

SOME COMPUTER ORIENTED MATERIALS FOR
USE IN ELEMENTARY CALCULUS AND
AN EXPERIMENT WITH THEIR USE

By

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CHAPTER I

INTRODUCTION AND BACKGROUND

Impetus for Entrance of Computers into Mathematics Instruction

Since the middle sixties considerable interest has been generated within the mathematical community, particularly among persons involved with undergraduate mathematics education, concerning the impact modern computers and computer based methods have had and should have on the instruction of mathematics. There have been many suggestions for alterations of both the content of modern mathematics curricula and methods of teaching it which are attributable to the availability of computers.¹

The authors of the influential Committee on the Undergraduate Program in Mathematics (CUPM) publication, A General Curriculum in Mathematics for Colleges, observed in 1965 that the prevalence of the high-speed computer necessarily affects the teaching of mathematics in a significant way. Many mathematically trained students will work closely with computers, and even those who do not should be taught to

¹This discourse as well as all subsequent comments refer to the content of mathematics courses and not the technique of using computers as mechanical teaching devices such as for computer assisted instruction (CAI) or automated programmed instruction.

Later when the terms "computer calculus" and "computer oriented calculus" are used they will refer to a calculus course revised in some way so as to make use of the computer's ability to manipulate symbols and do rapid numerical calculations. In general, an old subject is being approached with a new weapon.

appreciate the type of algorithm approach that enables a problem to be handled by a machine. Such a point of view should, therefore, be presented, along with the more classical one, at appropriate places in calculus, differential equations, and linear algebra. (7, p. 14)

In a report to the 1967 conference on "Computers in Undergraduate Education: Mathematics, Physics, Statistics and Chemistry," the pre-selected Mathematics Panel gave a strong recommendation for computer usage in mathematics teaching. They indicated that the availability of computers has created new challenges for curriculum development in mathematics. Specifically, the undergraduate mathematics curriculum must be changed to take into account new ways for solving old problems and the host of new problems which have arisen with recent computer applications. The Panel expressed the feeling that students can be enormously stimulated by the introduction of computing into the curriculum, and that, for many it can turn the all-too-often totally passive college experience into one of active participation. It can provide a powerful problem-solving tool, and enable students to do genuine research at an early period in their training, whereas only the truly exceptional student is capable of anything worthy of the name research in the classical areas of mathematics (5).

The Panel stated

"If the basic undergraduate mathematics courses are not appropriately modified to reflect the new points of view which are associated with computer applications in mathematics, these courses will lose much of their relevance for the coming generation of college and university students. This could lead to the unfortunate situation in which instruction in certain basic aspects of mathematics would be incorporated in the programs of other subject areas by default. Thus, it is urgent that experimental programs emphasizing this type of curriculum development be initiated ..." (5, p. 1)

The Panel stated that computers have clearly changed the nature of the solution process in mathematics. In addition to existence proofs, it has now become important to provide constructive methods for obtaining solutions. From a general point of view, many traditional special techniques have been rendered obsolete while the need for an algorithmic approach to mathematical thinking has been strongly reinforced. At the same time, the use of computing methods provides the student with a greater understanding of the concepts involved as well as with greater problem solving ability. The Panel stated that "computing activities" and relevant areas of numerical analysis should be integrated into the earliest college mathematics courses and in subsequent courses "when appropriate and natural." By way of illustration they noted that it is important in the study of calculus for a student to understand that a definite integral is a number. The computer can contribute effectively to this understanding. The role of the Fundamental Theorem of the Calculus is not likely to be misunderstood after this approach (5, p. 2)

Scholarly information about computer usage was gathered and published in 1966 in the National Academy of Sciences - National Research Council publication Digital Computer Needs in Universities and Colleges (10, p. 123). Their recommendations concerning mathematics were similar to the Panel's recommendations above.

Philip Davis of Brown University remarked,

"A case can be made out that all undergraduates should (a) pass a swimming test, (b) pass a driving test, (c) take a course in computers. To remain ignorant of computers is to remain ignorant of the world as it is, in fact, rapidly becoming. Computer calculus offers one possibility for elementary contact with this aspect of the New World, Brave or otherwise." (8, p. 15)

Davis does not recommend wholesale changes in the calculus as he states, "In my view only slight changes in the traditional calculus curriculum and problem assignments are necessary in order to convert them to a profitable computer calculus course." (8, p. 1)

Other conferences in which "calculus with computers" was on the agenda are listed in an article by E. P. Miles which also contains a brief history of the Center for Research in College Instruction of Science and Mathematics (CRICISAM) calculus project which will be mentioned from time to time in this discourse (26).

The Computer and Elementary Calculus

The first year elementary calculus course, because of the nature of the subject and its placement in the undergraduate curriculum, is a rather natural course for the introduction of computers into the students' domain of experience. An increasing number of students who enroll in such a course have had either calculus or computer experience, frequently both, in their high school mathematics curricula. Having learned of its relevance there, they can be interested and somewhat self-motivated in a college calculus course if it uses computers.

The knowledge obtained in calculus really rests upon the understanding of limit processes, and an association with computers can deepen such an understanding. Kuller states,

"... explicit computation definitely makes things clearer, and even the epsilon-delta of convergence can be gotten across by considering examples in which the epsilon serves as a test for termination of the execution of a computer program." (23, p. 4.2)

The machine setting imposes a discipline of logical thought in preparing a program while, in processing a program, the student is

supplied with rapid reinforcement, experiencing success or failure then and there, a cornerstone of many modern theories of learning. Porter suggests using the computer to help the student arrive at some of the definitions and proofs of calculus.

"The precision required in order to communicate with a computer forces students to approach problems in a manner more conducive to the teaching of mathematics. This method of thinking is more analytic and should be an advantage in other subjects as well as mathematics."
(28, p. 4.17)

There appears to be sufficient reasons for expecting that overall student comprehension, interest, insight, and motivation can be increased by giving more emphasis to constructive and algorithmic aspects of the calculus. For example, suppose $f(x)$ and $g(x)$ are functions of x and suppose the limit of each at zero is zero. Does the limit of $f(x)/g(x)$ at zero exist? If it does, what is it equal to? To a beginning calculus student the determination of such a limit may be quite difficult, and, even after applying L'Hospital's Rule (when possible), the result may mean little to him. However, with the aid of a computer this limit can be approximated, if it exists, and the problem can become very stimulating and real to the student as he watches the sequence of numbers converge.

Other reasons advanced for introducing computers into college calculus include the opportunity they provide for experimentation with certain important calculus concepts such as summation definitions of the integral, infinite series, improper integrals, and indeterminate forms of limits. Also, many calculus problems are very laborious when done by hand. Such problems can be more instructive, interesting, and much less tedious when programmed for a computer.

A summary of rationale and purposes for using the computer in elementary calculus instruction includes the following statements:

- (a) To motivate and improve learning by giving students a laboratory-type involvement with some of the numerics of calculus.
- (b) To encourage the development of insight into the "whys" of calculus theory and techniques.
- (c) To introduce computers into students' domain of experience in a reasonably natural setting at a relatively early time in their college career.
- (d) Provide an opportunity for students to do instructive problems that are too tedious to be done by hand.
- (e) To stimulate overall student interest, comprehension, and motivation by turning learning into a more active and less passive experience by giving more emphasis to the constructive and algorithmic aspect of the calculus.
- (f) To react to the profound effect the computer has had on how mathematics is applied by modifying the way it is taught.

How Has the Computer Been Used?

A result of recommendations by groups and individuals as stated above has been the creation of many programs devised to integrate computers and computer methods into traditional elementary calculus courses. Philip Davis and Charles Strauss of Brown University stated in a report, dated May 1969, that there were then about "ten to fifteen" computer oriented calculus courses given in a variety of

colleges throughout the nation (9, p. 1). In the August, 1969 CUPM Newsletter, computer calculus programs of fourteen colleges and universities were briefly described (24). In March, 1971, Miles reported that the computer-oriented calculus text (6) had undergone "successful test use" on about fifty campuses by the end of the 1969-70 academic year (26). A National Science Foundation Summer Institute was held at Florida State University in 1969 and again in 1970 to train college teachers (mostly with doctorates) for using computer-oriented calculus materials. The use of such materials, thus, seems to be gaining widespread attention.

The nature of computer-oriented calculus programs varies from minimal revisions in which a student is expected to independently use the computer to solve a few selected problems, to complete revisions of the traditional calculus content. The approaches at most institutions seem to be experimental, beginning with one or two sections of "Calculus with Computers" under the direction of instructors especially enthusiastic about such an assignment. This may be extended the following year to a larger body of students with the whole freshman calculus class the eventual objective (24, p. 2).

Not much published material has appeared yet, although apparently some is in the process of being published (8, p. 11). The limited amount of material that has been published appears to take essentially one of three forms. The first and most typical is in the form of a supplemental booklet such as Richard W. Hamming's, Calculus and the Computer Revolution (15). This short booklet examines the general relationship between computing and mathematics, particularly calculus topics, indicating some of the possibilities and limitations of

computers. Secondly, the material may be part of a calculus text written with computer applications in mind but which contains the traditional calculus approach. Computer related material is usually included in exercises, section-ending comments and a part of the appendix. An example of this type is Henriksen and Lee's Single Variable Calculus (18). This type may have an accompanying exercise book such as the Elementary Computer Programming With Illustrative Problems by Lisa and Judah Rosenblatt (32) written to be used with the Henriksen and Lee text. Thirdly, the material may be a complete revision of the traditional calculus such as the text Calculus: A Computer Oriented Presentation, developed by the CRICISAM at Florida State University. A team of writers sponsored by a National Science Foundation grant produced the text in the period 1967 to 1969.

An Approach to Using the Computer in Calculus Instruction

The author of this thesis, having taught courses in elementary calculus for several years, finds the list of topics to be "covered" in the course already so time and effort consuming for students that the addition of supplementary material, like that of Hamming (15) and the Rosenblatts (32), would too severely deplete time allotted for topics the author considers necessary in a beginning course. CRICISAM authors found it expedient to delete or minimize several traditional topics including parametric equations, polar coordinates, partial fractions and indeterminate forms. Many mathematicians, including this author, are not prepared to modify the curriculum this severely.

It is also true that a majority of students are now being taught elementary calculus from one of a handful of calculus texts which have evolved through several editions into their present forms. Such texts have become more or less complete and error free in their list of topics, presentations, and exercises. Most instructors of the subject prefer to use such texts and probably this situation will continue for several years. As a result, the typical calculus instructor's problem is how to improve his teaching of calculus while not deviating too much from the particular text and list of necessary topics. In particular, if there is a computer available, how might he harness its capabilities for improving his calculus teaching efficiency? He may envision several possible benefits to students who get an introduction to computing while taking his class, but he probably lacks experience or training suitable for making a sound judgment on the real benefits. He does know that he would like his students to learn faster, obtain insight into the subject and some of its applications, to develop some appreciation of the subject, and modern uses. The teaching of anything of significance about computer hardware, software or programming is precluded, however, since by our assumption the instructor does not know very much about these subjects. Also, he is pressed by time to do justice to the topics he considers essential in calculus. He simply sees promise in using the computer to increase understanding and insight through actual numerical calculation, with a fringe benefit being the students' exposure to a small sample of the capabilities of the digital computer.

The author has developed "calculus with computer" materials for use by instructors having the objectives and needs as described above.

The materials have been devised to motivate, develop insight, reduce tedious computations, provide for student experimentation, support modern text material without replacing large portions of it, and increase the effectiveness and relevance of elementary calculus instruction. These materials are intended to be usable by teachers of limited computer experience and knowledge. It is imperative that computer time usage and student programming be kept within very reasonable limits so that the expense can be minimized and so that students' time for theoretical study is not depleted. The tendency toward "baby numerical analysis" must be resisted, keeping numerical methods in proper perspective with beginning calculus. But at the same time, basic numerical methods necessarily receive more emphasis than is traditionally the case.

These materials will not substantially change the traditional content, but will provide an opportunity for a student to experience success and failure in solving several pertinent exercises which have been introduced to him in the calculus. This would not be possible without a computer.

Abundant opportunity is provided for individual instructors to choose from several key areas the particular flow charts programs and exercises which best meet the needs of their course objectives and student population. The materials are described in a manner which makes them independent of any specific programming language. However, examples of programs written in Conversational Programming System BASIC (CPS BASIC), the language all of the programs have been developed and tested in, will be presented. The purpose for such examples will be to give the prospective user some perspective for what a given

computer usage will entail. Flow charting is gaining popularity as a device for teaching and learning efficient analytical habits of problem solving. Hence, the author encourages its use as such, in addition to the usual preprogramming function. The use of flow charts also provides a means for making the materials computer-language-independent.

In general, the intent will be to present a reasonable way to use the computer in a calculus course for the average college in the present and near future, with the assumption that the average college mathematics student and instructor will have at least limited access to a computer.

An Experimental Use of the Materials

By way of providing an example of how such materials can be used, as well as what effect they have on student learning, description of an experiment with their use is provided in Chapter III. A portion of the materials was chosen and adapted for use in a course given at Oklahoma State University during the first semester of 1970-71. Data was collected and analyzed on students using the materials, along with a comparable group of students not introduced to the computer.

The plan for the remainder of this discourse, after having provided an introduction to computer use in calculus instruction, is to proceed with a description of the author's ideas for such computer use. Chapter II will consist of this description. In Chapter III the experimental study will be presented. Finally, a set of recommendations for further use of the computer and possible items for study will be listed in the last chapter, Chapter IV.

CHAPTER II
ON USING THE COMPUTER, FLOW CHARTS AND
RELATED EXERCISES WITH CALCULUS

As at the beginning of any new subject, symbolism, rules for operations, and procedures must be introduced. If one plans to use flow charts during the course, it is necessary to spend some time discussing their nature, purpose, and use. The amount of detailed discussion will depend, of course, on student background as well as how extensively flow charts are to be used. Much programming is done by people who do not use flow charts, so they are not necessarily basic to computer usage, much less to elementary calculus. Also, flow charts can detract attention from the algorithms they serve to diagram. However, this writer sees value in their use as a programming aid and for instructive thought organizing.

Since flow charting involves considerable diagramming, it is helpful to standardize commonly used symbols. This reduces the amount of writing and description necessary to transmit the intended ideas. Symbols used in flow charting vary from person to person and may depend on the nature of routines to be diagrammed. The author has derived a set of symbols which he finds adequate for flow charting of the type done in these materials. The set is listed in Appendix A.

An important technique when developing a flow chart with the class, as well as discussing one already completed, is to emphasize the crucial

step or steps within the logical structure. Attention to the somewhat routine parts such as input, output, and function definition should be put off as long as possible. The instructor should construct the flow chart around the "big step." For example, in the flow chart illustrated in Figure 6 on page 29 the important steps occur in the box marked with an asterisk.

The method by which the computer is presented to a class of students depends on how it is to be used as well as on the students' background. One of the first instructor decisions (if he has an option) is what programming language he will use. The most common languages are BASIC, FORTRAN, PL/1, and ALGOL. Some installations have their own individual languages, but most have at least FORTRAN. The author has used versions of all the above except ALGOL and presently feels BASIC to be most suitable for students in their first computer experience. The language was developed for such use, and its conversational nature is relatively easier to learn than most others. The author has found two or three hours of BASIC programming experience to be adequate if individual guidance can be provided for the initial few weeks of programming. There are several adequate BASIC manuals available which students can use as reference material. The author has found a simple general introduction-to-computer booklet helpful. This can be instructor-written fairly easily. A good example can be found in Elementary Computer Programming With Illustrative Problems (32) as well as SMSG's textbook (35).

Another decision to be made is what the ratio will be between instructor-provided programs and student written programs. While student programming is instructive, it is also very time consuming for a

student to write, debug, and execute a program. In the author's opinion time spent on flow charting and analyzing data from provided programs is more instructive, in general. But course objectives must be the principal deciding factor. Also, if a laboratory period is adjoined to the calculus course, as is possible in some situations, more programming will probably be beneficial.

Some criteria for consideration when programming problems for calculus are:

- (a) Can the computer be used effectively to do the given type of problem?
- (b) How complicated will the problem of programming become?
- (c) How much computer time is involved?
- (d) Can a general program for all problems of this type be written?
- (e) What degree of accuracy is required?
- (f) How should the output look?

Flow Charting, Computers, and Analytic Geometry²

The author has found it rather efficient to combine the initial programming instruction and introduction with topics of analytic

²An explanation about the format for presenting the author's ideas is probably needed here. Under each of the eight main topics of this chapter, the author will list several subtopics which will delineate how he suggests using the computer. These subtopics will include examples, flow charts, computer programs, and sample exercises. One important purpose of these subtopics is to stimulate the reader to improve on the author's approach as well as develop other ways to utilize the computer in his calculus instruction.

A reference matrix is presented in Appendix C to provide a convenient source of reference information for the interested reader. The author's materials are cross-referenced with eight other text sources.

geometry. Since many calculus courses begin with consideration of analytic geometry and simple function topics, two objectives can be accomplished simultaneously. Namely, the computer, computer language, programming, and flow charting can be introduced by using examples from analytic geometry, thus enabling students to learn about all of these together.

n Factorial. A reasonably easy but helpful place to start programming and flow charting is the computation of $n!$ for n up to at least ten. The program should be about eight to ten statements long, but students can be required to use certain techniques such as iteration segments and if statements. Data output in integer, decimal, and exponential forms can be required so that experience is obtained in determining output format. The nature of what is happening should be mathematically transparent.

Absolute Value of a Number. Another very simple programming exercise is to compute the absolute value of a (real) number. Experience with data input and output can be gained as well as a review of the nature of the important absolute value function. For some students, their definition of absolute value as "leaving off the sign" will be quite inadequate.

Table of Ordered Pairs for a Function. Another simple programming exercise is to write a program which will produce a table of function values for a given function at partition points of an interval $[a,b]$. It is handy for each student to have this program available during the calculus course. He can then investigate the nature of any function he meets by looking at some function values in a given interval.

Finding the Slope and Equation of a Line Through Two Points. A slightly more involved program is required to find the slope and slope-intercept equation of the line through two given points (x_1, y_1) and (x_2, y_2) . This serves to review the often used slope-intercept equation form $y = mx + b$ and the algebraic definition of slope (including nondefinition for a vertical line). The program should check for a vertical slope first, then compute the slope m if it is defined. The intercept is easily shown to be given by $b = y_1 - mx_1$.

Finding An Equation of the Perpendicular Bisector for a Line Segment.

Another slightly involved program is required to find the slope m and midpoint of a line segment through two points (x_1, y_1) , (x_2, y_2) and then compute the slope and slope-intercept equation of the perpendicular bisector of the line segment. Such a programming exercise can review the ideas of midpoint, slope, slope-intercept equation of a line, line length formula, and slope relationship of two mutually perpendicular lines.

Other Possible Introductory Programming And Flow Charting Exercises.

- (a) Write a program to read ten numbers, find their sum, their mean, and then sort into increasing order.
 - (b) Write a program to input coordinates of three (nonlinear) points and then compute the area of the triangle having these three points as vertices. The only formula for area allowed being $A = bh$. (Hint: consider two cases, one side parallel to x-axis and no side parallel to x-axis.)
- This exercise is more difficult than the preceding ones.

- (c) Write a program to determine if points P, Q, and R all lie on a single line.

The Limit of a Function

Since the limit concept is so fundamental to calculus, one is hopeful that the computer can be used in a vital way towards developing insight and comprehension. Many textbook presentations of the limit try to use a spiral approach, beginning with simple detailed examples using an intuitive approach and building slowly to the rigorous epsilon-delta definition. In an analogous manner, computer estimation and illustration of the limit of a function at a point can be spiraled. At first the computer is programmed to perform a fixed number of iterations while computing function values for points approaching a fixed number x_0 from below and above. Then a tolerance stop is added to the program so that it will stop when a reasonable estimate for the limit is reached. Another step would be to have the computer estimate suitable values of delta corresponding to values of a given epsilon. Such an approach encourages students to consider the characteristic behavior of function values for a function f whose limit at x_0 does or does not exist.

An Initial Limit Estimator. When considering the limit of a function f at a point x_0 , we are examining the nature of the function values $f(x)$ as x approaches (but does not equal) x_0 . Of course, x can approach x_0 in many ways. Since the computer works only with discrete rational numbers, we must decide on a reasonably illustrative but general way for obtaining a set of such numbers approaching the fixed

number x_0 . The author has usually chosen to form sequences of the type

$$L_n = x_0 - D^n \quad \text{and} \quad R_n = x_0 + D^n, \quad n = 0, 1, 2, 3, \dots$$

If D is given the value $1/2$, for example, then

$$\{L_n\} = \{x_0 - 1, x_0 - 1/2, x_0 - 1/4, \dots\}$$

and

$$\{R_n\} = \{x_0 + 1, x_0 + 1/2, x_0 + 1/4, \dots\}.$$

See Figure 1 for an illustration.

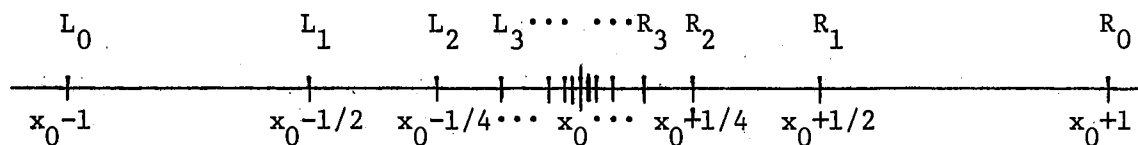


Figure 1: Diagram of Sequences $\{L_n\}$ and $\{R_n\}$

Of course, $L_n < L_{n+1} < x_0$ for all n and $\{L_n\}$ has limit x_0 , while $x_0 < R_{n+1} < R_n$ for all n and $\{R_n\}$ has limit x_0 also. Clearly, so long as D is a positive number less than one, the sequences $\{L_n\}$ and $\{R_n\}$ will have the property of approaching x_0 from the left and from the right, respectively. The "rate" at which the sequences converge to x_0 is dependent upon the value of D .

Logical steps for obtaining function values at numbers in the two sequences $\{L_n\}$ and $\{R_n\}$ is illustrated by the flow chart in Figure 2.

A computer program can be easily written which will iteratively produce a table of the two sets of ordered pairs $(L_n, f(L_n))$ and $(R_n, f(R_n))$ corresponding to $n = 0, 1, 2, 3, \dots, N$. Students will be able to investigate the limit of several functions at a fixed point x_0 from the left and right. For reference let us refer to such a program as LIM1.

An immediate question should come to mind: When will this process be ended? In the routine illustrated above there is no provision for stopping. When should it be stopped? Well, in using it to estimate the limit of f at x_0 , we are concerned with the two sets of numbers $\{f(L_n)\}$ and $\{f(R_n)\}$ as n increases. We will recognize that we have an estimate k for

$$\lim_{x \rightarrow x_0} f(x)$$

if and only if (a) $f(L_n)$ approaches the number k as n increases (and hence L_n approaches x_0) and (b) $f(R_n)$ approaches k as n increases (and R_n approaches x_0). Symbolically, this is indicated by

$$\lim_{x \rightarrow x_0^-} f(x) = \lim_{n \rightarrow \infty} f(L_n) = k = \lim_{n \rightarrow \infty} f(R_n) = \lim_{x \rightarrow x_0^+} f(x)$$

Figure 3 illustrates output from the author's version of LIM1 on two limits.

LIM1 can also be used to examine the continuity of a function at a point x_0 , since a function f is continuous at x_0 if and only if

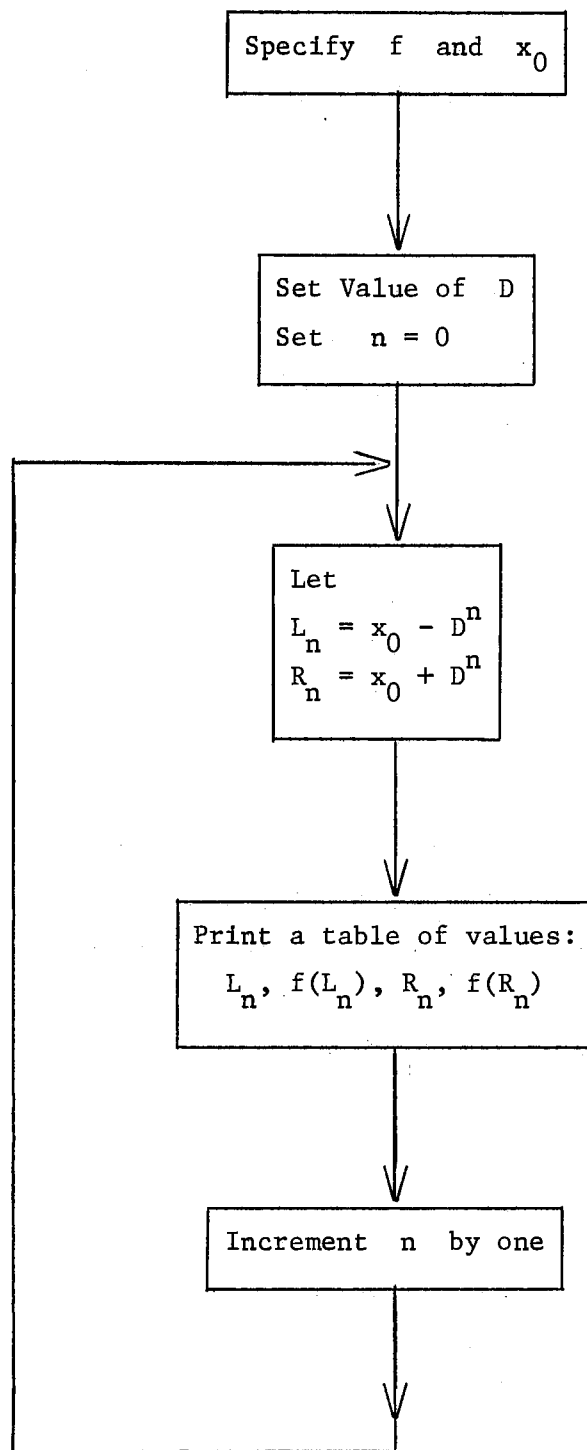


Figure 2. Flow Chart for Estimating the Limit of f at x_0 .

An Investigation of $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ using LIM1

i	$x_0 - DL^{**i}$	$f(x_0^-)$	$x_0 + DL^{**i}$	$f(x_0^+)$
0	-1.0000000000	.8414709848	1.0000000000	.8414709848
1	-.1000000000	.9983341665	.1000000000	.9983341665
2	-.0100000000	.9999833334	.0100000000	.9999833334
3	-.0010000000	.9999983333	.0010000000	.9999983333
4	-.0001000000	.9999999983	.0001000000	.9999999983

An Investigation of $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ using LIM1

i	$x_0 - DL^{**i}$	$f(x_0^-)$	$x_0 + DL^{**i}$	$f(x_0^+)$
0	-1.0000000000	-.8414709848	1.0000000000	.8414709848
1	-.5000000000	-.9092974268	.5000000000	.9092974268
2	-.2500000000	.7568024953	.2500000000	-.7568024953
3	-.1250000000	-.9893582466	.1250000000	.9893582466
4	-.0625000000	.2879033167	.0625000000	-.2879033167
5	-.0312500000	-.5514266812	.0312500000	.5514266812
6	-.0156250000	-.9200260382	.0156250000	.9200260382

Figure 3. Output from LIM1

$$(1) \quad \lim_{x \rightarrow x_0^-} f(x) = \lim_{x \rightarrow x_0^+} f(x).$$

An Improved Limit Estimator. Let us see how LIM1 could be extended such that it will be self-stopping and produce estimates for the two limits on line (1) to a given tolerance. The new program will be called LIM2. One way to accomplish the modification is as follows: After printing a line of values for $L_n, f(L_n), R_n, f(R_n)$ for a given n , compute $|f(L_n) - f(L_{n-1})|$. If this number is not less than the specified tolerance (for example, .00001) then increase the subscript by one and produce another line of values, examining the absolute value again for $n + 1$. If $|f(L_n) - f(L_{n+1})|$ is less than the tolerance, consider $|f(R_n) - f(R_{n+1})|$. Continue until the tolerance is met by both absolute values simultaneously. Then $f(L_1)$ and $f(R_1)$ of the last line printed are estimates for the limits of the limits in (1) to the specified tolerance. A maximum for n can be specified in case the tolerance is never met.

By now students will know that

$$\lim_{x \rightarrow x_0} f(x)$$

is the common value of the left hand and right hand limits (estimated by LIM2) if such exist. (If the programming language being used has a "greatest integer" library routine, as BASIC does, an easy example can be given for which the left hand and right hand limits exist but are not equal.)

The tolerance check for LIM2 could be set up to compare $f(L_n)$ and $f(R_n)$ for a given n , and thus the process would be stopped when

estimates for the limits in (1) appear to be sufficiently close or after a maximum number of iterations.

A sample of some limits to which students could apply LIM2 instructively includes:

- (a) $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ [1]*
- (b) $\lim_{x \rightarrow 1} \frac{1}{x-1}$ [left hand limit is $-\infty$, right hand limit is ∞].
- (c) $\lim_{x \rightarrow 0} \sin 1/x$ [neither one-sided limits exist]
- (d) $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4}$ [1/4]

One-sided Limit Estimator. The author has found it instructive to provide students with a program for estimating the limits in (1) independently. The program is capable of estimating either or both one-sided limits, has a tolerance stop for one-sided limits which are finite and has a tolerance stop in case the limits are infinite. Such a program is particularly useful for experimenting with certain limits usually found in the text section called "indeterminate forms."

Examples are

$$\lim_{x \rightarrow 0^+} (1/x + \ln x) \quad \text{and} \quad \lim_{x \rightarrow 0^+} x^x.$$

Figure 4 is a flow chart for such a routine which will be called ONESID for reference. Symbols in the flow chart have the following

* Answers and hints to exercises will be placed in brackets following the given exercises.

meaning:

TL is a tolerance stop for a finite limit estimate.

N is the maximum number of iterations allowed.

IC is analogous to D in the LIM1 and LIM2 programs discussed previously.

L and R are flags. For an estimate of the left hand limit at x_0 we set $L = -1$, otherwise let $L = 0$. Similarly for R, except we let $R = 1$ when the right hand limit is wanted.

K is a holder for the value of L and R during limit estimation.

When successive function values at XI differ by more than 1000 we will estimate the limit to be (plus or minus) infinity.

Estimator for $\lim_{x \rightarrow \infty} f(x)$. Another type of limit is that denoted by

$$\lim_{x \rightarrow \pm\infty} f(x).$$

This occurs, for example, when sequences are being studied. A routine for estimating such limits can be obtained by creating a sequence of numbers $\{X_i\}$ where $X_i = K \cdot D^i$ for $i = 1, 2, 3, \dots$. The terms in this sequence are always positive, and they become arbitrarily large if $K = 1$ and $D > 1$. The terms in this sequence are always negative and become arbitrarily small, that is, $X_i \rightarrow -\infty$ if $K = -1$ and $D > 1$. Again, the rate at which X_i increases or decreases can be determined by the choice of a value for D. (When examining sequences, X_i is simply defined to be $X_i = i$.) The set of numbers $\{f(X_i)\}$ indicates an estimate for the particular limit at hand.

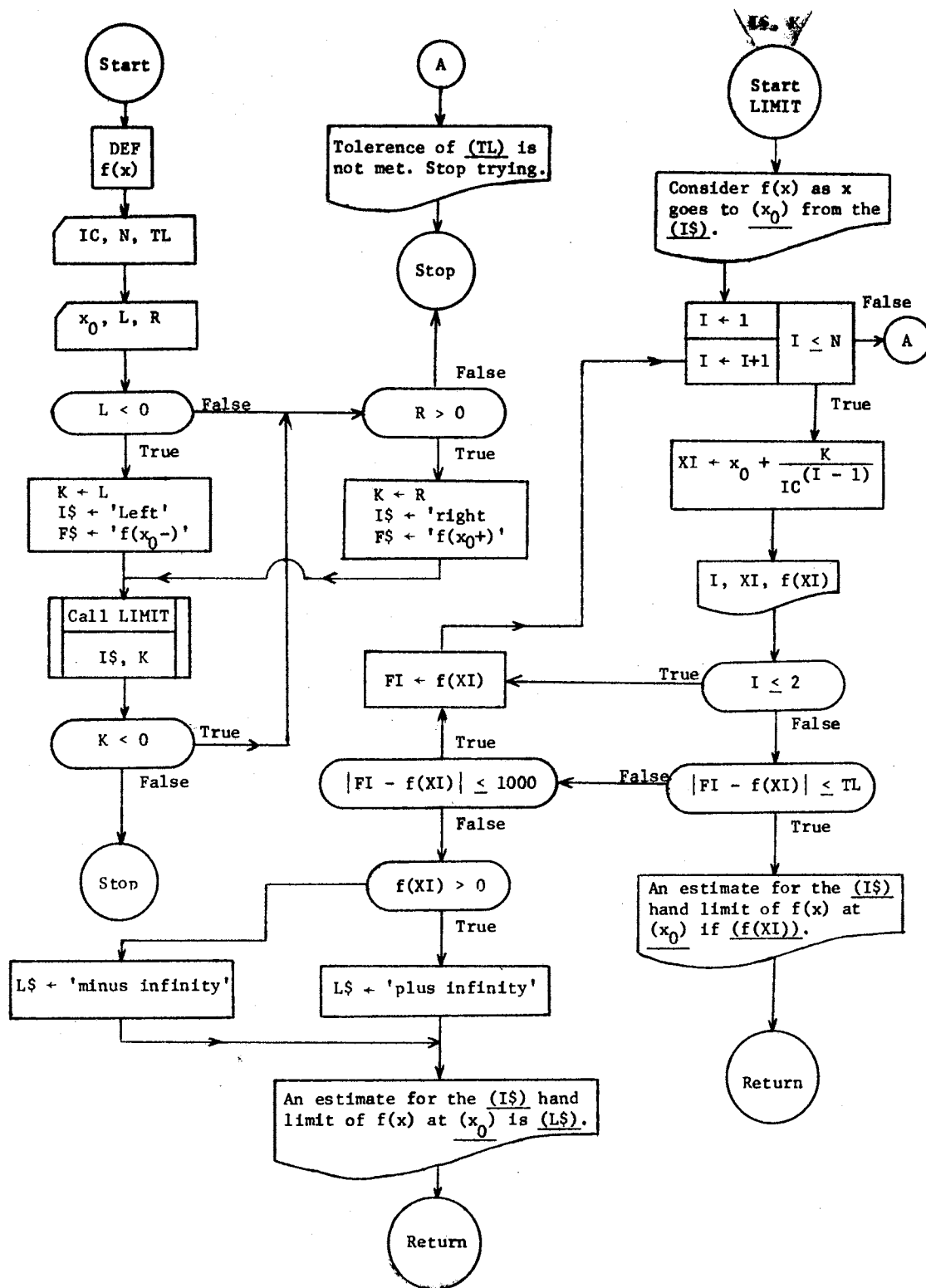


Figure 4. Flow Chart for ONESID

A logical flow chart is provided in Figure 5. For reference the program written from this flow chart will be called LIMINF.

The author has made a point of avoiding subscripted variables up to this point to keep the programming easier to understand. In some cases the programming would be improved by using subscripted variables. When and if they are to be used is a matter left to the instructors' judgment.

Problems to which the routine LIMINF might be applied include:

- (a) $\lim_{x \rightarrow \infty} (1 + 1/x)^x$ with $D = 2$ in the above routine. [Most computers will have trouble when x gets sufficiently large --- after about $x = 2^{48}$. Such an example can be used to discover some limitations of the computer.]
- (b) $\lim_{x \rightarrow \infty} x^5 e^{-x}$ [0]
- (c) $\lim_{x \rightarrow \infty} \frac{\sin x - x}{x}$ [no limit, oscillation]

Another idea which the author has not yet pursued, but which some (22, p. 4.39) have is that of a delta finder program. That is, a routine which, when a function f , number x_0 , and value for epsilon ϵ are specified, will find a value for δ such that

$$|f(x) - f(x_0)| < \epsilon$$

for x satisfying $|x - x_0| < \delta$. Such a program may well have value for teaching the epsilon-delta associated with a precise theoretical approach to limits.

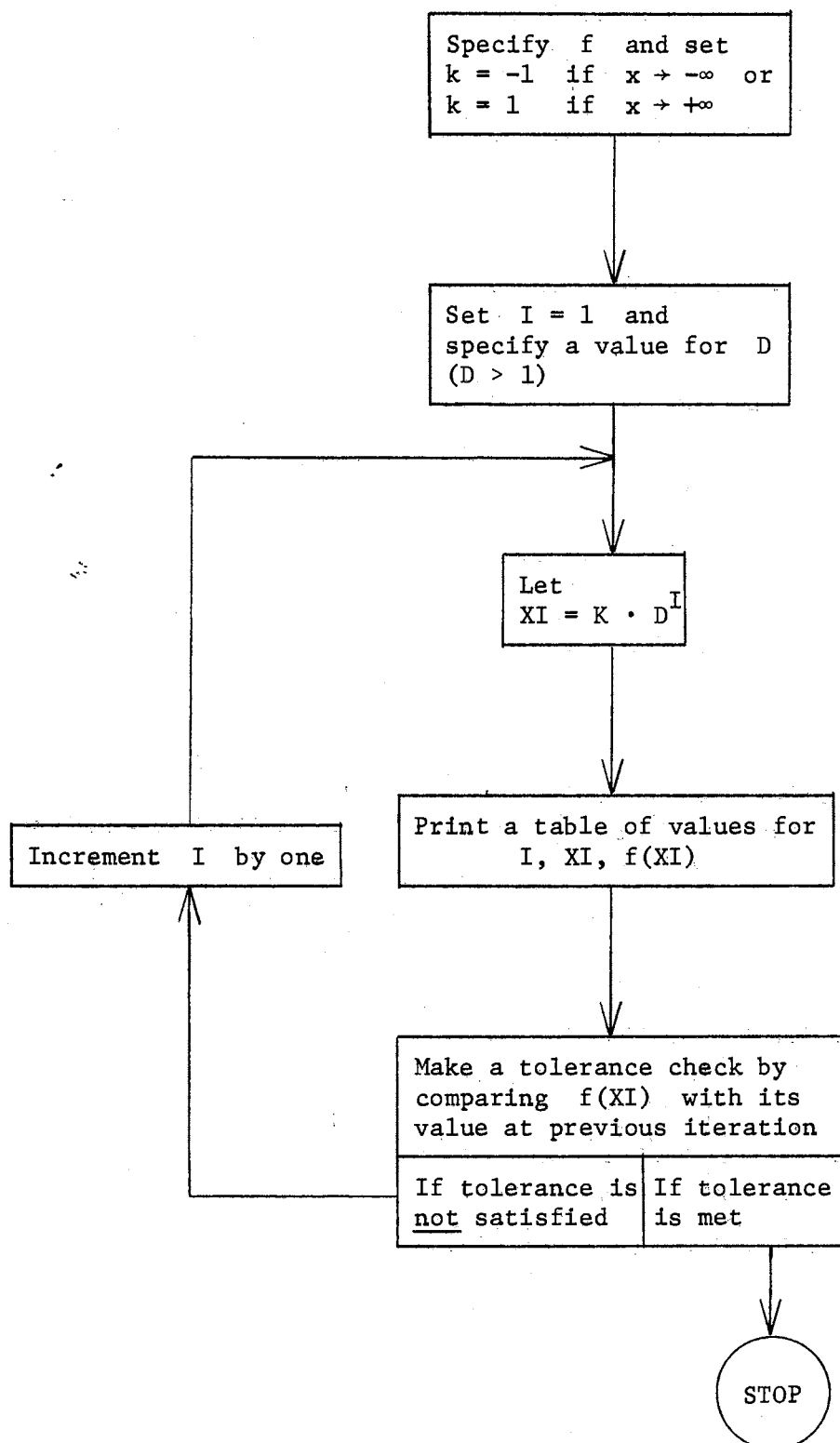


Figure 5. Flow Chart for Estimating $\lim_{x \rightarrow \pm\infty} f(x)$.

The Derivative And Derived Function.

Estimating the Derivative. Using the program LIM2 discussed in the preceding section, students can be given experience estimating the limit of a function f at a number x_0 by computing one-sided limits simultaneously until successive function values from both sides meet a specified tolerance. Since the derivative of f at x_0 , $f'(x_0)$, is the limit of the expression

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \quad (\text{called the difference quotient of } f \text{ at } x_0)$$

as $\Delta x \rightarrow 0$, LIM2 could be used to estimate $f'(x_0)$. We could simply define a function g by

$$g(\Delta x) = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

and use LIM2 to find

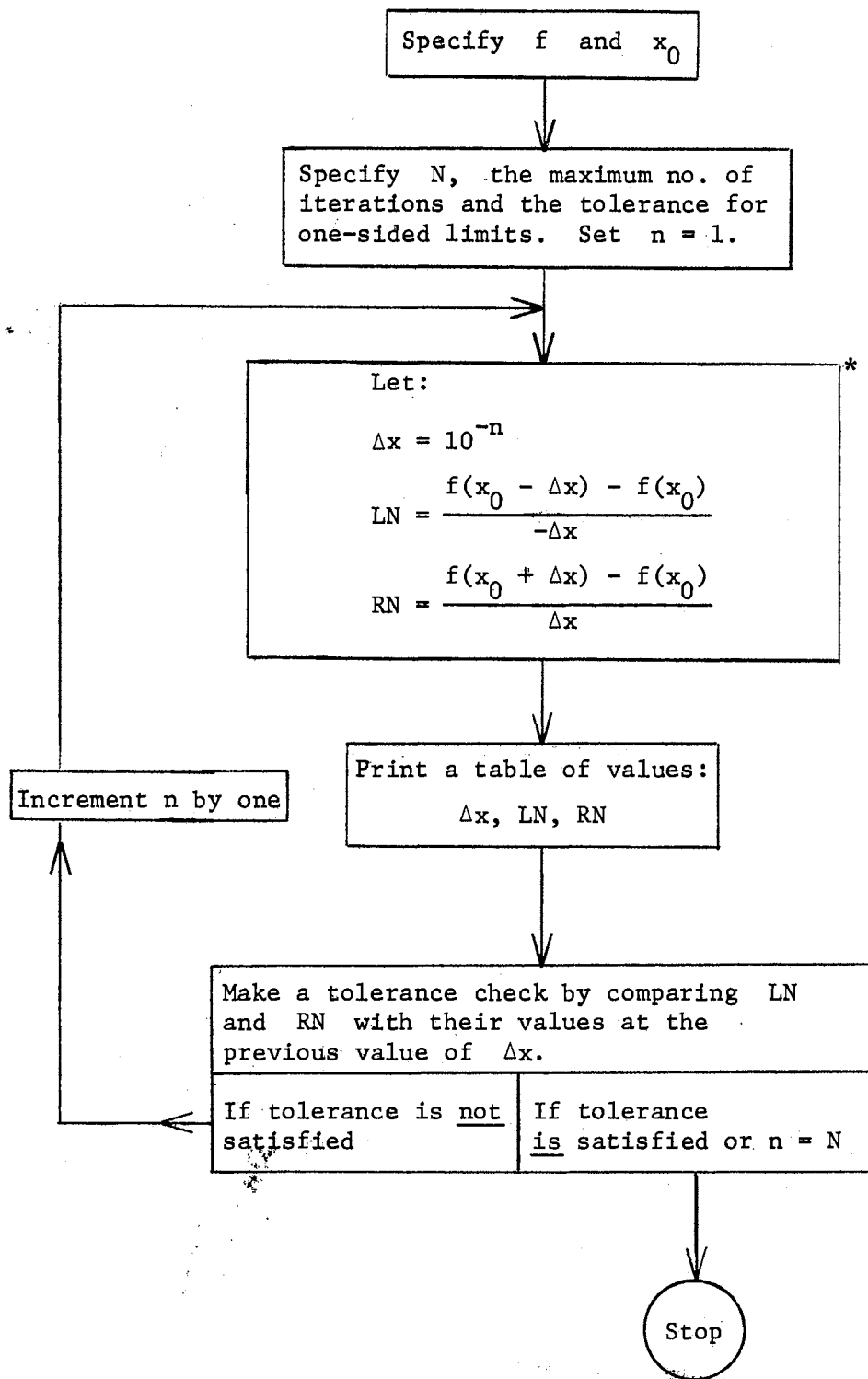
$$\lim_{\Delta x \rightarrow 0} g(\Delta x)$$

which equals $f'(x_0)$, if the limit exists.

The author has chosen to write a separate program, which will be called DERIV for reference, to estimate $f'(x_0)$. In principal it is the same as LIM2. Figure 6 is a flow chart of the routine.

Exercises using DERIV could be of the following type:

- (a) Estimate $f'(x_1)$, $f'(x_2)$, and $f'(x_3)$ where f is a linear function such as $f(x) = 2x + 3$.

Figure 6. Routine to Estimate $f'(x_0)$

* See page 14.

- (b) Estimate $f'(x_1)$, $f'(x_2)$, $f'(x_3)$ where f is a quadratic function such as $f(x) = x^2 - x + 2$.
- (c) Estimate $f'(0)$ where $f(x) = |x|$.

Discovering the Derived Function f' . After using DERIV to obtain $f'(x_0)$, a number, for a given function f and point x_0 , a routine can be developed from it to provide for student discovery of the derived function, f' . We modify DERIV so that it can estimate $f'(x)$ at partition points of an interval $[a,b]$. For reference we will call the modified program SHTDER (printout from DERIV is shortened to save time and paper). From this program a set of ordered pairs $(x, f'(x))$ will be obtained. Students can be directed to graph these pairs and then guess at the defining equation for the curve. If they guess correctly, of course, they obtain the defining equation for f' .

The detailed flow chart of SHTDER is illustrated in Figure 7. Symbols in the flow chart have the following meaning:

- [A,B] denotes the interval on which derivatives of f are to be estimated.
- IN denotes increment between points of derivative estimation.
- XO is holder for value of point at which derivative is being estimated.
- TL is tolerance stop for derivative estimate.
- N is maximum number of iterations in a derivative estimate.
- DX denotes Δx is the difference quotient.

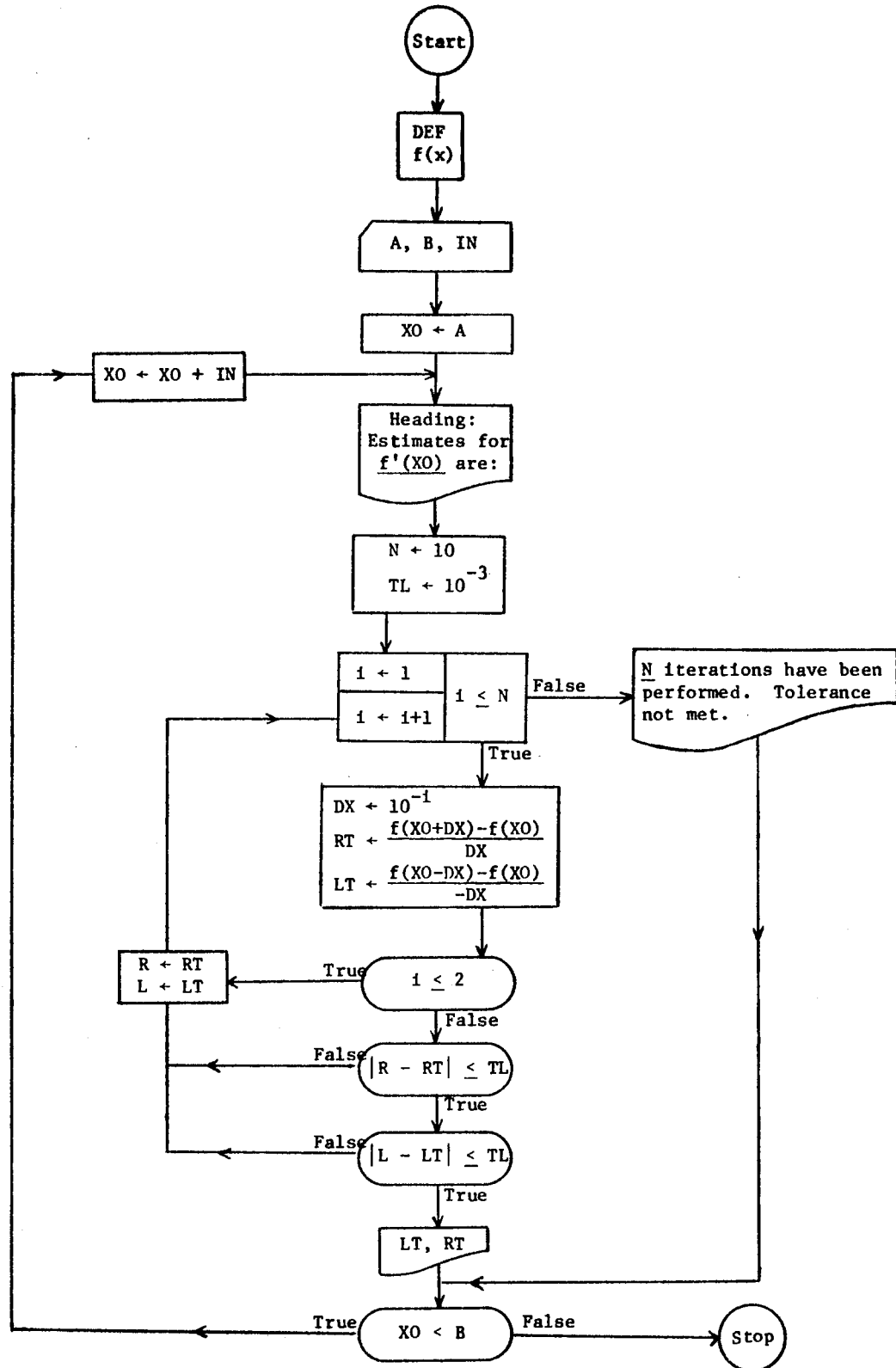


Figure 7. Flow Chart for Estimating Derivatives on [A,B]

RT and LT are holders for right and left hand limit estimates of $f'(x_0)$.

Figure 8 is an illustration of output from SHTDER with the associated graph.

Some rather obvious functions for students to apply SHTDER to include $\sin x$, $\cos x$, e^x , $\ln x$, and $1/x$.

Chain Rule. To help students learn the chain rule, obtain some insight into functional composition, and experience some feedback from the computer, a routine called CHNRUL has been used. The program DERIV for estimating $f'(x_0)$, discussed at the beginning of this section, is incorporated as a subroutine in CHNRUL.

Figure 9 is a flow chart illustrating the logic of the program.

A student must find and define for the computer the functions f , f' , g , g' , and $(f \circ g)'$, the latter using the chain rule. The computer evaluates these functions and some pertinent composed functions at x_0 and then, using f and g independently, estimates $(f \circ g)'(x_0)$. The computer then "checks" his use of the chain rule. He must find three derivatives correctly, and the printout is such that mistakes can usually be pinpointed by noting how the three values outputted for $(f \circ g)'(x_0)$ are produced. A sample of output from CHNRUL is illustrated in Figure 10.

Rate of Change Exercise. Since the ideas of distance traveled, velocity, and acceleration are frequently used to motivate the derivative, a reasonably instructive problem will be suggested. It can be easily programmed.

Estimates for $f'(-4.000)$ are:

-.757 and -.757

Estimates for $f'(-3.500)$ are:

-.351 and -.351

Estimates for $f'(-3.000)$ are:

.141 and .141

Estimates for $f'(-2.500)$ are:

.598 and .599

Estimates for $f'(-2.000)$ are:

.909 and .909

.....

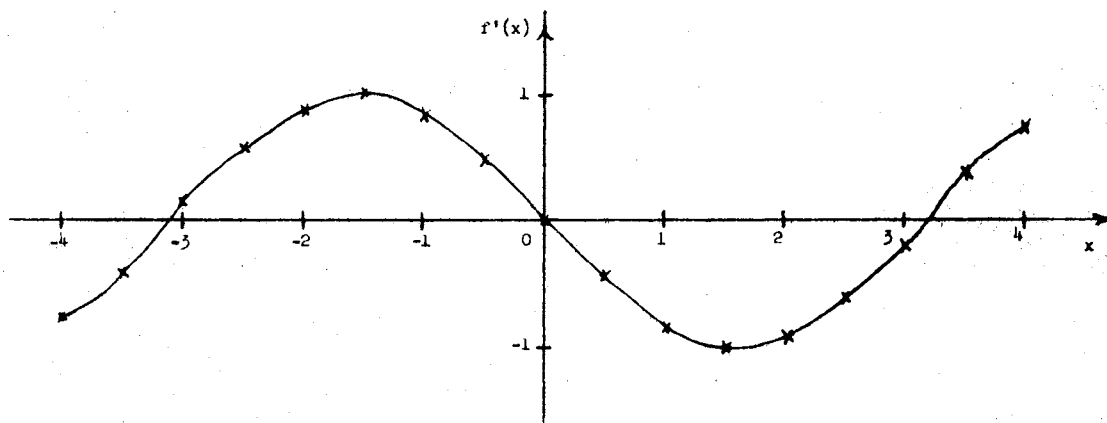
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Estimates for $f'(4.000)$ are:

.757 and .757

$$f(x) = \cos x$$

for $x \in [-4, 4]$



Student's guess: $y = -\sin x$?

Figure 8. Sample Output and Graph From SHTDER

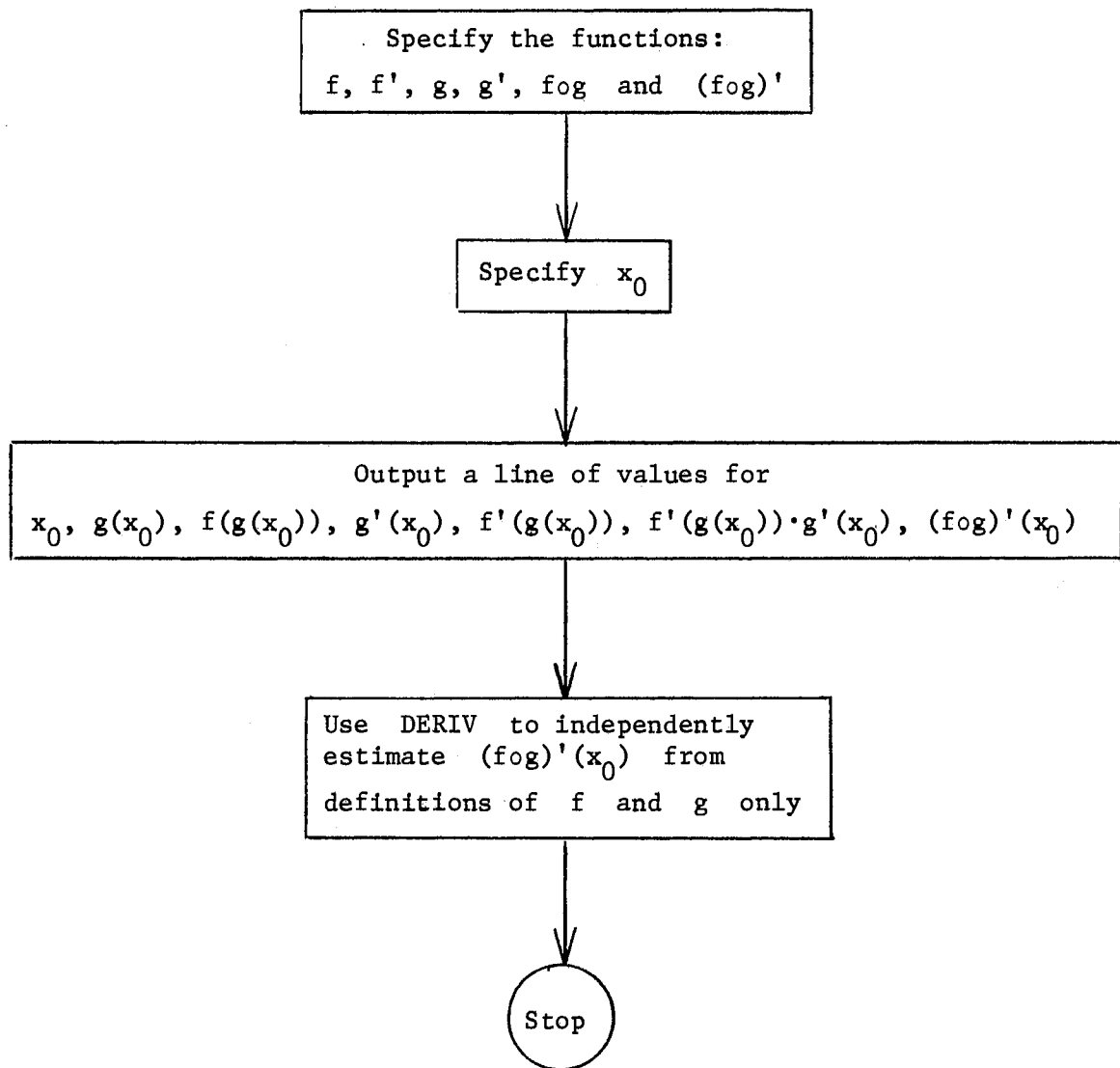


Figure 9. Flow Chart for CHNRUL

For $x = 1$, $f(x) = \sqrt{x}$, and $g(x) = 2x + x^2$, the computer output is:

Using the student's answers for f' , g' , and $(f \circ g)'$, the following results:

x	g(x)	f(g(x))	g'(x)	f'(g(x))	f'(g(x))*g'(x)	(f \circ g)'(x)
1.00	3.000000	1.732051	4.000000	.288675	1.154701	1.154701

Now using DERIV, estimate the derivative of $f \circ g$.

Estimates for $(f \circ g)'(1.0000)$ are as follows:

dx	Df(x ₀ -)	Df(x ₀ +)
.10E00	1.1650136543	1.1456772369
.1E-01	1.1556692496	1.1537446578
.1E-02	1.1547963276	1.1546043774
.1E-03	1.1547101615	1.1546909165
.1E-04	1.1547015006	1.1546995761

Both 1.154702 and 1.154700 seem to be good estimates for $(f \circ g)'(1.000)$.

Tolerance is .1E-04.

Figure 10. Sample Output From CHNRUL

Exercise: Suppose the position of a certain automobile moving along a straight highway at time t is $f(t) = \sqrt{t}$. For each value of $t = 0, .2, .4, \dots, 5$ calculate and print out the average speed of the car over the time interval extending from t to $t + \Delta t$ where $\Delta t = .1, .01, \text{ and } .001$ successively. Also print $1/(2\sqrt{t})$ for each value of t . Explain the results.

[As Δt gets smaller, the average speed over the interval from t to $t + \Delta t$, for fixed t , approaches the value of $1/(2\sqrt{t})$ since these are estimating $f'(t)$ which is defined by $f'(t) = 1/(2\sqrt{t})$. The approximations get better as t approaches 5 since the slope of $f(t)$ is decreasing.]

Applications of the Derivative

Tangent and Normal Lines. A common and reasonably useful application of the derivative is to find equations of the tangent and normal lines to the graph of a function at a point (x_0, y_0) . The programming of a routine to find these is likewise a reasonable student exercise. Logic for a routine to do this is given in Figure 11.

Differential Approximation. Another application of the derivative is the use of differentials in the approximation of certain numerical expressions. Justification for differential use briefly goes like this:

$\Delta y = f(x + \Delta x) - f(x)$ and $dy = f'(x)dx$ by definition. Rewriting the first equation and choosing $\Delta x = dx$ in the second equation, we get $f(x + \Delta x) = f(x) + \Delta y$ and $dy = f'(x)\Delta x$. Now using the fact that Δy approximately equals dy for Δx relatively small, we have the useful expression $f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x)\Delta x$.

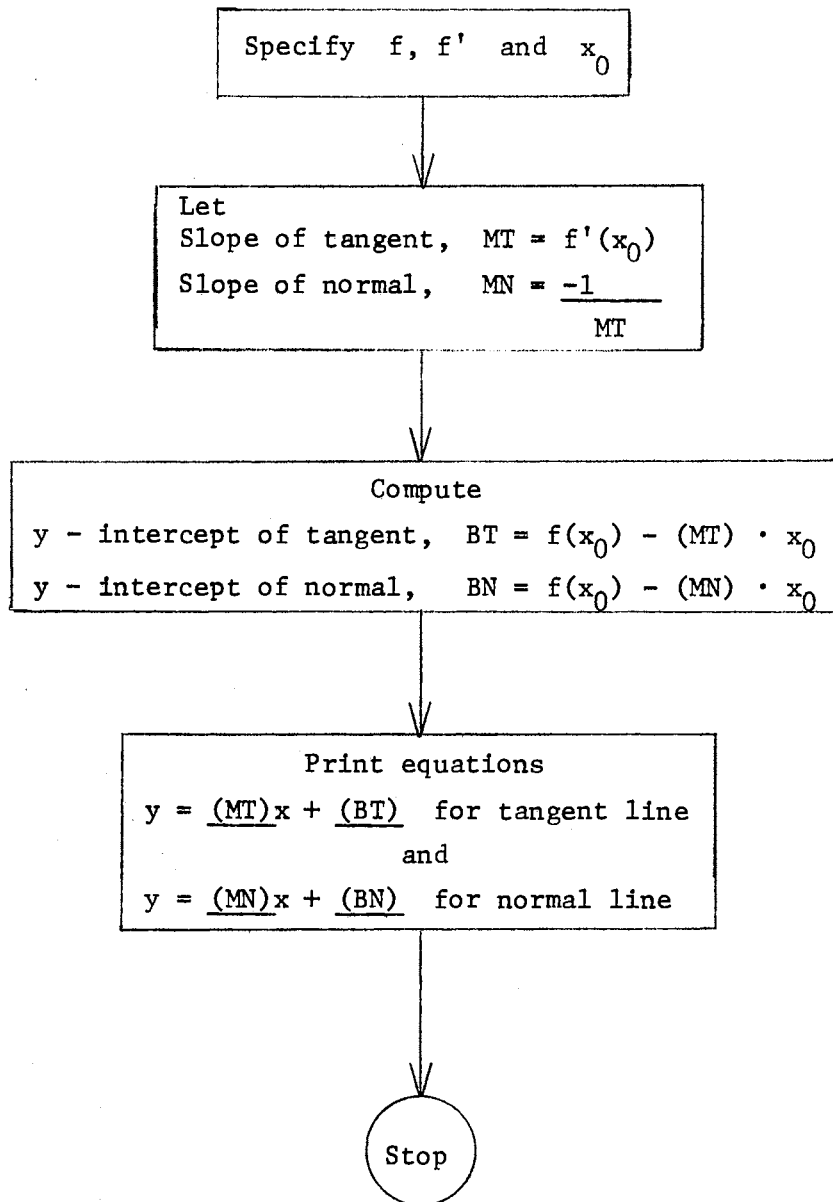


Figure 11. Flow Chart for Tangent and Normal Lines

To approximate $\sqrt{25.1}$, for example, we choose $f(x) = \sqrt{x}$,
 $x = 25$ and $\Delta x = .1$, then $f'(x) = 1/(2\sqrt{x})$.

$$\sqrt{25.1} = f(5 + .1) \approx \sqrt{25} + 1/(2\sqrt{25})(.1) = 5.01.$$

An instructive programming exercise is to have students do the above derivation and then write a program which computes

$$A = f(x) + f'(x)\Delta x$$

and

$$B = f(x + \Delta x)$$

for given f , f' , x , and Δx . The error in the approximation is, of course, $B - A$.

A sample of expressions which can be approximated includes:

$$\sqrt[3]{214}, [5.981]; \quad \sqrt[7]{126}, [1.9955];$$

$$\cos 61^\circ, [0.4849] \quad \text{and} \quad \text{Arc tan } 0.99, [0.7804]$$

Newton's Method. Newton's Method, a relatively important numerical method for approximating the roots of an equation $f(x) = 0$, can be interesting and instructive to a beginning calculus student. Convergence is usually rapid, if it occurs, and students see value in its potential use. (Other root approximation methods might also be used. See pages 61 - 96 of Henrici (16). The temptation to overemphasize such topics must be resisted at this level.)

Newton's method can be viewed in terms of differentials. Referring to Figure 12, suppose that x_0 is an initial approximation to a root of $f(x) = 0$, and that r is the actual root. The tangent line at $(x_0, f(x_0))$ meets the x -axis at x_1 . By definition of dy , $dy = f'(x_0)dx$. From the figure, $dy = f(x_0)$ and $dx = x_0 - x_1$. Thus,

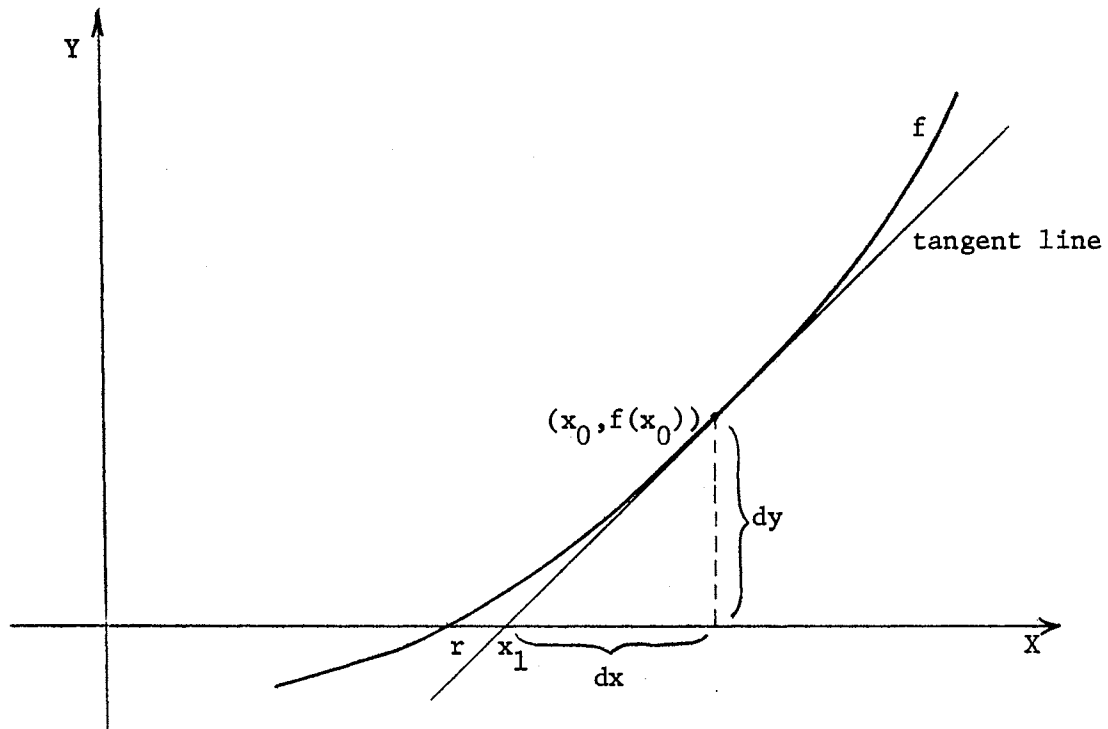


Figure 12. Illustration for Newton's Method

$$dy = f(x_0) = f'(x_0)(x_0 - x_1)$$

so

$$f(x_0)/(f'(x_0)) = x_0 - x_1.$$

Rewriting this equation we obtain $x_1 = x_0 - f(x_0)/(f'(x_0))$. Repeating the process with x_1 in place of x_0 , we get $x_2 = x_1 - f(x_1)/(f'(x_1))$ and, in general,

$$x_n = x_{n-1} - f(x_{n-1})/(f'(x_{n-1})).$$

Under certain conditions the sequence $\{x_n\}$ has r as its limit.

(16, pp. 77 - 83). The conditions can not be stated very simply and,

in general, involve the first guess x_0 , so any rigorous study of Newton's Method would take a calculus class too far afield.

Figure 13 is a detailed flow chart for a routine using Newton's Method. Symbols in the flow chart have the following meaning:

f and f' denote the function f and its derivative.

X_0 denotes first guess for a zero of $f(x)$.

N denotes maximum number of iterations.

TL denotes desired tolerance for estimate of a zero.

X_1 is a computed estimate for a zero of $f(x)$.

An exercise which can be instructive is the following: Verify the Mean Value Theorem for Derivatives where $f(x) = e^{-x^2/2}$, $a = 0$ and $b = 1$, that is, on $[0,1]$ using Newton's Method. [Note that $f(0) = 1$ and $f(1) = e^{-1/2}$ and use Newton's Method to find root of

$$g(x) = -xe^{-x^2/2} - (e^{-1/2} - 1)$$

between 0 and 1.]

Finding Relative Extrema. Techniques for finding relative extrema are given considerable attention in a beginning calculus course. It is important that students develop and understand a sequence of logical steps which one by one apply the various tests and criteria until the necessary information has been extracted.

Flow charting such a routine, even though the routine may be rather limited, can be instructive. The author has joined Newton's Method described previously with a common theorem, often called the Second Derivative Test for Relative Maxima and Minima, to obtain a routine called RELEXT for reference.

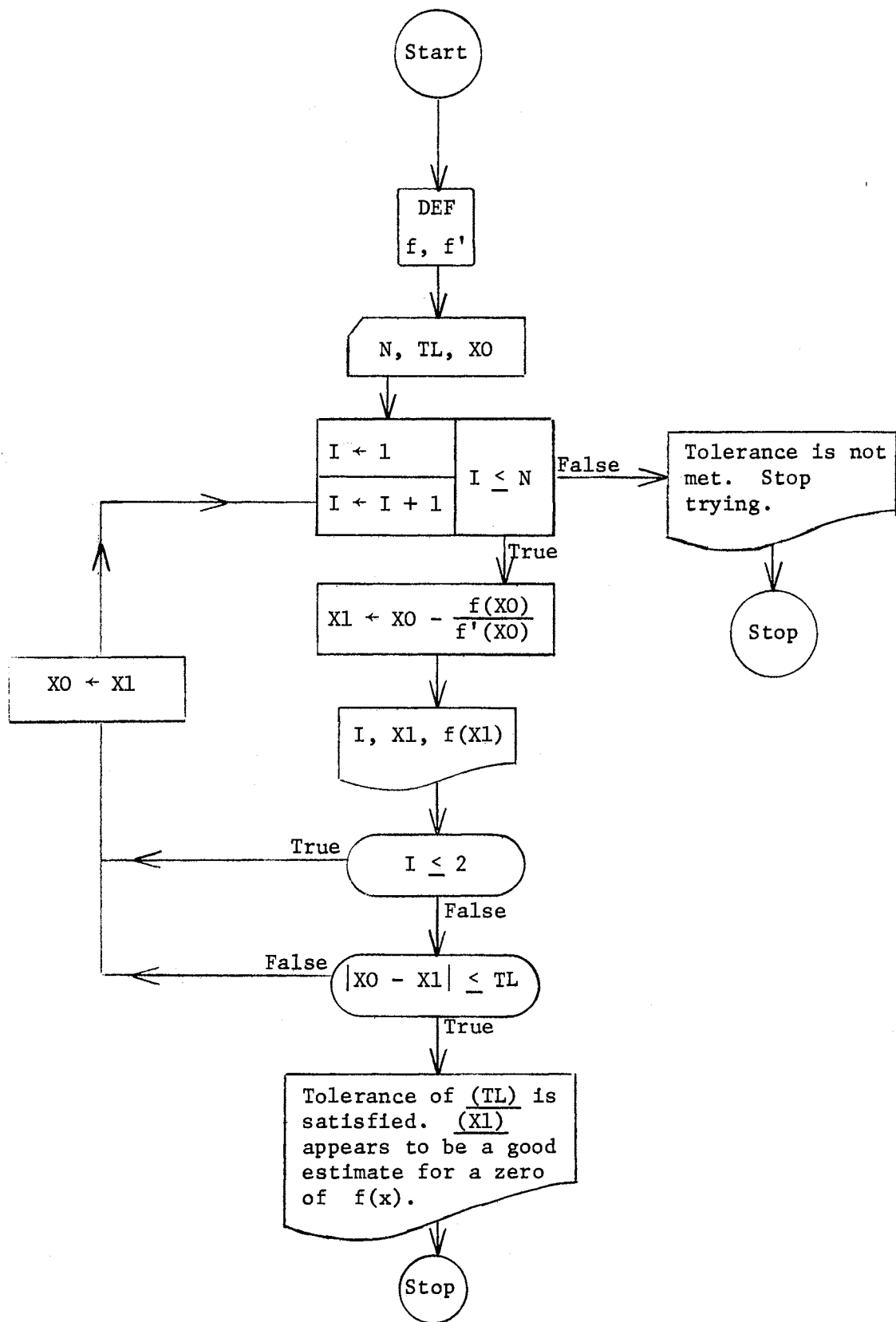


Figure 13. Flow Chart for Newton's Method

Figure 14 is a flow chart of the logic for the routine. In practice, several first guesses for zeroes of f' were specified and the routine performed for each guess.

An example exercise using RELEXT would be the following: Examine the function f defined by $f(x) = x^5 - 20x^2 + 4$ over the interval $[-5,5]$ for relative extrema. [relative maximum at $(0,4)$, relative minimum at $(2,-44)$, and flex point at $x = \sqrt[3]{2}$]

Indeterminate Forms and Taylor's Formula

Indeterminate Forms. An important topic in first year calculus is the study of limits of functions for which the initial attempted evaluation (using the elementary limit theorems) results in one of several particular forms, called indeterminate forms. A very powerful tool for handling such limits is L'Hospital's Rule. Two programs written by the author for examining such limits have been discussed previously. ONESID, discussed on page 23, is designed for estimating one-sided limits, but can be used on ordinary (two-sided) limits. LIMINF, discussed on page 24, is designed to estimate limit of $f(x)$ as x goes to either $+\infty$ or $-\infty$. Both programs have a tolerance stop for a finite limit and an infinite limit.

Some limits, in addition to those suggested earlier, for which ONESID may be instructive are:

$$(a) \quad \lim_{x \rightarrow 0^-} \frac{e^{2x} - 1}{\cos x - 1} \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{e^{2x} - 1}{\cos x - 1} \quad [+\infty \quad \text{and} \quad -\infty, \\ \text{respectively}]$$

$$(b) \quad \lim_{x \rightarrow 0^+} x^{\sqrt{x}} \quad [\text{quite slow convergence to 1}]$$

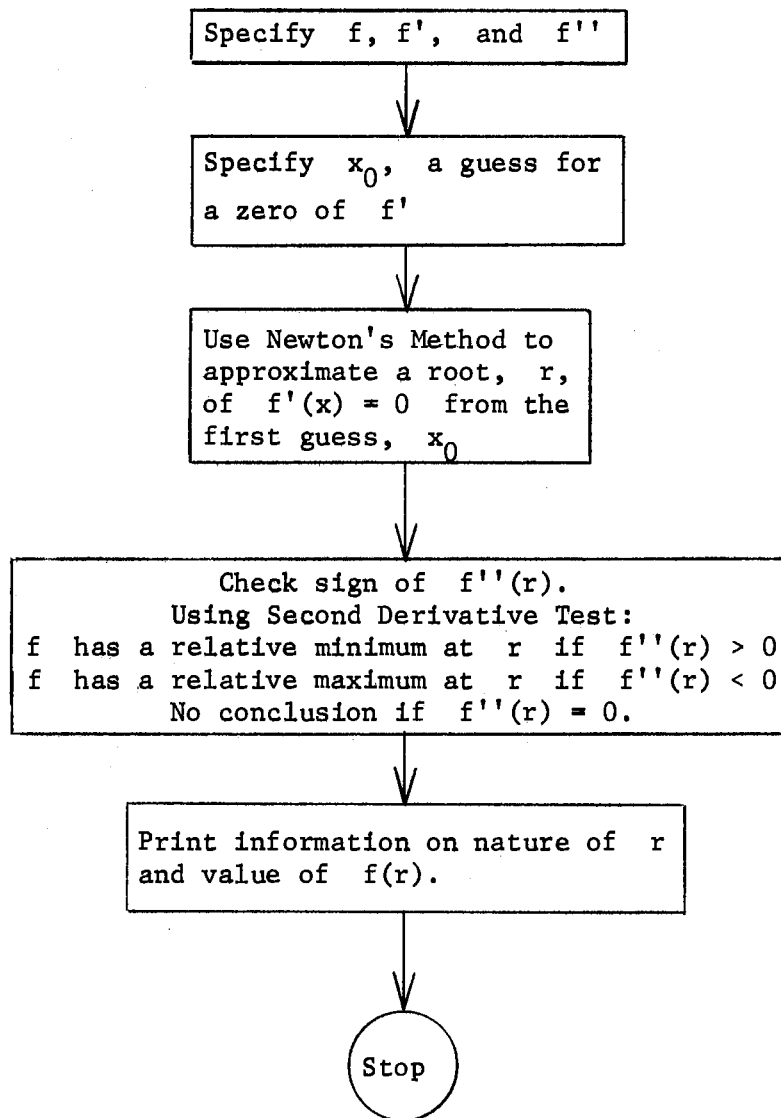


Figure 14. Flow Chart for Investigating Relative Extrema

$$(c) \lim_{x \rightarrow 0^+} x^{(x-c)} \quad [\infty \text{ if } c > 0, \quad 0 \text{ if } c < 0]$$

$$(d) \lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2 \sin x}{x - \sin x} \quad [4 \text{ is the limit, but most computers will have trouble getting the limit if } x \rightarrow 0 \text{ rapidly.}] \text{ (from either side)}$$

Some limits for which LIMINF may be instructive are:

$$(a) \lim_{x \rightarrow -\infty} x^2 e^x \quad [0]$$

$$(b) \lim_{x \rightarrow \infty} (1 + c/x)^x \quad [e^c]$$

$$(c) \lim_{x \rightarrow \infty} \frac{\sin x - x}{x} \quad [-1, \text{ note that L'Hospital's rule cannot be applied directly.}]$$

Taylor Polynomial Approximation. The application of a Taylor polynomial

$$P_n(x) = f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x - a)^n$$

(called the n th-degree Taylor's polynomial of f at a) to estimate the value of a function f at a point x_0 is a good example of where hand computation often obscures the pedagogical objective to such an extent that effective exercises are difficult to obtain. Computer usage can improve the situation. A program called TAYLOR has been written which computes the coefficients

$$\frac{f^{(n)}(a)}{n!} \quad \text{for } n = 0, 1, 2, \dots, 5$$

(5 was arbitrarily chosen for the maximum degree of $P_n(x)$)

for a given function f and number a . Then, for a given x_0 , the program evaluates the Taylor polynomial $P_n(x)$ at x_0 for $n = 0, 1, 2, \dots, K$ ($0 \leq K \leq 5$) successively. Values of $f(x_0)$ and

$$D = |f(x_0) - P_n(x_0)|$$

are also computed and printed out for each n . D is $|R_n(x_0)|$ where $R_n(x)$ is the n th remainder term of f at a . Students can thus investigate how well $P_n(x)$ estimates $f(x)$ as n and x_0 are varied independently.

A detailed flow chart for TAYLOR is illustrated by Figure 15.

Symbols in the flow chart have the following meaning:

$f(x)$ is function to be considered.

a is number about which $f(x)$ is expanded in Taylor polynomials.

n is maximum degree of $P_n(x)$.

$K, C(1), \dots, C(n)$ are coefficients of $P_n(x)$.

x is number where P_n are always evaluated.

$P(I)$ is value of P_I at x .

D is absolute value of difference between $f(x)$ and $P_I(x)$.

An example of output from the author's program on two problems is provided in Figure 16.

Instructive exercises using TAYLOR include:

- (a) Compute the value of $P_n(x)$ for $n = 0, 1, 2, 3, 4$ for $f(x) = e^x$, $a = 0$ and $x = 0.1$. [$P_4(.01) = 1.105171$, $R_4 < .0000003$]

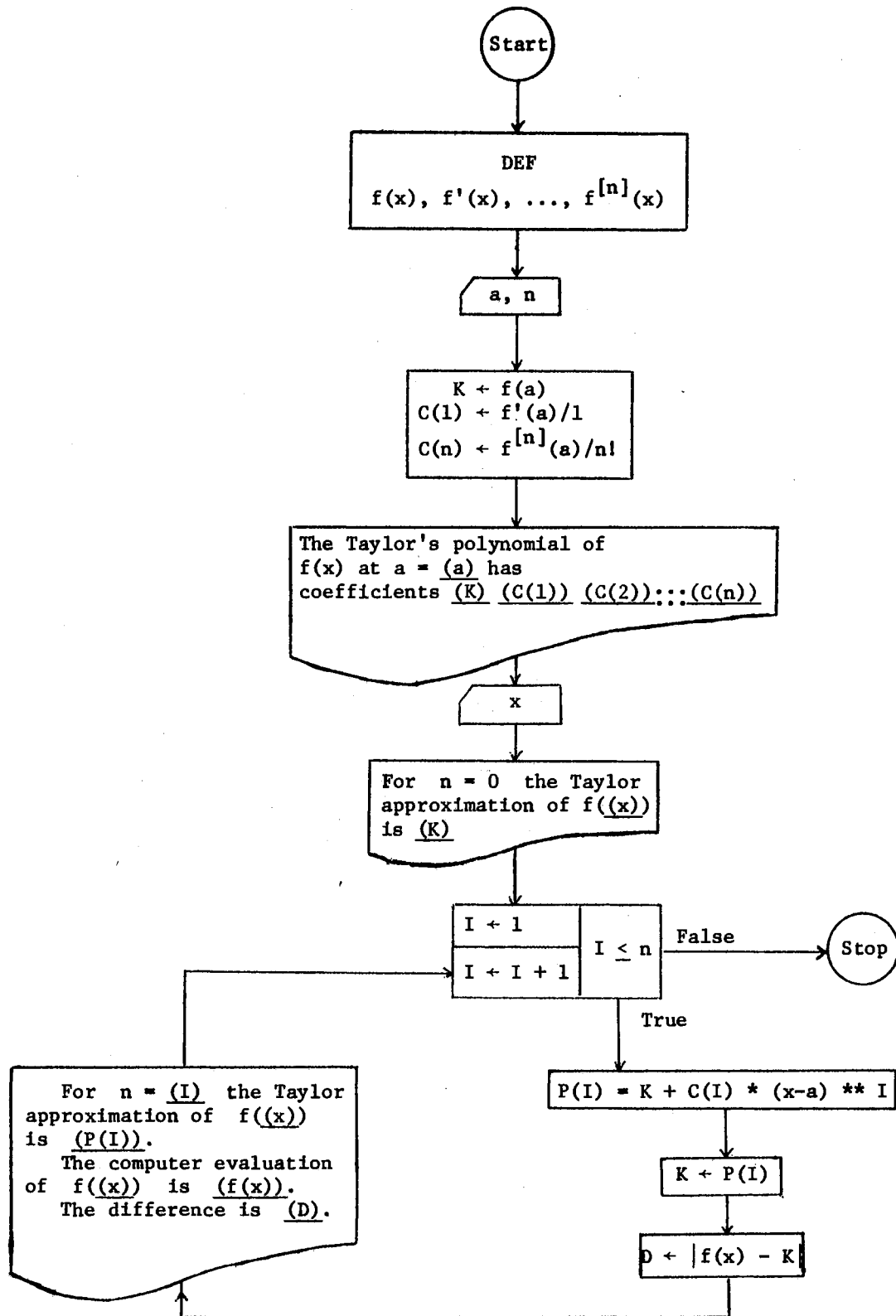


Figure 15. Flow Chart for Taylor Polynomial Approximation.

With $f(x) = \cos x$, $a = \pi/3$, $x = 1$

Coefficients for the Taylor's polynomial of $f(x)$ at $a = 1.05$ and $n = 3$ are:

.50900 -.86603 -.25000 .14434 .02083 -.00722

For $n = 0$, the Taylor approximation of $f(1.00)$ is .4999978793

For $n = 1$, the Taylor approximation of $f(1.00)$ is .5408743361

The computer evaluation of $f(1.00)$ is .5403023059

The difference is .0005720303

For $n = 2$, the Taylor approximation of $f(1.00)$ is .5403173785

The computer evaluation of $f(1.00)$ is .5403023059

The difference is .0000150726

For $n = 3$, the Taylor approximation of $f(1.00)$ is .54030220088

The computer evaluation of $f(1.00)$ is .5403023059

The difference is .0000001051

With $f(x) = \ln(1 + x)$, $a = 0$, $x = .95$

Coefficients for the Taylor's polynomial of $f(x)$ at $a = 0$ and $n = 4$ are:

0. 1.00000 -.50000 .33333 -.25000 .20000

For $n = 0$, the Taylor approximation of $f(.95)$ is 0.

For $n = 1$, the Taylor approximation of $f(.95)$ is .9500000000

The computer evaluation of $f(.95)$ is .6678293726

The difference is .2821706274

For $n = 2$, the Taylor approximation of $f(.95)$ is .4987500000

The computer evaluation of $f(.95)$ is .6678293726

The difference is .1690793726

For $n = 3$, the Taylor approximation of $f(.95)$ is .7845416667

The computer evaluation of $f(.95)$ is .6678293726

The difference is .1167122941

For $n = 4$, the Taylor approximation of $f(.95)$ is .5809151042

The computer evaluation of $f(.95)$ is .6678293726

The difference is .0869142684

Figure 16. Sample Output from TAYLOR

- (b) Compute $\sqrt[3]{26}$ to four decimal places. (Choose $a = 27$ and $f(x) = x^{1/3}$.) $[|R_2| < .00002]$
- (c) An example where the first few Taylor polynomials (that is, for n small) do not approximate the function very well is $\ln(1+x)$ with $|x|$ near 1. That is, $f(x) = \ln(1+x)$, $a = 0$ and x near to $+1$ or -1 .
 [For $x = .95$, $|P_4(.95) - f(.95)| > .08]$

The Integral

Riemann Sums. Rather early in a study of integration the concept of a Riemann sum for the function f and the interval $[a,b]$ is introduced. For a partition $P_n = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$, and z_i a number in the i th subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \dots, n$, a Riemann sum (for f over P_n) is defined by

$$R(P_n) = \sum_{i=1}^n f(z_i)(x_i - x_{i-1}).$$

Either by definition or theorem (depending upon the instructor's logical beginning)

$$\int_a^b f(x) dx = \lim_{\text{mesh } P_n \rightarrow 0} R(P_n)$$

where

$$\text{mesh } P_n = \max \{(x_i - x_{i-1}) \mid i = 1, 2, \dots, n\}.$$

In order to demonstrate the nature of a Riemann sum the author has required students to write a flow chart for a routine to compute $R(P_n)$ for a specified f , $[a,b]$, and n .

The author's detailed flow chart is illustrated in Figure 17.

Symbols in the flow chart have the following meaning:

- $[a,b]$ is interval of integration.
- TL is tolerance for approximation.
- J is a counter.
- SM is total Riemann sum after a given iteration.
- N is number of subintervals.
- DN is length of subintervals.
- $[XL, XR]$ is i th subinterval.
- CI is midpoint of i th subinterval.
- $S(J)$ is holder for Riemann sums after J th iteration.

In the flow chart the midpoint of $[x_{i-1}, x_i]$ is chosen for z_i , also, each partition is regular, that is, produces equal subintervals. If one's programming system has a random number library function, it might be instructive to choose z_i from the subinterval in a random fashion since this is the way a Riemann sum is formed. A tolerance stop has been included with the flow chart. For reference let us call the program written from this flow chart RIEMSM.

Possible exercises using RIEMSM include:

- (a) Estimate

$$\int_0^2 \sqrt{4-x^2} dx$$

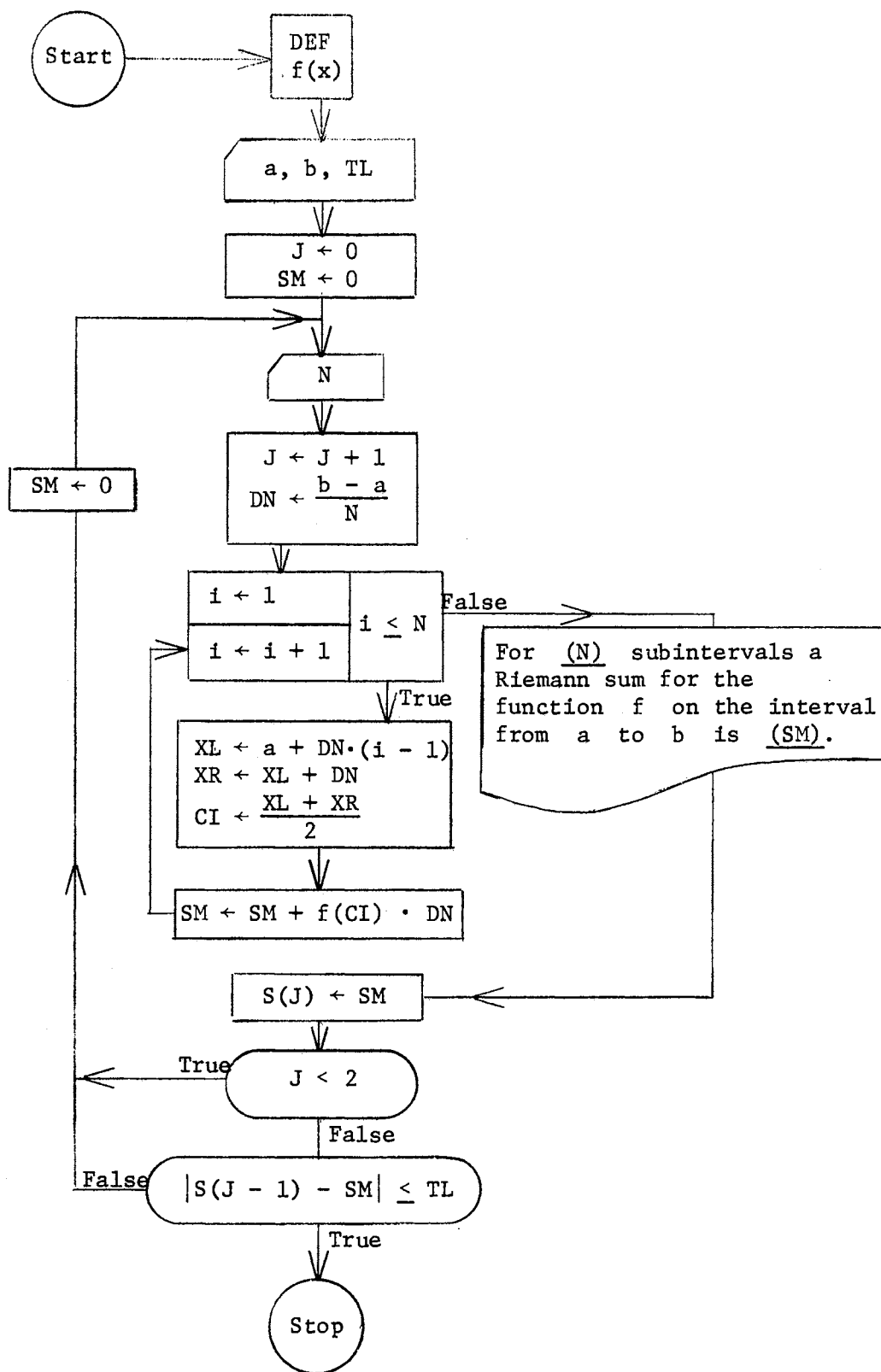


Figure 17. Riemann Sum Approximation to the Integral

using $n = 4, 8, 64, 128$ (for example). Notice that the area of one-fourth of a disc of radius 2 is being approximated, that is, we are approximating π . [The last approximation should be accurate to the third decimal place. All sums will be too large.]

- (b) Most texts list several Riemann sums for students to compute by hand. After doing so, they can investigate such problems further with RIEMSM.
- (c) Riemann sum approximations to a definite integral should be compared with approximations by other techniques such as the trapezoidal rule and Simpson's rule when these are studied.
- (d) Use RIEMSM on

$$\int_{-1}^1 \frac{t}{t^2 + 1} dt.$$

[The result is zero due to cancellation.]

Upper and Lower Sums. Many texts begin integration theory with the consideration of upper sums and lower sums (sometimes called Riemann upper and lower sums) where z_i for $[x_{i-1}, x_i]$ in the above discussion is taken to be the point of the subinterval at which f is a maximum, for upper sums, and minimum, for lower sums with $i = 1, 2, \dots, n$, of course.

The flow chart of Figure 17 and resulting program RIEMSM can be modified easily to find these sums for functions which are always monotone increasing (or, with another modification, for functions always

monotone decreasing). CI is simply taken to be XL (or XR for monotone decreasing functions).

Of course, a flow chart and program can be written to find upper and lower sums for a function f over an interval $[a,b]$ where the function need not be monotone. Provision must be made for dividing $[a,b]$ into subintervals $[c_i, d_i]$ say, on which f is always monotone. The program then applies the appropriate modified version of RIEMSM to each $[c_i, d_i]$. The flow charting of such a routine is of moderate difficulty. For reference the program for such a routine will be called UPLWSM.

Exercises for which the use of UPLWSM may be instructive include:

- (a) The same integrals to which students have applied RIEMSM, followed by a comparison of the results.
- (b) On

$$\int_0^{100} x^2 dx$$

to find $1^2 + 2^2 + \dots + 100^2$. [The sum is simply an upper sum for the integral with regular partition of mesh 1 unit.]

- (c) Similarly for $2^{-2} + 3^{-2} + \dots + 100^{-2}$ using

$$\int_0^{100} x^{-2} dx.$$

[Use lower sum as above.]

(d) Show that

$$\int_0^{10} \frac{x dx}{x^3 + 16} < 1.$$

[Break integration into two intervals [0,2] and [2,10] on which the function is increasing and decreasing, respectively.]

To illustrate the relationship of integration with concepts of displacement, speed and acceleration, an exercise of the following nature is instructive:

The speed of a certain automobile at time t is \sqrt{t} . For each value of $t = .1, .2, \dots, 1.0$ calculate and print the distance traveled by the car in the first t seconds. Do this by splitting up the time interval from 0 to t into 10 pieces, 50 pieces, and 100 pieces and summing the distances traveled in each piece on the assumption of constant speed in that piece. Also print $(2/3)t^{3/2}$ for each value of t . Explain what you see. [Riemann sums for

$$\int_0^T \sqrt{t} dt,$$

$T = .1, .2, \dots, 1.0$ are being computed with 10, 50, and 100 subdivisions. These sums should approximate $(2/3)t^{3/2}$ for each value of T .]

The Fundamental Theorem of Calculus. In an attempt to give meaning to the Fundamental Theorem of Integral Calculus, the author has written a program (FNDTHM) which approximates the integral

$$\int_a^b f(t)dt$$

using Simpson's Rule. Students are asked to consider this technique simply a "black box" and believe that it works if the Simpson's Rule has not been studied yet, which is usually the case. They are instructed to use the procedure to obtain values of F , where

$$F(x) = \int_a^x f(t)dt,$$

for suitable choices of $x \in [a,b]$. Then, from the set of ordered pairs $(x, F(x))$, a graph is sketched. Students are then told "see if you can find the defining equation for F ." Of course, they find

$$\int f(x)dx + f(a)$$

if they are correct. Next, they differentiate this function and obtain f , finding that $F'(x) = f(x)$. One naturally begins with f defined to be of the type $2x$, $\sin x$, e^x , or $1/x$, and preferably with $f(a)$ nearly zero, so the defining equation for F is not too difficult to guess. This is another example of where hand computation of

$$\int_a^x f(t)dt$$

would be too laborious for such an exercise, but which is easily done by machine.

Students, having experienced the above computer treatment of the Fundamental Theorem, can usually give the correct response to a question like:

$$\text{What do you think } Dx \left[\int_0^x \sqrt{\sin t^2} dt \right] = ?$$

Improper Integrals. Using an approximation technique of integrals (usually Simpson's Rule) the author has devised a way for students to investigate improper integrals.

One type of improper integral occurs, for example, if one has an integral of the type

$$\int_0^1 f(x) dx$$

with f having the y -axis as an asymptote. Its graph might look like that of Figure 18, for example.

In general, this type of improper integral has the form

$$\int_a^b f(x) dx$$

with f having $x = a$ or $x = b$ as asymptote. (A related but more involved improper integral occurs if f is allowed to have $x = c$ for $c \in (a, b)$ as an asymptote.)

By definition, the integral for the situation illustrated in Figure 18 is given by

$$\int_0^1 f(x) dx = \lim_{t \rightarrow 0} \int_t^1 f(x) dx$$

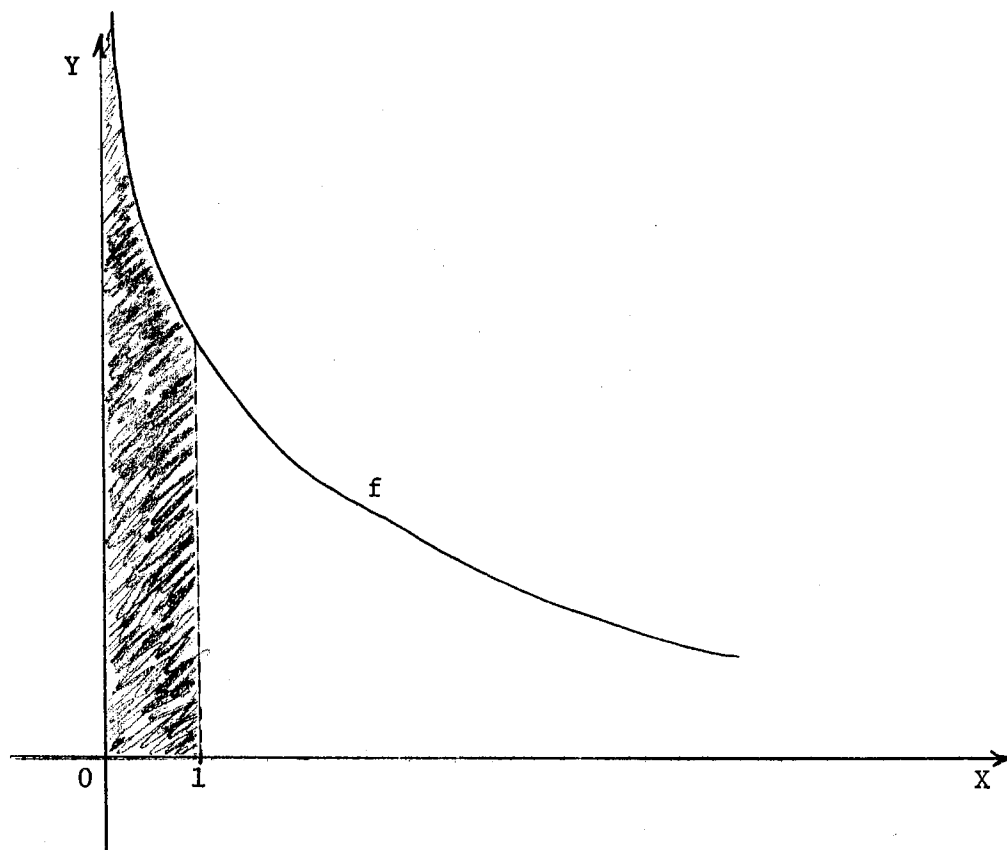


Figure 18. Graph of a Function f .

if this limit exists and is finite. Using the area interpretation of a definite integral,

$$\int_0^1 f(x) dx$$

should be representable as the shaded area of Figure 18. For some functions f the integral exists (that is, the limit on the right exists) while for some functions it does not exist (limit is infinite). That is, geometrically sometimes the "area" is finite and sometimes infinite. The surprise, of course, is that it is ever finite!

Similar comments hold for integrals of the form

$$\int_a^{\infty} f(x)dx$$

where f has a graph of the form illustrated in Figure 19 (that is, having a horizontal asymptote such as the x -axis).

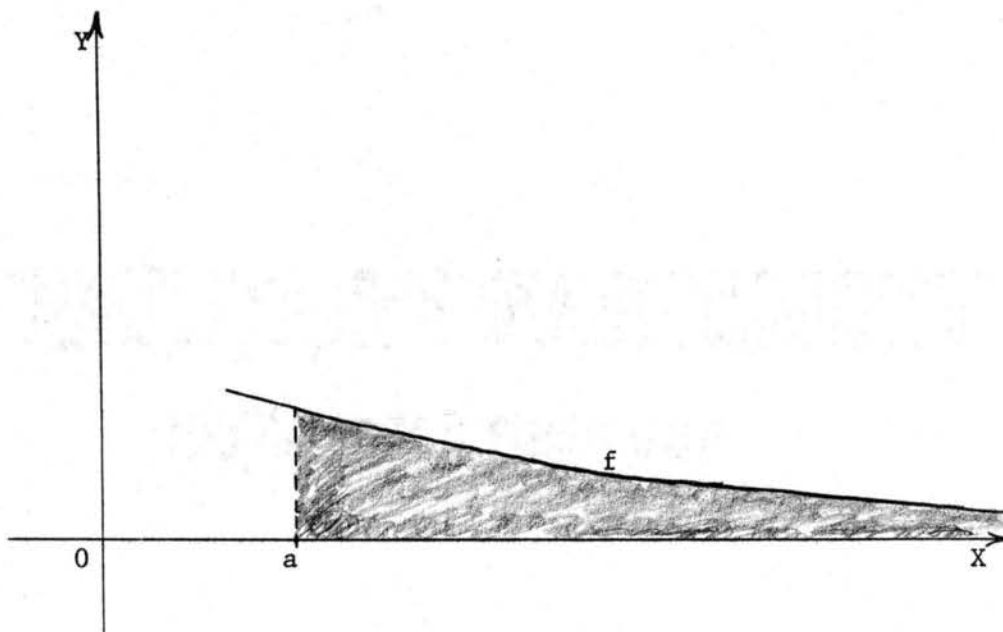


Figure 19. Graph of a Function f .

The author has written a program, called IMPROP for reference, which uses Simpson's Rule to investigate such improper integrals.

To consider integrals of the first type,

$$\int_0^1 f(x) dx,$$

as illustrated in Figure 18, for example, the routine approximates

$$\int_n^1 f(x) dx$$

setting $n = .1, .01, .001, .0001, \dots$ successively. The results of these approximations are then examined with the intent of determining whether the sequence of numbers seems to be converging or diverging.

It is instructive to use IMPROP to examine

$$\int_0^2 \frac{dx}{\sqrt{x}}, \quad \int_0^2 \frac{dx}{x}, \quad \text{and} \quad \int_0^2 \frac{dx}{x\sqrt{x}},$$

successively. [The first exists while the last two do not.]

To consider integrals of the second type, IMPROP is used to approximate

$$\int_a^n f(x) dx$$

for $n = 10, 100, 1000, \dots$. The results of these approximations are examined as above to determine, if possible, whether the sequence of numbers converges or diverges.

It is instructive to consider the improper integrals related to those suggested above, namely

$$\int_1^{\infty} \frac{dx}{\sqrt{x}}, \quad \int_1^{\infty} \frac{dx}{x}, \quad \text{and} \quad \int_1^{\infty} \frac{dx}{x\sqrt{x}}.$$

[For this type of integral the results are inverted for the first and third integrals previously mentioned and the same for the second. That is, the first and second do not exist while the third does.]

After doing these exercises, students can usually give a correct answer to questions of the type

(a) If

$$\int_0^1 \frac{dx}{x^{a+1}}$$

exists, what about

$$\int_0^1 \frac{dx}{x^a} ?$$

(b) If

$$\int_1^{\infty} \frac{dx}{x^a}$$

exists, what about

$$\int_1^{\infty} \frac{dx}{x^{a+1}} ?$$

(c) For what values of a do the following improper integrals exist?

$$(i) \int_0^1 x^a dx \quad (ii) \int_1^{\infty} x^a dx$$

Other improper integrals which may be instructive to consider are

$$\int_a^{\infty} \frac{dx}{e^x}, \quad \int_a^{\infty} \sin x \, dx, \quad \int_0^1 \ln x \, dx.$$

[exists, does not exist, exists]

Since the integration routine used is only an approximation process, errors will always occur, and these get significant as n gets near zero or large in the previous suggested use of IMPROP. This must be pointed out and discussed with students. An assumption is made that, on a given interval $[a,b]$, errors for integrals of a given type are about the same.

Application and Approximation of the Integral

In the previous section programs to compute Riemann sums as well as upper and lower sums for a given function f and interval $[a,b]$ were described. These can be used to approximate the integral

$$\int_a^b f(x) dx,$$

but more precise methods are usually used for this. Two such methods are Simpson's Rule and Trapezoidal Rule, the former being most used, but both are worth some consideration.

The Trapezoidal Rule. The derivation of this common approximation is well known (see (20, pp. 268 - 270), for example) and will not be given here. The formula is:

$$\int_a^b f(x) dx \approx \frac{b-a}{2n} \left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n) \right]$$

where $P_n = \{x_0, x_1, \dots, x_n\}$ is a regular partition of $[a, b]$. The error term, R_n , such that

$$\int_a^b f(x) dx = \frac{b-a}{2n} \left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n) \right] + R_n$$

is given by

$$R_n = -\frac{(b-a)^3}{n^2} \frac{f''(z)}{12}$$

for $z \in (a, b)$ (30, p. 118).

A flow chart for the Trapezoidal Rule is given in Figure 20. For reference the program written from this flow chart will be called TRAPEZ. Symbols in the flow chart have the following meaning:

- $f(x)$ is integrand function.
- $[A, B]$ is the interval of integration.
- N is the number of subintervals (max 50).
- DX is length of the subintervals.
- I is a subscript counter.
- SM is sum after the i -th iteration.
- IN is the final integral estimate.

Simpson's Rule. Simpson's Rule is given by

$$\int_a^b f(x) dx \approx \frac{b-a}{3n} \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]$$

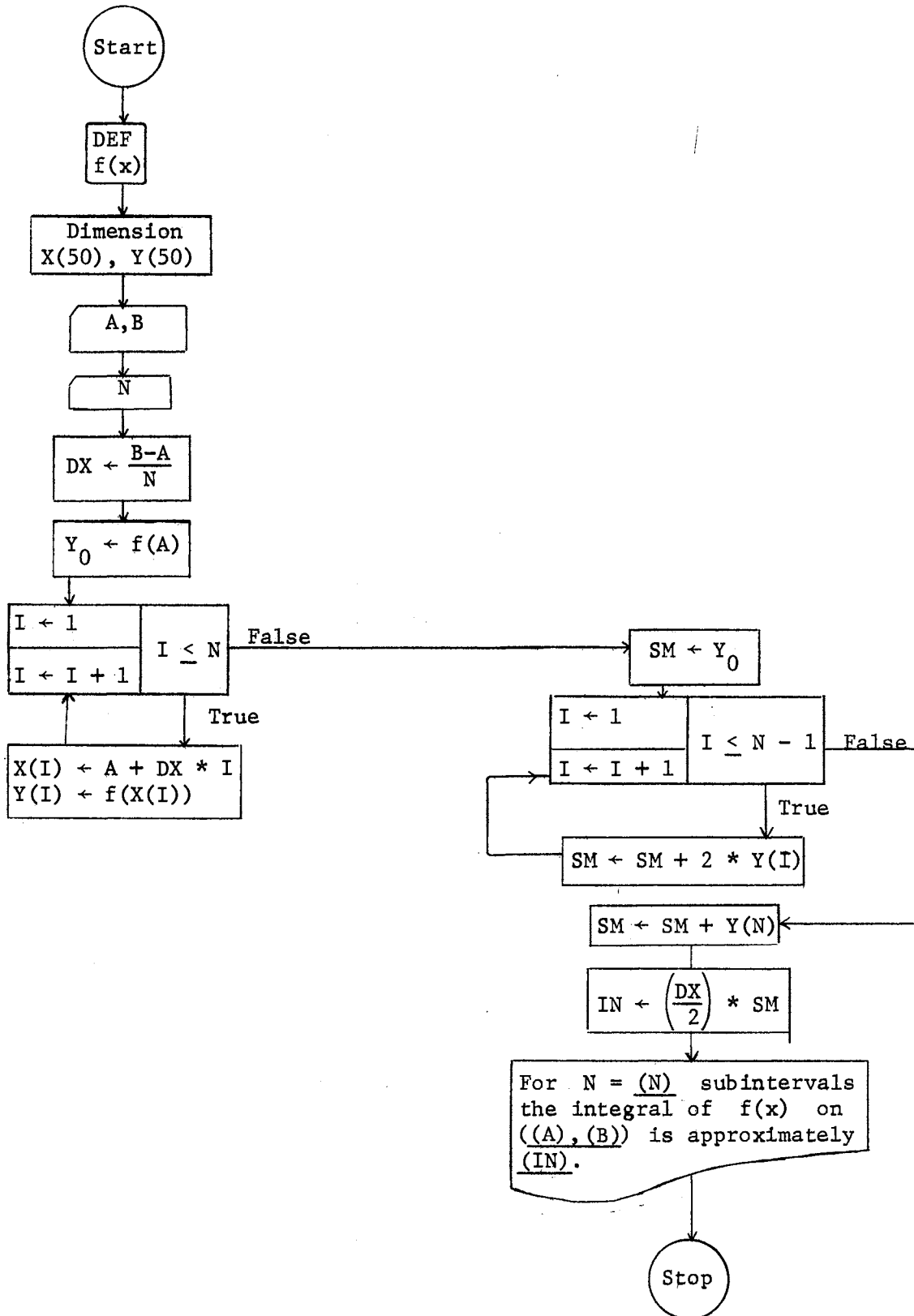


Figure 20. Flow Chart for the Trapezoidal Rule

where $P_n = \{x_0, x_1, \dots, x_n\}$ is a regular partition of $[a,b]$, and n is even. (See (20, pp. 271 - 275).

The error term is given by

$$R_n = -\frac{(b-a)^5}{180 n^4} f^{(4)}(z)$$

where $z \in (a,b)$. (30, p. 119)

A flow chart for Simpson's Rule is given in Figure 21. For reference the program written for this flow chart will be called SIMPSN. Symbols in the flow chart have the following meaning:

DX is mesh of the (regular) partition having n sub-intervals.

[A,B] is the interval of integration.

N must be even and ≤ 100 (unless dimension for X(I) and Y(I) is changed).

Y(1), Y(2), ..., Y(N) denote y_1, y_2, \dots, y_N , respectively,

($y_i = f(x_i)$, of course).

Similarly for X(1), ..., X(N).

SM is sum after n th iteration.

IN is final approximation for the integral.

Exercises on which students might instructively use RIEMSM, UPLWSM, TRAPEZ, and/or SIMPSN include:

(a) $\int_1^2 \frac{dx}{x}$ [ln 2]

(b) $\int_0^1 4\sqrt{1-x^2} dx$ [π]

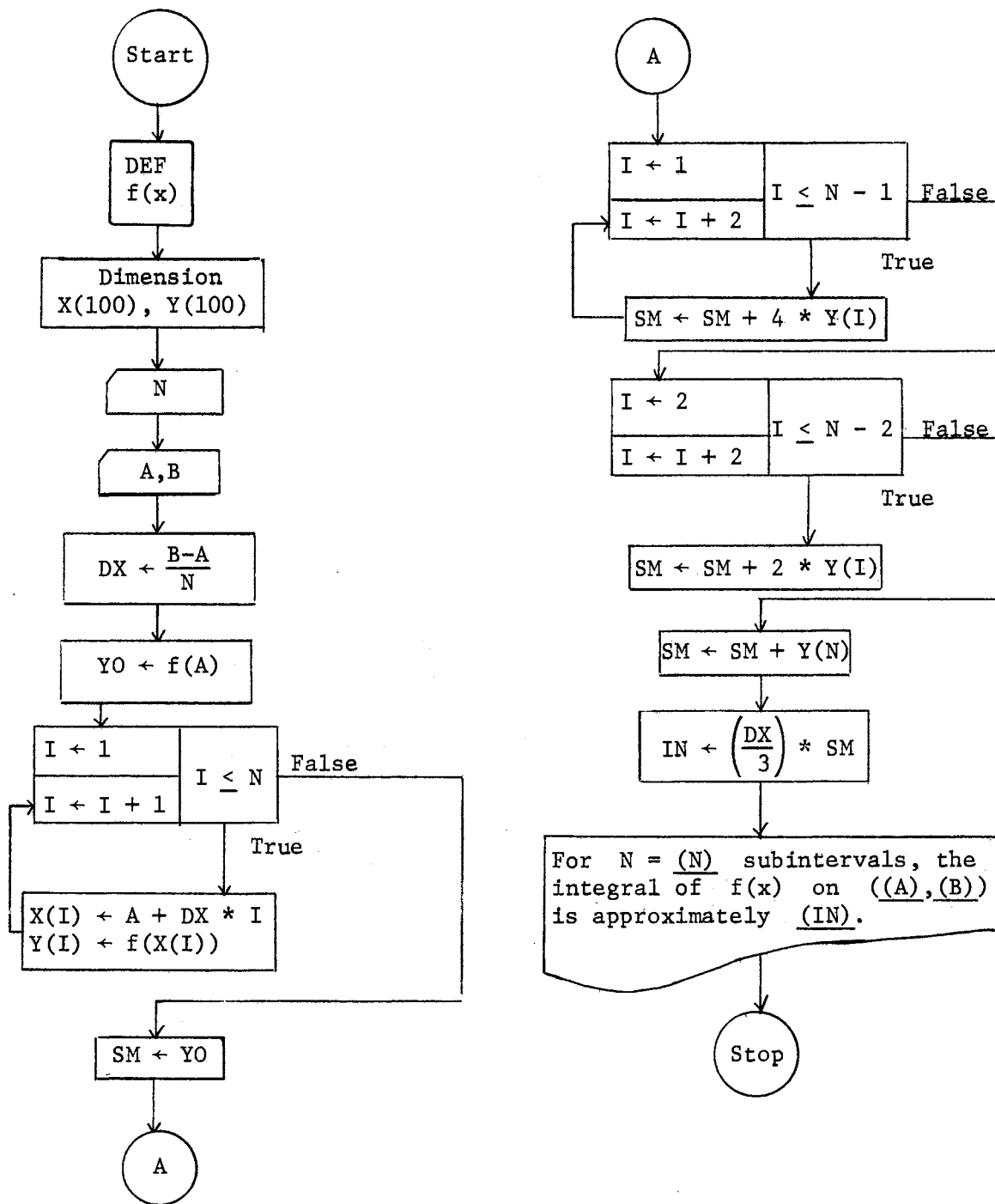


Figure 21. Flow Chart for Simpson's Rule

$$(c) \int_0^1 \frac{4}{1+x^2} dx \quad [\pi]$$

$$(d) \int_0^2 x^3 dx \quad [4, \text{illustrates the fact that Simpson's Rule is exact on polynomials of degree } \leq 3]$$

Arc Length. For a function f such that f' is continuous on $[a,b]$, the arc of the graph of f joining points $(a,f(a))$ and $(b,f(b))$ is rectifiable and has length

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

For many simple functions f the evaluation of this integral in closed form is difficult.

An exercise of the type which involves both the arc length formula and numerical approximation is the following:

Approximate the arc length of the graph of $y = 1/x$ between the points $(1,1)$ and $(5,1/5)$. [Simpson's Rule with $n = 4$ approximates the length to be about 4.187.]

Infinite Series

A major hurdle when teaching (and learning) infinite series is obtaining a reasonable understanding of the definition of convergence, divergence, and sum of a series. The author has found it instructive to have students draw a flow chart for finding the n th partial sum

$$S_n = \sum_{K=1}^n a_K,$$

for a series

$$\sum_{K=1}^{\infty} a_K,$$

having it produce values of n , S_n , and a_n .

Figure 22 is such a flow chart. For reference let us call a program written from this flow chart SERIES. Symbols in the flow chart have the following meaning:

$a(K)$ denotes a_K of the series

$$\sum a_K.$$

N is number of terms to be added (to get S_N)

SN is the N th partial sum.

Students are instructed to run SERIES (on a given series

$$\sum a_K)$$

with values of N such as 10, 50, 100, 200. And then, after looking at the successive values of S_n and perhaps at a_n decide whether

(1) they think the series is converging or diverging, (2) what its sum is if the series is converging, and (3) whether the convergence or divergence is rapid or slow.

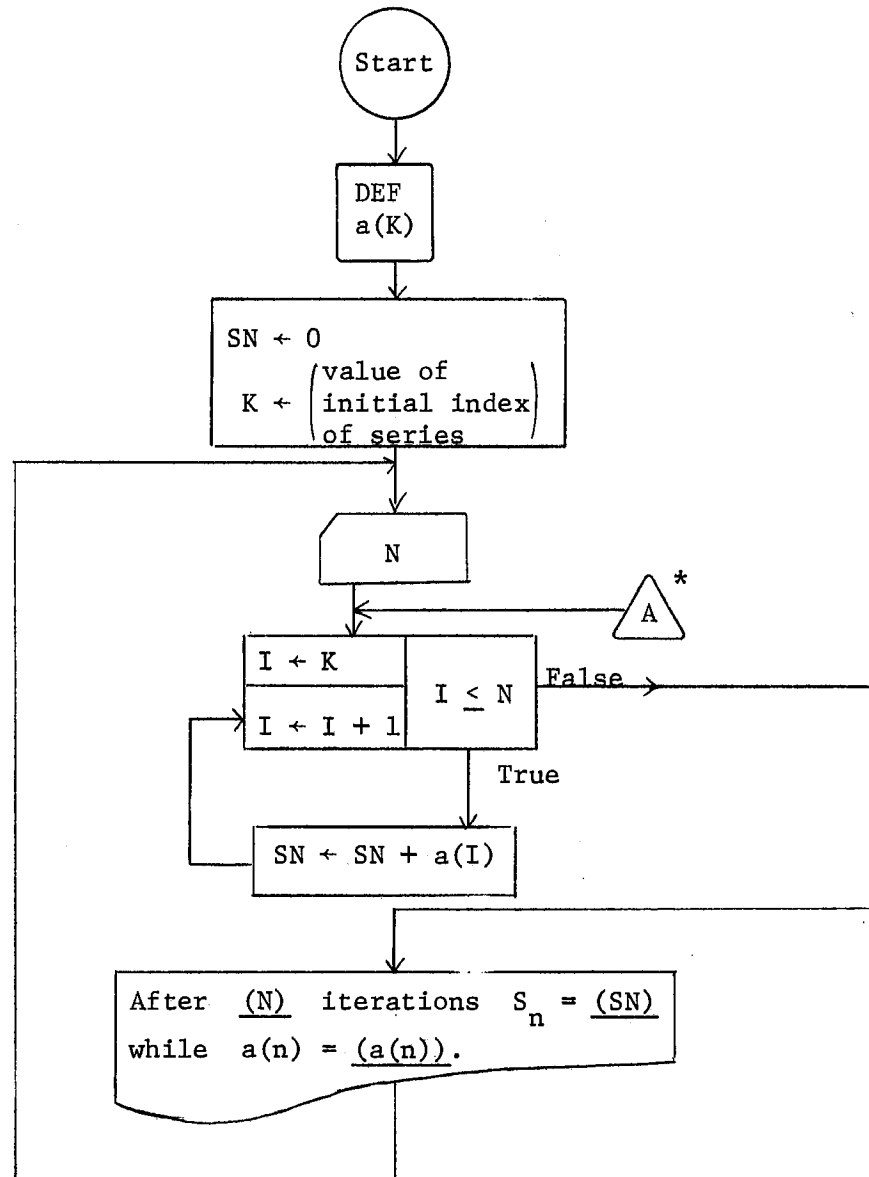


Figure 22. Flow Chart for Finding Partial Sums

* It may be wise to add a check at $\triangle A$ to make sure that an unreasonable value of N is not read in causing the routine to use up too much machine time.

Another, possibly better, way of writing a program to investigate series would be to put a tolerance stop into the routine such that successive values of S_n are checked until $|S_{n+1} - S_n|$ is less than a specified tolerance or a maximum number of iterations has been performed. The last value of S_n would be the estimated sum with accuracy according to the specified tolerance.

Example exercises which might be used include:

(a) $\sum_{K=0}^{\infty} r^n$ for values of $|r| > 1$ and $|r| < 1$. [This is a geometric series, of course, and for $|r| < 1$ the sum has a closed form of $1/(1 - r)$.]

(b) $\sum_{K=1}^{\infty} \frac{1}{K}$ [Here a thousand or more terms can be added to illustrate rather slow divergence.]

(c) $\sum_{K=1}^{\infty} \frac{1}{\sqrt{K}}$ [rather rapid divergence]

(d) $\sum_{K=1}^{\infty} \frac{(2K-1)(-1)^K}{5K+1}$ [very, very slow divergence]

(e) $\sum_{K=1}^{\infty} \frac{(-1)^{K+1}}{K}$ [$\ln 2$]

(f) $\sum_{K=1}^{\infty} 4 \frac{(-1)^{K+1}}{(2K-1)}$ [π]

CHAPTER III

AN EXPERIMENT

Statement of the Problem

In Chapter II a set of topics from elementary calculus was described along with specific suggestions for utilizing the computer to improve students' learning of ideas from these topics. The purpose of this chapter is to describe and report the results of an experiment involving the use of materials derived from Chapter II.

The problem to be examined is: What effect does this use of a computer have on achievement in an elementary calculus course?

Review of Related Literature

Subjective evaluations of the effectiveness of using the computer in an elementary calculus course have been given by several persons in their published and unpublished notes. Instructor's reports include such comments as: "The course has been a great success and next year's should be even better." (9, p. 14); "We think it would be foolish to claim that a student learns calculus better in a computer calculus section than in a conventional section." (8, p. 13); "The course has been successfully carried out." (1). In the 1967-68 academic year, a computer oriented calculus course was taught at the University of Minnesota by Warren Sternberg. The students (all honor students) took the same final exam as other calculus students at the University, and

the results were judged "highly successful" but apparently no empirical study was performed (13, p. 94). During the 1968-69 year H.C. Griffith and Emile Roth at Florida State University used the CRICISAM text in a team teaching effort with a group of high ability freshmen through an honors calculus sequence carrying credit for calculus, computer programming, and laboratory work. E.P. Miles reported,

"The class responded very favorably during the sequence, giving the best overall performance on proofs Dr. Griffith had observed in a long teaching career. His follow-up study of these honor students during 1969-70 indicates that they continued to excel after re-entering the traditional calculus sequence." (26, p. 289)

A limited number of empirical studies involving aspects of computer usage have been conducted. Wallace (39) and Kieren (21) investigated the effect of computer use in conjunction with courses from the high school curriculum.

Wallace's research hypotheses were (a) the learning of trigonometry will be facilitated by introducing algorithmic and flow charting techniques from computer mathematics, into the one-semester trigonometry course; (b) the learning of trigonometry in a one-semester course by eleventh and twelfth grade students will be facilitated by prior completion of a one-semester course in computer mathematics; (c) the learning of first semester high school physics will be facilitated by the concurrent completion of a one-semester course in computer mathematics. The research concluded that the use of flow charts and algorithmic method in teaching mathematics appears to fortify conventional teaching methods, with the result that higher learning rates are attained. No evidence was presented indicating an influence of learning computer techniques concurrently with learning physics. Finally, no significant difference was found to indicate that the learning of trigonometry in a

one-semester course by eleventh and twelfth grade students will be facilitated by prior completion of a one-semester course in computer mathematics (39, pp. 1 - 63).

Kieren's study was to determine if there is a difference in means of mathematics achievement tests between students in intermediate high school mathematics who use a computer in their study of mathematics and those who do not. The conclusion was that the results could not be considered as strong evidence that the computer has a great effect on mathematics learning. Kieren concluded, "while the evidence is not incontrovertible, it appears that use of the computer has some positive effects on mathematics learning," (21, pp, 121 - 128)

Stannard's (37) study concerned the use of computer assisted instruction in conjunction with conventional instruction in calculus. He attempted to determine the effect on final achievement in a beginning college calculus course resulting from the use of programmed materials written to supplement regular classroom instruction. The programmed materials were on topics which ordinarily cause learning problems in the introductory college calculus. Stannard's investigation concluded that statistically there is very little reason to believe that programmed materials have an effect on the final achievement of students in a beginning college calculus course.

Fiedler's (12) investigation treated the problem:

"Does a student gain a greater understanding of a mathematical concept by programming it for a digital computer than when he studies that concept in the usual course structure?"

The topics which were investigated were chosen from analytic geometry and calculus. Control and experimental groups were selected from analytic geometry and Calculus I at Black Hawk College, Moline, Illinois.

The conclusion was that there is no apparent difference in achievement between students who learn mathematical concepts by computer programming and those who learn the concept in the usual homework structure.

Bitter (2) studied the effect of computer use in introductory college calculus classes at three private liberal arts colleges. Control and experimental groups were selected from introductory calculus students at Colorado College, Colorado Springs, Colorado, Regis College, and the University of Denver both at Denver, Colorado. Students in the experimental group were required to write computer programs to solve selected homework exercises which sometimes replaced normal homework assignments. A significant difference in favor of the experimental group was found on topics in differential calculus and also in favor of female students on these same topics.

Thus, evaluations of using a computer to improve the teaching of mathematics are limited. Studies and personal evaluations mentioned above provide evidence bearing on the effect of computer use on calculus instruction. However, the evidence is neither conclusive nor complete, and further study is merited.

Hypothesis

The experiment will involve three groups of students, one control group and two experimental groups. The experimental groups will be referred to as "Group One-half" and "Group One" for reasons to be provided later.

Hypotheses to be examined are the following:

Hypothesis 1. There will be no significant difference in the level of achievement and understanding of calculus, as measured by the Calculus Cooperative Mathematics Test (C.C.M.T.) between the control group and the two experimental groups.

Hypothesis 2. There will be no significant difference in the level of achievement and understanding of calculus, as measured on a pre-chosen selection of questions from the C.C.M.T., between the control group and the two experimental groups.

Experimental Procedure and Design

The experiment was conducted on the campus of Oklahoma State University, Stillwater, Oklahoma, during the first semester of the 1970-71 school year. Dr. Craig Wood, the author's thesis adviser, was responsible for sections 5, 6, and 7 of Mathematics 2055 Calculus, and the author was assigned as his teaching assistant. The class met Monday, Wednesday, and Friday at 9:30 a.m. for conventional lectures by the principal instructor. Tuesday and Thursday classes were met by the assistant instructor, with students enrolling in one of the three sections for recitation, group discussion, and problem solving. Section 5, which had recitation meetings at 7:30 a.m., was designated as the control group. Section 6, having recitation meetings at 8:30 a.m. was called experimental Group One-half, and Section 7, having recitation meetings at 9:30 a.m., was called experimental Group One. Choice of which section would be control and which experimental was determined by lot.

Treatment of the groups was as follows:

Control Group. Recitation meetings were used in the conventional manner, namely, to answer questions, work and discuss problems. Homework was collected, graded, and returned weekly.

Experimental Group One-half. Recitation meetings were used to answer questions, work and discuss problems as in the control group. In addition, the students were taught flow charting as an approach to problem solving, were supplied with computer programs written by the author for their use, and had one-half hour of time per week provided for their use on the university's time sharing computer terminals. Students were required to hand in some flow charts and computer output with analysis from five sets of selected exercises, in addition to their regular homework.

Experimental Group One. This group was treated like Group One-half except that they were taught CPS BASIC, a simple programming language, required to write some of the programs provided for Group One-half, and were given one hour of time per week for their use on the terminals.

All three groups received the same Monday, Wednesday, and Friday lecture presentations and took the same in-class examinations.

To test the hypotheses, a non-equivalent control group design as described by Campbell and Stanley (4, pp. 47 - 50) was employed. This design was dictated by the fact that the control group and experimental groups did not have pre-experimental sampling equivalence. Due to student self-scheduling into the three sections comprising the three groups, assignment of students to the course and to individual sections on a random basis was not possible.

Assignment of control and experimental status was random among the three class sections. Analysis of covariance was employed to analyze the results of the experiment because it has the property of statistically adjusting for initial differences between groups. Two covariate scores, the pretest and American College Test in Mathematics (A.C.T.M.) scores, were used as the control variables with only one criterion variable, the post-test scores. A description of the evaluation instruments is given on pages 81 - 83.

Sample

The sample for this study consisted of eighty-six students who enrolled in Mathematics 2055 for the fall semester of 1970, excluding those who withdrew from the course during the first week of class. Thirty-one of these students were in the Control Group, twenty-five in the Group One-half, while thirty were in Group One. Students who withdrew later, or who lacked scores on one of the instruments were excluded from the analysis. (A.C.T.M. scores were not available on students who transferred to Oklahoma State University.)

Excluded from the analysis in the Control Group were two who withdrew and nine for whom related data was not available. In Group One-half three withdrew during the semester, and ten had some of the related data missing and were excluded from the analysis. In Group One the respective exclusions were one withdrawal and eight for unavailable data. As a result of these omissions, the analysis involved twenty students in the Control Group, twelve students in Group One-half and twenty-one students in Group One. From this point on when reference is

made to the sample, the author shall be referring only to subjects included in the statistical analysis.

Mathematics 2055 is designed principally for beginning engineering students and prospective science majors, but enrollment is not restricted. In the Control Group sixteen were engineering majors, three were arts and sciences majors, and one was an agriculture major. Group One-half had nine engineering majors, two arts and sciences majors, and one technical institute major. Group One had fifteen engineering majors, two arts and science majors, two agriculture majors, one each from business and the technical institute. Table I summarizes characteristics of the three groups. Three facts from the table which stand out are the large proportion of sophomores, consequent higher mean age in Group One as compared with the other two groups, and the wide range of mean A.C.T.M. scores. The Control Group A.C.T.M. mean is eleven percent higher than that of Group One-half and twenty-three percent higher than Group One.

One student in the Control Group indicated that he had classroom experience using a computer, in Group One-half two indicated some experience, while in Group One nine students said they had experience. In the author's judgment, the disproportionate amount of experience is more a result of mean age and class level than of a special interest or aptitude.

Subject Matter

Subject matter content of the control and experimental classes was the same traditional elementary calculus material, with the experimental groups using the computer to investigate certain ideas and verify some

TABLE I
 SUMMARY OF THE CONTROL AND EXPERIMENTAL
 GROUP CHARACTERISTICS
 (N = 53)

Characteristics	Control Group (N = 20)	Group One-half (N = 12)	Group One (N = 21)
Time of Recitation Class	7:30 a.m.	8:30 a.m.	9:30 a.m.
Number of Subjects	20	12	21
Male	19	12	18
Female	1	0	3
Freshmen	16	7	2
Sophomores	4	5	16
Juniors	0	0	2
Seniors	0	0	1
Mean A.C.T.M. Score	30.45	27.33	24.81
Mean Age	17.9	18.5	19.1
Mean Hours Enrolled for Credit (In-class Hours are in Parentheses)	15.8 (18.7)	15.75 (18.5)	15.24 (18.6)

concepts while solving some assigned problems. Calculus of the Elementary Functions by Shanks and Gambill (34) was the required class text. The text tends more toward a problem solving approach intended for "development of students' computational power" (34, p. v) than toward a theoretical presentation of the topics. Few complete proofs are provided in the text. Topics covered in Mathematics 2055 Calculus include the derivative, technique of differentiation, implicit differen-

tiation and higher derivatives, applications of the derivative, the mean value theorems, indeterminate forms, Taylor's formula and theorem, series, introduction to the definite integral and its properties, the Fundamental Theorem of Calculus, elementary applications of the integral, and elements and techniques of formal integration. This material is found on pages one to two hundred fifty-nine of the class text. Approximately forty-five hours were used for class lectures and thirty hours for recitation.

Method of Computer Use

Each of the experimental groups was given five exercise sets during the term which required use of the computer. These were introduced at approximately three-week intervals. They were, like other weekly homework assignments, graded and returned to the student. The exercise sets were produced by the author from topics listed in Chapter II and were written to correspond in timing and terminology with the class text and lecture. Appendix D presents the exercise sets for Group One-half and Appendix E presents the exercise sets for Group One. The two include essentially the same topics with the difference being that Group One was required to write several programs and do several more problems than Group One-half. Group One had twice as much terminal time as Group One-half available for use weekly. The groups did not use the maximum allotted computer terminal time. Group One-half used five hours per student, and Group One used eight and one-half hours per student during the course. Maximum allotted time was seven and fourteen hours, respectively.

Topics included in the computer exercise sets were: computation of $n!$, computation of absolute value of a real number, function evaluation and graphing, finding the equation of a secant line, computing one-sided limits, computing the derivative of a function at a point, finding the derived function, investigating function composition and the chain rule, finding tangent and normal lines to the graph of a function at a point, differential approximation, estimating relative extrema, Newton's rule for root approximation, estimating limits of functions at a point and at infinity, Taylor's polynomial approximation to a function at a point, computation of Riemann sums, computation of upper and lower sums, finding the anti-derivative of a function, and investigating the Fundamental Theorem of Integral Calculus. As these topics were covered in class, the experimental groups would write flow charts and programs using the computer to solve the special computer homework assignments. At the same time, they and the control group also solved traditional homework assignments from the class text. On two of the five exercise sets the experimental groups had the option of choosing a traditional homework assignment or exercise set. Most all chose the exercise set. An average of approximately eight minutes per recitation session was used with computer materials in Group One-half. This figure was approximately twelve minutes in Group One. This time replaced that used for discussion of conventional problems with the Control Group.

Oklahoma State University's computer time sharing terminals were available for use by the experimental groups. Students were permitted to sign up weekly for a thirty-minute period of terminal time at a terminal in the Mathematical Sciences Building. They were also

permitted to use terminals located elsewhere around the campus so long as they held to the weekly maximum time limit of either one or one-half hour. No obvious abuses of this limit were detected.

Computer time sharing is a remote control system permitting several persons to run programs simultaneously on a single computer. Each individual communicates directly with the computer by means of a teletype which, at Oklahoma State University, is connected to the computer by telephone. Direct, real time communication with the computer is thus possible, and a student can input his program, correct it while on-line, input his data, and then receive immediate results at the terminal console. At Oklahoma State University the terminals connect with an IBM 360 Model 65 computer.

During the experiment students were scheduled on the terminals at mid-afternoon or evening times during weekdays and Sunday, and throughout the day on Saturday. Computer Center hours were approximately 9:00 a.m. to 12:00 p.m. Monday through Friday, 9:00 a.m. to 6:00 p.m. on Saturday, and 2:00 p.m. to 12:00 p.m. on Sunday.

All programs were written in an IBM time sharing version of BASIC called CPS BASIC. The language, BASIC, is conversational and was developed specifically for time sharing communication. It is simple but sufficiently potent enough to serve the purpose of its use in classes of the type being described. Brief booklets produced by the author and introducing elementary programming concepts along with BASIC were made available to the experimental groups at a nominal cost. Two extra sessions for Group One and one extra session for Group One-half were held the first and second week of the semester to demonstrate BASIC programming and techniques of flow charting. The first exercise

set had problems designed to help students learn programming. A small portion of recitation sessions the first month were used to answer questions on programming and terminal use. The author, having an office in the Mathematical Sciences Building, encouraged students to seek aid with terminal and programming difficulties as these occurred. Students sought out such aid rather frequently throughout the term.

Evaluation Instruments

Two instruments were used to measure levels of achievement and understanding. They were the American College Test in Mathematics (A.C.T.M.) and the Calculus Cooperative Mathematics Test Form A (C.C.M.T.).

The A.C.T.M. test was one of four parts of the A.C.T. test developed by the American College Testing Program. It is a mathematical aptitude test considered to be a good predictor of future achievement in college mathematics (3, p. 9). The test consists of forty multiple choice questions that sample aptitudes related to pre-college mathematics. Shana'a studied several variables for use as placement guidelines for freshmen at the University of Oklahoma and found that "the A.C.T.M. appears to be the best single variable for use as a placement guideline." (33, p. 85) Further, Shana'a states:

Discriminant functions dependent on the A.C.T.M., the A.C.T.C., the high school mathematics grade point average, and the number of semesters of high school mathematics are of value in distinguishing membership in different mathematics courses at the five percent level of significance. However, they do not prove significantly better than the A.C.T.M. at this level as placement tools. (33, p. 85)

It was thus decided to use the A.C.T.M. as one of the covariates in the statistical analysis.

For a second covariate, results of the C.C.M.T. produced by the Cooperative Text Division of Educational Testing Service were used as the pretest in this experiment. The subjects were administered this test at the first class meeting. This same test was used for the post-test and was given as the final examination during the regular university final exam period. The test was administered to a national sample of 450 students in 1962. The reliability of Form A was .87. The content validity was determined by an advisory staff consisting of ten leaders in mathematics education (11, pp. 55 - 63). The test was selected for use because it was thought to be the only commercially produced test directly related to the objectives and content covered in the experiment. Not all topics covered in the computer exercise sets, however, are covered by the test. For example, the test does not include one-sided limits, Taylor polynomials, Riemann sums, or the elementary chain rule formula.

The C.C.M.T. consists of Part I and Part II with thirty multiple choice items for each part. Each test item has four possible solution choices.

Part I of the C.C.M.T. evaluates mathematical concepts of function, graphs, slope of line, limits, continuity, derivative definition, applications of the derivative, and differential approximations.

Part II evaluates the mathematical concepts of relation of e^x to the natural logarithm of x , chain rule and related rates, velocity and acceleration, mean value theorem, properties of the integral, formal integration by substitution, area volume, arc length, and simple differential equations (11, pp. 38 - 39).

Of the sixty items on the C.C.M.T., most are quite indirectly related to topics covered in the computer exercise sets. In the author's judgment, at most ten items can be considered directly related to the computer materials. It thus appeared that some actual differences in achievement could be masked due to the gross criterion measure. Because of this fact, the author decided to investigate the achievement of the subjects on the ten items as a separate part of the experiment. The items were pre-selected before the C.C.M.T. had been scored. Information on the ten items is listed in Appendix F.

Limitations

There are several limiting factors present in the experimental study that could place certain restrictions on the findings and conclusions. These limitations include the following:

1. The sample for this study was restricted to students enrolled in Mathematics 2055 Calculus, Sections 5, 6, and 7 at Oklahoma State University. These were predominately students planning to major in engineering. Also, students who had transferred from another college had to be excluded from the analysis because their A.C.T.M. scores were not available. Thus, caution should be exercised when attempting to generalize the results to other schools and different groups of students.
2. Intact groups of students, who had self-chosen the course, were used. As has been noted, a rather large difference in ability, as measured by the A.C.T.M., existed between groups. There was also a substantial age difference between groups. An analysis of covariance technique was used in comparing the groups so

that statistical adjustment has been made for much, but probably not all, of the initial group differences.

3. Different recitation times for the three groups of subjects could have resulted in a difference between groups which is not reflected in the pretest and A.C.T.M. covariates. This limitation is increased by the relatively early meeting times: 7:30 a.m., 8:30 a.m., and 9:30 a.m.
4. The experimental groups were aware (after their enrollment) that they were part of a study, and the results could have been affected by the Hawthorne effect.
5. Finally, since only two standardized tests were given to each group, and the same form was used for both pretest and post-test, the effect of taking the pretest may have affected the post-test results.

Analysis of the Data

The purpose of this section is to report the findings of the statistical tests used to determine the significance of the three methods of instruction as demonstrated on the post-test scores. The .05 level of probability was used to judge the significance of the statistics associated with each hypothesis.

Data for the groups was prepared for an analysis of covariance program (BMD04V--Analysis of Covariance with Multiple Covariates) supplied by the Oklahoma State University Computing Center. The program was then executed on the University's computer.

Analysis of Covariance -- Complete C.C.M.T. for Post-test. Table II presents the results of analysis of covariance where the criterion measure, the post-test, consisted of the complete C.C.M.T.

TABLE II
ANALYSIS OF COVARIANCE -- COMPLETE C.C.M.T.

Source of Variation	df	Sum of Squares	Mean Sum of Squares	F
Between	2	9.56	4.78	
Within	50	2038.05	42.46	
Total	52	2047.61		0.11*

*Not significant at .05 level. Critical value for $F(2,48) = 3.19$

Table III presents a summary of group means for the three groups when the complete C.C.M.T. was used as criterion measure.

From Table III it can be seen that the Control Group scored significantly higher on both the A.C.T.M. and pretest. Likewise on the post-test, this group scored better than both experimental groups. After adjusting the post-test means using the A.C.T.M. and pretest scores as covariates, the adjusted means were much nearer the same. However, the control group still held a .71 edge over Group One and 1.22 edge over Group One-half.

Table II indicates a calculated F value of 0.11 from analysis of covariance. The critical F value given by Hoel (19, p. 406) for the

TABLE III
GROUP MEANS -- ALL ITEMS OF C.C.M.T.

	Number	A.C.T.M.	Pretest	Post-test	Adjusted Post-test
Control Group	20	30.45	7.10	33.20	30.18
Group One-half	12	27.33	4.50	27.67	28.96
Group One	21	24.81	5.38	27.33	29.47
Total	53	27.53	5.66	29.40	29.54

given degrees of freedom at the .05 level was 3.19. Since the calculated F value was smaller than this, no significant difference is observed between the three groups on the adjusted post-test results. Thus, Hypothesis 1 must be accepted, concluding there was no difference among treatments.

Analysis of Covariance -- Ten Items from C.C.M.T. Used for Post-test.

Table IV presents the results of analysis of covariance where the criterion measure, the post-test, consisted of ten pre-selected items from C.C.M.T. which relate better than other items to the computer materials used in the experimental treatment.

Table V presents a summary of group means for the three groups when ten items of C.C.M.T. were used as criterion measure.

From Table V it can be seen that the Control Group scored higher on the ten items at the pretest and again at the post-test. However, after adjusting the post-test means using the A.C.T.M. and pretest scores as covariates, the adjusted means had a different relationship.

TABLE IV
ANALYSIS OF COVARIANCE -- TEN ITEMS OF C.C.M.T.

Sources of Variation	df	Sums of Squares	Mean Sum of Squares	F
Between	2	3.54	1.7710	
Within	50	117.83	2.4549	
Total	52	121.38		0.72*

*Not significant at .05 level. Critical value for $F(2,48) = 3.19$

TABLE V
GROUP MEANS -- TEN-ITEM SUBSET OF C.C.M.T.

	Number	A.C.T.M.	Pretest	Post-test	Adjusted Post-test
Control Group	20	30.45	1.70	6.65	6.10
Group One-half	12	27.33	1.08	5.50	5.61
Group One	21	24.81	1.48	5.86	6.31
Total	53	27.53	1.42	6.00	6.01

The Control Group and Group One-half differed by 0.49 with the Control Group still higher, but the Control Group and Group One means were reversed with Group One going from a 0.79 deficit to a 0.21 edge. There seems to have been a change in the positive direction.

The calculated F value of 0.72 shown in Table IV is still not significant, however, as the critical F value remains at 3.19. Since the calculated F value is less than 3.19, no significant difference between the groups can be claimed. Thus, Hypothesis 2 must be accepted, concluding that there was no difference among treatments.

Summary of Statistical Analysis

Included in this section is a summary of the results of the statistical analysis related to the two hypothesis of the experiment. Other conclusions and recommendations are presented in Chapter IV.

The experiment was separated into two main parts. First, an analysis comparing the three groups on results of a post-test criterion which was the Calculus Cooperative Mathematics Test. Second, an analysis comparing the three groups on a ten-item pre-selected subset of the C.C.M.T. was conducted. The ten items were selected for their pertinence to the computer extended treatment of the experimental groups.

In both analyses it was revealed that the null hypothesis should be accepted implying that there was no significant difference in the level of achievement and understanding between the three groups: Control Group, Group One-half, and Group One,

In the first analysis the Control Group held a large advantage in A.C.T.M. and pretest mean scores over both experimental groups. On the post-test mean scores the advantage was narrowed slightly and on adjusted mean scores the three groups were nearly the same with the Control Group still highest.

In the second analysis the Control Group again held a large advantage in A.C.T.M. mean and somewhat smaller advantage in pretest mean as compared with the two experimental groups. On the post-test mean scores, the advantage was narrowed slightly, and on the adjusted mean scores, the advantage went over to experimental Group One. No difference between experimental Group One and Group One-half could be detected.

CHAPTER IV

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Summary

In preceding chapters the author has traced the relatively recent entrance of computers in mathematics instruction, particularly calculus instruction, at the college level. The use of computers has resulted in large part from the expanding applications and availability of computers in today's world. Many individuals and groups have begun to experiment with calculus and the computer applications of calculus. The author has presented his approach to the marriage of calculus instruction and the computer. Basically, this is to use the computer to extend calculus instruction, that is, broaden it to include computation, numerical experimentation and application, without substantial deletion of traditional topics. The purpose for computer use is, thus, to stimulate, motivate, and broaden students' learning of calculus by adding a new dimension, numerical computation.

Attempting to obtain some measure of the effect of such a computer extended approach on students' understanding and achievement of the calculus, the author devised an experiment. A control group receiving the conventional lecture-discussion treatment was compared with two experimental groups who had the same lecture, but who used some computer calculus materials adapted from Chapter II. One experimental group wrote flow charts and used the author's programs while the second

experimental group wrote flow charts, used the author's programs, and in addition, wrote some of their own programs. Students in both experimental groups had computer-related assignments in addition to the regular assignments. Statistical results of the experiment indicated that there was no significant difference in achievement for the three groups. There was some evidence that students in the experimental group which did some programming may have achieved better on those calculus topics which were directly related to their computer experiences.

Conclusions

Learning and understanding of traditional calculus concepts by students in the sample did not seem to be substantially changed by the experimental approach to calculus with a computer. Hence, the approach described here has not been demonstrated to improve instruction of such concepts. On the other hand, since overall achievement of the groups was not changed by introduction of computer materials, apparently the added dimension of computer application need not diminish learning of the calculus concepts. Thus, skepticism of many mathematicians concerning the matter of maintaining a high level of conceptual learning should be reduced by the author's experiment. Apparently, an instructor can proceed with this type of modification of his instruction process without serious worry of doing students harm,

An instructor of calculus, perhaps even an entire department of mathematics, will have to examine the objectives of its various calculus courses in light of whether or not the computer should be used. Considerable money for computer time and supplies can be involved, as well

as student and instructor time. For a liberal arts student computer calculus may be his only serious encounter with the computer, and thus, the added expense can be justified as contributing to his liberal education. For an engineering student, who will meet the computer often in his undergraduate studies, the added expense might not be justifiable in this way. But such a student will most likely apply his calculus via a computer so computer extension is surely relevant. Also, of course, there is a prerequisite of adequate computing capability and instructor interest. Availability of these seems to be growing as was indicated in both Chapter I and III.

Recommendations for Further Study

There are several implications for further study. Some of these are:

1. Duplicate the study with a larger sample, with groups that are more alike on ability measures, and with an attempt to more thoroughly integrate computer materials with the traditional calculus. This would probably require a different experimental design so that computer applications could be made a part of most class lectures.
2. To investigate the effect computer use in mathematics instruction has on achievement in relation to student ability. There is some evidence in the author's study that lower ability students may respond favorably to the computer extension.
3. To study the effect computer extension has on the more theoretical oriented calculus students, for example, prospective mathema-

tics majors. The study described in this thesis had a sample of mostly beginning engineering students who are traditionally problem oriented.

4. Investigate the effects of computer use on student motivation, interest, and attitude, both for a short term and over an extended period of time.
5. Determine what effect, if any, the computer applications have on algorithmic understanding and skills and ability to generalize from experimental results to general patterns and conclusions.

Recommendations for Use of the Computer in Calculus

Recommendations for possible use of the computer in calculus instruction are the following:

1. Organize classes into teams of students which could meet weekly or oftener, divide up their programming, computing, and flow charting needs and then meet to compare and discuss their results as a group.
2. Obtain a time sharing terminal along with a closed circuit television system for demonstration during class. Such a system can be devised for much less than equipment catalogs indicate if some improvisation is practised.
3. Provide copies of output for students to analyze after they have discussed and examined the computer program to be used. Students need hands-on experience, but much duplication by the computer can be reduced, resulting in a considerable saving of time and money.

4. Integrate the elementary calculus course with a beginning computer science course. Programming instruction and time for calculus applications would be available from the computer science course, while application exercises would be available from calculus.
5. Integrate the use of flow charts into calculus instruction in a substantial way, using them to teach analytical techniques applicable to conceptual understanding, problem solving, and theoretical proof construction. A principal use of computer programming would be to verify the logical accuracy of flow charts and the reasoning put into their construction.

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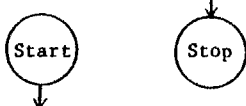

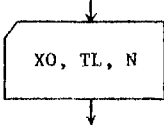
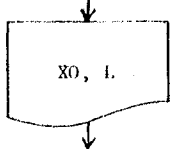
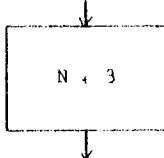

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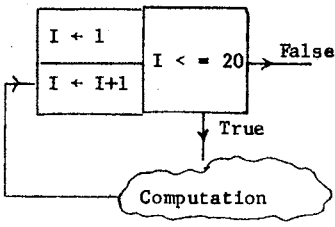
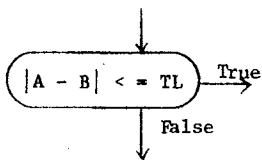
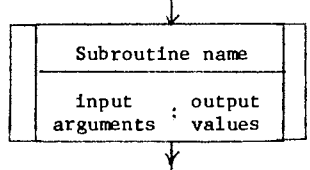
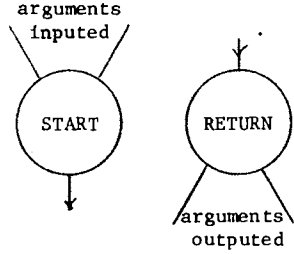
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APPENDIX A

FLOW CHARTING SYMBOLS

Symbol Name	Symbol	Effect in Program; Example BASIC Statement
Start and Stop		<p>Indicates the initial and terminal points, respectively.</p> <p>-----</p> <p>XEQ or RUN</p> <p>300 STOP or 300 END</p>
Function Definition		<p>Defines a function, f, that is to be used in the program.</p> <p>-----</p> <p>20 DEF FNF(x) = x + 1</p>
Input	 <p>(looks like an IBM card)</p>	<p>Causes values to be read into the computer for the variables listed in the box.</p> <p>-----</p> <p>20 READ XO, TL, N</p> <p>or</p> <p>20 READ XO, TL, N 30 DATA 2, .0001, 10</p>
Output	 <p>(looks like a torn page of paper)</p>	<p>Causes values of variables in the box to be printed out.</p> <p>-----</p> <p>50 PRINT XO, L</p> <p>or</p> <p>50 PRINT USING 55, XO, L 55 IMAGE At XO = ---, the value of L =</p>
Assignment		<p>Causes a position labeled "N" inside the computer to be occupied by the number at right of arrow.</p> <p>-----</p> <p>50 LET N = 3</p> <p>or</p> <p>50 N = 3</p>
Connector		<p>Used in flow charting to separate flow chart into segments. (to conserve space, for example).</p> <p>-----</p> <p>no computer statement</p>

Symbol Name	Symbol	Effect in Program; Example BASIC Statement
Iteration		<p>Causes computations to be performed with I = 1, then 2, then 3, ..., then 20. After I = 20 the computer moves on to the statement after NEXT I.</p> <hr style="border-top: 1px dashed black;"/> <pre>20 For I = 1 to 20 ... other statements ... 50 Next I</pre> <p style="text-align: center;">⋮</p>
Decision		<p>Causes the computer to branch (to right here) if statement inside the box is true, otherwise continue on by box.</p> <hr style="border-top: 1px dashed black;"/> <pre>50 IF ABS(A - B) ≤ TL THEN 100 ... other statements ... 100 PRINT A</pre> <p style="text-align: center;">⋮</p>
Subroutine Call		<p>Causes a set of statements somewhere in the program to be executed.</p> <hr style="border-top: 1px dashed black;"/> <pre>70 GOSUB 300 ... other statements ... 300 LET L = XO * 5 310 RETURN</pre> <p style="text-align: center;">⋮</p>
Subroutine Entrance and Exit		<p>First symbol denotes start of subroutine with whatever is in hopper used as data. Second symbol indicates end of subroutine with value for whatever is in chute to be returned to the program.</p> <hr style="border-top: 1px dashed black;"/> <p>No special start statement other than GOSUB.</p> <pre>310 RETURN</pre> <p style="text-align: center;">⋮</p>

APPENDIX B

EXAMPLE BASIC PROGRAM

WITH SAMPLE OUTPUT

Program SIMPSN

```

10 REM      This program approximates the integral of f(x) on (a,b) using
20 REM      Simpson's Rule. N is the number of subintervals.
30 REM      Change statements 50, 90 and 120 to change the problem.
40 REM      CAUTION: N must always be even!
50 DEF FNF(X) = EXP(X)
60 RESTORE
70 DIM X(100), Y(100)
80 READ A,B
90 DATA -10, 1
110 READ N
120 DATA 20, 40
130 LET DX = (B - A)/N
140 LET YO = FNF(A)
150 FOR I = 1 TO N
160 LET X(I) = A + DX * I
170 LET Y(I) = FNF(X(I))
180 NEXT I
190 LET SM = YO
200 FOR I = 1 TO N - 1 STEP 2
210 LET SM = SM + 4 * Y(I)
220 NEXT I
230 FOR I = 2 TO N - 2 STEP 2
240 LET SM = SM + 2 * Y(I)
250 NEXT I
260 LET SM = SM + Y(N)
270 LET IN = DX/3 * SM
280 PRINT "
290 PRINT USING 300, N,A,B
300 IMAGE
   For N = ---- subintervals, the integral of f(x) on (----,----,----)
310 PRINT USING 320, IN
320 IMAGE
      is approximately ----- by Simpson's Rule.
330 PRINT "
335 GO TO 110
340 END

```

Sample Output from SIMPSN

```

For N = 20 subintervals, the integral of f(x) on (-10.000, 1.000)
is approximately 2.71957006 by Simpson's Rule.

For N = 40 subintervals, the integral of f(x) on (-10.000, 1.000)
is approximately 2.71832202 by Simpson's Rule.

```

APPENDIX C

REFERENCE MATRIX

	Thesis	CRICISAM	Flanders	Henriksen Lees	Johnson Kiekemeister	Protter Morrey	Randolph	Schwartz	Thomas
Analytic Geometry	14-17	1 - 64			1 - 72, 374 - 414	1 - 70, 279 - 320	1 - 47	116 - 253	1 - 22
Limit of Functions	17-27	125 - 205, 355 - 390	888 - 894	75 - 94 -	73 - 105	71 - 130	48 - 61	376 - 412	34 - 58
Derivatives	28-36	391 - 422, 479 - 523	1 - 31, 225 - 244	52 - 74, 95 - 156	106 - 135	131 - 152	62 - 83	1 - 56	23 - 33, 59 - 98
Derivative Applications	36-42	423 - 478, 524 - 540, 613 - 665	32 - 72, 202 - 224, 245 - 264	157 - 224	136 - 184	153 - 205	84 - 133	423 - 457	99 - 143
Indeterminate forms - Taylor's Formula	42-48	648 - 665, 449 - 462	900 - 905, 215 - 224, 593 - 621		415 - 447		432 - 440, 416 - 426	819 - 823, 789 - 823	593 - 597, 578 - 584
Integration	48-60	207 - 324, 541 - 574, 666 - 756	108 - 127, 327 - 372, 666 - 684	225 - 292, 427 - 458	185 - 245, 342 - 373	201 - 278, 453 - 489	167 - 237	57 - 115, 329 - 369, 413 - 422, 504 - 508	144 - 185, 279 - 313
Applications of Integration	60-65	325 - 354, 757 - 851	128 - 151, 373 - 411	293 - 348	246 - 279	490 - 548	426 - 432	370 - 375, 458 - 503	186 - 228, 307 - 313
Infinite Series	65-68	852 - 937	622 - 660	371 - 397	448 - 499		448 - 481	750 - 823	563 - 610
Flow Charts & Programming	12-14 98-100	65 - 124	148 - 151						

Complete bibliographical information on the following page.

1. CRICISAM. Calculus: A Computer-Oriented Presentation, Parts 1 - 5, Florida State University, 1968.
2. Flanders, Harley, Robert Korflage and Justin Price. Calculus, Academic Press, 1970. (Contains computer applications and flow charts.)
3. Henriksen, Melvin and Milton Lees. Single Variable Calculus, Worth Publishers, 1970. (Contains some comments relating to computers.)
4. Johnson, R.E. and F.L. Kiekemeister. Calculus with Analytic Geometry, Fourth Edition, Allyn and Bacon, 1969.
5. Protter, Murray and Charles Morrey, Jr. Calculus with Analytic Geometry, Second Edition, Addison Wesley, 1970.
6. Randolph, John. Calculus with Analytic Geometry, Second Edition, Wadsworth, 1965.
7. Schwartz, Abraham. Calculus and Analytic Geometry, Second Edition, Holt, Rinehart and Winston, 1967.
8. Thomas, George B., Jr. Calculus, Third Edition, Addison-Wesley, 1969.

APPENDIX D

EXERCISE SETS FOR

GROUP ONE-HALF

Exercise Set 1

1. Load, list, and execute FACTN. (No handin required.)
2. Write a flow chart to read numbers from a data list, find the absolute value of each, and then print out this value. Execute the lab instructor's program, ABSVAL, on the list -5 , -2.1 , $-\text{sqr}(2)$, 0 , $\sin(-3.1416)$, $.25$, $3/4$, and 521 .

Hand in flow chart and output on the given number list.

3. Frequently throughout the course it will be necessary to sketch the graph of a function f over an interval $[a,b]$. A computer program can produce the table of ordered pairs $(x,f(x))$ needed for graphing f . A flow chart and computer program written by the lab instructor to produce this table will be discussed in class. Using this program, called FCNEVA, have the computer produce a table of values for the two functions of Problems 1 and 2 on Page 8 of your text.³ Divide $(-2,2)$ into subintervals of length (a) $.5$ and (b) $.25$.

Hand in only the four tables the program produced.

4. The lab instructor has written a program, called SECLIN, which computes the slope and equation of a secant line through a point (x_0, y_0) on the graph of a function f . A flow chart and program will be discussed in class. This program is very helpful on Problems 1 and 2 of page 8 of your text. Use the program on these problems.

Hand in your output on the two problems.

³The class text was Calculus of the Elementary Functions by Shanks and Gambill (34).

5. An essential idea of calculus is the effect upon $f(x)$ as x approaches some number x_0 . We allow x to approach x_0 from the right ($x \rightarrow x_0^+$) and from the left ($x \rightarrow x_0^-$), but x is not allowed to actually equal x_0 (since then it could not continue to "approach" x_0). Of special interest is the case where the $f(x)$ s (numbers) approach some fixed number L as the x 's approach x_0 . We then say L is the "limit of $f(x)$ at x_0 " and denote this by

$$\lim_{x \rightarrow x_0} f(x) = L.$$

(See pages 5 and 527 - 529 of your text.) We require that

$$\lim_{x \rightarrow x_0^+} f(x)$$

equal

$$\lim_{x \rightarrow x_0^-} f(x)$$

before we will say that

$$\lim_{x \rightarrow x_0} f(x)$$

exists (and it is then equal to the equal "one-sided limits").

Write a flow chart to estimate $\lim_{x \rightarrow x_0^+} f(x)$ and as $x \rightarrow x_0^-$. (Note that $x = x_0 \pm (DL)^i \rightarrow x_0$ if $DL < 1$ and i takes successive values of 1, 2, 3, ...) The lab instructor has written LIM1 from such a flow chart for your use.

Hand in your flow chart and the output from the program LIM1 applied to the following. Write on your output what you think

$$\lim_{x \rightarrow x_0} f(x)$$

is (if anything.)

$$\begin{array}{lll} \text{(a) } \lim_{x \rightarrow 3} \frac{1}{x} & \text{(b) } \lim_{x \rightarrow 2} (x^2 - x) & \text{(c) } \lim_{x \rightarrow 0} \sin \frac{1}{x} \\ \text{(d) } \lim_{x \rightarrow 0} \frac{1}{x} & \text{(e) } \lim_{x \rightarrow 2} \frac{x - 2}{(x^2 - 4)} \end{array}$$

6. As the definition on page 9 of your text indicates, the derivative of a function f at x_0 is simply the limit of a certain expression involving x , namely

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

(called the difference quotient of f). To find $f'(x_0)$, then, we need only find

$$\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

We could define a function g by

$$g(\Delta x) = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

and use the program from Exercise 5 above, to find

$$f'(x_0) = \lim_{\Delta x \rightarrow 0} g(\Delta x).$$

The lab instructor has written a flow chart and program from it to estimate $f'(x_0)$ in essentially this way. It is called DERIV

and will be discussed in class. (Note that we are finding a number $f'(x_0)$ and not the function f' .) Use DERIV to find the following:

- (a) $f'(1)$ where $f(x) = 2x + 3$
- (b) $f'(2)$ where $f(x) = x - x^2 + 2$
- (c) v of Problem 5 on page 19 of the text
- (d) $f'(0)$ where $f(x) = |x|$ (???)

Hand in your output from DERIV applied to the four problems.

Exercise Set 2

1. In Chapter 2 of your text you have encountered a set of functions called elementary functions (see page 65 for a convenient table listing these functions). Frequently it is quite helpful if one can have clearly in mind or be able to recall a sketch of the graph of these functions. To help you recall some of the graphs, use FCNEVA to obtain graph points for sketching a graph of each of the following functions on the given interval. Evaluate on values of x that are Δx units apart. You may put two graphs on the same set of axes if you want. Since you have adequate data, draw the graphs with reasonable care so they can be used later; however, you need not plot all of the points available.

(a) $f(x) = \log x$ on $[.1,1]$ with $\Delta x = .1$ and on $[1,10]$ with $\Delta x = 1$.

(b) $f(x) = e^x$ on $[-4,4]$ with $\Delta x = .5$.

(c) $f(x) = \tan x$ on $[-4,4]$ with $\Delta x = .5$.

(d) $f(x) = \sec^2 x$ on $[-1.5, 1.5]$ with $\Delta x = .5$.

Hand in your four graphs.

[At your option, time permitting, you could graph $f(x) = \sin x$ and $f(x) = \cos x$ on $[-4,4]$ with $\Delta x = .5$. These are two functions with whose graph you should be very familiar.]

2. In problem 6 of Exercise Set 1 the program DERIV was used to obtain $f'(x_0)$, a number, for a function f at a point x_0 . This same program can be used to find the derived function, f' , in the following way. If we know the value of $f'(x)$ at enough values of x , we "know" f' . By repeatedly using DERIV, $f'(x)$ can be

obtained for several values of x . A shortened form of DERIV, called SHTDER, which has less printout (an operation that considerably hinders a computer's work) and which evaluates f' on an interval, can be used. Use SHTDER on the following five functions on the indicated interval. (Instructions will be given in class.) Sketch a graph of f' for each f using the output, and see if you can determine from this graph what f' is. (Hint: the graphs of problem 1 above may be helpful.)

- | | |
|---------------------------------|--|
| (a) $f(x) = \sin x$ on $[-4,4]$ | (d) $f(x) = \log x$ on $ [.2,10]$ (??) |
| (b) $f(x) = \cos x$ on $[-4,4]$ | (e) $f(x) = \tan x$ on $[-1.5,1.5]$ |
| (c) $f(x) = e^x$ on $[-4,4]$ | |

Hand in the five graphs with your statement of what the defining equation for f' is.

3. The operation of composition on functions is discussed on page 66 of your text. The rule for differentiation of composed functions is so important that it has a special name, Chain Rule (page 66). The lab instructor has written a flow chart and program to verify numerically the chain rule at a point x_0 and to perform a partial check of your computed derivative. The flow chart and program, CHNRUL, will be explained in class. Use CHNRUL on the following:

- (a) Problem 2 on page 69 of the text with $x_0 = 2$.
- (b) Problem 5 on page 69 of the text with $x_0 = 2$.
- (c) Problem 1 on page 69 of the text with $x_0 = 1$.
- (d) Find $(dy/dx)(1/2)$ if $y = \sqrt{u}$ and $u = (1-x)/(1+x)$.

Hand in your output on the four problems, noting anything interesting that may have turned up.

Exercise Set 3

1. This exercise is on tangent and normal lines, refer to page 93 of your text. Write flow chart for a program which will input f , f' , and a number x_0 and then compute equations, in slope-intercept form, of the tangent and normal lines to the graph of f at $(x_0, f(x_0))$. The lab instructor has written a flow chart and program from it called TANNRM which does these things. Use it to work (a) Problem 1 and (b) Problem 16 on pages 94 - 95 of your text.

Hand in your flow chart and output on the two problems.

2. Write a flow chart that uses differentials (as in example 2 and 3 on page 116 of your text) to approximate certain expressions. Recall that justification for differential approximation goes like this: $\Delta y = f(x + \Delta x) - f(x)$ and $dy = f'(x)dx$ by definition. Rewriting the first equation and using $\Delta x = dx$ in the second equation, we get $f(x + \Delta x) = f(x) + \Delta y$ and $dy = f'(x)\Delta x$. Now using the fact that Δy approximately equals dy for Δx "relatively small," we have the useful expression

$$f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x)\Delta x.$$

Have your flow chart output the approximation $A = f(x) + f'(x)\Delta x$, the actual value $B = f(x + \Delta x)$, and the difference, D , between A and B . The lab instructor has written a program called DIFAPX to do this. Use it on the following:

- (a) Problem 9 on page 117.
- (b) Problem 10 on page 117.
- (c) Problem 11 on page 117. ($60^\circ = \pi/3$, $\pi \approx 3.14159$)

Hand in your flow chart and output on the three problems.

3. A flow chart and computer program called RELEXT which attempt to use the Second Derivative Test for Maxima and Minima (page 103) have been written by the lab instructor and will be discussed in class. The program is quite limited since finding maxima, minima, critical values, flex points, and so forth is rather difficult in general. However, RELEXT, will work on some problems (principally where f'' is continuous and f' is well-behaved). Use it on the following problems with the indicated numbers as first guesses for the zeros of f' .

- (a) Problem 24 on page 108 using -6, -4, 4, 6
- (b) Problem 29 on page 108 using -4, -2, -, 2, 4
- (c) Problem 35 on page 108 using -4, -2, 0, 2, 4

Hand in your computer output on the three problems.

4. Newton's method for approximating the roots for an equation $f(x) = 0$ was used as a "subroutine" (to find zeros of f') in RELEXT of exercise 3 above. Problem 23 on page 118 of your text explains very briefly how the method works. It is a useful procedure for finding roots (if one has a suitable first guess for a root). The lab instructor has written a program called NEWTON which uses the method. Use it on (a) and (b) of Problem 23.

Hand in your output on the two parts.

Exercise Set 4

Sections 2 and 3 in Chapter 7 of your text are concerned with limits of functions for which the initial attempted evaluation results in one of several particular forms, called "indeterminate forms." A very useful tool for handling such limits, L'Hospital's Rule, is stated (in several forms) and proven. The lab instructor has written two programs specifically for these two sections. ONESID is designed for estimating one-sided limits (see example 2 on p. 182 and example 1 on p. 185), but can be used on ordinary ("two-sided") limits. LIMINF is designed to estimate $\lim f(x)$ as x goes to either $+\infty$ or $-\infty$. Both programs have a tolerance stop for a finite limit and an infinite limit. The programs will be discussed in class before use.

[Note: Parts of problems below marked with a Δ are optional -- do only if required problems have been completed and extra time is available.]

1. After you have evaluated the following limits by your own means, use ONESID to obtain a partial check of your result.

(a) 4 on p. 183 (IC = 2)

(b) 5 on p. 183 (IC = 2)

(c) 6 on p. 183 (IC = 10)

(d) 17 on p. 183 (IC = 10. Change 340 to 340 For
I = 2 to N)

Δ (e) 26 on p. 183 (IC = 10 and IC = 2) ??

Hand in your output.

2. After you have evaluated the following one-sided limits by your own means, use `ONESID` to obtain a partial check of your result.

(a) 5 on p. 186 (IC = 10)

(b) 9 on p. 186 (IC = 10)

(c) $\lim_{x \rightarrow (\pi/2)^-} \frac{1}{\sin x}$ (IC = 10 and $\pi/2 \approx 1.5708$)

Δ (d) 28a on p. 187 with $C = 5$ (IC = 10)

Δ (e) 28b on p. 187 with $C = -5$ (IC = 2)

Hand in your output.

3. The following limits have x going to $+\infty$ or $-\infty$. Evaluate by your own means and then use `LIMINF` to obtain a partial check of your result.

(a) 2 on p. 186 (IN = 2)

(b) 10 on p. 186 (c = 1, IN = 10)

(c) 23 on p. 187 (n = 5, IN = 2)

Δ (d) 37 on p. 188 show the limit is -1 (IN = 2)

(e) $\lim_{x \rightarrow -\infty} x^2 e^x$ (IN = 2)

Hand in your output.

4. A theorem which is usually called Taylor's Formula is introduced in Section 4 (p. 188) of Chapter 7. The polynomials $P_n(x)$, called the Taylor polynomials (p. 189), provide good approximation to $f(x)$ if the remainder term $R_n(x)$ is small. The lab instructor has written a program called `TAYLOR` which computes the first 6 coefficients of $P_n(x)$ for a fixed x . Values of $P_n(x)$ are printed out for $n = 0, 1, \dots, 5$ along with the computer evaluation of

$f(x)$ and the differences between $P_n(x)$ and $f(x)$ for successive n . The program will be discussed in class. Do the following:

- (a) Use TAYLOR to partially check your answer on 11 of p. 192.
- (b) Use TAYLOR to partially check your answer on 15 of p. 193.
- (c) Use TAYLOR to partially check your answer on 13 of p. 193.
- Δ(d) In Example 4 of p. 202 the authors describe the convergence of Maclaurin's series for $\log(1+x)$ to be slow for $|x|$ near 1. Verify this by using TAYLOR with $n = 5$ and $x = 1/10, 1/2, 3/4, 9/10$ successively. (Recall that Maclaurin's series has $a = 0$.)

Hand in your output.

Analysis of Exercise Set 4

Problem 1

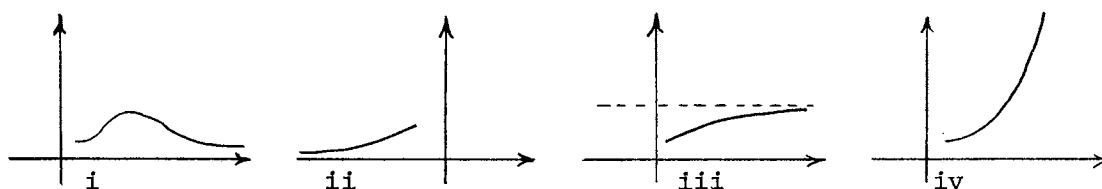
1. In (a) and (b) which function increases faster as $x \rightarrow x_0^-$?
2. Using (b) which function $N(x) = e^{2x} - 1$ or $D(x) = \cos x - 1$ approaches zero faster as $x \rightarrow 0^+$?
3. In (c) and (d) which function apparently approaches its limit faster as $x \rightarrow 0$?

Problem 2

4. Without regard to sign, which function $f(x) = 1/x$ or $g(x) = \log x$ has the steepest (numerically) graph as $x \rightarrow x_0^+$?
5. In parts (a), (b), and (c) which function approaches its limit slowest?

Problem 3

6. Match the graphs with parts (a), (b), (c), or (e) using your numerical output:



Problem 4

7. In (a) does the computed result for R_4 using TAYLOR seem correct as compared with the text's answers?
8. What is true of the Taylor polynomial coefficients printed out by the computer?

Exercise Set 5

1. The definite integral of a function f from a to b is defined (p. 224) as the limit of Riemann sums. Write a flow chart to compute a Riemann sum for a function f on an interval $[a,b]$ using n subintervals of equal length and choosing the midpoint of each subinterval as the ξ_i for that i -th subinterval (see p. 223).

The lab instructor has written such a flow chart and program called RIEMSM. The program is to be used on the following:

- (a) Estimate

$$\int_0^2 \sqrt{4 - x^2} dx$$

using $n = 4, 8, 64, 128,$ and 256 successively. Notice that you have approximated the area of one fourth of a disc whose radius is 2 , that is, you have been approximating π . (Also note how long it takes the computer on the last sum -- be glad we don't have to do it by hand! We shall eventually have a much better way to evaluate this integral.)

- (b) 4 on p. 227 ($n = 5$ and 10)
 (c) 5 on p. 227 ($n = 8$ and 16)
 (d) 6 on p. 227 ($n = 4, 8, 16$)

(If you wish, examine some of the above integrals using more division points.)

Hand in your flow chart and output on the four problems.

2. The lab instructor has written a flow chart and program called UPLWSM which finds upper and lower Riemann sums. Use it on the following problems:

- (a) 3 on p. 227 (n = 1, 8, 16, 64)
 (b) 4 on p. 227 (n = 5, 10, 20, 100)
 (c) 6 on p. 227 (n = 4, 8, 64)

[Note that $\sin x$ is increasing on $[0, \pi/2)$ and decreasing on $(\pi/2, \pi]$. $\pi/(2\sqrt{2}) \approx 1.11072$ and $\pi(\sqrt{2} + 1)/(2\sqrt{2}) \approx 2.68151.$]

Hand in your output on the three problems.

3. The lab instructor has written a flow chart and program called FNDTHM which approximates the integral

$$\int_a^b f(t) dt$$

using a numerical procedure called Simpson's Rule. You will have a chance to learn how this procedure works when we reach Section 3 of Chapter 12 (p. 312); however, for now you will have to simply believe that it works. Using it we can obtain a table of values for the function F defined as in Section 4 of Chapter 9 (p. 232) by

$$F(x) = \int_a^x f(t) dt$$

for x in $[a, b]$. Graphing the pairs $(x, F(x))$ should help you to understand the integral (and the Fundamental Theorem of Calculus) better. For each of the following obtain a table of ordered pairs $(x, F(x))$ for f and x as indicated, sketch a graph of $F(x)$, and see if you can find the defining equation for the graph.

- (a) $f(x) = 2x$, $a = 0$, $b = 2$, and $IC = .25$
 (b) $f(x) = \cos x$, $a = 0$, $b = 6$, and $IC = .5$

(c) $f(x) = e^x$, $a = -5$, $b = 4$, and $IC = 1$

(d) $f(x) = 1/x$ with $a = 1$, $b = .25$, and $IC = .25$;

then $a = 1$, $b = 6$, and $IC = .5$

Hand in your output and graphs for the four problems with your guess of the defining equation.

Analysis of Exercise Set 5

Problem 1

1. After examining the output of (a) which do you think is the most likely result for a Riemann sum for $f(x)$ when n is bigger than 256 ($n = 512$, for example): 3.14168111, 3.14161111 or 3.14141111?
2. In part (c) explain why the Riemann sums are zero while the area between the curve and x -axis is clearly not zero.

Problem 2

3. In part (a): (i) does the computed answer agree with the text's?
 (ii) using ξ_i' and ξ_i'' as on p. 225 of the text with $N = 1$, what are the values of ξ_1' and ξ_1'' ?
4. Comparing part (b) with part (b) of Problem 1, how does the Riemann sum compare with the lower and upper Riemann sums when $N = 5$? When $N = 10$?
5. As in 4 compare part (c) with part (d) of Problem 1 when $N = 8$. How do the sums compare?
6. In each part of Problem 2 what happens to the lower sums as N increases? What happens to the upper sums as N increases?

Problem 3

7. After doing this problem what do you think the defining equation for $F(x)$ would be if

$$F(x) = \int_0^x (-\sin t) dt?$$

8. What do you think

$$D_x \left[\int_0^x 2t \, dt \right] = ?$$

APPENDIX E

EXERCISE SETS FOR

GROUP ONE

Exercise Set 1

1. Load, list, and execute FACTN. (No handin required.)
2. Write a flow chart and computer program to read numbers from a data list, find the absolute value of each, and then print out this value. Execute the program on the list -5 , -2.1 , $-\text{sqr}(2)$, 0 , $\text{sin}(-3.1416)$, $.25$, $3/4$, and 521 .

Hand in your flow chart, program, and output on the given number list.

3. Frequently throughout the course it will be necessary to sketch the graph of a function f over an interval $[a,b]$. A computer program can produce the table of ordered pairs $(x,f(x))$ needed for graphing f . A flow chart and computer program written by the lab instructor to produce this table will be discussed in class. Using this program, called FCNEVA, have the computer produce a table of values for the two functions of Problems 1 and 2 on page 8 of your text.⁴ Divide $(-2,2)$ into subintervals of length (a) $.5$ and (b) $.25$.

Hand in only the four tables the program produced.

4. The lab instructor has written a program, called SECLIN, which computes the slope and equation of a secant line through a point (x_0, y_0) on the graph of a function f . A flow chart and program will be discussed in class. This program is very helpful on Problems 1 and 2 on page 8 of your text. Use the program on these problems.

Hand in your output on the two problems.

⁴The class text was Calculus of the Elementary Functions by Shanks and Gambill (34).

5. An essential idea of calculus is the effect upon $f(x)$ as x approaches some number x_0 . We allow x to approach x_0 from the right ($x \rightarrow x_0^+$) and from the left ($x \rightarrow x_0^-$), but x is not allowed to actually equal x_0 (since then it could not continue to "approach" x_0). Of special interest is the case where the $f(x)$ s (numbers) approach some fixed number L as the x 's approach x_0 . We then say L is the "limit of $f(x)$ at x_0 " and denote this by

$$\lim_{x \rightarrow x_0} f(x) = L.$$

(See pages 5 and 527 - 529 of your text.) We require that

$$\lim_{x \rightarrow x_0^+} f(x)$$

equal

$$\lim_{x \rightarrow x_0^-} f(x)$$

before we will say that

$$\lim_{x \rightarrow x_0} f(x)$$

exists (and it is then equal to the equal "one-sided limits").

Write a flow chart and computer program from it to estimate

$\lim_{x \rightarrow x_0} f(x)$ as $x \rightarrow x_0^+$ and as $x \rightarrow x_0^-$. (Note that

$$x = x_0 \pm (DL)^i \rightarrow x_0$$

if $DL < 1$ and i takes successive values of 1, 2, 3, ...).

Hand in your flow chart, program, and the output from the program applied to the following. Write on your output what you think

$$\lim_{x \rightarrow x_0} f(x)$$

is (if anything).

$$(a) \lim_{x \rightarrow 3} \frac{1}{x}$$

$$(b) \lim_{x \rightarrow 2} (x^2 - x)$$

$$(c) \lim_{x \rightarrow 0} \sin \frac{1}{x}$$

$$(d) \lim_{x \rightarrow 0} \frac{1}{x}$$

$$(e) \lim_{x \rightarrow 2} \frac{x - 2}{(x^2 - 4)}$$

6. As the definition on page 9 of your text indicates, the derivative of a function f at x_0 is simply the limit of a certain expression involving x , namely

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

(called the difference quotient of f). To find $f'(x_0)$, then, we need only find

$$\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

We could define a function g by

$$g(\Delta x) = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

and use the program from Exercise 5 above to find

$$f'(x_0) = \lim_{\Delta x \rightarrow 0} g(\Delta x).$$

The lab instructor has written a flow chart and program from it to estimate $f'(x_0)$ in essentially this way. It is called DERIV and will be discussed in class. (Note that we are finding a number $f'(x_0)$ and not the function f' .) Use DERIV to find the following:

- (a) $f'(1)$ where $f(x) = 2x + 3$
- (b) $f'(2)$ where $f(x) = x - x^2 + 2$
- (c) v of Problem 5 on page 19 of the text
- (d) $f'(0)$ where $f(x) = |x|$ (???)

Hand in your output from DERIV applied to the four problems.

obtained for several values of x . A shortened form of `DERIV`, called `SHTDER`, which has less printout (an operation that considerably hinders a computer's work) and which evaluates f' on an interval, can be used. Use `SHTDER` on the following five functions on the indicated interval. (Instructions will be given in class.) Sketch a graph of f' for each f using the output, and see if you can determine from this graph what f' is. (Hint: the graphs of problem 1 above may be helpful.)

- (a) $f(x) = \sin x$ on $[-4,4]$
 (b) $f(x) = \cos x$ on $[-4,4]$
 (c) $f(x) = e^x$ on $[-4,4]$
 (d) $f(x) = \log x$ on $[.2,10]$ (??)
 (e) $f(x) = \tan x$ on $[-1.5,1.5]$

Hand in the five graphs with your statement of what the defining equation for f' is.

3. The operation of composition on functions is discussed on page 66 of your text. The rule for differentiation of composed functions is so important that it has a special name, Chain Rule (page 66). The lab instructor has written a flow chart and program to verify numerically the chain rule at a point x_0 and to perform a partial check of your computed derivative. The flow chart and program, `CHNRUL`, will be explained in class. Use `CHNRUL` on the following:

- (a) Problem 2 on page 69 of the text with $x_0 = 2$
 (b) Problem 5 on page 69 of the text with $x_0 = 2$
 (c) Problem 1 on page 69 of the text with $x_0 = 1$
 (d) Find $(dy/dx)(1/2)$ if $y = \sqrt{u}$ and $u = (1-x)/(1+x)$.

Hand in your output on the four problems, noting anything interesting that may have turned up.

4. Write a flow chart and computer program that will input the coordinates of two points, P and Q and then find (a) the slope of the line through P and Q, and (b) the slope-intercept form equation of the line through P and Q. (Also print out coordinates of P and Q.)

Hand in your flow chart, program, and output on pairs of points:

- (a) (-1,1) and (3,5)
- (b) (-,1) and (5,1)
- (c) (2,-3) and (2,0)

Exercise Set 3

1. This exercise is on tangent and normal lines, refer to page 93 of your text. Write a flow chart and computer program from it which will input f , f' , and a number x_0 and then compute equations, in slope-intercept form, of the tangent and normal lines to the graph of f at $(x_0, f(x_0))$. Use the program to work (a) Problem 1 and (b) Problem 16 on pages 94 - 95 of your text.

Hand in your flow chart, program, and output on the two problems.

2. Write a flow chart and computer program that uses differentials (as in examples 2 and 3 on page 116) to approximate certain expressions. Recall that justification for differential approximation goes like this: $\Delta y = f(x + \Delta x) - f(x)$ and $dy = f'(x)dx$ by definition. Rewriting the first equation we get $f(x + \Delta x) = f(x) + \Delta y$ and, using $\Delta x = dx$ in the second equation we get $dy = f'(x)\Delta x$. Now using the fact that Δy approximately equals dy for Δx relatively small, we have the useful expression

$$f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x)\Delta x.$$

Have your program output the approximation $A = f(x) + f'(x)\Delta x$, the actual value $B = f(x + \Delta x)$, and the difference, D , between A and B . Use your program on the following:

- (a) Problem 9 on page 117
- (b) Problem 10 on page 117
- (c) Problem 11 on page 117 ($60^\circ = \pi/3$, $\pi \approx 3.14159$)

Hand in your flow chart, program, and output on the three problems.

3. A flow chart and computer program called RELEXT which attempt to use the Second Derivative Test for Maxima and Minima (page 103) have been written by the lab instructor and will be discussed in class. The program is quite limited since finding maxima, minima, critical values, flex points, and so forth is rather difficult in general. However, RELEXT, will work on some problems (principally where f'' is continuous and f' is well-behaved). Use it on the following problems with the indicated numbers as first guesses for the zeros of f' .

(a) Problem 24 on page 108 using $-6, -4, 4, 6$

(b) Problem 29 on page 108 using $-4, -2, 0, 2, 4$

(c) Problem 35 on page 108 using $-4, -2, 0, 2, 4$

Hand in your computer output on the three problems.

4. Newton's method for approximating the roots for an equation $f(x) = 0$ was used as a "subroutine" (to find zeros of f') in RELEXT of exercise 3 above. Problem 23 on page 118 of your text explains very briefly how the method works. It is a useful procedure for finding roots (if one has a suitable first guess for a root). The lab instructor has written a program called NEWTON which uses the method. Use it on (a) and (b) of Problem 23.

Hand in your output on the two parts.

Exercise Set 4

Sections 2 and 3 in Chapter 7 of your text are concerned with limits of functions for which the initial attempted evaluation results in one of several particular forms, called "indeterminate forms." A very useful tool for handling such limits, L'Hospital's Rule, is stated (in several forms) and proven. The lab instructor has written two programs specifically for these two sections. ONESID is designed for estimating one-sided limits (see example 2 on p. 182 and example 1 on p. 185), but can be used on ordinary ("two-sided") limits. LIMINF is designed to estimate $\lim_{x \rightarrow \pm\infty} f(x)$ as x goes to either $+\infty$ or $-\infty$. Both programs have a tolerance stop for a finite limit and an infinite limit. The programs will be discussed in class before use.

1. After you have evaluated the following limits by your own means, use ONESID to obtain a partial check of your result.
 - (a) 4 on p. 183 (IC = 2)
 - (b) 5 on p. 183 (IC = 2)
 - (c) 6 on p. 183 (IC = 10)
 - (d) 17 on p. 183 (IC = 10. Change 340 to 340 For I = 2 to N)
 - (e) 26 on p. 183 (IC = 10 and IC = 2) ??

Hand in your output.

2. After you have evaluated the following one-sided limits by your own means, use ONESID to obtain a partial check of your result.
 - (a) 5 on p. 186 (IC = 10)
 - (b) 9 on p. 186 (IC = 10)

(c) $\lim_{x \rightarrow (\pi/2)^-} \frac{1}{\sin x}$ (IC = 10 and $\pi/2 \approx 1.5708$)

(d) 28a on p. 187 with $C = 5$ (IC = 10)

(e) 28b on p. 187 with $C = -5$ (IC = 2)

Hand in your output.

3. The following limits have x going to $+\infty$ or $-\infty$. Evaluate by your own means and then use LIMINF to obtain a partial check of your result.

(a) 2 on p. 186 (IN = 2)

(b) 10 on p. 186 (c = 1, IN = 10)

(c) 23 on p. 187 (n = 5, IN = 2)

(d) 37 on p. 188 show the limit is -1 (IN = 2)

(e) $\lim_{x \rightarrow -\infty} x^2 e^x$ (IN = 2)

Hand in your output.

4. A theorem which is usually called Taylor's Formula is introduced in Section 4 (p. 188) of Chapter 7. The polynomials $P_n(x)$, called the Taylor polynomials (p. 189), provide good approximation to $f(x)$ if the remainder term $R_n(x)$ is small. The lab instructor has written a program called TAYLOR which computes the first 6 coefficients of $P_n(x)$ for a fixed x . Values of $P_n(x)$ are printed out for $n = 0, 1, \dots, 5$ along with the computer evaluation of $f(x)$ and the differences between $P_n(x)$ and $f(x)$ for successive n . The program will be discussed in class. Do the following:

(a) Use TAYLOR to partially check your answer on 11 of p. 192.

(b) Use TAYLOR to partially check your answer on 15 of p. 193.

- (c) Use TAYLOR to partially check your answer on 13 of p. 193.
- (d) In Example 4 of p. 202 the authors describe the convergence of Maclaurin's series for $\log(1 + x)$ to be slow for $|x|$ near 1. Verify this by using TAYLOR with $n = 5$ and $x = 1/10, 1/2, 3/4, 9/10$ successively. (Recall that Maclaurin's series has $a = 0$.)

Hand in your output.

Analysis of Exercise Set 4

Problem 1

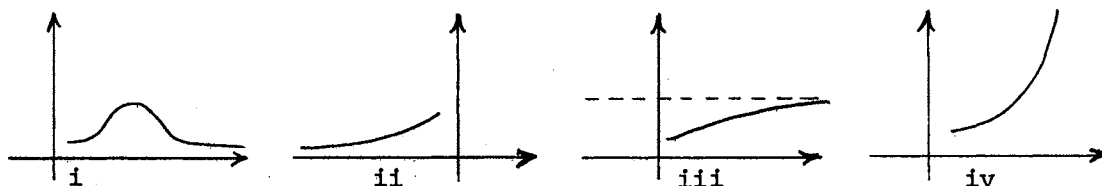
1. In (a) and (b) which function increases faster as $x \rightarrow x_0^-$?
2. Using (b) which function $N(x) = e^{2x} - 1$ or $D(x) = \cos x - 1$ approaches zero faster as $x \rightarrow 0^+$?
3. In (c) and (d) which function apparently approaches its limit faster as $x \rightarrow 0$?

Problem 2

4. Without regard to sign, which function $f(x) = 1/x$ or $g(x) = \log x$ has the steepest (numerically) graph as $x \rightarrow x_0^+$?
5. In parts (a), (b), and (c) which function approaches its limit slowest?

Problem 3

6. Match the graphs with parts (a), (b), (c), or (e) using your numerical output:



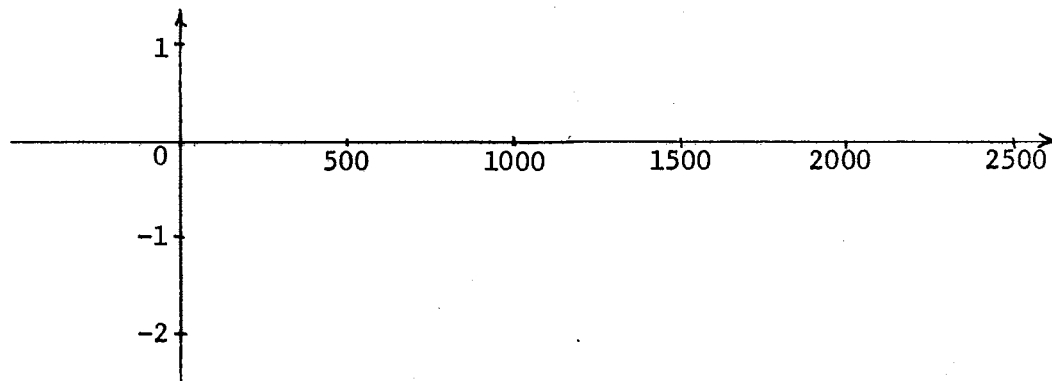
Problem 4

7. In (a) does the computed result for R_4 using TAYLOR seem correct as compared with the text's answers?
8. What is true of the Taylor polynomial coefficients printed out by the computer?
9. In Problem 1(e), for which value of IC does ONESID work best?
10. After doing Problem 2(e) what do you think

$$\lim_{t \rightarrow 0^+} t^{t+5}$$

is?

11. Sketch a general picture of what you think the graph of $f(x)$ looks like for Problem 3(d).



12. Using Problem 4(d), which TAYLOR approximation of $f(x) = \log(1 + x)$, with $n = 5$, would you expect to be most accurate: one for $f(.01)$ or one for $f(.99)$?

Exercise Set 5

1. The definite integral of a function f from a to b is defined (p. 224) as the limit of Riemann sums. Write a flow chart to compute a Riemann sum for a function f on an interval $[a,b]$ using n subintervals of equal length and choosing the midpoint of each subinterval as the ξ_i for that i -th subinterval (see p. 223). The lab instructor has written such a flow chart and program called RIEMSM. The program is to be used on the following:

- (a) Estimate

$$\int_0^2 \sqrt{4-x^2} dx$$

using $n = 4, 8, 64, 128,$ and 256 successively. Notice that you have approximated the area of one fourth of a disc whose radius is 2 , that is, you have been approximating π . (Also note how long it takes the computer on the last sum -- be glad you don't have to do it by hand! We shall eventually have a much better way to evaluate this integral.)

- (b) 4 on p. 227 ($n = 5$ and 10)
 (c) 5 on p. 227 ($n = 8$ and 16)
 (d) 6 on p. 227 ($n = 4, 8, 16$)

(If you wish, examine some of the above integrals using more division points.)

Hand in your flow chart and output on the four problems.

2. The lab instructor has written a flow chart and program called UPLWSM which finds upper and lower Riemann sums. Use it on the following problems:

- (a) 3 on p. 227 (n = 1, 8, 16, 64)
 (b) 4 on p. 227 (n = 5, 10, 20, 100)
 (c) 6 on p. 227 (n = 4, 8, 64)

[Note that $\sin x$ is increasing on $[0, \pi/2)$ and decreasing on $(\pi/2, \pi]$. $\pi/(2\sqrt{2}) \approx 1.11072$ and $\pi(\sqrt{2} + 1)/(2\sqrt{2}) \approx 2.68151.$]

Hand in your output on the three problems.

3. The lab instructor has written a flow chart and program called FNDTHM which approximates the integral

$$\int_a^b f(t) dt$$

using a numerical procedure called Simpson's Rule. You will have a chance to learn how this procedure works when we reach Section 3 of Chapter 12 (p. 312); however, for now you will have to simply believe that it works. Using it we can obtain a table of values for the function F defined as in Section 4 of Chapter 9 (p. 232) by

$$F(x) = \int_a^x f(t) dt$$

for x in $[a, b]$. Graphing the pairs $(x, F(x))$ should help you to understand the integral (and the Fundamental Theorem of Calculus) better. For each of the following obtain a table of ordered pairs $(x, F(x))$ for f and x as indicated, sketch a graph of $F(x)$, and see if you can find the defining equation for the graph.

- (a) $f(x) = 2x$, $a = 0$, $b = 2$, and IC = .25
 (b) $f(x) = \cos x$, $a = 0$, $b = 6$, and IC = .5

(c) $f(x) = e^x$, $a = -5$, $b = 4$, and $IC = 1$

(d) $f(x) = 1/x$ with $a = 1$, $b = .25$, and $IC = .25$;

then $a = 1$, $b = 6$, and $IC = .5$

Hand in your output and graphs for the four problems with your guess of the defining equation.

Analysis of Exercise Set 5

Problem 1

1. After examining the output of (a) which do you think is the most likely result for a Riemann sum for $f(x)$ when n is bigger than 256 ($n = 512$, for example): 3.14168111, 3.14161111 or 3.14141111?
2. In part (c) explain why the Riemann sums are zero while the area between the curve and x-axis is clearly not zero.

Problem 2

3. In part (a): (i) does the computed answer agree with the text's?
 (ii) using ξ_1' and ξ_1'' as on p. 225 of the text with $N = 1$, what are the values of ξ_1' ? ξ_1'' ?
4. Comparing part (b) with part (b) of Problem 1, how does the Riemann sum compare with the lower and upper Riemann sums when $N = 5$? When $N = 10$?
5. As in 4 compare part (c) with part (d) of Problem 1 when $N = 8$. How do the sums compare?
6. In each part of Problem 2 what happens to the lower sums as N increases? What happens to the upper sums as N increases?

Problem 3

7. After doing this problem what do you think the defining equation for $F(x)$ would be if

$$F(x) = \int_0^x (-\sin t) dt?$$

8. What do you think

$$D_x \left[\int_0^x 2t \, dt \right] = ?$$

APPENDIX F

TEN PRE-SELECTED ITEMS FROM THE CALCULUS

COOPERATIVE MATHEMATICS TEST

(C.C.M.T.)

Five items were selected from each part of the C.C.M.T. (Form A).

These items were:

Part I: Questions 6, 10, 13, 15, and 19

Part II: Questions 3, 7, 21, 24, and 28

APPENDIX G

INDIVIDUAL SCORES OF 53 SUBJECTS PARTICIPATING
IN THE STATISTICAL ANALYSIS

Control Group

A.C.T.M.	Pretest	Post-test	10-Item Pretest	10-Item Post-test
34	5	35	1	7
32	6	41	1	8
29	9	31	2	7
21	7	20	3	5
33	7	33	3	6
31	10	41	2	9
28	3	38	1	6
21	1	27	0	4
32	4	19	1	4
33	7	30	2	7
30	6	30	1	8
34	11	44	3	7
29	9	26	2	5
33	3	29	0	7
34	13	50	3	10
34	10	34	2	6
32	8	47	1	8
32	14	39	4	7
31	3	27	1	7
26	6	23	1	5

Group One-half

A.C.T.M.	Pretest	Post-test	10-Item Pretest	10-Item Post-test
24	3	27	0	7
28	4	21	1	6
25	6	14	1	3
36	8	40	2	7
19	3	23	1	4
28	6	31	2	6
30	3	23	2	4
29	6	37	2	6
28	3	35	0	6
28	4	30	1	5
23	4	20	0	4
30	4	31	1	8

Group One

A.C.T.M.	Pretest	Post-test	10-Item Pretest	10-Item Post-test
26	4	27	2	8
18	2	20	0	6
25	5	38	3	9
24	5	14	0	2
21	2	31	1	7
28	4	30	2	4
26	5	30	2	7
29	4	36	1	9
26	9	30	2	6
27	4	26	1	5
28	4	19	0	5
30	4	30	0	6
29	9	29	2	7
29	5	24	2	8
22	5	19	2	3
26	8	31	4	6
22	6	21	1	4
20	7	36	1	7
30	8	30	0	5
18	5	28	2	6
17	8	25	3	3

VITA ←

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Doctor of Education

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CALCULUS AND AN EXPERIMENT WITH THEIR USE

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