USE OF ORDER STATISTICS FOR GRAPHICAL ESTIMATION

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CHAPTER I

INTRODUCTION

If the observed values $x_1, x_2, \ldots, x_n$ of a sample of size n drawn from a population $f(x)$ are rearranged according to ascending order of magnitude and relabled $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$, then $x_{(i)}$ is denoted as the "ith order statistic" of the sample.

One of the earliest problems on the sampling theory of order statistics was published in 1902 by Karl Pearson [1] in his study of the "Galton difference problem." Thus, the subject of order statistics is nearly as old as the entire field of statistical inference. However, the major portion of the current technology of the use of order statistics has been developed since 1948.

An important use of order statistics has been for estimating parameters of the normal and other populations. Simple and effective practical methods based on order statistics have been developed in recent years. Among these are several graphical procedures. These have become known as "short-cut procedures" or "quick and dirty" methods of analysis. Their justification lies in an extreme reduction of computational labor. However, as Hartley points out, many of these procedures sacrifice the statistical efficiency of the analysis [2].

Normal Plotting Procedures

Most of these graphical methods rely on a special type of paper, usually probability paper. Normal probability paper is so constructed that the cumulative distribution function of a normally distributed random variable appears as a straight line when plotted on this special graph paper.

Chernoff and Lieberman [3], proposed a method of estimating the parameters in a normal distribution using normal probability paper. The order statistics $u_{(1)}, u_{(2)}, \ldots, u_{(n)}$ from a sample of size n from a normal distribution $N(\mu, \sigma^2)$ are plotted on the abscissa versus the empirical cumulative probabilities $p_1, p_2, \ldots, p_n$ on the ordinate scale. A straight line is then fitted to the points $(u_{(i)}, p_i)$, $i = 1, 2, \ldots, n$. The probability axis p can be constructed from the relationship

$$p_i = \int_{-\infty}^{v_i} N(x; 0, 1)\, dx \qquad (1.1)$$

where the points v_i represent a linear scale.

Since the variate U is distributed $N(u; \mu, \sigma^2)$, then

$$\int_{-\infty}^{\mu} N(u; \mu, \sigma^2)\, du = .5 \qquad (1.2)$$

and

$$\int_{-\infty}^{\mu+\sigma} N(u; \mu, \sigma^2)\, du = .8413 \qquad (1.3)$$

Equation (1.2) suggests estimating the mean μ by the abscissa where the fitted straight line intersects the horizontal line $p = .5$, and equation (1.3) suggests estimating the standard deviation σ by the difference in the abscissas where the fitted straight line intersects the horizontal lines $p = .8413$ and $p = .5$. The estimates are denoted by $\hat{\mu}$ and $\hat{\sigma}$ respectively and illustrated in Figure 1.

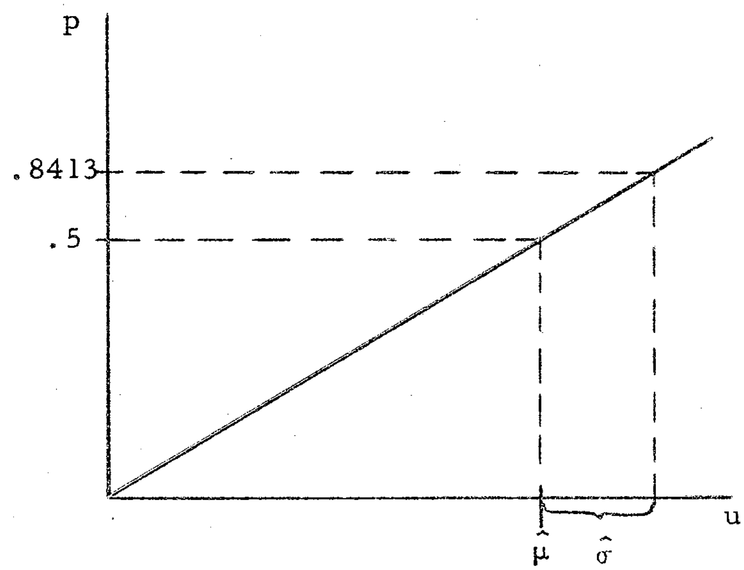

Figure 1: Estimates of μ and σ using Normal Probability Paper

In terms of the linear scale v, which is related to p by equation (1.1), the fitted line can be represented by

$$v = (u - \hat{\mu}) / \hat{\sigma}$$

or

$$u = \hat{\mu} + \hat{\sigma} v$$

The problem now is to find the values of the p_i and their associate values v_i such that $\hat{\mu}$ and $\hat{\sigma}$ are "good" estimates of μ and σ in some sense.

Several plotting conventions are in common use, though few have an obvious rationale. One method is to plot $p_i = i/n$ on the ordinate scale. However, since the point 1.0 does not appear on the probability scale, this method does not seem to be a good prospect.

Another alternative is to take the mean value of the points $(i - 1)/n$ and i/n for the point p_i, thus

$$p_i = (i - 1/2)/n. \qquad (1.4)$$

This method is a very popular one and has been used for distributions other than the normal as will be pointed out later in this chapter.

If $F(y)$ denotes the cumulative distribution function and $w_{(i)}$ is a standardized or reduced normal order statistic $w_{(i)} = (u_{(i)} - \mu)/\sigma$, then as Mood [4] has shown:

$$E[F(w_{(i)})] = i/(n + 1).$$

This suggests quite logically the ordinate values

$$p_i = i/(n + 1). \qquad (1.5)$$

Blom [5] has recommended another method for obtaining the ordinate values on normal probability paper. Although it was not derived as a direct method of plotting according to the order number as in equations (1.4) and (1.5), an average value was determined that follows the method of equation (1.4):

$$p_i = (i - 3/8)/(n + 1/4) \tag{1.6}$$

Kimball [6] suggests three other methods, although their inclusion in his list of plotting procedures is primarily for the purpose of comparison since they involve a large amount of computation:

$p_i = F[E(w_{(i)})]$

$p_i = F[\text{median of } w_{(i)}]$

$p_i = F[\text{Mode of } w_{(i)}]$

Chernoff and Lieberman [3] give the ordinate values p_i for sample sizes two to ten that yield minimum variance linear in the ordered observations unbiased estimates of the standard deviation σ. They also derive and tabulate the plotting positions for sample sizes up to ten that yield estimates of σ having minimum mean square deviation among all estimates which are linear in the ordered observations and whose bias is independent of the mean μ.

Kimball [6] in comparing the above described methods to determine which plotting convention could be used with some degree of assurance has based a criterion for choice on mean square deviation. He points out there may be some objection to such a criterion for situations where a significant bias is present.

Of the methods presented that allow for plotting and estimating of the parameters quickly and without heavy computational labor, Kimball recommends the method of equation (1.6) originally proposed by Blom.

Half-Normal Plotting Procedure

Daniel [7] in a paper published in 1959 developed another plotting procedure using a related although different type of probability paper. The parent distribution of the random variable is the half-normal or chi distribution with one degree of freedom :

$$f(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sigma^2} \exp - x^2/2\sigma^2, \text{ for } x \geq 0. \qquad (1.7)$$

An implicit assumption has been made that the location parameter of the distribution is zero as is evident in the form of the density function in equation (1.7). The reason for this assumption comes from the primary application of the procedure, which is to single degree of freedom contrast in factorial experiments. In this situation it is required that there are a majority of error contrasts with mean zero and only a small number of effects or interactions with non-zero mean. One purpose in the use of the half-normal plotting procedure then is to detect these real effects and interactions.

The half-normal probability paper is easily prepared from normal probability paper by the following transformation of the ordinate scale:

$$P' = 2P - 1$$

The graph then of the theoretical half-normal cumulative distribution is a straight line through the origin.

The plotting procedure developed by Daniel is to arrange the contrast-sums in order of absolute magnitude with the largest values indicating the effects most likely to be non-zero. For the corresponding ordinate values, Daniel tacitly recommends the convention of equation (1.4). A straight line through the origin is graphed by the line that best fits the points $(u_{(i)}, p_i)$. If the largest contrast in absolute value deviates significantly from the line, it is judged real and removed from consideration as an error contrast. The remaining points are replotted with the sample size decreased by one. This plotting procedure is continued until all remaining contrasts are considered to be estimates of the error in the experiment.

The parameter σ of the half-normal distribution can then be estimated by the abscissa value where the last fitted line intersects the probability:

$$P' = 2P - 1. = 2(.8413) - 1. = .6826$$

as illustrated in Figure 2.

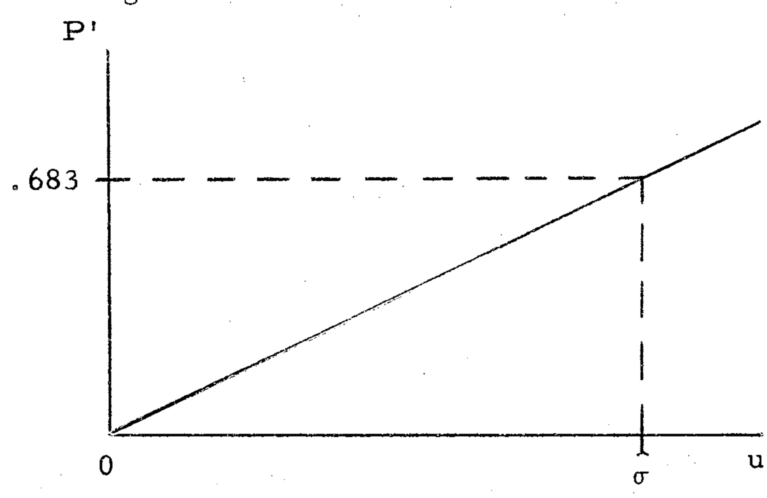

Figure 2: Estimate of σ Using Half-Normal Probability Paper

The main objection to the use of the half-normal plot is the subjective bias inherent in the process of deciding what values are error contrasts. However, on the basis of Kimball's work there also seems to be some question as to the appropriateness of the ordinate value convention.

Weibull Plotting Procedure

Kao [8] describes a method of estimating the parameters from the Weibull distribution using graphical procedures. The cumulative distribution function is given by

$$F(x) = 1 - \exp(-x^{\delta}/\theta), \quad x \geq 0 \qquad (1.8)$$

where θ is the scale parameter and δ the shape parameter. In a situation of reliability testing the Weibull gives the distribution of times to failure. That is, the observed variate $z_{(j)}$ is the cumulative failure age at the end of the jth inspection period and thereby admits to a natural arrangement of ascending order of magnitude. $F(z_{(j)})$ is the probability that any item under test will fail on or before time $z_{(j)}$. In terms of the empirical data equation (1.8) becomes after some rearrangement:

$$1/[1 - F(z_{(j)})] = \exp z_{(j)}^{\delta}/\theta \quad . \qquad (1.9)$$

Taking the natural logarithm (ℓn) twice of both sides of equation (1.9) yields

$$\ell n\, \ell n \left[\frac{1}{1 - F(z_{(j)})}\right] = -\ell n\, \theta + \delta\, \ell n\, z_{(j)} \quad . \qquad (1.10)$$

Defining the following values:

F_j = total number of failed items observed at end of jth inspection period.

S_j = total number of survived items observed at end of jth inspection period

n = total number of items in the samples

the minimum variance unbiased estimate of the probability $F(z_{(j)})$ is F_j/n. Since $S_j = n - F_j$, the left side of equation (1.10) becomes

$$\ln \ln \left[\frac{1}{1-F(z_{(j)})}\right] = \ln \ln \left[\frac{1}{1-F_j/n}\right]$$

$$= \ln \ln \left[\frac{n}{n-F_j}\right] = \ln \ln \; n/S_j$$

and results in equation (1.11)

$$\ln \ln \; S_j/n = -\ln \theta + \delta \ln z_{(j)} \quad . \qquad (1.11)$$

The results of the reliability test can be plotted on $\ln$ versus $\ln \ln$ paper with the cumulative failure age $z_{(j)}$ on the abscissa ($\ln$) scale and the inverse of the fraction of survived items n/S_j on the ordinate ($\ln \ln$) scale. If a straight line is fitted to the plotted points, then the ordinate intercept is an estimate of $-\ln \theta$ and the slope of the fitted line an estimate of the shape parameter δ as evident from equation (1.11).

Linear by Linear Plotting Procedure

All of the above described graphical procedures require a special type of paper on which to plot the results of an experiment. A great deal of the work published to date has been done trying to refine the rather simple and easily performed procedures. Much of this refinement has been in the choice of plotting positions in order to better estimate the parameters in the assumed parent distribution of the order statistics. Although tabular values of the ordinates are not necessary, they greatly reduce the time involved in arriving at the estimates and are helpful for the routine application of the procedures.

Another graphical procedure for estimating parameters of a distribution is presented in the remainder of this paper. Although somewhat similar in application, this method does not require the use of a special type of plotting paper. Both scales are linear; therefore, the procedure will be termed the "linear by linear procedure."

If $u_{(i)}$ $(i = 1, 2, \ldots, n)$ denote the order statistics of a sample of size n from a distribution with location parameter μ and scale parameter σ, then the reduced or standardized variate $z_{(i)}$ is derived from the relationship $z_{(i)} = (u_{(i)} - \mu)/\sigma$ and the expected value of the ordered observations becomes $E(u_{(i)}) = \mu + \sigma E(z_{(i)})$. The general procedure then is to plot the expected values of the tabulated standardized variate on the abscissa, the order statistics of the sample on the ordinate scale, and a straight line fitted to the points. The u-intercept is an estimate of the location parameter and the slope of the fitted line an estimate of the scale parameter.

If the joint distribution of the standardized variates $z_{(i)}$, $i = 1, 2, \ldots, n$, are known, then by the application of the generalized least squares theorem the straight line yielding the minimum variance linear unbiased estimates of the parameters can be obtained. The details of this theorem and its application to order statistics are presented in Chapter II.

In order to reduce the computational labor in obtaining the estimators, it is important that a visually fitted line be a very good approximation to the optimum line described above. After a detailed presentation of the linear by linear procedure, a comparison of these two types of estimators is given in Chapter III.

Chapter IV develops the method of obtaining the first and second moments of the order statistics based on the half-normal parent distribution and gives the tabulated values of these moments up to sample size twenty. The relationship between the estimates based on the half-normal and the normal parent distribution is given including the relative efficiencies of the methods.

The exponential and Weibull distributions are pertinent for testing in reliability situations. Chapter V applies the linear by linear procedure to the exponential case, describes the alterations necessary for application to the Weibull, and compares the method with other estimating procedures.

Finally the results of this paper are summarized in Chapter VI and possible extensions of the procedures and further problems are examined.

CHAPTER II

GENERALIZED LEAST SQUARES

For the linear matrix model

$$Y = X\beta + e \tag{2.1}$$

Y is an $n \times 1$ random vector, X a known $n \times p$ matrix, β a $p \times 1$ vector of unknown parameters, and e an $n \times 1$ random vector with

$$E\,e = \phi \qquad \text{(null vector)} \tag{2.2}$$

and $\mathrm{Cov}(e) = E\,ee' = V w^2$.

With V known, the best (minimum variance) linear (linear function of the observed vector Y) unbiased estimate of the vector β is given by

$$\hat{\beta} = (X'V^{-1}X)^{-1} X'V^{-1}Y \tag{2.3}$$

and the covariance matrix of $\hat{\beta}$ is given by

$$\mathrm{Cov}(\hat{\beta}) = (X'V^{-1}X)^{-1} w^2 . \tag{2.4}$$

Application to Order Statistics

It may be noted that the expected values of unordered observations are independent of the scale parameter. If the unordered observations are used, therefore, the method of estimating parameters by least squares is incapable of providing an optimum linear estimate of the scale parameter.

Suppose that we have a continuous variate U whose distribution function is of the form $F[(U - \mu)/\sigma]$. That is, if the transformation $Z = (U - \mu)/\sigma$ is made, then the distribution of Z will be parameter free. The order statistics of a sample of size n from U are denoted by $u_{(1)}, u_{(2)}, \ldots, u_{(n)}$. The standardized order statistics

$$z_{(i)} = (u_{(i)} - \mu)/\sigma$$

can be regarded as the ordered observations from the variate Z with known distribution.

Define the vector of standardized order statistics as

$$Z = \begin{bmatrix} z_{(1)} \\ z_{(2)} \\ \cdot \\ \cdot \\ \cdot \\ z_{(n)} \end{bmatrix} \qquad (2.5)$$

then the moments in matrix notation become

$$E(Z) = \alpha$$
$$\mathrm{Cov}(Z) = V \quad . \qquad (2.6)$$

Also let $W = V^{-1}$ and j be an n x 1 vector with each element equal to unity. Then the observed vector of order statistics becomes

$$U = \mu j + \sigma Z$$

and the moments are

$$E(U) = \mu j + \sigma a$$

$$\mathrm{Cov}\,(U) = \sigma^2 \,\mathrm{Cov}\,(Z) = \sigma^2 V \qquad (2.7)$$

The linear matrix model corresponding to (2.1) becomes

$$U = X\beta + e$$

where

$$E e = \phi$$
$$E ee' = \sigma^2 V \qquad (2.8)$$

$$X = (j, a)$$

and

$$\beta = \binom{\mu}{\sigma}$$

then by equation (2.3)

$$\hat{\beta} = (X'V^{-1}X)^{-1}X'V^{-1}U$$

$$= (X'WX)^{-1}X'WU$$

and by equation (2.4)

$$\mathrm{Cov}\,\hat{\beta} = (X'V^{-1}X)^{-1}\sigma^2 = (X'WX)^{-1}\sigma^2.$$

Partitioning $X'WX$ by the relationship $X = (j, a)$

$$X'WX = \begin{bmatrix} j'Wj & j'Wa \\ j'Wa & a'Wa \end{bmatrix} \qquad (2.9)$$

Lloyd [9] has shown for a symmetric parent distribution (such as the normal distribution) that

$$j'Wa = 0.$$

This reduces the 2×2 matrix $X'WX$ of equation (2.9) to a diagonal matrix with elements $j'Wj$ and $a'Wa$. The inverse is then easily obtained as

$$(X'WX)^{-1} = \mathrm{diag}\,(1/j'Wj,\ 1/a'Wa)$$

and the estimate of β symplifies to

$$\hat{\beta} = \begin{bmatrix} \hat{\mu} \\ \hat{\sigma} \end{bmatrix} = \begin{bmatrix} X'WU/j'Wj \\ X'WU/\alpha'W\alpha \end{bmatrix}$$

The variances and covariance become

$$\mathrm{Var}\,(\hat{\mu}) = \sigma^2/j'Wj$$

$$\mathrm{Var}\,(\hat{\sigma}) = \sigma^2/\alpha'W\alpha$$

$$\mathrm{Cov}\,(\hat{\mu}, \hat{\sigma}) = 0 \ .$$

In Chapters IV and V the half-normal distribution and the negative exponential distribution both assumed dependent on a single parameter will be discussed. As might be expected simplifications of equations (2.3) and (2.4) can be effected in the case of one-parameter distributions.

The standardized variate $Z = U/\sigma$ will again determine the vector of standardized order statistics as in equation (2.5) with moments given by equations (2.6). The expectation of the ordered observations becomes

$$E\,(U) = \sigma\alpha \ .$$

However, the covariance matrix of U remains as equation (2.7). The linear matrix model for the one-parameter case is $U = \sigma\alpha + e$, where the distribution of e is as in equations (2.8). The best linear unbiased estimate of σ is then

$$\hat{\sigma} = \alpha'WU/\alpha'W\alpha \tag{2.10}$$

and the variance of this estimate is

$$\text{Var}\ \hat{\sigma} = \sigma^2 / \alpha' W \alpha \ . \tag{2.11}$$

In the remaining chapters a comparison will be made of the simple least squares estimates and the generalized least squares estimates. For the one-parameter parent distributions mentioned above the simple least squares estimate of σ based on the order statistics is

$$\sigma^* = \alpha' U / \alpha' \alpha \tag{2.12}$$

and the variance of σ^* is

$$\text{Var}\ \sigma^* = \alpha' V \alpha\ \sigma^2 / (\alpha' \alpha)^2 . \tag{2.13}$$

CHAPTER III

ESTIMATION OF μ AND σ FROM THE NORMAL DISTRIBUTION

The purpose of this chapter is to describe a graphical procedure for estimating the parameters in a normal distribution from n independent observations on the variate U. The estimates will be compared with the best linear unbiased estimates obtained by the generalized least squares procedure stated in the previous chapter.

Given the order statistics $u_{(i)}$, $i = 1, 2, \ldots, n$, of a sample from a normal distribution with mean μ and variance σ^2, the standardized order statistics $z_{(i)} = (u_{(i)} - \mu)/\sigma$ can be considered the order statistics of a sample from a normal distribution with mean zero and variance one. The expectation of the $u_{(i)}$ in terms of the reduced ordered observations becomes

$$E[u_{(i)}] = \mu + \sigma E[z_{(i)}], \quad (i = 1, 2, \ldots, n). \tag{3.1}$$

Equation (3.1) is the equation of a straight line with u-intercept μ and slope σ. The parameters μ and σ can be estimated by the following procedure:

Plot the points $(u_{(i)}, Ez_{(i)})$ on graph paper where both axes are linear and fit a straight line to the plotted points. Estimates of the parameters are then $\hat{\mu}$, the u-intercept and $\hat{\sigma}$, the slope of the fitted line. The expected values of the standardized variate Z for samples of size fifty and less are tabulated by Fisher and Yates [10] to two decimal places. These values are sufficient for plotting even on very large graph paper.

The problem remains of fitting a straight line to the plotted points. The easiest and most practical procedure is to fit the line visually. If it can be assumed that the visually fitted line closely approximates the line fitted by a simple least squares procedure, then the precision of the estimates and the efficiencies relative to the best linear unbiased estimates can be computed. The relative efficiency will be calculated as the percentage ratio of the variance of the best linear unbiased estimate to the variance of the simple least squares estimate. Since the estimate of μ in both cases is the mean of the observations and has variance σ^2/n, the relative efficiency is 100%.

The coefficients c(i) of the $u_{(i)}$ that give the best linear unbiased estimates of σ,

$$\hat{\sigma} = \sum_{i=1}^{n} c(i)\, u_{(i)} ,$$

and the variances of these estimates are given by Sarhan and Greenberg [2]. The coefficients that correspond to the visual fit by the above assumption are given in Table I. Table II gives the variances of these estimates calculated from equation (2.13).

TABLE I

THE COEFFICIENTS OF THE SIMPLE LEAST SQUARES ESTIMATES OF THE STANDARD DEVIATION IN SAMPLES UP TO SIZE TWENTY FROM A NORMAL DISTRIBUTION*

N	I	C(I)	N	I	C(I)	N	I	C(I)	N	I	C(I)
2	2	.886	10	6	.015	13	12	.108	16	13	.056
3	3	.591	10	7	.047	13	13	.154	16	14	.072
4	3	.129	10	8	.083	14	8	.007	16	15	.093
4	4	.448	10	9	.127	14	9	.023	16	16	.128
5	4	.155	10	10	.194	14	10	.039	17	10	.010
5	5	.364	11	7	.025	14	11	.056	17	11	.020
6	4	.049	11	8	.052	14	12	.076	17	12	.031
6	5	.156	11	9	.082	14	13	.102	17	13	.042
6	6	.308	11	10	.120	14	14	.144	17	14	.055
7	5	.070	11	11	.179	15	9	.013	17	15	.070
7	6	.150	12	7	.010	15	10	.026	17	16	.090
7	7	.268	12	8	.032	15	11	.040	17	17	.122
8	5	.025	12	9	.055	15	12	.056	18	10	.004
8	6	.079	12	10	.081	15	13	.074	18	11	.013
8	7	.142	12	11	.113	15	14	.098	18	12	.022
8	8	.237	12	12	.165	15	15	.136	18	13	.032
9	6	.039	13	8	.018	16	9	.006	18	14	.042
9	7	.082	13	9	.036	16	10	.017	18	15	.054
9	8	.134	13	10	.056	16	11	.029	18	16	.068
9	9	.214	13	11	.079	16	12	.041	18	17	.086

* NEGATIVE AND ZERO ELEMENTS NOT RECORDED. THESE VALUES ARE OBTAINED BY THE RELATIONSHIPS C(I)=-C(N-I+1) AND FOR N ODD C((N+1)/2)=0.

TABLE I (CONTINUED)

N	I	C(I)	N	I	C(I)	N	I	C(I)	N	I	C(I)
18	18	.116	19	15	.042	20	11	.004	20	16	.042
19	11	.008	19	16	.053	20	12	.011	20	17	.052
19	12	.016	19	17	.066	20	13	.018	20	18	.064
19	13	.024	19	18	.083	20	14	.025	20	19	.080
19	14	.033	19	19	.110	20	15	.033	20	20	.106

TABLE II

THE VARIANCES OF THE SIMPLE LEAST SQUARES ESTIMATES

N	VARIANCE/σ^2	N	VARIANCE/σ^2
2	.57079634	12	.04693010
3	.27548200	13	.04292970
4	.18012853	14	.03955544
5	.13342104	15	.03667132
6	.10580203	16	.03417799
7	.08759051	17	.03200120
8	.07469396	18	.03008440
9	.06508882	19	.02838370
10	.05766120	20	.02686456
11	.05174801		

The covariances necessary for these calculations were derived from the expected values and expected values of products of normal order statistics given by Teichroew [11].

The relative efficiencies of the simple least squares estimates decrease slightly as the sample size increases. These values range from 100% for sample size two to 99.91% for sample size twenty. The closeness of the two types of estimators is also evident in comparing the coefficients for a particular sample size. When rounded to two decimal places, the coefficients are nearly the same for each sample size considered.

To illustrate the linear by linear procedure the following example is presented:

Twenty random normal deviates were generated by a Monte Carlo technique from a distribution with actual mean -1.0 and standard deviation 1.0. The generated data is given in Table III along with the ordered values and the expectations of the standardized order statistics for sample size twenty. Figure 3 shows the plotted values with the true line of slope 1.0 and intercept -1.0 drawn. The average value of the sample is -1.079 and the sample standard deviation is 1.042.

The linear by linear procedure may be used for purposes other than estimating parameters. If the plotted points quite obviously do not tend to fall along a straight line, then there is reason to doubt that the assumed normal distribution is the correct one. Although a probability

TABLE III

NORMAL DATA

INDEX	GENERATED DATA	ORDERED VALUES (ORDINATES)	EXPECTED VALUES (ABSCISSAS)
**	******	******	*****
1	-.103	-3.033	-1.87
2	-3.033	-2.561	-1.41
3	-1.934	-2.244	-1.13
4	-.952	-2.187	-.92
5	-1.323	-1.934	-.75
6	.367	-1.652	-.59
7	-1.439	-1.439	-.45
8	.018	-1.323	-.31
9	-1.321	-1.321	-.19
10	-.704	-1.236	-.06
11	-2.187	-.952	.06
12	-.592	-.908	.19
13	-.699	-.704	.31
14	-1.236	-.669	.45
15	-1.652	-.592	.59
16	-.908	-.308	.75
17	-2.561	-.103	.92
18	-2.244	.018	1.13
19	-.303	.367	1.41
20	1.217	1.217	1.87

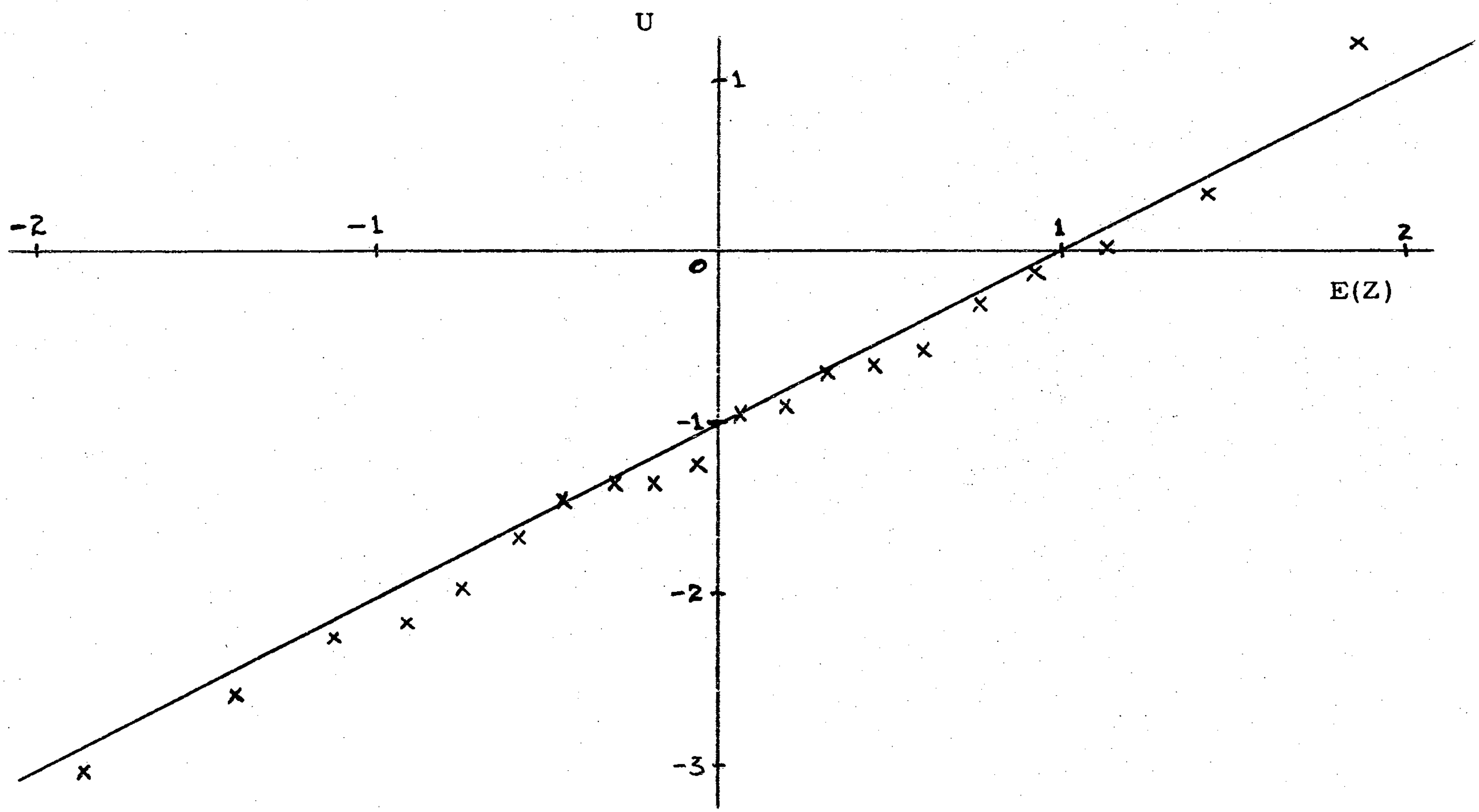

Figure 3. Linear by Linear Graph of Normal Data

level can not be attached to such a decision, the procedure does allow a quick test of compliance with the assumed distribution and gives a basis for determining whether a more rigorous and exact test is necessary. Another purpose served by the graphical procedure is to determine outliers in the observed data. If a point deviates "significantly" from a straight line through the remaining data, that point may be judged an erroneous one due to measurement error, incorrect reporting, or other sources of variation. Again, the graphical procedure may be used as a preliminary investigation of the data to determine whether further testing is necessary.

CHAPTER IV

ESTIMATION OF σ FROM THE HALF-NORMAL DISTRIBUTION

The Half-Normal Plot described in Chapter I can be used as a graphical procedure for estimating the scale parameter in the single parameter half-normal distribution. In this chapter the application of the linear by linear procedure to estimate the scale parameter will be discussed.

If the variate U has the half-normal probability density function as given in equation (1.7) and the transformation $Z = U/\sigma$ is applied, then the variate Z is distributed as a half-normal with the parameter σ equal to one, and therefore, the distribution Z is completely known. The order statistics of a sample of n observations on the variate U have expectation

$$E[u_{(i)}] = \sigma \; E[z_{(i)}] , \quad (i = 1, 2, \ldots, n).$$

If the points $[u_{(i)}, E(z_{(i)})]$ are plotted on linear-linear graph paper and a straight line visually fitted through the origin and the plotted points, then an estimate of the parameter σ is the slope of the fitted line. In order to compare the estimate of σ determined by the linear by linear procedure with the best linear unbiased estimate for a particular sample size it will again be assumed that the straight line fitted visually to the points closely approximates the simple least squares line.

The values of $E(z_{(i)})$ are necessary for the application of the linear by linear procedure, and the covariance matrix V of the standardized order statistics for each size of sample is necessary for the calculation of the best linear unbiased estimates for use in the evaluation of the efficiencies. Since these values are not currently tabulated in the literature for the half-normal distribution, the purpose of the following section of this chapter is to describe the calculations and give the results in tabular form.

Moments of the Order Statistics

The probability density function of the ith order statistic from a sample of size N is given by

$$f_i(x) = \frac{N!}{(i-1)!\,(N-i)!} f(x)\,[F(x)]^{i-1}\,[1-F(x)]^{N-i}, \quad x \geq 0$$

$$= 0 \qquad \text{otherwise,}$$

where f(x) is the density function and F(x) the cumulative distribution function of the parent distribution. The joint density function of the ith and jth order statistics for $i < j$ is

$$f_{i,j}(x, y) = \frac{N!}{(i-1)!\,(j-i-1)!\,(N-j)!} f(x)\,f(y)\,[1-F(x)]^{i-1}$$

$$\cdot\,[F(y)-F(x)]^{j-i-1}\,[F(y)]^{N-j}, \quad 0<x<y<\infty$$

$$f_{i,j}(x, y) = 0 \qquad \text{otherwise.}$$

The first, second, and mixed moments about the origin of the order statistics are defined as

$$\mu_{i,N} = E(x_{(i)}) = \int_0^{\infty} x f_i(x)\, dx$$

$$\mu_{i,N}^{(2)} = E(x_{(i)}^2) = \int_0^{\infty} x^2 f_i(x)\, dx$$

$$\mu_{\substack{i,j,N \\ i<j}} = E(x_{(i)} x_{(j)}) = \int_0^{\infty} \int_0^{y} xy f_{i,j}(x, y)\, dx\, dy.$$

Govindarajulu [12] has calculated these moments for sample sizes two, three, and four and the second and mixed moments only for sample size five. In order to tabulate the above moments for sample sizes twenty and less, it would be necessary to evaluate 420 single integrals and 1330 double integrals. However, Govindarajulu gives a recurrence relation that exists among the mixed moments that allows for a significant reduction in the number of double integrals that are necessary to compute:

$$(i-1)\mu_{i,j,N} + (j-i)\mu_{i-1,j,N} + (N-j+1)\mu_{i-1,j-1,N} = N\mu_{i-1,j-1,N-1}. \qquad (4.1)$$

Use of the following notation

$$\mu_{i,j,N} = \frac{N!}{(i-1)!\,(j-i-1)!\,(N-j)!}\, G(i-1, N-j, j-i-1) \qquad (4.2)$$

for substitution in equation (4.1) results in

$$\frac{(i-1)N!\;\; G(i-1, N-j, j-i-1)}{(i-1)!\;\; (j-i-1)!\; (N-j)!} + \frac{(j-i)N!\;\; G(i-2, N-j, j-i)}{(i-2)!\;\; (j-i)!\; (N-j)!}$$

$$+ \frac{(N-j+1)N!\;\; G(i-2, N-j+1, j-i-1)}{(i-2)!\; (j-i-1)!\; (N-j+1)!}$$

$$= \frac{N(N-1)!\;\; G(i-2, N-j, j-i-1)}{(i-2)!\; (j-i-1)!\; (N-j)!} \,.$$

Since all coefficients cancel, the above equation can be written in the form

$$G(i-2, N-j, j-i) = G(i-2, N-j, j-i-1) - G(i-1, N-j, j-i-1)$$

$$- G(i-2, N-j+1, j-i-1). \qquad (4.3)$$

Letting

$$p = j - i - 1$$

$$m = i - 2$$

$$n = N - j$$

equation (4.3) becomes

$$G(m, n, p+1) = G(m, n, p) - G(m+1, n, p) - G(m, n+1, p) \,. \qquad (4.4)$$

On the basis of the recurrence relation (4.4) if the values for G(m, n, 0) can be calculated for the parameter range m = 0, 1, 2, . . . , 18 and n = 0, 1, 2, . . . , 18 - m, then the number of double integrals necessary to compute numerically is 190. The integral G(m, n, p) has the following form

$$G(m,n,p) = \int_0^{\infty}\int_0^{y} xy\, f(x)\, f(y)\, [F(x)]^m\, [1-F(y)]^n$$

$$\cdot\ [F(y) - F(x)]^p\ dxdy\ . \qquad (4.5)$$

and for p = 0 reduces to

$$G(m,n,0) = \int_0^{\infty} y\, f(y)\, [1-F(y)]^n$$

$$\cdot \int_0^{y} x\, f(x)\, [F(x)]^m\ dxdy\ . \qquad (4.6)$$

In order to obtain a more accurate evaluation of the integrals represented by equation (4.6) than possible with standard numerical integration techniques, it is advantageous to rewrite the equation in three terms. Two of these terms are one-dimensional integrals and the third term is a two-dimensional integral of simpler form than equation (4.6).

Integrating the following integral by parts with respect to x:

$$\int_0^{y} x\, f(x)\, [F(x)]^m\ dx = \frac{y[F(y)]^{m+1}}{m+1} - \int_0^{y} \frac{[F(x)]^{m+1}}{m+1}\ dx,$$

and substituting in equation (4.5) yields:

$$G(m,n,0) = \frac{1}{m+1} \int_0^{\infty} y^2 f(y) [1-F(y)]^n [F(y)]^{m+1} dy$$

$$- \frac{1}{m+1} \int_0^{\infty} y f(y) [1-F(y)]^n$$

$$\cdot \int_0^{y} [F(x)]^{m+1} dxdy. \qquad (4.7)$$

The second term in equation (4.7) can be integrated by parts with respect to y by letting

$$u = y \int_0^{y} [F(x)]^{m+1} dx$$

and

$$dv = f(y) [1-F(y)]^n dy,$$

then

$$v = - \frac{[1-F(y)]^{n+1}}{n+1}$$

and

$$du = \int_0^{y} [F(x)]^{m+1} dxdy + y [F(y)]^{m+1} dy,$$

giving

$$\int_0^\infty \int_0^y y\, f(y)\, [1-F(y)]^n\, [F(x)]^{m+1}\, dxdy =$$

$$- \frac{[1-F(y)]^{n+1}}{n+1}\; y \int_0^y [F(x)]^{m+1}\, dx \;\Big|_0^\infty$$

$$+ \frac{1}{n+1} \int_0^\infty y[F(y)]^{m+1}\, [1-F(y)]^{n+1}\, dy$$

$$+ \frac{1}{n+1} \int_0^\infty \int_0^y [1-F(y)]^{n+1}\, [F(x)]^{m+1}\, dxdy. \qquad (4.8)$$

The first term on the right hand side of equation (4.8) is to be evaluated at the limits zero and infinity. At the lower limit the value of the term is obviously zero. The upper limit can be evaluated in the following manner:

Since $0 \leq F(y) \leq 1$ and all factors are non-negative

$$\lim_{y \to \infty} \frac{[1-F(y)]^{n+1}}{n+1}\; y \int_0^y [F(x)]^{m+1}\, dx$$

$$\leq \lim_{y\to\infty} [1-F(y)]\; y \int_0^y F(x)\, dx$$

$$= \lim_{y\to\infty} \frac{y \int_0^y F(x)\, dx}{[1-F(y)]^{-1}} \qquad = L \text{ (say)}.$$

Applying L'Hôpital's rule to the indeterminant form we get

$$L = \lim_{y \to \infty} \frac{y\ F(y) + \int_0^y F(x)\,dx}{f(y)\,[1-F(y)]^{-2}} \quad . \tag{4.9}$$

This limit is again an indeterminant form. By considering the form of the frequency function,

$$f(y) = c \exp - y^2/2$$

where c is a constant, the relationship

$$\frac{df(y)}{dy} = -y\,f(y)$$

is determined and can be used in the application of L'Hôpital's rule to the limit (4.9),

$$L = \lim_{y \to \infty} \frac{y\,f(y) + 2\,F(y)}{2f^2(y)\,[1-F(y)]^{-3} - y\,f(y)\,[1-F(y)]^{-2}} \quad . \tag{4.10}$$

Considering the four individual terms of the limit (4.10) and successively applying the rule for evaluating indeterminant forms when necessary gives

(a) $\lim_{y \to \infty} y\,f(y) = \lim_{y \to \infty} \dfrac{y}{c \exp y^2/2} = \lim_{y \to \infty} \dfrac{1}{c\,y \exp y^2/2} = 0$

(b) $\lim_{y \to \infty} 2F(y) = 2$

(c) $$\lim_{y \to \infty} \frac{2 f^2(y)}{[1-F(y)]^3} = \lim_{y \to \infty} \frac{-4 y f^2(y)}{-3[1-F(y)]^2 f(y)} = \frac{4}{3} \lim_{y \to \infty} \frac{y f(y)}{[1-F(y)]^2}$$

$$= \frac{4}{3} \lim_{y \to \infty} \frac{-y^2 f(y) + f(y)}{-2[1-F(y)] f(y)} = \frac{2}{3} \lim_{y \to \infty} \frac{y^2 - 1}{[1-F(y)]}$$

$$= \infty$$

(d) $$-\lim_{y \to \infty} \frac{y f(y)}{[1-F(y)]^2} = -\lim_{y \to \infty} \frac{-y^2 f(y) + f(y)}{-2f(y)[1-F(y)]}$$

$$= -\lim_{y \to \infty} \frac{y^2 - 1}{2[1-F(y)]} = -\infty.$$

Therefore, the denominator of equation (4.10) is in an indeterminant form, but can be evaluated by rewriting it as

$$\lim_{y \to \infty} \frac{2f(y) - y[1-F(y)]^3}{f^{-1}(y)[1-F(y)]^3}$$

$$= \lim_{y \to \infty} \frac{-2yf(y) - [1-F(y)] + yf(y)[1-F(y)]^{-1}}{-3[1-F(y)]^2 + y[1-F(y)]^3 f^{-1}(y)} = L'\text{(say)}.$$

The value of the first term in the numerator of L' is zero by (a) above. The second term is obviously zero. The limit of the third term is

$$\lim_{y \to \infty} \frac{y f(y)}{[1-F(y)]} = \lim_{y \to \infty} \frac{-y^2 f(y) + f(y)}{-f(y)}$$

$$\lim_{y\to\infty} \frac{y\, f(y)}{[1-F(y)]} = \lim_{y\to\infty} y^2 - 1 = \infty .$$

The first term in the denominator of L' is zero. The second term can be evaluated by

$$\lim_{y\to\infty} \frac{y[1-F(y)]^3}{f(y)} = \lim_{y\to\infty} \frac{-3yf(y)[1-F(y)]^2 + [1-F(y)]^3}{-y\, f(y)}$$

$$= \lim_{y\to\infty} 3[1-F(y)]^2 - \lim_{y\to\infty} \frac{[1-F(y)]^3}{y\, f(y)}$$

$$= 0 - \lim_{y\to\infty} \frac{-3\, f(y)\, [1-F(y)]^2}{-y^2\, f(y) + f(y)}$$

$$= - \lim_{y\to\infty} \frac{3[1-F(y)]^2}{y^2 + 1} = 0.$$

Therefore, the value of L' the denominator of L is infinite, the value of L is zero, and the value of the first term on the right hand side of equation (4.8) is zero.

The function $G(m,n,o)$ can now be written as

$$G(m,n,o) = \frac{1}{m+1} \int_0^\infty y^2 f(y)\, [1-F(y)]^n\, [F(y)]^{m+1}\, dy$$

$$- \frac{1}{(m+1)(n+1)} \int_0^\infty y\, [F(y)]^{m+1}\, [1-F(y)]^{n+1}\, dy$$

$$- \frac{1}{(m+1)(n+1)} \int_0^\infty \int_0^y [1-F(y)]^{n+1} [F(x)]^{m+1}$$

$$dxdy \; . \qquad (4.11)$$

This result is similar to the integral expansions obtained by Teichroew [11] and Godwin [13] for the normal parent distribution.

The numerical evaluation of all the moments required in the first section of this chapter is based on the following set of four functions:

$$M(a,b) = \int_0^\infty y f(y) [1-F(y)]^a [F(y)]^b \, dy$$

$$D(a,b) = \int_0^\infty y^2 f(y) [1-F(y)]^a [F(y)]^b \, dy$$

$$B(a,b) = \int_0^\infty y [F(y)]^a [1-F(y)]^b \, dy$$

$$Q(a,b) = \int_0^\infty [1-F(y)]^a \, H(y,b) \, dy \, ,$$

where $H(y,b) = \int_0^y [F(x)]^b \, dx. \qquad (4.12)$

The first moments about the origin of the order statistics from a half-normal parent distribution are calculated from the equation

$$\mu_{i,N} = \frac{N!}{(i-1)!\,(N-i)!} M(N-i, i-1). \qquad (4.13)$$

The second moments about the origin are calculated from the equation

$$\mu^{(2)}_{i,N} = \frac{N!}{(i-1)!\,(N-i)!} D(N-i, i-1) . \qquad (4.14)$$

The relationship

$$G(m,n,o) = \frac{D(n,m+1)}{(m+1)} - \frac{B(m+1,n+1)}{(m+1)(n+1)} - \frac{Q(n+1,m+1)}{(n+1)(m+1)}$$

upon which the application of recurrence relation (4.4) results in the values necessary for equation (4.2) is the basis for the calculations of the mixed moments.

For the numerical evaluation of the functions (4.12) Simpson's rule was used throughout with the upper limit of integration truncated at 10.0. Eighteen significant digits were retained in all calculations. Since all integrands involve factors of the cumulative distribution function $F(x)$, it was necessary to obtain very precise values of this function. Values of $F(x)$ were computed at x intervals of .002 with four ordinates of the probability density function appearing in each value and checked at intervals of .02 with existing tables of the normal cumulative distribution function $\phi(x)$ to fifteen decimal places by the relationship:

$$F(x) = 2\phi(x) - 1.0.$$

From the 5001 computed values of $F(x)$ were generated the ordinates for the functions $[F(x)]^a$, $x[F(x)]^a$, $[1-F(x)]^a$, $xf(x)[1-F(x)]^a$, and $x^2 f(x)[1-F(x)]^a$, which represent factors of the integrands of equations (4.12). The first of these functions was generated at intervals of .002, the others at intervals of .02. The range on the parameter a for the first three functions is 1 to 19 and for the remaining two the range is 0 to 19. The values of the function $H(y, a)$ were then computed with ten ordinates of the integrand $[F(x)]^a$ involved in each value. Pairs of the above functions were combined according to equations (4.12) by multiplying corresponding ordinates and numerically integrating for each value of the parameters a and b. Each integration involved 501 ordinates in the application of Simpson's rule.

Multiplying by the appropriate coefficients according to equations (4.13) and (4.14) gives the first and second moments of the order statistics. These values are tabulated in Table IV and checked using formulas given by Govindarajulu [12]:

$$\sum_{i=1}^{N} \mu_{i,N} = N\,E(X)$$

$$\sum_{i=1}^{N} \mu_{i,N}^{(2)} = N\,E(X^2)$$

TABLE IV

FIRST AND SECOND MOMENTS OF HALF-NORMAL ORDER STATISTICS

N	I	$\mu_{I,N}$	$\mu^{(2)}_{I,N}$	N	I	$\mu_{I,N}$	$\mu^{(2)}_{I,N}$
2	1	.46738	.36338	9	1	.12692	.02984
2	2	1.12837	1.63661	9	2	.25728	.09243
3	1	.33490	.19279	9	3	.39334	.19293
3	2	.73236	.70454	9	4	.53819	.34009
3	3	1.32638	2.10265	9	5	.69636	.54889
4	1	.26208	.12070	9	6	.87526	.84669
4	2	.55336	.40908	9	7	1.08883	1.28982
4	3	.91136	.99999	9	8	1.36966	2.02246
4	4	1.46472	2.47021	9	9	1.83508	3.63680
5	1	.21569	.08307	10	1	.11515	.02470
5	2	.44764	.27120	10	2	.23289	.07611
5	3	.71195	.61591	10	3	.35485	.15770
5	4	1.04430	1.25605	10	4	.48316	.27513
5	5	1.56983	2.77374	10	5	.62074	.43753
6	1	.18344	.06083	10	6	.77198	.66025
6	2	.37694	.19428	10	7	.94411	.97098
6	3	.58902	.42502	10	8	1.15086	1.42647
6	4	.83487	.80680	10	9	1.42436	2.17146
6	5	1.14902	1.48067	10	10	1.88071	3.79962
6	6	1.65399	3.03236	11	1	.10538	.02079
7	1	.15967	.04654	11	2	.21277	.06380
7	2	.32604	.14657	11	3	.32338	.13148
7	3	.50420	.31358	11	4	.43875	.22764
7	4	.70212	.57361	11	5	.56088	.35824
7	5	.93443	.98170	11	6	.69258	.53268
7	6	1.23485	1.68027	11	7	.83814	.76655
7	7	1.72385	3.25771	11	8	1.00467	1.08780
8	1	.14140	.03679	11	9	1.20568	1.55347
8	2	.28751	.11476	11	10	1.47295	2.30879
8	3	.44163	.24199	11	11	1.92149	3.94870
8	4	.60849	.43289	12	1	.09716	.01774
8	5	.79575	.71433	12	2	.19590	.05429
8	6	1.01764	1.14211	12	3	.29715	.11139
8	7	1.30725	1.85965	12	4	.40208	.19175
8	8	1.78336	3.45743	12	5	.51210	.29943

TABLE IV (CONTINUED)

N	I	$\mu_{I,N}$	$\mu^{(2)}_{I,N}$	N	I	$\mu_{I,N}$	$\mu^{(2)}_{I,N}$
12	6	.62916	.44058	15	2	.15835	.03579
12	7	.75600	.62479	15	3	.23928	.07281
12	8	.89681	.86781	15	4	.32212	.12397
12	9	1.05859	1.19780	15	5	.40750	.19084
12	10	1.25471	1.67202	15	6	.49621	.27565
12	11	1.51660	2.43614	15	7	.58925	.38147
12	12	1.95829	4.08620	15	8	.68793	.51273
13	1	.09012	.01532	15	9	.79405	.67590
13	2	.18153	.04677	15	10	.91020	.88087
13	3	.27493	.09564	15	11	1.04041	1.14371
13	4	.37123	.16389	15	12	1.19144	1.49294
13	5	.47148	.25442	15	13	1.37627	1.98638
13	6	.57709	.37144	15	14	1.62555	2.77063
13	7	.68991	.52124	15	15	2.05089	4.44446
13	8	.81265	.71354	16	1	.07406	.01044
13	9	.94941	.96423	16	2	.14886	.03170
13	10	1.10712	1.30161	16	3	.22476	.06438
13	11	1.29899	1.78315	16	4	.30223	.10935
13	12	1.55616	2.55487	16	5	.38179	.16783
13	13	1.99181	4.21381	16	6	.46406	.24148
14	1	.08405	.01337	16	7	.54980	.33259
14	2	.16914	.04072	16	8	.63998	.44432
14	3	.25585	.08305	16	9	.73588	.58115
14	4	.34489	.14180	16	10	.83929	.74960
14	5	.43707	.21911	16	11	.95275	.95963
14	6	.53343	.31798	16	12	1.08025	1.22738
14	7	.63530	.44273	16	13	1.22850	1.58147
14	8	.74453	.59976	16	14	1.41037	2.07982
14	9	.86374	.79888	16	15	1.65629	2.86932
14	10	.99701	1.05609	16	16	2.07720	4.54947
14	11	1.15117	1.39981	17	1	.06991	.00933
14	12	1.33930	1.88769	17	2	.14046	.02828
14	13	1.59231	2.66606	17	3	.21191	.05735
14	14	2.02254	4.33287	17	4	.28470	.09721
15	1	.07874	.01177	17	5	.35922	.14882

TABLE IV (CONTINUED)

N	I	$\mu_{I,N}$	$\mu^{(2)}_{I,N}$	N	I	$\mu_{I,N}$	$\mu^{(2)}_{I,N}$
17	6	.43597	.21346	19	6	.38917	.17055
17	7	.51556	.29285	19	7	.45892	.23263
17	8	.59871	.38935	19	8	.53110	.30708
17	9	.68640	.50616	19	9	.60629	.39570
17	10	.77987	.64780	19	10	.68518	.50093
17	11	.88088	.82085	19	11	.76870	.62603
17	12	.99196	1.03532	19	12	.85809	.77561
17	13	1.11705	1.30740	19	13	.95505	.95630
17	14	1.26280	1.66579	19	14	1.06207	1.17819
17	15	1.44199	2.16854	19	15	1.18302	1.45758
17	16	1.68486	2.96276	19	16	1.32446	1.82328
17	17	2.10172	4.64864	19	17	1.49901	2.33356
18	1	.06620	.00838	19	18	1.73654	3.13594
18	2	.13296	.02539	19	19	2.14624	4.83185
18	3	.20047	.05142	20	1	.05986	.00688
18	4	.26912	.08701	20	2	.12014	.02080
18	5	.33923	.13292	20	3	.18096	.04203
18	6	.41120	.19016	20	4	.24264	.07093
18	7	.48552	.26006	20	5	.30537	.10801
18	8	.56276	.34439	20	6	.36945	.15389
18	9	.64366	.44555	20	7	.43520	.20943
18	10	.72914	.56677	20	8	.50299	.27571
18	11	.82045	.71263	20	9	.57328	.35413
18	12	.91933	.88973	20	10	.64663	.44652
18	13	1.02827	1.10812	20	11	.72373	.55533
18	14	1.15119	1.38405	20	12	.80550	.68388
18	15	1.29469	1.74629	20	13	.89315	.83676
18	16	1.47145	2.25299	20	14	.98838	1.02066
18	17	1.71154	3.05148	20	15	1.09365	1.24570
18	18	2.12467	4.74259	20	16	1.21281	1.52820
19	1	.06287	.00757	20	17	1.35238	1.89705
19	2	.12622	.02292	20	18	1.52488	2.41059
19	3	.19021	.04637	20	19	1.76006	3.21654
19	4	.25518	.07835	20	20	2.16656	4.91687
19	5	.32139	.11948				

where

$E(X)$ = mean of half-normal distribution

$$= (2/\pi)^{1/2} = .79788456$$

and

$E(X^2)$ = second moment about the origin of half-normal distribution

$$= 1.0$$

The mixed moments of the order statistics were then computed according to equation (4.2) and checked by another formula given by Govindarajulu [12]:

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \mu_{i,j,N} = \binom{N}{2} [E(X)]^2 .$$

The mixed moments are tabulated in Table V.

From the moments now tabulated, the covariance matrix of the order statistics for each sample size less than twenty was computed. The upper right triangular portion of each of the 19 symmetric covariance matrices is tabulated in Table VI.

Application of Generalized Least Squares

The best linear unbiased estimate of σ,

$$\hat{\sigma} = \sum_{i=1}^{n} c'(i)\, u_{(i)} ,$$

TABLE V

MIXED MOMENTS OF HALF-NORMAL ORDER STATISTICS

N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$
2	1	2	.63661	7	2	6	.42792	9	1	6	.12002
3	1	2	.31156	7	2	7	.58287	9	1	7	.14625
3	1	3	.49563	7	3	4	.40738	9	1	8	.18094
3	2	3	1.10265	7	3	5	.51864	9	1	9	.23884
4	1	2	.18955	7	3	6	.66401	9	2	3	.12536
4	1	3	.27624	7	3	7	.90325	9	2	4	.16068
4	1	4	.41349	7	4	5	.72794	9	2	5	.19952
4	2	3	.59091	7	4	6	.92970	9	2	6	.24373
4	2	4	.87929	7	4	7	1.26203	9	2	7	.29684
4	3	4	1.47021	7	5	6	1.24876	9	2	8	.36713
5	1	2	.12863	7	5	7	1.68922	9	2	9	.48443
5	1	3	.18148	7	6	7	2.25771	9	3	4	.24684
5	1	4	.24905	8	1	2	.05600	9	3	5	.30613
5	1	5	.35775	8	1	3	.07646	9	3	6	.37363
5	2	3	.38040	8	1	4	.09879	9	3	7	.45475
5	2	4	.52015	8	1	5	.12406	9	3	8	.56210
5	2	5	.74516	8	1	6	.15424	9	3	9	.74131
5	3	4	.83679	8	1	7	.19392	9	4	5	.42103
5	3	5	1.19300	8	1	8	.25969	9	4	6	.51322
5	4	5	1.77374	8	2	3	.15628	9	4	7	.62403
6	1	2	.09342	8	2	4	.20163	9	4	8	.77071
6	1	3	.12963	8	2	5	.25295	9	4	9	1.01567
6	1	4	.17216	8	2	6	.31423	9	5	6	.66785
6	1	5	.22717	8	2	7	.39484	9	5	7	.81087
6	1	6	.31676	8	2	8	.52848	9	5	8	1.00029
6	2	3	.26848	8	3	4	.31149	9	5	9	1.31676
6	2	4	.35564	8	3	5	.39016	9	6	7	1.02582
6	2	5	.46846	8	3	6	.48413	9	6	8	1.26320
6	2	6	.65227	8	3	7	.60778	9	6	9	1.66006
6	3	4	.56066	8	3	8	.81285	9	7	8	1.58370
6	3	5	.73634	8	4	5	.54097	9	7	9	2.07556
6	3	6	1.02284	8	4	6	.67013	9	8	9	2.63680
6	4	5	1.05437	8	4	7	.84015	10	1	2	.03742
6	4	6	1.45866	8	4	8	1.12227	10	1	3	.05070
6	5	6	2.03236	8	5	6	.88261	10	1	4	.06475
7	1	2	.07109	8	5	7	1.10426	10	1	5	.07990
7	1	3	.09769	8	5	8	1.47232	10	1	6	.09665
7	1	4	.12754	8	6	7	1.42411	10	1	7	.11582
7	1	5	.16292	8	6	8	1.89297	10	1	8	.13895
7	1	6	.20911	8	7	8	2.45743	10	1	9	.16972
7	1	7	.28507	9	1	2	.04530	10	1	10	.22135
7	2	3	.20075	9	1	3	.06157	10	2	3	.10292
7	2	4	.26159	9	1	4	.07901	10	2	4	.13132
7	2	5	.33372	9	1	5	.09819	10	2	5	.16194

TABLE V (CONTINUED)

N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$
10	2	6	.19579	11	2	10	.32301	12	2	4	.09274
10	2	7	.23453	11	2	11	.41694	12	2	5	.11341
10	2	8	.28129	11	3	4	.16697	12	2	6	.13549
10	2	9	.34348	11	3	5	.20474	12	2	7	.15951
10	2	10	.44787	11	3	6	.24568	12	2	8	.18628
10	3	4	.20091	11	3	7	.29113	12	2	9	.21716
10	3	5	.24751	11	3	8	.34336	12	2	10	.25472
10	3	6	.29904	11	3	9	.40670	12	2	11	.30508
10	3	7	.35800	11	3	10	.49129	12	2	12	.39039
10	3	8	.42920	11	3	11	.63401	12	3	4	.14112
10	3	9	.52390	11	4	5	.27882	12	3	5	.17244
10	3	10	.68287	11	4	6	.33429	12	3	6	.20591
10	4	5	.33848	11	4	7	.39589	12	3	7	.24233
10	4	6	.40853	11	4	8	.46669	12	3	8	.28291
10	4	7	.48871	11	4	9	.55255	12	3	9	.32972
10	4	8	.58555	11	4	10	.66724	12	3	10	.38667
10	4	9	.71436	11	4	11	.86076	12	3	11	.46303
10	4	10	.93066	11	5	6	.42904	12	3	12	.59239
10	5	6	.52734	11	5	7	.50767	12	4	5	.23408
10	5	7	.63016	11	5	8	.59806	12	4	6	.27932
10	5	8	.75437	11	5	9	.70769	12	4	7	.32855
10	5	9	.91964	11	5	10	.85417	12	4	8	.38342
10	5	10	1.19726	11	5	11	1.10137	12	4	9	.44671
10	6	7	.78778	11	6	7	.62958	12	4	10	.52373
10	6	8	.94190	11	6	8	.74100	12	4	11	.62699
10	6	9	1.14705	11	6	9	.87616	12	4	12	.80196
10	6	10	1.49179	11	6	10	1.05680	12	5	6	.35697
10	7	8	1.15883	11	6	11	1.36175	12	5	7	.41960
10	7	9	1.40901	11	7	8	.90105	12	5	8	.48941
10	7	10	1.82970	11	7	9	1.06426	12	5	9	.56994
10	8	9	1.73004	11	7	10	1.28246	12	5	10	.66795
10	8	10	2.24103	11	7	11	1.65098	12	5	11	.79937
10	9	10	2.79962	11	8	9	1.28282	12	5	12	1.02211
11	1	2	.03145	11	8	10	1.54367	12	6	7	.51740
11	1	3	.04250	11	8	11	1.98447	12	6	8	.60304
11	1	4	.05409	11	9	10	1.86514	12	6	9	.70185
11	1	5	.06642	11	9	11	2.39234	12	6	10	.82213
11	1	6	.07978	11	10	11	2.94870	12	6	11	.98345
11	1	7	.09461	12	1	2	.02681	12	6	12	1.25690
11	1	8	.11165	12	1	3	.03616	12	7	8	.72750
11	1	9	.13231	12	1	4	.04590	12	7	9	.84603
11	1	10	.15990	12	1	5	.05615	12	7	10	.99034
11	1	11	.20644	12	1	6	.06711	12	7	11	1.18393
11	2	3	.08609	12	1	7	.07903	12	7	12	1.51219
11	2	4	.10947	12	1	8	.09231	12	8	9	1.00808
11	2	5	.13435	12	1	9	.10763	12	8	10	1.17891
11	2	6	.16130	12	1	10	.12627	12	8	11	1.40816
11	2	7	.19123	12	1	11	.15125	12	8	12	1.79702
11	2	8	.22563	12	1	12	.19357	12	9	10	1.39883
11	2	9	.26732	12	2	3	.07313	12	9	11	1.66873

TABLE V (CONTINUED)

N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$
12	9	12	2.12681	13	5	10	.54488	14	2	8	.13449
12	10	11	1.99059	13	5	11	.63342	14	2	9	.15412
12	10	12	2.53175	13	5	12	.75251	14	2	10	.17614
12	11	12	3.08620	13	5	13	.95511	14	2	11	.20167
13	1	2	.02313	13	6	7	.43420	14	2	12	.23292
13	1	3	.03116	13	6	8	.50272	14	2	13	.27507
13	1	4	.03946	13	6	9	.57936	14	2	14	.34700
13	1	5	.04814	13	6	10	.66804	14	3	4	.10486
13	1	6	.05732	13	6	11	.77632	14	3	5	.12753
13	1	7	.06716	13	6	12	.92199	14	3	6	.15130
13	1	8	.07790	13	6	13	1.16982	14	3	7	.17652
13	1	9	.08991	13	7	8	.60304	14	3	8	.20364
13	1	10	.10380	13	7	9	.69453	14	3	9	.23333
13	1	11	.12075	13	7	10	.80042	14	3	10	.26662
13	1	12	.14355	13	7	11	.92972	14	3	11	.30523
13	1	13	.18232	13	7	12	1.10371	14	3	12	.35249
13	2	3	.06292	13	7	13	1.39979	14	3	13	.41624
13	2	4	.07963	13	8	9	.82111	14	3	14	.52502
13	2	5	.09711	13	8	10	.94563	14	4	5	.17234
13	2	6	.11559	13	8	11	1.09771	14	4	6	.20436
13	2	7	.13540	13	8	12	1.30240	14	4	7	.23834
13	2	8	.15702	13	8	13	1.65082	14	4	8	.27488
13	2	9	.18120	13	9	10	1.10937	14	4	9	.31488
13	2	10	.20917	13	9	11	1.28669	14	4	10	.35973
13	2	11	.24331	13	9	12	1.52543	14	4	11	.41176
13	2	12	.28922	13	9	13	1.93194	14	4	12	.47544
13	2	13	.36729	13	10	11	1.50778	14	4	13	.56134
13	3	4	.12094	13	10	12	1.78547	14	4	14	.70794
13	3	5	.14739	13	10	13	2.25858	14	5	6	.25966
13	3	6	.17536	13	11	12	2.10767	14	5	7	.30268
13	3	7	.20535	13	11	13	2.66102	14	5	8	.34895
13	3	8	.23808	13	12	13	3.21381	14	5	9	.39961
13	3	9	.27468	14	1	2	.02017	14	5	10	.45641
13	3	10	.31703	14	1	3	.02713	14	5	11	.52231
13	3	11	.36871	14	1	4	.03430	14	5	12	.60297
13	3	12	.43822	14	1	5	.04176	14	5	13	.71178
13	3	13	.55645	14	1	6	.04957	14	5	14	.89751
13	4	5	.19957	14	1	7	.05787	14	6	7	.37042
13	4	6	.23731	14	1	8	.06678	14	6	8	.42684
13	4	7	.27777	14	1	9	.07654	14	6	9	.48861
13	4	8	.32194	14	1	10	.08748	14	6	10	.55788
13	4	9	.37133	14	1	11	.10018	14	6	11	.63824
13	4	10	.42848	14	1	12	.11571	14	6	12	.73662
13	4	11	.49823	14	1	13	.13666	14	6	13	.86936
13	4	12	.59205	14	1	14	.17241	14	6	14	1.09593
13	4	13	.75163	14	2	3	.05472	14	7	8	.50983
13	5	6	.30229	14	2	4	.06915	14	7	9	.58332
13	5	7	.35363	14	2	5	.08414	14	7	10	.66573
13	5	8	.40968	14	2	6	.09987	14	7	11	.76135
13	5	9	.47236	14	2	7	.11655	14	7	12	.87843

TABLE V (CONTINUED)

N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$
14	7	13	1.03640	15	3	4	.09183	15	8	9	.58353
14	7	14	1.30611	15	3	5	.11149	15	8	10	.66124
14	8	9	.68575	15	3	6	.13199	15	8	11	.74861
14	8	10	.78219	15	3	7	.15354	15	8	12	.85023
14	8	11	.89413	15	3	8	.17647	15	8	13	.97495
14	8	12	1.03119	15	3	9	.20120	15	8	14	1.14368
14	8	13	1.21616	15	3	10	.22833	15	8	15	1.43265
14	8	14	1.53201	15	3	11	.25882	15	9	10	.76548
14	9	10	.91057	15	3	12	.29427	15	9	11	.86619
14	9	11	1.04022	15	3	13	.33776	15	9	12	.98335
14	9	12	1.19900	15	3	14	.39659	15	9	13	1.12716
14	9	13	1.41333	15	3	15	.49727	15	9	14	1.32175
14	9	14	1.77941	15	4	5	.15042	15	9	15	1.65506
14	10	11	1.20542	15	4	6	.17800	15	10	11	.99611
14	10	12	1.38834	15	4	7	.20700	15	10	12	1.13020
14	10	13	1.63533	15	4	8	.23785	15	10	13	1.29483
14	10	14	2.05735	15	4	9	.27112	15	10	14	1.51763
14	11	12	1.61043	15	4	10	.30763	15	10	15	1.89935
14	11	13	1.89493	15	4	11	.34866	15	11	12	1.29666
14	11	14	2.38127	15	4	12	.39637	15	11	13	1.48448
14	12	13	2.21741	15	4	13	.45490	15	11	14	1.73875
14	12	14	2.78152	15	4	14	.53406	15	11	15	2.17453
14	13	14	3.33287	15	4	15	.66958	15	12	13	1.70745
15	1	2	.01774	15	5	6	.22570	15	12	14	1.99795
15	1	3	.02384	15	5	7	.26237	15	12	15	2.49607
15	1	4	.03010	15	5	8	.30137	15	13	14	2.32070
15	1	5	.03658	15	5	9	.34343	15	13	15	2.89440
15	1	6	.04333	15	5	10	.38958	15	14	15	3.44446
15	1	7	.05043	15	5	11	.44146	16	1	2	.01573
15	1	8	.05798	15	5	12	.50179	16	1	3	.02112
15	1	9	.06612	15	5	13	.57581	16	1	4	.02664
15	1	10	.07505	15	5	14	.67592	16	1	5	.03232
15	1	11	.08509	15	5	15	.84731	16	1	6	.03822
15	1	12	.09676	15	6	7	.32025	16	1	7	.04437
15	1	13	.11108	15	6	8	.36771	16	1	8	.05087
15	1	14	.13045	15	6	9	.41888	16	1	9	.05779
15	1	15	.16359	15	6	10	.47505	16	1	10	.06527
15	2	3	.04805	15	6	11	.53818	16	1	11	.07350
15	2	4	.06063	15	6	12	.61160	16	1	12	.08276
15	2	5	.07365	15	6	13	.70169	16	1	13	.09356
15	2	6	.08722	15	6	14	.82355	16	1	14	.10683
15	2	7	.10149	15	6	15	1.03218	16	1	15	.12481
15	2	8	.11667	15	7	8	.43776	16	1	16	.15569
15	2	9	.13304	15	7	9	.49847	16	2	3	.04253
15	2	10	.15100	15	7	10	.56512	16	2	4	.05362
15	2	11	.17118	15	7	11	.64003	16	2	5	.06503
15	2	12	.19464	15	7	12	.72715	16	2	6	.07688
15	2	13	.22343	15	7	13	.83407	16	2	7	.08925
15	2	14	.26236	15	7	14	.97869	16	2	8	.10229
15	2	15	.32900	15	7	15	1.22634	16	2	9	.11620

TABLE V (CONTINUED)

N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$
16	2	10	.13123	16	6	14	.67058	17	1	4	.02374
16	2	11	.14776	16	6	15	.78316	17	1	5	.02877
16	2	12	.16638	16	6	16	.97643	17	1	6	.03397
16	2	13	.18807	16	7	8	.38063	17	1	7	.03937
16	2	14	.21474	16	7	9	.43187	17	1	8	.04503
16	2	15	.25088	16	7	10	.48724	17	1	9	.05100
16	2	16	.31293	16	7	11	.54815	17	1	10	.05739
16	3	4	.08111	16	7	12	.61677	17	1	11	.06430
16	3	5	.09835	16	7	13	.69673	17	1	12	.07191
16	3	6	.11622	16	7	14	.79507	17	1	13	.08051
16	3	7	.13489	16	7	15	.92839	17	1	14	.09054
16	3	8	.15459	16	7	16	1.15730	17	1	15	.10290
16	3	9	.17558	16	8	9	.50389	17	1	16	.11969
16	3	10	.19827	16	8	10	.56829	17	1	17	.14857
16	3	11	.22322	16	8	11	.63913	17	2	3	.03792
16	3	12	.25132	16	8	12	.71894	17	2	4	.04776
16	3	13	.28406	16	8	13	.81195	17	2	5	.05786
16	3	14	.32432	16	8	14	.92635	17	2	6	.06830
16	3	15	.37889	16	8	15	1.08146	17	2	7	.07914
16	3	16	.47256	16	8	16	1.34780	17	2	8	.09050
16	4	5	.13251	16	9	10	.65511	17	2	9	.10250
16	4	6	.15653	16	9	11	.73648	17	2	10	.11532
16	4	7	.18163	16	9	12	.82815	17	2	11	.12920
16	4	8	.20810	16	9	13	.93501	17	2	12	.14450
16	4	9	.23632	16	9	14	1.06645	17	2	13	.16176
16	4	10	.26682	16	9	15	1.24469	17	2	14	.18191
16	4	11	.30036	16	9	16	1.55079	17	2	15	.20673
16	4	12	.33813	16	10	11	.84227	17	2	16	.24045
16	4	13	.38215	16	10	12	.94670	17	2	17	.29846
16	4	14	.43627	16	10	13	1.06844	17	3	4	.07219
16	4	15	.50964	16	10	14	1.21819	17	3	5	.08743
16	4	16	.63557	16	10	15	1.42130	17	3	6	.10317
16	5	6	.19815	16	10	16	1.77018	17	3	7	.11952
16	5	7	.22984	16	11	12	1.07797	17	3	8	.13666
16	5	8	.26326	16	11	13	1.21596	17	3	9	.15476
16	5	9	.29889	16	11	14	1.38574	17	3	10	.17410
16	5	10	.33739	16	11	15	1.61605	17	3	11	.19504
16	5	11	.37974	16	11	16	2.01174	17	3	12	.21812
16	5	12	.42744	16	12	13	1.38352	17	3	13	.24416
16	5	13	.48303	16	12	14	1.57567	17	3	14	.27455
16	5	14	.55137	16	12	15	1.83639	17	3	15	.31200
16	5	15	.64402	16	12	16	2.28449	17	3	16	.36286
16	5	16	.80308	16	13	14	1.79942	17	3	17	.45040
16	6	7	.27997	16	13	15	2.09525	17	4	5	.11767
16	6	8	.32056	16	13	16	2.60394	17	4	6	.13881
16	6	9	.36383	16	14	15	2.41823	17	4	7	.16077
16	6	10	.41061	16	14	16	3.00056	17	4	8	.18378
16	6	11	.46206	16	15	16	3.54947	17	4	9	.20809
16	6	12	.52001	17	1	2	.01404	17	4	10	.23406
16	6	13	.58754	17	1	3	.01884	17	4	11	.26219

TABLE V (CONTINUED)

N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$
17	4	12	.29318	17	9	12	.71068	18	1	18	.14212
17	4	13	.32815	17	9	13	.79477	18	2	3	.03402
17	4	14	.36898	17	9	14	.89298	18	2	4	.04282
17	4	15	.41928	17	9	15	1.01400	18	2	5	.05183
17	4	16	.48760	17	9	16	1.17843	18	2	6	.06110
17	4	17	.60518	17	9	17	1.46151	18	2	7	.07069
17	5	6	.17547	17	10	11	.72452	18	2	8	.08068
17	5	7	.20317	17	10	12	.80909	18	2	9	.09117
17	5	8	.23218	17	10	13	.90454	18	2	10	.10227
17	5	9	.26284	17	10	14	1.01603	18	2	11	.11415
17	5	10	.29560	17	10	15	1.15342	18	2	12	.12704
17	5	11	.33108	17	10	16	1.34012	18	2	13	.14126
17	5	12	.37017	17	10	17	1.66159	18	2	14	.15734
17	5	13	.41428	17	11	12	.91625	18	2	15	.17614
17	5	14	.46578	17	11	13	1.02393	18	2	16	.19934
17	5	15	.52922	17	11	14	1.14972	18	2	17	.23092
17	5	16	.61541	17	11	15	1.30475	18	2	18	.28538
17	5	17	.76374	17	11	16	1.51546	18	3	4	.06468
17	6	7	.24705	17	11	17	1.87833	18	3	5	.07826
17	6	8	.28225	17	12	13	1.15639	18	3	6	.09223
17	6	9	.31944	17	12	14	1.29783	18	3	7	.10669
17	6	10	.35918	17	12	15	1.47220	18	3	8	.12176
17	6	11	.40221	17	12	16	1.70922	18	3	9	.13756
17	6	12	.44963	17	12	17	2.11749	18	3	10	.15430
17	6	13	.50315	17	13	14	1.46637	18	3	11	.17221
17	6	14	.56563	17	13	15	1.66238	18	3	12	.19164
17	6	15	.64260	17	13	16	1.92889	18	3	13	.21309
17	6	16	.74718	17	13	17	2.38810	18	3	14	.23733
17	6	17	.92716	17	14	15	1.88682	18	3	15	.26567
17	7	8	.33444	17	14	16	2.18744	18	3	16	.30066
17	7	9	.37839	17	14	17	2.70568	18	3	17	.34826
17	7	10	.42535	17	15	16	2.51064	18	3	18	.43038
17	7	11	.47621	17	15	17	3.10076	18	4	5	.10523
17	7	12	.53226	17	16	17	3.64864	18	4	6	.12398
17	7	13	.59551	18	1	2	.01261	18	4	7	.14339
17	7	14	.66936	18	1	3	.01692	18	4	8	.16360
17	7	15	.76036	18	1	4	.02130	18	4	9	.18481
17	7	16	.88398	18	1	5	.02579	18	4	10	.20727
17	7	17	1.09676	18	1	6	.03040	18	4	11	.23131
17	8	9	.44035	18	1	7	.03518	18	4	12	.25739
17	8	10	.49484	18	1	8	.04016	18	4	13	.28617
17	8	11	.55386	18	1	9	.04538	18	4	14	.31870
17	8	12	.61890	18	1	10	.05091	18	4	15	.35675
17	8	13	.69231	18	1	11	.05683	18	4	16	.40370
17	8	14	.77803	18	1	12	.06325	18	4	17	.46760
17	8	15	.88365	18	1	13	.07034	18	4	18	.57783
17	8	16	1.02715	18	1	14	.07834	18	5	6	.15654
17	8	17	1.27417	18	1	15	.08771	18	5	7	.18099
17	9	10	.56857	18	1	16	.09927	18	5	8	.20646
17	9	11	.63617	18	1	17	.11499	18	5	9	.23319

TABLE V (CONTINUED)

N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$
18	5	10	.26149	18	9	18	1.38386	19	1	15	.07627
18	5	11	.29178	18	10	11	.63166	19	1	16	.08505
18	5	12	.32464	18	10	12	.70207	19	1	17	.09589
18	5	13	.36091	18	10	13	.77981	19	1	18	.11068
18	5	14	.40190	18	10	14	.86770	19	1	19	.13624
18	5	15	.44985	18	10	15	.97052	19	2	3	.03070
18	5	16	.50902	18	10	16	1.09744	19	2	4	.03862
18	5	17	.58955	18	10	17	1.27024	19	2	5	.04670
18	5	18	.72847	18	10	18	1.56843	19	2	6	.05500
18	6	7	.21978	18	11	12	.79177	19	2	7	.06355
18	6	8	.25063	18	11	13	.87915	19	2	8	.07242
18	6	9	.28302	18	11	14	.97796	19	2	9	.08167
18	6	10	.31731	18	11	15	1.09356	19	2	10	.09140
18	6	11	.35401	18	11	16	1.23627	19	2	11	.10172
18	6	12	.39382	18	11	17	1.43059	19	2	12	.11278
18	6	13	.43777	18	11	18	1.76595	19	2	13	.12479
18	6	14	.48745	18	12	13	.98752	19	2	14	.13808
18	6	15	.54555	18	12	14	1.09810	19	2	15	.15312
18	6	16	.61725	18	12	15	1.22749	19	2	16	.17073
18	6	17	.71485	18	12	16	1.38725	19	2	17	.19250
18	6	18	.88321	18	12	17	1.60481	19	2	18	.22217
18	7	8	.29647	18	12	18	1.98034	19	2	19	.27347
18	7	9	.33468	18	13	14	1.23161	19	3	4	.05829
18	7	10	.37514	18	13	15	1.37613	19	3	5	.07047
18	7	11	.41845	18	13	16	1.55461	19	3	6	.08297
18	7	12	.46544	18	13	17	1.79768	19	3	7	.09586
18	7	13	.51731	18	13	18	2.21736	19	3	8	.10922
18	7	14	.57594	18	14	15	1.54554	19	3	9	.12317
18	7	15	.64452	18	14	16	1.74501	19	3	10	.13783
18	7	16	.72916	18	14	17	2.01675	19	3	11	.15338
18	7	17	.84436	18	14	18	2.48605	19	3	12	.17004
18	7	18	1.04311	18	15	16	1.97007	19	3	13	.18815
18	8	9	.38865	18	15	17	2.27504	19	3	14	.20817
18	8	10	.43552	18	15	18	2.80195	19	3	15	.23083
18	8	11	.48569	18	16	17	2.59842	19	3	16	.25737
18	8	12	.54012	18	16	18	3.19565	19	3	17	.29018
18	8	13	.60021	18	17	18	3.74259	19	3	18	.33490
18	8	14	.66814	19	1	2	.01139	19	3	19	.41222
18	8	15	.74759	19	1	3	.01527	19	4	5	.09468
18	8	16	.84566	19	1	4	.01922	19	4	6	.11145
18	8	17	.97915	19	1	5	.02325	19	4	7	.12873
18	8	18	1.20946	19	1	6	.02738	19	4	8	.14665
18	9	10	.49911	19	1	7	.03164	19	4	9	.16535
18	9	11	.55645	19	1	8	.03606	19	4	10	.18502
18	9	12	.61865	19	1	9	.04067	19	4	11	.20587
18	9	13	.68734	19	1	10	.04552	19	4	12	.22822
18	9	14	.76498	19	1	11	.05066	19	4	13	.25251
18	9	15	.85581	19	1	12	.05617	19	4	14	.27936
18	9	16	.96792	19	1	13	.06216	19	4	15	.30975
18	9	17	1.12053	19	1	14	.06878	19	4	16	.34535

TABLE V (CONTINUED)

N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$
19	4	17	.38935	19	8	17	.81151	19	15	18	2.10042
19	4	18	.44935	19	8	18	.93627	19	15	19	2.57894
19	4	19	.55306	19	8	19	1.15197	19	16	17	2.04956
19	5	6	.14058	19	9	10	.44230	19	16	18	2.35846
19	5	7	.16234	19	9	11	.49172	19	16	19	2.89332
19	5	8	.18490	19	9	12	.54472	19	17	18	2.68202
19	5	9	.20845	19	9	13	.60230	19	17	19	3.28577
19	5	10	.23320	19	9	14	.66597	19	18	19	3.83185
19	5	11	.25946	19	9	15	.73806	20	1	2	.01034
19	5	12	.28760	19	9	16	.82251	20	1	3	.01386
19	5	13	.31818	19	9	17	.92692	20	1	4	.01743
19	5	14	.35199	19	9	18	1.06929	20	1	5	.02107
19	5	15	.39026	19	9	19	1.31546	20	1	6	.02479
19	5	16	.43508	19	10	11	.55674	20	1	7	.02862
19	5	17	.49050	19	10	12	.61658	20	1	8	.03257
19	5	18	.56604	19	10	13	.68161	20	1	9	.03668
19	5	19	.69664	19	10	14	.75352	20	1	10	.04097
19	6	7	.19689	19	10	15	.83494	20	1	11	.04549
19	6	8	.22420	19	10	16	.93033	20	1	12	.05029
19	6	9	.25270	19	10	17	1.04827	20	1	13	.05544
19	6	10	.28266	19	10	18	1.20910	20	1	14	.06105
19	6	11	.31444	19	10	19	1.48722	20	1	15	.06726
19	6	12	.34851	19	11	12	.69311	20	1	16	.07429
19	6	13	.38553	19	11	13	.76600	20	1	17	.08254
19	6	14	.42645	19	11	14	.84661	20	1	18	.09276
19	6	15	.47278	19	11	15	.93789	20	1	19	.10671
19	6	16	.52705	19	11	16	1.04484	20	1	20	.13086
19	6	17	.59413	19	11	17	1.17708	20	2	3	.02785
19	6	18	.68560	19	11	18	1.35743	20	2	4	.03501
19	6	19	.84372	19	11	19	1.66932	20	2	5	.04231
19	7	8	.26482	19	12	13	.85689	20	2	6	.04978
19	7	9	.29841	19	12	14	.94679	20	2	7	.05746
19	7	10	.33372	19	12	15	1.04859	20	2	8	.06539
19	7	11	.37118	19	12	16	1.16788	20	2	9	.07363
19	7	12	.41134	19	12	17	1.31539	20	2	10	.08224
19	7	13	.45497	19	12	18	1.51658	20	2	11	.09130
19	7	14	.50321	19	12	19	1.86457	20	2	12	.10093
19	7	15	.55783	19	13	14	1.05623	20	2	13	.11127
19	7	16	.62180	19	13	15	1.16940	20	2	14	.12252
19	7	17	.70089	19	13	16	1.30203	20	2	15	.13497
19	7	18	.80872	19	13	17	1.46605	20	2	16	.14909
19	7	19	.99515	19	13	18	1.68981	20	2	17	.16564
19	8	9	.34593	19	13	19	2.07687	20	2	18	.18614
19	8	10	.38677	19	14	15	1.30386	20	2	19	.21412
19	8	11	.43010	19	14	16	1.45115	20	2	20	.26259
19	8	12	.47655	19	14	17	1.63333	20	3	4	.05281
19	8	13	.52702	19	14	18	1.88189	20	3	5	.06381
19	8	14	.58282	19	14	19	2.31198	20	3	6	.07506
19	8	15	.64600	19	15	16	1.62134	20	3	7	.08662
19	8	16	.72002	19	15	17	1.82394	20	3	8	.09857

TABLE V (CONTINUED)

N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$	N	I	J	$\mu_{I,J,N}$
20	3	9	.11098	20	6	14	.37739	20	10	18	1.00452
20	3	10	.12395	20	6	15	.41566	20	10	19	1.15494
20	3	11	.13761	20	6	16	.45905	20	10	20	1.41555
20	3	12	.15212	20	6	17	.50993	20	11	12	.61314
20	3	13	.16769	20	6	18	.57293	20	11	13	.67520
20	3	14	.18463	20	6	19	.65896	20	11	14	.74275
20	3	15	.20339	20	6	20	.80798	20	11	15	.81755
20	3	16	.22465	20	7	8	.23812	20	11	16	.90236
20	3	17	.24959	20	7	9	.26792	20	11	17	1.00186
20	3	18	.28046	20	7	10	.29907	20	11	18	1.12507
20	3	19	.32262	20	7	11	.33188	20	11	19	1.29336
20	3	20	.39564	20	7	12	.36672	20	11	20	1.58495
20	4	5	.08567	20	7	13	.40413	20	12	13	.75289
20	4	6	.10075	20	7	14	.44483	20	12	14	.82801
20	4	7	.11625	20	7	15	.48990	20	12	15	.91119
20	4	8	.13227	20	7	16	.54098	20	12	16	1.00552
20	4	9	.14890	20	7	17	.60091	20	12	17	1.11619
20	4	10	.16629	20	7	18	.67511	20	12	18	1.25324
20	4	11	.18459	20	7	19	.77643	20	12	19	1.44045
20	4	12	.20404	20	7	20	.95194	20	12	20	1.76486
20	4	13	.22491	20	8	9	.31013	20	13	14	.91996
20	4	14	.24763	20	8	10	.34612	20	13	15	1.01211
20	4	15	.27277	20	8	11	.38401	20	13	16	1.11661
20	4	16	.30127	20	8	12	.42426	20	13	17	1.23923
20	4	17	.33471	20	8	13	.46748	20	13	18	1.39108
20	4	18	.37609	20	8	14	.51451	20	13	19	1.59854
20	4	19	.43261	20	8	15	.56657	20	13	20	1.95808
20	4	20	.53050	20	8	16	.62560	20	14	15	1.12251
20	5	6	.12698	20	8	17	.69485	20	14	16	1.23802
20	5	7	.14648	20	8	18	.78057	20	14	17	1.37358
20	5	8	.16663	20	8	19	.89765	20	14	18	1.54147
20	5	9	.18756	20	8	20	1.10046	20	14	19	1.77087
20	5	10	.20944	20	9	10	.39511	20	14	20	2.16850
20	5	11	.23247	20	9	11	.43828	20	15	16	1.37332
20	5	12	.25693	20	9	12	.48413	20	15	17	1.52312
20	5	13	.28320	20	9	13	.53336	20	15	18	1.70868
20	5	14	.31178	20	9	14	.58693	20	15	19	1.96225
20	5	15	.34342	20	9	15	.64625	20	15	20	2.40187
20	5	16	.37928	20	9	16	.71350	20	16	17	1.69402
20	5	17	.42135	20	9	17	.79239	20	16	18	1.89947
20	5	18	.47343	20	9	18	.89007	20	16	19	2.18028
20	5	19	.54455	20	9	19	1.02347	20	16	20	2.66727
20	5	20	.66774	20	9	20	1.25458	20	17	18	2.12561
20	6	7	.17748	20	10	11	.49517	20	17	19	2.43811
20	6	8	.20185	20	10	12	.54686	20	17	20	2.98027
20	6	9	.22716	20	10	13	.60236	20	18	19	2.76183
20	6	10	.25361	20	10	14	.66275	20	18	20	3.37157
20	6	11	.28147	20	10	15	.72962	20	19	20	3.91687
20	6	12	.31106	20	10	16	.80544				
20	6	13	.34283	20	10	17	.89439				

TABLE VI

VARIANCES AND COVARIANCES OF HALF-NORMAL ORDER STATISTICS

N	I	J	VAR/COV	N	I	J	VAR/COV	N	I	J	VAR/COV
2	1	1	.144927	6	2	5	.035343	8	1	4	.012752
2	1	2	.109227	6	2	6	.028804	8	1	5	.011542
2	2	2	.363380	6	3	3	.078073	8	1	6	.010337
3	1	1	.080639	6	3	4	.068899	8	1	7	.009063
3	1	2	.066297	6	3	5	.059544	8	1	8	.007513
3	1	3	.051423	6	3	6	.048594	8	2	2	.032100
3	2	2	.168187	6	4	4	.109794	8	2	3	.029308
3	2	3	.131260	6	4	5	.095089	8	2	4	.026683
3	3	3	.343356	6	4	6	.077789	8	2	5	.024156
4	1	1	.052015	6	5	5	.160431	8	2	6	.021640
4	1	2	.044530	6	5	6	.131887	8	2	7	.018978
4	1	3	.037393	6	6	6	.296659	8	2	8	.015736
4	1	4	.029616	7	1	1	.021050	8	3	3	.046954
4	2	2	.102875	7	1	2	.019038	8	3	4	.042768
4	2	3	.086596	7	1	3	.017185	8	3	5	.038736
4	2	4	.068769	7	1	4	.015434	8	3	6	.034716
4	3	3	.169417	7	1	5	.013721	8	3	7	.030460
4	3	4	.135312	7	1	6	.011943	8	3	8	.025269
4	4	4	.324783	7	1	7	.009821	8	4	4	.062632
5	1	1	.036554	7	2	2	.040266	8	4	5	.056765
5	1	2	.032081	7	2	3	.036363	8	4	6	.050908
5	1	3	.027919	7	2	4	.032672	8	4	7	.044697
5	1	4	.023805	7	2	5	.029059	8	4	8	.037109
5	1	5	.019152	7	2	6	.025304	8	5	5	.081118
5	2	2	.070818	7	2	7	.020817	8	5	6	.072824
5	2	3	.061706	7	3	3	.059359	8	5	7	.064009
5	2	4	.052680	7	3	4	.053371	8	5	8	.053208
5	2	5	.042440	7	3	5	.047502	8	6	6	.106511
5	3	3	.109044	7	3	6	.041393	8	6	7	.093791
5	3	4	.093307	7	3	7	.034082	8	6	8	.078134
5	3	5	.075362	7	4	4	.080641	8	7	7	.150736
5	4	4	.165483	7	4	5	.071853	8	7	8	.126115
5	4	5	.134364	7	4	6	.062685	8	8	8	.277036
5	5	5	.309371	7	4	7	.051679	9	1	1	.013735
6	1	1	.027184	7	5	5	.108529	9	1	2	.012649
6	1	2	.024273	7	5	6	.094868	9	1	3	.011647
6	1	3	.021585	7	5	7	.078391	9	1	4	.010706
6	1	4	.019013	7	6	6	.155406	9	1	5	.009809
6	1	5	.016400	7	6	7	.129004	9	1	6	.008933
6	1	6	.013356	7	7	7	.286042	9	1	7	.008049
6	2	2	.052199	8	1	1	.016803	9	1	8	.007100
6	2	3	.046452	8	1	2	.015350	9	1	9	.005925
6	2	4	.040945	8	1	3	.014011	9	2	2	.026238

TABLE VI (CONTINUED)

N	I	J	VAR/COV	N	I	J	VAR/COV	N	I	J	VAR/COV
9	2	3	.024164	10	2	4	.018796	11	1	5	.007311
9	2	4	.022216	10	2	5	.017379	11	1	6	.006791
9	2	5	.020358	10	2	6	.016010	11	1	7	.006283
9	2	6	.018543	10	2	7	.014656	11	1	8	.005776
9	2	7	.016709	10	2	8	.013271	11	1	9	.005252
9	2	8	.014741	10	2	9	.011766	11	1	10	.004677
9	2	9	.012302	10	2	10	.009875	11	1	11	.003945
9	3	3	.038216	10	3	3	.031789	11	2	2	.018534
9	3	4	.035148	10	3	4	.029459	11	2	3	.017289
9	3	5	.032218	10	3	5	.027245	11	2	4	.016118
9	3	6	.029355	10	3	6	.025104	11	2	5	.015008
9	3	7	.026460	10	3	7	.022987	11	2	6	.013941
9	3	8	.023351	10	3	8	.020820	11	2	7	.012899
9	3	9	.019496	10	3	9	.018464	11	2	8	.011858
9	4	4	.050441	10	3	10	.015502	11	2	9	.010782
9	4	5	.046256	10	4	4	.041688	11	2	10	.009599
9	4	6	.042163	10	4	5	.038566	11	2	11	.008095
9	4	7	.038021	10	4	6	.035545	11	3	3	.026902
9	4	8	.033567	10	4	7	.032556	11	3	4	.025086
9	4	9	.028040	10	4	8	.029495	11	3	5	.023362
9	5	5	.063972	10	4	9	.026164	11	3	6	.021705
9	5	6	.058349	10	4	10	.021974	11	3	7	.020086
9	5	7	.052649	10	5	5	.052212	11	3	8	.018469
9	5	8	.046513	10	5	6	.048142	11	3	9	.016797
9	5	9	.038883	10	5	7	.044111	11	3	10	.014959
9	6	6	.080611	10	5	8	.039980	11	3	11	.012621
9	6	7	.072808	10	5	9	.035480	11	4	4	.035139
9	6	8	.064387	10	5	10	.029813	11	4	5	.032731
9	6	9	.053891	10	6	6	.064297	11	4	6	.030416
9	7	7	.104256	10	6	7	.058949	11	4	7	.028153
9	7	8	.092358	10	6	8	.053460	11	4	8	.025891
9	7	9	.077461	10	6	9	.047473	11	4	9	.023551
9	8	8	.146491	10	6	10	.039919	11	4	10	.020977
9	8	9	.123366	10	7	7	.079634	11	4	11	.017701
9	9	9	.269286	10	7	8	.072285	11	5	5	.043659
10	1	1	.011444	10	7	9	.064252	11	5	6	.040582
10	1	2	.010610	10	7	10	.054091	11	5	7	.037573
10	1	3	.009838	10	8	8	.101985	11	5	8	.034563
10	1	4	.009114	10	8	9	.090800	11	5	9	.031448
10	1	5	.008426	10	8	10	.076591	11	5	10	.028020
10	1	6	.007761	10	9	9	.142657	11	5	11	.023652
10	1	7	.007105	10	9	10	.120803	11	6	6	.053013
10	1	8	.006433	10	10	10	.262530	11	6	7	.049101
10	1	9	.005703	11	1	1	.009687	11	6	8	.045185
10	1	10	.004787	11	1	2	.009030	11	6	9	.041129
10	2	2	.021876	11	1	3	.008423	11	6	10	.036661
10	2	3	.020287	11	1	4	.007852	11	6	11	.030961

TABLE VI (CONTINUED)

N	I	J	VAR/COV	N	I	J	VAR/COV	N	I	J	VAR/COV
*****	*****	*****	*****	*****	*****	*****	*****	*****	*****	*****	*****
11	7	7	.064070	12	3	12	.010473	13	1	2	.006777
11	7	8	.058993	12	4	4	.030083	13	1	3	.006381
11	7	9	.053727	12	4	5	.028179	13	1	4	.006006
11	7	10	.047920	12	4	6	.026354	13	1	5	.005651
11	7	11	.040499	12	4	7	.024582	13	1	6	.005311
11	8	8	.078444	12	4	8	.022836	13	1	7	.004983
11	8	9	.071504	12	4	9	.021074	13	1	8	.004663
11	8	10	.063834	12	4	10	.019236	13	1	9	.004344
11	8	11	.054008	12	4	11	.017196	13	1	10	.004021
11	9	9	.099793	12	4	12	.014572	13	1	11	.003681
11	9	10	.089226	12	5	5	.037181	13	1	12	.003301
11	9	11	.075634	12	5	6	.034780	13	1	13	.002808
11	10	10	.139195	12	5	7	.032449	13	2	2	.013820
11	10	11	.118432	12	5	8	.030149	13	2	3	.013010
11	11	11	.256574	12	5	9	.027829	13	2	4	.012246
12	1	1	.008308	12	5	10	.025406	13	2	5	.011521
12	1	2	.007782	12	5	11	.022717	13	2	6	.010828
12	1	3	.007295	12	5	12	.019255	13	2	7	.010159
12	1	4	.006836	12	6	6	.044735	13	2	8	.009504
12	1	5	.006401	12	6	7	.041748	13	2	9	.008854
12	1	6	.005985	12	6	8	.038799	13	2	10	.008193
12	1	7	.005581	12	6	9	.035823	13	2	11	.007499
12	1	8	.005183	12	6	10	.032713	13	2	12	.006722
12	1	9	.004782	12	6	11	.029259	13	2	13	.005713
12	1	10	.004364	12	6	12	.024808	13	3	3	.020052
12	1	11	.003901	12	7	7	.053247	13	3	4	.018877
12	1	12	.003305	12	7	8	.049504	13	3	5	.017763
12	2	2	.015914	12	7	9	.045723	13	3	6	.016697
12	2	3	.014918	12	7	10	.041771	13	3	7	.015667
12	2	4	.013981	12	7	11	.037375	13	3	8	.014660
12	2	5	.013092	12	7	12	.031704	13	3	9	.013660
12	2	6	.012240	12	8	8	.063539	13	3	10	.012644
12	2	7	.011414	12	8	9	.058717	13	3	11	.011576
12	2	8	.010599	12	8	10	.053670	13	3	12	.010381
12	2	9	.009779	12	8	11	.048050	13	3	13	.008830
12	2	10	.008923	12	8	12	.040788	13	4	4	.026084
12	2	11	.007973	12	9	9	.077172	13	4	5	.024547
12	2	12	.006752	12	9	10	.070596	13	4	6	.023076
12	3	3	.023090	12	9	11	.063259	13	4	7	.021655
12	3	4	.021642	12	9	12	.053756	13	4	8	.020265
12	3	5	.020269	12	10	10	.097718	13	4	9	.018884
12	3	6	.018954	12	10	11	.087691	13	4	10	.017480
12	3	7	.017677	12	10	12	.074652	13	4	11	.016004
12	3	8	.016419	12	11	11	.136060	13	4	12	.014353
12	3	9	.015151	12	11	12	.116244	13	4	13	.012208
12	3	10	.013828	12	12	12	.251271	13	5	5	.032123
12	3	11	.012361	13	1	1	.007206	13	5	6	.030204

TABLE VI (CONTINUED)

N	I	J	VAR/COV	N	I	J	VAR/COV	N	I	J	VAR/COV
*****************				*****************				*****************			
13	5	7	.028348	14	1	5	.005024	14	5	6	.026516
13	5	8	.026532	14	1	6	.004743	14	5	7	.025008
13	5	9	.024727	14	1	7	.004472	14	5	8	.023539
13	5	10	.022892	14	1	8	.004209	14	5	9	.022092
13	5	11	.020962	14	1	9	.003949	14	5	10	.020644
13	5	12	.018802	14	1	10	.003690	14	5	11	.019162
13	5	13	.015995	14	1	11	.003425	14	5	12	.017593
13	6	6	.038410	14	1	12	.003144	14	5	13	.015825
13	6	7	.036057	14	1	13	.002828	14	5	14	.013507
13	6	8	.033754	14	1	14	.002415	14	6	6	.033431
13	6	9	.031464	14	2	2	.012118	14	6	7	.031534
13	6	10	.029134	14	2	3	.011450	14	6	8	.029686
13	6	11	.026683	14	2	4	.010818	14	6	9	.027865
13	6	12	.023939	14	2	5	.010219	14	6	10	.026042
13	6	13	.020370	14	2	6	.009646	14	6	11	.024177
13	7	7	.045259	14	2	7	.009094	14	6	12	.022200
13	7	8	.042379	14	2	8	.008557	14	6	13	.019972
13	7	9	.039514	14	2	9	.008028	14	6	14	.017049
13	7	10	.036598	14	2	10	.007499	14	7	7	.039118
13	7	11	.033528	14	2	11	.006958	14	7	8	.036833
13	7	12	.030087	14	2	12	.006385	14	7	9	.034580
13	7	13	.025610	14	2	13	.005739	14	7	10	.032324
13	8	8	.053141	14	2	14	.004893	14	7	11	.030014
13	8	9	.049565	14	3	3	.017588	14	7	12	.027566
13	8	10	.045924	14	3	4	.016620	14	7	13	.024803
13	8	11	.042086	14	3	5	.015702	14	7	14	.021179
13	8	12	.037782	14	3	6	.014824	14	8	8	.045435
13	8	13	.032175	14	3	7	.013978	14	8	9	.042667
13	9	9	.062844	14	3	8	.013155	14	8	10	.039893
13	9	10	.058256	14	3	9	.012344	14	8	11	.037051
13	9	11	.053416	14	3	10	.011534	14	8	12	.034037
13	9	12	.047980	14	3	11	.010705	14	8	13	.030634
13	9	13	.040887	14	3	12	.009827	14	8	14	.026167
13	10	10	.075886	14	3	13	.008839	14	9	9	.052830
13	10	11	.069635	14	3	14	.007544	14	9	10	.049411
13	10	12	.062602	14	4	4	.022857	14	9	11	.045906
13	10	13	.053402	14	4	5	.021595	14	9	12	.042187
13	11	11	.095772	14	4	6	.020390	14	9	13	.037984
13	11	12	.086220	14	4	7	.019227	14	9	14	.032459
13	11	13	.073678	14	4	8	.018096	14	10	10	.062064
13	12	12	.133209	14	4	9	.016982	14	10	11	.057690
13	12	13	.114225	14	4	10	.015867	14	10	12	.053041
13	13	13	.246511	14	4	11	.014727	14	10	13	.047782
14	1	1	.006311	14	4	12	.013520	14	10	14	.040859
14	1	2	.005957	14	4	13	.012159	14	11	11	.074625
14	1	3	.005629	14	4	14	.010377	14	11	12	.068663
14	1	4	.005319	14	5	5	.028081	14	11	13	.061906

TABLE VI (CONTINUED)

N	I	J	VAR/COV	N	I	J	VAR/COV	N	I	J	VAR/COV
****************				****************				****************			
14	11	14	.052990	15	3	14	.007614	15	8	11	.032874
14	12	12	.093955	15	3	15	.006519	15	8	12	.030599
14	12	13	.084823	15	4	4	.020210	15	8	13	.028172
14	12	14	.072728	15	4	5	.019161	15	8	14	.025416
14	13	13	.130607	15	4	6	.018157	15	8	15	.021770
14	13	14	.112358	15	4	7	.017192	15	9	9	.045388
14	14	14	.242206	15	4	8	.016255	15	9	10	.042729
15	1	1	.005574	15	4	9	.015337	15	9	11	.040048
15	1	2	.005277	15	4	10	.014428	15	9	12	.037285
15	1	3	.005004	15	4	11	.013513	15	9	13	.034336
15	1	4	.004744	15	4	12	.012572	15	9	14	.030985
15	1	5	.004497	15	4	13	.011569	15	9	15	.026549
15	1	6	.004261	15	4	14	.010431	15	10	10	.052394
15	1	7	.004034	15	4	15	.008929	15	10	11	.049123
15	1	8	.003814	15	5	5	.024789	15	10	12	.045748
15	1	9	.003599	15	5	6	.023493	15	10	13	.042144
15	1	10	.003386	15	5	7	.022245	15	10	14	.038044
15	1	11	.003171	15	5	8	.021035	15	10	15	.032612
15	1	12	.002951	15	5	9	.019849	15	11	11	.061248
15	1	13	.002716	15	5	10	.018674	15	11	12	.057066
15	1	14	.002450	15	5	11	.017491	15	11	13	.052596
15	1	15	.002099	15	5	12	.016274	15	11	14	.047504
15	2	2	.010715	15	5	13	.014977	15	11	15	.040747
15	2	3	.010158	15	5	14	.013505	15	12	12	.073407
15	2	4	.009629	15	5	15	.011561	15	12	13	.067704
15	2	5	.009127	15	6	6	.029420	15	12	14	.061197
15	2	6	.008647	15	6	7	.027861	15	12	15	.052544
15	2	7	.008185	15	6	8	.026348	15	13	13	.092260
15	2	8	.007737	15	6	9	.024866	15	13	14	.083502
15	2	9	.007299	15	6	10	.023396	15	13	15	.071812
15	2	10	.006864	15	6	11	.021916	15	14	14	.128221
15	2	11	.006427	15	6	12	.020393	15	14	15	.110628
15	2	12	.005977	15	6	13	.018770	15	15	15	.238286
15	2	13	.005497	15	6	14	.016927	16	1	1	.004959
15	2	14	.004953	15	6	15	.014493	16	1	2	.004708
15	2	15	.004235	15	7	7	.034254	16	1	3	.004478
15	3	3	.015560	15	7	8	.032398	16	1	4	.004257
15	3	4	.014752	15	7	9	.030580	16	1	5	.004048
15	3	5	.013985	15	7	10	.028776	16	1	6	.003848
15	3	6	.013252	15	7	11	.026960	16	1	7	.003656
15	3	7	.012547	15	7	12	.025089	16	1	8	.003470
15	3	8	.011863	15	7	13	.023095	16	1	9	.003289
15	3	9	.011193	15	7	14	.020832	16	1	10	.003110
15	3	10	.010530	15	7	15	.017839	16	1	11	.002933
15	3	11	.009862	15	8	8	.039484	16	1	12	.002753
15	3	12	.009175	15	8	9	.037275	16	1	13	.002567
15	3	13	.008443	15	8	10	.035083	16	1	14	.002368

TABLE VI (CONTINUED)

N	I	J	VAR/COV	N	I	J	VAR/COV	N	I	J	VAR/COV
*****	*****	*****	*****	*****	*****	*****	*****	*****	*****	*****	*****
16	1	15	.002142	16	5	8	.018918	16	10	10	.045195
16	1	16	.001841	16	5	9	.017930	16	10	11	.042640
16	2	2	.009544	16	5	10	.016957	16	10	12	.040051
16	2	3	.009075	16	5	11	.015987	16	10	13	.037367
16	2	4	.008627	16	5	12	.015006	16	10	14	.034488
16	2	5	.008202	16	5	13	.013990	16	10	15	.031195
16	2	6	.007795	16	5	14	.012903	16	10	16	.026805
16	2	7	.007404	16	5	15	.011661	16	11	11	.051886
16	2	8	.007026	16	5	16	.010010	16	11	12	.048750
16	2	9	.006657	16	6	6	.026129	16	11	13	.045498
16	2	10	.006293	16	6	7	.024828	16	11	14	.042005
16	2	11	.005931	16	6	8	.023568	16	11	15	.038007
16	2	12	.005564	16	6	9	.022339	16	11	16	.032673
16	2	13	.005185	16	6	10	.021128	16	12	12	.060424
16	2	14	.004779	16	6	11	.019922	16	12	13	.056417
16	2	15	.004315	16	6	12	.018700	16	12	14	.052110
16	2	16	.003698	16	6	13	.017436	16	12	15	.047174
16	3	3	.013870	16	6	14	.016082	16	12	16	.040578
16	3	4	.013187	16	6	15	.014536	16	13	13	.072240
16	3	5	.012539	16	6	16	.012479	16	13	14	.066770
16	3	6	.011920	16	7	7	.030311	16	13	15	.060492
16	3	7	.011325	16	7	8	.028777	16	13	16	.052083
16	3	8	.010749	16	7	9	.027279	16	14	14	.090678
16	3	9	.010187	16	7	10	.025803	16	14	15	.082255
16	3	10	.009633	16	7	11	.024332	16	14	16	.070934
16	3	11	.009082	16	7	12	.022842	16	15	15	.126025
16	3	12	.008524	16	7	13	.021301	16	15	16	.109020
16	3	13	.007948	16	7	14	.019648	16	16	16	.234698
16	3	14	.007330	16	7	15	.017762	17	1	1	.004442
16	3	15	.006626	16	7	16	.015251	17	1	2	.004227
16	3	16	.005690	16	8	8	.034748	17	1	3	.004031
16	4	4	.018010	16	8	9	.032944	17	1	4	.003843
16	4	5	.017126	16	8	10	.031166	17	1	5	.003664
16	4	6	.016280	16	8	11	.029393	17	1	6	.003493
16	4	7	.015467	16	8	12	.027597	17	1	7	.003328
16	4	8	.014680	16	8	13	.025738	17	1	8	.003169
16	4	9	.013913	16	8	14	.023745	17	1	9	.003015
16	4	10	.013157	16	8	15	.021468	17	1	10	.002863
16	4	11	.012404	16	8	16	.018436	17	1	11	.002714
16	4	12	.011641	16	9	9	.039621	17	1	12	.002564
16	4	13	.010853	16	9	10	.037489	17	1	13	.002412
16	4	14	.010009	16	9	11	.035362	17	1	14	.002253
16	4	15	.009045	16	9	12	.033208	17	1	15	.002083
16	4	16	.007764	16	9	13	.030976	17	1	16	.001889
16	5	5	.022065	16	9	14	.028582	17	1	17	.001628
16	5	6	.020977	16	9	15	.025846	17	2	2	.008557
16	5	7	.019931	16	9	16	.022202	17	2	3	.008157

TABLE VI (CONTINUED)

N	I	J	VAR/COV
17	2	4	.007775
17	2	5	.007411
17	2	6	.007063
17	2	7	.006729
17	2	8	.006405
17	2	9	.006091
17	2	10	.005782
17	2	11	.005477
17	2	12	.005171
17	2	13	.004860
17	2	14	.004537
17	2	15	.004189
17	2	16	.003789
17	2	17	.003253
17	3	3	.012444
17	3	4	.011863
17	3	5	.011310
17	3	6	.010782
17	3	7	.010274
17	3	8	.009783
17	3	9	.009305
17	3	10	.008837
17	3	11	.008373
17	3	12	.007910
17	3	13	.007438
17	3	14	.006948
17	3	15	.006421
17	3	16	.005817
17	3	17	.005009
17	4	4	.016158
17	4	5	.015406
17	4	6	.014686
17	4	7	.013994
17	4	8	.013325
17	4	9	.012674
17	4	10	.012036
17	4	11	.011404
17	4	12	.010772
17	4	13	.010129
17	4	14	.009461
17	4	15	.008742
17	4	16	.007917
17	4	17	.006813
17	5	5	.019781
17	5	6	.018858
17	5	7	.017971
17	5	8	.017113
17	5	9	.016277
17	5	10	.015458
17	5	11	.014648
17	5	12	.013836
17	5	13	.013011
17	5	14	.012154
17	5	15	.011230
17	5	16	.010171
17	5	17	.008753
17	6	6	.023387
17	6	7	.022288
17	6	8	.021225
17	6	9	.020190
17	6	10	.019176
17	6	11	.018172
17	6	12	.017166
17	6	13	.016143
17	6	14	.015080
17	6	15	.013935
17	6	16	.012622
17	6	17	.010863
17	7	7	.027057
17	7	8	.025769
17	7	9	.024515
17	7	10	.023285
17	7	11	.022068
17	7	12	.020848
17	7	13	.019608
17	7	14	.018318
17	7	15	.016929
17	7	16	.015334
17	7	17	.013199
17	8	8	.030892
17	8	9	.029392
17	8	10	.027920
17	8	11	.026463
17	8	12	.025003
17	8	13	.023518
17	8	14	.021973
17	8	15	.020308
17	8	16	.018398
17	8	17	.015838
17	9	9	.035016
17	9	10	.033267
17	9	11	.031535
17	9	12	.029799
17	9	13	.028032
17	9	14	.026194
17	9	15	.024213
17	9	16	.021938
17	9	17	.018890
17	10	10	.039603
17	10	11	.037547
17	10	12	.035486
17	10	13	.033388
17	10	14	.031203
17	10	15	.028849
17	10	16	.026143
17	10	17	.022515
17	11	11	.044908
17	11	12	.042452
17	11	13	.039950
17	11	14	.037345
17	11	15	.034535
17	11	16	.031304
17	11	17	.026968
17	12	12	.051337
17	12	13	.048326
17	12	14	.045188
17	12	15	.041801
17	12	16	.037903
17	12	17	.032667
17	13	13	.059608
17	13	14	.055760
17	13	15	.051603
17	13	16	.046813
17	13	17	.040371
17	14	14	.071129
17	14	15	.065869
17	14	16	.059800
17	14	17	.051618
17	15	15	.089201
17	15	16	.081080
17	15	17	.070094
17	16	16	.123995
17	16	17	.107522
17	17	17	.231394
18	1	1	.004002
18	1	2	.003816
18	1	3	.003648
18	1	4	.003486
18	1	5	.003332
18	1	6	.003184
18	1	7	.003042

TABLE VI (CONTINUED)

N	I	J	VAR/COV	N	I	J	VAR/COV	N	I	J	VAR/COV
*****	***	***	********	*****	***	***	********	*****	***	***	********
18	1	8	.002905	18	4	7	.012724	18	7	15	.015925
18	1	9	.002772	18	4	8	.012150	18	7	16	.014742
18	1	10	.002642	18	4	9	.011592	18	7	17	.013379
18	1	11	.002514	18	4	10	.011046	18	7	18	.011542
18	1	12	.002387	18	4	11	.010509	18	8	8	.027694
18	1	13	.002260	18	4	12	.009975	18	8	9	.026429
18	1	14	.002129	18	4	13	.009438	18	8	10	.025191
18	1	15	.001993	18	4	14	.008890	18	8	11	.023972
18	1	16	.001847	18	4	15	.008318	18	8	12	.022759
18	1	17	.001678	18	4	16	.007699	18	8	13	.021539
18	1	18	.001451	18	4	17	.006986	18	8	14	.020292
18	2	2	.007716	18	4	18	.006025	18	8	15	.018990
18	2	3	.007373	18	5	5	.017845	18	8	16	.017582
18	2	4	.007044	18	5	6	.017054	18	8	17	.015957
18	2	5	.006730	18	5	7	.016294	18	8	18	.013767
18	2	6	.006430	18	5	8	.015559	18	9	9	.031253
18	2	7	.006141	18	5	9	.014845	18	9	10	.029792
18	2	8	.005862	18	5	10	.014147	18	9	11	.028353
18	2	9	.005591	18	5	11	.013459	18	9	12	.026921
18	2	10	.005326	18	5	12	.012776	18	9	13	.025480
18	2	11	.005065	18	5	13	.012089	18	9	14	.024008
18	2	12	.004806	18	5	14	.011387	18	9	15	.022470
18	2	13	.004545	18	5	15	.010654	18	9	16	.020805
18	2	14	.004278	18	5	16	.009862	18	9	17	.018885
18	2	15	.004000	18	5	17	.008948	18	9	18	.016296
18	2	16	.003699	18	5	18	.007718	18	10	10	.035126
18	2	17	.003351	18	6	6	.021073	18	10	11	.033433
18	2	18	.002882	18	6	7	.020134	18	10	12	.031748
18	3	3	.011231	18	6	8	.019227	18	10	13	.030053
18	3	4	.010731	18	6	9	.018346	18	10	14	.028320
18	3	5	.010255	18	6	10	.017484	18	10	15	.026509
18	3	6	.009800	18	6	11	.016635	18	10	16	.024548
18	3	7	.009363	18	6	12	.015791	18	10	17	.022285
18	3	8	.008941	18	6	13	.014943	18	10	18	.019234
18	3	9	.008530	18	6	14	.014076	18	11	11	.039479
18	3	10	.008129	18	6	15	.013171	18	11	12	.037496
18	3	11	.007734	18	6	16	.012192	18	11	13	.035499
18	3	12	.007342	18	6	17	.011064	18	11	14	.033457
18	3	13	.006948	18	6	18	.009544	18	11	15	.031323
18	3	14	.006545	18	7	7	.024331	18	11	16	.029011
18	3	15	.006125	18	7	8	.023237	18	11	17	.026341
18	3	16	.005671	18	7	9	.022173	18	11	18	.022740
18	3	17	.005147	18	7	10	.021133	18	12	12	.044560
18	3	18	.004443	18	7	11	.020108	18	12	13	.042196
18	4	4	.014584	18	7	12	.019089	18	12	14	.039778
18	4	5	.013938	18	7	13	.018065	18	12	15	.037248
18	4	6	.013319	18	7	14	.017018	18	12	16	.034506

TABLE VI (CONTINUED)

N	I	J	VAR/COV	N	I	J	VAR/COV	N	I	J	VAR/COV
*****	*****	*****	*****	*****	*****	*****	*****	*****	*****	*****	*****
18	12	17	.031339	19	2	7	.005627	19	5	6	.015504
18	12	18	.027062	19	2	8	.005384	19	5	7	.014847
18	13	13	.050770	19	2	9	.005149	19	5	8	.014212
18	13	14	.047873	19	2	10	.004919	19	5	9	.013595
18	13	15	.044841	19	2	11	.004693	19	5	10	.012994
18	13	16	.041553	19	2	12	.004470	19	5	11	.012403
18	13	17	.037751	19	2	13	.004247	19	5	12	.011820
18	13	18	.032613	19	2	14	.004022	19	5	13	.011237
18	14	14	.058810	19	2	15	.003792	19	5	14	.010649
18	14	15	.055107	19	2	16	.003550	19	5	15	.010046
18	14	16	.051088	19	2	17	.003286	19	5	16	.009414
18	14	17	.046435	19	2	18	.002981	19	5	17	.008728
18	14	18	.040139	19	2	19	.002567	19	5	18	.007933
18	15	15	.070073	19	3	3	.010189	19	5	19	.006857
18	15	16	.065004	19	3	4	.009756	19	6	6	.019100
18	15	17	.059126	19	3	5	.009343	19	6	7	.018291
18	15	18	.051154	19	3	6	.008948	19	6	8	.017509
18	16	16	.087819	19	3	7	.008569	19	6	9	.016750
18	16	17	.079972	19	3	8	.008202	19	6	10	.016010
18	16	18	.069294	19	3	9	.007847	19	6	11	.015283
18	17	17	.122112	19	3	10	.007500	19	6	12	.014565
18	17	18	.106123	19	3	11	.007160	19	6	13	.013847
18	18	18	.228341	19	3	12	.006823	19	6	14	.013123
19	1	1	.003624	19	3	13	.006487	19	6	15	.012381
19	1	2	.003462	19	3	14	.006149	19	6	16	.011602
19	1	3	.003318	19	3	15	.005802	19	6	17	.010757
19	1	4	.003177	19	3	16	.005438	19	6	18	.009778
19	1	5	.003043	19	3	17	.005043	19	6	19	.008453
19	1	6	.002915	19	3	18	.004587	19	7	7	.022019
19	1	7	.002791	19	3	19	.003969	19	7	8	.021080
19	1	8	.002672	19	4	4	.013233	19	7	9	.020168
19	1	9	.002557	19	4	5	.012674	19	7	10	.019277
19	1	10	.002444	19	4	6	.012138	19	7	11	.018403
19	1	11	.002334	19	4	7	.011623	19	7	12	.017538
19	1	12	.002225	19	4	8	.011125	19	7	13	.016675
19	1	13	.002116	19	4	9	.010642	19	7	14	.015804
19	1	14	.002006	19	4	10	.010171	19	7	15	.014911
19	1	15	.001894	19	4	11	.009709	19	7	16	.013974
19	1	16	.001776	19	4	12	.009251	19	7	17	.012957
19	1	17	.001649	19	4	13	.008795	19	7	18	.011778
19	1	18	.001501	19	4	14	.008335	19	7	19	.010183
19	1	19	.001302	19	4	15	.007863	19	8	8	.025004
19	2	2	.006994	19	4	16	.007368	19	8	9	.023924
19	2	3	.006698	19	4	17	.006831	19	8	10	.022869
19	2	4	.006412	19	4	18	.006209	19	8	11	.021834
19	2	5	.006139	19	4	19	.005367	19	8	12	.020809
19	2	6	.005878	19	5	5	.016188	19	8	13	.019786

TABLE VI (CONTINUED)

N	I	J	VAR/COV
*****	*****	*****	*****
19	8	14	.018754
19	8	15	.017695
19	8	16	.016584
19	8	17	.015378
19	8	18	.013980
19	8	19	.012087
19	9	9	.028121
19	9	10	.026884
19	9	11	.025669
19	9	12	.024466
19	9	13	.023265
19	9	14	.022053
19	9	15	.020809
19	9	16	.019505
19	9	17	.018088
19	9	18	.016445
19	9	19	.014220
19	10	10	.031457
19	10	11	.030037
19	10	12	.028633
19	10	13	.027230
19	10	14	.025814
19	10	15	.024360
19	10	16	.022835
19	10	17	.021178
19	10	18	.019257
19	10	19	.016654
19	11	11	.035123
19	11	12	.033485
19	11	13	.031849
19	11	14	.030195
19	11	15	.028498
19	11	16	.026718
19	11	17	.024782
19	11	18	.022538
19	11	19	.019494
19	12	12	.039282
19	12	13	.037368
19	12	14	.035434
19	12	15	.033448
19	12	16	.031363
19	12	17	.029095
19	12	18	.026465
19	12	19	.022896
19	13	13	.044176
19	13	14	.041897
19	13	15	.039556

N	I	J	VAR/COV
*****	*****	*****	*****
19	13	16	.037098
19	13	17	.034424
19	13	18	.031319
19	13	19	.027104
19	14	14	.050196
19	14	15	.047405
19	14	16	.044471
19	14	17	.041277
19	14	18	.037566
19	14	19	.032523
19	15	15	.058036
19	15	16	.054466
19	15	17	.050574
19	15	18	.046048
19	15	19	.039890
19	16	16	.069071
19	16	17	.064175
19	16	18	.058473
19	16	19	.050697
19	17	17	.086524
19	17	18	.078926
19	17	19	.068531
19	18	18	.120360
19	18	19	.104812
19	19	19	.225505
20	1	1	.003298
20	1	2	.003156
20	1	3	.003030
20	1	4	.002907
20	1	5	.002790
20	1	6	.002678
20	1	7	.002570
20	1	8	.002466
20	1	9	.002365
20	1	10	.002267
20	1	11	.002171
20	1	12	.002076
20	1	13	.001982
20	1	14	.001888
20	1	15	.001793
20	1	16	.001695
20	1	17	.001593
20	1	18	.001481
20	1	19	.001352
20	1	20	.001176
20	2	2	.006370
20	2	3	.006112

N	I	J	VAR/COV
*****	*****	*****	*****
20	2	4	.005862
20	2	5	.005623
20	2	6	.005394
20	2	7	.005174
20	2	8	.004961
20	2	9	.004755
20	2	10	.004554
20	2	11	.004357
20	2	12	.004163
20	2	13	.003971
20	2	14	.003778
20	2	15	.003582
20	2	16	.003380
20	2	17	.003168
20	2	18	.002937
20	2	19	.002667
20	2	20	.002298
20	3	3	.009287
20	3	4	.008909
20	3	5	.008549
20	3	6	.008204
20	3	7	.007872
20	3	8	.007552
20	3	9	.007242
20	3	10	.006939
20	3	11	.006643
20	3	12	.006351
20	3	13	.006062
20	3	14	.005772
20	3	15	.005479
20	3	16	.005177
20	3	17	.004860
20	3	18	.004514
20	3	19	.004113
20	3	20	.003567
20	4	4	.012065
20	4	5	.011577
20	4	6	.011110
20	4	7	.010660
20	4	8	.010225
20	4	9	.009804
20	4	10	.009394
20	4	11	.008992
20	4	12	.008596
20	4	13	.008203
20	4	14	.007810
20	4	15	.007411

TABLE VI (CONTINUED)

N	I	J	VAR/COV	N	I	J	VAR/COV	N	I	J	VAR/COV
20	4	16	.007001	20	7	18	.011479	20	11	19	.019542
20	4	17	.006570	20	7	19	.010451	20	11	20	.016934
20	4	18	.006100	20	7	20	.009053	20	12	12	.035041
20	4	19	.005554	20	8	8	.022712	20	12	13	.033456
20	4	20	.004810	20	8	9	.021781	20	12	14	.031867
20	5	5	.014757	20	8	10	.020872	20	12	15	.030255
20	5	6	.014162	20	8	11	.019983	20	12	16	.028595
20	5	7	.013589	20	8	12	.019106	20	12	17	.026847
20	5	8	.013035	20	8	13	.018235	20	12	18	.024939
20	5	9	.012499	20	8	14	.017363	20	12	19	.022717
20	5	10	.011976	20	8	15	.016479	20	12	20	.019689
20	5	11	.011464	20	8	16	.015570	20	13	13	.039037
20	5	12	.010959	20	8	17	.014613	20	13	14	.037188
20	5	13	.010459	20	8	18	.013569	20	13	15	.035312
20	5	14	.009957	20	8	19	.012355	20	13	16	.033380
20	5	15	.009449	20	8	20	.010703	20	13	17	.031344
20	5	16	.008927	20	9	9	.025477	20	13	18	.029121
20	5	17	.008377	20	9	10	.024416	20	13	19	.026531
20	5	18	.007778	20	9	11	.023377	20	13	20	.023000
20	5	19	.007081	20	9	12	.022352	20	14	14	.043769
20	5	20	.006133	20	9	13	.021335	20	14	15	.041570
20	6	6	.017401	20	9	14	.020316	20	14	16	.039302
20	6	7	.016698	20	9	15	.019283	20	14	17	.036912
20	6	8	.016019	20	9	16	.018220	20	14	18	.034302
20	6	9	.015360	20	9	17	.017101	20	14	19	.031259
20	6	10	.014718	20	9	18	.015881	20	14	20	.027106
20	6	11	.014089	20	9	19	.014461	20	15	15	.049626
20	6	12	.013469	20	9	20	.012528	20	15	16	.046931
20	6	13	.012854	20	10	10	.028394	20	15	17	.044089
20	6	14	.012239	20	10	11	.027188	20	15	18	.040982
20	6	15	.011615	20	10	12	.025998	20	15	19	.037358
20	6	16	.010973	20	10	13	.024817	20	15	20	.032408
20	6	17	.010297	20	10	14	.023633	20	16	16	.057290
20	6	18	.009561	20	10	15	.022433	20	16	17	.053840
20	6	19	.008705	20	10	16	.021198	20	16	18	.050067
20	6	20	.007540	20	10	17	.019898	20	16	19	.045659
20	7	7	.020038	20	10	18	.018480	20	16	20	.039632
20	7	8	.019224	20	10	19	.016829	20	17	17	.068120
20	7	9	.018434	20	10	20	.014582	20	17	18	.063383
20	7	10	.017664	20	11	11	.031546	20	17	19	.057843
20	7	11	.016910	20	11	12	.030169	20	17	20	.050250
20	7	12	.016167	20	11	13	.028801	20	18	18	.085308
20	7	13	.015430	20	11	14	.027429	20	18	19	.077937
20	7	14	.014691	20	11	15	.026039	20	18	20	.067804
20	7	15	.013943	20	11	16	.024608	20	19	19	.118724
20	7	16	.013173	20	11	17	.023100	20	19	20	.103582
20	7	17	.012362	20	11	18	.021456	20	20	20	.222863

for each sample of size twenty and less can now be computed and compared with the simple least squares or visual estimate of σ,

$$\sigma^* = \sum_{i=1}^{n} c(i)\, u_{(i)} .$$

The coefficients of the observed values for the linear estimates are given in Table VII and the variance of each estimate is given in Table VIII. The efficiencies of the simple least squares estimates relative to the best linear unbiased estimates range from 100 % for sample size two to 99.95% for sample size twenty.

Comparison of the Normal and the Half-Normal Estimating Procedures

If the variate X is distributed $N(0, \sigma^2)$, $Z = X/\sigma$ is distributed N(0, 1), and a sample of size n is drawn from the population of X with observations $x_1, x_2, \ldots, x_n$, then as described in Chapter III the parameter σ can be estimated by the linear by linear procedure, plotting

$$x_{(i)} = \hat{\sigma}\, E\, z_{(i)} .$$

However, if the transformation $Y = |X|$ is made, then the distribution of Y is half-normal with the same scale parameter σ. Following the method of this chapter, the parameter σ can be estimated by plotting

TABLE VII

THE COEFFICIENTS OF THE SIMPLE LEAST SQUARES AND BEST LINEAR UNBIASED ESTIMATES OF THE STANDARD DEVIATION IN SAMPLES UP TO SIZE TWENTY FROM A HALF-NORMAL DISTRIBUTION

N	I	C(I)	C'(I)	N	I	C(I)	C'(I)	N	I	C(I)	C'(I)
2	1	.313	.313	10	3	.039	.040	14	3	.019	.020
2	2	.756	.756	10	4	.053	.052	14	4	.026	.027
3	1	.139	.155	10	5	.068	.066	14	5	.033	.033
3	2	.304	.280	10	6	.084	.081	14	6	.041	.040
3	3	.551	.560	10	7	.103	.098	14	7	.048	.047
4	1	.078	.094	10	8	.125	.119	14	8	.057	.055
4	2	.165	.160	10	9	.155	.148	14	9	.066	.063
4	3	.272	.250	10	10	.205	.217	14	10	.076	.073
4	4	.437	.450	11	1	.010	.015	14	11	.088	.084
5	1	.050	.064	11	2	.021	.025	14	12	.102	.098
5	2	.104	.105	11	3	.032	.033	14	13	.121	.117
5	3	.165	.156	11	4	.043	.043	14	14	.154	.164
5	4	.242	.224	11	5	.055	.054	15	1	.006	.007
5	5	.364	.379	11	6	.068	.066	15	2	.011	.011
6	1	.035	.046	11	7	.082	.079	15	3	.017	.016
6	2	.071	.075	11	8	.099	.094	15	4	.023	.024
6	3	.112	.108	11	9	.118	.113	15	5	.029	.029
6	4	.158	.149	11	10	.145	.139	15	6	.035	.035
6	5	.218	.203	11	11	.189	.201	15	7	.042	.041
6	6	.314	.328	12	1	.009	.013	15	8	.049	.047
7	1	.026	.035	12	2	.018	.022	15	9	.056	.054
7	2	.052	.056	12	3	.027	.028	15	10	.064	.062
7	3	.081	.080	12	4	.036	.037	15	11	.074	.071
7	4	.112	.107	12	5	.046	.045	15	12	.084	.081
7	5	.150	.141	12	6	.056	.055	15	13	.097	.094
7	6	.198	.186	12	7	.068	.065	15	14	.115	.112
7	7	.276	.290	12	8	.080	.077	15	15	.145	.155
8	1	.020	.028	12	9	.095	.091	16	1	.005	.005
8	2	.040	.044	12	10	.112	.108	16	2	.010	.010
8	3	.061	.061	12	11	.136	.131	16	3	.015	.013
8	4	.084	.082	12	12	.175	.187	16	4	.020	.022
8	5	.110	.105	13	1	.007	.010	16	5	.025	.026
8	6	.141	.133	13	2	.015	.019	16	6	.031	.031
8	7	.181	.171	13	3	.023	.023	16	7	.036	.036
8	8	.247	.260	13	4	.031	.032	16	8	.042	.041
9	1	.015	.022	13	5	.039	.039	16	9	.049	.047
9	2	.031	.035	13	6	.047	.047	16	10	.056	.054
9	3	.048	.049	13	7	.057	.055	16	11	.063	.061
9	4	.066	.064	13	8	.067	.065	16	12	.071	.069
9	5	.085	.082	13	9	.078	.075	16	13	.081	.078
9	6	.107	.102	13	10	.091	.088	16	14	.093	.090
9	7	.133	.126	13	11	.107	.103	16	15	.110	.106
9	8	.167	.159	13	12	.128	.124	16	16	.137	.147
9	9	.224	.237	13	13	.164	.175	17	1	.004	.003
10	1	.013	.018	14	1	.006	.008	17	2	.009	.009
10	2	.025	.029	14	2	.013	.018	17	3	.013	.010

TABLE VII (CONTINUED)

N	I	C(I)	C'(I)	N	I	C(I)	C'(I)	N	I	C(I)	C'(I)
17	4	.018	.020	18	11	.048	.047	19	17	.083	.080
17	5	.022	.023	18	12	.054	.052	19	18	.096	.094
17	6	.027	.027	18	13	.060	.058	19	19	.119	.127
17	7	.032	.032	18	14	.067	.065	20	1	.003	.001
17	8	.037	.036	18	15	.076	.073	20	2	.006	.006
17	9	.043	.042	18	16	.086	.083	20	3	.009	.010
17	10	.048	.047	18	17	.100	.098	20	4	.013	.016
17	11	.055	.053	18	18	.124	.133	20	5	.016	.017
17	12	.062	.060	19	1	.003	.002	20	6	.019	.020
17	13	.069	.067	19	2	.007	.007	20	7	.023	.023
17	14	.078	.076	19	3	.011	.012	20	8	.026	.026
17	15	.090	.087	19	4	.014	.017	20	9	.030	.030
17	16	.105	.102	19	5	.018	.018	20	10	.034	.033
17	17	.130	.139	19	6	.022	.022	20	11	.038	.037
18	1	.004	.003	19	7	.025	.025	20	12	.042	.041
18	2	.008	.008	19	8	.029	.029	20	13	.047	.046
18	3	.012	.013	19	9	.034	.033	20	14	.052	.050
18	4	.016	.018	19	10	.038	.037	20	15	.057	.056
18	5	.020	.020	19	11	.042	.041	20	16	.064	.062
18	6	.024	.024	19	12	.047	.046	20	17	.071	.069
18	7	.028	.028	19	13	.053	.051	20	18	.080	.078
18	8	.033	.032	19	14	.059	.057	20	19	.092	.090
18	9	.038	.037	19	15	.065	.063	20	20	.113	.121
18	10	.043	.042	19	16	.073	.071				

TABLE VIII

VARIANCES OF THE ESTIMATES OF THE STANDARD DEVIATION

N	VAR(SIMPLE)	VAR(BEST)
2	.27393282	.27393281
3	.17878761	.17873428
4	.13235666	.13229049
5	.10495391	.10488988
6	.08690294	.08684526
7	.07412643	.07407568
8	.06461264	.06456829
9	.05725595	.05721715
10	.05139885	.05136480
11	.04662605	.04659601
12	.04266239	.04263576
13	.03931856	.03929480
14	.03645985	.03643854
15	.03398798	.03396879
16	.03182954	.03181216
17	.02992849	.02991265
18	.02824145	.02822693
19	.02673419	.02672080
20	.02537951	.02536700

$$y_{(i)} = \hat{\sigma}' E z'_{(i)} ,$$

where the variate $Z' = Y/\sigma$ is the standardized half-normal variate.

It is not surprising that the best linear unbiased estimate $\hat{\sigma}$ based on the normal order statistics and given by Sarhan and Greenberg [2] is different from the best linear unbiased estimate $\hat{\sigma}'$ based on the half-normal order statistics as evident upon comparison with Table VII. However, it is perhaps surprising that the variance of the half-normal estimate is less than the variance of the normal estimate for each sample size considered. This is due in part to the fact that the estimate $\hat{\sigma}'$ is not linear in the original observations on the variate X. The efficiencies of the normal estimates relative to the half-normal estimates are given in Table IX.

TABLE IX

EFFICIENCIES OF NORMAL ESTIMATES RELATIVE TO HALF-NORMAL ESTIMATES

N	% R.E.	N	% R.E.	N	% R.E.
2	47.99	9	88.01	16	93.17
3	64.88	10	89.18	17	93.57
4	73.47	11	90.15	18	93.92
5	78.67	12	90.95	19	94.23
6	82.16	13	91.63	20	94.51
7	84.66	14	92.22		
8	86.54	15	92.73		

CHAPTER V

ESTIMATION OF PARAMETERS IN OTHER DISTRIBUTIONS

Exponential Distribution

If the variate X is distributed as an exponential with probability density function

$$f(x) = \frac{1}{\theta} \exp - x/\theta \ , \quad x \geq 0,$$

and the transformation $X = \theta Z$ is made, then the variate Z is distributed as an exponential with density function given by the equation

$$f(z) = \exp - z \ , \qquad z \geq 0 .$$

The transformation to the standardized variate suggests applying the linear by linear procedure to the plotting equation

$$x_{(r)} = \hat{\theta} \ E \ z_{(r)} \ ,$$

and estimating the parameter θ by the slope $\hat{\theta}$ of the line fitted through the origin and to the points $(x_{(r)}, E\ z_{(r)})$, $r = 1, 2, \ldots, n.$

Sarhan and Greenberg [2] give formulas for the expected values and variances and covariances of standardized exponential order statistics:

$$E \ z_{(r)} = \sum_{i=1}^{r} 1/(n-i-1)$$

$$\text{Var}(z_{(r)}) = \underset{r<s}{\text{Cov}}(z_{(r)}, z_{(s)}) = \sum_{i=1}^{r} 1/(n-i-1)^2.$$

The expected values are tabulated for sample sizes twenty and less in Table X.

The best linear unbiased estimate of θ is the mean of the observations and the variance of this estimate is θ^2/n. The simple least squares estimate of θ based on the order statistics is given by equation (2.12) with its variance given by equation (2.13). A comparison of the two estimates results in percentage relative efficiencies ranging from 96.15% from sample size two to 86.88% for sample size twenty.

Weibull Distribution

The two-parameter Weibull distribution function given in equation (1.8) has the following probability density function:

$$f(x) = \frac{\delta}{\theta} x^{\delta-1} \exp - x^{\delta}/\theta \ , \ x \geq 0.$$

If the transformation $X = (\theta Z)^{1/\delta}$ is made, then the distribution of Z is the standardized Weibull or exponential distribution with density function:

$$f(z) = \exp - z \ , \quad z \geq 0.$$

Taking the natural logarithm (ℓn) of the transformation stated above, the plotting equation in terms of the order statistics becomes:

$$\ell n \, E \, z_{(i)} = - \ell n \, \hat{\theta} + \hat{\delta} \ \ell n \, x_{(i)} \ .$$

TABLE X

EXPECTED VALUES OF STANDARDIZED EXPONENTIAL ORDER STATISTICS

N	I	$\mu_{I,N}$	N	I	$\mu_{I,N}$	N	I	$\mu_{I,N}$	N	I	$\mu_{I,N}$	N	I	$\mu_{I,N}$
2	1	.50	9	8	1.83	13	8	.90	16	8	.66	18	17	2.50
2	2	1.50	9	9	2.83	13	9	1.10	16	9	.79	18	18	3.50
3	1	.33	10	1	.10	13	10	1.35	16	10	.93	19	1	.05
3	2	.83	10	2	.21	13	11	1.68	16	11	1.10	19	2	.11
3	3	1.83	10	3	.34	13	12	2.18	16	12	1.30	19	3	.17
4	1	.25	10	4	.48	13	13	3.18	16	13	1.55	19	4	.23
4	2	.58	10	5	.65	14	1	.07	16	14	1.88	19	5	.30
4	3	1.08	10	6	.85	14	2	.15	16	15	2.38	19	6	.37
4	4	2.08	10	7	1.10	14	3	.23	16	16	3.38	19	7	.44
5	1	.20	10	8	1.43	14	4	.32	17	1	.06	19	8	.53
5	2	.45	10	9	1.93	14	5	.42	17	2	.12	19	9	.62
5	3	.78	10	10	2.93	14	6	.53	17	3	.19	19	10	.72
5	4	1.28	11	1	.09	14	7	.66	17	4	.26	19	11	.83
5	5	2.28	11	2	.19	14	8	.80	17	5	.34	19	12	.95
6	1	.17	11	3	.30	14	9	.97	17	6	.42	19	13	1.10
6	2	.37	11	4	.43	14	10	1.17	17	7	.51	19	14	1.26
6	3	.62	11	5	.57	14	11	1.42	17	8	.61	19	15	1.46
6	4	.95	11	6	.74	14	12	1.75	17	9	.72	19	16	1.71
6	5	1.45	11	7	.94	14	13	2.25	17	10	.85	19	17	2.05
6	6	2.45	11	8	1.19	14	14	3.25	17	11	.99	19	18	2.55
7	1	.14	11	9	1.52	15	1	.07	17	12	1.16	19	19	3.55
7	2	.31	11	10	2.02	15	2	.14	17	13	1.36	20	1	.05
7	3	.51	11	11	3.02	15	3	.22	17	14	1.61	20	2	.10
7	4	.76	12	1	.08	15	4	.30	17	15	1.94	20	3	.16
7	5	1.09	12	2	.17	15	5	.39	17	16	2.44	20	4	.22
7	6	1.59	12	3	.27	15	6	.49	17	17	3.44	20	5	.28
7	7	2.59	12	4	.39	15	7	.60	18	1	.06	20	6	.35
8	1	.13	12	5	.51	15	8	.73	18	2	.11	20	7	.42
8	2	.27	12	6	.65	15	9	.87	18	3	.18	20	8	.49
8	3	.43	12	7	.82	15	10	1.03	18	4	.24	20	9	.58
8	4	.63	12	8	1.02	15	11	1.23	18	5	.31	20	10	.67
8	5	.88	12	9	1.27	15	12	1.48	18	6	.39	20	11	.77
8	6	1.22	12	10	1.60	15	13	1.82	18	7	.48	20	12	.88
8	7	1.72	12	11	2.10	15	14	2.32	18	8	.57	20	13	1.00
8	8	2.72	12	12	3.10	15	15	3.32	18	9	.67	20	14	1.15
9	1	.11	13	1	.08	16	1	.06	18	10	.78	20	15	1.31
9	2	.24	13	2	.16	16	2	.13	18	11	.90	20	16	1.51
9	3	.38	13	3	.25	16	3	.20	18	12	1.05	20	17	1.76
9	4	.55	13	4	.35	16	4	.28	18	13	1.21	20	18	2.10
9	5	.75	13	5	.46	16	5	.36	18	14	1.41	20	19	2.60
9	6	1.00	13	6	.59	16	6	.45	18	15	1.66	20	20	3.60
9	7	1.33	13	7	.73	16	7	.55	18	16	2.00			

On graph paper with both scales logarithmic, if the standardized expected values of the order statistics are plotted on the ordinate scale versus the sample order statistics on the abscissa, then the ordinate intercept of the fitted line is an estimate of $-\ln \theta$ and the slope of the fitted line an estimate of the shape parameter δ.

This procedure is somewhat simpler than the one described in Chapter I and makes use of a more readily available graph paper.

CHAPTER VI

SUMMARY AND FURTHER PROBLEMS

Several graphical procedures for estimation of unknown parameters have been discussed in this paper which greatly reduce the amount of computational labor even though statistical efficiency of the analysis is often sacrificed. Most of these methods require a special type of graph paper on which to plot the results of an experiment and there is some variation as to the plotting convention used for obtaining the best possible estimates. The methods have been applied in the literature to different assumed populations, such as the normal, half-normal, exponential, Weibull, extreme-value, and Gamma to name a few.

The linear by linear graphical procedure was described in which the plotting positions are well defined. This procedure was applied to estimating the location and scale parameters from the order statistics of a sample from an assumed normal distribution. The method of obtaining the best linear unbiased estimates of the parameters was presented and these best estimates were compared with the estimates obtained by visually fitting a straight line to the plotted points on the linear by linear graph. On the basis of the relative efficiency of the estimating procedures, it is concluded that for samples of twenty and less the visual fitting of a straight line through the plotted points is more than 99.9% as efficient as fitting the line by generalized least squares.

The linear by linear procedure was also applied to estimating the scale parameter in a half-normal distribution where the location parameter was assumed to be zero. The moments of the standardized half-normal order statistics were tabulated for use with the estimating procedure and for comparison of the resulting estimates with the best linear unbiased estimates. The results of this comparison showed the visual estimates to be at least 99.95% as efficient for estimating the scale parameter as the best linear unbiased estimates.

A relationship between the normal and half-normal distributions was described for the case where the mean of the normal distribution is zero. For this case the standard deviation of the normal distribution can be estimated by applying the linear by linear procedure to the ordered absolute values of the normal sample. The variance of the resulting estimates is less than the variance of the estimates obtained from the original order statistics. The efficiency of the estimates based on normal order statistics relative to the estimates based on half-normal order statistics increases as the sample size increases and ranges from 48% for sample size two to 95% for sample size twenty.

A problem not investigated in this paper is the estimation of the parameters in a normal distribution from the ordered absolute values of the sample when the mean of the distribution is not zero.

The remaining portion of this paper described procedures for estimating the parameters in the single parameter negative exponential distribution and the related two parameter Weibull distribution.

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