theta = \theta_1$, $v = v_1$ and $c_1 = c + 1$ into $\theta_1 = k + c_1 - v_1$ then in (S,R,M), $\theta = (k + 1) + c - v$. Now suppose edge $[a,b]_j$ is contained in a cycle of (S,R,M). Thus $c_1 = c$. By Definition 4.9 the cyclomatic number of (S,R,M) is the minimum number of edges that when deleted eliminates all cycles. Since $[a,b]_j$ is in a cycle then one edge in the cycle must be in the set of those eliminated. If $[a,b]_j$ is the edge then $\theta_1 = \theta - 1$. If a different edge $[c,d]_i$ of the cycle is deleted and it is not involved in any other cycle then still $\theta_1 = \theta - 1$. If $[c,d]_i$ is contained in a cycle other than the one containing $[a,b]_j$ then when deleted $[a,b]_j$ is still contained in a cycle in (S,R,M). It will eventually follow that

$\theta_1 = \theta - 1$. By substituting $\theta_1 = \theta - 1$, $v_1 = v$ and $c_1 = c$ into $\theta = k + c_1 - v_1$ then $\theta - 1 = k + c - v$ or $\theta = (k + 1) + c - v$.

There are many very interesting applications of Theorem 4.35. For instance, suppose the M-graph in Figure 13. represents the fenced-in land of a farm. The farmer wishes to put all his land in pasture. What is

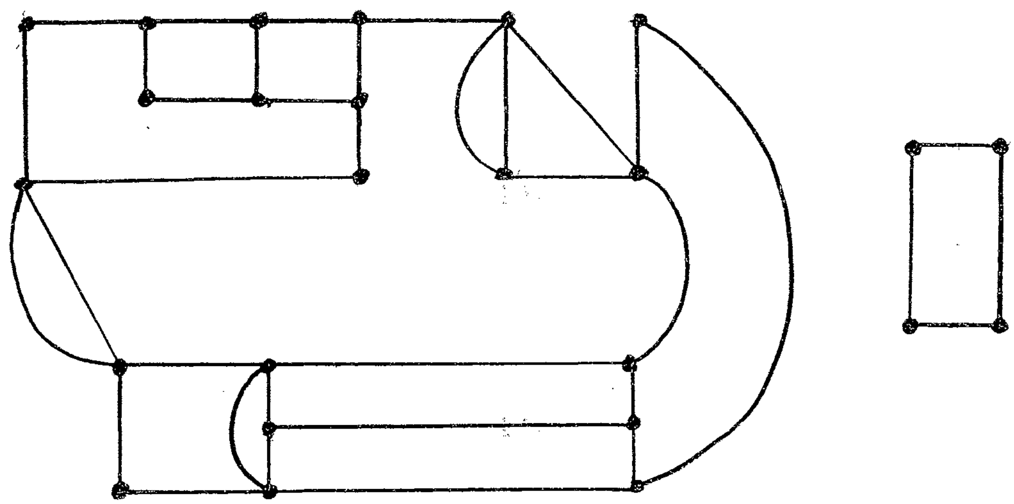


Figure 13.

the minimum number of fences he needs to cut in order that the cattle can freely roam the entire farm? The answer is the cyclomatic number of the M-graph and by Theorem 4.36 it is given by $e + c - v = 37 + 2 - 25 = 14$.

Theorem 4.37. If (S,R) is a finite irreflexive complete graph, then the cyclomatic number $\theta = (v - 1)(v - 2)/2$.

By Theorem 4.25, $2e = v(v - 1)$ and by Theorem 4.12, $c = 1$. By Theorem 4.36 the cyclomatic number $\theta = e + c - v$, so $2\theta = 2e + 2c - 2v = v(v - 1) + 2 - 2v = (v - 1)(v - 2)$.

Corollary 4.38. If (S,R,M) is a finite irreflexive complete M-graph then the cyclomatic number $\Theta \geq (v - 1)(v - 2)/2$.

The Shortest Path

Another function that is of obvious significance associates with every path (chain) or circuit (cycle) of a graph or M-graph a non-negative integer called its "length".

Definition 4.11. The <u>length</u> of a path (chain) $u = x_1x_2\cdots x_nx_{n+1}$ or circuit (cycle) $u = x_1x_2\cdots x_nx_1$ is $\ell\,(u) = n$.

It should be noted that the length of a path (chain) is not just the number of arcs (edges) involved in the path (chain) for some arcs (edges) may be used more than once.

Between any two vertices a and b of a finite graph or M-graph there can be only a finite number of simple paths (chains). Hence, if there is a path, there is at least one path (chain) of minimum length from a to b; moreover, such a path (chain) is elementary.

Definition 4.12. A <u>shortest path (chain)</u> from vertex a to b is a path (chain) of minimum length from a to b.

It should be clear that a shortest path (chain) need not be unique in a graph or M-graph.

Definition 4.13. The length of a shortest path from a to $b \neq a$, if such exists, is called the <u>distance</u> from a to b and is denoted by $d(a,b)$. Also, $d(a,b) = 0$ if and only if $a = b$.

Furthermore, it will be convenient to designate that there is no path from a to b $\neq$ a, by $d(a,b) = \infty$.

Before considering problems dealing directly with the length of paths (chains), the more immediate problem regarding the existence of a path from vertex a to a vertex b in a graph should be discussed. Is there a path in a graph going from a to b? If there is such a path how can one find it?

There are several algorithms for finding a path (if it exists) from a vertex a to vertex b. First consider a special case of a symmetric graph, one that can be represented on a plane with no edges intersecting except at the vertices. Attention is given this special case since a maze of passageways as used in experimenting with mice and the House of Mirrors (Fun House) at carnivals are examples of this kind of graph. Figure 14. represents a typical maze of seventy rooms where one tries to go from a to b.

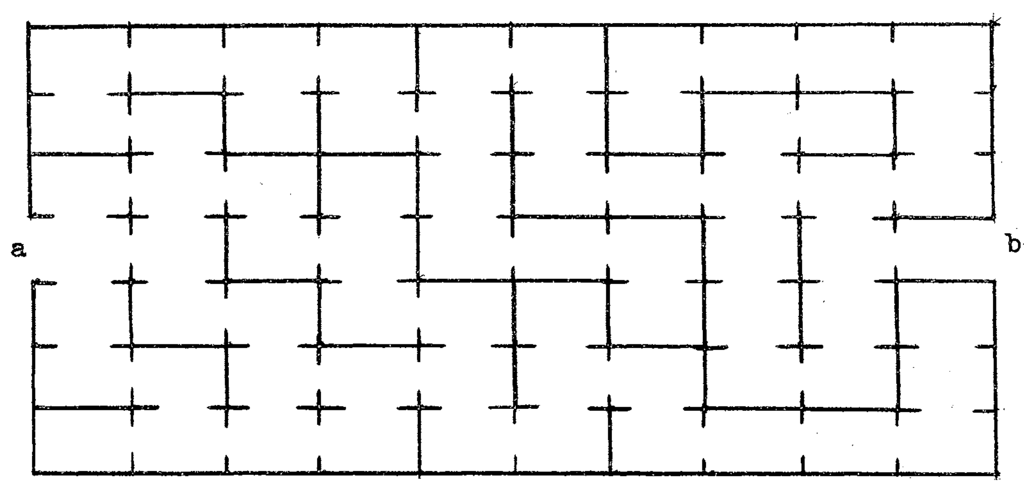


Figure 14.

The following algorithm is sufficient for finding a path from a to b in a finite planar symmetric graph. The algorithm is due to G. Tarry.

Algorithm I (128). At each vertex x take an edge not previously traveled in that direction, but only take the edge by which one first arrived at x when no other choice is available.

This procedure has an advantage in that one need not have any prior knowledge of the graph, but the disadvantage in that the path traveled need not be the shortest path.

There still remains the problem of finding a path of minimum length from a to b. An algorithm for finding a shortest path from a to b will obviously be another way of finding a path a to b. The next algorithm is for finding a shortest path from a to b (if one exists) in any finite graph (S,R).

Algorithm 2. Label vertex a with an 0. Label all vertices in $A_1 = aR$ except possibly a with a 1. Next label all vertices in $A_2 = \{xR \mid x \in A_1\}$ with a 2 if not already labeled. Now label all vertices in $\{yR \mid y \in A_2\}$ with a 3 if not already labeled and so on as far as possible. If b is labeled, say with n, then there is a shortest path with length n. One can find a shortest path by starting at b, backing one edge to a vertex x_{n-1}, labeled n - 1, backing one edge from x_{n-1} to a vertex x_{x-2}, labeled n - 2, and so forth until reaching a. The path $ax_1x_2\cdots b$ is a shortest path.

The proof of Algorithm 2 is quite simple. If there were a shorter path $ay_1y_2\cdots y_i = b$, $i < n$, then y_1 would have been labeled by 1, y_2 by 2,...b by i, contradicting that n was the first label given b.

Figure 15. shows a finite graph with vertices labeled following such a procedure. A shortest path will be of length 8 and there are only two such paths.

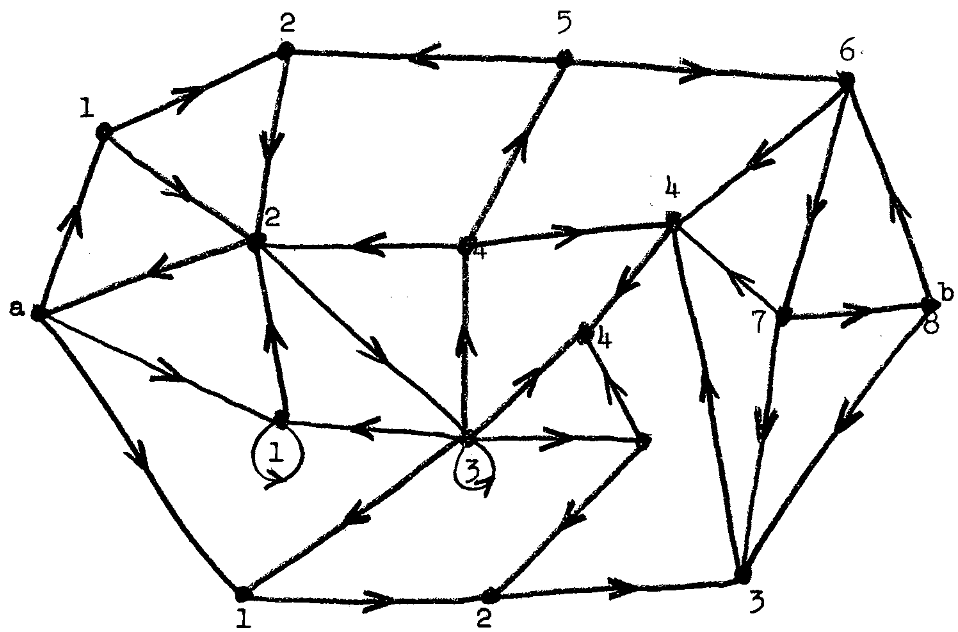


Figure 15.

The following three theorems reveal some properties and character-izations of a shortest path between vertices in a graph or M-graph.

Theorem 4.39. A shortest path is an elementary path.

The theorem is obvious since if a path is not elementary, then part of it can be deleted to obtain a shorter path. Hence, it would not be a shortest path. It should be clear that the converse of Theorem 4.39 is not true.

Theorem 4.40. If $x_1x_2\cdots x_n$ is a shortest path between x_1 and x_n, then the portion of the path $x_i x_{i+1} \cdots x_j$ is a shortest path between x_i and x_j.

The proof of Theorem 4.40 is immediate.

Theorem 4.41 (7). A path $x_1 x_2 \cdots x_n$ in a graph (S,R) is a shortest path from x_1 to x_n if and only if for sets $S_i = \{x \mid x \in S$ and there exist a shortest path $x_i x_{i+1} a_1 a_2 \cdots x\}$, $\bigcap_{i=1}^{n-1} S_i \neq \emptyset$.

If $x_1 x_2 \cdots x_n$ is a shortest path then each set S_i contains the vertex x_n. Hence $\bigcap_{i=1}^{n-1} S_i \neq \emptyset$. Conversely, if $\bigcap_{i=1}^{n-1} S_i \neq \emptyset$ then there exists an element $y \in \bigcap_{i=1}^{n-1} S_i$. For $i = 1$ there is a shortest path $x_1 x_2 a_{11} a_{12} \cdots y$ by definition of S_1. If $i = 2$ then there is a shortest path $x_2 x_3 a_{21} a_{22} \cdots y$. Hence $x_1 x_2 x_3 a_{21} a_{22} \cdots y$ is a shortest path. If $i = 3$ then there is a shortest path $x_3 x_4 a_{31} a_{32} \cdots y$ and therefore $x_1 x_2 x_3 x_4 a_{31} a_{32} \cdots y$ is a shortest path. By continuing this process $n - 1$ times then $x_1 x_2 x_3 \cdots x_n a_{(n-1)1} a_{(n-1)2} \cdots y$ is a shortest path. By Theorem 4.39, then $x_1 x_2 \cdots x_n$ is a shortest path.

Theorem 4.42. In a finite M-graph (S,R,M) every cycle is of even length if and only if every elementary cycle is of even length.

If every cycle is of even length then it follows that every elementary cycle is of even length. Conversely, suppose (S,R,M) contains a cycle $x_1 x_2 \cdots x_n$, $x_1 = x_n$, of odd length. If cycle $x_1 x_2 \cdots x_n$ is an elementary cycle then the theorem follows. If not, then some $x_i = x_j$ for $i < j$. If $j < n$ then the cycle $x_1 x_2 \cdots x_n$ can be divided into two cycles $x_i \cdots x_j$ and $x_1 \cdots x_i x_{j+1} \cdots x_n$. If $j = n$ then the cycle $x_1 x_2 \cdots x_n$ can also be divided into two cycles $x_i \cdots x_j$ and $x_1 \cdots x_i$. Therefore, any cycle that is not elementary can be divided into two cycles whose lengths total that of the given cycle. Since cycle $x_1x_2\cdots x_n$ has odd length then one of the two cycles must also be of odd length. If the one cycle that is odd is not elementary, then it can be divided into two cycles one of which must be odd. Since (S,R,M) is finite then this procedure would eventually arrive at an elementary cycle that is of odd length. This proves the theorem.

Theorem 4.43. In a finite M-graph (S,R,M) every cycle is of even length if and only if S can be partitioned into two sets A and B such that the M-subgraphs (A,X,M) and B,Y,M) are null.

Let (S,R,M) be a finite M-graph such that every cycle has an even number of edges. By Theorem 4.10 one can partition (S,R,M) into disjoint connected M-subgraphs. Furthermore, since (S,R,M) is finite then there are only a finite number of these M-subgraphs (C_{v_1},W_1,M), $(C_{v_2},W_2,M),\cdots(C_{v_n},W_n,M)$. Let A be the set of all vertices of S that are connected to a vertex of $\{v_1,v_2,\cdots,v_n\}$ by a chain of even length and B the set of all vertices of S that are connected to a vertex of $\{v_1,v_2,\cdots,v_n\}$ by a chain of odd length. Since every vertex of S is contained in one of the connected M-subgraphs, then $A \cup B = S$. Also, $A \cap B = \emptyset$ for if $a \in A$ then there is an even chain from a to some $v_i \in \{v_1,v_2\cdots,v_n\}$ and any other chain from a to v_i cannot be odd or otherwise together they form an odd cycle. Hence sets A and B partition the set S.

It will now be shown that the M-subgraphs (A,X,M) and (B,Y,M) are null. Assume (A,X,M) is not null. Hence there exists $a_1 \in A$, $a_2 \in A$ such that $[a_1,a_2]_j \in (S,R,M)$. Also since the connected M-subgraphs

$(C_{v_1},W_1,M)\cdots(C_{v_n},W_n,M)$ is a partition of (S,R,M) then a_1 and a_2 are contained in one and only one M-subgraph (C_{v_i},W_i,M). But $a_1,a_2 \in A$ means there are even chains from a_1 to v_i and a_2 to v_i. These chains together with $[a_1,a_2]_j$ is a cycle of odd length which is a contradiction. Hence M-subgraph (A,X,M) is null. A similar contradiction is reached by assuming (B,Y,M) is not null.

Conversely, suppose S is partitioned into two sets A and B such that M-subgraphs (A,X,M) and (B,Y,M) are null. Let $x_1x_2\cdots x_nx_1$ be any cycle of (S,R,M). The cycle must alternate back and forth from the sets A and B since M-subgraphs (A,X,M) and (B,Y,M) are null. Hence the cycle is even.

Corollary 4.44. In a finite connected M-graph (S,R,M) with every cycle of even length, S can be partitioned into unique sets A and B such that the M-subgraphs (A,X,M) and (B,Y,M) are null.

Suppose the government is interested in placing a group of scientist at two different testing grounds. For security reasons they prefer to divide the group such that no two at the same testing ground could each understand what the other is doing. This situation is a graph if one considers a scientist a related to a scientist b if and only if a can understand what b is doing. Theorem 4.43 indicates that it can be done if and only if there are no odd cycles. Also, if such a division is possible, then the proof of Theorem 4.43 shows a way of finding such a division. For example, suppose Figure 16 A represents a graph with the lines indicating who can understand whom. Notice that every cycle is even. Figure 16 B shows the same graph, but with the two sets of scientists represented by vertices at the top and bottom of the figure the desired groups.

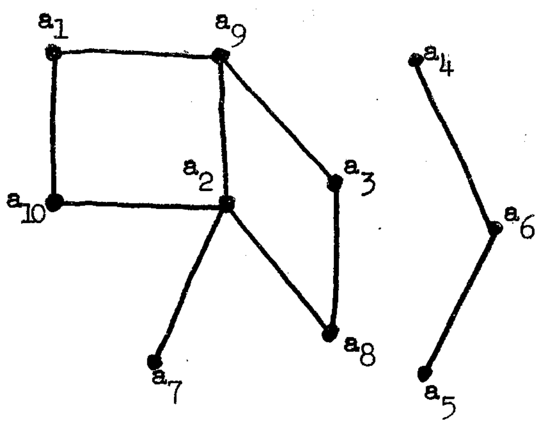


Figure 16 A.

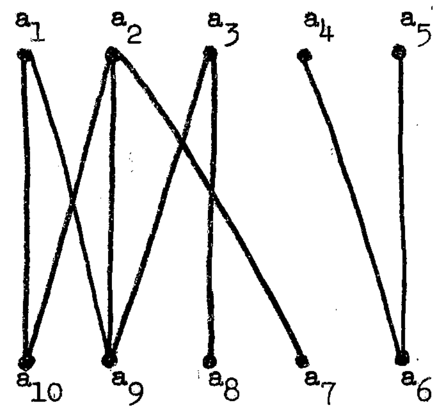


Figure 16 B.

Theorem 4.45. If every cycle of a finite M-graph (S,R,M) is of even length and the order of each vertex is at most k, with $k \leq e$, then there exists a partition of the edges into k cells such that no two edges in the same cell have a common vertex.

The theorem will be proved by induction on the number of edges. If in (S,R,M) $e = k$, then the theorem follows by merely considering every edge a cell. Let (S,R,M) be a M-graph satisfying the hypotheses with $e > k$. Assume the theorem holds for any M-graph satisfying the hypotheses with the number of edges e' such that $e' < e$. Consider the partial M-graph $(S,R,M) - [a,b]_j$ of (S,R,M) obtained by deleting an edge $[a,b]_j$. In the partial M-graph every cycle is still of even length and every vertex has order at most k. Since the number of edges $e' = e - 1 < e$, then by assumption there is a partition $P = \{A_1, A_2 \cdots A_k\}$ of the edges in the desired manner. The order of vertex a, and of b, in the partial

M-graph is at most $k - 1$ and there are k cells in P, so there exist a cell such that a does not belong to any of its edges and a cell such that b does not belong to any of its edges. If there is one cell meeting both conditions, then the edge $[a,b]_j$ can be included in that cell to have an admissible partition of (S,R,M). If there is not one cell meeting both conditions, then there are two cells, A_i and A_j, such that a is not in an edge of A_j, but is in A_i, and b is not in an edge of A_i, but is in A_j. Let $a_o = a$ and consider a chain $a_o a_1 a_2 \cdots a_n$ starting at vertex a with distinct edges alternately in A_i and A_j as far as possible. In such a chain, edges $[a_{2k},a_{2k+1}] \in A_i$ and $[a_{2k+1},a_{2k+2}] \in A_j$. If the chain stops with an edge in A_j then the edge cannot include a. If it stops with an edge in A_i then the edge cannot include vertex a since two edges of A_i cannot have a common vertex. Hence the chain cannot return to a. Nor can the chain return to any other a_i in the chain since there would be two edges either in A_i or A_j with a common vertex, namely a_i. Furthermore, the chain cannot end at b, since the chain would be of even length making the chain together with edge $[a,b]_j$ in (S,R,M) a cycle of odd length. Now consider a new partition of the edges of the partial M-graph by shifting the edges $[a_{2k},a_{2k+1}] \in A_i$ in partition P to A_j and edges $[a_{2k+1},a_{2k+2}] \in A_j$ to A_i. It can be shown with the above noted properties of the chain that such a repartitioning of the edges is in the desired manner. Therefore, adding edge $[a,b]_j$ to the new A_i gives a desired partitioning of the edges in (S,R,M).

Theorem 4.46.(48) If $v = pq$ vertices of an M-graph are partitioned in two ways with p cells of q vertices each, then there exists a set of p vertices that includes a vertex of each cell, in both partitions.

Let $P_1 = \{A_1, A_2 \cdots, A_p\}$ and $P_2 = \{B_1, B_2 \cdots, B_p\}$ be the two partitions. Consider a new M-graph (S,R,M) with $S = \{A_1, A_2 \cdots, A_p, B_1, B_2 \cdots, B_p\}$ and the number of edges between $X \in S$ and $Y \in S$, $X \neq Y$, defined by the cardinality of $X \cap Y$. Notice that there are no edges between elements of P_1 since $A_i \cap A_j = \emptyset$ and likewise no edges between elements of P_2. Also, since there are q vertices in each cell, then there are pq edges in (S,R,M). It follows that S is partitioned into two sets P_1 and P_2 such that M-subgraphs (P_1, R_1, M) and (P_2, R_2, M) are both null. Thus by Theorem 4.42, in (S,R,M) every cycle is of even length. Now the order of each vertex in (S,R,M) is q since each cell (vertex) of S had q vertices. Therefore by Theorem 4.45 the edges of (S,R,M) can be partitioned into q cells such that no two edges in the same cell have a common vertex. Thus in each of the q cells every vertex of S is included in some edge in the cell and hence consists of exactly p edges. Now each one of the p edges in any one such cell of edges must connect an $A_i \in P_1$ with a $B_j \in P_2$. Furthermore, each of the p edges must connect different cells in P_1 and P_2 since no two edges in the same cell have a common vertex. By definition, an edge $[A_i, B_j]_\ell$ exists only if $A_i \cap B_j \neq \emptyset$. Thus, if for each of the p edges one considers a vertex a_i from the intersection of the cells making up the edge, then the p such vertices satisfy the condition needed.

Corollary 4.47. If v = pq vertices of a complete M-graph are partitioned in two ways with p cells of q vertices each, then there exists a chain of length p - 1 that includes exactly one vertex of each cell, in both partitions. Likewise there exists an elementary cycle of length p that includes a vertex of each cell, in both partitions.

By Theorem 4.46 there are p vertices that include a vertex in each cell of both partitions. Since the graph is complete then there exists a chain of length p - 1 containing the p vertices and a elementary cycle containing the p vertices.

It should be pointed out that Theorem 4.46 deals in pure combinatorics even though it is stated in the language of graphs. For example, in Figure 17. there are 24 points with two partitions into 6 cells of 4 points each. The cells of one partition are indicated with closed

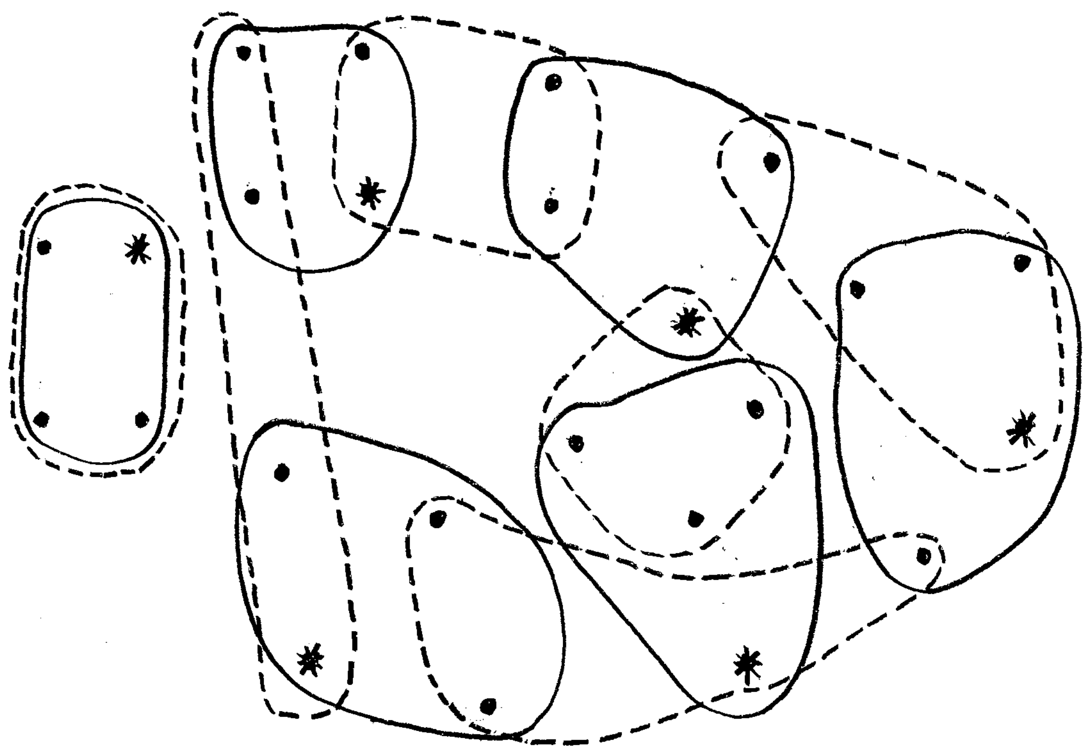


Figure 17.

solid lines and cells in the other partition with dotted lines. Theorem 4.46 states that there are 6 points such that each closed curve contains one of these points. Six such points are indicated with asterisks.

Diameter and Radius

Many problems of graph theory deal with a path of maximum length from the set of all shortest paths in a graph or M-graph. Problems of economy in communication are sometimes concerned with keeping the length of the shortest paths at a minimum or below certain bounds.

Definition 4.14. In an M-graph (S,R,M) the max $\{d(x,y) | x,y \in S$ and $d(x,y) \neq \infty\}$ is called the <u>diameter</u> of (S,R,M), denoted by D.

In a finite M-graph the diameter is the length of a longest shortest path in the graph. For example, the graph represented in Figure 18. has a diameter D = 6, since the longest path fecdbag has length 6.

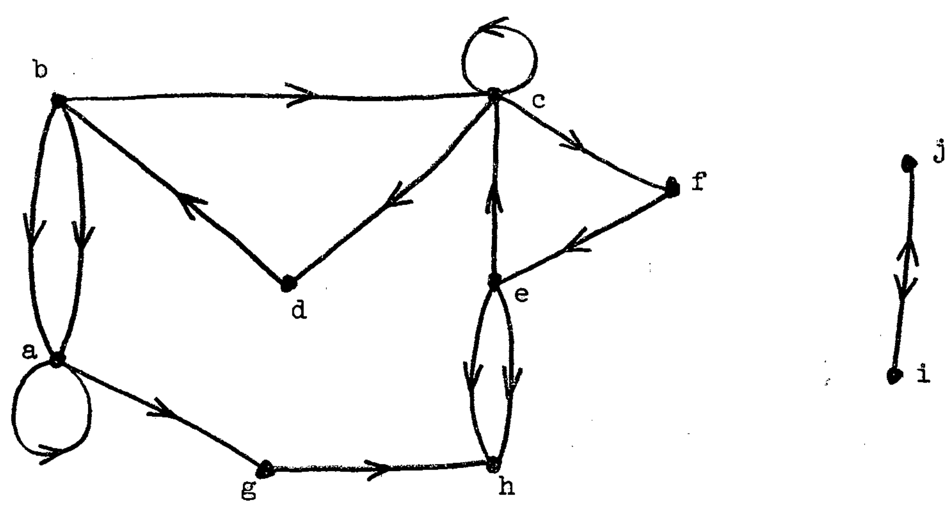


Figure 18.

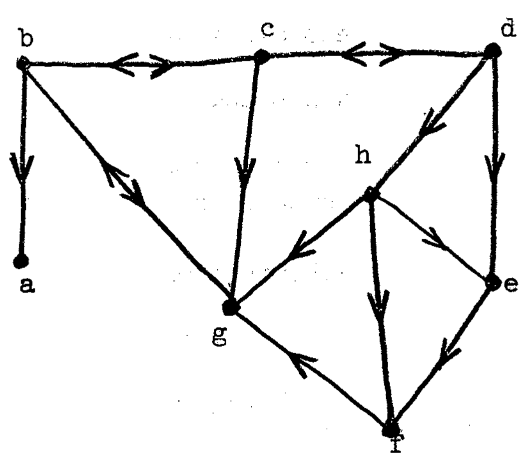
Another problem in graph theory is to find a vertex from which all other vertices can most easily be reached. The criteria for "easily" maybe either the maximum of the distances or some average. The former criterion will be used. Such a vertex would represent what one might mean by the "center" of a graph.

With each vertex x of a M-graph (S,R,M) one can associate

$$g(x) = \left\{\begin{matrix} \text{Max } \{d(x,y) \mid y \in S\} \text{ if } d(x,y) \neq \infty \text{ for all } y \in S \\ \infty \qquad \text{otherwise} \end{matrix}\right\}$$

Definition 4.15. The <u>center</u> of a M-graph (S,R,M) is any vertex c of (S,R,M) such that $g(c) \neq \infty$ and $g(c) = \min \{g(x) \mid x \in S \text{ and } g(x) \neq \infty\}$. The number g(c) is called the <u>radius</u> of (S,R,M), denoted by R.

One should note that an M-graph need not have a center as an null graph or could have several centers. For example, the graph represented in Figure 19. has two centers, namely, vertex c and vertex d. The radius is 3 and one can reach any vertex in the graph from either c or d on a path of length 3 or less.



$g(a) = \infty$

$g(b) = 4$

$g(c) = 3$

$g(d) = 3$

$g(e) = 6$

$g(f) = 5$

$g(g) = 4$

$g(h) = 4$

$\min \{g(x) \mid x \in S, g(x) \neq \infty\} = 3$

Figure 19.

The following four theorems follow directly. The first gives the obvious relation between the radius and diameter.

Theorem 4.48. If a finite M-graph has a center c, then $g(c) = R \leq D$.

Theorem 4.49. In any finite M-graph $D \leq v - 1$.

Theorem 4.50. A complete strongly connected graph is symmetric if and only if $D = 1$.

Theorem 4.51. A graph is complete and symmetric if and only if it is strongly connected and $D = 1$.

The next rather natural question to ask is: when does a M-graph have a center? One answer follows:

Theorem 4.52. A finite M-graph (S,R,M) has a center c if and only if for every pair of vertices x and y there is a vertex z from which paths lead to x and y. (If z = x or y then only the path to the other vertex is needed).

If c is the center, then there must exist paths from c to any pair of vertices x and y or otherwise $g(c) = \infty$ and c would not be a center. Conversely, if $S = \{x_1, x_2, \cdots x_n\}$ then there exists a vertex z_2 with paths leading to x_1 and x_2. There is a vertex z_3 with paths leading to z_2 and x_3 and hence also paths from z_3 to x_1 and x_2. The process may be continued until vertex z_n provides paths leading to every vertex of (S,R,M). Hence $\min \{g(x) \mid x \in S\}$ is finite and for some vertex (center) c, $g(c) = \min \{g(x) \mid x \in S\}$.

Theorem 4.53. Every finite complete M-graph has a center.

Theorem 4.53 follows directly since given any two vertices x and y there is a path (arc) from x to y or y to x. Hence by Theorem 4.52 it has a center.

Theorem 4.54 (7). In a finite complete M-graph, $R \leq 2$.

By Theorem 4.53 the M-graph has a center c. Assume $R > 2$, that is, $g(c) > 2$. Since $g(c) = \min \{g(x) \mid x \in S\}$, then $g(x) > 2$ for all $x \in S$. Now since the graph is finite, then there exists a vertex x_o such that $|x_oR - \{x_o\}|$ is a maximum. But since $g(x_o) > 2$, then there exists a vertex y such that $d(x_o,y) > 2$ by definition of $g(x_o)$. Furthermore, any vertex $z \in x_oR - \{x_o\}$ cannot be y and cannot have $(z,y) \in R$. So for completeness, $z \in yR - \{y\}$. Hence $x_oR - \{x_o\} \subset yR - \{y\}$. Moreover, $x_o \in yR - \{y\}$ and $x_o \notin x_oR - \{x_o\}$. Therefore $x_oR - \{x_o\}$ is a proper subset of $yR - \{y\}$ and $|x_oR - \{x_o\}| < |yR - \{y\}|$, contradicting that $|x_oR - \{x_o\}|$ is a maximum and that $R > 2$.

Theorem 4.54 indicates that in any finite complete M-graph there is at least one vertex (center) from which one can reach all other vertices by paths of length two or less. This result is quite significant and somewhat non-intuitive. Figure 20. shows a complete graph with centers a, c and d and $R = 2$.

Theorem 4.55. In a finite complete reflexive graph (S,R), if $|x_oR| = \max \{|xR| \mid x \in S\} = p$, then x_o is a center.

By Theorem 4.54 either $R = 1$ or $R = 2$. If $R = 1$, then $p = v$. Since $|x_oR| = p = v$, then $d(x_o,x) = 1$ for all $x \in S$, $x \neq x_o$, and therefore

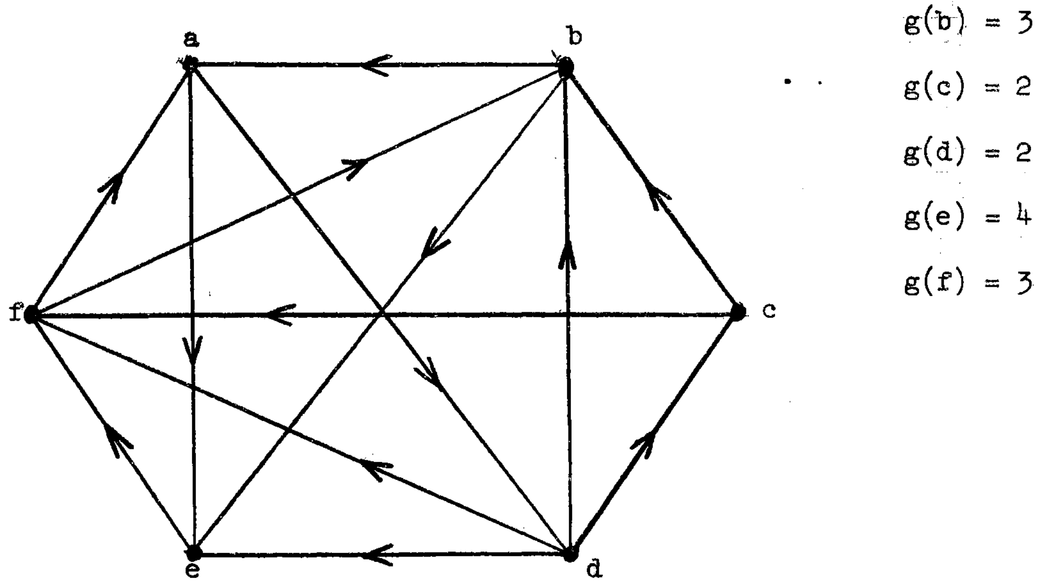


Figure 20.

$g(x_o) = \min\{g(x) \mid x \in S\} = 1$. Hence x_o is a center. If $R = 2$, then consider any $x \in Rx_o$. There must exist some $y \in x_oR$ such that $(y,x) \in R$ or otherwise $|x_oR| < |xR|$ contradicting that $|x_o R| = p$. Thus there is a path of length two or less from x_o to any $x \in S$. This implies that x_o is a center.

Corollary 4.56. In a finite complete irreflexive graph (S,R), if $|x_oR| = \max\{|xR| \mid x \in S\} = p$, then x_o is a center.

The proof of the corollary is almost identical to the argument used in Theorem 4.55, except in the case where $R = 1$, $p = v - 1$.

Theorem 4.57 (7). In a finite M-graph (S,R,M) with a center c and $p = \max\{|xR| \mid x \in S\} > 1$, the radius R,

$$R \geq \frac{\log (vp - v + 1)}{\log p} - 1.$$

The number of vertices of distance 1 from the center c is less than or equal to p since $p = \max \{ | xR \| \; x \in S \}$. The number of vertices of distance 2 from c is less than or equal to p^2; the number of vertices of distance 3 from c is less than or equal to p^3; and so forth. Now since c is the center, there are paths from c to every vertex of (S,R,M) except perhaps c. Also, there are no vertices of distance greater than R from c by definition 4.15 of R. Consequently,

$v \leq 1 + p + p^2 + p^3 + \cdots + p^R$ (The one is added to take care of center c)

$$v \leq p^{R+1} - 1/p - 1,$$

$$v(p - 1) \leq p^{R+1} - 1,$$

$$\log [v(p - 1) + 1] \leq \log p^{R+1},$$

$$\log (vp - v + 1) \leq (R + 1) \log p,$$

and

$$R \geq \frac{\log (vp - v + 1)}{\log p} - 1.$$

Theorem 4.50. If a center c of a finite symmetric (connected) graph (S,R) is a separating vertex, then all other centers are in the same component of the subgraph $(S - \{c\}, W)$.

Since (S,R) has a center then (S,R) has a radius $R = g(c)$. The subgraph $(S - \{c\}, W)$ has two or more components $C_1, C_2, \cdots, C_n$ since c is a separating vertex. Since c is a center with radius R then there exists at least one vertex x such that $d(c,x) = R$. The vertex x must lie in one of the components, say C_i. Now any vertex $z \notin (C_i \cup \{c\})$

could not be a center since a shortest path from z to x must go through c and thus $d(z,x) > R$. Therefore any other center necessarily must be a vertex of component C_i.

Theorem 4.59. If a center c of a finite symmetric (connected) graph (S,R) is a simple separating vertex, then either c is the only center or only one other center exists adjacent to c.

It should be clear that if two vertices x_1 and x_2 are in different components of the subgraph $(S - \{c\},W)$ such that $d(c,x_1) = d(c,x_2) = R$, then c is the only center. So let C_i be the only component containing a vertex x such that $d(c,x) = R$. Suppose $\max \{d(c,x) \mid x \in C_i \cup \{c\}\} < R - 1$. Since c is a simple separating vertex, then there is only one vertex $x_i \in C_i$ such that $[c,x_i]$ is an edge. Hence, the supposition implies that $\max \{d(x_1,x) \mid x \in S\} < R$ which contradicts R being the radius. Therefore the supposition is incorrect and $\max \{d(c,x) \mid x \notin C_i \cup \{c\}\} \geq R - 1$. Now let c_1 be any other center of (S,R). By the previous argument there exists a vertex $y \notin C_i \cup \{c\}$ such that $d(c,y) \geq R - 1$. By Theorem 4.58, the center $c_1 \in C_i$. Hence for $d(c_1,y) = R$, then c_1 must be adjacent to c. But there is only one vertex of component C_i adjacent to c by definition 3.3 of a simple separating vertex and the theorem follows.

Chromatic Function

The next integral valued function defined on the set of all finite graphs or M graphs to be discussed is called the chromatic function. There exists at least one partition of S in an M-graph (S,R,M) such that no two vertices in the same cell have a common edge (adjacent), since one can always consider the partition with each vertex a cell. Consider the set of all partitions of X such that no two vertices in the same cell are adjacent. Since (S,R,M) is finite, then there is only a finite number of partitions in . Hence there is at least one partition P ϵ that has a minimum number p of cells.

Definition 4.16. The function that assigns to each finite M-graph the number p is called the <u>chromatic function</u>. The number p is called the <u>chromatic number</u> of (S,R,M) and (S,R,M) is said to be <u>p-chromatic</u>.

The reason that such a number p is called the "chromatic" number is due to its significance in the problem of coloring the regions of a map with a minimum number of colors such that no two adjacent regions have the same color. For example, consider the map of the United States in Figure 21. Consider each state as a vertex with (state A) R (state B) if and only if state A has a boundary in common with state B. Such a graph is obviously symmetric. Now the chromatic number of such a graph is the least number of colors needed to color the map. It is in connection to such coloring problems that the chromatic junction is of interest. A more complete discussion of map-coloring will be given in the last chapter on Planar Graphs.

Theorem 4.60. An M-graph with $v > 1$ is 2-chromatic if and only if every cycle is of even length.

Theorem 4.60 follows directly from Theorem 4.43.

Theorem 4.61. Of the three following properties of a finite M-graph

(S,R,M) any two imply the third:

(1) Completeness

(2) v vertices

(3) v-chromatic.

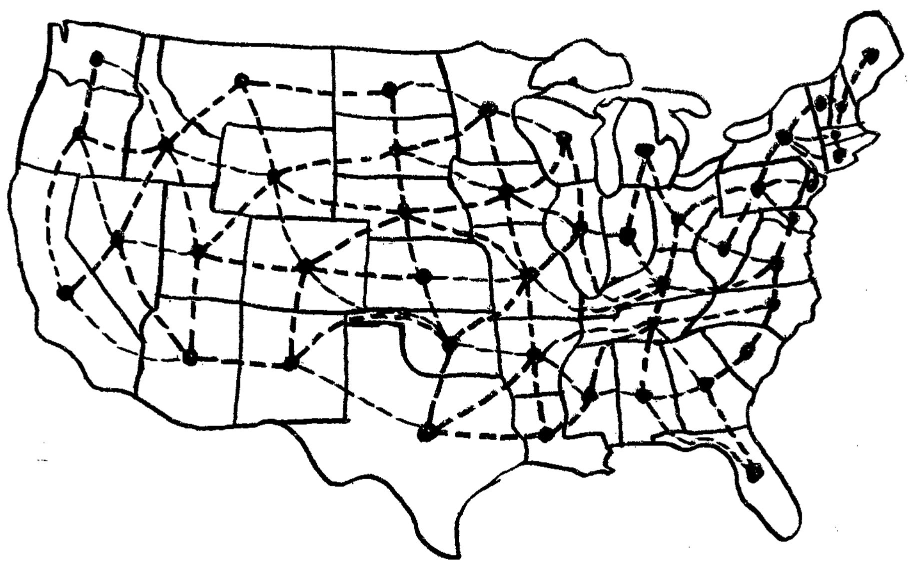


Figure 21.

There is one and only one partition of S in a complete graph such that no two vertices in the same cell have a common edge, namely, the partition with each vertex a cell.

Theorem 4.61. In an M-graph (S,R,M) with $q = \max \{\text{order of } x \mid x \in S\}$

if (1) When $q = 2$, every cycle is on even length,

(2) No M-subgraph of (S,R,M) with $q + 1$ vertices is complete,

then (S,R,M) is p-chromatic where $p \leq q$.

If an M-graph (S,R,M) has $q = 1$ and satisfies condition (2) then (S,R,M) must be a null graph. Hence (S,R,M) is 1-chromatic. If (S,R,M) has $q = 2$ and satisfies condition (1), then by Theorem 4.59 (S,R,M) is 2-chromatic. Let (S,R,M) be a graph with $q > 2$ satisfying conditions (2) and assume $p > q$. By definition of q there exists a vertex x_{q+1} connected by an edge to each of q vertices $x_1, x_2 \cdots x_q$. Since $p > q$ then $p = q + 1$. But by Theorem 4.60 the M-subgraph is complete, contradicting condition (2).

Definition 4.17. In an M-graph (S,R,M) a subset $T \subset S$ such that no two vertices are contained in a common edge is called internally stable.

Definition 4.18. The internal stability number, denoted by s, of an M-graph is the maximum number of vertices in an internally stable set.

Theorem 4.62. If a finite M-graph with v vertices is p-chromatic and s is the internal stability number, then $ps \geq v$.

It is possible to partition S into p internally stable cells. Since s is the maximum cardinality of all the internally stable sets, then $ps \geq v$.

CHAPTER V

TRAVERSING A GRAPH

There are basically two different meanings that one might give to "traversing a graph." For example, consider a graph illustrated by a road map of Germany. One might mean by "traversing" Germany the traveling by road through every town of Germany and not including any town twice, except maybe ending at the initial town. However, one might mean traveling over every road in Germany just once. Both meanings present interesting problems worthy of attention. A terminology will be developed to discuss both situations.

Definition 5.1. An elementary path (chain) in a graph (S,R) that includes every vertex of (S,R) is called a <u>Hamiltonian path</u> (<u>chain</u>). If a Hamiltonian path (chain) is a circuit (cycle) then it is called a <u>Hamiltonian circuit</u> (<u>cycle</u>).

Definition 5.2. A simple path (chain) in a graph (S,R) that includes every arc (edge) is called an <u>Euler path</u> (<u>chain</u>). If the Euler path (chain) is a circuit (cycle) then it is called an <u>Euler circuit</u> (<u>cycle</u>).

Hamiltonian Chains

Hamiltonian paths (chains) and circuits (cycles) are named after the 19th century Irish mathematician Sir William Rowan Hamilton. He was the first known to have studied such problems. He proved that a Hamiltonian cycle could be traced along the edges of each of the Platonic solids. In fact, a puzzle he devised based on finding Hamiltonian excursions along the edges of a dodecahedron was manufactured.

Theorem 5.1. A complete graph with $v > 1$ contains a Hamiltonian path.

The proof of Theorem 5.1 is by mathematical induction. In a complete graph (S,R) consisting of exactly two vertices a and b it is obvious a Hamiltonian path exists since to be complete either $(a,b) \in R$ or $(b,a) \in R$. Assume the theorem holds for a complete graph with v vertices. Let (S,R) be a complete graph with $v + 1$ vertices. Consider any subgraph $(S - \{x\},W)$ with v vertices. By Theorem 3.5, $(S - \{x\},W)$ is a complete graph. Therefore, by assumption, there exists an elementary path $a_1a_2\cdots a_{v-1}a_v$ containing v of the $v + 1$ vertices with x being the only vertex of (S,R) not included. Now either (1) $(a_i,x) \notin R$ for all $i = 1,2,\cdots v$ or (2) $(a_i,x) \in R$ for some $i = 1,2\cdots v$. If (1) occurs then necessarily $(x,a_1) \in R$ since (S,R) is complete and $xa_1a_2\cdots a_v$ is a Hamiltonian path in (S,R). If (2) occurs then let j be the largest index such that $(a_j,x) \in R$. Thus $(x,a_i) \in R$ for all $i = j + 1, j + 2, \cdots v$ unless $j = v$, and $a_1a_2\cdots a_jxa_{j+1}a_{j+2}\cdots a_v$ is a Hamiltonian path. When $j = v$ this path terminates at x.

An interesting but somewhat misleading application of Theorem 5.1 is in connection to the league play of football teams. Let (S,R) be a graph where S is a set of football beams and aRb if and only if team a wins over team b. If there are no tie ball games during the season then the graph (S,R) is a complete graph since in league play every team plays every other team. Theorem 5.1 implies that at the end of the season one can always list the teams in an order such that any team has won over the succeeding team on the list. The reason that such a list is misleading is that the team at the top need not be the league's winning football team, according to the usual method of determining the winning team.

Theorem 5.2. A finite irreflexive graph with $2e = v(v - 1)$, $v > 1$, contains a Hamiltonian path.

Theorem 5.2 follows directly from Theorem 4.25 and Theorem 5.1.

Theorem 5.3. A complete graph with $v > 2$ vertices contains $(v - 1)!/2$ Hamiltonian cycles.

Consider $(v - 1)$ things out of v things. There are $(v - 1)!$ different permutations of the $v - 1$ things. Hence there are $(v - 1)!$ different arrangements of v things on a circle. However, half of the $(v - 1)!$ arrangements are just reversed in direction to others. Thus, in a complete graph with $v > 2$ vertices there are $(v - 1)!/2$ different Hamiltonian cycles.

Theorem 5.4. An irreflexive finite graph with the order of each vertex $\geq (v - 1)/2$, $v > 1$, contains a Hamiltonian chain.

First consider a graph (S,R) with an odd number of vertices, say $2k + 1$, with the order of each vertex $\geq k$. Since the graph is finite then there exists an elementary chain $u = x_1x_2\cdots x_n$ containing a maximum number of vertices. It will be shown that u contains all the vertices of (S,R) and hence is a Hamiltonian chain. Assume that u does not include all the vertices of (S,R). Then there exists a vertex a_1 not in u and u has at most 2k vertices. Now a_1 is not adjacent to either x_1 or x_n for if so then u would not be of maximum length. Also, a_1 cannot be adjacent to consecutive vertices x_i and x_{i+1} of u for then $x_1 \cdots x_1 a_1 x_{i+1} \cdots x_n$ would be an elementary chain longer than u. Since u contains at most 2k vertices then a_1 is adjacent to at most $(2k - 2)/2 = k - 1$ vertices of u. But since the order of a_1 is $\geq k$ and $k - 1 < k$ then a_1 must be adjacent to at least one other vertex a_2 not in u. Now u has at most $2k - 1$ vertices. Since a_1 and a_2 form an elementary chain then a_2 cannot be adjacent to x_1, x_2, x_3 or x_4 of u for if so then a longer elementary chain would exist. Again, a_2 cannot be adjacent to consecutive vertices of u. Thus a_2 is adjacent to at most $(2k - 4)/2 = k - 2$ of u. But the order of a_2 is $\geq k$, so a_2 is adjacent to a vertex $a_3 \neq a_1$ not in u. Hence $a_1 a_2 a_3$ is an elementary chain. Now u has at most $2k - 2$ vertices. Vertex a_3 cannot be adjacent to either x_1, x_2, x_3, x_{n-2}, x_{n-1} or x_n or consecutive vertices in u; hence, a_3 is adjacent to at most $(2k - 6)/2$ vertices in u. But $(2k - 6)/2 = k - 3 < k$ and a_3 is adjacent to another vertex $a_4 \neq a_1$, $a_4 \neq a_2$ not in u. Hence $a_1 a_2 a_3 a_4$ is an elementary chain not in u. This process can be containued until an elementary chain $a_1 a_2 \cdots a_{k+1}$ is formed, with u having at most k vertices contradicting that u is of maximum length.

Now consider a graph (S,R) with an even number of vertices, say 2k. The order of each vertex is $(2k - 1)/2 = k - \frac{1}{2}$ and hence the order is $\geq k$. This permits one to use a similar argument as for the case of an odd number of vertices, to prove the theorem.

Corollary 5.5. An irreflexive symmetric graph (S,R) with $xR \geq (v - 1)/2$ for all $x \in S$ contains a Hamiltonian path.

The symmetry changes the chain of Theorem 5.4 to a Hamiltonian path.

Theorem 5.6 A graph with $v = 2n$ vertices is complete if and only if there is a set of n edge-wise disjoint Hamiltonian chains with distinct endpoints.

Consider a complete graph with $v = 2n$. The easiest way to show that there are n edge-wise disjoint Hamiltonian chains with distinct endpoints is to consider the vertices equally spaced in a circular pattern (Figure 21.) and a particular Hamiltonian chain that when rotated about the circular pattern reveals the other n disjoint Hamiltonian chains. For example, consider the Hamiltonian chain in the graph of 10 vertices in Figure 22. If the Hamiltonian chain is

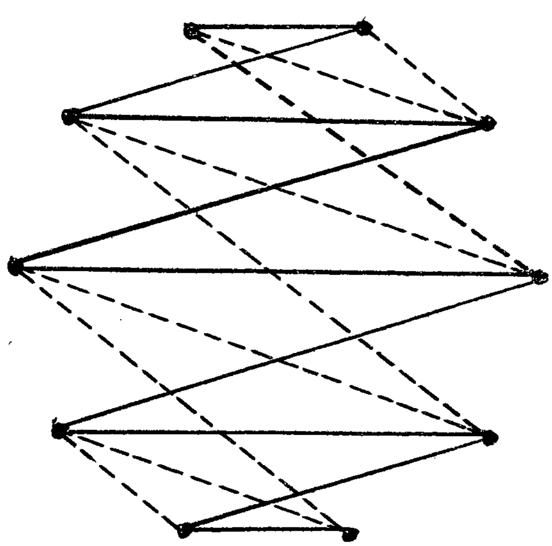


Figure 22.

rotated to the position indicated by the dotted lines, a new Hamiltonian chain is formed disjoint from the first and with distinct endpoints. It should be clear that one could continue to rotate the Hamiltonian chain three more times to reveal the five disjoint Hamiltonian chains with distinct endpoints. If there were 2n vertices then one could rotate such a Hamiltonian chain n - 1 times to reveal the n disjoint Hamiltonian chains.

Conversely, suppose there are n disjoint Hamiltonian chains with distinct endpoints in a graph with $v = 2n$. Each Hamiltonian chain contains $2n - 1$ distinct edges. Since there are n such Hamiltonian chains then there are at least $n(2n - 1) = v(v - 1)/2$ edges that are not loops. By Theorem 4.25 the graph is complete.

Theorem 5.7 A graph with $v = 2n - 1$ vertices is complete if and only if there are n edge-wise disjoint Hamiltonian cycles.

Let (S,R) be a complete graph with $v = 2n + 1$. If x is any vertex of (S,R), then the subgraph $(S - \{x\},W)$ by Theorem 3.5 is complete. Hence by Theorem 5.6 the subgraph $(S - \{x\},W)$ is covered by n disjoint Hamiltonian chains with distinct endpoints. But x is adjacent to each of the endpoints since (S,R) is complete. Therefore from each Hamiltonian chain $x_1x_2\cdots x_n$ in $(S - \{x\},W)$ one can form a Hamiltonian cycle $xx_1x_2\cdots x_nx$ in (S,R). It should be clear that all n Hamiltonian chains in $(S - \{x\},W)$ are edge-wise disjoint.

Conversely, each Hamiltonian cycle contains $2n + 1$ edges. Since there are n disjoint Hamiltonian cycles, then there are at least $n(2n + 1) = v(v - 1)/2$ edges. By Theorem 4.25 (S,R) is complete.

Theorem 5.8. A complete strongly connected graph with $v > 1$ contains a Hamiltonian circuit.

Let (S,R) be a complete strongly connected graph. Consider any two vertices a and b. Since (S,R) is complete then $(a,b) \in R$ or $(b,a) \in R$ or both. Suppose without loss of generality that $(a,b) \in R$. Since (S,R) is strongly connected there exists a path $bb_1 \cdots b_n a$ from b to a. Since $(a,b) \in R$ then $bb_1 \cdots b_n ab$ is a circuit. By Corollary 3.3 every circuit contains an elementary circuit. It is therefore known that every complete strongly connected graph contains an elementary circuit. Denote this elementary circuit in (S,R) as $a_1 a_2 \cdots a_m a_1$. If $S = \{a_1, a_2 \cdots a_m\}$ then, of course, the theorem follows. If not, than let x be any other vertex in (S,R) distinct from those in the circuit. It will be shown that there is an elementary circuit through all of $\{x, a_1, a_2 \cdots a_m\}$. This will enable one to use an inductive argument to prove the theorem.

Now either (1) $(x,a_i) \in R$ for all $i = 1,2 \cdots m$; (2) $(a_i,x) \in R$ for all $i = 1,2 \cdots m$ or; (3) $(x,a_i) \in R$ for some but not all $i = 1,2 \cdots m$ and $(a_i,x) \in R$ for the rest. Consider case (1). Since (S,R) is strongly connected then there exist a path from a_1 to x. By Theorem 3.2 there exist an elementary path from a_1 to x. This elementary path may contain vertices that are contained in the elementary circuit. Nevertheless, there is an elementary path $a_j c_1 c_2 \cdots c_k x$ from some vertex a_j of the circuit to x that contains no other vertex of the circuit. If $j < m$, then $a_1 a_2 \cdots a_j c_1 c_2 \cdots c_k x a_1$ is the desired circuit. The argument for case (2) is similar to that for case (1) except one must find an elementary path from x to a vertex of the circuit that involves no other vertices of the circuit and proceed as before. If case (3) occurs then there must exist a vertex a_j of the circuit such that for $j < m$, $(a_j,x) \in R$ and $(x,a_{j+1}) \in R$ or for $j = m$, $(a_m,x) \in R$ and $(x,a_1) \in R$. Hence $a_1a_2\cdots a_jxa_{j+1}\cdots a_ma_1$ if $j < m$ or $a_1a_2\cdots a_jxa_1$ if $j = m$ is the desired elementary circuit.

It should be noted that the proof of Theorem 5.8 indicates a detailed scheme, but nevertheless workable way of finding a Hamiltonian circuit. In order to reveal the complexity of such a method consider the strongly connected complete graph illustrated in Figure 23. A Hamiltonian circuit is bdaefcb.

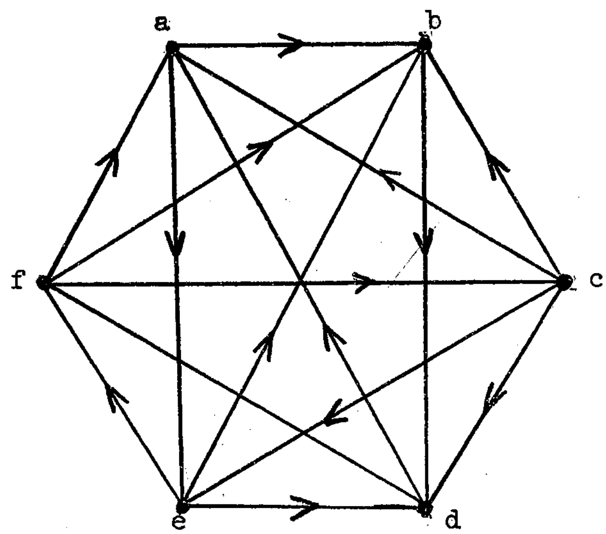


Figure 23.

Theorem 5.9. Every vertex of a graph containing a Hamiltonian cycle is a non-separating vertex.

The theorem is immediate since the deletion of any vertex leaves a chain of the Hamiltonian cycle connecting the remaining vertices.

It should be noted that only sufficient conditions for the existence of Hamiltonian paths (chains) and circuits (cycles) have been presented. Mathematicians have not yet found a necessary and sufficient condition for their existence.

Euler Chains

A classic illustration of a problem dealing with Euler chains is the famous Königsberg Bridge Problem. The problem was solved by the brilliant Swiss mathematician Leonhard Euler (1707-1783). In fact, Euler published the first paper on graphs in 1756 with the Königsberg Bridge Problem as the first topic. The problem dealt with seven bridges in the town of Königsberg, Prussia. The bridges were located as illustrated in Figure 24. The question arose whether one could reach each section by walking across each bridge once and only once. If one considers

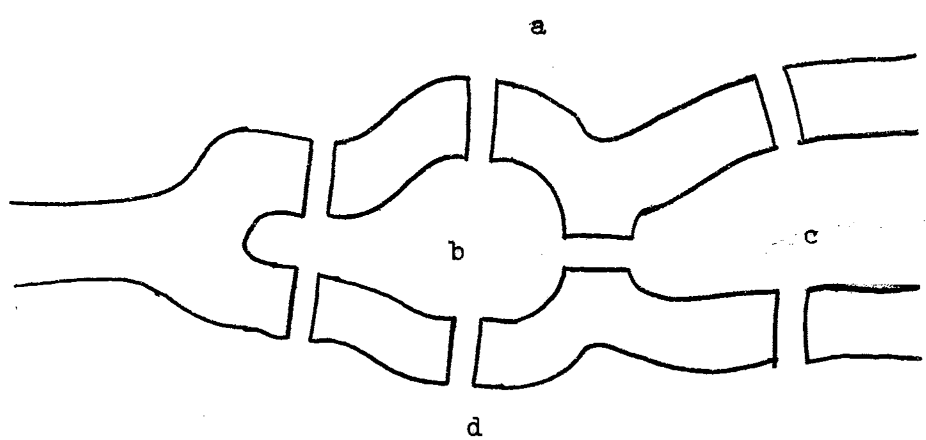


Figure 24.

the sections of town a, b, c, and d as vertices and the bridges as edges, then every one of the four sections (vertices) is odd. By the next theorem (Euler's theorem) it is impossible to make such a walk.

Theorem 5.10. (Euler's theorem) An irreflexive graph containing an Euler chain has every vertex even if the chain is a cycle or exactly two odd vertices otherwise.

Suppose that an irreflexive graph (S,R) contains an Euler chain $a_1a_2\cdots a_{n-1}a_n$. If a vertex x is not contained in an edge of the Euler chain then x is not in any edge of (S,R) since the Euler chain includes all the edges of (S,R). Therefore the vertex x is even. Now consider the vertices a_i, $i = 2,3\cdots n - 1$. Every time such a vertex a_i is included in the Euler chain it appears in two distinct edges. Hence vertices a_i are even. If the Euler chain is not a cycle then vertices a_1 and a_n are odd. If the Euler chain is a cycle, which makes it a Euler cycle, then a_1 and a_n are even by the same argument that a_i, $i = 2,$ $3\cdots n - 1$ are even.

It may be a little clearer for application purposes to consider the contrapositive of Theorem 5.10 which, of course, is equivalent.

Corollary 5.11. (Contrapositive of Theorem 5.10) An irreflexive graph which has one or more than two odd vertices does not contain an Euler chain.

Theorem 5.12. A finite irreflexive connected graph with each vertex of even order contains an Euler cycle.

Let (S,R) be any irreflexive connected graph with no odd vertices.

It will first be shown that (S,R) contains a simple cycle. Consider any vertex a_1 of (S,R). There must exist an edge $[a_1,a_2]$ containing a_1 since (S,R) is connected. Note that $a_1 \neq a_2$ since (S,R) is irreflexive. Since each vertex is even then there must exist another edge containing a_2, call it $[a_2,a_3]$, with $a_2 \neq a_3$. Likewise, there must exist another edge containing a_3. If this edge is distinct from the rest call it $[a_3,a_4]$. This process, when continued far as possible, gives a simple chain $a_1a_2\cdots a_{n-1}a_n$. It should be noted that such a simple chain need not be elementary. However, such a chain is a cycle, that is, $a_1 = a_n$. The chain is a cycle for if $a_n \notin \{a_1,a_2\cdots a_{n-1}\}$, then since a_n is even there would exist another distinct edge containing a_n and the process could have been continued. If $a_n = a_i, 1 < i < n$ then since a_i is even there would still exist another distinct edge containing a_i and the process could have been continued. Thus $a_n = a_1$.

If the constructed simple cycle includes all the edges of the graph (S,R) then the proof is complete. If not, then there exist other edges distinct from those in the cycle. However, since the graph is connected then there is an edge $[a_i,b]$ distinct from those of the cycle but having one vertex a_i in the cycle. One can continue as before and construct a new simple cycle $a_ib_1b_2\cdots b_ma_i$ of edges distinct from those of the first cycle. Join these two simple cycles to form a new simple cycle $a_1a_2\cdots a_ib_1b_2\cdots b_ma_ia_{i+1}\cdots a_na_1$. Since there are only a finite number of edges then such a process can be continued until it contains all edges. Such a cycle would be an Euler cycle.

It should be noted that the construction type proof of Theorem

5.12 indicates exactly how such an Euler cycle can be obtained. The proof also reveals that the Euler cycle need not be unique. Since the Euler cycle is not necessarily unique, it may be for certain problems that one Euler cycle could have advantages over another. Problems of this nature will be discussed later. Another fact worth noting is that since the graph is traversed by an Euler cycle then one could start at any vertex. The Euler cycle that was constructed for the proof started at an arbitrary vertex.

In order to illustrate Theorem 5.12 consider the following question. Is it possible to trace the drawing in Figure 25. without lifting the pencil and retracing any part? The answer is "yes" since every vertex is of even order.

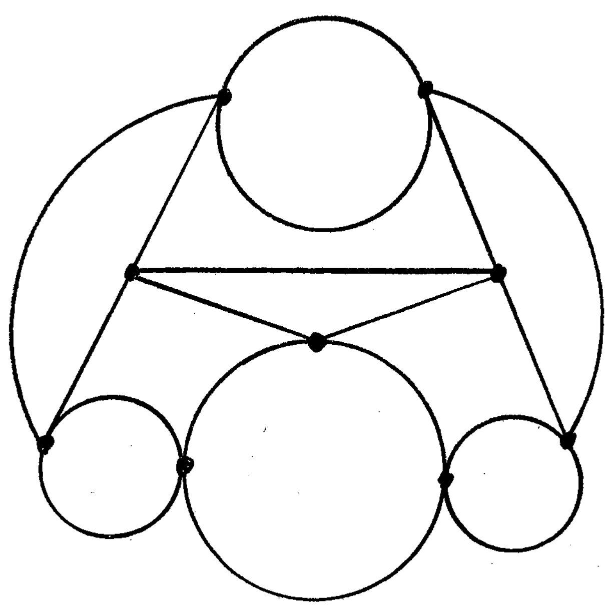


Figure 25.

Theorem 5.13. A finite irreflexive connected graph with exactly two odd vertices contains an Euler chain.

Let (S,R) be a finite irreflexive connected graph with exactly two odd vertices a and b. Now the graph (S,R) either contains an edge [a,b] or does not. If there is no edge [a,b] in (S,R), then by adding the edge each vertex becomes even. Thus by Theorem 5.12 the graph $(S,R \cup \{(a,b)\})$ contains a Euler cycle. This means the graph (S,R) contains a Euler chain. On the other hand, if there is an edge [a,b] in (S,R), then by deleting the edge the partial graph (S,R) - [a,b] has each vertex even. By Theorem 3.21 (S,R) - [a,b] has two components C_a and C_b or is connected. If connected then by Theorem 5.12 it contains a Euler cycle. Thus (S,R) contains a Euler chain. If (S,R) - [a,b] has two components C_a and C_b, necessarily with even vertices, then by Theorem 5.12 each component contains Euler cycles $aa_1a_2\cdots a_na$ and $bb_1b_2\cdots b_mb$. Combining these cycles and edge [a,b] gives the Euler chain $aa_1a_2\cdots a_mabb_1b_2\cdots b_mb$ in (S,R). This completes the proof.

It should be noted that the Euler chain begins and ends at the two odd vertices.

Notice that part of Theorem 5.10 and Theorem 5.13 are converses of each other. Furthermore, each of these theorems can be generalized and the generalizations conversely related. The following theorem tells the whole story. The following theorem also illustrates what so often happens in mathematics, namely, that a generalization stimulates the construction of a kind of proof that is simpler than the kinds of proofs used for specific cases.

Theorem 5.14. A finite irreflexive connected graph has 2k odd vertices if and only if there exist a minimum of k edgewise disjoint simple chains which together, contain all edges of the graph.

Let (S,R) be any finite irreflexive connected graph with odd vertices. Since (S,R) is irreflexive then by Theorem 3.17 there is an even number of odd vertices, say 2k. Since (S,R) is finite then there exists a minimum number of edgewise disjoint simple chains that together include all edges of (S,R). Now every vertex at either end of any one of the simple chains is odd or otherwise the original set of disjoint simple chains is not a minimum. Furthermore, every odd vertex of (S,R) is some end of one of the simple chains or otherwise it would be even. This is not to say that an odd vertex cannot appear in the middle of one of the chains. However, no two vertex ends of simple chains coincide or otherwise the original set of simple chains is not a minimum. Thus, there is a one-to-one correspondence between the odd vertices and the ends of the simple chains. This proves the theorem.

Theorem 5.15. If a graph (S,R) has an Euler circuit then for each $a \in S$, the sets Ra and aR are equivalent.

A graph (S,R) such that for each $a \in S$ the sets Ra and aR are equivalent means that there are the same number of arcs in (S,R) with a as the initial vertex as arcs with a as the terminal vertex. Each vertex $a \in S$ appears in the Euler circuit accounts for two unique arcs, one with a as a terminal vertex and one with a as an initial vertex. If a vertex $a \in S$ and is not contained in the Euler circuit, then the vertex is not contained in any arc. Thus, for each vertex $a \in S$, the sets aR and Ra are equivalent.

Theorem 5.16. If a finite graph (S,R) is connected and for each vertex $a \in S$, the sets aR and Ra are equivalent, then (S,R) contains an Euler circuit.

If (S,R) consists of only one vertex and no arcs then the theorem follows trivially. If (S,R) has more than one vertex then every vertex must be contained in some arc since it is connected. Consider any vertex $a_1 \in S$. Since there is an arc containing a_1 and $aR \cong Ra$ then there exists an arc (a_1,a_2). If $a_1 \neq a_2$ then there exists an arc (a_2,a_3) since $a_2R \cong Ra_2$. If $a_3 = a_1$ or $a_3 = a_2$ then (S,R) contains a simple circuit $a_1a_2a_3$ or a_2a_3. If $a_3 \neq a_1$ and $a_3 \neq a_2$ then there exists an arc (a_3,a_4) since $a_3R \cong Ra_3$. If $a_4 = a_i$ for some $i = 1,2,3$ then (S,R) contains a simple circuit $a_ia_{i+1}\cdots a_4$. If not then by continuing such a process a simple circuit will be obtained since (S,R) is finite. Hence (S,R) does contain a simple circuit, call it $a_1a_2\cdots a_na_1$. If such a circuit includes all the arcs of (S,R) then the theorem follows. If not, then since (S,R) is connected and Ra aR for all $a \in S$, it follows that there exists an arc (a_ib_1) distinct from those in the circuit but with a_i a vertex in the circuit. One can continue as before to obtain a simple circuit $a_ib_1b_2\cdots b_ma_i$ distinct from those of the first simple circuit. Hence $a_1a_2\cdots a_ib_1b_2\cdots b_ma_ia_{i+1}\cdots a_na_1$ is a simple circuit. One can continue this process until all arcs are included since (S,R) is finite. Thus (S,R) contains an Euler circuit.

CHAPTER VI

TREES

A graph with no cycles is of special importance in graph theory for various reasons. First, such a graph has many interesting unique properties as will be pointed out in this chapter. Secondly, graphs with no cycles have numerous applications. For instance, the connected graph illustrated in Figure 26 may represent an administrative organization of an institution with v_o the president, $v_i, i = 1,2,3$ the vice-presidents, etc. The figure could also represent a sorting process starting at v_o. An investigation of the fundamental properties and applications of finite connected graphs with no cycles is the purpose of this chapter.

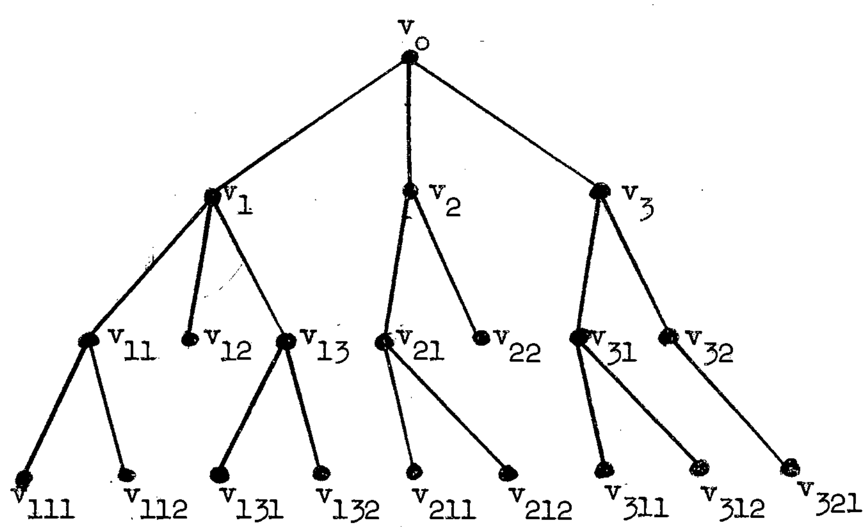


Figure 26.

Definition 6.1. A finite connected graph (S,R) with no cycles (or loops) is called a tree.

It is obvious why the word "tree" was selected, if we look at a geometric representation of a graph as in Figure 25. Trees were first introduced and studied by Cayley.

Theorem 6.1. A finite graph (S,R) is a tree if and only if it is connected and every edge is a separating edge.

Theorem 6.1 follows directly from the definitions.

Theorem 6.2. A graph (S,R) is a tree if and only if it is connected and $e = v - 1$.

By Theorem 4.36 the cyclomatic number θ of (S,R) is $\theta = e + c - v$. If (S,R) is a tree then it is connected and by Theorem 4.20, $c = 1$. Also, since being a tree it contains no cycles and by Theorem 4.34, $\theta = 0$. Therefore, $o = e + 1 - v$ and $e = v - 1$. Conversely, if (S,R) is connected and $e = v - 1$, then $\theta = e + 1 - v = v - 1 + 1 - v = 0$. Since $\theta = 0$ then by Theorem 4.34 (S,R) has no cycles. Therefore (S,R) is a tree.

Theorem 6.3. The sum of the orders of all the vertices in a tree is $2(v - 1)$.

Theorem 6.3 follows directly from Theorem 4.31 and Theorem 6.2.

Theorem 6.4. A tree contains at least two vertices of order one.

Assume that only one vertex or none has an order of one. If only one vertex has order one then $2(v - 1) + 1 \leq$ the sum of the orders of the vertices. But by Theorem 6.3 the sum of the orders is $2(v - 1)$ which gives the obvious contradiction $2(v - 1) + 1 \leq 2(v - 1)$. Likewise if no vertices are of order one then $2v \leq$ the sum of the orders which gives the contradiction $2v \leq 2(v - 1)$. Hence the theorem follows.

Theorem 6.5. A graph (S,R) is a tree if and only if every edge is a separating edge and $e = v - 1$.

If (S,R) is a tree then by definition 6.1 it is connected and has no loops. Therefore every edge is a separating edge and by Theorems 4.21, 4.34 and 4.36, $0 = e + 1 - v$ and $e = v - 1$. Conversely, since every edge is a separating edge then (S,R) contains no cycles. Furthermore, $0 = v - 1 + c - v$ and $c = 1$. Hence (S,R) is connected and therefore a tree.

Theorem 6.6. A finite graph (S,R) with $e > 0$ is a tree if and only if any two vertices are connected by one and only one chain.

Let (S,R) be a tree. Since (S,R) is connected then there exists at least one chain between any two vertices. If there were more than one chain between any two vertices then there would exist a cycle contradicting that (S,R) is a tree. Hence any two vertices are connected by one and only one chain. Conversely, (S,R) contains no cycles for if so then there exist two vertices that are connected by more than one chain. Since (S,R) is connected then it is a tree.

Theorem 6.7. A finite graph (S,R) has a partial graph that is a tree if and only if (S,R) is connected.

Let (S,W) be a partial graph of (S,R) that is a tree. Thus (S,W) is connected and therefore (S,R) is connected. Now suppose (S,R) is connected. If every edge is a separating edge then by Theorem 6.1 (S,R) is a tree and the theorem would follow. If some edge is a non-separating edge then deleting such an edge leaves a partial graph which is still connected. If the connected partial graph has a non-separating edge then delete it and so on until a connected partial graph is obtained that has no non-separating edges. By Theorem 6.1 such a partial graph is a tree.

Theorem 6.8. Every tree is 2-chromatic.

Theorem 6.8 follows directly from Theorem 4.60.

Theorem 6.9. Every connected subgraph or partial graph of a tree is a tree.

Since no subgraph or partial graph can contain a cycle then either must be a tree.

An interesting question arises in the study of graphs, namely, are there necessary and sufficient conditions for a connected graph to have the same number of vertices as edges? Problems of this nature are quite common. For instance, what kind of telephone system would permit each terminal to have the responsibility of maintaining a unique telephone line and thereby maintain all lines? When can the same set of names be used for streets and intersections such that every street has the same name as an intersection involving that street? Are there any deductions one can make about an organization when it is known that there are people and conversely.

The following theorem is useful in answering these questions.

Theorem 6.10. (87). A connected graph with $e > 0$ has $e = v$ if and only if it is an elementary cycle or consists of an elementary cycle and a set of disjoint trees such that each tree contains only one unique vertex of the cycle.

Let (S,R) be a connected graph with $e = v$. Since (S,R) is connected then $c = 1$. By Theorem 4.36 $\theta = e + c - v$ and therefore $\theta = e + 1 - e = 1$. Thus by Theorem 4.35, (S,R) contains exactly one elementary cycle. If this elementary cycle includes all of S then the theorem follows. If not, then let (S,W) be the partial graph of (S,R) obtained by deleting the edges of the cycle. By Theorem 3.11, (S,W) can be partitioned into disjoint connected subgraphs. Each such subgraph can not contain a cycle since (S,R) contains only one and therefore each subgraph is a tree. No two of these trees are connected in (S,W) since they are disjoint. However, since (S,R) is connected then each tree must contain a vertex in the cycle. Furthermore, each tree can contain only one vertex in the cycle or otherwise a new cycle would be formed in (S,R).

Now consider the converse. Let (S,R) be a connected graph with $e > 0$ that is an elementary cycle or consists of an elementary cycle and a set of disjoint trees such that each tree is connected to a unique vertex of the cycle. In each case (S,R) contains one and only one cycle and $\theta = 1$. Since (S,R) is connected then $c = 1$. Hence by Theorem 4.36, $1 = e + 1 - v$ and $v = e$.

Theorem 6.11. The cyclomatic number Θ of a graph with e edges that contains a partial subgraph tree with e' edges is $e - e'$.

By Theorem 6.2, $e' = v - 1$. By Theorem 4.35, $\Theta = e + 1 - v$. Thus, $\Theta = e + 1 - v = e - e'$.

An interesting question arises in connecting a set of points with the least number of lines such as connecting a set of towns with the least number of highways. It should be clear that such a graph will be a tree. However, given n vertices how many trees can be constructed connecting the n vertices? The following theorem is due to Cayley and gives the answer.

Theorem 6.12 (105). A complete graph (S,R) with v vertices contains v^{v-2} distinct partial subgraph trees.

Label the set of vertices with $1,2,\cdots v$. Any tree T in (S,R) can be uniquely indicated by a sequence of $v - 2$ numbers in the following manner: Let the first number a_1 in the sequence be the label of the vertex adjacent to the vertex b_1 labeled with the smallest number in the set of all numbers that correspond to vertices of T with order one. Delete vertex b_1 from T. Since b_1 had order one, then the resulting graph is also a tree, call it T'. Let the second number a_2 in the sequence be the index of the vertex adjacent to the vertex b_2 labeled with the smallest number in the set of all numbers that correspond to vertices of T' with order one. Delete vertex b_2 from T'. Continue this process until a sequence $u = a_1, a_2 \cdots a_{v-2}$ of $v - 2$ terms is obtained. There is no need to extend the sequence to $v - 1$ terms since the last term will always be v. Any tree covering S uniquely determines such a sequence.

Conversely, it can also be shown that any sequence of $v - 2$ numbers from $\{1,2,\cdots v\}$ uniquely determines a tree covering S. One can associate with each sequence $u = a_1,a_2\cdots a_{v-2}$ a unique tree by first connecting the vertex a_1 to the vertex b_1 labeled with the smallest number not in the sequence. Next connect a_2 to a vertex b_2 labeled with the smallest number not in the sequence $a_2,a_3\cdots a_{v-2}$ or the number labeling b_1. Connect a_3 to a vertex labeled with the smallest number not in the sequence $a_3,a_4\cdots a_{v-2}$ or the numbers labeling b_1 or b_2 and so on. The last edge will be determined by connecting the vertex labeled with v to the vertex other than $b_1,b_2,\cdots b_{v-2}$ with the smallest label. It can be shown that this procedure gives a 1 - 1 correspondence between the set of all trees and the set of all sequencs of $v - 2$ terms. But since there are $v - 2$ terms in the sequence and each term could be one of v numbers, then there are v^{v-2} different sequences each corresponding to a unique tree covering S in (S,R).

CHAPTER VII

PLANAR GRAPHS

Throughout this paper a graph (S,R) has been represented by considering the vertices as points in a plane and the arc $(a,b) \in R$ as an oriented geometric simple arc from the point a to b. The question arises as to whether a graph may be represented on a plane such that the arcs do not intersect except at their endpoints. For instance, the complex system of electrical circuits needed for today's electronic equipment has lead to the printing of circuits. Obviously a printed circuit would be "shorted out" if two paths crossed. It therefore becomes necessary to determine whether or not a particular circuit (graph) is planar. Fortunately a Polish mathematician Kuratowski (1930) has given a necessary and sufficient condition for a graph to be planar. A discussion of Kuratowski's theorem will be given later. A graph with a representation having this planar property will be called a planar graph. However, a little more consideration is needed in order to have a good general definition.

Topological Graphs

It should be clear that any M-graph (S,R,M) may be represented in three-dimensional Euclidean space, E^3, such that all vertices are distinct and no two arcs intersect except maybe at their endpoints.

Definition 7.1. A representation of an M-graph in E^3 such that all vertices are distinct and no two arcs intersect except at their endpoints is called a <u>topological graph</u>, denoted by $(S,R,M)_t$.

Definition 7.2. If π is a surface in E^3 and there exists a one-to-one open continuous mapping (homeomorphism) of $(S,R,M)_t$ into π, then $(S,R,M)_t$ is said to be a π-<u>topological graph</u>. When π is a plane then $(S,R,M)_t$ is called a <u>planar topological graph</u>.

Definition 7.3. An M-graph (S,R,M) is <u>planar</u> if and only if there exists a planar topological graph $(S,R,M)_t$.

The use of topological graphs offers the rather obvious advantage that one is able to use the known tools of topology in Euclidean space to investigate properties of graphs. Indeed, the subject of graphs is topological in nature. Some of the problems that gave rise to the subject of topology were in the area of graph theory.

The following material will involve a few basic concepts and theorems of point-set topology. It is assumed that the reader is familiar with such basic topological concepts. The first two topological concepts used are that of a boundary point and that of a component (topological). A <u>boundary point</u> of a set S in E^2 is any point of E^2 such that every neighborhood of the point contains at least one point of S and at least one point not belonging to S. A <u>component</u> A of a topological space S is any nonempty connected subset of S such that if B is any connected subset of S and $A \cap B \neq \emptyset$, then B is a subset of A.

Definition 7.4. Let $(S,R,M)_t$ be a planar topological M-graph. Every open component (region) of $E^2 - (S,R,M)_t$ is called a face of $(S,R,M)_t$.

Definition 7.5. A face is said to be bounded or unbounded according to whether the face is a bounded or unbounded set.

Definition 7.6. A boundary edge of a face is any edge of $(S,R,M)_t$ all of whose points are boundary points of the face.

Definition 7.7. The contour of a face is the union of all the boundary edges of a face.

Definition 7.8. Any two faces whose contours have a common edge are said to be adjacent.

In order to illustrate the above definitions consider the planar topological graph in Figure 27. There are three faces A, B and C denoted by the shaded regions. The contour of face A is {[b,c],[c,e],

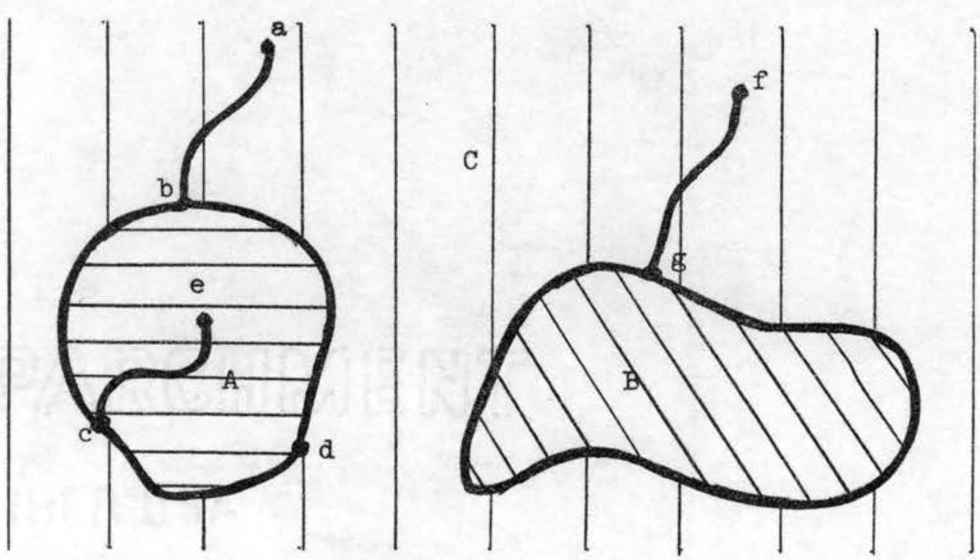


Figure 27.

[c,d],[d,b]}; of face B is {[g,g]}; of face C is {[a,b],[b,c],[c,d], [d,b],[f,g],[g,g]}. Notice that the contour of a face is not necessarily a set of edges bounding the face, but the set of boundary edges of the face. Some edges are boundary edges to only one face, such as edge [c,e] is a boundary edge to only the face A. Also, some faces have only one boundary edge as face B. Faces A and B are bounded faces whereas face C is unbounded. Faces A and B are not adjacent whereas any other pair of faces are adjacent.

It should be mentioned that any planar graph could just as well be represented on a sphere and conversely, any graph that can be represented on a sphere such that the edges intersect only at vertices is planar. One of the simplest ways to establish this result is by a stereographic projection. Consider a sphere tangent to a plane at point T. (Figure 28.) For every point p in the plane the point

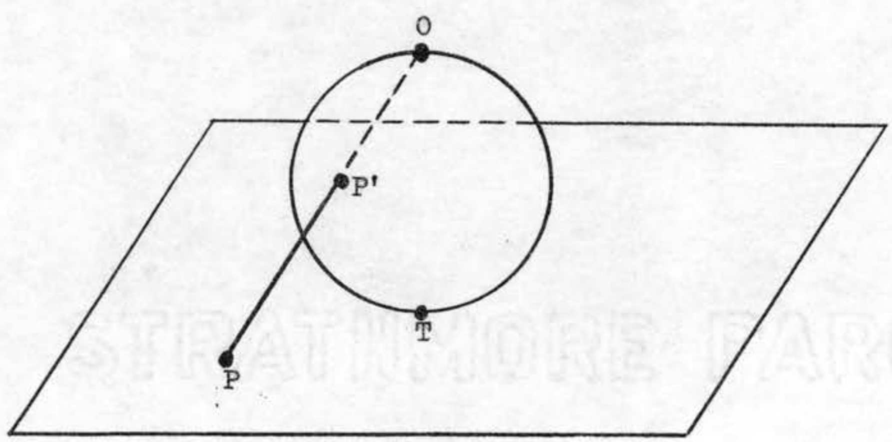


Figure 28.

O diametrically opposite T determines a line intersecting the sphere at one and only one point P' (other than point O). Thus every point on the plane corresponds to one and only one point on the sphere and every point of the sphere other than point O corresponds to one and only one point on the plane. It can be shown that such a mapping is a homoeomorphism. The following theorem would follow by choosing a representation of the graph in the plane that does not contain the point T.

Theorem 7.1. An M-graph is planar if and only if it is a π-topological where π is a sphere.

It should be clear that a tree (S,R) is a planar graph. Moreover, since the tree is connected and contains no cycles (simple closed curve), then $E^2 - (S,R)_t$ is an unbounded open connected component. That is, $(S,R)_t$ has only one face which is unbounded. In fact, the following is true:

Theorem 7.2. There is one and only one unbounded face in a finite planar topological graph $(S,R,M)_t$.

Since $(S,R,M)_t$ is finite then there are only a finite number of edges and consequently $(S,R,M)_t$ is a bounded set in E^2. Hence there exists a simple closed curve C such that $(S,R,M)_t$ is contained in the interior of C. By the Jordan Curve Theorem[1], $E^2 - C = A \cup B$ where

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[1] Jordan Curve Theorem. Let C be a simple closed curve in E^2 and let $S = E^2 - C$. The $S = A \cup B$ where A and B are open regions, exactly one of which is bounded. Furthermore, the boundary of A, boundary of B and the boundary of C are equal.

unbounded region and since $B \cap (S,R,M)_t = \emptyset$, then only one face F of $(S,R,M)_t$ contains B. Hence F is unbounded. No other face of $(S,R,M)_t$ is unbounded since it must be in the interior of the bounded open region A.

Theorem 7.3. No edge of a finite planar topological M-graph is on the contour of more than two faces.

Every arc is a closed set. Consider any edge $[a,b]_j$ of the planar topological M-graph. Since there are only a finite number of edges, then the union of all edges other than $[a,b]_j$ is closed. If x is a point of $[a,b]_j$ other than the endpoint, then there exists a neighborhood of x containing no points of the other edges. Hence, there exists an arc that is a subset of the edge with endpoints on the closure of the neighborhood with all other points in the neighborhood. It is known that such an arc is a boundary of exactly two components of the neighborhood.[1] Hence, the entire edge is on the contour of at most two faces, since every point of the edge must be a boundary point to a face if it is a boundary edge.

Theorem 7.4. If (S,R) is a tree, then in $(S,R)_t$, $v - e + f = 2$.

By Theorem 6.2, $v - e = 1$ in $(S,R)_t$. Since $f = 1$ (only one face), then $v - 3 + f = 2$.

Theorem 7.4 can be generalized by the following classical theorem

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[1]The theorem referred to is: Any arc with only the endpoints on the boundary of a connected region in E^2 is a boundary to at most two components of the region.

due to Euler.

Theorem 7.5. (Euler's formula) In any finite connected planar topological M-graph, $v - e + f = 2$.

The theorem will be proved by mathematical induction on the number of faces. Let $(S,R,M)_t$ be a finite connected planar topological graph. Suppose $(S,R,M)_t$ has exactly one face. $(S,R,M)_t$ has no cycles or otherwise there would be at least two faces. Hence $(S,R,M)_t$ is a tree and by Theorem 7.4 satisfies the theorem. Assume the theorem holds for a connected planar topological graph with f faces and let $(S,R,M)_t$ have $f + 1$ faces. Since $(S,R,M)_t$ has more than one face then by Theorem 7.2 at least one face is bounded. The contour of a bounded face will contain a simple cycle of edges (simple closed curve) or a loop that bounds the face. Any edge $[a,b]_j$ of such a cycle is on the contour of two faces or otherwise it would not be in the cycle. By Theorem 7.3, $[a,b]_j$ is on the contour of exactly two faces. Hence, if edge $[a,b]_j$ is deleted from $(S,R,M)_t$ then $(S,R,M)_t - [a,b]_j$ has one less face than $(S,R,M)_t$. Also since $[a,b]_j$ is in a cycle of $(S,R,M)_t$ then $(S,R,M)_t - [a,b]_j$ is still connected. By assumption, in $(S,R,M)_t - [a,b]_j$, $v - e + f = 2$. Thus, in $(S,R,M)_t$ when $[a,b]_j$ is added, $v - (e + 1) + (f + 1) = 2$.

Theorem 7.6. The number of bounded faces f_b of a finite connected planar topological graph is equal to the cyclomatic number Θ.

By Theorem 7.5, $v - e + f = 2$ and $f = e + 2 - v$. By Theorem 4.36 the cyclomatic number $\Theta = e + c - v$ and since it is connected $\Theta =$

$e + 1 - v$. But by Theorem 7.2 the number of bounded faces $f_b = f - 1$. Hence, $f = f_b + 1 = e + 2 - v$ and $f_b = e + 1 - v = \Theta$.

Corollary 7.7. The number of faces of a finite connected planar topological graph is equal to $\Theta + 1$.

Theorem 7.8. The contour of every face of an irreflexive planar topological graph with $e \geq 3$ contains at least three edges.

Any bounded face is bounded by a simple cycle of edges from its contour. Since it is irreflexive then the simple cycle is not a loop and has at least three edges. If there is only one face, necessarily unbounded, then each edge is a boundary edge and hence on its contour. Since $e \geq 3$ then the contour has at least three edges.

Theorem 7.9. The contour of every face of an irreflexive planar topological M-graph with $e \geq 2$ contains at least 2 edges.

Theorem 7.9 follows by an argument similar to that for Theorem 7.8. The difference being that the least number of edges in an irreflexive M-graph bounding a face is two.

Theorem 7.10. In every irreflexive planar topological graph with $e \geq 3$, $f \leq 2e/3$.

Let $X = \{(a,b) \mid$ a is a face and b a boundary edge of a$\}$. The cardinal number of X, $|X| \leq 2e$ since by Theorem 7.3 no edge is on the contour of more than two faces. Also, $3f \leq |X|$ since by Theorem 7.8 the contour of each face has at least three edges. Therefore, $3f \leq |X| \leq 2e$ and $f \leq 2e/3$.

Theorem 7.11. End every irreflexive planar topological M-graph with $e \geq 1$, $f \leq e$.

Using a similar argument with $e > 1$ as given for Theorem 7.10, $|X| \leq 2e$. But $2f \leq |X|$ since by Theorem 7.9 the contour of each face has at least 2 edges. Therefore, $2f \leq |X| \leq 2e$ and $f \leq e$. For $e = 1$, $f = 1 \leq e$ anyway.

Theorem 7.12. Every finite irreflexive planar graph (S,R) has at least one vertex of order less than six.

Theorem 7.12 will follow for any irreflexive planar graph if it can be shown that every connected irreflexive planar graph has at least one vertex of order less than six.

Assume that the order of each vertex of $(S,R)_t$ is at least six. Let $Y = \{(a,b) \mid a$ is an edge and b a vertex of $a\}$. The cardinal number of Y, $|Y| \leq 2e$ since no edge contains more than two vertices. Also, $6v \leq |Y|$ since by assumption every vertex is contained in at least six edges. Therefore, $6v \leq |Y| \leq 2e$ and $v \leq e/3$. By Theorem 7.10, $f \leq 2e/3$. By Theorem 7.5, $2 = v - e + f$ and $2 = v - e + f \leq e/3 - e + 2e/3 = 0$ which obviously is a contradiction.

Corollary 7.13. Every planar graph has at least one vertex of order less than seven.

Theorem 7.14. A connected planar topological M-graph with each vertex of order three or more has at least one face whose contour contains less than six edges.

By Theorem 4.30, $v \leq 2e/3$. Suppose the contour of every face contains at least six edges. Let $X = \{(a,b) \mid a \text{ is a face and } b \text{ a boundary edge of } a\}$. The cardinal number of X, $|X| \leq 2e$, since by Theorem 7.3 every edge is on the contour of at most two faces. Also $6f \leq |X|$, since by assumption every face has at least six edges in its contour. Therefore, $6f \leq |X| \leq 2e$ and $f \leq e/3$. By Theorem 7.5, $2 = v - e + f$. Hence, $2 = v - e + f \leq 2e/3 - e + e/3 = 0$ which is a contradiction.

Theorem 7.15. The complete graph of five vertices is not planar.

If one can show that an irreflexive complete graph (S,R) of five vertices is not planar, then the theorem follows. Assume that (S,R) is planar. Hence there is a planar topological graph $(S,R)_t$. By Theorem 4.25, $e = v(v - 1)/2 = 10$. Since $(S,R)_t$ is connected, then by Theorem 7.5, $v - e + f = 2$ and $f = 2 - v + e = 2 - 5 + 10 = 7$. Also, by Theorem 7.10, $f \leq 2e/3$. Therefore, $7 \leq 2 \cdot 10/3$, which is a contradiction. Hence (S,R) is not planar.

The British cosmologist G. J. Whitrow in his book The Structure and Evolution of the Universe argued that man could not have evolved from a space of two dimensions because of graph theory. He indicated that any brain requires a vast number of nerve cells to be connected in pairs by nerves that do not intersect. When he said graph theory ruled out the possibility of a two-dimensional creature he was refering specifically to Theorem 7.15. That is, Theorem 7.15 indicates the maximum number of complete nerve cells of a two-dimensional animal would be four.

Theorem 7.16. An irreflexive graph (S,R) of six vertices each of order three with S partitioned into two cells such that no two vertices in the same cell are adjacent, is not planar.

First note that each cell must contain three vertices since the order of each vertex is three. Therefore each vertex in a cell is connected by an edge to each vertex in the other cell. Hence, a graph satisfying the hypothesis of the theorem is unique and illustrated in Figure 29.

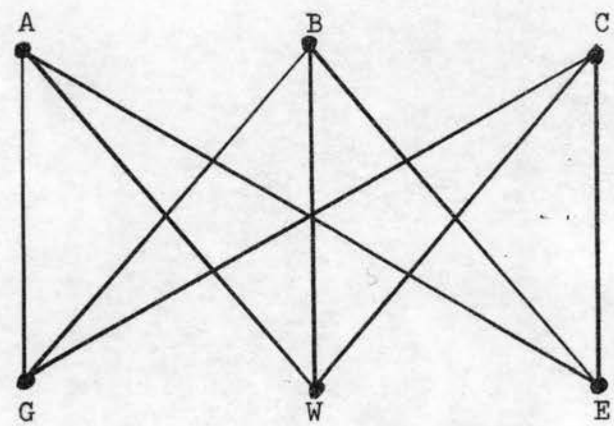

Figure 29.

Assume that a graph (S,R,) meeting the hypothesis of Theorem 7.16 is planar. Hence there exists a planar topological graph $(S,R)_t$. Let $X = \{(a,b) \mid a \text{ is a face and } b \text{ a boundary edge of } a\}$. The cardinal number of X, $|X| \leq 2e$, since by Theorem 7.3 each edge is a boundary edge to at most two faces. By Theorem 4.43 every cycle is of even length. Therefore, no face has a contour of just three edges and $|X| \geq 4f$. But $4f \leq |X| \leq 2e$ and $2f \leq e$. By Theorem 7.5, $v - e + f = 2$ and $f = 2 + e - v = 2 + 9 - 6 = 5$. Hence $2 \cdot 5 \leq 9$ which is a contradiction.

Theorem 7.16 gives the answer to the classical gas-water-electricity problem. The problem is to connect three houses A, B and C (Figure 28) with gas (G), water (W) and electricity (E) such that none of the utility lines cross. Theorem 7.16 indicates that this is impossible in a plane.

The non-planar graphs indicated in Theorem 7.15 and Theorem 7.16 take on special significance. We shall refer to these graphs as the <u>hexagonal</u> and <u>pentagonal graphs</u>. Obviously an M-graph that has a partial subgraph that is either hexagonal or pentagonal is non-planar. It may come as a surprise that the converse is also true. This result is stated in the very important and difficult theorem due to the Polish mathematician Kuratowski (1930).

Theorem 7.17. (Kuratowski's theorem) A graph is planar if and only if it contains no pentagonal or hexagonal partial subgraph.

It is easily seen by Kuratowski's theorem that an M-graph (S,R,M) is planar if and only if its graph (S,R) is planar.

An interesting problem pertaining to planar graphs arises in designing certain patterns for prints, tile floors or mosaics in general. It is always possible to start with a piece of tile of x boundary edges and continuously add pieces with x boundary edges around it keeping the order of the interior vertices the same. For instance, could one continue adding pieces with five edges keeping interior vertices of order three to the pattern started in Figure 30? The answer is "no". Problems of this nature are revealed by the following theorem.

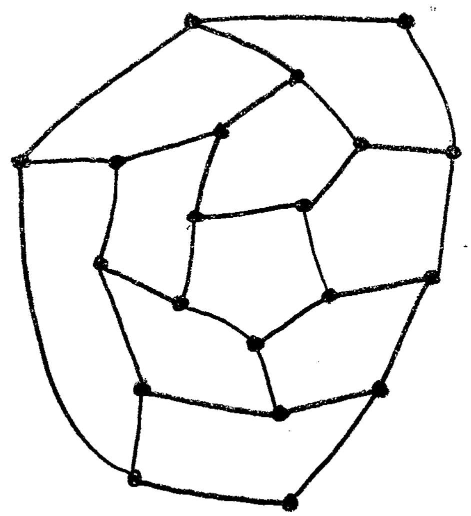

Figure 30.

Theorem 7.18 (87) If sets of edges can be repeatedly added to an elementary cycle of x edges such that at each step:

(1) An irreflexive connected planar graph is formed.

(2) $\lim_{v \to \infty} v_b/v = 0$, where v_b is the number of vertices of the unbounded face.

(3) Every bounded face has exactly x boundary edges.

(4) Every vertex perviously on the contour of the unbounded face becomes an interior vertex of order y.

(5) Every edge is in a cycle.

then $x = 3, y = 6, x = 6, y = 3$, or $x = 4, y = 4$.

Since by condition (4) every vertex eventually has order y then at any given step no vertex could have an order greater than y. Furthermore, at no step could all the vertices have order y for then no more edges could be added. Hence $2e \leq yv$. But $y(v - v_b) < 2e$.

Therefore,

$$y(v - v_b) < 2e < yv,$$

$$y(v - v_b)/2v < 2e/2v < yv/2v \quad \text{and}$$

(A) $$y/2 - yv_b/2v < e/v < y/2.$$

By condition (2) $\lim_{v \to \infty} v_b/v = 0$ and

(B) $$\lim_{v \to \infty} (y/2 - yv_b/2v) = \lim_{v \to \infty} y/2 - (y/2)\lim_{v \to \infty} v_b/v = y/2.$$

Steps (A) and (B) imply that $\lim_{v \to \infty} e/v = y/2$.

The number of edges at each step could be counted in another way. By Theorem 7.2 there is only one unbounded face and by condition (3) each of the f - 1 bounded faces has exactly x boundary edges. Hence $(f - 1)x$ counts each edge, not on the contour of the unbounded face, twice and each edge on the contour once. Furthermore v_b is the number of edges on the contour of the unbounded face. Therefore,

$$2e = (f - 1)x + v_b,$$

$$2e/xv = (f - 1)x/xv + v_b/xv \quad \text{and}$$

$$f/v = 2e/xv + 1/v - v_b/xv.$$

Hence, $\lim_{v \to \infty} f/v = (2/x)\lim_{v \to \infty} e/v + \lim_{v \to \infty} 1/v - (1/x)\lim_{v \to \infty} v_b/v = y/x.$

By Theorem 7.5,

$$v - e + f = 2 \quad \text{and}$$

$$1 - e/v + f/v = 2/v.$$

Hence, $\lim_{v \to \infty}(1 - e/v + f/v) = \lim_{v \to \infty} 2/v$ and

$$1 - y/2 + y/x = 0,$$

$$2x - xy + 2y = 0,$$

$$-2x + xy - 2y + 4 = 4,$$

$$x(y - 2) - 2(y - 2) = 4,$$

(C) $$(x - 2)(y - 2) = 4.$$

The only positive integral solutions to (C) are the ones stated in the theorem.

Notice that the pattern started in Figure 28 could never be continuously expanded under the conditions of Theorem 7.18 since $x = 5$ regardless of the order of interior vertices.

Dual Graphs

For every planar undirected M-graph (S,R,M) one can construct another undirected M-graph, called its dual, in the following manner:

(1) Let every face of $(S,R,M)_t$ correspond to one and only one vertex in the dual.

(2) Let every edge $[a,b]_j$ of $(S,R,M)_t$ correspond an edge in the dual connecting the vertices or vertex that correspond to the faces or face for which $[a,b]_j$ is a boundary edge.

The dual of $(S,R,M)_t$ can be drawn by placing a dot in each face of $(S,R,M)_t$ to represent the vertices of the dual and dashed lines connecting these points through the edges of $(S,R,M)_t$ between the faces to represent the edges in the dual (Figure 31 . When an edge of $(S,R,M)_t$ has the same face on either side, that is not involved in a cycle, then a loop will be formed in the dual. Likewise, a loop in $(S,R,M)_t$ will result in an edge of the dual not involved in a cycle. Also notice that when two faces of $(S,R,M)_t$ have several common boundary edges, a new edge in the dual is formed for each one. This is one reason for considering M-graphs.

There are several other observations that one should make. In Figure 30, $(S,R,M)_t$ is connected as is the dual graph. The multiplicity of each edge of $(S,R,M)_t$ is one, whereas in the dual one edge has a multiplicity of three. The dual of $(S,R,M)_t$ is also a planar M-graph. Also note that the number of edges in $(S,R,M)_t$ and its dual are the same, whereas the number of faces in one is the number of vertices in the other. The latter remark about duality will be true in general if $(S,R,M)_t$ is connected. The following theorems reveal in general some of the remarks made above.

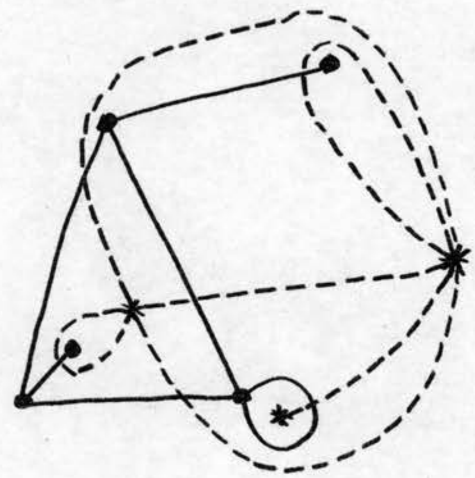

Figure 31.

Theorem 7.19. An edge not in a cycle of a planar graph corresponds to a loop in its dual.

If an edge [a,b] is not in a cycle of a planar graph then it is a boundary edge to only one face. Hence by part (2) of the definition of a dual graph the edge [a,b] will correspond to a loop with the vertex corresponding to the face.

Theorem 7.20. A loop in a planar graph will correspond to an edge in the dual not in a cycle.

A loop in a planar graph is the only boundary to the face in its interior. It is also a boundary to another face. Hence the vertex in the dual corresponding to the interior face is contained in only one edge of the dual graph connecting the interior face to the other.

Theorem 7.21. The dual graph of a tree with $e = n$ is a planar M-graph with only one vertex and a loop of multiplicity n.

There is only one face in a tree and consequently only one vertex in its dual. Since each edge of the tree is a boundary edge and only one face, then corresponding to each edge is a loop in its dual. Figure 32 illustrates a tree and its dual. Notice the relationship

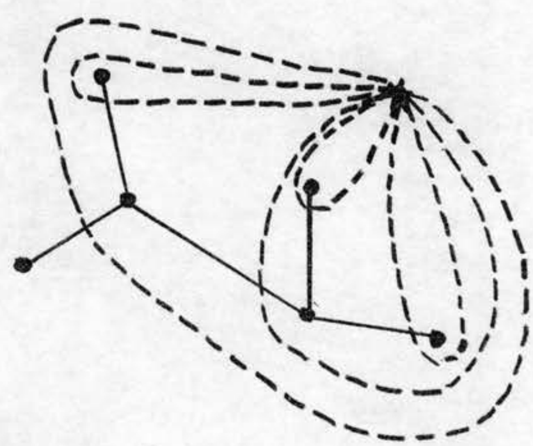

Figure 32.

between inclusion of the loops and the branch pattern in the tree. It can also be shown that the converse of Theorem 7.21 is true.

Theorem 7.22. The dual of any planar M-graph is connected.

If the dual is not connected then its vertices could be partitioned into two or more cells such that no two vertices in different cells belong to the same edge. This means that the faces of the original graph could be partitioned such that no two faces in different cells have a common boundary edge. This is impossible in E^2. It is impossible since the closures of the faces in each cell are disjoint and hence $E^2 = \bigcup_{i=1}^{n} A$ such that each A is closed and connected.

Theorem 7.23. The dual of a planar M-graph is planar.

The Theorem will follow by Kuratowski's theorem if it can be shown that it is impossible for a planar graph to (1) contain 5 mutually adjacent faces or (2) contain two groups of three faces with each group mutually edgewise disjoint and every face adjacent to the faces in the other group.

In case (1) order the faces F_1, F_2, F_3, F_4 and F_5 such that $i > j$ implies F_i is on the exterior of F_j. If the common edges of F_1 and F_2 are deleted then the remaining edges on the contour of F_1 and F_2 form one simple cycle or edgewise disjoint cycles separated by F_1 and F_2. Also, the edges from F_2 in any such cycle are connected as well as the edges from F_1. F_3 must be interior or exterior to one of these cycles. But F_3 is adjacent to such a cycle in at least two edges, one from F_1 and the other from F_2. If these edges are deleted then a cycle or disjoint cycles are formed with the various edges from each face always connected in the cycle and the cycles separated edgewise by F_1, F_2 and F_3. F_4 must be interior or exterior to one of these cycles with edges from F_1, F_2 and F_3. F_4 is adjacent to such a cycle in at least three edges, one from each of the first three faces. If these edges are deleted then the remaining part of the contour of F_4 and the cycle form disjoint cycles separated edgewise by F_1, F_2, F_3 and F_4. It can be shown that no such cycle contains edges from all four faces. Hence F_5 is separated edgewise from at least one of the first four faces.

In case (2) let F_1, F_2, F_3 and F_4, F_5, F_6 be the groups of mutually edgewise disjoint faces. F_4, F_5 and F_6 are all adjacent to F_1 and F_2. If the common edges are deleted then the remaining edges form at least three edgewise disjoint cycles separated by F_1, F_2, F_4, F_5 and F_6 and none of the cycles include edges from F_4, F_5 and F_6. Hence F_3 being adjacent only to one such cycle cannot be adjacent to F_4, F_5 and F_6 as was originally assumed.

Theorem 7.24. The planar dual M-graph of a connected planar M-graph with v vertices, e edges and f faces has f vertices, e edges and v faces.

Let (S,R,M) be a connected planar M-graph. By definition of a dual graph we know it has f vertices and e edges, that is, $v' = f$ and $e' = e$. By Theorem 7.5, $v - e + f = 2$. But by Theorem 7.22 and Theorem 7.23 the dual is connected and planar. Hence by Theorem 7.5, $v' - e' + f' = 2$ and $f - e + f' = 2$. But $v - e + f = 2 = f - e + f'$ and $f' = v$. Hence the dual graph has v faces.

Polyhedral Graphs

The rest of the discussions will pertain to a particular kind of planar topological graphs called polyhedral graphs.

Definition 7.9. A graph (S,R) is called a <u>polyhedral graph</u> if and only if:

(1) (S,R) is connected and planar.

(2) Every vertex of (S,R) has order greater than two.

(3) Every face of $(S,R)_t$ has more than two boundary edges.

(4) (S,R) is irreflexive. (no loops)

(5) Every edge of (S,R) is in some cycle. (non-separating)

One should note that in the definition of polyhedral graphs that conditions (2) and (3) are dual statements as well as conditions (4) and (5). Conditions (4) and (5) are dual statements because of Theorems 7.19 and 7.20. By Theorems 7.22 and 7.23 the dual of a connected planar graph is also connected and planar. Hence, polyhedral graphs have the following important property.

Theorem 7.25. The dual graph of a polyhedral graph is a polyhedral graph.

Theorem 7.25 is very important in that any theorem about the vertices, faces, and edges of a polyhedral graph has a dual statement (theorem) by Theorem 7.24 and that because of Theorem 7.25 is about polyhedral graphs. This is known as the <u>principle of duality</u>. In other words, any theorem about faces, edges and vertices of a polyhedral graph has a dual statement obtained by interchanging the words faces and vertices that is also true of polyhedral graphs. For instance, the statement that each two vertices are joined by an edge is the dual of the statement that each two faces have a common boundary edge.

Theorem 7.26. Each face of a polyhedral graph has at most $v - 1$ boundary edges.

The boundary edges of a face form a simple cycle. If there were v boundary edges to a face then there could be no other edges in the polyhedral graph. Hence the order of each vertex is two, which contradicts the definition of a polyhedral graph.

Theorem 7.27. (Dual of Theorem 7.26) Each vertex of a polyhedral graph has order of at most $f - 1$.

Theorem 7.28. Every finite polyhedral graph has one vertex of order less than six.

Theorem 7.28 follows directly from Theorem 7.12.

Theorem 7.29. (Dual of Theorem 7.28) Every finite polyhedral graph has one face whose contour has less than six edges.

Theorem 7.30. In every polyhedral graph, $v \leq 2e/3$.

Theorem 7.30 follows directly from Theorem 4.30.

Theorem 7.31. (Dual of Theorem 7.30) In every polyhedral graph, $f \leq 2e/3$.

Theorem 7.32. In every polyhedral graph, $e \geq 6$.

By Theorem 7.5, $v - e + f = 2$ and $3v - 3e + 3f = 6$. By Theorems

7.30 and 7.32, $3v \leq 2e$ and $3f \leq 2e$. Hence $6 = 3v - 3e + 3f \leq 2e - 3e + 2e = e$ and $e \geq 6$.

Theorem 7.33. No polyhedral graph has $e = 7$.

By Theorem 7.5, $v - e + f = 2$ and $3f = 6 + 3e - 3v$. By Theorem 7.31, $f \leq 2e/3$ and $3f \leq 2e$. Hence, $2e \geq 6 + 3e - 3v$ and $v \geq (6 + e)/3$. By Theorem 7.30 $v \leq 2e/3$. Therefore, $(6 + e)/3 \leq v \leq 2e/3$. If $e = 7$, then $4\frac{1}{3} \leq v \leq 4\frac{2}{3}$ which is absurd. Hence $e \neq 7$.

One may wonder if it is impossible for a polyhedral graph to have a number of edges greater than five other than seven. The answer is "no". Such a theorem is found in Vorlesungen über die Theorie der Polyeder by Ernst Steinetz (Springer, 1934), edited by Hans Rademacher.

Theorem 7.34. (16). In every polyhedral graph there are at least four vertices each belonging to less than six edges.

By Theorem 7.27 the order of each vertex is less than f. Thus, if v_i, $i = 3,4,\ldots,(f - 1)$, denotes the number of vertices with order i, then $v = v_3 + v_4 + \ldots + v_{f-1}$. Since a polyhedral graph is irreflexive then each edge has exactly two vertices and $2e = 3v_3 + 4v_4 + \ldots + (f - 1)v_{f-1}$. Hence by subtracting,

$$\begin{array}{rl} 6v = & 6v_3 + 6v_4 + 6v_5 + 6v_6 + 6v_7 + \ldots + 6v_{f-1} \\ 2e = & 3v_3 + 4v_4 + 5v_5 + 6v_6 + 7v_7 + \ldots + (f - 1)v_{f-1} \\ \hline 6v - 2e = & 3v_3 + 2v_4 + v_5 \qquad - v_7 - \ldots - (f - 7)v_{f-1} \end{array}$$

and

$$6v - 2e \leq 3v_3 + 2v_4 + v_5.$$

But, it was shown in the proff of Theorem 7.33 that $3v \geq 6 + e$ and $6v - 2e \geq 12$. Hence, $12 \leq 3v_3 + 2v_4 + v_5 \leq 3(v_3 + v_4 + v_5)$ and

$4 \leq (v_3 + v_4 + v_5)$. This proves the theorem.

Theorem 7.35. (Dual of Theorem 7.34) In every polyhedral graph there are at least four faces each having less than six boundary edges.

Theorem 7.36. (16). In a polyhedral graph the number of faces with exactly three boundary edges plus the number of vertices of order three is greater than seven.

By Theorem 7.26 each face has at most $v - 1$ boundary edges. Thus, if F_i denotes the number of faces with i boundary edges, then $f = F_3 + F_4 + \ldots + F_{v-1}$. Also, $v = v_3 + v_4 + \ldots + v_{f-1}$. Furthermore, since each edge is a boundary edge of two faces $2e = 3F_3 + 4F_4 + \ldots + (v - 1)F_{v-1}$ and since every edge has two vertices $2e = 3v_3 + 4v_4 + \ldots + (f - 1)v_{f-1}$. Therefore, $4e = 3(F_3 + v_3) + 4(F_4 + v_4) + \ldots$. By Theorem 7.4, $2 = v - e + f$ and $8 = 4v + 4f - 4e$. Substituting,

$$8 = 4v + 4f - 4e = 4(v_3 + v_4 + \ldots + v_{f-1}) + 4(F_3 + F_4 + \ldots + F_{v-1})$$
$$-3(F_3 + v_3) - 4(F_4 + v_4) - \ldots - \ldots \quad \text{and}$$
$$8 = (F_3 + v_3) - (F_5 + v_5) - (F_6 + v_6) - \ldots - \ldots \quad \text{and}$$
$$8 \leq F_3 + v_3.$$

Hence $F_3 + v_3 > 7$.

Corollary 7.37. No polyhedral graph can have every face with more than three boundary edges and every vertex of order greater than three.

The reader is no doubt aware that there are only five regular polyhedrons (solids), sometimes called the Platonic solids. They are the tetrahedron, cube, octahedron, dodecahedron and icosahedron.

It was required that each face be a regular polygon and all faces be congruent. A proof that there are only five such solids need not be based on the fact that the faces are regular and all faces congruent. In other words, these things are superfluous as the problem is a topological one as revealed by the following theorem.

Theorem 7.38. There are only five possible polyhedral graphs such that each face has the same number of boundary edges and each vertex the same order.

Let Q be the order of each vertex and B the number of boundary edges of each face. By definition of a polyhedral graph $Q \geq 3$ and $B \geq 3$. By Theorems 7.28 and 7.29, $Q \leq 5$ and $B \leq 5$. Thus $3 \leq Q \leq 5$ and $3 \leq B \leq 5$. But by Corollary 7.37 either $Q = 3$ or $B = 3$. Hence there are only five possible regular polyhedral graphs. It is not hard to show that the five possibilities correspond to the five regular polyhedrons.

Theorem 7.39 (7). Every planar M-graph is at most 5-chromatic.

If any irreflexive planar graph is at most 5-chromatic then it will follow that any planar M-graph is at most 5-chromatic since the addition of loops or multiplicity of edges will not affect the chromatic number. Thus, consider any irreflexive planar graph. The proof is by mathematical induction on the number of vertices. If $v \leq 5$ in an irreflexive planar graph it obviously follows that each vertex could be colored with a different color satisfying the theorem. Assume that every irreflexive planar graph with $v - 1$ vertices is at most

5-chromatic. Let (S,R) be an irreflexive planar graph with v vertices. By Theorem 7.12 one vertex of (S,R) has order less than six. By assumption the subgraph (S - {x},W) is 5-chromatic. If x has order less than five then x could be colored with a color other than the colors of the adjacent vertices and the theorem would follow. Even if x has order five and less than five colors are used to color the adjacent vertices, then x could still be colored with the remaining color to prove the theorem. Hence, let $x_i, i = 1,2,3,4,5$ denote the five adjacent vertices to x each with a distinct color i. Also let the order in the indexing of the adjacent vertices agree with their clockwise order about x (Figure 33).

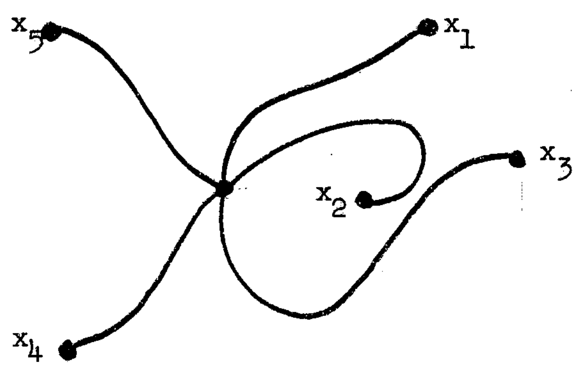

Figure 33.

Let ${}_iG_j$ be the subgraph of (S - {x},W) with all vertices of color i or j. It is impossible for both x_1 and x_3 to be connected in ${}_1G_3$ and also x_2 and x_3 to be connected in ${}_2G_4$. For if x_1 and x_3 are connected in ${}_1G_3$ then any chain connectin x_2 and x_4 would necessarily include a vertex colored 1 or 3. Hence the chain could not be in ${}_2G_4$.

Hence, two of the vertices, say x_1 and x_3, are not connected in ${}_1G_3$. Consider the component C_{x_1} in ${}_1G_3$. Obviously $x_3 \notin C_{x_1}$ since x_1 is not connected to x_3 in ${}_1G_3$. The colors 1 and 3 can be interchanged in C_{x_1} without destroying the property that two adjacent vertices in $(S - \{x\},W)$ are of different colors. Thus x_1 and x_3 are now both the same color 3 and x can be colored by 1 which proves the theorem.

Corollary 7.40. Every polyhedral graph is at most 5-chromatic.

The relationship between the chromatic number of a planar graph and the least number of colors needed to color the faces such that no two adjacent faces have the same color was mentioned in Chapter IV. Theorem 7.39 is the key theorem in determining the number of colors needed to color a planar M-graph. Now it is not true that for every planar M-graph there exists a planar M-graph such that its dual is the given M-graph. For instance, since by Theorem 7.22 the dual of any planar M-graph is connected then there is no planar M-graph whose dual is a non-connected planar M-graph. However, given any planar M-graph (S,R,M) its dual is planar by Theorem 7.23 and is at most 5-chromatic by Theorem 7.39. Now its dual being at most 5-chromatic is sufficient to know that (S,R,M) can be colored with at most five colors since two vertices in the dual of the same color are not connected by an edge and thus the two faces in (S,R,M) corresponding to such vertices are not adjacent. Therefore, the faces of a planar M-graph can be colored with five or less colors by coloring each face a color corresponding to the vertex color in its at most 5-chromatic dual. The following famous theorem is established.

Theorem 7.41. Every planar M-graph can be colored with five or less colors.

The problem of determining the least number of colors needed to color a planar M-graph has had a long history. Map makers have experienced that any map on a globe could be colored with four or less colors. In fact, no one has constructed a planar M-graph that required five colors to color it. However, no one has ever proved that four colors is sufficient. This problem is referred to as the Four-Color Problem. A rough history of the Four-Color Problem is: Mobuis first mentioned it formally in a lecture in 1840; in 1850 De-Morgan made some remarks about it; in 1878, Cayley stated that he could not prove it; in 1880 Kempe presented a proof for it; in 1890 P. J. Heawood exposed a flaw in Kempe's proof.

CHAPTER VIII

SUMMARY AND EDUCATIONAL IMPLICATIONS

This thesis presents an abstract sequential deductive development of basic theory pertaining to graphs. Included are numerous examples and applications that reveal the significance and basic nature of graphs; comments about problems of historical interest; statements regarding unsolved problems in the area of graph theory; and an extensive list of publications that are notable contributions to the subject.

Summary

In Chapter I is the statement of the problem and discussions on the justifications, procedures, limitations, and expected outcomes of this paper. Basic definitions that underlie the subject of graphs are formulated in Chapter II. The close connection between graphs and relations, especially the carry-over of terminology from relations to graphs, is discussed in this chapter. In the chapter is a dual set of definitions that enables one to discuss directed or undirected graphs with a terminology that clearly distinguishes the two situations. Chapter III includes some very basic theorems relating some of the terms presented in Chapter II and that will be useful for discussions to follow. Some of these theorems are important in themselves, but most of them reveal basic properties that eventually lead to more important relationships.

Chapter IV contains an introduction to some important integral-valued functions of graph theory. The functions introduced in this chapter lead to applicable concepts and many lead to important characterizations of terms and properties already discussed. A large number of the significant theorems of graph theory are presented in this chapter. In Chapter V, theory is developed pertaining to the special problems of "transversing" a graph. Much of the history of graph theory is revealed in this chapter. Chapter VI also includes theory regarding special kinds of problems dealing with trees. Chapter VII includes the significant part of graph theory known as planar graphs. The natural relationship between graph theory and topology is revealed in this chapter. The all important topics of polyhedral graphs, dual graphs and the Four Color Problem are contained in this chapter. Kuratowski's Theorem and Euler's Theorem, two of the most notable theorems of graph theory, are also included in this chapter.

Educational Implications

Theory regarding graphs has been developed in numerous disciplines and scattered throughout the literature under many disguises. Any person wishing to gain an understanding of graphs would need to make a search through the literature in various disciplines and then interrelate the different definitions and concepts. The material in this study offers an undergraduate a means to become abreast of the theory using one comprehensive set of definitions and a sequential deductive development of the basic theory. The explicit precise definitions and rigor used in proving the theorems should merit the use of this material as a sound basis for further research. The extensive selected bibliography is intended to facilitate such research.

SELECTED BIBLIOGRAPHY

1. Arnold, B. H., Intuitive Concepts in Elementary Topology, Englewood Cliffs, N. J.: Prentice-Hall Inc., 1962.

2. Avondo-Bodino, G., Economic Applications of the Theory of Graphs, New York: Gordon and Breach, 1962.

3. Beckmann, M., McGuire, C. B. and Winsten, C. B., Studies in the Economics of Transportation, New Haven: Yale University Press, 1956.

4. Belck, H. B., Regulare Faktoren von Graphen, J. reine angeu. Math., 1950, pp. 228-229.

5. Bellman, R., On a Routing Problem, Quart. Appl. Math 16 (1958), pp. 87-90.

6. Berge, C., Theorie der Graphes et ses Applications, Paris: Dunod, 1958.

7. _____, The Theory of Graphs and Its Application, New York: John Wiley and Sons, Translation by Alison Doig, 1962.

8. _____, Graph Theory, The Am. Math. Monthly, Vol. 71, No. 7 (1964), pp. 471-481.

9. _____, Surl' isovalence et la regularite' des transformateurs, C. R. Acad. Sciences, 1950.

10. _____, Two Theorems in Graph Theory, Proc. Nat. Acad. Sce. U.S.A., 1957, pp. 842.

11. Birkhoff, G. P., On the Number of Ways of Colouring a Map, Proceedings of the Edinbourgh Math. Soc., 1930, pp. 83-91.

12. Bratton, P., Efficient Communication Networks, Cowles Comm. Disc. Paper, 1955, p. 2119.

13. Brooks, R. L., On Coloring the Nodes of a Network, Proceedings of the Cambridge Philosophical Society, Vol. 37, 1941, pp. 194-197.

14. Burge, W. H., Sorting, Trees, and Measures of Order, Information and control, Vol. 1 (1958), pp. 181-197.

15. Cahn, A. S., The Warehouse Problem, Bull. Amer. Math. Soc., 1948, p. 1073.

16. Cairns, S. S., Introductory Topology, New York: The Ronald Press Co., 1961.

17. Carlitz, L. and Riordan, J., The Number of Labeled Two Terminal Series-Parallel Networks, Duke Math. J. 23 (1956), pp. 435-446.

18. Cayley, A., A Theorem on Trees, Quarterly Journal of Pure and Appl. Math., V. 23 (1889), pp. 376-378.

19. Clarke, L. E., On Cayley's Formula for Counting Trees, London Math. Soc. (1958), pp. 471-474.

20. Coxeter, H. S. M., Map Coloring Problems, Scripta Mathematics, Vol. 23, 1957, pp. 11-25.

21. _____, Self-dual Configurations and Regular Graphs, Bull. Amer. Math. Soc., V. 56 (1950), pp. 413-455.

22. Culik, K., Zur Theorie der Grophen, Casopies Pest. Mat., V. 83 (1958), pp. 133-155.

23. Dantzig, G., On the Shortest Route through a Network, The Rand Corporation, Santa Monica, Calif. 1959.

24. _____, Application of the Simplex Method to a Transportation Problem, New York: John Wiley and Sons, p. 359.

25. Davis, R. L., The Number of Structures of Finite Relations, Proc. Amer. Math. Soc. 4 (1953), pp. 486-495.

26. _____, Structures of Dominance Relations, Bull. Math. Biophysics (1953), pp. 131-140.

27. Dawson, R. and Good, I. J., Exact Markov Probabilities from Oriented Linear Graphs, Ann. Math. Statist., 1957, p. 838.

28. DeBruijn, N. G. and Erdos, P., A Colour Problem for Infinite Graphs and a Problem in the Theory of Relations, Proc. Kon. Ned. Akad. Wetensch, 1951, p. 371.

29. Dennis, J. B., Mathematical and Electrical Networks, New York: Wiley, 1959.

30. Dirac, G. A., The Structure of K-chromatic Graphs, Fund. Math., 1953, p. 50.

31. _____, Some Theorems on Abstract Graphs, Proc. London Math. Soc., (3) V. 2 (1952), pp. 69-81.

32. _____, Connectivity Theorems for Graphs, Quarterly J. Math., Oxford Ser. (2) V. 3 (1952), pp. 171-174.

33. _____, Note on the Coloring of Graphs, Math. Z., V. 54 (1951), pp. 347-353.

34. _____, Map Colour Theorems Related to the Heawood Colour Theorem, J. London, Math. Soc., 1956, p. 460.

35. Euler, L., Solutio Problematis ad Geometriam situs Pertimentis, Commentarii Academiae Petropolitian, V. 8 (1736), pp. 128-140.

36. Flood, M. M., On the Traveling Salesman's Problem, Journal Operational Research Soc. of America., V. 4 (1956), pp. 61-75.

37. Ford, G. W. and Uhlenbeck, G. E., Combinatorial Problems in the Theory of Graphs, Proc. Natl. Acad. Sci. U.S.A., 1956, pp. 1220128, 203-208, 529-535.

38. Ford, L. R., Jr. and Fulkerson, D. R., Flows in Networks, Princeton, N. J.: Princeton University Press, 1962.

39. Franklin, P., The Four Color Problem, New York: Scripta Mathematica, Yeshiva University, 1961.

40. Frucht, R., Graphs of Degree 3 with a Given Abstract Group, Canadian J. Math. 1949, pp. 365-378.

41. _____, Die Gruppe des Petersenschen Graphen, Comment. Math. Helr., V. 9 (1937), pp. 217-223.

42. _____, On Graphs of Repeated Graphs, Bull. Amer. Math. Soc., V. 55 (1949), pp. 418-420.

43. Gallai, T., On Factorization of Graphs, Acta. Math. Gcad. Sci. Hung., 1950, p. 133.

44. Gale, D., A Theorem on Flows in Networks, Pacific J. Math., 1957, p. 1073.

45. _____, Transient Flows in Networks, Michigan Math. J. G. (1959), pp. 59-63.

46. Gardner, M., The Second Scientific American Book of Mathematical Puzzles and Diversions, New York: Simon and Schuster, 1961.

47. Gilbert, E. N., Enumeration of Labeled Graphs, Canadian J. Math., 1956, pp. 405-411.

48. _____, Gray Codes and Paths on the n-Cube, Bell System Tech. J., 1958, pp. 815-826.

49. Hadwiger, H. and Debrunner, H., Combinatorial Geometry in the Plane, New York: Holt, Rinehart and Winston, 1964.

50. Harary, F., On the Notion of Balance of a Signed Graph, Michigan Math. J., 1953-54, pp. 143-146.

51. _____, The number of Linear, Directed, Rooted and Connected Graphs, Trans. Amer. Math. Soc., 1955, pp. 445-463.

52. _____, On the Number of Dissimilar Line-subgraphs of a Given Graph, Pacific J. Math., 1956, pp. 57-64.

53. _____, The Number of Dissimilar Supergraphs of a Linear Graph, Pacific J. Math., 1957, pp. 903-911.

54. _____, The Number of Oriented Graphs, Michigan Math. J., 1957, pp. 221-224.

55. _____, On the Number of Dissimilar Graphs between a Given Graph Subgraph Pair, Canadian J. Math., 1958, pp. 513-516.

56. _____, On the Number of Bicolored Graphs, Pacific J. Math., 1958, pp. 743-755.

57. _____, The Number of Functional Digraphs, Math. Annalen., 1959, pp. 203-210.

58. _____, Unsolved Problems in the Enumeration of Graphs, Magyar Tud. Akad. Mat. Kutato. Int. Kosl., 1960, pp. 63-95.

59. Harary, F., Norman, R. Z. and Cartwright, D., Structual Models, New York: John Wiley and Sons, 1965.

60. Harary, F. and Prins, G., Enumeration of Bicolourable Graphs, Canadian J. Math., 1963, pp. 237-248.

61. Harary, F. and Norman, R. Z., Graph Theory as a Mathematical Model in Social Science, University of Michigan, Ann Arbor, 1953.

62. Heller, I., The Traveling Salesman Problem, George Washington Univ. Logistics Research Project, 1954.

63. Higgins, P. J., Disjoint Transversals of Subsets, Canada J. Math., V. 11 (1959), pp. 280-285.

64. Hilbert, D. and Cohn-Vossen, S., Geometry and the Imagination, New York: Chelsea, 1952.

65. Kaloba, R., On Some Communication Network Problems, Providence Rhode Island: American Math. Soc., 1960.

66. Kelley, J. B. and Kelley, L. M., Paths and Circuits in Critical Graphs, Amer. J. Math., 1954, p. 228.

67. Konig, D., Theorie der Endlichen und Unendlichen Graphen, New York: Chelsea Publishing Co., 1950.

68. _____, Uber Trennende Knotenpunkte in Graphen, Acta. Litt., Sci. Sjeyed., V. 6 (1933), pp. 155-179.

69. Kotzig, A., The Significance of the Skeleton of a Graph for the Construction of Composition Basis of some Subgraphs, Mat. Fyz. Cosopes Slouensk. Akad. Vied., V. 6 (1956), pp. 68-77.

70. Kraitchik, M., Mathematical Recreations, New York: Dover Publications Inc., 1953.

71. Kruskal, J. B., On the Shortest Spanning Subtree of a Graph, Proc. Amer. Math. Soc., 7 (1956), pp. 48-50.

72. Kuhn, H. W., The Hungarian Method for the Assignment Problem, Naval, Res. Logist. Quart., 1955, p. 83.

73. Kuratowski, C., Sur le Probeme des Courbes Gauches en Topologie, Fundamental Mathematical, Vol. 15, 1930, pp. 271-83.

74. Luce, R. D., Two Decomposition Theorems for a Class of Finite Oriented Graphs, Amer. J. Math., V. 74 (1952), pp. 701-722.

75. MacLane, S., Some Unique Separation Theorems for Graphs, Amer. J. Math., V. 57 (1935), pp. 805-820.

76. Minty, G. J., A Comment on the Shortest Route Problem, Op. Res. 5 (1957), 724.

77. Moon, J. W., On Some Combinatorial and Probabilistic Aspects of Bipartite Graphs, Doctorial Dissertation, University of Alberta, 1962.

78. Mycielski, J., Sur le Coloriage des Graphes, Colloq. Math., V. 3 (1955), pp. 161-162.

79. Neville, E. H., The Codifying of Tree Structure, Proc. Cambridge Philos. Soc., V. 49 (1953), pp. 381-385.

80. Newman, D. J., A Problem in Graph Theory, Amer. Math. Monthly, V. 65 (1958), p. 611.

81. Newman, J. R., The World of Mathematics, Vol. 1, New York: Simon and Schuster, 1956.

82. Nordhaus, E. A. and Goddin, J. W., On Complementary Graphs, Amer. Math. Monthly., V. 63 (1956), pp. 175-177.

83. Norman, R. Z., On the Number of Linear Graphs with Given Blocks, Doctoral Dissertation, University of Michigan, 1954.

84. Norman, R. Z. and Robin, M. O., An Algorithm for a Minimum Cover of a Graph, Notices of the Amer. Math. Soc., 1958, p. 36.

85. Okada, S., Algebraic and Topological Foundations of Network Synthesis, Proc. Symp. on Modern Network Synthesis, New York, 1955, p. 283.

86. Orden, A., The Transshipment Problem, Management Sci., 1956, p. 276.

87. Ore, O., Graphs and Their Uses, New York: Random House, Inc., 1963.

88. _____, Sex in Graphs, Proc. Amer. Math. Soc., V. 11 (1960), pp. 533-539.

89. _____, Studies on Directed Graphs I, Ann. of Math. 63 (1956), pp. 383-408.

90. _____, An Excursion into Labyrinths, Math. Teacher, V. 52 (1959), pp. 367-370.

91. _____, Graphs and Subgraphs, Trans. American Math. Soc., 84 (1957), pp. 109-137.

92. _____, Note on Hamiltonian Circuits, Amer. Math. Monthly, V. 67 (1960), p. 55.

93. _____, A Problem Regarding the Traciny of Graphs, Elem. Math., V. 6 (1951), pp. 49-53.

94. Ore, O., Theory of Graphs, Amer. Math. Soc. Colloq. Publs., American Mathematical Society, Providence, Rhode Island, 1962.

95. _____, Graphs and Matching Theorems, Duke Math. J., 1955, p. 625.

96. Otter, R., The Number of Trees, Ann. of Math., 1948, pp. 583-599.

97. Peterson, J., Die Theorie der Regularen Graphs, Acta Math., 1851, p. 193.

98. Pollack, M., The Maximum Capacity through a Network, Operations Res., 1960, pp. 733-736.

99. _____, Solutions of the Kth Best Route Through a Network, J. Math. Anal. Appl., 1961, pp. 547-559.

100. _____, The Kth Best Route Through a Network, Operations Res., 1961, pp. 578-580.

101. Polya, G., Sur les Types des Propositions Composees, J. Symbolic Logic, 1940, pp. 98-103.

102. _____, Kombinatousche Anzahlbestimmungen fur Gruppen, Acta. Math. V. 68 (1937), pp. 145-254.

103. _____, On Picture Writing, Amer. Math. Monthly, 1956, pp. 689-697.

104. Prim, R., Shortest Connection Networks and Some Generalizations, Bell System Tech. J., 1957, pp. 1389-1401.

105. Prufer, H., Nevei Beweis eines Satzes ubei Permutationen, Archir der Math. und Phys., 3 V 37 (1918), pp. 142-144.

106. Radner, P. and Tritter, A., Communication in Networks, Cowles Comm. Paper, 1954, p. 2098.

107. Rado, R., Factorization of Even Graphs, Quart. J. Math., 1949, p. 95.

108. Read, R. C., The Enumeration of Locally Restricted Graphs, J. London Math. Soc., 1959, pp. 417-436.

109. _____, Maximal Circuits in Critical Graphs, J. London Math. Soc., V. 32 (1957), pp. 456-462.

110. _____, The Number of K-colored Graphs on Labeled Nodes, Canadian J. Math., 1960, pp. 409-414.

111. _____, A Note on the Number of Functional Diagraphs, Math. Ann., 1961, pp. 109-110.

112. _____, Contributions to the Cell-Growth Problem, Canadian J. Math., 1962, pp. 1-20.

113. _____, On the Number of Self-Complementary Graphs and Diagraphs, J. London Math., 1963, pp. 98-104.

114. Redfield, J. H., The Theory of Group-reduced Distributions, Amer. J. Math., 1927, pp. 433-455.

115. Reordan, J., An Introduction to Combinatorial Analysis, New York: John Wiley and Sons, 1958.

116. Riordan, J. and Shannon, C. E., The Number of Two-Terminal Series-Parallel Networks, J. Math. Phys., 1942, pp. 83-93.

117. _____, The Numbers of Labeled Colored and Chromatic Trees, Acta Math., 1957, pp. 211-225.

118. _____, An Introduction to Combinatorial Analysis, New York: John Wiley and Sons, 1958.

119. Richardson, M., Relativization and Extension of Solutions of Irreflexive Relations, Pacific J. Math., 1955, p. 557.

120. Robacker, J. T., On Network Theory, The Rand Corp., Research Memorandum RMp1498, Many 26, 1955.

121. Robbins, H. E., A Theorem on Graphs, Amer. Math. Monthly, V. 46 (1939), pp. 281-283.

122. Rouse Ball, W., Mathematical Recreations and Essays, London and New York: MacMillian, 1962.

123. Sabedussi, G., Graph Derivatives, Math. Z, 1961, pp. 385-401.

124. Samte-Lague, Les Reseaux (on Graphs), Memoi. Math., V. 18, 1926.

125. Senior, J. K. Partitions and Their Representative Graphs, Amer. J. Math., 1951, pp. 663-689.

126. Stein, S. K., Mathematics the Man-made Universe, San Francisco: W. H. Freeman and Company, 1963.

127. Supnick, F., Extreme Hamilton Lines, Ann. of Math. (2) V. 66, (1957), pp. 179-201.

128. Tarrey, G., Le Probleme des Labyrinthes, Nouvelles Annales de Math., 1895, p. 187.

129. Tranque, G. T., The Type in Cubic Graphs, Gaceta Mat., (1) V. 5 (1953), pp. 11-23.

130. Turan, P., On the Theory of Graphs, Collog. Math., V. 3 (1954), pp. 19-30.

131. Tutte, W. T., On Hamiltonian Circuits, J. London Math. Soc., 1946, p. 99.

132. _____, A Ring in Graph Theory, Proc. Cambridge Philos. Soc., V. 43 (1947), pp. 26-40.

133. _____, A Census of Planar Triangulations, Canadian J. Math., 1962, pp. 21-28.

134. _____, The Factors of Graphs, Canadian J. Math. 4 (1952), pp. 314-329.

135. _____, A New Branch of Enumerative Graph Theory, Bull. Amer. Math. Soc., 1962, pp. 500-504.

136. _____, Symmetrical Graphs and Coloring Problems, Scripta Mathematica, Bol. 25, No. 4, 1961, pp. 305-316.

137. Uhlenbeck, G. E., and Ford, G. W., Theory of Linear Graphs with Application to the Theory of the Virial Development of the Properties of Gases, Amsterdam: North Holland Publishing Company, 1962.

138. Whitney, H., Congruent Graphs and the Connectivity of Graphs, Amer. J. Math., 1932, pp. 150-168.

139. _____, Non-separable and Planar Graphs, Tans. Amer. Math. Soc., 1932, pp. 339-362.

140. _____, The Coloring of Graphs, Ann. of Math. (2) V. 33 (1932), pp. 688-718.

141. _____, A Set of Topological Invariants for Graphs, Amer. J. Math., V. 55 (1933), pp. 231-235.

142. Whitney, H., Planar Graphs, Fund. Math., 1933, p. 73.

143. Wilson, J. C., On the Transversing of Geometrical Figures, New York: Oxford University Press, 1905.

144. Wine, R. L. and Freund, J. E., On the Enumeration of Decision Patterns Involving n Means, Ann. M. Stat., 1957, pp. 256-259.

145. Zeide, B., Uber 4-und 5-chrome Graphen, Monatsh. Math., V. 62 (1958), pp. 212-218.

APPENDIX

INDEX

VITA

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Biographical:

Personal Data: Born in West Mineral, Kansas, December 3, 1930, the son of Jules M. and Flossie May Bruyr.

Education: Attended grade school in West Mineral, Kansas, Montrose, Colorado and Vallejo, California; graduated from Cherokee County Rural High School at Columbus, Kansas in 1948; received the Bachelor of Science degree from Kansas State College at Pittsburg, Kansas, with a major in mathematics, in 1951; received a Master of Science degree in mathematics from Kansas State College at Pittsburg, Kansas in 1955; completed requirements for the Doctor of Education degree at Oklahoma State University, Stillwater, Oklahoma, in May, 1965.

Professional Experience: Taught mathematics in high schools from 1951-1960 of which the last five years were at Cherokee County Rural High School at Columbus, Kansas; was an instructor of mathematics at Kansas State Teachers College, Emporia, Kansas, from 1960-1962; was a graduate assistant at Oklahoma State University in 1962 and a staff assistant from 1963-1965 while completing work for the doctorate.

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