SOME PROPERTIES OF THE CARTESIAN PRODUCT
OF TWO SETS OF POINTS

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PREFACE

In this paper some properties which occur in the special type of linear programming problem known as the transportation problem are explored in a generalized form. A self-contained sketch of this problem is included in Chapter II. In this investigation an important role is played by the property of unique representation of elements of an abstract Cartesian product of two sets of points. As part of the development of these properties a known result for finite Cartesian products is reformulated and investigated in terms of graphs of functions in an infinite Cartesian product which is also a topological space.

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CHAPTER I

INTRODUCTION

It is the purpose of this paper to present a generalized development of certain properties encountered in network flow problems and to further investigate some of these properties in terms of topological structure. Much of the theory for network flow is considered as part of graph theory. (1). However, the development here proceeds directly from a type of network flow problem in which material is allocated from sources to destinations in an optimal way. This problem is known generally as a transportation problem, and methods for its solution are widely treated in the field of linear programming, since it may be expressed as a special type of linear programming problem. (2). (3). (4). In Chapter II a well-known general solution method for a transportation problem is discussed to illustrate the notion of a basis for a transportation problem and to indicate how it can be used to obtain an optimal solution. Even though the basis presented has a direct correspondence with the basis for the associated linear programming problem, the treatment here does not require the structure of linear algebra used in the linear programming method. This simplified treatment results from special properties of the problem some of which are generalized in this study.

In the third chapter the setting of the transportation problem is generalized to a Cartesian product of two sets of points which are identifiable as the two index sets for the sources and destinations of the transportation problem. A network is defined on a general Cartesian product and various properties stemming from the transportation problem are investigated in terms of chains of arcs belonging to networks. One property introduced is that of cross connectivity which is a rudimentary form of the property known as spanning in a linear space. A basis is defined to be a minimal cross connected network on a Cartesian product and the existence of a basis on an arbitrary Cartesian product is established in Theorem 3.2. A number of other theorems are proved which indicate relations between chains, cross connectivity and bases. One important property of a basis is that of giving unique representations for elements of a Cartesian product. All proofs given for the theorems as well as the generalized definitions of the properties they involve are original with the author although some of these definitions and theorems have equivalents in terms of finite networks. Examples are included to illustrate all properties discussed. The known result for finite networks which appears in Theorem 3.9 shows that the number of routes joining sources and destinations in a basis for a transportation problem is one less than the number of sources and destinations. A reformulation of this property of a basis for a finite network suggests questions which are investigated in the fourth chapter.

In Chapter IV the elements of a Cartesian product which belong to a basis are represented as points of the graphs of two functions defined on the two factor sets. When considered with this representation the result of Theorem 3.9 shows that for a finite Cartesian product there is a common point on the graphs of the two functions which represent a basis. It is an extension of this property of a basis with this representation that is investigated for an infinite Cartesian product. In particular the Cartesian product is taken to be the topological space which is the product of the unit interval in the space of real numbers with itself. In this Cartesian product space the graphs of functions mapping the unit interval into itself are investigated with regard to being closed and connected. The main result obtained is that if the union of the graphs of two functions mapping the unit interval into itself is both closed and connected, then the two graphs have a common point. In this case the functions have the same domain and range sets. An open question is whether or not the theorem holds when the domain of one function is the range of the other.

CHAPTER II

SOME PROPERTIES OF THE TRANSPORTATION PROBLEM

In this chapter appears a sketch of the transportation problem. The description here suffices to illustrate several notions to be developed in succeeding chapters, but for a more complete and detailed discussion of this problem reference should be made to other treatments, e.g. Dantzig (2).

The classical Hitchcock (7) transportation problem was formulated originally as a problem in economics and more recently has been considered to be one of the larger class known as mathematical programming problems. The situation which gave rise to the transportation problem was that of distributing homogeneous material from several sources to numerous destinations. Here it was assumed that there were various routes joining sources to destinations to which material could be allocated at a constant cost per unit of material for a given route. The problem, then, was that of finding a set of allocations to various routes such that all requirements of supply and demand were satisfied and such that the total cost of the set of allocations which is the sum of the allocation costs for individual routes was a minimum cost.

For a more precise definition of a transportation problem (T) let there be given the sets of real numbers

$$\left\{ a_i : a_i \geq 0,\ i = 1,\ldots,m \right\},$$

$$\left\{ b_j : b_j \geq 0,\ j = 1,\ldots,n \right\},$$

and

$$\left\{ c_{ij} : i = 1,\ldots,m;\ j = 1,\ldots,n \right\},$$

where these sets represent, respectively, the number of units of material available at the sources, the number of units of material required at the destinations, and the costs per unit of material for transporting it on routes joining sources to destinations. Then determine a set of real numbers

$$\left\{ x_{ij} : i = 1,\ldots,m;\ j = 1,\ldots,n \right\},$$

representing allocations to routes, which satisfies the conditions

(i) $\sum_{j=1}^{n} x_{ij} = a_i$ for $i = 1,\ldots,m$,

(ii) $\sum_{i=1}^{m} x_{ij} = b_j$ for $j = 1,\ldots,n$,

(iii) $x_{ij} \geq 0$ for all i, j, $i = 1,\ldots,m$; $j = 1,\ldots,n$,

and for which the expression

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

is a minimum. An additional condition

$$\text{(iv)} \quad \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

may be assumed without losing generality.

As indicated earlier this problem is one of the class of linear programming problems and may be solved with the simplex method of Dantzig (2). However the special conditions (i) through (iv) above permit a simplified method for the solution of (T). In describing (T) and in following its solution process it is convenient to use the matrix tableau of Figure 1. A position (i,j) of this matrix represents a

	1	2	. . .	n	
1	x_{11} c_{11}	x_{12} c_{12}	. . .	x_{1n} c_{1n}	a_1
2	x_{21} c_{21}	x_{22} c_{22}	. . .	x_{2n} c_{2n}	a_2
⋮	⋮	⋮		⋮	⋮
m	x_{m1} c_{m1}	x_{m2} c_{m2}	. . .	x_{mn} c_{mn}	a_m
	b_1	b_2	. . .	b_n	

Matrix Tableau.

Figure 1.

route joining the i'th source and the j'th destination. The values c_{ij} and x_{ij} of the cost and allocation, respectively, associated with route (i,j) are indicated by entries at that position. The sums of the x_{ij}'s in rows are the a_i's and the b_j's are the sums of the x_{ij}'s in columns.

The solution process is described by first determining a subset R of the set of all routes joining sources and destinations such that R has both of the two following properties designated as property F and property B. The first of these is known as feasibility and belongs to a subset of routes if allocations assigned to routes of the subset satisfy conditions (i) through (iv) when all other allocations assigned to routes not belonging to the subset have the value zero. And the second is the property of being a basis and belongs to a subset of routes if every route (i,j) in the matrix tableau has a unique representation as a finite sequence of routes (matrix positions) such that the sequence has the form $((i_1,j_1),(i_2,j_1),(i_2,j_2),(i_3,j_2),(i_3,j_3),\ldots,(i_k,j_{k-1}),(i_k,j_k))$ where $i_1 = i$, $j_k = j$, and every route of the sequence belongs to the subset.

Next in the solution process the cost of every route not belonging to the set R is compared with a cost evaluation of the unique representation of the route in terms of routes of R. Then if there exists some route whose cost is less than that of its representation, an allocation is assigned to this route and previous allocations assigned to routes in its representation in terms of R are adjusted to yield a new set R' of routes which has properties F and B. This procedure is repeated until a set R_o of routes is obtained for which every route's cost is not less than that of its representation in terms of R_o. Then the values of the x_{ij}'s associated with routes of R_o give an optimal solution of (T) with the x_{ij}'s for routes not belonging to R_o having values of zero.

The steps of this procedure are illustrated in the sample problem of Figure 2. The matrix positions which contain entries in their upper left-hand corners represent routes of a set R which has properties F and B. Property F for this set is easily verified directly, and property B can be shown by consideration of the equivalent linear programming problem as in Gass (3), p. 193ff.. It is necessary that the set R contain m + n - 1 elements, or six in the case of the sample problem.

	1	2	3	4	
1	$\boxed{6}$ + 5	$\boxed{4}$ - 2	2	4	10
2	7	$\boxed{3}$ + 8	$\boxed{4}$ 5	$\boxed{2}$ - 5	9
3	2	4	6	$\boxed{6}$ + 3	6
	6	7	4	8	

Sample Problem.

Figure 2.

The "northwest corner rule", as described in Gass (3) on pp. 196-198, was used to obtain the initial allocations shown in Figure 2. According to this rule $x_{11} = \min \{a_1, b_1\}$ and either the row or the column corresponding to a_1 or b_1, respectively, has its equality in (i) or (ii) satisfied by x_{11} having the value given by this rule and the remaining x_{ij}'s in that equality having values of zero. Then the larger of a_1 and b_1 is reduced by x_{11}, (1,1) is included in R, and a similar allocation is assigned to the x_{ij} for the row and column of least index which contain some x_{ij} whose value is not assigned. This procedure is continued from the "northwest corner" to the "southeast corner" where the final allocation x_{mn} is assigned. The resulting set of x_{ij}'s satisfies (i) through (iv) and an associated set R of routes is obtained that has properties F and B.

The sequence of routes (matrix positions) of R which represents a given route can be obtained by finding a sequence of indices starting with the row index of the given route and ending with its column index such that the sequence elements alternate between row and column indices and such that matrix elements identified by adjacent indices of the sequence belong to R. This is illustrated in Figure 2 in which route (3,1) is represented by the sequence ((3,4), (2,4),(2,2),(1,2),(1,1)) belonging to R. It is observed that the combined effect of allocation increases of one unit to routes (3,4),(2,2) and (1,1), and allocation decreases of one unit to routes (2,4) and (1,2) is just that of an allocation increase of one unit to route (3,1). This can be expressed as

(a) $1_{(3,1)} = 1_{(3,4)} - 1_{(2,4)} + 1_{(2,2)} - 1_{(1,2)} + 1_{(1,1)}$

or equivalently as

(b) $1_{(3,1)} - 1_{(3,4)} + 1_{(2,4)} - 1_{(2,2)} + 1_{(1,2)} - 1_{(1,1)} = 0$

where expression (b) shows that for such unit changes of allocations conditions (i), (ii) and (iv) are satisfied. Determining costs for each of these modes of allocating and comparing gives

$$c_{31} = 2,$$

$$\begin{aligned} c_{34} - c_{24} + c_{22} - c_{12} + c_{11} &= 3 - 5 + 8 - 2 + 5 \\ &= 9 \end{aligned}$$

and

$$c_{31} < 9.$$

Since the cost of route (3,1) is less than that of its representation in terms of R, an allocation x_{31} to route (3,1) can be made and the allocations to routes in its representation adjusted such that conditions (i) and (ii) are satisfied and the value of the expression

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}$$

is reduced. For adjusted allocations to satisfy condition (iii) the value x_{31} is the smallest value of an allocation to a route upon which $l_{(3,1)}$ has positive dependence in expression (a). In the sample problem this is $x_{22} = 3$ for route (2,2). Making the appropriate adjustments to the allocations by respectively decreasing and increasing by three units those allocations upon which $l_{(3,1)}$ has positive and negative dependence in expression (a) gives

Allocation	x_{11}	x_{21}	x_{22}	x_{23}	x_{24}	x_{34}	x_{31}
Old Value	6	4	3	4	2	6	0
New Value	3	7	0	4	5	3	3

and a new set R' of routes is obtained which consists of the routes of R with the exception that route (2,2) has been replaced by route (3,1).

The new set R' of routes also has properties F and B, and the procedure may be repeated until an optimal set R_o of routes is obtained. This will not be done here since the purpose of this section has been to illustrate several notions which occur in the solution process of a transportation problem. In particular the concepts involved in property B illustrated here are to be investigated and developed in a more general setting.

CHAPTER III

SOME PROPERTIES OF NETWORKS ON CARTESIAN PRODUCTS

In the preceding chapter a notion was described whereby a subset of a set had the property of being a minimal subset in terms of which every element of the set had a unique representation. This concept is related notably to that of a linear basis in the treatment of linear spaces in which algebraic field structure is required to formulate the notion of linear independence. (6). The present chapter is concerned with a development in which a type of basis is defined in the general setting of a Cartesian product and then some theorems pertaining to various properties of this basis are proved.

The generalization here proceeds from the setting of the previous chapter in which a matrix representation of a network was illustrated. A network, linear graph or graph is usually defined to be a system consisting of a collection of points (generally finite) called nodes or vertices together with branches or arcs joining various pairs of nodes or vertices. (1). If the branches are not directed, the term "edge" is frequently used instead of "arc" which implies the existence of an order between the pair of vertices which are its end points. Although for some of the considerations in this paper an order of this type is not necessary, the term "arc" will be used uniformly in the treatment here to mean an ordered pair of vertices with order being emphasized for those situations in which it is essential. There will also be occasions when functional relationships between two sets are considered as well as the point sets which are the graphs of these relationships. In view of these preliminary remarks the following general definitions are made and the particular terminology will be selected for the situation being studied.

Definition 3.1. Let A and B be two non-empty sets and let $A \times B$ denote the Cartesian product $\{(a,b) : a \in A$ and $b \in B\}$ of A and B. The sets A and B are assumed to be disjoint unless stated otherwise, in which case the common elements of A and B are distinguished to indicate membership in A or B. A network (binary relation, graph) on $A \times B$ is defined to be a system consisting of the sets A and B and a subset of $A \times B$. The sets A and B are called, respectively, the first and second factor sets and their elements are called vertices. The ordered pairs of vertices which are the elements of $A \times B$ are called arcs.

According to this formal definition an arc is a point in a Cartesian product. However, it may be that for such a point, which is an ordered pair of vertices, the vertices may be identified with two points of some topological space in which there exists a topological arc from the first point to the second point. In the case of an electrical network an arc of this type may represent a wire or connection joining two terminals. But in the case of a binary relation on two sets an arc just indicates that the first vertex is related to the second. In the case of a graph of a function an arc is a point in the Cartesian product of the domain and range sets and indicates that a pair of points from these sets are related by the function. For all of these situations the terms "vertices" and "arcs" will be used although the term "end point" will occasionally be used to mean a vertex.

Some definitions pertaining to networks are now given. These are essentially equivalent to similar definitions for finite networks as in Hadley (4) on pp. 284-291.

Definition 3.2. A chain in a network is defined to be a finite sequence of arcs $((a_1,b_1),\ldots,(a_k,b_k))$ belonging to the network such that either $a_i = a_{i+1}$ or $b_i = b_{i+1}$, but not both, for $1 \leq i < i+1 \leq k$. The first and last arcs of a chain are called end arcs of the chain and two arcs (a_i,b_i) and (a_j,b_j) are said to be adjacent arcs of the chain if $|i - j| = 1$. A vertex of an end arc of a chain which does not belong to an adjacent arc of the chain is called an end vertex or end point of the chain. Hence a chain has a first end point and a last end point belonging to the first and last end arcs, respectively. A vertex is said to belong to a chain if it belongs to some arc of the chain. A simple chain is a chain in which a vertex belongs to two arcs only if the arcs are two adjacent arcs of the chain. A chain is said to be a <u>closed chain</u> if the first end point and the last end point of the chain are the same point, and a <u>simple closed chain</u> is a closed chain in which each vertex of the chain belongs to exactly two arcs of the chain.

It is implicit in the definition of a chain that the chain is ordered from its first end point to its last end point and in this way order is assigned to its vertices and arcs. From the definitions of simple chains and simple closed chains the following remarks are immediate. Every simple closed chain contains an even number of arcs and contains at least four arcs. Each of the sequences $((a_1,b_1),(a_2,b_2),\ldots,(a_k,b_k))$ and $((a_2,b_2),\ldots,(a_k,b_k),(a_1,b_1))$ is a simple closed chain if and only if the other is. If the sequence of arcs $((a_1,b_1),(a_2,b_2),\ldots,(a_k,b_k))$ is a simple chain, then the reverse sequence $((a_k,b_k),\ldots,(a_2,b_2),(a_1,b_1))$ is also a simple chain.

<u>Example 3.1</u>. Let $A = \{a,c,e,g\}$ and $B = \{b,d,f\}$ and let N be the network on $A \times B$ consisting of A and B and the subset of $A \times B$ indicated in Figure 3. The chain $((g,d),(a,d),(a,f),(e,f),(e,b))$ is a simple chain with end points g and b. An example of a closed chain in the network N is the chain $((a,d),(c,d),(e,d),(e,f),(a,f))$. The chain $((a,d),(e,d),(e,f),(a,f))$ is a simple closed chain contained in the preceding closed chain. The chain $((e,b),(e,d),(a,d),(a,f))$ is an example in network N of a simple chain both end points of which belong to the same factor set.

B					
	b		(c,b)	(e,b)	
	d	(a,d)	(c,d)	(e,d)	(g,d)
	f	(a,f)		(e,f)	
		a	c	e	g
			A		

Network on $A \times B$.

Figure 3.

<u>Theorem 3.1</u>. In a network every chain with distinct end points contains a simple chain with the same end points.

<u>Proof</u>. Let $C = ((a_1,b_1),\ldots,(a_k,b_k))$ be a chain in a network on $A \times B$ with end points x and y belonging to the first arc (a_1,b_1) and last arc (a_k,b_k), respectively. Let i_1 be the largest index for $1 \leq i_1 \leq k$ such that x is an end point of the arc (a_{i_1},b_{i_1}) belonging to the chain C and let z_1 be the other vertex of arc (a_{i_1},b_{i_1}). If $z_1 = y$, then $((a_{i_1},b_{i_1}))$ is a simple chain in C and has the same end points as C. If $z_1 \neq y$, then let i_2 be the largest index for $i_1 < i_2 \leq k$ such that z_1 is a vertex of the arc (a_{i_2},b_{i_2}) belonging to the chain C and let z_2 be the other vertex of (a_{i_2},b_{i_2}). If $z_2 = y$, then $((a_{i_1},b_{i_1}),(a_{i_2},b_{i_2}))$ is a simple chain in C and has the same end points as C. If $z_2 \neq y$, then this procedure is repeated at most k times until an index i_n is found such that z_{i_n} is an end point of

the arc (a_{i_n}, b_{i_n}) belonging to C such that $z_{i_n} = y$. By construction the sequence $((a_{i_1}, b_{i_1}), \ldots, (a_{i_n}, b_{i_n}))$ is a simple chain contained in C having the same end points as C.

Chains may be considered as being a sequence of relationships between vertices belonging to them. For example the simple chain ((g,d),(a,d),(a,f),(e,f),(e,b)) of Example 3.1 could indicate that the vertex g is related to vertex d, which in turn is related to f, f to e, and e to b. Thus g is related to b by this sequence of relationships. In simple chains this sequence of relationships is unique. For simple closed chains a vertex is related to itself through a single cycle of relationships since the first and last end points of these chains are the same point. A type of network in which every distinct pair of vertices is related by a finite sequence of relationships is defined next.

Definition 3.3. A network N is said to be cross connected if for every pair of distinct vertices of the network, there exists a simple chain belonging to N such that each vertex of the pair is an end point of this simple chain. The two vertices may belong to the same or to different factor sets.

Definition 3.4. A subnetwork S of a network N on $A \times B$ is defined to be a system consisting of subsets A_1 and B_1 of A and B, respectively, together with a subset of the set of arcs belonging to N provided S is a network on $A_1 \times B_1$.

Example 3.2. Referring to the network in Figure 3 it is seen that the subnetwork S consisting of the sets $A_1 = A - \{g\}$ and $B_1 = B - \{f\}$ and the subset $\{(a,d),(c,b),(e,b)\}$ of $A_1 \times B_1$ is not cross connected because for the vertices a and b of S there does not exist a simple chain in S having these vertices as end points. However, if the arc (e,d), for example, is included with S, then the resulting system is a cross connected network on $A_1 \times B_1$.

If a network N on $A \times B$ includes the entire set $A \times B$, then it is apparent that N is a cross connected network. One necessary requirement for a network to be cross connected is that for every vertex of the network there exists in the network an arc containing the vertex. For otherwise there is some vertex not belonging to any arc of the network and for such a vertex there exists no simple chain in the network having this vertex for one of its end points. On the other hand if for every vertex there are many arcs to which it belongs, as in the case of a network N on $A \times B$ which includes the set $A \times B$ and in which neither A nor B is degenerate, then there may be many simple chains in the network joining a given pair of distinct vertices. A network in which a unique simple chain joins each ordered pair of distinct vertices is described next.

Definition 3.5. A basis for $A \times B$ is defined to be a cross connected network on $A \times B$ that is minimal with respect to being cross connected.

Example 3.3. The network shown in Figure 4 is cross connected since for every pair of distinct vertices of the network there is a simple chain belonging to the network such that the vertices of this pair are the end points of the simple chain. If an ordered pair of vertices is one of

a	(a,1)			(a,4)
b		(b,2)		(b,4)
c	(c,1)		(c,3)	
	1	2	3	4

Basis for a Cartesian Product.

Figure 4.

the arcs of this network, then the simple chain consisting of that single arc is a simple chain belonging to the network such that each of the vertices of the pair is an end point of this simple chain. The network in the figure is minimal with respect to being cross connected because for every network on this Cartesian product whose set of arcs is a proper subset of those in the figure there is some pair of vertices which are not the end points of a simple chain contained in the network. Hence, the network in the figure is a basis for that Cartesian product.

The following theorem establishes the existence of a basis for the Cartesian product of two non-empty sets.

Theorem 3.2. If $A \times B$ is the Cartesian product for two non-empty sets, then if (a,b) belongs to $A \times B$, the network N consisting of A and B together with the subset $(\{a\} \times B) \cup (A \times \{b\})$ of $A \times B$ is a basis for $A \times B$.

Proof. If x and y are two distinct vertices belonging to A or B, then by considering the possible cases, it is shown that x and y are the end points of one of the following simple chains belong to N:

(i) $((x,b),(y,b))$,

(ii) $((x,b),(a,y))$,

(iii) $((a,x),(a,y))$,

(iv) $((x,b),(a,b),(a,y))$,

(v) $((a,b))$.

Hence the network N on $A \times B$ is cross connected.

Suppose that N properly contains a cross connected subnetwork S on $A \times B$. Then there exists an arc (c,d) belonging to $N - S$. If $(c,d) = (a,b)$ and if $x \in A$ and $y \in B$, then a simple chain in S having x and y for its end points has the form $((x,b),(a,y))$ and either $x = a$ or $y = b$ and (a,b) belongs to S which is a contradiction. If $(c,d) \neq (a,b)$, then either $c \neq a$ or $d \neq b$ and there exists no arc belonging to S such that either d or c, respectively, is a vertex belonging to it because every arc belonging to S is of the form (x,b) or (a,x). Consequently S must be empty and it follows that the network N is a basis for $A \times B$.

The next theorems establish the uniqueness of simple chains from one vertex to another when the arcs of the simple chains belong to a basis and they also indicate the relationships between a basis for a Cartesian product, simple chains, and simple closed chains.

<u>Theorem 3.3</u>. A cross connected network N on $A \times B$ contains no simple closed chain if and only if for every pair of distinct vertices x and y of N there exists a unique simple chain belonging to N for which the vertices x and y are, respectively, the first and last end points.

<u>Proof</u>. First let every pair of distinct vertices of N have a unique simple chain belonging to N for which these vertices are the first and last end points in some order. Suppose that there exists a simple closed chain $((a_1,b_1), \ldots,(a_k,b_k))$ belonging to N and let x be the vertex which is both the first end point and the last end point of this simple closed chain. Let y be the other vertex belonging to arc (a_1,b_1). Then the chains $((a_2,b_2),\ldots,(a_k,b_k))$ and $((a_1,b_1))$ each are simple chains belonging to N from y to x contradicting the assumption of uniqueness for such chains. Therefore, N contains no simple closed chain.

Next, let a cross connected network N contain no simple closed chain and suppose $((a_1,b_1),\ldots,(a_k,b_k))$ and $((u_1,v_1), \ldots,(u_n,v_n))$ are two distinct simple chains in N each having a pair of vertices x and y for its first and last end points, respectively. Then the vertex x belongs to both arcs (a_1,b_1) and (u_1,v_1). Let i_1 be the smallest index, $i_1 \leq \min\{k,n\}$, for which arcs (a_{i_1},b_{i_1}) and (u_{i_1},v_{i_1}) have only one common vertex and let i_2 be the smallest index greater than i_1, $i_1 < i_2 \leq \min\{k,n\}$, for which arcs (a_{i_2},b_{i_2}) and an arc (u_{i_3},v_{i_3}) of the simple chain $((u_1,v_1),\ldots,(u_n,v_n))$ have a common vertex. Then the chain $((a_{i_1},b_{i_1}),\ldots,(a_{i_2},b_{i_2}),$ $(u_{i_3},v_{i_3}),\ldots,(u_{i_1},v_{i_1}))$ in N has first and last end points which are the same point and each vertex of the chain belongs to exactly two arcs of the chain. Therefore, this chain is a simple closed chain in N. From this contradiction it follows that there do not exist two distinct simple chains in N having both the same first end points and the same last end points. And hence, for every distinct pair of vertices x and y of N there exists a unique simple chain belonging to N having x and y as the first and last end points, respectively.

This last theorem is illustrated by referring to Figure 4 where it is seen that there are no simple closed chains in the network indicated there. For the vertices b and c of this network the unique simple chain in the network having b and c for its first and last end points, respectively, is ((b,4),(a,4),(a,1),(c,1)). Likewise, for vertices 3 and a the chain ((c,3),(c,1),(a,1)) is the unique simple chain in the network having vertices 3 and a, respectively, for its first and last end points.

Theorem 3.4. If the network N on $A \times B$ is a basis for $A \times B$, then N contains no simple closed chain.

Proof. Suppose the basis N for $A \times B$ contains a simple closed chain $((a_1,b_1),\ldots,(a_k,b_k))$ and let z be the vertex belonging to each of the end arcs (a_1,b_1) and (a_k,b_k) of this simple closed chain. To show that $S = N - \{(a_1,b_1)\}$ is a cross connected network on $A \times B$ let x and y be a pair of distinct vertices belonging to N such that there exists a simple chain C contained in N from end point x to end point y and such that C contains the arc (a_1,b_1). If $C = ((a_1,b_1))$, then $((a_2,b_2),\ldots,(a_k,b_k))$ is a simple chain in S with end points x and y. On the other hand, if $C \neq ((a_1,b_1))$, then C is a simple chain $((u_1,v_1),\ldots,(u_i,v_i),(a_1,b_1),(u_{i+1},v_{i+1}),\ldots,(u_n,v_n))$ from end point x to end point y. The vertex z of the simple closed chain belongs to either (u_i,v_i) or (u_{i+1},v_{i+1}) since it is a vertex of (a_1,b_1). Suppose that z belongs to (u_i,v_i). (The argument for the other case where z belongs to (u_{i+1},v_{i+1}) is similar.) Then replacing the arc (a_1,b_1) by the sequence $((a_2,b_2),\ldots,(a_k,b_k))$ in the sequence of C gives the sequence of arcs $C' = ((u_1,v_1),\ldots,(u_i,v_i),(a_2,b_2),\ldots,(a_k,b_k),(u_{i+1},v_{i+1}),\ldots,(u_n,v_n))$ in S and in which adjacent arcs have at least one common vertex. Let C_0' be the subsequence of C' which is obtained by successively deleting one arc of adjacent arcs which have both vertices in common, i.e. by replacing an arc which is repeated consecutively in C' by a single arc of the repeated set. Then C_0' is a chain in

S from end point x to end point y and by Theorem 3.1 C_0' contains a simple chain C_0 with end points x and y. Since in either of the above cases a simple chain is contained in S, it follows that N is not minimal with respect to being cross connected. From this contradiction the conclusion of the theorem follows.

Theorem 3.5. If N is a cross connected network on $A \times B$ such that for every pair of distinct vertices x and y of N there exists a unique simple chain belonging to N from end point x to end point y, then N is a basis for $A \times B$.

Proof. Suppose N satisfies the hypothesis of the theorem but is not a basis for $A \times B$. Then N is not minimal with respect to being cross connected and properly contains a cross connected subnetwork S on $A \times B$. Let (x,y) belong to $N - S$. There exists a simple chain $((a_1,b_1),\ldots,(a_k,b_k))$ in S with end points x and y. But this simple chain and the simple chain $((x,y))$ are distinct simple chains in N from end point x to end point y. This contradiction shows that N is a basis for $A \times B$.

The results of the three preceding theorems can be combined to give the following results.

Theorem 3.6. The following properties of a cross connected network N on $A \times B$ are equivalent:

(i) N is a basis for $A \times B$.

(ii) N contains no simple closed chain.

(iii) For every pair of distinct vertices x and y of N there exists a unique simple chain from end point x to end point y belonging to N.

To show that every cross connected network on a Cartesian product of two non-empty sets contains a subnetwork which is a basis for the Cartesian product the following form of Zorn's lemma is used.

Theorem 3.7. (Zorn's lemma). If the partially ordered set S is such that every linearly ordered subset of S has an upper bound, then S has at least one maximal element.

Theorem 3.8. Let A and B be two non-empty sets. Then every cross connected network on $A \times B$ contains a basis for $A \times B$.

Proof. Let N be a cross connected network on $A \times B$ and let S be the set of all subnetworks E of N such that if E is a network on $C \times D$ where $C \subset A$ and $D \subset B$, then E is a basis for $C \times D$. Assuming neither A nor B is empty, then N is not empty, so there exist an arc (a,b) belonging to N, two sets $C = \{a\}$ and $D = \{b\}$, and a basis E for $C \times D$ consisting of C, D and $\{(a,b)\}$. Hence S is not empty. Let a basis E_1 for $C_1 \times D_1$ be defined to be $\leq$ a basis E_2 for $C_2 \times D_2$ in S if E_1 is a subnetwork of E_2. Then S is partially ordered by $\leq$.

Let $K = \{E_t \text{ on } C_t \times D_t : t \in T\}$ be a linearly ordered subset of S. Then if $C = \bigcup_{t \in T} C_t$ and $D = \bigcup_{t \in T} D_t$ and if E is the system consisting of C and D and the set of arcs $\{(a,b):$ $(a,b) \in E_t$ and $t \in T\}$, it is seen that E is a subnetwork of N. The following argument shows that E is a basis for $C \times D$ and belongs to S, so every linearly ordered subset of S has an upper bound. E is cross connected because if x and y are distinct vertices belonging to C or D, then they belong to some C_{t_x} or D_{t_x} and C_{t_y} or D_{t_y}, respectively, and hence to some C_o or D_o where C_o contains C_{t_x} and C_{t_y} and D_o contains D_{t_x} and D_{t_y}. It follows that there exists a simple point chain belonging to E_o, and hence E, such that each of the vertices x and y is an end point of this simple chain.

Suppose, now, that E is not a basis for $C \times D$. This means that E is not minimal with respect to being a cross connected network on $C \times D$, so for some arc (a,b) belonging to E the network $E - \{(a,b)\}$ is a cross connected network on $C \times D$. Now, for some index $t \in T$, (a,b) belongs to E_t. Since E_t is a basis for $C_t \times D_t$, there is a pair of distinct vertices x and y belonging to C_t or D_t for which no simple chain in $E_t - \{(a,b)\}$, hence in any $E_{t_1} - \{(a,b)\}$ where $E_{t_1} \leq E_t$ in K, has end points x and y. Suppose x and y are the end points of a simple chain in $E_{t_2} - \{(a,b)\}$ for some E_{t_2} belonging to K such that $E_t \leq E_{t_2}$ and $E_t \neq E_{t_2}$. This chain can not be the same as the simple chain in E_t having end points x and y. But both of these distinct simple chains having end points x and y belong to E_{t_2} contradicting the fact that E_{t_2} is a basis for $C_{t_2} \times D_{t_2}$. Thus, E is minimal with respect to being cross connected and hence is a basis for $C \times D$. Therefore, E belongs to S and is an upper bound for K.

By Zorn's lemma S has a maximal element M on $A_1 \times B_1$ where $A_1 \subset A$ and $B_1 \subset B$ and M is a basis for $A_1 \times B_1$. Suppose there exists an arc (a,b) belonging to N and not contained in $A_1 \times B_1$. Then either $a \in A - A_1$ or $b \in B - B_1$ and there exists an arc (a_1,d) or (c,b_1) belonging to N where $a_1 \in A_1$, $d \in B - B_1$, $c \in A - A_1$ and $b_1 \in B_1$. For otherwise there would be no simple chain belonging to N one end point of which is a vertex of A_1 or B_1 and the other end point a vertex of $A - A_1$ or $B - B_1$, contradicting the hypothesis that N is a cross connected network on $A \times B$. It follows that either $M \cup \left\{(c,b_1)\right\}$ or $M \cup \left\{(a_1,d)\right\}$ is a basis for $(A_1 \cup \left\{c\right\}) \times B_1$ or $A_1 \times (B_1 \cup \left\{d\right\})$, respectively. Since either of these belongs to S and is $\geq M$, the existence of either one contradicts the fact that M is a maximal element for S. Therefore, $A_1 = A$, $B_1 = B$ and M is a basis for $A \times B$ contained in N.

As a particular situation involving properties (i) and (iii) of Theorem 3.6 every element (a,b) of the Cartesian product $A \times B$ has a unique representation in terms of the elements which belong to a basis for $A \times B$. The number of arcs required for a basis giving this unique representation is considered for the finite case in the next theorem. First a lemma is established to aid in the proof the theorem.

Lemma 3.1. If a network N is a basis for the Cartesian product of two non-empty finite factor sets, then there exists a vertex of N which belongs to exactly one arc of N.

Proof. Let A and B be two non-empty finite sets and let N be a basis for $A \times B$. Since N is a cross connected network on $A \times B$, for every vertex of N there exists at least one arc belonging to N and containing that vertex. Suppose that for every vertex of N there exist at least two arcs belonging to N which contain the vertex. Let x_1 be a vertex of N. Then there exists a simple chain C_k of the form $((a_1,b_1),\ldots,(a_k,b_k))$ belonging to N for which x_1 is the first end point since N is cross connected. If x_k is the end point of C_k contained in arc (a_k,b_k), then there exists another arc (a_{k+1}, b_{k+1}) belonging to N for which x_k is a vertex. Either the chain $C_{k+1} = ((a_1,b_1),\ldots,$ $(a_k,b_k),(a_{k+1},b_{k+1}))$ is a simple chain or the end point of C_{k+1} belonging to (a_{k+1},b_{k+1}) is a vertex belonging to C_k and hence C_{k+1} contains a simple closed chain. The process of successively adding arcs belonging to N to simple chains of N in the former case may be repeated until the latter case holds since N is finite and hence N contains a simple closed chain. By Theorem 3.6 this is a contradiction because N is a basis for $A \times B$. Consequently there exists a vertex belonging to N which belongs to exactly one arc in N.

Theorem 3.9. If $A = \{a_i: i = 1,\ldots,m\}$ and $B = \{b_j:$ $j = 1,\ldots,n\}$ are two non-empty finite sets, then a cross connected network N on $A \times B$ is a basis for $A \times B$ if and only if N contains $m + n - 1$ arcs.

Proof. Let N be a cross connected network on $A \times B$ and first suppose that N is a basis for $A \times B$. From Lemma 3.1 there exists some vertex x of N which belongs to exactly one arc (a,b) belonging to N. The subnetwork of N which consists of all vertices and arcs of N except x and (a,b) is a basis for the associated Cartesian product. With this result a monotone decreasing sequence $\{N_k \text{ on } A_k \times B_k\}$ of subnetworks of N is obtained such that either $A_{k+1} = A_k$ or $B_{k+1} = B_k$, but not both, and $B_{k+1} = B_k - \{b_{j_k}\}$ or $A_{k+1} = A_k - \{a_{i_k}\}$, respectively, where b_{j_k} or a_{i_k} is a vertex belonging to exactly one arc contained in N_k. Also N_k is a basis for $A_k \times B_k$. Let $N_k(L)$ denote a basis N_k containing L vertices. Then if $N_o = N$, the sequence above is $N_o\ (m+n) \supset N_1\ (m+n-1) \supset \ldots \supset N_{m+n-2}(2)$ which terminates with one arc joining two vertices in the final network since a non-empty network contains at least two vertices. Since only one arc belongs to N_k and not to N_{k+1}, it follows that the basis $N_o = N$ contains $m + n - 1$ arcs.

Next let a cross connected network N on $A \times B$ contain $m + n - 1$ arcs. An inductive procedure is used to show that N is a basis for $A \times B$. For $k = 1$ let $a_{i_1} \in A$. Since N is cross connected, there exists an arc (a_{i_1}, b_{i_1}) belonging to N for which a_{i_1} is a vertex. Let S_1 denote the subnetwork of N consisting of the subsets $A_1 = \{a_{i_1}\}$ of A and

$B_1 = \{b_{i_1}\}$ of B together with the subset $\{(a_{i_1},b_{i_1})\}$ of $A_1 \times B_1$. By Theorem 3.6 the network S_1 is a basis for $A_1 \times B_1$ since it is cross connected and contains no simple closed chain. Suppose S_k is a subnetwork of N consisting of $k + 1$ vertices of the subsets A_k of A and B_k of B together with the subset of arcs $\{(a_{i_1},b_{i_1}),\ldots,(a_{i_k},b_{i_k})\}$ of $A_k \times B_k$ such that S_k is a basis for $A_k \times B_k$. If $k = m + n - 1$, then $S_k = N$ and N is a basis for $A \times B$. If $k < m + n - 1$, then there exist a vertex $x \in N - S_k$, a simple chain belonging to N having x and a vertex belonging to S_k for its end points, and an arc $(a_{i_{k+1}},b_{i_{k+1}})$ one of whose vertices belongs to S_k and the other vertex y belongs to $N - S_k$. If $y \in A$, let $A_{k+1} = A_k \cup \{y\}$. Otherwise $y \in B$ and in this case let $B_{k+1} = B_k \cup \{y\}$. Let S_{k+1} be the subnetwork of N consisting of $k + 2$ vertices of the subsets A_{k+1} of A and B_{k+1} of B together with the subset of arcs $\{(a_{i_1},b_{i_1}),\ldots,(a_{i_{k+1}},b_{i_{k+1}})\}$ of $A_{k+1} \times B_{k+1}$. The network S_{k+1} is cross connected since the vertex y is an end point of a simple chain in S_{k+1} whose other end point belongs to $S_k \subset S_{k+1}$ and hence y is the end point of a simple chain in S_{k+1} whose other end point is some other vertex belonging to S_{k+1}. Since there is a single arc $(a_{i_{k+1}},b_{i_{k+1}})$ in S_{k+1} to which the vertex y belongs, this arc is contained in no simple closed chain in S_{k+1}. And since S_k is a basis and contains no simple closed chain, it follows that S_{k+1} contains no simple closed chain and therefore S_{k+1} is a basis for $A_{k+1} \times B_{k+1}$ by Theorem 3.6.

This procedure is continued until a subnetwork S_{m+n-1} contained in N is obtained such that S_{m+n-1} consists of m + n vertices of the subsets A_{m+n-1} of A and B_{m+n-1} of B together with the subset of m + n - 1 arcs $\{(a_{i_1}, b_{i_1}), \ldots, (a_{m+n-1}, b_{m+n-1})\}$ of $A_{m+n-1} \times B_{m+n-1}$ on which S_{m+n-1} is a basis. Hence $S_{m+n-1} \supset N$ and since $S_{m+n-1} \subset N$, it follows that $S_{m+n-1} = N$ and, therefore, that N is a basis on $A \times B$.

From Theorem 3.6 it is seen that for every pair of distinct vertices x and y of a network on a Cartesian product $A \times B$ such that the network is a basis for $A \times B$ there exists a unique simple chain in the network for which x and y are, respectively, the first and last end points. In particular it is noted that the elements of $A \times B$ are ordered pairs of distinct vertices and for every one of these elements there exists a unique representation in terms of the elements of $A \times B$, this representation being the unique simple chain in the basis whose first and last end points are, respectively, the first and second vertices of the ordered pair. It is seen that for representations of this type the simple chains contain an odd number of arcs and also that if a chain consists of an arc together with the arcs of its unique representation chain, then the chain so formed is a simple closed chain. The next two examples illustrate bases in both finite and infinite Cartesian products.

<u>Example 3.4</u>. Consider the group table of the cyclic group of order four in Figure 5. In this case the binary operation on the set $A = B = \{e, a, b, c\}$ gives a unique element of A for every ordered pair of elements of A. Let S be the sets A and B together with the subset of $A \times B$ indicated by the elements with asterisks in Figure 5. Since S is a cross connected network on $A \times B$ and contains $4 + 4 - 1 = 7$ elements, it is a basis for $A \times B$. The element (c,a), for example, has for its unique representation in S the simple chain $((c,c),(e,c),(e,a))$ since c and a are, respectively, the first and last end points of this simple chain whose arcs belong to S. And by using the group operation to identify unique elements of the group for ordered pairs of vertices, the unique simple chain for (c,a) is expressed uniquely as (b,c,a). It is observed that alternating direct and inverse relationships for the group operation in the two expressions for the unique representation and combining terms gives, respectively,

$$\begin{aligned}(c,c)(e,c)^{-1}(e,a) &= (cc)(ec)^{-1}(ea) \\ &= ccc^{-1}e^{-1}ea \\ &= ceea \\ &= ca \\ &= e\end{aligned}$$

and

$$\begin{aligned}bc^{-1}a &= baa \\ &= bb \\ &= e.\end{aligned}$$

In this manner every simple chain representation in terms of S for arcs of $A \times B$ can be transformed into an element of the group.

$\circ$	e	a	b	c
e	e	*a	b	*c
a	*a	b	c	e
b	b	*c	e	a
c	*c	e	*a	*b

(Rows: A; columns: B)

$A \times B$

Basis for a Group Table.

Figure 5.

Example 3.5. For an example of a basis for an infinite Cartesian product let $A = \{x: 0 \leq x\}$ and let $B = \{y: 0 \leq y\}$ so $A \times B$ is the closed positive quadrant of the plane. Let S be the system consisting of the sets A and B together with the subset of $A \times B$ which is the union of the graphs of $xy = 1$ and $y = 0$ restricted to $A \times B$. Then S is a basis for $A \times B$ and every point (arc) of $A \times B$ has a unique representation in terms of arcs of S. For if (x,y) is an arc of S, then the simple chain $((x,y))$ is the unique representation in S for (x,y). And if (x,y) is an element of $A \times B$ not belonging to S, then $((x,0),(1/y,0),(1/y,y))$ is the unique simple chain belonging to S representing (x,y).

The basis for a transportation problem, which was described in the preceding chapter as a set of routes having property B, was a finite basis of the same type as the one illustrated in Example 3.4. It is also noted that the arcs (routes) of simple chains in the transportation problem basis which were the unique representations of arcs belonging to the associated Cartesian product could be combined by using an operation somewhat analogous to the group operation of Example 3.4 to give single arcs. The development here stemming from the transportation problem has given notions of cross connectivity and of a basis which are defined in completely general Cartesian products. Some theorems proved in this chapter relate certain properties of these networks. In particular Theorem 3.9 shows that the number of arcs belonging to a basis for a finite Cartesian product is one less than the number of vertices contained in its two factor sets. It is an investigation of this property in terms of topological structure that follows in the next chapter.

CHAPTER IV

SOME PROPERTIES OF GRAPHS OF FUNCTIONS MAPPING THE UNIT INTERVAL INTO ITSELF

The property of Theorem 3.9 was seen to relate the number of vertices and the number of arcs in a basis for a finite Cartesian product. This property is reformulated in this chapter in a way that suggests some questions about similar properties for the Cartesian product space formed by the product of the unit interval in the space of real numbers with itself. For this investigation it is convenient to treat some of the subsets of this Cartesian product space as the graphs of functions mapping the unit interval into itself. Here it is recalled from the preceding chapter that the definition of a graph is the same as that of a network on a Cartesian product. In this discussion the functions involved will be single-valued functions and for them the expression "graph of a function" will be used to mean the subset of the Cartesian product of the domain and range sets which is determined by the function. Also for the purposes here a continuum will mean a closed and connected subset of a topological space.

In Lemma 3.1 it was shown that in a basis for the Cartesian product of two non-empty finite factor sets there exists a vertex belonging to exactly one arc of the basis. It was with this property that a monotone decreasing sequence $\{N_k \text{ on } A_k \times B_k\}$ of subnetworks of a basis N for $A \times B$ was obtained in Theorem 3.9 such that for consecutive terms of this sequence either $A_{k+1} = A_k$ or $B_{k+1} = B_k$, but not both, and $B_{k+1} = B_k - \{b_{j_k}\}$ or $A_{k+1} = A_k - \{a_{i_k}\}$, respectively, where b_{j_k} or a_{i_k} is a vertex belonging to exactly one arc of the basis N_k for $A_k \times B_k$. By this sequence there is induced on each of the factor sets A and B functions f and g, respectively, whose respective ranges are subsets of B and A. Thus, if a_{i_k} or b_{j_k} is the vertex of N_k belonging to exactly one arc $(a_{i_k}, b_{j_{i_k}})$ or $(a_{i_{j_k}}, b_{j_k})$, respectively, of N_k, then define $f(a_{i_k})$ to be $b_{j_{i_k}}$ or $g(b_{j_k})$ to be $a_{i_{j_k}}$, respectively. In this manner the $m + n$ vertices of N correspond one-to-one with the $m + n - 1$ arcs of N except that the two vertices of the final basis N_{m+n-2} of the sequence correspond to the single arc of this basis and every vertex in the domains of the functions f and g has an image in the domain of the other function. This shows that for a basis N for a Cartesian product of two non-empty finite factor sets A and B there exist two functions, f mapping A into B and g mapping B into A, having respective graphs $F = \{(x, f(x)) : x \in A\}$ and $G = \{(g(y), y) : y \in B\}$ in $A \times B$ such that the union of F and G is the subset of arcs of $A \times B$ which belong to N. With this formulation of the conditions of Theorem 3.9 it follows from that theorem that the number of arcs contained in N is one less than the number of vertices belonging to N and hence that there exists a single point belonging to both F and G. This is illustrated in the next example.

Example 4.1. Let two functions, f and g mapping D into A and A into D, respectively, have graphs whose union is indicated by the subset of arcs in the basis for $D \times A$ illustrated in Figure 6. It is seen in the figure that the point (arc) $(2,f(2)) = (g(d),d)$ is the single point belonging to both of the graphs.

A		1	2	3	4
	a	(1,f(1))		(g(a),a)	
	b	(g(b),b)			(4,f(4))
	c				(g(c),c)
	d		(2,f(2)) =(g(d),d)	(3,f(3))	

D

Basis as the Union of Graphs of Functions.

Figure 6.

It is noted that if one of the two functions mapping one factor set into the other is one-to-one, then it has an inverse function and in this case there exist graphs of two functions having the same domain and range sets such that the union of the two graphs is the subset of arcs which belong to a basis. The function g of Example 4.1 is one-to-one and hence has an inverse function g^{-1} whose graph in $D \times A$ is the same set of points as the graph of g. The example then illustrates that the graphs of f and g^{-1} have a single point in common.

The next two examples illustrate bases for infinite Cartesian product spaces in which the set of arcs of the basis is the union of the graphs of two functions one of which is defined on the first factor set with the second factor set for its range and the other is defined on the second factor set with the first factor set for its range. These examples show that it is not necessary for such graphs to have a common point if either the Cartesian product space is not compact or if the set of arcs belonging to the basis is not a continuum.

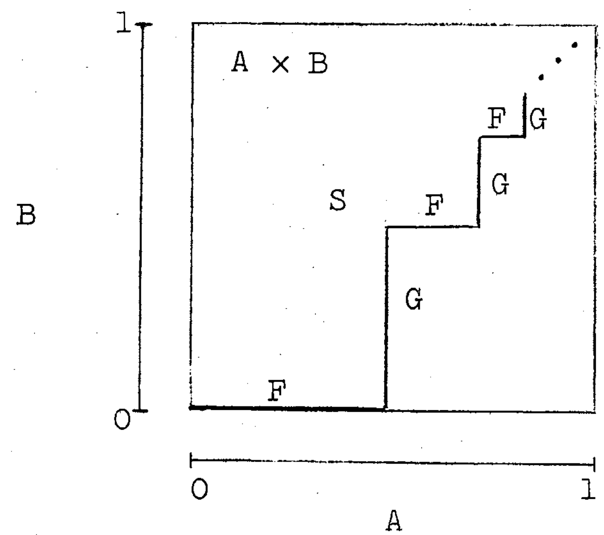
Example 4.2. Let A and B each be the half-open interval $[0,1)$ in the space of real numbers. The Cartesian product space $A \times B$ is not compact. A subset S of $A \times B$ is now defined by means of graphs of two functions, f mapping A into B and g mapping B into A. Let $[x]$ denote the greatest integer in x for a real number x. Then let

$$f(x) = [x/(1-x)] \;/\; (1 + [x/(1-x)]) \text{ for } x \in A$$

and let

$$g(y) = [1/(1-y)] \;/\; (1 + [1/(1-y)]) \text{ for } y \in B.$$

The respective graphs of these functions in $A \times B$ are $F = \{(x,f(x)): x \in A\}$ and $G = \{(g(y),y): y \in B\}$. Let $S = F \cup G$. The system S^* consisting of A, B and S is seen to be a basis for $A \times B$. In this example S is a relatively closed and connected subset of the continuum $A \times B$ and $F \cap G = \emptyset$. See Figure 7.



Basis for an Infinite
Cartesian Product Space.

Figure 7.

Example 4.3. Let A and B each be the closed unit interval [0,1] in the space of real numbers. Then the Cartesian product space $A \times B$ is a compact continuum. Let f and g be the corresponding functions of Example 4.2 extended to the closed unit interval such that $f(1_A) = 1_B$ and $g(1_B) = 0_A$. Refer to Figure 7. Then if F and G denote the respective graphs of f and g in $A \times B$, the union S of F and G together with A and B is seen to be a basis for $A \times B$. In this example S is a closed and hence compact subset of $A \times B$, but S is not connected and $F \cap G = \emptyset$.

The preceding examples suggest that if the union of graphs of the type illustrated forms a basis and if the union of these graphs is a compact subcontinuum of $I \times I$ where I is the closed unit interval [0,1] of real numbers, then the basis may have the property that two of its vertices correspond to the same arc or, equivalently, that there is a common point on the graphs of the two functions. This leads to the following question if the requirement that the union of the graphs forms a basis is omitted. If there are two functions mapping I into itself such that the range set of one is the domain set of the other and the union of their graphs is a subcontinuum of $I \times I$, do their graphs have a common point? The observation of Example 4.1 suggests that if one of the functions has an inverse function, then two functions having the same domain and range sets could be considered in the preceding question. A second question, then, arising from this is whether or not the graphs of two functions mapping a common domain I into itself have a common point if their union is a subcontinuum of $I \times I$. This last question is settled in the affirmative in Theorem 4.4. It is an investigation stemming from this question that follows. Some properties of connected graphs of functions mapping I into itself are developed in the next theorems.

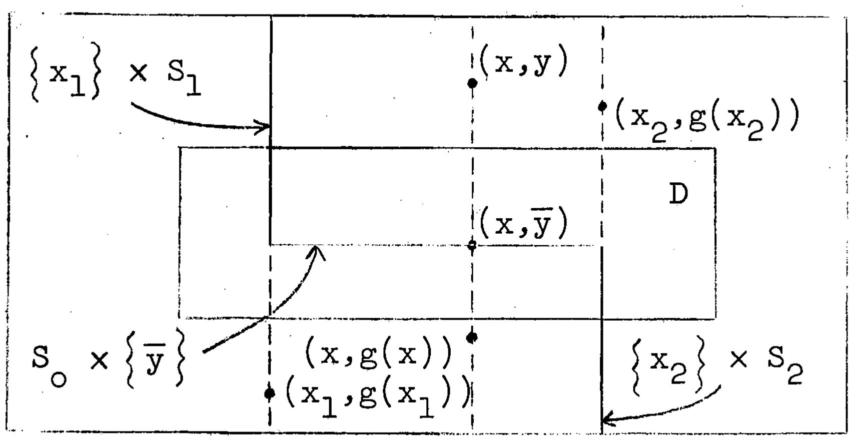
<u>Theorem 4.1</u>. Let a function f defined on the unit interval I in the space of real numbers and mapping I into itself have a graph F in I $\times$ I such that F is closed. Then F is a subcontinuum of I $\times$ I.

<u>Proof</u>. Let $\{x_n\}$ be a sequence in I converging to x_o. The limit x_o of this sequence belongs to I since I is closed. Every subsequence of the sequence $\{(x_n, f(x_n))\}$ in the graph F has a convergent subsequence with a limit in F of the form (x_o, y) since F is compact and every subsequence of $\{x_n\}$ converges to x_o. But since f is a single-valued function, it follows that $y = f(x_o)$ and hence that the sequence $\{(x_n, f(x_n))\}$ converges to $(x_o, f(x_o))$. Consequently $\{f(x_n)\}$ converges to $f(x_o)$ and f is continuous. By Theorem 4.21 of (5, p. 76) the subspace F of I $\times$ I is homeomorphic to I if and only if f is continuous, and by this homeomorphism it follows that F is a compact subcontinuum of I $\times$ I.

<u>Theorem 4.2</u>. Let g be a real valued function on a non-degenerate closed interval of the real numbers R and suppose its graph G in R $\times$ R is connected and not closed. Then G contains a limit point of $\overline{G}$ - G.

<u>Proof</u>. Since G is not closed, G is a proper subset of its closure $\overline{G}$, and since G is connected, $\overline{G}$ is connected. Let F = $\overline{G}$ - G. Since $\overline{G} = F \cup G$ is closed and connected, either G contains a limit point of F or F is closed. Suppose it is the latter condition which holds. Since F is non-empty, there exists a point $(x,y) \in F$ and it follows

that $(x,g(x)) \notin F$. Consider the closed subinterval L of $\{x\} \times R$ which has (x,y) and $(x,g(x))$ for its endpoints. The subset of this interval not contained in $F \cup G$ is non-empty and open (relatively) and contains a point $(x,\overline{y})$ not belonging to $F \cup G$. Since $(x,\overline{y})$ is not a limit point of the closed set $F \cup G$, there exists an open connected rectangular neighborhood $D = U \times V$ of $(x,\overline{y})$ such that $D \cap (F \cup G) = \emptyset$. Since (x,y) and $(x,g(x))$ are both limit points of G, there exist points x_1 and x_2 belonging to U such that $(x_1,\overline{y})$ and $(x_2,\overline{y})$ belong to D and $\overline{y}$ separates $g(x_1)$ and $g(x_2)$ in R. See Figure 8.



R × R

Illustration for Theorem 4.2.

Figure 8.

Let S_1 and S_2 be the closures of the components of $R - \{\overline{y}\}$ containing $g(x_2)$ and $g(x_1)$, respectively. And let

S_o be the closed subinterval of U which has x_1 and x_2 for its endpoints. Let S be the union of $\{x_1\} \times S_1$, $\{x_2\} \times S_2$, and $S_o \times \{\overline{y}\}$ in $R \times R$. Since g is a single-valued mapping, neither of the sets $\{x_1\} \times S_1$ and $\{x_2\} \times S_2$ contains a point of G. The set $S_o \times \{\overline{y}\}$ is a subset of D and hence contains no point of G. Therefore the set S contains no point of G and, furthermore, S separates $R \times R$ into two mutually separated subsets, one containing $(x_1,g(x_1))$ and the other containing $(x_2,g(x_2))$, whose union contains G. Since this contradiction follows from the assumption that F is closed, it must be that G contains a limit point of F.

The theorem just proved shows that if the graph G of a function mapping I into I is connected but not closed, then it contains a limit point of $\overline{G} - G$. The next theorem shows that the graph G of a function mapping I into I contains a point of every closed subset of $I \times I$ whose union with G is a subcontinuum of $I \times I$. The following lemma is needed for the proof of the theorem.

Lemma 4.1. Let L be a closed and bounded interval $[x_o,y_o]$ in the space of real numbers. If H is a decomposition of L such that the elements of H are subintervals of L, then there exists an element of H which is a closed subinterval of L.

Proof. Suppose that the collection H contains no member which is a closed subinterval of L. Then every member of H is a non-degenerate subinterval of L which does not contain both of its end points. Let $K = \{x : [x,y) \in H\}$. The set K is not empty since it contains x_0, and since K is bounded there exists a least upper bound $\overline{x}$ for K. It follows that $\overline{x} \in L$ since L is closed and bounded, and hence $\overline{x}$ belongs to some member H_1 of H. If $H_1 = [x_1,y_1)$ or if $H_1 = (x_1,y_1)$, then there exists an interval $[y_1,z)$ belonging to H, and hence $y_1 \in K$ and $y_1 > \overline{x}$ contradicting the fact that $\overline{x}$ is an upper bound for K. If $H_1 = (x_1,y_1]$ and $x \in K$, then $x < x_1 < \overline{x}$ which contradicts that $\overline{x}$ is a least upper bound for K. Therefore $\overline{x}$ is contained in no member of H which is not a closed subinterval of L. Consequently, H contains some closed, possibly degenerate, subinterval of L.

<u>Theorem 4.3</u>. If g is a mapping of the closed unit interval I in the space of real numbers into itself and G is the graph of g in $I \times I$, and if F is a non-empty closed subset of $I \times I$ such that the union of F and G is a subcontinuum of $I \times I$, then there exists a point belonging to $F \cap G$.

<u>Proof</u>. Suppose that $F \cap G = \emptyset$. Let $\{G_a : a \in A\}$ be the collection of components of G for some index set A where $a = b$ if and only if $G_a = G_b$. For every $a \in A$ let $H_a = \{x : x \in I \text{ and } (x,g(x)) \in G_a\}$, i.e. H_a is the projection of G_a in the domain of its first coordinate. Since a projection mapping is continuous, it follows that for every $a \in A$, the image H_a of G_a under the projection is a connected subset of I. Furthermore, the members of the collection

$H = \left\{H_a\colon a \in A\right\}$ are pairwise disjoint; for if $x \in H_a \cap H_b$, then $(x,g(x)) \in G_a \cap G_b$ and $a = b$. And since $I = \bigcup_{a \in A} H_a$, the collection H is a decomposition of I whose elements are connected subsets of I. From Lemma 4.1 there exists an element H_b of the collection H such that H_b is a subcontinuum of I.

With the assumption that $F \cap G = \emptyset$, the set $F = (F \cup G) - G$. The conditions (1) F is a closed proper subset of the continuum $F \cup G$, (2) $\overline{G}$ is compact, and (3) G_a is a component of G for every $a \in A$, satisfy the hypothesis for Theorem 51 of (8, p. 18) from which it follows that F contains a limit point of G_a for every $a \in A$. Hence for every $a \in A$ the component G_a of G is not closed. In particular the component G_b whose projection H_b is a subcontinuum of I is not closed. Since G_b is non-degenerate, it follows that H_b is a non-degenerate closed subinterval of I and, with respect to H_b, the conditions of Theorem 4.2 are satisfied. By that theorem G_b contains a limit point (x,y) of $\overline{G}_b - G_b$. Since $(\overline{G}_b - G_b) \subset F$, the point (x,y) belongs to $G_b \cap F$ which contradicts the assumption that $F \cap G = \emptyset$.

The following theorem concerns two functions each mapping the unit interval I into itself and shows that if the union of their graphs in $I \times I$ is a continuum, then for some point of their common domain I both functions have the same value.

Theorem 4.4. Let I be the unit interval [0,1] in the space of real numbers. If F and G are the graphs in $I \times I$

of two functions f and g each of which maps I into itself and if the union of F and G is a continuum, then $F \cap G \neq \emptyset$.

Proof. In the proof of this theorem a property is defined for certain subsets of $F \cup G$, and a subset of $F \cup G$ having this property is shown to be not irreducible with respect to this property if $F \cap G = \emptyset$. Also shown is that this property is inductive whereby a subset of $F \cup G$ is obtained which is irreducible relative to this property. Then it is for this subset that the condition $F \cap G = \emptyset$ gives a contradiction.

Let two functions f and g each mapping I into itself have graphs F and G, respectively. A non-empty subset of $F \cup G$ has property P if it is the union of the graphs of f and g restricted to a compact subinterval of I and is a compact continuum. Under the hypothesis of the theorem the set $F \cup G$ has property P, so the collection of subsets of $F \cup G$ having property P is non-empty. Next suppose that $F \cap G = \emptyset$ and let $[x_0,x_1]$ be the compact subinterval of I associated with a set K which has property P. Since $F \cap G = \emptyset$, the interval $[x_0,x_1]$ is non-degenerate. Otherwise $x_0 = x_1$ and the continuum K consists of $(x_0,f(x_0)) \cup (x_0,g(x_0))$ which must be a single point which belongs to $F \cup G$ which is a contradiction. For convenience of notation suppose that $f(x) < g(x)$ for $x_0 \leq x \leq x_1$. This results in no loss of generality since an interchange of points of F and G does not affect either $F \cup G$ or $F \cap G$ since $F \cap G = \emptyset$. The closed subinterval of $\{x_o\} \times I$ having end points $(x_o,f(x_o))$ and $(x_o,g(x_o))$ is nondegenerate and has a midpoint (x_o,y) which does not belong to K. See Figure 9.

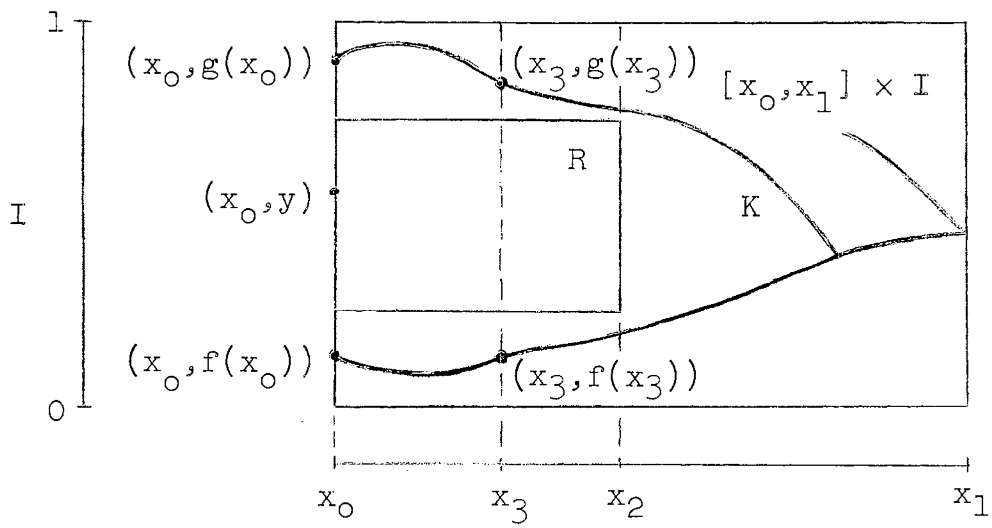


Illustration for Theorem 4.4.

Figure 9.

Since (x_o,y) is not a limit point of the closed set K, there exists a rectangular region $R = [x_o,x_2) \times (y_1,y_2)$ open in $[x_o,x_1] \times I$ where $y_1 < y < y_2$ such that $R \cap K = \emptyset$. If there exists an $x \in [x_o,x_2)$ such that $f(x) > y$ or $g(x) < y$, then for such an x the set $(\{x\} \times [0,y]) \cup ([x_o,x] \times \{y\})$ or the set $(\{x\} \times [y,1]) \cup ([x_o,x] \times \{y\})$, respectively, separates K in $[x_o,x_1] \times I$. From this contradiction it follows that for every $x \in [x_o,x_2)$ $f(x) < y < g(x)$ and that if $\{z_n\}$ is a sequence of points belonging to $[x_o,x_2)$ which converges to x, then the sequences $\{(z_n,f(z_n))\}$ and

$\{(z_n, g(zn))\}$ converge to $(x, f(x))$ and $(x, g(x))$, respectively, since K is compact. Consequently the functions f and g are continuous on $[x_o, x_2)$ and for every point $x \in (x_o, x_2)$, the graphs of the restrictions of f and g to $[x_o, x]$ are homeomorphic to the arc $[x_o, x]$ by a Theorem 4.21 of (5, p. 76). Let $x_3 \in (x_o, x_2)$. Then the points $(x_3, f(x_3))$ and $(x_3, g(x_3))$ are cut points of K and each of them separates K into two components. It follows from Theorem 60 of (8, p. 25) that the set $K_1 = K \cap \{[x_3, x_1] \times I\}$ is connected. Hence K_1 is a non-empty compact proper subcontinuum of K and has property P. Thus, the subset K is not irreducible with respect to property P.

Next let $K_1 \supset K_2 \supset \ldots \supset K_n \supset \ldots$ be a monotone decreasing sequence of non-empty compact continua every one of which has property P. Let D_n be the compact subinterval of I which is associated with K_n by property P for $n = 1, 2, \ldots$. Then $\{D_n\}$ is also a monotone decreasing sequence of non-empty compact continua. Let $K = \bigcap_n K_n$ and $D = \bigcap_n D_n$. By Theorem 9.4 of (9, p. 15) both of the sets K and D are non-empty compact continua. If $x \in D$, then $x \in D_n$ and $(x, f(x)) \cup (x, g(x)) \subset K_n$ for $n = 1, 2, \ldots$ from which $(x, f(x)) \cup (x, g(x)) \subset K$. Also if $(x, f(x))$ or $(x, g(x))$ belongs to K, then $(x, f(x)) \cup (x, g(x))$ is contained in K_n and $x \in D_n$ for $n = 1, 2, \ldots$ and hence $x \in D$. Therefore, K is the union of the graphs of f and g restricted to the compact subinterval D of I. This shows that property P is an inductive property.

Since the non-empty compact set $F \cup G$ has the inductive property P, it follows from the Brouwer Reduction Theorem (9, p. 17) that there exists a non-empty closed subset K_0 of $F \cup G$ which is irreducible with respect to property P. But with the assumption that $F \cap G = \emptyset$ it was shown above that a set with property P is not irreducible relative to property P. Hence, for the subset K_0 of $F \cup G$, the assumption that $F \cap G = \emptyset$ results in a contradiction and therefore the conclusion of the theorem is obtained.

CHAPTER V

SUMMARY

The initial part of this study is concerned with the notion of unique representation which is illustrated in Chapter II as a property belonging to a basis for a transportation problem. This unique representation property is generalized to Cartesian products in the absence of the algebraic field structure of linear spaces required in the usual treatment of the transportation problem. A generalized basis is developed as a network on a Cartesian product and its existence is shown for arbitrary Cartesian products. The previously known result of Theorem 3.9 for finite networks shows that the number of vertices of a basis for a Cartesian product is one larger than the number of arcs belonging to the basis. This result is seen to be an extension of the property that a single arc has two end points.

The subsequent portion of this paper is an investigation concerning an extension of the property given in Theorem 3.9 for a basis for a finite Cartesian product to an infinite Cartesian product. Examples indicate that additional conditions must be imposed if a similar property is to be obtained. The Cartesian product for the investigation is taken to be the topological space which is the Cartesian product of the unit interval in the space of real numbers with itself. With a basis formed as the union of graphs of two functions each mapping one factor space I into the other and with the conditions of being both closed and connected imposed on the union of the graphs in I × I the following question is considered: Does there exist a common point on the graphs of the two functions? It is an affirmative answer to this question without the condition that the union of the graphs be a basis and with the modified condition that both functions have the same factor space for their domains that is the main result of the investigation. If one of the functions has an inverse function, then this result applies with the condition that the domain space of one function is the range space of the other. It is an open question if neither of the functions has an inverse function.

Several other questions for future investigations are mentioned. The result obtained here applies if the union of the graphs of the functions forms a basis, but it is not known if every basis for I × I which is a compact continuum can be formed in this manner. Another problem is to characterize the Cartesian product spaces in which the property of the main result holds. Simple examples show that it does not hold on a cyclinder or torus, both of which are compact continua and which have a cyclic factor space.

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