CHARACTERIZATIONS OF CERTAIN CLASSES OF PLANAR DYNAMICAL SYSTEMS

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CHAPTER I

DYNAMICAL SYSTEMS

Introduction

Historically, dynamical systems developed from the qualitative theory of differential equations, and hence, is often considered to be a topic in the theory of differential equations. Some mathematicians prefer to consider dynamical systems as applied topology. Others consider dynamical systems to be an independent mathematical discipline. Due to the development of topological dynamics as an independent mathematical discipline, it seems feasible to consider dynamical systems as a topic of topological dynamics with results applicable to certain classes of differential equations. The author's point of view lies with the latter two concepts but is by no means fixed.

The theory of dynamical systems appears to have evolved from the pioneering studies of the topological properties of solutions of autonomus systems of ordinary differential equations with planar phase spaces by Poincaré and Bendixson at the turn of the century. At about the same time Lyapunov introduced his theory of stability of motions. Birkhoff in his 1927 monograph "Dynamical Systems" was the first to undertake a systematic development of the theory and may well be considered the founder. The definition of an abstract dynamical system has been attributed to the independent works of Markov and Whitney in the early 1930's. In 1947, Nemytskii and Stepanov published their

"Qualitative Theory of Differential Equations" which contains a survey of differential dynamical systems. Much effort in the late 1940's and 1950's was directed toward generalizing the concept of a dynamical system to transformation groups. In 1955, Gottschalk and Hedlund published their book "Topological Dynamics" which surveys this work. The recent developments in dynamical systems are surveyed in four recent monographs. The major feature of Hajek's book, "Dynamical Systems in the Plane," is the extension of the Poincaré-Bendixson theory to dichotomic 2-manifolds. Although Bhatia and Szego give a very adequate survey of recent thought on dynamical systems in "Stability Theory of Dynamical Systems," the treatment is limited to metric spaces. Even though somewhat limited in scope, the notes "Theory of Dynamical Systems" and "Local Semi-Dynamical Systems" by Bhatia and Hajek present the major developments of modern dynamical systems in the most general form.

In [14] Ura introduced his theory of prolongations and pointed out its connection with stability theory. He suggested the importance of studying dynamical systems of characteristic 0^+ which are flows satisfying a certain stability criterion expressed in terms of prolongation. Ahmad classified such planar flows in [1] in terms of their critical points.

In this paper we characterize planar flows of characteristic 0^+ as well as flows satisfying the bilateral concept of characteristic 0. In each case the characterization is given in terms of the set S of critical points and is based on three mutually exclusive and exhaustive cases: $S = \emptyset$, S nonempty and compact, and S noncompact. Examples of dynamical systems of characteristic 0^+ satisfying all of the properties obtained by Ahmad in [1] for the noncompact case are given including one example which nontrivially satisfies every property. Examples of the nontrivial flows of characteristic 0 are given. Finally, as a consequence of the characterization theorem for flows of characteristic 0, we extend a well-known result from differential planar flows to arbitrary planar flows.

This study essentially completes the planar versions of the characteristic 0^+, 0^-, $0^\pm$, and 0 problems.

Basic Definitions and Notations

We shall denote the real numbers, nonnegative real numbers, nonpositive real numbers, and Euclidean plane with the usual topology by R, R^+, R^-, and R^2, respectively.

Definition 1.1: A pair (X,π) consisting of a topological space X, called the phase space, and a continuous mapping $\pi\colon X \times R \to X$ from the product space $X \times R$ into X is called a dynamical system or (continuous) flow whenever the following conditions are satisfied.

(1) Identity axiom: $\pi(x,0) = x$ for each $x \in X$.

(2) Homomorphism axiom: $\pi(\pi(x,t),s) = \pi(x,t+s)$ for each $x \in X$ and $t,s \in R$.

(3) Continuity axiom: π is continuous on $X \times R$.

Throughout this paper the phase space X of a dynamical system (X,π) will be Hausdorff. We shall denote $\pi(x,t)$ by xt for brevity.

Unless otherwise specified, we shall let (X,π) denote a fixed but arbitrary flow throughout this chapter. When we refer to a point or a set without mention of the location, then they are assumed to be in X.

Definition 1.2: For each x, $C(x) = xR = \{xt: t \in R\}$, $C^+(x) = xR^+ = \{xt: t \in R^+\}$, and $C^-(x) = xR^- = \{xt: t \in R^-\}$ are called the trajectory (or orbit), positive semi-trajectory, and negative semi-trajectory through x, respectively.

Let $F: X \to P(X)$ be a function from X into the power set $P(X)$ of X. We denote $\bigcup\{F(x): x \in M\}$ by $F(M)$ for any set M. When $F(M)$ is a singleton $\{x\}$, we write $F(M) = x$. If M is a singleton $\{x\}$, we write $F(x)$.

Definition 1.3: A point x is called a critical or rest point if $C(x) = x$. If $C(x) \neq x$ but $xt = x$ for some $t > 0$ then x is called periodic.

Proposition 1.4: The set of all critical points in X is closed. (See [7], I, p.14 and [8], p.17.)

Remark: In each of the remaining sections of this chapter, we shall state definitions, propositions, and theorems which are basic to the development of the succeeding chapters. Since these results are all well-known, we shall not prove any of them; however, we shall give a reference for each. Several results are simple exercises, but they are included for completeness.

Almost every definition, proposition, and theorem in this chapter has a positive, negative, and bilateral version. Since the positive and negative versions are duals, it is customary to state only the positive versions of results. In this chapter, we shall state the positive versions except in definitions and shall usually note when bilateral versions hold.

Invariance

Definition 1.5: A set M is called invariant if $C(M) = M$ and positively (negatively) invariant if $C^+(M) = M$ $(C^-(M) = M)$.

Proposition 1.6: A set M is positively (bilaterally) invariant if and only if $X - M$ is negatively (bilaterally) invariant. Furthermore, M is positively (bilaterally) invariant if and only if each of its components is positively (bilaterally) invariant. (See [7], I, pp.26-27 and [8], p.13.)

Proposition 1.7: If $\{M_i: i \in I\}$ is a family of positively (bilaterally) invariant sets, then $\bigcup\{M_i: i \in I\}$ and $\bigcap\{M_i: i \in I\}$ are positively (bilaterally) invariant. (See [7], I, p.26 and [8], p.12.)

We shall denote the boundary, interior, and closure of a set M by ∂M, M^o, and $\overline{M}$, respectively.

Proposition 1.8: If a set M is positively (bilaterally) invariant, then $\overline{M}$ and M^o are positively (bilaterally) invariant. (See [7], I, p.27 and [8], p.13.)

Proposition 1.9: If a set M is invariant, then ∂M is invariant. The converse holds if M is open or closed. (See [7], I, p.28 and [8], p.13.)

For any simple closed curve C in R^2 we shall denote the bounded and unbounded components of $R^2 - C$ by $\text{int } C$ and $\text{ext } C$, respectively.

Theorem 1.10: Let $X = R^n$ be Euclidean n space and let M be a positively (bilaterally) invariant set homeomorphic to the closed unit ball in R^n. Then M contains a critical point. (See [7], I, p.30.)

Corollary 1.11: Let $X = R^2$ and x be a periodic point. Then int C(x) contains a critical point. (See [11], p.175.)

Theorem 1.12: Let X' be an invariant subset of X and $\pi' = \pi|X'$ (the restriction of π to X'). Then (X', π') is a dynamical system. (See [7], I, p.32.)

Limit Sets

For each x we let $K(x) = \overline{C(x)}$, $K^+(x) = \overline{C^+(x)}$, and $K^-(x) = \overline{C^-(x)}$.

Proposition 1.13: For each x,

(1) $K^+(x)$ is closed and positively invariant;

(2) $K^+(xt) \subset K^+(x)$ for each $t \in R^+$, and

(3) $K(x) = K^+(x) \cup K^-(x)$.

The bilateral versions hold for (1) and (2). (See [6], p.50; [7], I, p.30; and [8], pp.22-23.)

Definition 1.14: The positive (negative) limit set of x is

$$L^+(x) = \{y: xt_i \to y \text{ for some net } t_i \to +\infty\}$$

$$(L^-(x) = \{y: xt_i \to y \text{ for some net } t_i \to -\infty\}).$$

The limit set of x is $L(x) = L^+(x) \cup L^-(x)$.

Proposition 1.15: For each x,

(1) $L^+(x) = \bigcap \{K^+(xt): t \in R\}$,

(2) $L^+(x)$ is closed and invariant,

(3) $L^+(x) = L^+(xt)$ for each $t \in R$,

(4) $K^+(x) = C^+(x) \bigcup L^+(x)$,

(5) $K(x) = L^+(x) \bigcup C(x) \bigcup L^-(x)$, and

(6) $K(x) = K(xt)$ for each $t \in R$.

The bilateral versions hold for (1) through (4). (See [6], p.50; [7], I, p.35; and [8], pp.22-23,58.)

We shall let $\eta(x)$ and $\eta(M)$ denote the neighborhood filters of the point x and set M, respectively.

Definition 1.16: A space Y is called rim-compact if for each $y \in Y$ and $V \in \eta(y)$ there is a $U \in \eta(y)$ such that $U \subset V$ and ∂U is compact.

Proposition 1.17: If X is rim-compact, then for any point x, $K^+(x)$ is compact if and only if $L^+(x)$ is nonempty and compact. (See [6], p.54 and [7], I, p.36.)

Theorem 1.18: Let X be a subspace of R^2. Then, for any point x, $x \in L^+(x)$ if and only if x is either periodic or critical. (See [13].)

Proposition 1.19: If x is a point of $X = R^2$ and $L^+(x) \neq \emptyset$, then either $L^+(x)$ is a periodic trajectory or $L^+(y)$ and $L^-(y)$ consist of critical points for each $y \in L^+(x)$. (See [11], p.184.)

Theorem 1.20: Let X be locally compact and $X^* = X \cup \{\infty\}$ be the one point compactification of X. Then there is a uniquely determined dynamical system (X^*, π^*) such that $\pi = \pi^* | X$; furthermore, ∞ is a critical point. (See [7], I, p.16.)

We shall refer to (X^*, π^*) as the extended flow. For each point $x \in X^*$, we let $K^{*+}(x)$ and $L^{*+}(x)$ denote $\overline{C^+(x)}$ and the positive limit set relative to the extended flow, respectively. Similar notation will be used in the next section for prolongation sets and prolongational limit sets.

Proposition 1.21: Let X be locally compact. For each $x \in X$,

$$K^{*+}(x) = \begin{cases} K^+(x) & \text{if } K^+(x) \text{ is compact} \\ K^+(x) \cup \{\infty\} & \text{if } K^+(x) \text{ is not compact} \end{cases}$$

and

$$L^{*+}(x) = \begin{cases} L^+(x) & \text{if } L^+(x) \text{ is compact} \\ L^+(x) \cup \{\infty\} & \text{if } L^+(x) \text{ is not compact} \end{cases}$$

Furthermore, the bilateral versions of these statements hold. (See [7], I, p.36.)

Prolongation

Definition 1.22: For each point x, the positive (negative) prolongation of x is

$$D^+(x) = \{y \in X\colon x_i t_i \to y \text{ for some nets } x_i \to x \text{ and } t_i \geq 0\}$$

$$(D^-(x) = \{y \in X\colon x_i t_i \to y \text{ for some nets } x_i \to x \text{ and } t_i \leq 0\}).$$

The prolongation of x is $D(x) = D^+(x) \cup D^-(x)$.

<u>Proposition 1.23</u>: For each x,

(1) $D^+(x)$ is closed and positively invariant,

(2) $D^+(xt) \subset D^+(x)$ for each $t \in R^+$,

(3) $K^+(x) \subset D^+(x)$, and

(4) $D^+(x) = \bigcap \{\overline{C^+(M)}: M \in \eta(x)\}$.

The bilateral versions of these statements hold. (See [6], p.60; [7], I, pp.42-44; and [8], p.26.)

<u>Definition 1.24</u>: For each point x, the positive (negative) prolongational limit set of x is

$$J^+(x) = \{y \in X: x_i t_i \to y \text{ for some nets } x_i \to x \text{ and } t_i \to +\infty\}$$

$$(J^-(x) = \{y \in X: x_i t_i \to y \text{ for some nets } x_i \to x \text{ and } t_i \to -\infty\}).$$

The prolongational limit set of x is $J(x) = J^+(x) \cup J^-(x)$.

<u>Proposition 1.25</u>: For each x,

(1) $J^+(x)$ is closed and invariant,

(2) $J^+(xt) = J^+(x)$ for each $t \in R$,

(3) $L^+(x) \subset J^+(x)$,

(4) $J^+(x) = \bigcap \{D^+(xt): t \in R\}$, and

(5) $D^+(x) = C^+(x) \cup J^+(x)$.

The bilateral versions of these statements hold. (See [6], p.60; [7], I, pp.44-45; and [8], pp. 26,58.)

<u>Proposition 1.26</u>: If $y \in K(x)$, then $J^+(x) \subset J^+(y)$. (See [6], p.72 and [7], I, p.51.)

<u>Proposition 1.27</u>: If $y,z \in L^+(x)$, then $y \in J^+(z)$. (See [6], p.71.)

<u>Proposition 1.28</u>: $y \in D^+(x)$ if and only if $x \in D^-(y)$. Moreover, $y \in J^+(x)$ if and only if $x \in J^-(y)$. (See [7], I, p.46; [8], p.29; and [14], p.127.)

<u>Theorem 1.29</u>: Let X be locally compact and $x \in X$. If $D^+(x)$ is compact, then it is connected. Furthermore, if $D^+(x)$ is not connected, then it has no compact components. The bilateral versions of these statements hold. (See [7], I, pp.46-48 and [8], pp.26-27.)

<u>Proposition 1.30</u>: Let X be locally compact. Then

$$D^{*+}(x) = \begin{cases} D^+(x) & \text{if } D^+(x) \text{ is compact} \\ D^+(x) \bigcup \{\infty\} & \text{if } D^+(x) \text{ is not compact} \end{cases}$$

and

$$J^{*+}(x) = \begin{cases} J^+(x) & \text{if } J^+(x) \text{ is compact} \\ J^+(x) \bigcup \{\infty\} & \text{if } J^+(x) \text{ is not compact.} \end{cases}$$

The bilateral versions of these statements hold. (See [7], I, pp.45-46.)

<u>Definition 1.31</u>: A set M is called positively d-invariant or d^+-invariant if $D^+(M) = M$. The negative and bilateral versions are defined similarly.

<u>Proposition 1.32</u>: Let $\{M_i : i \in I\}$ be a family of d^+-invariant (d-invariant) sets. Then $\bigcup\{M_i : i \in I\}$ and $\bigcap\{M_i : i \in I\}$ are d^+-invariant (d-invariant). (See [7], II, p.3.)

<u>Proposition 1.33</u>: If X is locally compact, then a compact subset is d^+-invariant (d-invariant) if and only if each of its components is such. (See [7], II, p.4.)

Proposition 1.34: A set M is d^+-invariant (d-invariant) if and only if $X - M$ is d^--invariant (d-invariant). (See [7], II, p.4.)

Dispersion and Parallelizability

Definition 1.35: A point x is called dispersive if $J^+(x) = \emptyset$. The flow (X,π) is called dispersive if each of its points is dispersive.

Theorem 1.36: The flow (X,π) is dispersive if and only if $D^+(x) = C^+(x)$ for each $x \in X$ and there are no periodic or critical points. (See [7], I, p.79 and [8], pp.44,47.)

Definition 1.37: Two flows (X,π) and (X',π') are dynamically isomorphic if and only if there exists a homeomorphism $f: X \to X'$ such that $f(xt) = f(x)t$ for each $x \in X$ and $t \in R$. Let $g: X \to Y$ be a homeomorphism from X to Y. We call (Y,π'') where $\pi''(y,t) = g(\pi(g^{-1}(y),t))$ for each $y \in Y$ and $t \in R$ the flow induced on Y from (X,π) by g.

Definition 1.38: A dynamical system $(Y \times R,\pi')$ is called parallel if and only if $(y,s)t = (y,s + t)$ for each $y \in Y$ and $s,t \in R$.

Definition 1.39: The flow (X,π) is called parallelizable if and only if it is dynamically isomorphic to a parallel flow.

Theorem 1.40: If X is a locally compact separable metric space, then (X,π) is parallelizable if and only if it is dispersive. (See [3], p.548 and [5], p.91.)

Stability

<u>Definition 1.41</u>: A set M is called positively stable if for every $U \varepsilon \eta(M)$ there exists a $V \varepsilon \eta(M)$ such that $C^+(V) \subset U$ (or equivalently, such that $V \subset U$ and $C^+(V) = V$). The negative and bilateral versions are defined in the obvious manner.

It is customary to drop the adjective "positive" but never the adjectives "negative" or "bilateral" when referring to stability. We shall adopt this procedure. We shall refer to a point x as (negatively, bilaterally) stable if $\{x\}$ is such.

The bilateral versions of the next seven results hold. This fact is stated in Theorem 1.46 for emphasis.

<u>Proposition 1.42</u>: Every open positively invariant set is stable. (See [7], II, p.6.)

<u>Proposition 1.43</u>: The union of stable sets is stable. (See [7], II, p.6.)

<u>Proposition 1.44</u>: If M is stable, then M is positively invariant. (See [7], II, p.6 and [8], p.60.)

<u>Proposition 1.45</u>: Let X be regular and M be a closed set. If M is stable then it is d^+-invariant. (See [7], II, p.8.)

<u>Theorem 1.46</u>: Let X be locally compact and ∂M be compact. Then M is stable (negatively stable) if and only if $D^+(M) = M$ $(D^-(M) = M)$. Furthermore, M is bilaterally stable if and only if $D(M) = M$. (See [6], p.77; [7], II, p.8; and [14], p.127.)

Proposition 1.47: If X is locally compact, then a compact set M is stable if and only if each of its components is such. (See [7], II, p.10 and [8], p.62.)

Proposition 1.48: If each component of a set M is a stable set, then so is M. Conversely, if M is compact and stable, then so is each of its components. (See [7], II, p.13.)

Attraction and Asymptotic Stability

Unless otherwise specified, we shall consider R to be a directed set directed by the usual order. Thus, for each x, the mapping π with x fixed is a net on R which we shall denote by (xt). The statement "(xt) is ultimately in V" means that there exists a $T \in R$ such that $xt \in V$ for $t \geq T$. Similarly, the negative net is defined relative to the reverse of the usual order on R.

The bilateral versions of the results which follow are valid through Proposition 1.56.

Definition 1.49: For any set M and point x we say x is positively attracted to M if and only if for any $V \in \eta(M)$ there exists a $T(V) \in R^+$ such that $xt \in V$ whenever $t \geq T(V)$. The negative version is defined relative to the negative net (xt) and the bilateral version is the conjunction of the positive and negative versions.

We shall drop the adjective "positive" as we did for stability. If $M = \{p\}$, then we say x is attracted to p rather than $\{p\}$.

Definition 1.50: If M is a set, then $A^+(M) = \{x \in X: x$ is attracted to $M\}$. The negative and bilateral versions are defined similarly and denoted by $A^-(M)$ and $A(M)$, respectively.

Proposition 1.51: If x is a point and M_1 and M_2 are sets, then

(1) $x \in A^+(M_1)$ if and only if $C(x) \subset A^+(M_1)$,

(2) $M_1 \subset M_2$ implies $A^+(M_1) \subset A^+(M_2)$,

(3) $C(M_1) = M_1$ implies $M_1 \subset A^+(M_1)$, and

(4) $A^+(M_1) \cup A^+(M_2) \subset A^+(M_1 \cup M_2)$.

(See [7], II, pp.17,25,26 and [6], p.81.)

Theorem 1.52: Let M be a closed set with ∂M compact. If $x \in A^+(M)$, then either (xt) is ultimately in M or $\emptyset \neq L^+(x) \subset M$. The converse holds whenever X is rim-compact. (See [6], p.84.)

Definition 1.53: A set M is called an attractor whenever $A^+(M) \in \eta(M)$. The term global attractor is used in case $A^+(M) = X$. The negative and bilateral versions are defined similarly.

Proposition 1.54: A set M is an attractor if and only if $A^+(M)$ is the smallest open invariant set containing M. (See [6], p.81 and [7], II, pp.27-28.)

Proposition 1.55: Let M be an attractor. Then every set M_1 with $M \subset M_1 \subset A^+(M)$ is also an attractor and $A^+(M_1) = A^+(M)$. (See [7], II, p.27.)

Proposition 1.56: For any set M, $J^+(A^+(M)) \subset J^+(M)$. (See [6], p.90 and [7], II, p.32.)

Definition 1.57: A set M is called asymptotically stable if and only if it is a stable attractor. The negative and bilateral versions are defined similarly.

Theorem 1.58: If $X = R^2$ and x is a periodic point of X with G either of the components of $X - C(x)$, then either

(1) relative to $\overline{G}$, $C(x)$ is asymptotically stable or negatively asymptotically stable, and there exists a $V \varepsilon \eta(C(x))$ such that $V \cap G$ contains no periodic points; or

(2) $C(x)$ is bilaterally stable relative to $\overline{G}$, and for each $V \varepsilon \eta(C(x))$ there exists a periodic trajectory in $V \cap G$. (See [11], p.196.)

Theorem 1.59: If $X = R^2$ and x is an isolated stable critical point, then x satisfies precisely one of the following conditions.

(1) x is a focus ($x \varepsilon L^+(y)$ for some point $y \neq x$).

(2) x is a Poincaré center (there is a neighborhood of periodic points surrounding x).

(3) x is a center focus (every neighborhood of x contains both periodic and nonperiodic trajectories).

(See [11], p.198.)

Flows of Characteristic 0^+, 0^-, $0^{\pm}$, and 0

Definition 1.60: The flow (X,π) is said to have characteristic 0^+ (0^-) if and only if $D^+(x) = K^+(x)$ $(D^-(x) = K^-(x))$ for each $x \varepsilon X$. If the flow has both characteristic 0^+ and 0^-, then it is said to have characteristic $0^{\pm}$. The flow has characteristic 0 if and only if $D(x) = K(x)$ for each $x \varepsilon X$.

Theorem 1.61: The flow (X,π) has characteristic 0^+ if and only if $J^+(x) = L^+(x)$ for each $x \in X$. The negative and bilateral versions hold. (See [6], p.138.)

As we have already noted in the first section, the concepts of characteristics 0^+, 0^-, and $0^\pm$ were introduced by Ahmad in [1] where he classified such planar flows and characterized planar flows of characteristic $0^\pm$.

In [1] Ahmad defined a point x to be attracted to a closed invariant set M if $\emptyset \neq L^+(x) \subset M$. This is not equivalent to Definition 1.49. However, under suitable conditions the definitions are equivalent. Since we shall use the results of [1] in this paper, we show by means of the following proposition that the definitions are equivalent for the sets and spaces which we study.

Proposition 1.62: Let X be regular and let M be a closed stable set with $\emptyset \neq L^+(x)$ for each point x in M. Then, $y \in A^+(M)$ if and only if $\emptyset \neq L^+(y) \subset M$. The negative and bilateral versions hold.

Proof: Let $y \in X$ such that $\emptyset \neq L^+(y) \subset M$ and let $U \in \eta(M)$. Then there exists a $V \in \eta(M)$ such that $C^+(V) = V \subset U$. Since $L^+(y) \subset V$, the net (yt) must eventually be in V. The invariance of V implies (yt) is ultimately in U.

Conversely, let $y \in A^+(M)$. By virtue of the regularity of X, there exist disjoint neighborhoods $U \in \eta(M)$ and $V \in \eta(z)$ for any $z \notin M$. The stability of M implies that (yt) is ultimately in U, so that $z \notin L^+(y)$. Hence, $\emptyset \neq L^+(y) \subset M$.

The negative and bilateral versions follow similarly. The proof is complete.

The definition of attraction in [1] is applied to sets and spaces satisfying the criterion of Proposition 1.62. Consequently, the results obtained in [1] are valid with respect to Definition 1.49.

The following theorem is a compilation of the results obtained in [1] for planar flows of characteristic 0^+.

<u>Theorem 1.63</u>: Let (R^2,π) be of characteristic 0^+ and S be the set of critical points. Then one of the following results.

(1) $S = \emptyset$ and the flow is parallelizable.

(2) S is compact and one of the following holds.

(a) S consists of a global Poincaré center.

(b) S consists of a local Poincaré center s. The set N consisting of s and the periodic orbits surrounding s is a globally asymptotically stable simply connected continuum.

(c) S is a globally asymptotically stable simply connected continuum.

(3) S is unbounded and $S = R^2$.

(4) S is unbounded and the following hold.

(a) $R^2 - S$ is unbounded.

(b) S is asymptotically stable.

(c) $A^+(S)$ has a countable number of components each being homeomorphic to R^2 and unbounded.

(d) $A^+(S_o)$ is a component of $A^+(S)$ if and only if S_o is a component of S.

(e) For each component S_o of S, $\partial A^+(S_o)$ consists of a countable number of trajectories $C(x)$ such that $L^{\pm}(x) = \emptyset$.

(f) S has a countable number of components, each being noncompact and simply connected.

(g) For each $s \in \partial S$ there is a nonperiodic noncritical point y with $L^+(y) = \{s\}$.

(h) For each $x \in R^2$, $L^+(x)$ is either empty or consists of a single rest point. Further, $L^+(x) = \emptyset$ for all $x \in R^2 - A^+(S)$ and $L^-(x) = \emptyset$ for all $x \in R^2 - S$.

Corollary 1.64: Under the conditions of Theorem 1.63 part (4)(e) we have $\partial A^+(S_o)$ dispersive.

Proof: By Theorem 1.61, $J^+(x) = L^+(x) = \emptyset$ for each $x \in \partial A^+(S_o)$.

The following theorem based on Theorem 5.1 of [1] characterizes planar flows of characteristic $0^{\pm}$.

Theorem 1.65: A flow (R^2, π) has characteristic $0^{\pm}$ if and only if one of the following holds where S is the set of critical points.

(1) $S = \emptyset$ and the flow is parallelizable.

(2) $S = R^2$.

(3) S consists of a global Poincaré center.

CHAPTER II

DYNAMICAL SYSTEMS OF CHARACTERISTIC 0^+

Characterization of Planar Flows of Characteristic 0^+

Throughout this chapter we shall denote the set of rest points for a given flow (R^2,π) by S.

The following proposition is given in order to sharpen Theorem 2.2 which completely characterizes dynamical systems of characteristic 0^+ on the plane. Dual results hold for flows of characteristic 0^-.

Proposition 2.1: Let (R^2,π) be a flow and $S = \{s_o\}$ where s_o is a local Poincaré center. If the set $N = \{x \in R^2: x$ is periodic or critical$\}$ is a connected global attractor, then $N = \overline{\text{int } C(x)}$ for some point x in N and N is globally asymptotically stable.

Proof: Suppose that for some point p of N we have $\text{int } C(p) \not\subset N$. If $y \in (\text{int } C(p)) - N$, then $C(y) \subset \text{int } C(p)$. There is a point z in $N - S$ such that $y \in (\text{int } C(p)) - \text{int } C(z)$ since s_o is a Poincaré center. Now $L^+(y) \subset \overline{\text{int } C(p)} - \text{int } C(z)$ and $s_o \in \text{int } C(z)$ (see Corollary 1.11), so that $L^+(y) \neq \emptyset$ and $L^+(y) \cap S = \emptyset$. Thus, $L^+(y)$ is a periodic orbit (see Proposition 1.19). The component G of $R^2 - L^+(y)$ to which y belongs does not contain periodic points near $L^+(y)$ (see Theorem 1.58). Thus, if $G = \text{int } L^+(y)$, then there is a simple closed curve C such that $\overline{G} \subset \text{int } C$ and

$(\overline{\text{int } C} - \overline{G}) \cap N = \emptyset$. Since the connected set N meets int C and ext C it meets C which is absurd. Similarly, $G = \text{ext } L^+(y)$ leads to a contradiction. Hence, int $C(p) \subset N$.

Next, we show that N is compact. Suppose that $N \cap \partial N = \emptyset$. Then N is an open invariant set, and hence, $N = A^+(N) = R^2$ (see Proposition 1.54) which is a contradiction. Hence, $N \cap \partial N \neq \emptyset$. Let $x \in N \cap \partial N$. If $y \in N - \text{int } C(x)$, then int $C(x) \subset$ int $C(y)$. Since int $C(y) \subset N^o$ and $C(x) \subset \partial N$ we have $C(x) = C(y)$. Hence, $N = \overline{\text{int } C(x)}$.

Finally, N is stable (see Theorem 1.58) and $A^+(N) = R^2$, so that it is globally asymptotically stable. The proof is complete.

In case (*i*) of the following theorem we characterize flows of characteristic 0^+ which have a compact set of rest points. Parts (a), (b), and (c) of case (*i*) are mutually exclusive and exhaustive. Properties (*i*)(a) and (*i*)(b) characterize those flows where S is empty (we regard $\emptyset$ as compact) and S consists of exactly one Poincaré center, respectively. All other flows are characterized by property (*i*)(c).

<u>Theorem 2.2</u>: A flow (R^2,π) is of characteristic 0^+ if and only if either case (*i*) or (*ii*) is satisfied.

(*i*) S is compact and one of the following holds.

(a) (R^2,π) is parallelizable.

(b) $S = \{s_o\}$ where s_o is either a global Poincaré center or a local Poincaré center such that the set N consisting of s_o and all the periodic points is a connected global attractor.

(c) $\{s\}$ is stable for each $s \in \partial S$ and S is a global attractor.

(*ii*) S is not compact and each of the following holds.

(a) $\{s\}$ is stable for each $s \in \partial S$.

(b) Each $x \in \partial A^+(S)$ is dispersive.

(c) The flow restricted to $R^2 - \overline{A^+(S)}$ is parallelizable.

Proof: By virtue of Theorem 1.63, the properties in case (*i*) are necessary. Parallelizability of (R^2,π) requires that $D^+(x) = K^+(x)$ for each $x \in R^2$ in view of Theorems 1.36 and 1.40 since $K^+(x) \subset D^+(x) = C^+(x)$; therefore, property (*i*)(a) is also sufficient. Next, if property (*i*)(b) holds, then Theorem 1.58 and Proposition 2.1 infer that every periodic or critical orbit $C(x)$ is stable, and hence, $D^+(x) = D^+(C(x)) = C(x) = K^+(x)$ (see Proposition 1.23 and Theorem 1.46). Furthermore, Proposition 2.1 implies that $L^+(x) = \partial N$ for each $x \in R^2 - N$ where N is the set of periodic and critical points. Also $J^+(x) \subset D^+(N) = N$ (see Proposition 1.56 and Theorem 1.46) and $J^+(x) \cap N^o = \emptyset$, so that we must have $J^+(x) = L^+(x)$. Hence, property (*i*)(b) is sufficient (see Theorem 1.61). Finally, suppose that (*i*)(c) holds. Then $D^+(s) = K^+(s)$ for any point s in S since $\{s\}$ is stable (see Theorem 1.46). For any point x in $A^+(S) - S$ we have $\emptyset \neq L^+(x) \subset S$ by virtue of Proposition 1.62. If s, $s_1 \in L^+(x)$, then $s_1 \in D^+(s) = \{s\}$ (see Proposition 1.27). Thus, $\{s\} = L^+(x) \subset J^+(x) \subset J^+(s) = L^+(s) = \{s\}$ (see Proposition 1.26), and hence, $L^+(x) = J^+(x)$. The proof of case (*i*) is now complete.

The necessity of case (*ii*) follows from Theorem 1.63 and Corollary 1.64, and so we assume that (*ii*) holds. For any point s in S, $D^+(s) = \{s\} = K^+(s)$. No point of S is in $\partial A^+(S)$ since $J^+(s) \neq \emptyset$

for each point s in S. Thus, S is an attractor. By an argument similar to the one used to prove the sufficiency of property (i)(c) we obtain $L^+(x) = J^+(x)$ for each point x in $A^+(S) - S$. For any point x in $\partial A^+(S)$ we have $J^+(x) = \emptyset$, so that $L^+(x) = J^+(x)$. The parallelizability of the flow restricted to $R^2 - \overline{A^+(S)}$ implies that, relative to $R^2 - \overline{A^+(S)}$, $D^+(x) = K^+(x)$ for each point x in $R^2 - \overline{A^+(S)}$ (see Theorems 1.36 and 1.40). Since $R^2 - \overline{A^+(S)}$ is open we also have $D^+(x) = K^+(x)$. The proof of the theorem is complete.

The following four examples show that all conditions given in properties (i)(b) and (i)(c) are needed. Since (R^2,π) is parallelizable if and only if it is dispersive (see Theorem 1.40) property (i)(a) cannot be weakened.

Example 2.3: The flow defined by the system of differential equations

$$\dot{r} = \begin{cases} 0 & \text{for } 0 \leq r \leq 1 \\ (r - 1) \ln (r - 1) & \text{for } 1 < r < 2 \\ 2r \ln \frac{2}{r} & \text{for } 2 \leq r \end{cases}$$

$$\dot{\theta} = 1 \quad \text{for } r \geq 0$$

(see Figure 2.1) shows that the connectedness of N in property (i)(b) is necessary. Note that $A^+(N) = R^2$ but, since $J^+((2,0)) \neq L^+((2,0))$, the flow is not of characteristic 0^+.

Example 2.4: The system of differential equations

$$\dot{r} = -r^2 \sin \theta$$

$$\dot{\theta} = 1$$

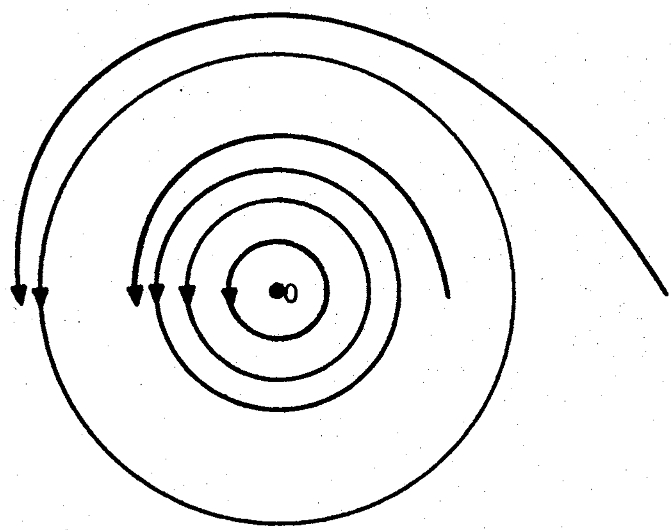

Figure 2.1

for $r \geq 0$ defines a flow for which N is connected but $A^+(N) \neq R^2$ (see Figure 2.2). Since $J^+(x) = C(x)$ and $L^+(x) = \emptyset$ for any point x in ∂N, the flow is not of characteristic 0^+.

Example 2.5: The dynamical system defined by

$$\dot{r} = \begin{cases} (r - 1) \ln (1 - r) & \text{for } 0 \leq r < 1 \\ -r \ln r & \text{for } 1 \leq r \end{cases}$$

$$\dot{\theta} = 0 \quad \text{for } r \geq 0$$

(see Figure 2.3) does not satisfy the stability condition of property (*i*)(c) since $\{(0,0)\}$ is not stable. However, the set of rest points is a global attractor. Note that $J^+((0,0)) \neq L^+((0,0))$ which implies that the flow is not of characteristic 0^+.

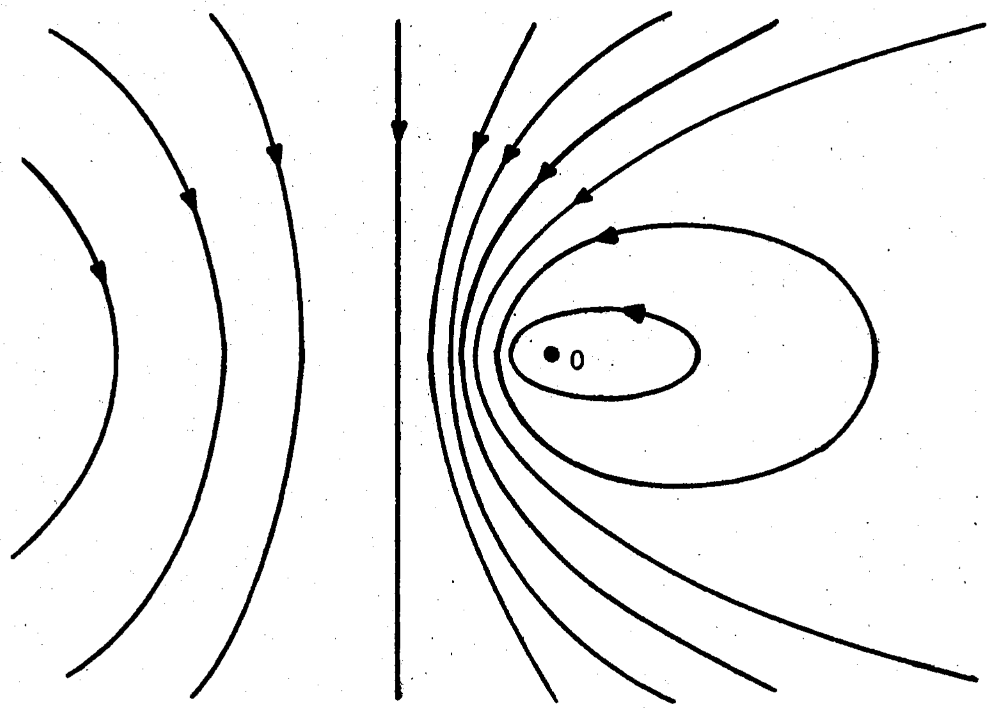

Figure 2.2

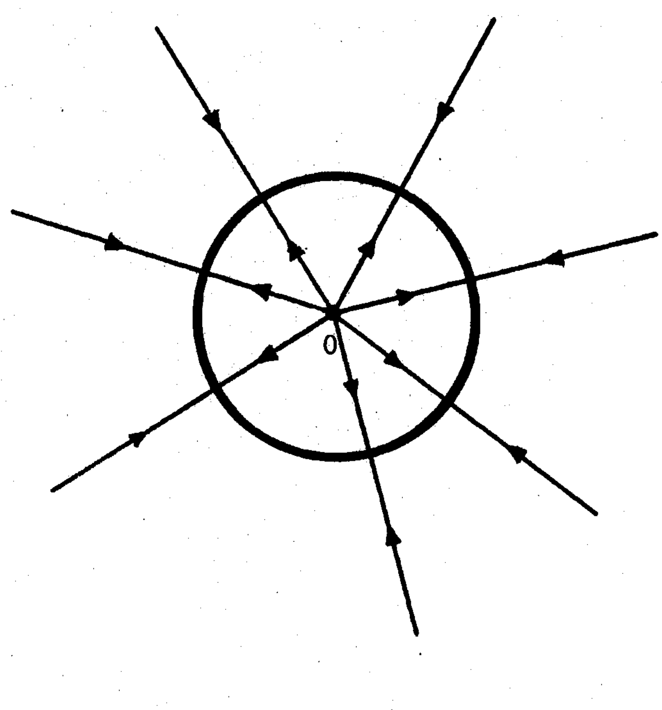

Figure 2.3

Example 2.6: For the flow (R^2,π) defined by the system of differential equations

$$\dot{r} = \begin{cases} (1 - r) \ln (1 - r) & \text{for } 0 \leq r < 1 \\ -r \ln r & \text{for } 1 \leq r \end{cases}$$

$$\dot{\theta} = 1 \quad \text{for } r \geq 0$$

(see Figure 2.4) each point of ∂S is stable but $A^+(S) \neq R^2$ and $J^+((1,0)) \neq L^+((1,0))$, whence global attraction is necessary in property (*i*)(c).

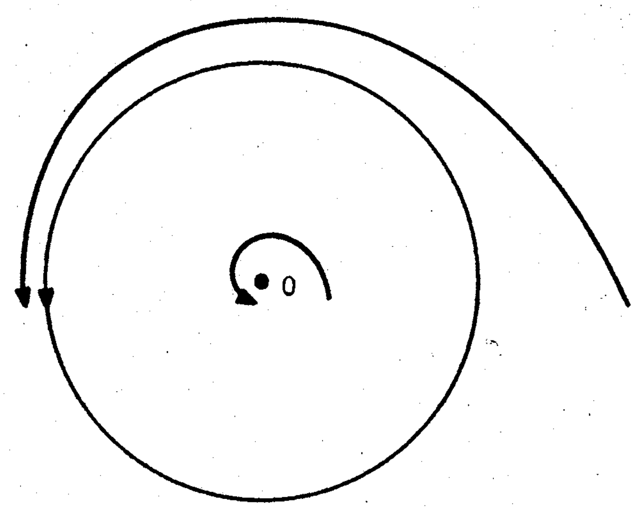

Figure 2.4

The fact that we cannot weaken the noncompact case for S in Theorem 2.2 follows by virtue of the next three examples.

<u>Example 2.7</u>: The flow (R^2,π) defined by the system of differential equations

$$\dot{r} = \begin{cases} r(1 - r) & \text{for } 0 \leq r \leq 1 \\ 0 & \text{for } 1 < r \end{cases}$$

$$\dot{\theta} = 0 \quad \text{for } r \geq 0$$

(see Figure 2.5) satisfies properties (*ii*)(b) and (*ii*)(c) vacuously since S is a global attractor. However, the point (0,0) in ∂S is not stable. Since $J^+((0,0)) \neq L^+((0,0))$, the flow is not of characteristic 0^+.

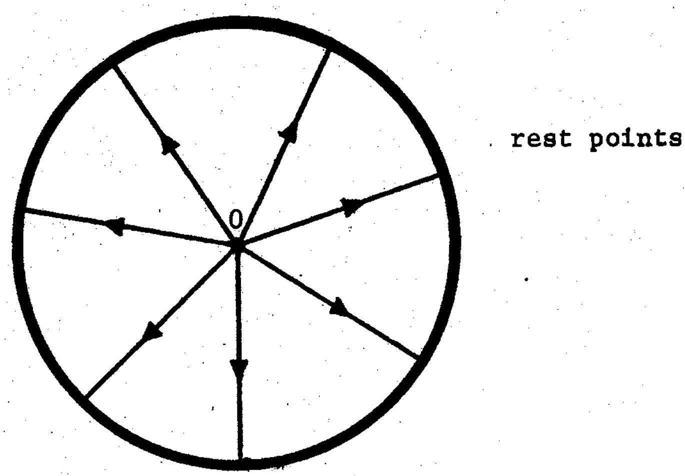

Figure 2.5

Example 2.8: Define a flow (R^2,π) by the system of differential equations

$$\dot{x} = \begin{cases} x & \text{for } x^2y^2 \geq 1 \text{ and } y > 0 \\ 2x^3y^2 - x & \text{for } x^2y^2 < 1 \text{ and } y > 0 \\ -x & \text{for } y \leq 0 \end{cases}$$

$$\dot{y} = \begin{cases} -y & \text{for } y > 0 \\ 0 & \text{for } y \leq 0 \end{cases}$$

(see Figure 2.6 and [4], p.118). Each point of S is stable and, since $J^+(x) = \emptyset$ for each point x in $R^2 - \overline{A^+(S)}$, the flow restricted to $R^2 - \overline{A^+(S)}$ is parallelizable. Thus, properties (*ii*)(a) and (*ii*)(c) are satisfied. However, for any point p in $\partial A^+(S) = \{(x,y): x^2y^2 = 1$ and $y > 0\}$ we have $(0,0) \in J^+(p)$ which implies that p is not

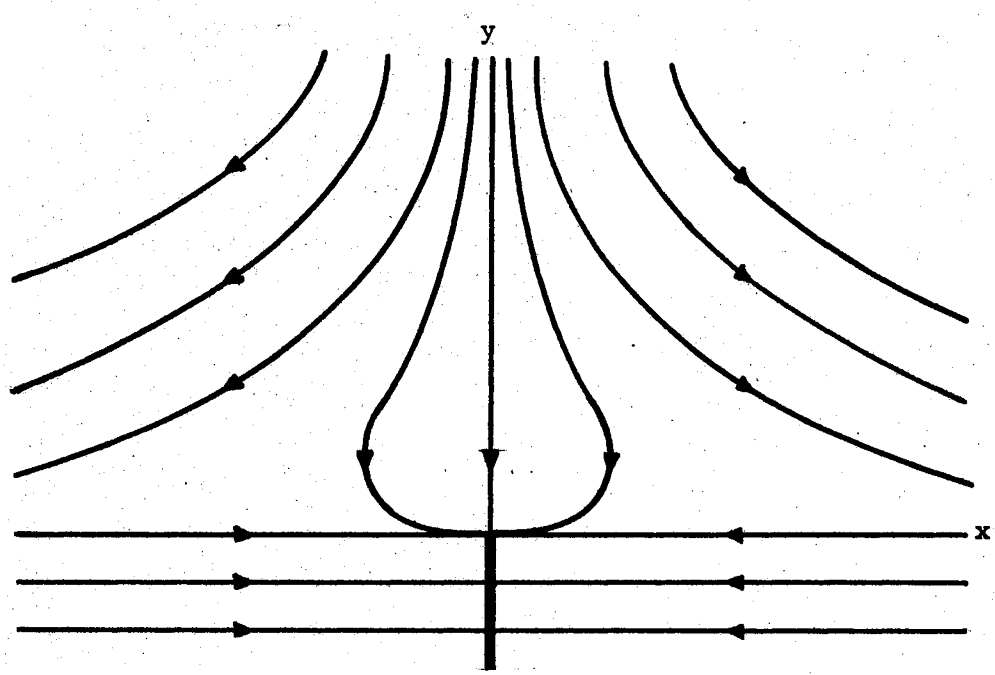

Figure 2.6

dispersive, and hence, that property (*ii*)(b) is not satisfied. Note that $J^+(p) \neq \emptyset = L^+(p)$, and so the flow is not of characteristic 0^+.

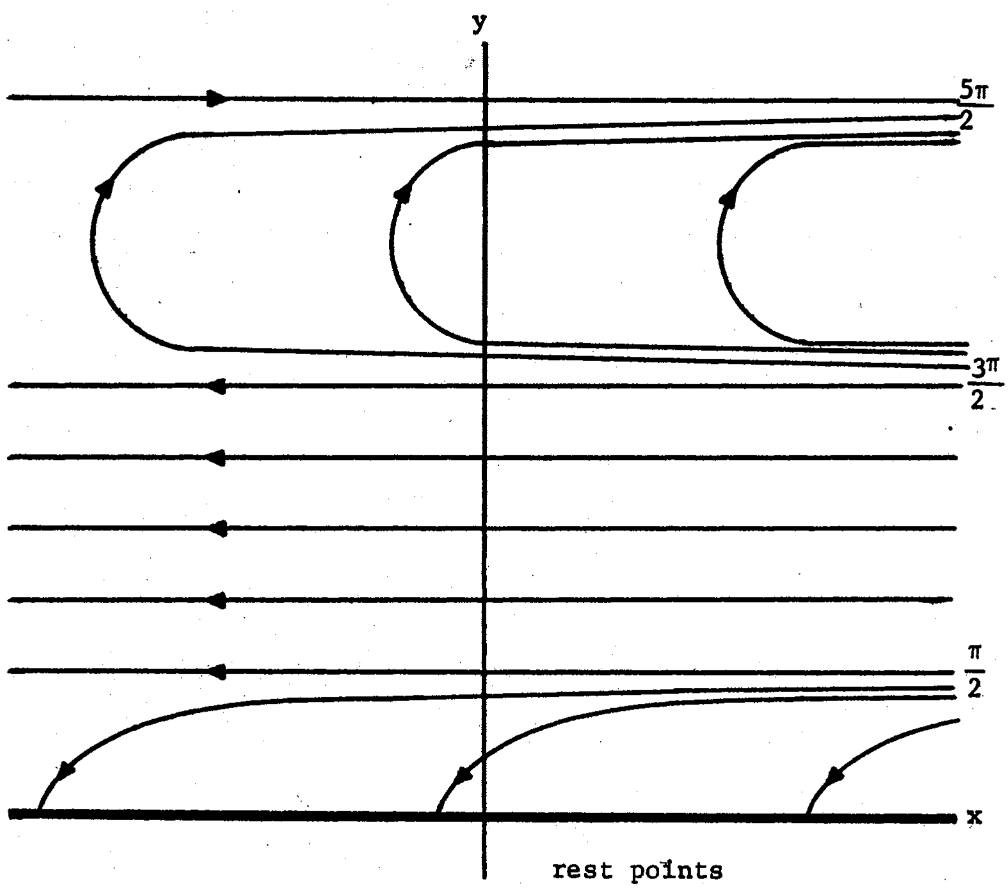

Figure 2.7

<u>Example 2.9</u>: If we define a dynamical system (R^2,π) by the system of differential equations

$$\dot{x} = \begin{cases} \frac{\pi}{2}\sin y & \text{for } \frac{3\pi}{2} \leq y \\ -\frac{\pi}{2} & \text{for } \frac{\pi}{2} \leq y < \frac{3\pi}{2} \\ -y & \text{for } 0 \leq y < \frac{\pi}{2} \\ 0 & \text{for } y < 0 \end{cases}$$

$$\dot{y} = \begin{cases} \frac{\pi}{2}\cos^2 y & \text{for } \frac{3\pi}{2} \leq y \\ 0 & \text{for } \frac{\pi}{2} \leq y < \frac{3\pi}{2} \\ -y\cos^2 y & \text{for } 0 \leq y < \frac{\pi}{2} \\ 0 & \text{for } y < 0 \end{cases}$$

(see Figure 2.7 and [12], pp.30-31), then case (*ii*) is satisfied except for property (*ii*)(c). Each point of S is stable and, since $J^+(p) = \emptyset$ for any point p in $\partial A^+(S)$, each point of $\partial A^+(S)$ is dispersive. The flow restricted to $R^2 - \overline{A^+(S)}$ is not parallelizable because $J^+((0, \frac{3\pi}{2})) \neq \emptyset$. Since $J^+((0, \frac{3\pi}{2})) \neq L^+((0, \frac{3\pi}{2}))$ the flow is not of characteristic 0^+.

Examples of Flows of Characteristic 0^+

The purpose of this section is to present nontrivial examples of dynamical systems of characteristic 0^+ on the plane. Since flows of characteristic 0^+ having compact sets of rest points are easily found, we will give examples for which the sets of rest points are not compact. In particular, the examples will satisfy the statements (a) through (h) in case (4) of Theorem 1.63 and Corollary 1.64.

Note that Example 2.8 satisfies all of the conditions given in case (4) of Theorem 1.63. However, it is not of characteristic 0^+. According to Theorem 2.2 the condition given in Corollary 1.64 must be satisfied in order for a flow to have characteristic 0^+.

The first three examples lead us to Example 2.13 which nontrivially satisfies all eight statements of Theorem 1.63 case (4) and Corollary 1.64.

Example 2.10: The flow defined by the system of differential equations

$$\dot{x} = \begin{cases} 2n\pi - y & \text{for } \frac{4n-1}{2}\pi \leq y \leq \frac{4n+1}{2}\pi \\ y - (2n+1)\pi & \text{for } \frac{4n+1}{2}\pi < y < \frac{4n+3}{2}\pi \end{cases}$$

$$\dot{y} = \begin{cases} (2n\pi - y)\cos^2 y & \text{for } \frac{4n-1}{2}\pi \leq y \leq \frac{4n+1}{2}\pi \\ [(2n+1)\pi - y]\cos^2 y & \text{for } \frac{4n+1}{2}\pi < y < \frac{4n+3}{2}\pi \end{cases}$$

for $n = 0, \pm 1, \pm 2, \ldots$ nontrivially satisfies all of the statements in case (4) of Theorem 1.63 except (e). Figure 2.8 illustrates the trajectories for $n = 0$.

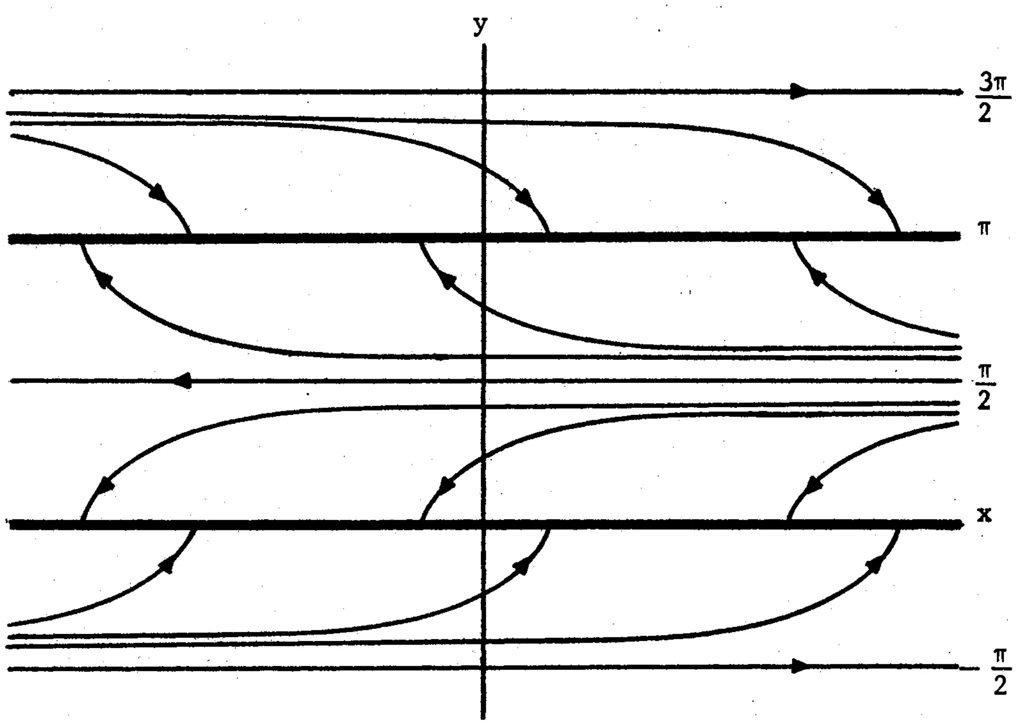

Figure 2.8

Example 2.11: We give this example in order to facilitate the construction of Example 2.12.

The flow defined by the system of differential equations

$$\dot{x} = \begin{cases} 0 & \text{for } \frac{\pi}{2} < y \\ \frac{\pi}{2} - y & \text{for } 0 \leq y \leq \frac{\pi}{2} \\ \frac{\pi}{2} & \text{for } y < 0 \end{cases}$$

$$\dot{y} = \begin{cases} (\frac{\pi}{2} - y) \sin^2 y & \text{for } 0 \leq y \leq \frac{\pi}{2} \\ 0 & \text{for } y < 0, \frac{\pi}{2} < y \end{cases}$$

is of characteristic 0^+ and its trajectories are illustrated in Figure 2.9.

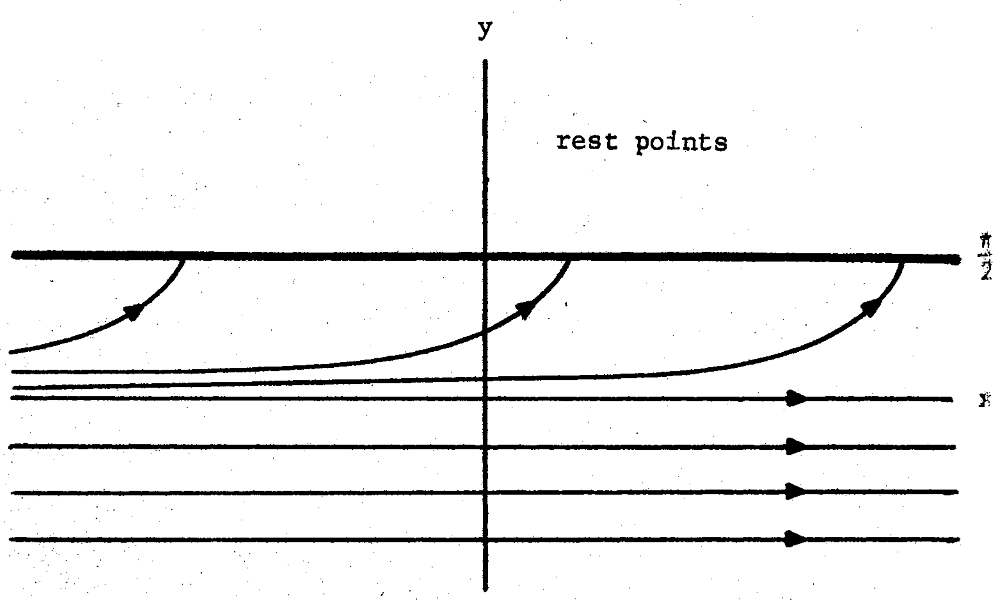

Figure 2.9

Example 2.12: The flow defined in this example nontrivially satisfies statement (4)(e) of Theorem 1.63. We will induce it on R^2 by means of homeomorphisms from the phase plane of the flow of Example 2.11.

Let $a = \frac{\pi(2+\pi)}{4}$ and $I_n = ((2n-1)a, (2n+1)a)$ for $n = 0, \pm1, \pm2, \ldots$. For each such n define mappings $f_n\colon R^2 \to I_n \times R$ by

$$f_n(x,y) = \begin{cases} (\tan^{-1}x + 2na, \sqrt{1+x^2} - y) & \text{for } y < 0 \\ ((1+y)\tan^{-1}x + 2na, \sqrt{1+x^2} - y) & \text{for } 0 \le y \le \frac{\pi}{2} \\ ((1+\frac{\pi}{2})\tan^{-1}x + 2na, \sqrt{1+x^2} - y) & \text{for } \frac{\pi}{2} < y. \end{cases}$$

The mapping f_n carries horizontal lines in R^2 bijectively onto nonintersecting secant curves filling the strip $I_n \times R$. Thus, by examining the images of basic open sets in R^2 consisting of rectangles having sides parallel to the coordinate axes, we can easily see that these mappings are homeomorphisms.

Let us denote the flow of Example 2.11 by (R^2,π_1). Define a flow (R^2,π) by

$$\pi((x,y),t) = \begin{cases} f_n(\pi_1(f_n^{-1}(x,y),t)) & \text{for } (x,y) \in I_n \times R \\ (x,y) & \text{for } x = (2n-1)a \end{cases}$$

for $n = 0, \pm1, \pm2, \ldots$. The trajectories of this flow are illustrated in Figure 2.10. Note that, although S is connected, $\partial A^+(S)$ contains infinitely many trajectories.

Example 2.13: We now describe a flow (R^2,π) that nontrivially satisfies all of the statements in case (4) of Theorem 1.63. The trajectories of the flow are illustrated in Figure 2.11.

In the regions $R \times (2n\pi, (2n+1)\pi)$ for $n = 0, \pm1, \pm2, \ldots$ define the flow by the system of differential equations of Example 2.10.

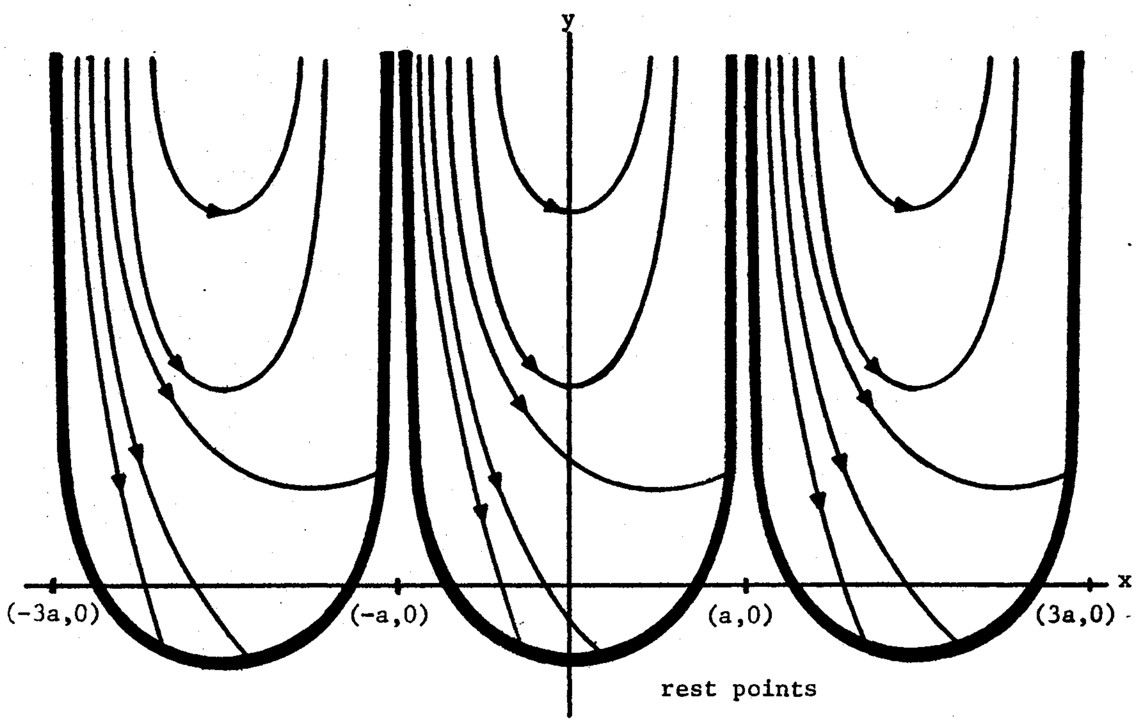

Figure 2.10

Next, induce the flow on $R \times [-\pi,0]$ from the region $[-a,a] \times R$ of the flow in Example 2.12 by means of the homeomorphisms

$$g_n: [-a,a] \times R \to R \times [(2^{-n} - 1)\pi,\ (2^{1-n} - 1)\pi]$$

for $n = 1, 2, 3, \ldots$ where

$$g_n(x,y) = \left(y + \frac{\pi}{2} + n - 1,\ \frac{\pi x}{2^{n+1}a} + \left(\frac{3}{2^{n+1}} - 1\right)\pi\right).$$

The effect of the g_n's is one of rotation, contraction, and translation of $[-a,a] \times R$. Finally, induce the flow on $R \times [(2n - 1)\pi,\ 2n\pi]$ for $n = \pm 1, \pm 2, \ldots$ by the translation mapping

$$h_n: R \times [-\pi,0] \to R \times [(2n - 1)\pi,\ 2n\pi]$$

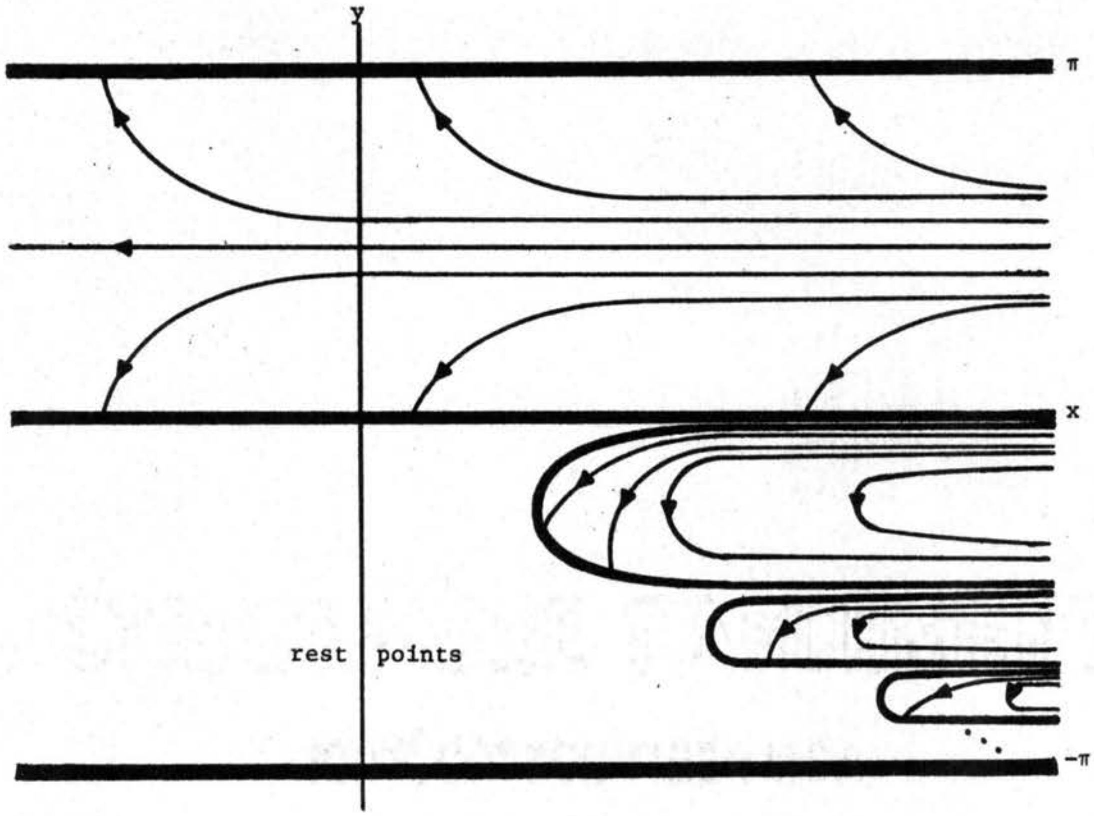

Figure 2.11

where

$$h_n(x,y) = (x,\ y + 2n\pi).$$

Thus, if we let (R^2,π_1) and (R^2,π_2) denote the flows of Examples 2.10 and 2.12, respectively, then for any $x \in R^2$ and $t \in R$ we have

$$\pi(x,t) = \begin{cases} \pi_1(x,t) & \text{for } x \in R \times (2n\pi,(2n+1)\pi),\ n \in A \\ g_n(\pi_2(g_n^{-1}(x),t)) & \text{for } x \in R \times [(2^{-n}-1)\pi,(2^{1-n}-1)\pi],\ n \in B \\ h_n g_n(\pi_2(g_n^{-1}h_n^{-1}(x),t)) & \text{for } x \in R \times [(2n-1)\pi,2n\pi],\ n \in C \end{cases}$$

where $A = \{0,\pm 1,\pm 2,\ldots\}$, $B = \{1,2,3,\ldots\}$, and $C = \{\pm 1,\pm 2,\pm 3,\ldots\}$.

CHAPTER III

DYNAMICAL SYSTEMS OF CHARACTERISTIC 0

Planar Flows of Characteristic 0

It seems natural to ask whether there is a connection between flows of characteristic 0 and flows of characteristic 0^+, 0^-, or $0^{\pm}$. Since $D^+(x) = K^+(x)$ and $D^-(x) = K^-(x)$ for each $x \in R^2$ implies $D(x) = K(x)$, any flow of characteristic $0^{\pm}$ is a flow of characteristic 0. A flow which has characteristic 0^+ (0^-) but not characteristic 0 is given below in Example 3.1. Examples 3.2 and 3.3 consist of flows of characteristic 0 that are not of characteristic 0^+, 0^-, or $0^{\pm}$.

Example 3.1: The system of differential equations

$$\dot{x} = -x$$

$$\dot{y} = -y$$

defines a flow of characteristic 0^+ in which the origin is a proper node. Note, however, that $D((0,0)) = R^2 \neq \{(0,0)\} = K((0,0))$, and so the flow does not have characteristic 0.

Similarly, the flow defined by $\dot{x} = x$ and $\dot{y} = y$ is of characteristic 0^- but not of characteristic 0.

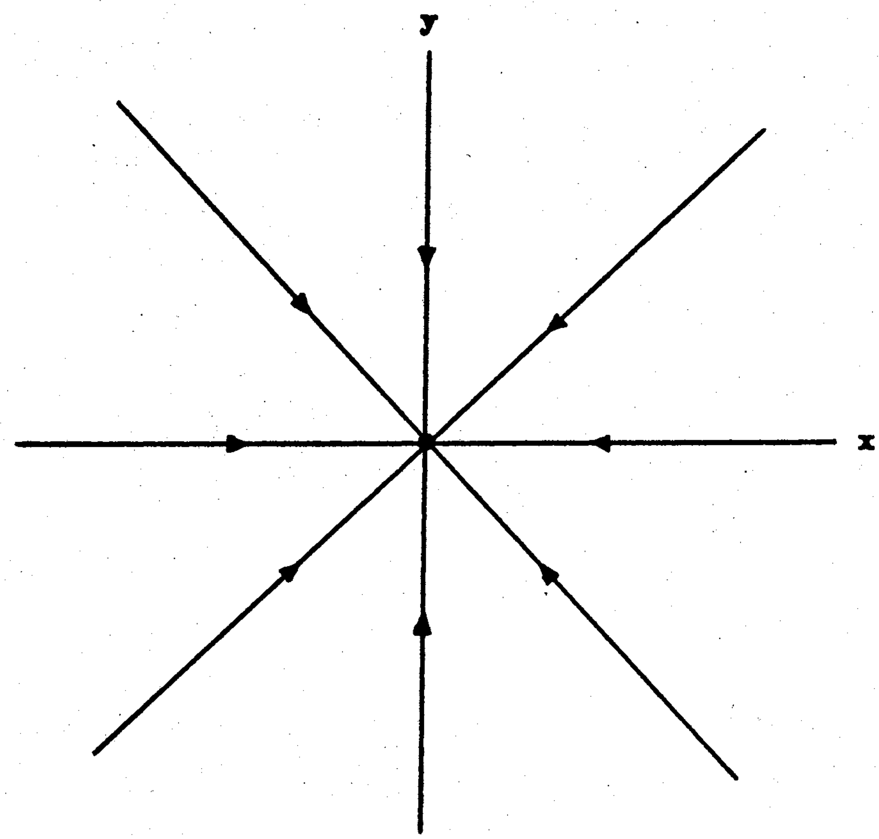

Figure 3.1

Example 3.2: Let a flow be defined by the system

(1) $$\dot{r} = -r^2 \sin \theta$$

$$\dot{\theta} = 1$$

for $r \geq 0$. Figure 2.2 illustrates the trajectories of the flow.

This flow is of characteristic 0 but not characteristic 0^+, 0^-, or $0^\pm$. For let x_o be a point on the parabolic boundary of the region consisting of the pole and the periodic orbits surrounding the pole. Then $D^\pm(x_o) = C(x_o)$, and hence, $D^\pm(x_o) \neq K^\pm(x_o)$ whereas $D(x) = C(x) = K(x)$ for each $x \varepsilon R^2$.

<u>Example 3.3</u>: The flow defined by the system of differential equations

$$(2) \qquad \dot{x} = -xy$$

$$\dot{y} = \begin{cases} x - 1 - y^2 & \text{for } x \geq 0 \\ -x - 1 - y^2 & \text{for } x < 0 \end{cases}$$

is of characteristic 0. System (2) can be obtained from system (1) by changing system (1) to Cartesian coordinates, translating to obtain the equations of (2) for $x \geq 0$, and then reflecting the trajectories of (2) for $x \geq 0$ in the y axis to obtain the trajectories of (2) for $x < 0$. The orbits of the flow defined by (2) are illustrated in Figure 3.2.

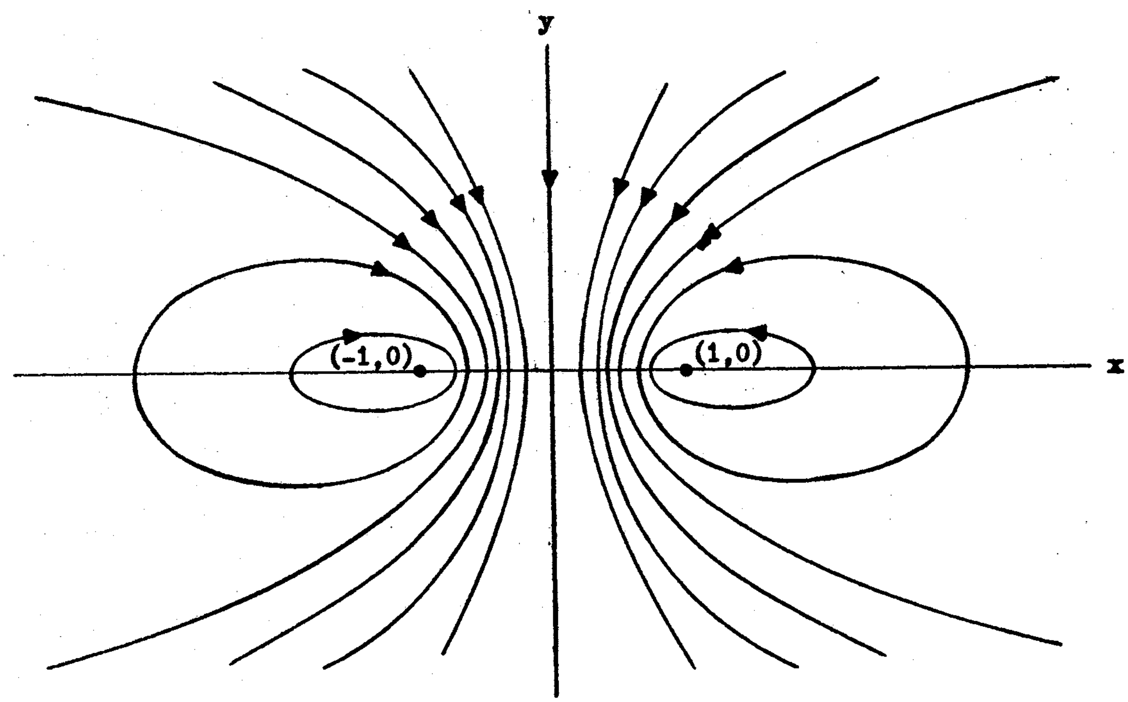

Figure 3.2

Characterization of Planar Flows

of Characteristic 0

The purpose of this section is to give necessary and sufficient conditions for a flow (R^2,π) to have characteristic 0. Unless otherwise specified, we shall let (R^2,π) be a fixed flow of characteristic 0 and S be the set of critical points. We shall first prove a few lemmas.

Lemma 3.4: If $L^+(x) \neq \emptyset$ $(L^-(x) \neq \emptyset)$ for some $x \in R^2$, then x is either periodic or critical.

Proof: Let $y \in L^+(x)$. Then $x \in J^-(y)$ since $y \in J^+(x)$ (see Proposition 1.28). Hence, $x \in D(y) = K(y) \subset L^+(x)$. By Theorem 1.18, $x \in L^+(x)$ if and only if x is either a critical point or a periodic point. The result for $L^-(x) \neq \emptyset$ follows similarly.

Lemma 3.5: If $x \in S$ or x is periodic, then $C(x)$ is bilaterally stable.

Proof: The proof follows from Theorem 1.46 since $D(xt) = K(xt) = K(x) = C(x)$ for each t in R implies $D(C(x)) = C(x)$.

Notation: For any $s \in S$ we shall henceforth let

$$N_s = \{x \in R^2 : x = s \text{ or } x \text{ is periodic and } S \cap \text{int } C(x) = \{s\}\}.$$

Lemma 3.6: If s_o is an isolated point of S, then s_o is a Poincaré center and N_{s_o} is an unbounded connected open set. If $N_{s_o} \neq R^2$, then ∂N_{s_o} is a single trajectory and N_{s_o} is a simply connected component of $R^2 - \partial N_{s_o}$.

Proof: Let C be a simple closed curve with $S \cap \text{int } C = \{s_o\}$. By virtue of Lemma 3.5, there exists a $V \varepsilon \eta(s_o)$ such that $C(V) \subset \text{int } C$. Since $L^+(x) \neq \emptyset$ for each $x \varepsilon V$, $V - \{s_o\}$ consists of periodic points. If $x \varepsilon V - \{s_o\}$, then

$$\emptyset \neq S \cap \text{int } C(x) \subset S \cap \text{int } C = \{s_o\}.$$

Thus, V consists of s_o and periodic orbits surrounding s_o implying that s_o is a Poincaré center.

Let $x \varepsilon N_{s_o} - \{s_o\}$ and $y \varepsilon (\text{int } C(x)) - \{s_o\}$. Since $L^+(y) \neq \emptyset$, y is periodic. We have

$$\emptyset \neq S \cap \text{int } C(y) \subset S \cap \text{int } C(x) = \{s_o\}$$

so that $y \varepsilon N_{s_o}$. Hence, $\text{int } C(x) \subset N_{s_o}$. Furthermore, N_{s_o} is connected since $N_{s_o} = \bigcup \{\overline{\text{int } C(x)}: x \varepsilon N_{s_o}\}$ is the union of connected sets each containing the point s_o.

If $\partial N_{s_o} = \emptyset$, then $N_{s_o} = R^2$ and s_o is a global Poincaré center. Suppose $\partial N_{s_o} \neq \emptyset$. Note that ∂N_{s_o} is invariant since N_{s_o} is invariant. We shall show that in this case ∂N_{s_o} contains no critical points or periodic points. First, suppose $s \varepsilon \partial N_{s_o} \cap S$. There is an open simply connected neighborhood U in $\eta(s)$ such that $s_o \notin U$. By Lemma 3.5 there exists a $V_1 \varepsilon \eta(s)$ such that $C(V_1) \subset U$. Let $x \varepsilon V_1 \cap N_{s_o}$. Then $C(x) \subset U$. Since U is simply connected and $\text{int } C(x) \subset N_{s_o}$, $s_o \varepsilon \text{int } C(x) \subset U$ which is a contradiction. Hence, $S \cap \partial N_{s_o} = \emptyset$. Next, suppose there is a periodic point x in ∂N_{s_o}. Let $S_o = S \cap \text{int } C(x)$. There is a simply connected neighborhood $U \varepsilon \eta(C(x))$ such that $S \cap U = S_o$. By Lemma 3.5 there is a $V_2 \varepsilon \eta(C(x))$ such that $V_2 = C(V_2) \subset U$. For any periodic point y

in $N_{s_o} \cap V_2$ we have $C(y) \subset V_2$. Since U is simply connected, int $C(y) \subset U$. Hence, $s_o \in S \cap U = S_o$. The sets S_o and $S - S_o$ are closed implying that there are simple closed curves C_1 and C_2 contained in int $C(x)$ and ext $C(x)$, respectively, such that $S \cap (\text{ext } C_1) \cap (\text{int } C_2) = \emptyset$. By Lemma 3.5 there is a $V_3 \in \eta(C(x))$ such that $C(V_3) \subset (\text{ext } C_1) \cap (\text{int } C_2)$. Now N_{s_o} is connected with $s_o \in N_{s_o}$ and $C(x) \subset \partial N_{s_o}$, so that we can select a point y from $N_{s_o} \cap V_3 \cap \text{int } C(x)$. Thus, $S_o \cap \text{int } C(y) \neq \emptyset$, $S_o \subset \text{int } C_1$, and $C(y) \subset V_3 \subset \text{ext } C_1$ imply $S_o \subset \text{int } C(y)$. Hence, $S_o = \{s_o\}$ and $x \in N_{s_o}$. Finally, for any point $z \in V_3 \cap \text{ext } C(x)$, $L^+(x) \neq \emptyset$ implying z is periodic. Since $C(z) \subset C(V_3) \subset \text{int } C_2$ we have $S \cap \text{int } C(z) = S_o$. The point z is in N_{s_o} and $C(x) \subset \text{int } C(z) \subset N^o_{s_o}$. This contradicts $x \in \partial N_{s_o}$. Therefore, the points of ∂N_{s_o} are neither periodic nor critical.

By virtue of Lemma 3.4 and the fact that ∂N_{s_o} contains no periodic or rest points, $L^{\pm}(x) = \emptyset$ for each $x \in \partial N_{s_o}$. Thus, ∂N_{s_o} is not bounded and hence N_{s_o} is an unbounded open set.

We now show that ∂N_{s_o} is a single trajectory. Let x and y be distinct points of ∂N_{s_o}. Let C_1 and C_2 be simple closed curves such that $x \in \text{int } C_1$, $y \in \text{int } C_2$, and $\overline{\text{int } C_1} \cap \overline{\text{int } C_2} = \emptyset$. For z in $N_{s_o} \cap \text{int } C_1$, we have $\overline{\text{int } C(z)} \subset N_{s_o}$, and so ext $C(z) \in \eta(y)$. Thus, $(\text{int } C_2) \cap (\text{ext } C(z)) \in \eta(y)$ and $(\text{int } C_2) \cap (\text{ext } C(z)) \cap N_{s_o} \neq \emptyset$. Let

$$w \in (\text{int } C_2) \cap (\text{ext } C(z)) \cap N_{s_o}.$$

Then $C(z) \subset \text{int } C(w) \subset N_{s_o}$. We have $z \in \text{int } C(w)$ and $x \in \text{ext } C(w)$. Since $x, z \in \text{int } C_1$ and int C_1 is connected, it follows that $C(w) \cap \text{int } C_1 \neq \emptyset$. Hence, we can find nets (w_i) and $(w_i t_i)$ converging to y and x, respectively. In other words, $x \varepsilon D(y) = K(y) = C(y)$.

Suppose N_{s_o} is not a component of $R^2 - \partial N_{s_o}$. Since N_{s_o} is connected, it is a subset of some component B. If $N_{s_o} \neq B$, then $\partial N_{s_o} \cap B \neq \emptyset$ contradicting $B \subset R^2 - \partial N_{s_o}$. Hence, N_{s_o} is a component of $R^2 - \partial N_{s_o}$.

Finally, let $R^2 \neq N_{s_o}$. Suppose that C is a simple closed curve lying in N_{s_o} with $\text{int } C \not\subset N_{s_o}$. Then $\text{int } C$ connected and $N_{s_o} \cap \text{int } C \neq \text{int } C$ imply that $\partial N_{s_o} \cap \text{int } C \neq \emptyset$. Furthermore, $\partial N_{s_o} \cap \text{ext } C \neq \emptyset$ since ∂N_{s_o} is unbounded. Thus, $C \cap \partial N_{s_o} \neq \emptyset$ contradicting $C \subset N_{s_o}$. Therefore, N_{s_o} is simply connected.

<u>Lemma 3.7</u>: If $S_o = S \cap \text{int } C(x_o)$ for some periodic point x_o, then S_o consists of exactly one Poincaré center.

Proof: Let $N = \{x \varepsilon \text{int } C(x_o): x \text{ is periodic and } S_o = S \cap \text{int } C(x)\}$ and $D = \bigcap \{\overline{\text{int } C(x)}: x \varepsilon N\}$. At least $x_o \varepsilon N$, and so, $D \neq \emptyset$. Also, D is the intersection of closed invariant sets containing S_o so that D is a closed invariant set and $S_o \subset D$. It also follows that ∂D is invariant.

In order to facilitate the argument we show that $V \varepsilon \eta(C(y))$ implies $V \cap N \neq \emptyset$ for all $y \varepsilon \partial D$. Suppose $V \cap N = \emptyset$ for some $V \varepsilon \eta(C(y))$. By Lemma 3.5 there is a connected set $U \varepsilon \eta(C(y))$ such that $C(U) \subset V$. For $x \varepsilon N$, $U \cap C(x) = \emptyset$. Since $y \varepsilon \text{int } C(x)$ and U is connected, we have $U \subset \text{int } C(x)$. The point x was arbitrary, so that $U \subset D$. But this implies $y \varepsilon D^o$ which contradicts $y \varepsilon \partial D$.

Suppose that D is not a singleton. We first show that there exists a point $y \in N$ such that $D = \overline{\text{int } C(y)}$. If $D = \overline{\text{int } C(x_o)}$, then we are done. Assume $D \neq \overline{\text{int } C(x_o)}$ and choose points x in D and y in ∂D such that $x \neq y$. Either $y \in S_o$ or y is periodic. Suppose there exists a simple closed curve C such that $x \in \text{ext } C$ and $C(y) \subset \text{int } C$. By Lemma 3.5 there is a $V \in \eta(C(y))$ such that $C(V) \subset \text{int } C$. We have shown that $V \cap N \neq \emptyset$. Let $z \in V \cap N$. Then $C(z) \subset C(V) \subset \text{int } C$. But this implies that

$$x \in \text{int } C(z) \subset \text{int } C$$

contradicting $x \in \text{ext } C$. Thus, y is periodic and $x \in \text{int } C(y)$. Since x was an arbitrary point of D, we have $D \subset \overline{\text{int } C(y)}$. Furthermore, $C(y) \subset \partial D \subset \text{int } C(z)$ for each $z \in N$ implying $\overline{\text{int } C(y)} \subset \bigcap \{\overline{\text{int } C(z)}: z \in N\} = D$. Hence, $D = \overline{\text{int } C(y)}$.

Since S_o is compact there exists a simple closed curve $C \subset \text{int } C(y)$ with $S_o \subset \text{int } C$. By Lemma 3.5 there is a $V \in \eta(C(y))$ such that $C(V) \subset \text{ext } C$. Each point z in $V \cap \text{int } C(y)$ is periodic by Lemma 3.4, and so, $S_o \cap \text{int } C(z) \neq \emptyset$. Since $C(z) \subset \text{ext } C$, $\text{int } C(z) \cap \text{int } C \neq \emptyset$, and $\text{int } C$ is connected, we have $S_o \subset \text{int } C \subset \text{int } C(z)$ and $z \in N$. Thus, $D \subset \overline{\text{int } C(z)}$ and $C(z) \subset \text{int } C(y)$ imply that $D \subset \text{int } C(y)$ which contradicts $y \in D$. Consequently, D must be a singleton.

Finally, $\emptyset \neq S_o \subset D$ implies that D is composed of an isolated critical point. By Lemma 3.6, S_o consists of a Poincaré center.

<u>Lemma 3.8</u>: If $S \neq \emptyset$ and $S \neq R^2$, then S consists of Poincaré centers.

Proof: Let S_o denote the set of Poincaré centers. We can select a point s from ∂S since $S \neq \emptyset$ and $S \neq R^2$. For any compact set $U \in \eta(s)$ there exists a $V \in \eta(s)$ such that $C(V) \subset U$ by Lemma 3.5. For any $x \in V \cap (R^2 - S)$, $L^+(x) \neq \emptyset$ implying that x is periodic. Thus, Lemma 3.7 implies $S_o \neq \emptyset$.

Suppose $s \in \partial(S - S_o)$. Since s is bilaterally stable, $\eta(s)$ contains a compact connected simply connected invariant set V. Either V contains a regular point or a Poincaré center. If it contains a regular point x, then x must be periodic so that int $C(x)$, and hence V, must contain a Poincaré center. Therefore, we can assume that V contains a Poincaré center s_o. Now, for each $x \in N_{s_o} - \{s_o\}$, $s_o \in \text{int } C(x)$ and, by Lemma 3.7, $s \in \text{ext } C(x)$. Thus, V must meet $C(x) = \partial \text{int } C(x)$ since it is connected. But this implies $C(x) \subset V$ and hence $N_{s_o} \subset V$, contradicting Lemma 3.6. Therefore, $\partial(S - S_o) = \emptyset$, and so $S = S_o$.

<u>Lemma 3.9</u>: If $S \neq \emptyset$ and $S \neq R^2$, then S consists of at most two Poincaré centers.

Proof: Suppose s_1, s_2, and s_3 are distinct points of S. We shall show that this supposition leads to a countable collection of mutually disjoint closed sets whose union is R^2 which is impossible. Unless explicitly stated, the remainder of the proof will be considered relative to the extended dynamical system on R^{2*}. We denote the closure of the trajectory through x in the extended system by $K^*(x)$.

Since the sets N_s are disjoint and open relative to R^2, $A = R^2 - \bigcup \{N_s : s \in S\}$ is nonempty. For each $x \in A$, $K^*(x) = C(x) \cup \{\infty\}$ is a simple closed curve. Let

$M = \{x \in A\colon N_{s_1} \subset A_x$ and $N_{s_2} \cup N_{s_3} \subset B_x$ where A_x and B_x are the components of $R^{2*} - K^*(x)\}$. By Lemma 3.6, $M \neq \emptyset$ since $\partial N_{s_1} - \{\infty\} \subset M$. Note that $\overline{A}_x = A_x \cup K^*(x)$ and let

$$F_{s_1} = \bigcup \{\overline{A_x}\colon x \in M\}.$$

Each set $\overline{A_x}$ is connected and contains N_{s_1}, and so F_{s_1} is connected.

For any point p_1 in $\partial F_{s_1} - \{\infty\}$ we have $F_{s_1} = \overline{A}_{p_1}$. For let p_1 and q_1 be distinct points in $\partial F_{s_1} - \{\infty\}$ and let C_1 and C_2 be simple closed curves in R^2 surrounding p_1 and q_1, respectively, such that $\overline{\text{int } C_1} \cap \overline{\text{int } C_2} = \emptyset$. There exists a point p for which $A_p \cap \text{int } C_1$, and hence $C(p) \cap \text{int } C_1$, are nonempty sets. Since $B_p \cap \text{int } C_2 \in \eta(q_1)$ there exists a point q such that

$$A_q \cap B_p \cap \text{int } C_2 \neq \emptyset;$$

hence, $C(q) \cap \text{int } C_2 \neq \emptyset$. Now, A_q meets A_p and B_p, so that $A_p \subset A_q$. Thus, A_q is a connected set which meets both int C_1 and ext C_1 implying that $C(q) \cap \text{int } C_1 \neq \emptyset$. We can find nets (x_i) and $(x_i t_i)$ converging to q_1 and p_1, respectively; hence, $p_1 \in D(q_1) = K(q_1) = C(q_1)$ and $\partial F_{s_1} - \{\infty\} = C(p_1)$. Now, $C(p_1) \not\subset N_s$ for any s in S since $N_s \cap F_{s_1} \neq \emptyset$ implies there exists an x in M such that $C(p_1) \subset N_s \subset A_x \subset F^o_{s_1}$ contradicting $C(p_1) \subset \partial F_{s_1}$. Thus, $C(p_1) \subset A$. Since F_{s_1} is an invariant set, either $C(p_1) \subset F_{s_1}$ or $C(p_1) \cap F_{s_1} = \emptyset$. Suppose

$$C(p_1) \cap F_{s_1} = \emptyset.$$

Then $F_{s_1} - \{\infty\}$ is the connected set $F^o_{s_1}$, and so it is a component of $R^{2*} - K^*(p_1) = R^{2*} - \partial F_{s_1}$. Also,

$$N_{s_2} \bigcup N_{s_3} \subset \bigcap\{B_x : x \in M\} = R^{2*} - F_{s_1}$$

which means $p_1 \in F_{s_1}$, contradicting $C(p_1) \bigcap F_{s_1} = \emptyset$. Hence, $F_{s_1} = \overline{A}_{p_1}$.

Analogously, for s_2 and s_3 there exist points p_2 and p_3 in A and sets F_{s_2} and F_{s_3} such that $F_{s_2} = \overline{A'_{p_2}}$ and $F_{s_3} = \overline{A''_{p_3}}$. Note that $F_{s_1} = \overline{A}_{p_1}$ and $F_{s_2} \subset B_{p_1}$. If $\partial F_{s_1} = K^*(p_1) = K^*(p_2) = \partial F_{s_2}$, then $F_{s_1} \bigcup F_{s_2} = R^{2*}$ which contradicts $s_3 \notin F_{s_1} \bigcup F_{s_2}$. Hence, $F_{s_1} \bigcap F_{s_2} = \{\infty\}$. Similarly, $F_{s_1} \bigcap F_{s_3} = F_{s_2} \bigcap F_{s_3} = \{\infty\}$.

Let $F = F_{s_1} \bigcup F_{s_2} \bigcup F_{s_3}$. Obviously, $R^{2*} \neq F$, and so $R^{2*} - F \neq \emptyset$. Suppose that $A \bigcap (R^{2*} - F) = \emptyset$. Then $R^{2*} - F$ must consist of periodic and rest points, so that $N_s \subset R^{2*} - F$ for some $s \in S$. Furthermore, $\partial N_s - \{\infty\} \subset A$ implies that

$$\partial N_s \subset \partial F = K^*(p_1) \bigcup K^*(p_2) \bigcup K^*(p_3).$$

By letting $\partial N_s = K^*(p_k)$ we have $R^{2*} = N_s \bigcup F_{s_k}$ since N_s and $F^o_{s_k}$ are components of $R^{2*} - K^*(p_k)$. But this implies that $s_i \in F_{s_k}$ for $i \neq k$ which is clearly not possible. Therefore,

$$A \bigcap (R^{2*} - F) \neq \emptyset.$$

For each point x in $A \bigcap (R^{2*} - F)$ one component of $R^{2*} - K^*(x)$ contains F since $K^*(x)$ does not separate any of the sets N_{s_1}, N_{s_2}, and N_{s_3} from the other two. Denote the components of $R^{2*} - K^*(x)$ by G_x and H_x where $F \subset H_x$. For any point y in $A \bigcap (R^{2*} - F)$, let $M_y = \{x \in A \bigcap (R^{2*} - F) : G_y \subset G_x\}$. Note that $M_y \neq \emptyset$ since $y \in M_y$. Let $F'_y = \bigcup\{G_x : x \in M_y\}$. By arguing as we did for F_{s_1}, we can find a point w in $\partial F'_y \cap A$ such that $F'_y = \overline{G}_w$. For each point p in $A \cap (R^{2*} - F)$ for which $F'_p = F'_y$, select a point y_o in $C(w)$ and denote F'_p by F_{y_o}. Let I' be the index set for all the F_{y_o} sets and let $I = I' \cup \{s_1, s_2, s_3\}$.

If x and z are distinct points in I, then $F_x \cap F_z = \{\infty\}$. For suppose $F_x \cap F_z \neq \{\infty\}$. The sets F_x^o and F_z^o are components of $R^{2*} - \partial F_x$ and $R^{2*} - \partial F_z$, respectively, where ∂F_x and ∂F_z are simple closed curves each consisting of $\{\infty\}$ and a single trajectory. Thus, either $\partial F_x - \{\infty\} \subset F_z^o$, $\partial F_z - \{\infty\} \subset F_x^o$, or $F_x^o \cap F_z^o = \emptyset$. The first two statements imply that $F_x = F_z$, and hence $x = z$, contradicting $x \neq z$. The third statement implies that $F_x \cup F_z = R^{2*}$ which is impossible. Therefore, $F_x \cap F_z = \{\infty\}$.

Next, $R^{2*} = \bigcup \{F_x : x \in I\}$. For let z belong to $R^{2*} - E$ where $E = \bigcup \{F_x : x \in I\}$. Since $A \subset E$, there is a point s in S such that $z \in N_s$. For some point y in E, $K^*(y) = \partial N_s$. Furthermore, there is a point x in I such that $K^*(y) = \partial F_x$ since $K^*(y) \subset \partial E$. The sets N_s and F_x^o are disjoint components of $R^{2*} - K^*(y)$, and so $R^{2*} = N_s \cup F_x$. This implies $F_x = E$, and thus $s_i \in F_x$ for $i = 1,2,3$, which is clearly impossible. Hence, $R^{2*} = E$.

The set $\{F_x : x \in I\}$ is a countable collection of closed sets such that $F_x \cap F_z = \{\infty\}$ for $x \neq z$. Hence, $\{F_x - \{\infty\} : x \in I\}$ is a countable collection of mutually disjoint sets closed in R^2 and $R^2 = \bigcup \{F_x - \{\infty\} : x \in I\}$. This is not possible as we indicated at the outset of our argument. Therefore, s_1, s_2, and s_3 are not distinct.

<u>Lemma 3.10</u>: Let $S \neq R^2$. Then the flow restricted to

$$R^2 - \bigcup \{N_s : s \in S\}$$

is parallelizable.

Proof: Let $x \in R^2 - \bigcup \{N_s : s \in S\}$. Then $L^{\pm}(x) = \emptyset$. Recalling that $J(y) = L(y)$ for each $y \in R^2$, we have $J^+(x) = \emptyset$. Hence, $D^+(x) = C^+(x)$. The result follows by Theorem 1.36.

<u>Theorem 3.11</u>: A flow (R^2, π) has characteristic 0 if and only if one of the following holds.

(1) $S = \emptyset$ and (R^2, π) is parallelizable.

(2) S consists of at most two Poincaré centers. For each $s \in S$, either s is a global Poincaré center or N_s is unbounded and ∂N_s is a single trajectory. The restriction of the flow to $R^2 - \bigcup \{N_s : s \in S\}$ is parallelizable.

(3) $S = R^2$.

Proof: The necessity of the conditions follows from the lemmas. Conversely, Theorem 1.36 shows that condition (1) is sufficient. Similarly, if condition (2) holds, we get $D(x) = K(x)$ for each $x \in R^2 - \bigcup \{N_s : s \in S\}$. For each $s \in S$, N_s is a component of $R^2 - \partial N_s$ since ∂N_s is a single trajectory. Thus, N_s is a connected simply connected set. Obviously, $x \in \overline{N}_s$ implies $D(x) = K(x)$. Hence, condition (2) is sufficient. Condition (3) is trivially sufficient.

<u>Corollary 3.12</u>: A flow (R^2, π) has characteristic 0 if and only if $D(x) = C(x)$ for each $x \in R^2$.

Remark: Theorem 3.11 implies that there are six basic types of planar flows having characteristic 0; namely,

(1) parallelizable flows,

(2) flows having a global Poincaré center,

(3) flows similar to Example 2.4,

(4) flows similar to Example 3.3,

(5) flows similar to Example 3.3 except that $\partial N_s = \partial N_t$ where $S = \{s,t\}$, and

(6) flows having only critical points.

CHAPTER IV

PERIODIC DYNAMICAL SYSTEMS

Many well known properties of differential flows have proven to be valid for general dynamical systems. In this chapter we show that certain properties of planar differential flows having only periodic and critical points generalize to planar dynamical systems.

Definition 4.1: We shall call a flow (X,π) having only periodic and critical points a periodic flow.

Throughout the remainder of this chapter we shall denote the set of critical points for a given flow by S.

Theorem 4.2: Let (R^2,π) be a periodic flow. If S_o is a compact component of S, then S_o is bilaterally stable.

Proof: For each point x in $R^2 - S$, $C(x)$ is bilaterally stable (see Theorem 1.58) yielding $D(x) = D(C(x)) = C(x)$ (see Theorem 1.46). Thus, $D(R^2 - S) = R^2 - S$, and hence, $D(S) = S$. Suppose s_o is a point of S_o such that $D(s_o) \not\subset S_o$. Since $D(s_o)$ is a subset of S which meets both the component S_o and the set $S - S_o$, it is not connected. The set $D(s_o) \cap S_o$ is a compact component of $D(s_o)$ which is absurd (see Theorem 1.29). Hence, $D(S_o) = S_o$ and S_o is bilaterally stable (see Theorem 1.46).

Corollary 4.3: Let (R^2,π) be a periodic flow. If s_o is an isolated point of S, then s_o is a Poincaré center.

Proof: In view of Theorem 1.59 we need only observe that s_o is an isolated bilaterally stable point.

Corollary 4.4: Let (R^2,π) be a periodic flow. If S is a finite set, then S consists of a global Poincaré center.

Proof: By Corollary 4.3, S consists of Poincaré centers. Each trajectory is bilaterally stable (see Theorem 1.58 and Theorem 4.2), so that $D(x) = D(C(x)) = C(x) = K(x)$ for each $x \varepsilon R^2$ (see Theorem 1.46). Thus, (R^2,π) has characteristic 0 and the desired result follows from Theorem 3.11.

Corollary 4.5: Let x be a periodic point of a flow (R^2,π). If the restriction of the flow to $\text{int } C(x)$ is periodic and $S \cap \text{int } C(x)$ is finite, then it consists of exactly one Poincaré center.

Proof: Define a flow (R^2,π') which agrees with (R^2,π) on $\overline{\text{int } C(x)}$ and consists of periodic trajectories surrounding $C(x)$ on $\text{ext } C(x)$. The proof follows from Corollary 4.4.

In Corollary 4.5 the components of $S \cap \text{int } C(x)$ are assumed to be a finite number of isolated critical points. The following example illustrates that there can be countably many isolated critical points if the set $S \cap \text{int } C(x)$ is not finite.

Example 4.6: Let (R^2,π) be defined by the system of differential equations

$$\dot{x} = \begin{cases} 0 & \text{for } (x,y) \varepsilon\ B - A \\ y\sqrt{\left(x - \frac{1}{2^n}\right)^2 + y^2}\left(\sqrt{\left(x - \frac{1}{2^n}\right)^2 + y^2} - \frac{1}{2^{n+2}}\right) & \text{for } (x,y) \varepsilon\ A_n,\ n = 1,2,3,\ldots \\ y\left(1 - \sqrt{x^2 + y^2}\right) & \text{for } (x,y) \varepsilon\ R^2 - B \end{cases}$$

$$\dot{y} = \begin{cases} 0 & \text{for } (x,y) \varepsilon\ B - A \\ \left(x - \frac{1}{2^n}\right)\sqrt{\left(x - \frac{1}{2^n}\right)^2 + y^2}\left(\frac{1}{2^{n+2}} - \sqrt{\left(x - \frac{1}{2^n}\right)^2 + y^2}\right) & \text{for } (x,y) \varepsilon\ A_n, n=1,2,3,\ldots \\ x\left(\sqrt{x^2 + y^2} - 1\right) & \text{for } (x,y) \varepsilon\ R^2 - B \end{cases}$$

where

$$B = \{(x,y):\ x^2 + y^2 \leq 1\},$$

$$A_n = \left\{(x,y):\ \sqrt{\left(x - \frac{1}{2^n}\right)^2 - y^2} < \frac{1}{2^{n+2}}\right\},$$

and

$$A = \bigcup_{n=1}^{\infty} A_n.$$

The phase plane is illustrated in Figure 4.1. Note that the points $\left(\frac{1}{2^n}, 0\right)$ for $n = 1,2,3,\ldots$ are local Poincaré centers.

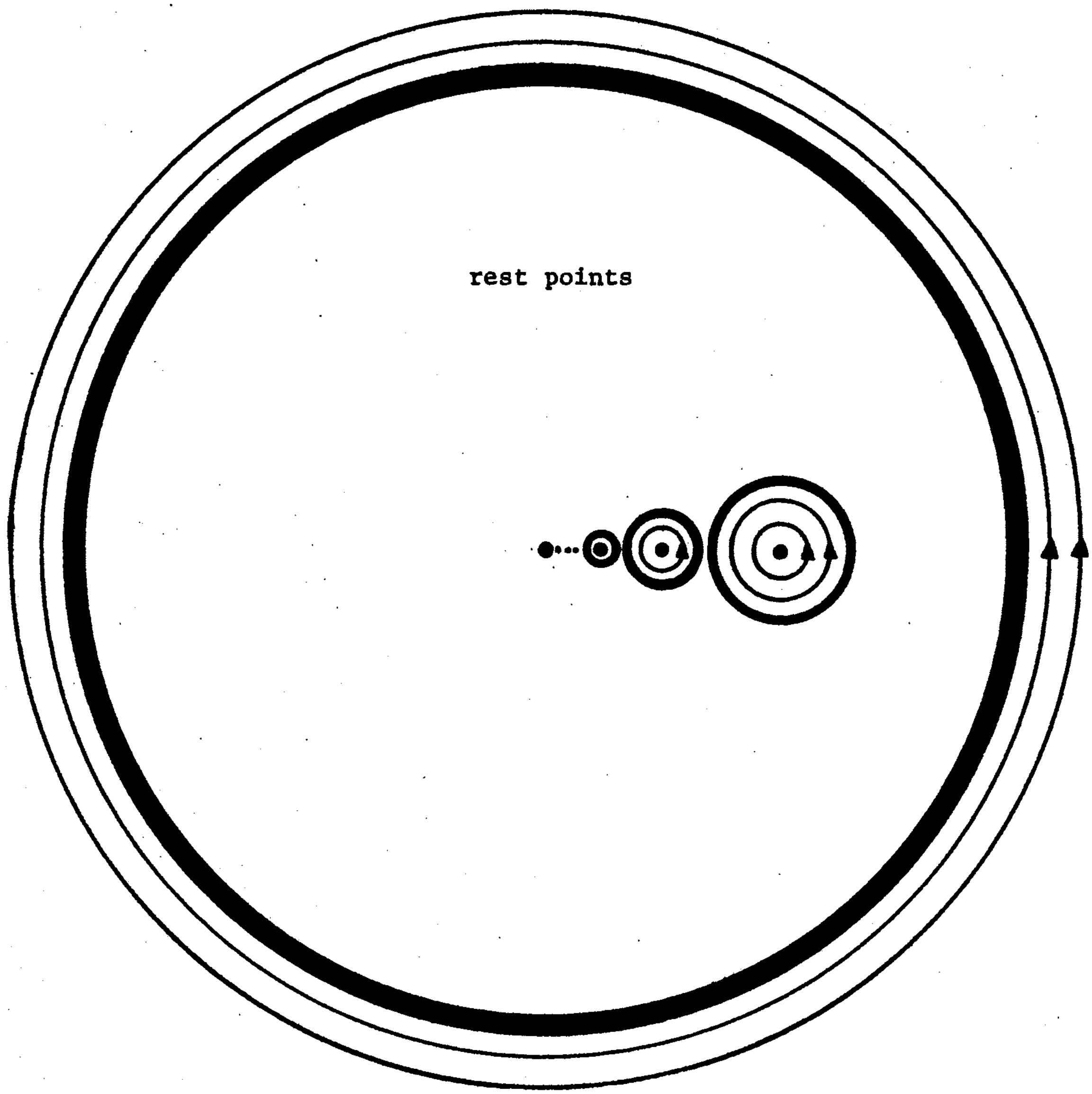

Figure 4.1

CHAPTER V

SUMMARY

In Chapter I we listed the basic properties of dynamical system theory used in this thesis. Included is a survey of the known results for planar dynamical systems having characteristic 0^+, 0^-, $0^{\pm}$, or 0.

Chapter II contains a characterization of dynamical systems of characteristic 0^+ (0^-) in terms of the set of critical points. Examples are given to show that the conditions of the characterization are sharp. Examples of dynamical systems satisfying all of the properties obtained by Ahmad in [1] for flows having noncompact sets of critical points are given including one which nontrivially satisfies all of the properties.

In Chapter III we characterize dynamical systems of characteristic 0 in terms of the set of critical points. Examples of the nontrivial types of planar flows having characteristic 0 are given.

Finally, in Chapter IV we show that the set of critical points of a planar periodic flow having a finite number of critical points consists of a global Poincaré center. Also, if the interior of a planar periodic orbit is a periodic flow and the set of critical points is finite, then the set of critical points consists of a single Poincaré center.

There are many questions suggested by the results of this thesis. One might attempt similar characterizations of planar dynamical systems of characteristic α^+, α^-, $\alpha^{\pm}$, and α where α is any ordinal number; that is, flows where $D^+_\alpha(x) = D^+_{\alpha-1}(x)$ for each x where $D^+_\alpha(x)$ represents the αth prolongation of x and so forth. Dynamical systems for which $D^+_\alpha(x) = D^+_\beta(x)$ for each x where α and β are fixed ordinal numbers as well as their negative, conjunctive, and bilateral versions can be studied. Any of these problems can be studied for arbitrary rather than planar phase spaces. Transformation groups having any of these properties can be studied. Planar periodic flows can be classified and characterized in terms of their critical points.

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