HOMOLOGICAL METHODS AND
ABELIAN GROUPS

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Submitted to the Faculty of the Graduate College
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in partial fulfillment of the requirements
for the degree of
DOCTOR OF EDUCATION
May, 1971

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PREFACE

It is the purpose of this thesis to give a detailed expository account of homological methods and their use in Abelian group theory. These techniques have had a unifying effect in algebra over the past decade, since they can be applied to modules over quite arbitrary rings. From the outset, however, I decided it was unproductive to generalize to arbitrary modules; many have done this, and the results still seem mysterious. What is required, I feel, is a natural body of examples from group theory upon which to base understanding, indeed appreciation, of deep structure theorems in a module setting. It is a tough task to generalize upon an empty base.

The first two chapters develop the elementary concepts of homological algebra and Abelian group theory we will need throughout our work. The later chapters develop the homological method proper, and the last chapter applies the method to some sophisticated problems in the structure of Abelian groups. I have tried to include some features which would give the reader a sense of continuity: I have credited the main theorems and have included numerous examples. The Summary contains an outline of the entire method along with directions for further study.

Without too much difficulty the topics given here could easily comprise a senior-beginning graduate level course for mathematics majors, preferably budding algebraists. It is really they I had in mind during the writing of this thesis. I envisioned such a reader with a healthy imagination in the formative years of his mathematical maturity. But I have only assumed that his working knowledge of algebra is confined to the first two chapters of I. N. Herstein's classic <u>Topics in Algebra</u>; this friendly survey of algebra is fast becoming a standby in the undergraduate mathematics curriculum. For anyone interested in algebra, further specialist or not, the examples here will serve him well; the techniques are typical of work being done in a spectrum of areas from Abelian groups to category theory.

It would be impossible for me to adequately acknowledge the scores of individuals who have had a part in helping me fulfill degree requirements. I especially wish to thank Dr. Dennis Bertholf for suggesting this topic to me. He has spent numerous hours both in conversing with me about the topics and in reading the various manuscripts. His enthusiasm and faith in the direction we were taking made this thesis a reality. I could not do much in mathematics, no matter how minor, without a pause to thank my former teacher at the University of Kansas, Professor Paul J. McCarthy. I shall always be grateful for the personal interest he showed in me.

My thanks go also to the other members of my committee for their efforts in my behalf: John Jewett, John Hoffman, Robert Alciatore and John Susky.

Finally, to my wife Judy I owe a special debt. Her patience and many sacrifices have sustained me throughout my graduate program. She read the entire manuscript carefully, and her many constructive criticisms of the thesis have helped to clean up more obscurities and errors I care to imagine. Her efforts made this thesis possible.

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CHAPTER I

INTRODUCTION

1.1. History

This thesis is concerned with the influence and interplay of the techniques of homological algebra in the theory of Abelian groups. This influence has been deep and far-reaching; the structure theorems that it has provided have so influenced the study of Abelian groups that few new studies in Abelian groups are being written today without some consideration of the homological aspects of the theory.

The profound work in Abelian groups can be traced to the 1940's in the works of R. Baer, H. Prüfer, L. Ya. Kulikov, and T. Szele. It was Baer who first discovered the role of divisible groups (1940) and proved that they were direct summands in every group in which they were contained as a subgroup. He did not have the word "injective," but he did prove the existence of injective envelopes for groups, as did Kulikov. Prüfer introduced the notion of pure subgroup in 1923 and many key theorems hinge upon them. For example, Kulikov's powerful basic subgroups (1945) are based, in part, on the notion of purity.

One underlying motivation for the study of Abelian groups, aside from their intrinsic beauty, is that they provide a principal impetus to further research. Each Abelian group is a Z-module, where Z is the ring of integers. For each new theorem that arises in Abelian groups, one natural question is: "Can this result be extended to arbitrary modules over a commutative ring with unity?" Sometimes the proofs can be restated with minor changes. Most times major revisions are necessary. But Abelian groups provide a storehouse of examples of facts which hold in greater generality. We have thus so far progressed: we use Abelian groups, once very abstract objects, as down-to-earth examples of more far-reaching results!

But from our point of view, the influence of homological algebra has been more striking. Simply put, the word "homological" refers to the general properties of homomorphisms and the role these properties play in determining the structure of the groups on which they are defined. These techniques have had a rich but comparatively short history. However vehemently some algebraists may deny it, the fact remains that we owe a debt to the workers in topology who first influenced an examination of the use of group theoretic invariants for the spaces they were studying. The notion of exact sequence has its roots in the exact differentials of DeRham's theorem.

The influential workers in the field have come to be Henri Cartan and Samuel Eilenburg with their powerful text <u>Homological Algebra</u> (1956). It is there that the words "projective" and "injective" as applied to modules appear. They also introduced the torsion product of two groups and the exact sequence for Tor. Finally, they showed the existence of projective resolutions for modules and how the process generalizes to a wide class of functors. But a warning: Cartan and Eilenburg is not for the about-to-be-initiated! One gets the impression that all that can be said has been said there; it is still a true reference work.

In <u>Homological Algebra</u>, Cartan and Eilenburg credit Künneth for investigations in the 1920's that formed a basic impetus for the use of algebra in topology. Eilenburg-MacLane introduced the notion of cohomology groups (1941) in the study of topological spaces. They were also the founders of the unifying notion of category. The use we make of "pushout" and "pullback" constructions comes from category theory and S. Lang and P. Freyd who popularized them. Finally, the father of homological algebra seems to be G. Hochschild; he recognized that the notions of the cohomology of groups could be applied to purely algebraic structures. His work in the 1940's in cohomology and in the 1950's in relative homological algebra form the foundation of today's work.

If there is one paper that has influenced the use of homological methods in Abelian group theory and indeed the writing of this thesis, it is D. K. Harrison's paper "Infinite Abelian groups and homological methods" (1959). There he introduced the exact sequence for Pext and exploited it successfully to determine the structure of the tensor product of torsion groups. Many of his suggestions are detailed here; for example, his remark that Pext(A,C) is a subgroup of Ext(A,C). Many of the techniques he used, however, namely topological considerations and Pontryagin duality for groups, had to be omitted as too far afield for our purpose.

1.2. Overview

A short time ago we mentioned that Abelian groups serve as a fund of examples for further work. One of the expressed purposes of this thesis is to provide a home base for the basic techniques and results of the homological method as applied to Abelian groups. Thus equipped the student can venture off in many different directions. The possibilities are immense. In the Summary a number of these paths are examined. The Summary also contains a review of the entire method; it might prove instructive to refer to it from time to time as we progress.

Two writers in particular have influenced the writing of this thesis: Jans [16] and Kaplansky [18]. A concerted effort has been made to give a style and presentation in their "folksy" way. But what Jans is to rings we are to groups. We also go beyond Jans and include the tensor product and Tor (along with Pext), which he admits is needed to be "fully educated" in the subject. Kaplansky served as a bouy in the deep waters of group theory. Many of the directions he takes have found their way here. Hence in the manner of presentation and the level of difficulty, our work falls somewhere between those of Herstein [15], Kaplansky and Jans. Beyond these, MacLane has contributed much here; his <u>Homology</u> [25] is a general reference for Chapters V and VI.

Our program is somewhat ambitious in that all of the major exact sequences of the homological method for Abelian groups are developed and exploited. However, many other interesting and powerful routes had to be omitted. Only once or twice do we succumb to the temptation to generalize results for modules. We introduce and use only Zorn's lemma; no knowledge of cardinal and ordinal numbers is assumed or used. The useful tools of topology (natural topologies for groups, compact groups, completions of groups) are bypassed. The elegant and unifying concepts of category and functor loom in the background. What the category theorists are trying to accomplish surely will be a little clearer in light of the work we outline here.

The basic flow of the paper is as follows. In Chapter II we start at the meagerest of beginnings with homomorphisms and their properties. The notion of exact sequence is introduced and its basic properties exposed. Direct sums and products of groups are fully represented along with their interaction with exact sequences.

The third chapter is purely a survey chapter of Abelian group theory. The choice of topics is for convenience only. Zorn's lemma is presented with examples. The notions of divisibility, purity and torsion for subgroups are developed at length. The use of homological methods is seen in the section on algebraically compact groups and their properites. We prove the important result of Kulikov that all torsion groups have basic subgroups. Finally it is proved that bounded groups are direct sums of cyclics. The group $Z(p^{\infty})$ is discussed, also.

In Chapter IV groups of homomorphisms are set forth along with the notion of induced homomorphisms. It is seen how an exact sequence of Abelian groups induces a left exact sequence of homomorphism groups. A natural motivation for the notions of projective and injective groups is seen in the desire to complete the exactness of the exact sequence of homomorphism groups. Finally, free groups are introduced and projective groups characterized in terms of them.

Chapter V is concerned with the notion of the group of group extensions, due to R. Baer. We show that an addition can be introduced in the collection of all group extensions of one group A by another C so as to make the resulting structure Ext(C,A) into an Abelian group. How these groups restore exactness to the exact sequence of homomorphism groups is fully explained. It is then that injective groups are characterized in terms of Ext(C,A) and the remaining notions of divisible groups postponed from Chapter III are proved. A modern approach to the notion of divisible hull (or injective envelope), due to Eckmann and Schopf, finishes the chapter. Many examples of Ext(C,A) are given.

The surprising tensor product, $A \otimes B$, and its properties are exposed in Chapter VI. Induced homomorphisms between tensor products are seen to give rise to induced right exact sequences of Abelian groups. The torsion product is introduced to rescue the reaminder of the exact sequence, and so we have the second major exact sequence of the method. A characterization of torsion-free groups is given along with examples of $A \otimes B$.

The notion of purity is revamped in a homological setting in Chapter VII. The subgroup Pext(C,A) of Ext(C,A) is introduced along with the third major exact sequence of the method.

It is Chapter VIII, Applications, where we hint at the power of the homological method. The notions of pure projective and pure injective are introduced and characterized through the triviality of Pext(C,A). Harrison's theorems about $A \otimes B$ and Hom(A,B) when both A and B are torsion groups are also given. The chapter closes with a discussion of one of Nunke's adjoint relations for Ext.

The Summary contains a review of the entire method along with advice for further work.

1.3. Notes to the Reader

To those hearty fellows still intent upon actually reading what follows after all that has been said already: observe the following notation and facts! For a group G and $x \in G$, $\langle x \rangle$ denotes the cyclic subgroup of G generated by x. If $x \in G$ with $n = o(x)$ and $mx = 0$ for some integer m, then $n|m$ (i.e., n divides m). The notation

(n,m) for integers n and m denotes the greatest common divisor of $\{m,n\}$.

The notation $\phi:G \to H$ means that ϕ is a homomorphism with domain G and range H. The image of ϕ is $\phi(G)$, and it may be properly contained in H. If G and H are isomorphic groups, we write $G \simeq H$. If $\phi:G \to H$ is an onto homomorphism, then $H \simeq G/K$, where K is the kernel of ϕ. Given any group G, 1_G (or simply 1 if no confusion can arise) denotes the identity homomorphism of G: $1(g) = g$ for all $g \in G$. If $\phi:G \to H$ and $\psi:H \to G$ are homomorphisms such that $\psi\phi = 1_G$ and $\phi\psi = 1_H$, then ϕ and ψ are isomorphisms, inverse to one another.

When euphony seems to call for it, we use "iff" to mean "if and only if." The symbol || signifies the end of a proof (if it isn't obvious that we've finished one). All of our homomorphisms are applied on the left of their arguments; hence the composite of two homomorphisms, which we write as $\phi\psi$, say, is applied this way: $\phi(\psi(g)) = (\phi\psi)(g)$ for all g in some group G.

One of the disadvantages of the method is that some of the proofs can become very intricate and quite computational. This might prove helpful to the reader: skip over a proof that seem tedious by trying to construct the details for himself using the exposition for a guide. The reading will probably go much smoother and quicker with greater understanding the result.

CHAPTER II

HOMOLOGICAL ALGEBRA

2.1. Homomorphisms and diagrams

For us the word "group" will always mean Abelian group. All groups will be written additively, so if G is a group with $g \in G$ and n is a non-negative integer, ng is that element of G consisting of n summands of g. As usual, we will abuse the notation and let 0 denote the (additive) identity of all groups. We adopt the (standard) notation of denoting by Z the (additive) group of integers and Q the group of rationals. Hence all of our groups will be modules over Z. We will have much to say about Abelian group theory in the next chapter, but all of our discussion of homological algebra will be housed in the language of groups. We will make one pronouncement: practically everything we say homologically can be (and has been) said for modules over arbitrary (commutative) rings with unity. The reader is invited to consider the necessary changes in the exposition which will achieve that level of generalization.

Let G and H be groups and $\kappa: G \to H$ a homomorphism. Recall that the rule of κ satisfies $\kappa(g_1 + g_2) = \kappa(g_1) + \kappa(g_2)$ for all g_1 and g_2 in G. We say κ is a <u>monomorphism</u> if $\kappa(g_1) = \kappa(g_2)$ implies $g_1 = g_2$ for all g_1 and g_2 in G. For variety, we may say that "κ is monic." Recall that κ is monic iff $\operatorname{Ker} \kappa$ is trivial, where $\operatorname{Ker} \kappa = \{g \in G \mid \kappa(g) = 0\}$. We say κ is an <u>epimorphism</u> if for all $h \in H$ there is $g \in G$ such that $\kappa(g) = h$; one also may say that "κ is epic."

If A is a subgroup of B, we have the following standard homomorphisms: $\iota : A \to B$ defined by $\iota(a) = a$ for all $a \in A$ and $\eta : B \to B/A$ defined by $\eta(b) = b + A$ for all $b \in B$. Whenever the reader sees these symbols used for maps, he may assume they are the ones above. The map ι is called the injection of A into B; it is monic. The map η is called the projection of B onto the quotient group B/A; it is epic.

Let A, B, C, and D be groups with homomorphisms as below:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \gamma\downarrow & & \downarrow\beta \\ C & \xrightarrow{\delta} & D \end{array}$$

We say that the <u>square commutes</u> (or the diagram commutes) if for all $a \in A$, $\beta(\alpha(a)) = \delta(\gamma(a))$.

2.2. Exact sequences and their properties

We are now ready to begin the study of homological concepts, or as Jans [16] puts it, the study of "anything with little arrows on it." Really, homological notions deal with homomorphisms between groups and their relationship to general structure properties of the group. We will examine some questions of this type in this chapter.

A collection $\{f_{n+1} : A_{n+1} \to A_n \mid n \in Z\}$ is called a sequence of Abelian groups and Abelian group homomorphisms, or just a <u>sequence</u> for short. Sometimes we write $\cdots \to A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \to \cdots$

for a sequence. We say a sequence is <u>exact</u> at A_n if $\text{Ker } f_n = \text{Im } f_{n+1}$; the entire sequence is exact if it is exact at each A_n. The sequence is called <u>quasi-exact</u> at A_n if we have a zero sequence at A_n; i.e., $f_n f_{n+1} = 0$. This means that $\text{Im } f_{n+1}$ is a subgroup of $\text{Ker } f_n$.

An exact sequence of the form $0 \overset{f}{\to} A \to B \to \cdots$ is called <u>left-exact</u>, whereas an exact sequence of the form $\cdots \to A \to B \overset{g}{\to} 0$ is called <u>right-exact</u>. We will always mean by 0 in this context the trivial group. In the above situation, f must be the <u>trivial map</u>; i.e., f must take zero only to zero if it is to be a homomorphism. Similarly, g must be the <u>zero map</u>; i.e., $g(b) = 0$ for all $b \varepsilon B$. After 2.2.1. the maps f and g will not be written since they are uniquely determined. Finally, a sequence which is exact and has the form $0 \to A \to B \to C \to 0$ is called a <u>short exact sequence</u>. The more useful facts about exact sequences are collected in the following proposition. The proofs are left for the reader.

2.2.1. <u>Proposition</u> Let A, B and C be groups.

(a) $0 \overset{f}{\to} A \overset{\kappa}{\to} B$ left-exact means κ is monic.

(b) $A \overset{\sigma}{\to} B \overset{g}{\to} 0$ right-exact means σ is epic.

(c) Suppose $0 \to A \overset{\kappa}{\to} B$ is exact. Then $A = 0$ if either $B = 0$ or κ is the zero homomorphism.

(d) Suppose $A \overset{\sigma}{\to} B \to 0$ is exact. Then $B = 0$ if either $A = 0$ or σ is the trivial homomorphism.

(e) Suppose $A \overset{\kappa}{\to} B \to C \to 0$ is exact with κ epic. Then $C = 0$.

(f) $0 \to A \overset{\sigma}{\to} B \to 0$ exact means that σ is an isomorphism.

(g) $\sigma : A \to B$ epic implies $0 \to \text{Ker } \sigma \overset{\iota}{\to} A \overset{\sigma}{\to} B \to 0$ is short exact.

(h) $\kappa : A \to B$ monic implies $0 \to A \overset{\kappa}{\to} B \overset{\eta}{\to} B/\text{Im } \kappa \to 0$ is exact.

(i) $0 \to A \overset{\kappa}{\to} B \overset{\sigma}{\to} C \to 0$ exact implies $\sigma\kappa = 0$ and C is isomorphic to $B/\text{Im}\ \kappa$.

We now investigate some conditions which give rise to certain properties of maps between exact sequences. The following proposition gives us most of the facts we will need.

2.2.2. Proposition (Strong 4 lemma) Let the following be a commutative diagram with exact rows, τ epic and ν monic:

$$\begin{array}{ccccccc} A & \overset{\gamma}{\to} & B & \overset{\lambda}{\to} & C & \overset{\rho}{\to} & D \\ \tau\downarrow & & \alpha\downarrow & & \beta\downarrow & & \nu\downarrow \\ E & \underset{\zeta}{\to} & F & \underset{\delta}{\to} & G & \underset{\sigma}{\to} & H \end{array}$$

Then: (a) $\text{Ker}\ \beta = \lambda(\text{Ker}\ \alpha)$ and (b) $\delta^{-1}(\text{Im}\ \beta) = \text{Im}\ \alpha$.

Proof: The arguments make strong use of the "diagram chase" technique. Let $x \in \lambda(\text{Ker}\ \alpha)$, so $\lambda(y) = x$ for some $y \in \text{Ker}\ \alpha$. Then $\beta(x) = \beta(\lambda(y)) = \delta(\alpha(y)) = 0$, so $x \in \text{Ker}\ \beta$. Now let $x \in \text{Ker}\ \beta$. Then $\nu(\rho(x)) = \sigma(\beta(x)) = 0$ implies $\rho(x) = 0$, since ν is monic. Hence $x \in \text{Ker}\ \rho = \text{Im}\ \lambda$, so $\lambda(w) = x$ for some $w \in B$. Let $w' = \alpha(w) \in F$. Then $\delta(w') = \delta(\alpha(w)) = \beta(\lambda(w)) = 0$, so $w' \in \text{Ker}\ \delta = \text{Im}\ \zeta$; i.e., $w' = \zeta(e)$ for some $e \in E$. Since τ is epic, there exists $a \in A$ such that $e = \tau(a)$. Set $b = \gamma(a)$. Then $\alpha(b) = \alpha(\gamma(a)) = \zeta(\tau(a)) = \zeta(e) = w'$. Finally, put $v = b - w$. Then $\alpha(v) = 0$, so $v \in \text{Ker}\ \alpha$, and $\lambda\gamma = 0$ implies that $\lambda(v) = x$. Thus $x \in \lambda(\text{Ker}\ \alpha)$, and (a) follows.

To show (b), let $f \in \text{Im}\ \alpha$. Then $f = \alpha(b)$ for some $b \in B$. Set $c = \lambda(b)$. Then $\delta(f) = \delta(\alpha(b)) = \beta(c)$; i.e., $f \in \delta^{-1}(\text{Im}\ \beta)$. Now let $f \in \delta^{-1}(\text{Im}\ \beta)$, so that $\beta(c) = \delta(f)$ for some $c \in C$. Then $\sigma(\delta(f)) = 0$ so that $\nu(\rho(c)) = \sigma(\beta(c)) = \sigma(\delta(f)) = 0$. But ν monic implies $\rho(c) = 0$, so $c = \lambda(b)$ for some $b \in B$. Let $f' = \alpha(b)$. Then $\delta(f' - f) = 0$, so by exactness $\zeta(e) = f' - f$ for some $e \in E$. Since τ is epic, $e = \tau(a)$ for some $a \in A$. Set $b' = \gamma(a)$. Then $\alpha(b - b') = f$, and so $f \in \mathrm{Im}\,\alpha$. ||

The above proposition is used to prove the following result, which is really the one which is useful for our work.

2.2.3. <u>Proposition</u> (3-lemma) Let the following diagram have commutative squares and exact rows:

$$\begin{array}{ccccccccc} 0 & \to & A & \overset{\alpha}{\to} & B & \overset{\beta}{\to} & C & \to & 0 \\ & & \lambda\downarrow & & \mu\downarrow & & \downarrow\nu & & \\ 0 & \to & D & \underset{}{\overset{\gamma}{\to}} & E & \overset{\delta}{\to} & F & \to & 0 \end{array}$$

Then: (a) λ and ν monic implies μ is monic.

(b) λ and ν epic implies μ is epic.

(c) λ and ν isomorphisms imply μ is an isomorphism.

<u>Proof</u>. Clearly (a) and (b) imply (c). To see (a), apply the previous proposition, where now τ of that result will be the trivial isomorphism between the trivial groups on the left end of the above diagram. We then have $\mathrm{Ker}\,\mu = \alpha(\mathrm{Ker}\,\lambda) = 0$ since λ is monic. To show (b), apply the proposition to the diagram:

$$\begin{array}{ccccccc} A & \to & B & \to & C & \to & 0 \\ \lambda\downarrow & & \mu\downarrow & & \nu\downarrow & & | \\ D & \to & E & \overset{\delta}{\to} & F & \to & 0 \end{array}$$

Since λ is epic and the map on the far right is monic, we have $\mathrm{Im}\,\mu = \delta^{-1}(\mathrm{Im}\,\nu)$. But δ and ν epic imply $F = \mathrm{Im}\,\nu$ and $F = \delta(E)$; i.e., $E = \mathrm{Im}\,\mu$. Thus μ is epic. ||

The following proposition describes the relationship between exact sequences connected with isomorphisms. Let the reader practice his "diagram chasing" on it.

2.2.4. <u>Proposition</u> Suppose the following is a commutative diagram with α, β, and γ isomorphisms.

$$\begin{array}{ccccc} A & \xrightarrow{\kappa} & B & \xrightarrow{\sigma} & C \\ \alpha\downarrow & & \beta\downarrow & & \downarrow\gamma \\ D & \xrightarrow{\rho} & E & \xrightarrow{\tau} & F \end{array}$$

Then the top row is exact if and only if the bottom row is exact.

The next few propositions deal with the construction of certain homomorphisms. The constructions are somewhat standard, and we will have occasion to refer to them later.

2.2.5. <u>Proposition</u> Let the following diagrams be given:

$$\begin{array}{ccccc} & & G & & \\ & & \downarrow\rho & & \\ A & \xrightarrow{\kappa} & B & \xrightarrow{\sigma} & C \end{array} \qquad (1) \qquad\qquad \begin{array}{ccccc} A & \xrightarrow{\kappa} & B & \xrightarrow{\sigma} & C \\ & & \downarrow\rho & & \\ & & G & & \end{array} \qquad (2)$$

In (1) if the row is exact at B, κ is monic and $\sigma\rho = 0$, then there is a unique homomorphism $\beta : G \to A$ such that $\kappa\beta = \rho$. In (2) if the row is exact at B, σ is epic and $\rho\kappa = 0$, then there is a unique homomorphism $\beta : C \to G$ such that $\beta\sigma = \rho$.

<u>Proof</u>. For (1), let $g \in G$. Then $\sigma(\rho(g)) = 0$ implies that $\rho(g) \in \operatorname{Ker} \sigma = \operatorname{Im} \kappa$, and so $\rho(g) = \kappa(a)$ for some $a \in A$. For each $g \in G$ set $\beta(g) = a$, where $\kappa(a) = \rho(g)$. Then β is a homomorphism; we show it is well-defined. If $g_1 = g_2$ in G, we have $\kappa(a_1) = \kappa(a_2)$, and so $a_1 = a_2$ because κ is monic. Thus $\beta(g_1) = \beta(g_2)$. With this definition of β the diagram commutes. If β' is another map with the same properties, then $\kappa\beta' = \rho = \kappa\beta$, and so $\kappa(\beta' - \beta) = 0$. Since is monic, this finally implies that $\beta = \beta'$. In (2), a similar proof

works: let $c \in C$. Then $\sigma(b) = c$ for some $b \in B$. Let $g = \rho(b)$ and set $g = \beta(c)$. This definition makes β a unique, well-defined homomorphism and makes the appropriate diagram commute. ||

2.2.6. Proposition. Let $A \overset{\kappa}{\to} B \overset{\sigma}{\to} C$ be quasi-exact. Then there is a homomorphism $\rho: B/\mathrm{Im}\ \kappa \to C$ such that $\rho\eta = \sigma$, where $\eta: B \to B/\mathrm{Im}\ \kappa$ is the natural projection.

Proof. Define $\rho(b + \mathrm{Im}\ \kappa) = \sigma(b)$. If $b - b' \in \mathrm{Im}\ \kappa$, we have $b - b' = \kappa(a)$ for some $a \in A$, and so $\sigma(b - b') = 0$; i.e., $\sigma(b) = \sigma(b')$, and ρ is therefore well-defined. Clearly, ρ is a homomorphism and $(\rho\eta)(b) = \rho(b + \mathrm{Im}\ \kappa) = \sigma(b)$. ||

2.2.7. Lemma Consider the following commutative diagram where the top row is exact and the bottom row is quasi-exact.

$$\begin{array}{ccccc} A & \overset{\alpha}{\to} & B & \overset{\beta}{\to} & C \\ \theta\downarrow & & \phi\downarrow & & \\ X & \overset{\rho}{\to} & Y & \overset{\pi}{\to} & D \end{array}$$

(a) If β is epic, there is a homomorphism $\psi: C \to D$ such that $\psi\beta = \pi\phi$.

(b) Suppose $\psi: C \to D$ is defined with the property that $\psi\beta = \pi\phi$ and β and θ are both epic. Then if the two rows are exact, we have $\mathrm{Ker}\ \psi \subseteq \beta(\mathrm{Ker}\ \phi)$ and $\mathrm{Im}\ \psi \supseteq \pi(\mathrm{Im}\ \phi)$.

(c) If ψ is as in (a) and the hypotheses of (b) are met, then π epic and ϕ an isomorphism together imply that ψ is an isomorphism.

Proof. (a) Let $c \in C$. Since β is epic, $c = \beta(b)$ for some $b \in B$. Set $\psi(c) = \pi\phi(b)$. When ψ is defined in this manner, it is clear that the right-hand square commutes and ψ is a homomorphism.

It remains to show that ψ is well-defined. If $\beta(b_1) = c = \beta(b_2)$, then $b_1 - b_2 = \alpha(a)$ by the exactness of the top row. Since $\pi\rho = 0$ and $\phi\alpha = \rho\theta$, we have: $\phi(\alpha(a)) = \phi(b_1 - b_2)$, and so $\phi(b_1 - b_2) = \rho\theta(a)$. Applying π yields $\pi\phi(b_1 - b_2) = 0$; i.e., $\pi\phi(b_1) = \pi\phi(b_2)$.

(b) To show $\text{Ker}\,\psi \subseteq \beta(\text{Ker}\,\phi)$, let $c \in \text{Ker}\,\psi$. Then $c = \beta(b')$ for some $b' \in B$. Let $y = \phi(b')$, so $\pi(y) = 0$. Thus $y \in \text{Im}\,\rho$ (by the exactness of the bottom row), and so $y = \rho(x)$ for some $x \in X$. Thus $\theta(a) = x$ for some $a \in A$. Let $b'' = \alpha(a)$ and set $b = b' - b''$. Immediately $\phi(b) = 0$, and so $b \in \text{Ker}\,\phi$, and $\beta(b) = c - \beta(\alpha(a)) = c$; thus $c \in \beta(\text{Ker}\,\phi)$. To show the remaining containment, let $d \in \pi(\text{Im}\,\phi)$. Then $d = \pi(y)$ for $y = \phi(b)$, where $b \in B$. Set $c = \beta(b)$. By the property of ψ, we have $\psi(c) = \pi(\phi(b)) = \pi(y)$, so $d \in \text{Im}\,\psi$.

(c) When ϕ is an isomorphism, $\text{Ker}\,\phi = \{0\}$, and $\text{Im}\,\phi = Y$. Since π is epic, $\pi(Y) = D$. By (b), $\text{Ker}\,\psi$ is contained in 0, and $\text{Im}\,\psi$ contains $\pi(Y) = D$. These two statements say ψ is an isomorphism.

2.3. Direct Sums and Direct Products of Groups

We begin to explore the tools necessary to break down groups and put them back together again. Of course, it is hoped that in the process something will be learned about the <u>structure</u> of the groups. The results of such a program will be given in greater detail throughout the exposition, especially in the next chapter. For now, we indulge in the basic concepts and their homological properties.

We begin with the simplest case. Given groups A and B, we define the <u>direct sum</u> of A and B, written $A \oplus B$, by $A \oplus B = \{(a,b) \mid a \in A \text{ and } b \in B\}$, where $(a_1,b_1) = (a_2,b_2)$ iff $a_1 = a_2$ and $b_1 = b_2$ and addition in $A \oplus B$ is defined componentwise:

$(a_1,b_1) + (a_2,b_2) = (a_1 + a_2, b_1 + b_2)$. With these definitions, one easily shows that $A \oplus B$ is a group. Since the action of a function can be viewed through ordered pairs, note that we could have defined $A \oplus B$ by $A \oplus B = \{f:\{1,2\} \to A \cup B \mid f(1) \in A$ and $f(2) \in B\}$. Then $A \oplus B$ has as objects all functions defined on the (index) set $\{1,2\}$, and we define addition of such functions by $(f + g)(i) = f(i) + g(i)$ for $i = 1,2$. It is this view of $A \oplus B$ that readily generalizes to arbitrary collections of groups.

Since we are interested in the homological properties of such groups, we need to know how to "imbed" and "project."

2.3.1. <u>Definition</u> Let A_1 and A_2 be groups. Define $\iota_i:A_i \to A_1 \oplus A_2$ $(i = 1,2)$ by $\iota_1(a_1) = (a_1,0)$ and $\iota_2(a_2) = (0,a_2)$ for all $a_1 \in A_1$ and $a_2 \in A_2$. The monomorphisms ι_i are called the <u>injections</u> of each A_i into the direct sum $A_1 \oplus A_2$. Similarly, define $\pi_i:A_1 \oplus A_2 \to A_i$ $(i = 1,2)$ by $\pi_1(a_1,a_2) = a_1$ and $\pi_2(a_1,a_2) = a_2$. The epimorphisms π_i are called the <u>projections</u> of the direct sum onto each A_i.

2.3.2. <u>Proposition</u> Let A_1 and A_2 be groups, and set $B = A_1 \oplus A_2$. The homomorphisms defined above satisfy the following:

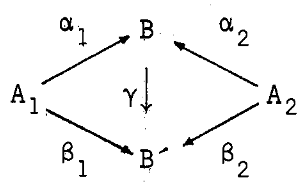
(1) $\pi_1\iota_1 = 1_{A_1}$ and $\pi_2\iota_2 = 1_{A_2}$

(2) $\iota_1\pi_1 + \iota_2\pi_2 = 1_B$.

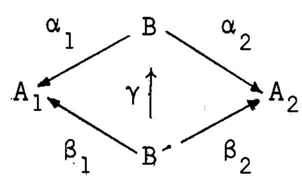
The proof follows directly from the definitions of the maps. We will see that in a very natural way these formulas characterize direct sums of arbitrary collections of groups.

To this end, we follow MacLane [25] and call a diagram

$A_1 \overset{\alpha_1}{\rightarrow} B \overset{\alpha_2}{\leftarrow} A_2$ universal (with ends A_i) if for all diagrams $A_1 \overset{\beta_1}{\rightarrow} B' \overset{\beta_2}{\leftarrow} A_2$ there is a <u>unique</u> homomorphism $\gamma: B \rightarrow B'$ such that diagram (1) below commutes. To form the "dual" concept, turn all of the arrows around in the above discussion. Then a diagram $A_1 \overset{\alpha_1}{\leftarrow} B \overset{\alpha_2}{\rightarrow} A_2$ is called <u>couniversal</u> (with ends A_i) if for all diagrams $A_1 \overset{\beta_1}{\leftarrow} B' \overset{\beta_2}{\rightarrow} A_2$ there is a <u>unique</u> homomorphism $\gamma: B' \rightarrow B$ such that the diagram (2) below commutes.



(1)



(2)

2.3.3. <u>Proposition</u> (a) Any two universal (or couniversal) diagrams with the same ends give an isomorphism between B and B'.

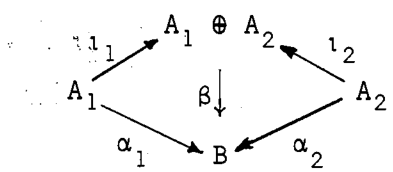
(b) $A_1 \overset{\iota_1}{\rightarrow} A_1 \oplus A_2 \overset{\iota_2}{\leftarrow} A_2$ is universal.

(c) $A_1 \overset{\pi_1}{\leftarrow} A_1 \oplus A_2 \overset{\pi_2}{\rightarrow} A_2$ is couniversal.

<u>Proof</u>. (a) Suppose in (1) we assume that both rows are universal. We have a homomorphism $\gamma': B' \rightarrow B$ such that $\gamma'\beta_1 = \alpha_1$ and $\gamma'\beta_2 = \alpha_2$. Combined with $\gamma\alpha_1 = \beta_1$ and $\gamma\alpha_2 = \beta_2$, we have $(\gamma'\gamma)\alpha_1 = \gamma'(\gamma\alpha_1) = \gamma'\beta_1 = \alpha_1$ and similarly $(\gamma'\gamma)\alpha_2 = \alpha_2$. Now what happens in (1) when $B = B'$? We have then that there is precisely one way (by the uniqueness assumption) of filling in the diagram with a homomorphism $\zeta: B \rightarrow B$ such that $\zeta\alpha_1 = \alpha_1$ and $\zeta\alpha_2 = \alpha_2$. But 1_B also has these properties, so by the uniqueness assumption, we must have that $1_B = \zeta$. Hence $\gamma'\gamma = 1_B$. Now repeat the argument to

show that $\gamma\gamma' = 1_{B'}$ using the uniqueness properties of the bottom row. Hence γ is a unique isomorphism between B' and B.

(b) We have the following diagram, where the bottom row is some arbitrary diagram:



Define $\beta = \alpha_1\pi_1 + \alpha_2\pi_2$; by the definition of the projections and injections, we have $\beta\iota_i = \alpha_i$ for $i = 1,2$. Now if β is any homomorphism with this property, we have $\beta = \beta(\iota_1\pi_1 + \iota_2\pi_2) = (\beta\iota_1)\pi_1 + (\beta\iota_2)\pi_2 = \alpha_1\pi_1 + \alpha_2\pi_2$. Hence (b) is universal.

(c) A similar argument in involved here, except now $\beta : B \to A_1 \oplus A_2$ is defined by $\beta = \iota_1\alpha_1 + \iota_2\alpha_2$. ||

2.3.4. <u>Corollary</u> For all groups A_1 and A_2, $A_1 \oplus A_2 \simeq A_2 \oplus A_1$.

<u>Proof</u>. The diagrams $A_1 \overset{\iota_1}{\rightarrow} A_1 \oplus A_2 \overset{\iota_2}{\leftarrow} A_2$ and $A_1 \overset{\iota_2}{\rightarrow} A_2 \oplus A_1 \overset{\iota_1}{\leftarrow} A_2$ are universal. Apply 2.3.3. (a). ||

2.3.5. <u>Proposition</u> Let $A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \underset{\delta}{\overset{\gamma}{\leftrightarrows}} C$ be any diagram with the following properties: $\beta\alpha = 1_A$, $\delta\gamma = 1_C$, $\delta\alpha = 0$, $\beta\gamma = 0$ and $\alpha\beta + \gamma\delta = 1_B$. Then $B \simeq A \oplus C$.

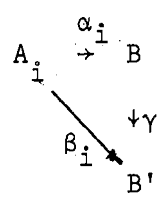
<u>Proof</u>. The diagram $A \overset{\alpha}{\rightarrow} B \overset{\gamma}{\leftarrow} C$ is universal: for let $A \overset{\sigma}{\rightarrow} B' \overset{\rho}{\leftarrow} C$ be any given diagram. Define $\lambda = \rho\delta + \sigma\beta$. Using the given properties of the maps, we have $\lambda\gamma = \rho$ and $\lambda\alpha = \sigma$. Thus $\lambda : B \to B'$ is a homomorphism which makes the diagram commute. If λ

is a homomorphism with these properties, we have $\lambda = \lambda 1_B = \lambda(\alpha\beta + \gamma\delta) = (\lambda\alpha)\beta + (\lambda\gamma)\delta = \sigma\beta + \rho\delta$. This completes the proof.

Such an object B above is called a biproduct, a term we have been taught by the category theorists.

We intend to proceed to the general case, but first we try for three groups! Define $\iota_i : A_i \to A_1 \oplus A_2 \oplus A_3$ by: $\iota_i(a_i)$ is an ordered triple with a_i in the $i^{\underline{th}}$ component and zeros elsewhere, $i = 1,2,3$. Also, set $\pi_i(a_1,a_2,a_3) = a_i$ for $i = 1,2,3$. These are the injections into and projections out of $A_1 \oplus A_2 \oplus A_3$.

We now consider a diagram similar to the one we used for the case of two groups, but for ease of display, we write the diagram in set theoretic form. We say a diagram $D = \{B \overset{\alpha_i}{\leftarrow} A_i \mid i = 1,2,3\}$ is universal (with ends A_i) if for all diagrams $D' = \{B' \overset{\beta_i}{\leftarrow} A_i \mid i=1,2,3\}$, there is a unique homomorphism $\gamma : B \to B'$ such that the following diagrams commute for all $i = 1,2,3$:



Counniversal diagrams are defined similarly, but all the arrows are turned around.

As before, one uses the same type of proof to establish an isomorphism between B and B' if both diagrams D and D' are universal. Also

$$D = \{A_1 \oplus A_2 \oplus A_3 \overset{\iota_i}{\leftarrow} A_i \mid i = 1,2,3\} \text{ is universal, and}$$

$$D = \{A_1 \oplus A_2 \oplus A_3 \xrightarrow{\pi_i} A_i \mid i = 1,2,3\} \text{ is couniversal.}$$

Using the same idea used in 2.3.4., we have that $A_1 \oplus A_2 \oplus A_3 \simeq A_2 \oplus A_3 \oplus A_1 \simeq A_3 \oplus A_1 \oplus A_2$, etc.

Let us now proceed to the general case. Let Λ serve as a quite arbitrary index set; for example, when $\Lambda = \{1,2,\cdots,n\}$, we have the arbitrary finite case, and when $\Lambda = Z$, we have the countable case. Let $\{G_\lambda\}_{\lambda\varepsilon\Lambda}$ be a collection (or "family") of groups G_λ. Let

$$P = \{f:\Lambda \to \bigcup_\lambda G_\lambda \mid f(\lambda) \varepsilon G_\lambda \text{ for all } \lambda \varepsilon \Lambda\}.$$

We have the usual definition for equality of functions: $f = g$ if $f(\lambda) = g(\lambda)$ for all $\lambda \varepsilon \Lambda$, and we define addition of functions so it will be compatible with the finite case; i.e., $(f + g)(\lambda) = f(\lambda) + g(\lambda)$ for all $\lambda \varepsilon \Lambda$. With these definitions, P becomes a group. We call P the (unrestricted, or full) <u>direct product</u> of the G_λ, and we denote it by $P = \prod_\lambda G_\lambda$. As we mentioned before, when $\Lambda = Z$, we have a countable product of groups G_λ. If we use our imaginations, we can see that P can very quickly become very "wild" for "large" Λ and "horrible" groups G_λ. We want to "cut down" P a little.

Single out a special subgroup of P: consider all those functions $f \varepsilon P$ such that $f(\lambda) = 0 \varepsilon G_\lambda$ almost all the time; i.e., $f(\lambda)$ is zero for all but finitely many $\lambda \varepsilon \Lambda$. Denote this subgroup of P by $S = \sum_\lambda G_\lambda$. Again,

$$\sum_\lambda G_\lambda = \{f:\Lambda \to \bigcup_\lambda G_\lambda \mid f(\lambda) \varepsilon G_\lambda \text{ and } f(\lambda) = 0 \text{ for all but finitely many } \lambda \varepsilon \Lambda\}.$$

Indeed, S is a subgroup of P (it inherits the addition of P) and is called the (external) <u>direct sum</u> of the groups G_λ.

2.3.6. <u>Definition</u> For each $\lambda \in \Lambda$ and $g \in G_\lambda$, define $\iota_\lambda : G_\lambda \to S$ by:

$$[\iota_\lambda(g)](\mu) = \begin{cases} g & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, define $\pi_\lambda : P \to G_\lambda$ by $\pi_\lambda(f) = f(\lambda)$. The monomorphisms ι_λ are called the <u>injections</u> of the groups G_λ into the direct sum S, and the epimorphisms π_λ are called the <u>projections</u> of the direct product P onto each G_λ.

2.3.7. <u>Proposition</u> Let P, S, ι_λ and π_λ be as above. Then:

(1) $P = S$ when Λ is a finite index set.

(2) $\pi_\lambda \iota_\mu = \begin{cases} 1_{G_\lambda} & \text{if } \mu = \lambda \\ 0 & \text{otherwise} \end{cases}$

(3) $\sum_{\lambda \in \Lambda} \iota_\lambda \pi_\lambda = 1_S$ (compare with 2.3.2.)

<u>Proof.</u> (1) Clear from the definition of P and S.

(2) Follows from the definitions of π_λ and ι_λ: $\pi_\lambda \iota_\lambda(g) = g$ for all $g \in G_\lambda$.

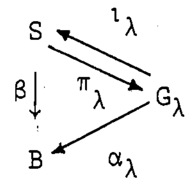
(3) A word is needed about the summation: for each $f \in S$, $(\sum_\lambda \iota_\lambda \pi_\lambda)(f) = \sum_\lambda \iota_\lambda(\pi_\lambda(f)) = \sum_\lambda \iota_\lambda(f(\lambda))$, and $f(\lambda)$ is zero "almost always"; i.e., $f(\lambda) = 0$ for all but finitely many $\lambda \in \Lambda$. Thus there are only finitely many (perhaps zero) nonzero summands in the summation as λ ranges over Λ. Now suppose that $f \in \sum_\lambda G_\lambda$. Then $\iota_\lambda(f(\lambda)) = h_\lambda$ has the property that $h_\lambda(\mu)$ equals $f(\lambda)$ if $\mu = \lambda$ and $h_\lambda(\mu)$ equals zero otherwise. Thus $f = \Sigma_\lambda h_\lambda$.

From the above calculations, then, we see that $(\sum_\lambda \iota_\lambda \pi_\lambda)(f) =$

$\sum_\lambda \iota_\lambda(f(\lambda)) = \sum_\lambda h_\lambda = f.$ ||

2.3.8. Proposition $D = \{\iota_\lambda : G_\lambda \to S \mid \lambda \in \Lambda\}$ is a universal diagram with ends G_λ, and $D' = \{\pi_\lambda : P \to G_\lambda \mid \lambda \in \Lambda\}$ is a couniversal diagram with ends G_λ.

Proof. If $D' = \{\iota_\lambda : G_\lambda \to B \mid \lambda \in \Lambda\}$ is a given diagram with ends G_λ, define $\beta : S \to B$ by $\beta = \sum_\lambda \alpha_\lambda \pi_\lambda$. Refer to the diagram below:



Note that the sum in the defining equation for β is finite for each $f \in S$. One easily checks that β is a homomorphism and $\beta\iota_\lambda = \alpha_\lambda$ for all $\lambda \in \Lambda$. Furthermore, if β' is any homomorphism from S to B such that $\beta'\iota_\lambda = \alpha_\lambda$ for all λ, then $\beta' = \beta'(1_S) = \beta'(\sum_\lambda \iota_\lambda \pi_\lambda) = \sum_\lambda (\beta'\iota_\lambda)\pi_\lambda = \sum_\lambda \alpha_\lambda \pi_\lambda$.

The second assertion is just as straightforward, except that $\beta : B \to P$ must be defined by $[\beta(b)](\mu) = \alpha_\mu(b)$, where $b \in B$ and $\mu \in \Lambda$. One easily shows that β can only be defined in this manner to achieve this property. ||

Using precisely the same type of proof we used in 2.3.3., we have the following proposition.

2.3.9. Proposition Any two universal diagrams with the same ends $\{\alpha_\lambda : G_\lambda \to B\}$ and $\{\alpha'_\lambda : G_\lambda \to B'\}$ yield an isomorphism between B and B'. The same statement holds for any two couniversal diagrams.

2.3.10. Corollary Let $\sigma : \Lambda \to \Lambda$ be any one-to-one and onto function (i.e., σ is a permutation of Λ) and set $\mu = \sigma(\lambda)$. Then

$\sum\limits_{\lambda} G_{\lambda} \simeq \sum\limits_{\mu} G_{\mu}$.

Proof. The diagrams $\{\iota_{\lambda}: G_{\lambda} \to \sum\limits_{\lambda} G_{\lambda}\}$ and $\{\iota_{\mu}: G_{\mu} \to \sum\limits_{\mu} G_{\mu}\}$ are both universal. By the previous result, we get the desired isomorphism.

Essentially, what the corollary states is that a direct sum of groups is unaffected by the order of its summands.

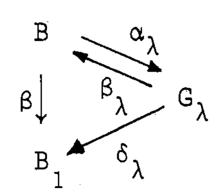
We are now in a position to give a useful result which characterizes the direct sum diagram. It is a generalization of 2.3.5.

2.3.11. Lemma Suppose $\{B \underset{\beta_{\lambda}}{\overset{\alpha_{\lambda}}{\rightleftarrows}} G_{\lambda}\}$ is a given diagram of Abelian groups. We impose a finiteness condition: for each fixed $b \in B$, we declare that $\alpha_{\lambda}(b)$ is nonzero for only finitely many $\lambda \in \Lambda$. In addition, suppose for all λ and μ in Λ we have:

$$\alpha_{\lambda}\beta_{\mu} = \begin{cases} 1_{G_{\lambda}} & \text{if } \mu = \lambda \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \sum_{\lambda} \beta_{\lambda}\alpha_{\lambda} = 1_B$$

Then $B \simeq \sum\limits_{\lambda} G_{\lambda}$.

Proof. We show that $\{B \overset{\beta_{\lambda}}{\leftarrow} G_{\lambda}\}$ is universal and then we may apply propositions 2.3.8. and 2.3.9. Let $\{B_1 \overset{\delta_{\lambda}}{\leftarrow} G_{\lambda}\}$ be any other diagram. The situation is pictured below:



Define $\beta = \sum\limits_{\lambda} \delta_{\lambda}\alpha_{\lambda}$ so that $\beta: B \to B_1$. The sum makes sense from our finiteness assumption. Using the properties of α_{λ} and β_{λ} above, one easily shows that $\beta\beta_\lambda = \delta_\lambda$; hence the diagram commutes when β is defined in this way. Now suppose β has been defined so that $\beta\beta_\lambda = \delta_\lambda$. Then $\beta = \beta 1_B = \beta(\sum_\lambda \beta_\lambda\alpha_\lambda) = \sum_\lambda (\beta\beta_\lambda)\alpha_\lambda = \sum_\lambda \delta_\lambda\alpha_\lambda$. ||

2.4. Internal Direct Sums

We now tackle the problem from the inside out. Given a family $\{G_\lambda\}_{\lambda\varepsilon\Lambda}$ of <u>subgroups</u> of G, the set $\bigcup_\lambda G_\lambda$ need not be a subgroup of G (except in that important case noted in 3.1.2.). However, there is a smallest group H containing $\bigcup_\lambda G_\lambda$, and H is merely the intersection of all subgroups of G which contain $\bigcup_\lambda G_\lambda$. In this case, we say H is <u>generated</u> by $\bigcup_\lambda G_\lambda$. Each element $h \varepsilon H$ can be written as $h = g_{\lambda_1} + \cdots + g_{\lambda_k}$, where $g_{\lambda_i} \varepsilon G_{\lambda_i}$ $(i = 1,\cdots,k)$. Let H_λ be the subgroup generated by $\bigcup_{\mu\neq\lambda} G_\mu$. If we know that $G_\lambda \cap H_\lambda = \{0\}$ for all $\lambda \varepsilon \Lambda$, then each such sum above is unique. What we really want to know is: when is G isomorphic to the direct sum $\sum_\lambda G_\lambda$ of its subgroups G_λ? An answer follows.

2.4.1. <u>Proposition</u> Let G be a group and $\{G_\lambda\}_{\lambda\varepsilon\Lambda}$ a collection of subgroups of G. Let H_λ be the subgroup of G defined above. Then the following are equivalent:

(1) The diagram $\{\iota:G_\lambda \to G\}$ is universal with ends G_λ.

(2) G is generated by $\bigcup_\lambda G_\lambda$ and $G_\lambda \cap H_\lambda = \{0\}$ for all λ.

<u>Proof</u>. (1) implies (2): Since $\{\iota:G_\lambda \to G\}$ is assumed universal, there is a unique isomorphism $\Theta:G \to \sum_\lambda G_\lambda$ such that $\Theta\iota = \iota_\lambda$ for all λ (2.3.8. and 2.3.9.). Hence $\pi_\lambda\Theta\iota = \pi_\lambda\iota_\lambda = 1_{G_\lambda}$ for all λ; i.e., $\sum_\lambda \pi_\lambda\Theta = 1_G$, where there are only finitely many nonzero summands in the

sum because $\Theta(b) \in \sum_\lambda G_\lambda$ for each $b \in G$. If $b \in G$, then $[\Theta(b)](\lambda) = g_\lambda \in G_\lambda$ and $b = 1_G(b) = (\sum_\lambda \pi_\lambda \Theta)(b) = \sum_\lambda \pi_\lambda(\Theta(b)) = \sum_\lambda g_\lambda$, where there are only finitely many nonzero terms in the last sum. Thus b is equal to a finite sum of (nonzero) elements in the subgroups G_λ. Note that $\text{Ker}\,\Theta = \{g \in G \mid \Theta(g)$ is the zero function of $\Sigma G_\lambda\}$. To show that $G_\lambda \cap H_\lambda = \{0\}$ for all λ, suppose there is $g_\lambda \neq 0$ in $G_\lambda \cap H_\lambda$ for some fixed λ. Then $g_\lambda = g_{\mu_1} + \cdots + g_{\mu_k}$, where none of the indices μ_i are equal to λ. Note first that $g_\rho \in G_\rho$ implies $\iota_\lambda(g_\rho) = 0$ if $\lambda \neq \rho$. We now show that $\Theta(g_\lambda) = 0$. First,

$$\Theta(g_\lambda)(\mu) = \Theta(\iota(g_\lambda))(\mu) = (\Theta\iota)(g_\lambda)(\mu) = \iota_\lambda g_\lambda(\mu) = \begin{cases} g_\lambda & \text{if } \mu = \lambda \\ 0 & \text{otherwise.} \end{cases}$$

However, $\Theta(g_\lambda)(\lambda) = \sum_i \Theta(g_{\mu_i})(\lambda) = 0$ by the above remark. This means that $g_\lambda \in \text{Ker}\,\Theta$, a contradiction to the fact that Θ is an isomorphism. Thus $G_\lambda \cap H_\lambda = \{0\}$ for all λ.

(2) implies (1). As we already pointed out, the two conditions say that each element of G is uniquely written as a finite sum of elements from the various groups G_λ. If $\{\alpha_\lambda : G_\lambda \to G'\}$ is any other diagram, define $\beta : G \to G'$ by $\beta(g_{\lambda_1} + \cdots + g_{\lambda_k}) = \alpha_{\lambda_1}(g_{\lambda_1}) + \cdots + \alpha_{\lambda_k}(g_{\lambda_k})$. Then a straightforward argument shows that β is the unique homomorphism such that $\beta\iota = \alpha_\lambda$ for all λ. ||

When either of the above conditions holds, we have that G is isomorphic to a direct sum of its subgroups G_λ. Then G is called the (internal) <u>direct sum</u> of its subgroups G_λ. When (2) holds, the internal and external direct sums are isomorphic by virtue of the universality of the direct sum diagram. In particular, G is the internal direct sum of two of its subgroups H and K iff G is generated by $H \cup K$ and $H \cap K = \{0\}$. In this case, then, $G \simeq H \oplus K$. Any subgroup G_λ in an internal direct sum $G \simeq \sum_\lambda G_\lambda$ is called a <u>direct summand</u> of G.

An important theorem concerning this internal direct sum is found in Herstein [15], Theorem 4.J, page 162. Letting $R = Z$ in the theorem, the result may be stated as follows.

2.4.2. <u>Proposition</u> Any finitely generated Abelian group is a direct sum (internal, of course) of a <u>finite</u> number of cyclic subgroups.

2.5. Direct Sums and Exact Sequences

Before we enter a short study of Abelian group theory, we want to see how the splitting off of a direct summand is reflected in an exact sequence. For example, $0 \to A_1 \xrightarrow{\iota_1} A_1 \oplus A_2 \xrightarrow{\pi_2} A_2 \to 0$ is always an exact sequence. The most important results of this section are 2.5.1., 2.5.2., and 2.5.8.

2.5.1. <u>Proposition</u> Suppose $A \xrightarrow{f} B \xrightarrow{g} A$ is a sequence of groups and group homomorphisms such that $gf = 1_A$. Then the following hold:

(1) f is monic and g is epic.

(2) $B \simeq A \oplus \text{Ker } g \simeq \text{Im } f \oplus \text{Ker } g$ (internal sum).

<u>Proof.</u> Since $a = g(f(a))$, we have that for all $a \in A$ there is $b \in B$ (namely $b = f(a)$) such that $g(b) = a$. Thus g is epic. If $f(a) = 0$, then $0 = g(f(a)) = a$, and hence f is monic.

(2) First note that $\text{Im } f \cap \text{Ker } g = \{0\}$, for if $x \in \text{Im } f \cap \text{Ker } g$, we have $g(x) = 0$ and $x = f(a)$ for some $a \in A$. Thus $0 = g(x) = g(f(a)) = a$, and so $x = f(a) = 0$. Now define $\phi: B \to G = \text{Im } f \oplus \text{Ker } g$

by $\phi(b) = (f(g(b)), b - f(g(b)))$ and $\psi:G \to B$ by $\psi(f(a),b) = (f(a) + b)$, where $g(b) = 0$ (since $b \in \text{Ker } g$). Note that all the definitions make sense and ϕ and ψ are homomorphisms. A short calculation shows that $\phi\psi(f(a),b) = (f(a),b)$ and that $\psi\phi(b) = b$ for all $b \in B$. Thus ϕ and ψ are isomorphisms. Since f is monic, f defines an isomorphism from A onto $\text{Im } f$, and finally, then, $\text{Im } f \oplus \text{Ker } g \simeq A \oplus \text{Ker } g$. ||

2.5.2. <u>Lemma</u> Suppose $f:A \to B$ is monic and $B \simeq \text{Im } f \oplus B_1 \simeq A \oplus B_1$, where B_1 is some subgroup of B. Then there is a homomorphism $g:B \to A$ such that $gf = 1_A$.

<u>Proof.</u> Note that this is a converse to the above result. Every $b \in B$ can be written uniquely as $b = f(a) + b_1$, where $a \in A$ and $b_1 \in B_1$. Define $g:B \to A$ by $g(b) = a$. Since f is monic, we have that g is a well-defined homomorphism. If $a \in A$, we have also $g(f(a)) = a$ (since $b_1 = 0$ in this case). ||

2.5.3. <u>Definition</u> Let $\cdots \to G \xrightarrow{\zeta} H \xrightarrow{\delta} J \to \cdots$ be a given exact sequence of groups and group homomorphisms. We say the sequence <u>splits at H</u> if $\text{Ker } \delta = \text{Im } \zeta$ is a direct summand of H; i.e., H is the internal direct sum of $\text{Ker } \delta = \text{Im } \zeta$ and some other subgroup of H.

Consider a short exact sequence $0 \to A \to B \to C \to 0$. It obviously splits at A and C, since 0 is a direct summand of A and C is a direct summand of itself. Thus we make the following definition.

2.5.4. <u>Definition</u> A short exact sequence $0 \to A \to B \to C \to 0$ is said <u>to split</u> (or <u>splits</u>, or is <u>split exact</u>) if the sequence splits at B.

2.5.5. <u>Lemma</u> Suppose $\cdots \to G \xrightarrow{\zeta} H \xrightarrow{\delta} J \to \cdots$ is an exact sequence which splits at H. Then $H \simeq \text{Im } \delta \oplus \text{Im } \zeta$.

Proof. By definition, $H \simeq H_1 \oplus \text{Im } \zeta$. It remains to show that $H_1 \simeq \text{Im } \delta$. To this end, let $\rho = \delta|_{H_1}$ (i.e., ρ is simply the restriction of δ to H_1). First, $H_1 \cap \text{Ker } \delta = H_1 \cap \text{Im } \zeta = \{0\}$; this means that $\text{Ker } \rho = \{0\}$, and so ρ is monic. We show $H_1 \simeq \text{Im } \rho = \text{Im } \delta$: certainly, $\text{Im } \delta$ contains $\text{Im } \rho$. Let $j \in \text{Im } \delta$, so $j = \delta(h)$ for some $h \in H$. Write $h = h_1 + \zeta(g)$, so $j = \delta(h) = \delta(h_1) + \delta(\zeta(g)) = \delta(h_1)$, using the fact that $\delta\zeta = 0$. Thus $\text{Im } \delta$ is contained in $\text{Im } \rho$. Since ρ is monic, it establishes an isomorphism between H_1 and $\text{Im } \delta$. ||

2.5.6. Corollary Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a split exact sequence. Then $B \simeq A \oplus C$.

Proof. We know $B \simeq \text{Im } f \oplus \text{Im } g$ by the previous lemma. But $\text{Im } g = C$ since g is epic, and $\text{Im } f = A$ since f is monic. ||

2.5.7. Corollary An exact sequence of groups

$\cdots \to G \xrightarrow{\zeta} H \xrightarrow{\delta} J \to \cdots$ splits at H if either of the following holds:

(1) There is a homomorphism $\sigma: H \to G$ such that $\sigma\zeta = 1_G$.

(2) There is a homomorphism $\rho: J \to H$ such that $\delta\rho = 1_J$.

Proof. (1) We have $H \simeq \text{Im } \zeta \oplus \text{Ker } \sigma$, by 2.5.1., and hence the sequence splits. Note that $\text{Im } \delta \oplus \text{Im } \zeta \simeq H \simeq G \oplus \text{Ker } \sigma$, using the previous lemma 2.5.5. and $G \simeq \text{Im } \zeta$.

(2) Using 2.5.1. again, we have $H \simeq J \oplus \text{Ker } \delta$ and so the sequence splits. Also note that $\text{Im } \delta \oplus \text{Im } \zeta \simeq H \simeq J \oplus \text{Ker } \delta$. ||

2.5.8. Corollary Let $0 \to A \xrightarrow{\kappa} B \xrightarrow{\sigma} C \to 0$ be a short exact sequence. Then the following are equivalent:

(1) The sequence splits.

(2) There is a homomorphism $\rho: B \to A$ such that $\rho\kappa = 1_A$.

(3) There is a homomorphism $\tau : C \to B$ such that $\sigma\tau = 1_C$.

<u>Proof</u>. The homomorphisms of (2) and (3) are commonly referred to as "splitting homomorphisms." We have already proven (2) implies (1) and (3) implies (1) in the previous corollary. The implication (1) implies (2) is precisely the statement of 2.5.2.

To show (1) implies (3), we have the argument of 2.5.2. as a guide. We have that σ is epic, and its kernel is disjoint from B_1, where $B \simeq \text{Im}\,\kappa \oplus B_1 \simeq \text{Ker}\,\sigma \oplus B_1$ for some subgroup B_1 of B, since the sequence splits. Hence the restriction of σ to B_1 defines an isomorphism ζ from B_1 onto C. If we take $\tau(c) = \zeta^{-1}(c)$, we have a well-defined homomorphism such that $\sigma\tau = 1_C$. ||

CHAPTER III

TOPICS IN ABELIAN GROUPS

3.1. Introduction and Zorn's Lemma

The purpose of this chapter is to lay a foundation for the Abelian group theory we will have occasion to use elsewhere in our work. By any standard, the choice of topics is highly utilitarian, and is not intended to be an exhaustive account of the subject. The reader would to well to consult the standard references, Fuchs [8] and Kaplansky [18], for more details. Much of the work we do can be found there. However, Fuchs' detailed treatise is an expansive account of the entire subject; it is, therefore, a true reference work. Kaplansky's little book, on the other hand, is concise and affable. In fact, Kaplansky's approach and style served as an inspiration for the writing of this chapter. His Notes at the end of the book read like a diary of a worker in the field, complete with confessions and suggestions. It is hoped that the Summary at the end of this work can capture a little of that flavor.

For convenience, the cyclic groups are given standard notation: Z denotes the group of integers and $Z(n)$ the finite cyclic group of order n. Any finite cyclic group is isomorphic to some $Z(n)$, for if G is such a group with $o(G) = n$, and g is a generator of G, the homomorphism $\psi: G \to Z(n)$ defined by $\psi(x) = [j]$, where $x = jg$ and $[j]$ is the equivalence class determined by j, is an isomorphism. Any non-finite cyclic group is isomorphic to Z. Referring to example 4, page 93 of Herstein [15], we have the usual isomorphism $Z/(n) \simeq Z(n)$, where (n) is the subgroup of Z consisting of all multiples of n. More incisive examples of groups will be given as we progress.

Two set theoretic tools are usually needed in the study of groups: an assumption equivalent to the so called Zorn's Lemma and a working knowledge of ordinal and cardinal numbers. We will limit ourselves only to Zorn's Lemma. For reference, we give the following outline.

A relation R is a set of ordered pairs; i.e., R is a set any member of which is an ordered pair (since we are eventually going to define "order relation" in terms of "ordered pair," this seeming circumlocution can be avoided by defining $(x,y) = \{\{x\}, \{x\} \cup \{y\}\}$; this is a reduction of the concept of ordered pair to concepts involving sets, and is due to N. Wiener [34]). The domain of R is the collection of all first components of all the elements of R, and the range of R is the collection of all second components of all the elements of R. If we let $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$ for any sets A and B, it is clear that given a relation R, $R \subseteq$ (domain R) $\times$ (range R).

Let R be a relation and set $X =$ (domain R) $\cup$ (range R). We say:

(1) R is reflexive if $(x,x) \in R$ for all $x \in X$.

(2) R is symmetric if for all $x \in X$ and $y \in X$, $(x,y) \in R$ implies $(y,x) \in R$.

(3) R is transitive if for all x, y, and z in X, $(x,y) \in R$ and $(y,z) \in R$ imply $(x,z) \in R$.

An equivalence relation is any relation with properties (1)-(3). An ordering is any transitive relation. A relation written $<$ orders a set X if $<$ is transitive on X. If $A \subseteq X$ and X is ordered by $<$, then $x \in X$ is an <u>upper bound</u> of A if for all $y \in A$, either $y < x$ or $y = x$. A <u>total ordering</u> is an ordering $<$ such that

(4) if $x < y$ and $y < x$, then $x = y$; and

(5) either $x < y$ or else $y < x$ when x and y are distinct members of (domain $<$) $\bigcup$ (range $<$).

Thus, for example, for familiar "less than" in Z has properties (3)-(5), and so Z is totally ordered by it. A set with a relation which totally orders it is called a <u>chain</u>.

Sometimes one says that in a totally ordered set T any two elements are comparable; i.e., given $x \in T$ and $y \in T$, either $x = y$ or else $x < y$ or else $y < x$. Hence the use of the word "chain": the elements of T are "locked in place."

We can now state Zorn's Lemma, which we assume is true:

3.1.1. <u>Zorn's Lemma</u> Let P be a nonempty partially ordered set. If every chain $C \subseteq P$ has an upper bound in P, then P contains a maximal element (i.e., there exists $u \in P$ such that $s \leq u$ for all $s \in P$). It is known that the statement of Zorn's Lemma is equivalent to many of the forms of the Axiom of Choice (one of which asserts that given any family $\{A_\lambda\}_{\lambda \in \Lambda}$ of sets, there is a function $f:\Lambda \to \bigcup_\lambda A_\lambda$ such that $f(\lambda) \in A_\lambda$ for all $\lambda \in \Lambda$; i.e., $\prod_\lambda A_\lambda \neq \emptyset$--of course, we have been tacitly assuming this all along!).

Let us practice the use of Zorn's Lemma as we will most often use it.

3.1.2. <u>Proposition</u> Let G be a group with subgroups H and K such that $H \bigcap K = (0)$. Then there is a subgroup M of G maximal with respect to the properties $K \subseteq M$ and $H \bigcap M = (0)$.

<u>Proof</u>. Take for P in 3.1.1. the set $P = \{S | S$ is a subgroup

of G and $H \cap S = (0)\}$. Then $P \neq \emptyset$, since $K \varepsilon P$, and P is partially ordered by set theoretic inclusion $\subseteq$. Let $C = \{A_\lambda\}_{\lambda \varepsilon \Lambda}$ be a chain in P. We must show that there is a subgroup T of G such that $T \varepsilon P$ and $A_\lambda \subseteq T$ for all $A_\lambda \varepsilon C$. Normally, one would never pick $T = \bigcup_\lambda A_\lambda$ (because it is rarely a subgroup of G!). But now it is a subgroup because C is a chain. First, T certainly contains 0. Let x and y be in T. Then there are elements μ and ν in Λ such that $x \varepsilon A_\mu$ and $y \varepsilon A_\nu$. But because we have a chain, either $A_\mu \subseteq A_\nu$ or $A_\nu \subseteq A_\mu$. Suppose $A_\mu \subseteq A_\nu$. Then both x and y are in A_ν, and so $x - y \varepsilon A_\nu \subseteq T$. Repeat the argument if really $A_\nu \subseteq A_\mu$. In any event, T is a subgroup of G, and $T \cap H = (\bigcup_\lambda A_\lambda) \cap H = \bigcup_\lambda (A_\lambda \cap H) = (0)$, since $H \cap A_\lambda = (0)$ for all $\lambda \varepsilon \Lambda$. Hence $T \varepsilon P$ and T is the required maximal element of the chain C. By Zorn's Lemma, then, there is $M \varepsilon P$ such that $M \supseteq S$ for all $S \varepsilon P$. In particular, $M \supseteq K$, $M \cap H = (0)$ and M is maximal with respect to these properties. ||

Notice how useful this result might prove to be in direct sum arguments: we produce a "large" subgroup which is as disjoint from a given subgroup as it possibly can be.

3.2. Divisible Groups

We now define and give important properties for a class of groups that has come to play a large role in Abelian group theory. The notion of divisibility is due to Reinhold Baer [1] who exploited it successfully in his study of direct summands of groups. He showed that divisible groups have the property that they are direct summands of every group which contains them as a subgroup. We prove this result in 5.5.7. The structure of divisible groups is completely known, and we state the main result along this line in 3.5.2.

If one were to conjecture the "basic" difference between Z and Q as additive groups, it might be that given $r \in Q$ and an integer $n \neq 0$ there exists a (unique) rational $s \in Q$ such that $ns = r$; furthermore, this kind of statement is plainly false in Z itself. We are thus led to the following definition.

3.2.1. Definition An Abelian group G is said to be divisible if for all $g \in G$ and $n \neq 0$ there exists $y \in G$ such that $ny = g$.

Formally, then, a given group G is not divisible if there exists $g \in G$ and an integer $n \neq 0$ such that for every element $y \in G$, $ny \neq g$. The element y of the definition need not be unique, for if y has finite order m and $(n,m) \neq 1$, select a nonzero common divisor d of m and n. Set $y' = (1 + m/d)y$. Then $y' \neq y$ but $ny' = g$. However, if no element of a divisible group has finite order (see 3.3. for a discussion of these groups), then given $n \in Z$ and $g \in G$, there is precisely one $y \in G$ such that $ny = g$ (just like in Q!).

Some immediate properties of divisible groups follow.

3.2.2. Proposition Let $n|x$ denote the fact that $ny = x$, where x and y are elements of a group G (given n and x, y always exists in a divisible group).

(1) If $m = o(x)$ and $(n,m) = 1$, then $n|x$.

(2) If $n|x$ and $n|y$, then $n|(rx + sy)$ for all integers r and s.

(3) If $G = A \oplus B$, then G is divisible if and only if both A and B are divisible.

(4) Let $G = \sum_{\lambda} A_{\lambda}$. Then G is divisible if and only if each A_{λ} is divisible.

(5) Let G and H be groups and $\psi: G \to H$ an epimorphism. Then

G divisible implies H divisible (i.e., the homomorphic image of a divisible is divisible).

(6) No nontrivial cyclic group is divisible.

Proof. (1) and (2) follow directly from the definition.

(3) It is enough to show if g is any element of G and $g = a + b$, with $a \in A$ and $b \in B$, then $n|g$ if and only if $n|a$ (in A) and $n|b$ (in B). Suppose $n|g$; then $ny = g$ for some $y \in G$. Write $y = a_1 + b_1$, so that $na_1 + nb_1 = ny = g = a + b$. Since the sum is direct, we have $na_1 = a$ and $nb_1 = b$, which is what we desired. The converse follows directly from (2).

(4) Suppose G is divisible, and write, for a fixed $\mu \in \Lambda$, $G = A_\mu \oplus \sum_{\lambda \neq \mu} A_\lambda$. Then A_μ is divisible. Conversely, suppose A_λ is divisible for all $\lambda \in \Lambda$. Let $f \in \sum_\lambda A_\lambda$ and select an integer $n \neq 0$. Let $\lambda_1, \cdots, \lambda_k$ be precisely those indices such that $f(\lambda_i) \neq 0$ for $i = 1, \cdots, k$. Then there exist $a_{\lambda_i} \in A_{\lambda_i}$ such that $na_{\lambda_i} = f(\lambda_i)$ for $i = 1, \cdots, k$. Define $f' \in \Sigma A_\lambda$ by

$$f'(\mu) = \begin{cases} a_{\lambda_j} & \text{if } \mu = \lambda_1, \cdots, \lambda_k \\ 0 & \text{otherwise.} \end{cases}$$

Then $nf' = f$, and so G is divisible.

(5) Let $n \in Z$, $h \in H$ and $g \in G$ such that $\psi(g) = h$. Select $g' \in G$ such that $ng' = g$. Then $n(\psi(g')) = h$, and so H is divisible.

(6) Let G be a finite cyclic group with generator g. Then $mg = 0$ for some integer $m \neq 0$. Let $y \in G$. Then $ng = y$ for some $n \in Z$, since G is cyclic. But then $my = n(mg) = 0$, and so for all

$y \in G$ there is an integer $m \in Z$ such that $my \neq g$; i.e., G is not divisible. The only infinite cyclic group is Z, and it is not divisible. ||

3.2.3. Corollary (1) Q and Q/Z are divisible.

(2) Neither Q nor Q/Z can be written as a direct sum of cyclic groups.

Proof. (1) Q is clearly divisible, and Q/Z is divisible by (5) of the previous result, using the projection $\eta: Q \to Q/Z$.

(2) If $Q \simeq \sum_\lambda Z(n_\lambda)$, then, by (4) from the above, each $Z(n_\lambda)$ is divisible, which contradicts (6) of 3.2.2. The same argument works for Q/Z because it too is divisible. ||

The corollary says something quite significant: although the cyclic groups play an important role as "building blocks" for Abelian groups (see, for example, 2.4.2.), not all groups can be represented as direct sums of cyclic groups (namely Q!). It is the divisibility property that prevents it, so we would hope we could "split" off the divisible portion of G. We prove this fact is indeed true in 5.5.9., but we record the result here for reference.

3.2.4. Theorem [Fuchs, 8] Every group G can be written as $G = D \oplus R$, where D is the unique maximal divisible subgroup of G and R is reduced; i.e., the only divisible subgroup of R is (0).

Because divisible groups enjoy a rather simple structure theory (see §19 of Fuchs [8]), the study of the structure of Abelian groups reduces to that of reduced groups. More properties of divisible groups will be developed in Chapter V. One more definition before we leave!

3.2.5. Definition A group G is said to be p-divisible (where p is a fixed prime) if for all $g \in G$ and $n \in Z$, there is $y \in G$

such that $p^n y = g$.

The following useful fact comes directly from the definitions.

3.2.6. Lemma A group G is divisible iff $G = nG$ for all $n \in Z$ (recall that $nG = \{ng \mid g \in G\}$ is a subgroup of G). Also a group G is p-divisible iff $G = pG$.

3.3. Pure Subgroups. Torsion Groups.

The two concepts of this section will be introduced together and developed simultaneously so that their interaction might be made more clear. The notion of purity (due to Prüfer [31] who called them "serving" subgroups) was first introduced as a "substitute" for direct summand; it has come to play a large role in the theory (namely, Chapter VII). Torsion groups admit a fairly simple structure theory.

3.3.1. Definition Let G be a group. Set

$$T(G) = \{x \in G \mid nx = 0 \text{ for some } n \in Z\}.$$

Then $T(G)$ is a subgroup of G; it consists of all the elements of G which have finite order. We call the subgroup $T(G)$ the torsion subgroup of G. A group G is called torsion-free if $T(G) = (0)$, and G is called a torsion group if $T(G) = G$. Groups G for which $(0) \subset T(G) \subset G$ are called mixed; i.e., they are neither torsion nor torsion-free.

Hence Z and Q are torsion-free, whereas every $Z(m)$ is a torsion group. The group $Z \oplus Z(m)$ is mixed.

3.3.2. Proposition Let G be a group. Then $T(G/T(G)) = (0)$; i.e., regardless of G, $G/T(G)$ is torsion-free. Hence every group G can be imbedded in an exact sequence $0 \to A \to B \to C \to 0$ with A a torsion group and B torsion-free.

Proof. Let $g + T(G)$ be an element of $G/T(G)$ and suppose that $n(g + T(G)) = 0$, where n is a nonzero integer. Then $ng \in T(G)$, and so $m(ng) = 0$ for some integer m. This means $(mn)g = 0$, and so $g \in T(G)$. Hence $g + T(G) = 0$, and $G/T(G)$ is torsion-free. The exact sequence of the proposition is therefore

$$0 \to T(G) \overset{\iota}{\to} G \overset{\eta}{\to} G/T(G) \to 0. \quad ||$$

3.3.3. Definition Let G be a group and p a fixed prime. Set

$$Tp(G) = \{x \in G \mid p^n x = 0 \text{ for some integer } n \geq 0\}.$$

Then $Tp(G)$ is a subgroup of G; in fact, it is a subgroup of $T(G)$, and is called the p-primary subgroup of G. We say G is a p-group if $G = Tp(G)$; hence, every element in a p-group has order a power of p. If G is known to be a torsion group to begin with, we write G_p instead of $Tp(G)$.

The groups G_p play an important role in the structure theory for torsion groups.

3.3.4. Theorem Every torsion group G has a unique decomposition into a direct sum of groups G_p for various primes p.

Proof. Let G be a torsion group and G_p its subgroups consisting of all elements of prime p power order for each prime p. Let $x \in G$ with $mx = 0$. We may factor m into a product of powers of primes:

$$m = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n} \quad ; \quad \text{set } m_j = m/p_j^{a_j}.$$

Then the greatest common divisor of $\{m_1, \cdots, m_n\}$ is 1, and so there must exist integers $b_1, \cdots, b_n$ such that $b_1 m_1 + \cdots + b_n m_n = 1$. Thus we may conclude the following:

$$x = b_1 m_1 x + \cdots + b_n m_n x \quad \text{and} \quad p_j^{a_j}(m_j x) = mx = 0.$$

This means that $m_j x \in G_{p_j}$ for $j = 1, \cdots, n$. Thus G is generated by $\{G_p\}_{p \in Z}$.

We must now show that every such representation is unique. First, note that $G_p \cap G_q = (0)$ for all primes $p \neq q$. For if $x \in G_p \cap G_q$, then $o(x) = p^j$ for some j and $o(x) = q^k$ for some k. Since $p \neq q$, we must have that $o(x) = 1$, and so $x = 0$. Hence suppose we have

$$x = a_1 + \cdots + a_k = b_1 + \cdots + b_k$$

where both a_i and b_i are elements of G_{p_i} for $i = 1, \cdots, k$.

Then, subtracting, we have $a_1 - b_1 = (b_2 - a_2) + \cdots + (b_k - a_k)$.

The left hand side has order a power of p_1, while the right hand side has order equal to a product of powers of $p_2, \cdots, p_n$. Since these two orders are relatively prime, an argument just like the one above gives that $o(a_1 - b_1) = 1$, and so $a_1 - b_1 = 0$; i.e., $a_1 = b_1$. Clearly the argument holds for any index i, and so we conclude that $a_i = b_i$ for all $i = 1, \cdots, k$. ||

Note that the theorem says that a torsion group can be expressed as a direct sum of its p-subgroups in precisely one way: a typical summand is the p-primary subgroup for some fixed prime p. For example, $Z(m)$ is a torsion group, and if m is factored into powers of various primes p as displayed on page 38, we reach the following conclusion.

3.3.5. For all m, $Z(m) \simeq Z(p_1^{a_1}) \oplus \cdots \oplus Z(p_n^{a_n})$.

The concept of purity is one of relative divisibility. In most cases pure subgroups are easy to handle, and there are many theorems, some of which we give, that state when pure subgroups are direct summands. Another pleasant feature: the union of an ascending chain of pure subgroups is pure, something that fails in general for direct summands (see Fuchs [8], page 77, for an example of this).

3.3.6. Definition Let G be a group and H a subgroup of G. We say H is a pure subgroup of G if $H \cap nG = nH$ for all integers n. Further, H is said to be a p-pure subgroup fo G (for a fixed prime p) if $H \cap p^nG = p^nH$ for all $n \in Z$.

In the pure case, the definition can be thought of in this way: let $h \in H$ and $n \in Z$. If there exists $g \in G$ such that $h = ng$, then there exists $h_1 \in H$ such that $ng = h = nh_1$. Thus every element which is divisible in G is divisible in H whenever H is pure in G. Notice how we also produce an element of G, namely $g - h_1$, which is annihilated by n; i.e., $n(g - h_1) = 0$.

The following proposition brings together many useful, yet immediate, properties these concepts enjoy. The proofs are easy for the most part, so many are omitted.

3.3.7. Proposition Let A, B, and G be groups.

(1) If A is a direct summand of B, then A is a pure subgroup of B.

(2) Pure subgroups of divisible groups are divisible subgroups, and, conversely, if A is a divisible subgroup of B, then A is a pure subgroup of B.

(3) T(G) is a pure subgroup of G, for all G. If G is divisible, then so is T(G).

(4) If G/A is torsion-free, then A is pure in G.

(5) If A is pure in B and B is pure in G, then A is

pure in G (thus purity is a transitive property of groups).

(6) The statement of (5) remains true if "pure" is replaced by "p-pure."

(7) A is pure in B iff A is p-pure in B for all primes p.

(8) In a p-group, purity is equivalent to p-purity.

(9) If A is a pure subgroup of G and B contains A such that B/A is a pure subgroup of G/A, then B is pure in G.

(10) The union of an ascending chain of pure subgroups of G is again a pure subgroup of G.

(11) Suppose $\{B_\lambda\}$ and $\{A_\lambda\}$ are families of groups. If B_λ is a pure subgroup of A_λ for all λ, then $\sum_\lambda B_\lambda$ is a pure subgroup of $\sum_\lambda A_\lambda$.

(12) Let H be a pure subgroup of G, and suppose $y \in G/H$ has finite order $n = o(y)$. Then there exists $x \in G$ such that $y = x + G$ and $o(x) = n$, also.

(13) $T(\sum_\lambda A_\lambda) \simeq \sum_\lambda T(A_\lambda)$ for any family of groups $\{A_\lambda\}$.

<u>Proof</u>. (9) Let $b \in B$ and $b = ng$ for $g \in G$. Set $b^* = b + A$ and $g^* = g + A$. Then $b^* = ng^*$, and so there exists $r \in B/A$ such that $b^* = nr$. We may write $r = b' + A$ for some $b' \in B$. Then $b = nb' + a$ for some element $a \in A$. But then $a = n(g - b')$, and because A is pure in G, we have $a = na'$ for some $a' \in A$. Hence $b = n(b' + a')$, and $b' + a' \in B$, as desired.

(11) Let $f \in \Sigma B_\lambda$ and $n \in Z$. Suppose there exists $g \in \Sigma A_\lambda$ such that $f = ng$. Let $\lambda_1, \cdots, \lambda_k$ be precisely those indices such that $f(\lambda_i) \neq 0$ for $i = 1, \cdots, k$. From the k equations $f(\lambda_i) = ng(\lambda_i)$ we may select $b_{\lambda_i} \in B_{\lambda_i}$ such that $f(\lambda_i) = nb_{\lambda_i}$. Set

$$f'(\mu) = \begin{cases} b_{\lambda_i} & \text{if } \mu = \lambda_1, \cdots, \lambda_k \\ 0 & \text{otherwise} \end{cases}$$

Then $f = nf'$, and the result follows.

(12) Since $y \in G/H$, there must exist $g \in G$ such that $y = g+H$. Hence $0 = ny = ng + H$, and so $ng \in H$. Thus $ng = h'$ for some $h' \in H$, and because H is pure in G, we may select $h \in H$ such that $nh = h' = ng$. Set $x = h - g$. Immediately, $o(x) = n$, and $x^* = y$. ||

Pure subgroups are very convenient to use and appear many places, sometimes unexpectedly. In the next proposition, which gives a homological property of a group, purity plays an important role.

3.3.8. <u>Proposition</u> Let G be any group. Then there is a direct sum of cyclic groups P and an epimorphism $\theta: P \to G$ such that $K = \operatorname{Ker} \theta$ is a pure subgroup of P (i.e., every G is the homomorphic image of a direct sum of cyclics with pure kernel).

<u>Proof.</u> Let $G = \{g_\lambda \mid \lambda \in \Lambda\}$, and set $P = \sum_\lambda \langle g_\lambda \rangle$; thus P is the direct sum of the cyclic groups generated by all the elements of G.

We define $\theta: P \to G$ in the following way. Let $f \in P$; then there are indices $\lambda_1, \cdots, \lambda_k \in \Lambda$ and elements $n_{\lambda_i} g_{\lambda_i} \in \langle g_{\lambda_i} \rangle$ such that

$$f(\mu) = \begin{cases} n_{\lambda_i} g_{\lambda_i} & \text{if } \mu = \lambda_1, \cdots, \lambda_k \\ 0 & \text{otherwise} \end{cases}$$

Define $\theta(f) = \sum_{i=1}^{k} n_{\lambda_i} g_{\lambda_i}$; this sum is in G. Clearly, θ is an epimorphism, for if $g_\rho \in G$ is given, we have $\theta(h) = g_\rho$, where $h(\mu) = g_\rho$ if $\mu = \rho$ and 0 elsewhere.

We wish to show that K is pure in P. Let f be as above with $nf \in \operatorname{Ker} \theta$. Then there is an index λ such that $\theta(f) = g_\lambda$, and $0 = \theta(nf) = ng_\lambda$. Define $h \in P$ by $h(\mu)$ equals g_λ if $\mu = \lambda$, and $h(\mu)$ is zero otherwise. Thus $nh = 0$. Let $y = f - h \in P$. Then $\theta(y) = \theta(f) - g_\lambda = 0$, and so $y \in K$. But $ny = nf - nh = nf$, and so $\operatorname{Ker} \theta$ is pure in P. ||

3.4. Algebraically Compact Groups

We introduce now an extremely useful class of groups which were first studied extensively by Kaplansky [18]. At the time he was studying compact groups, and to simplify the problem somewhat, he enlarged the class, calling the enlarged class algebraically compact. As it turns out, these groups admit a very elegant homological characterization. This will be given in Chapter VIII. For now we indulge in the elementary homological properties of these groups.

3.4.1. A group G is <u>algebraically compact</u> (AC) if G is a direct summand of any group in which it is contained as a pure subgroup.

At the outset, of course, we must prove that there are any such groups at all. This is accomplished in the first proposition. The last assertions of this section explore the homological techniques that can be brought to bear upon this class of groups.

3.4.2. <u>Proposition</u> Let C be a pure torsion cyclic subgroup of a group G. Then C is a direct summand of G (i.e., cyclic torsion groups are algebraically compact).

<u>Proof.</u> The proof proceeds by a maximality argument which is typical of many found in algebra. Let $C = \langle a \rangle$ be the subgroup in question. Since C is torsion, $na = 0$ for some positive integer n,

and so $nC = (0)$. By the purity of C in G, we have $C \cap nG = (0)$. Now apply the typical Zorn's lemma argument (3.1.1.) and select a subgroup H of G maximal with respect to the properties $H \supseteq nG$ and $H \cap C = (0)$. We show that $G = H \oplus C$. Since $H \cap C = (0)$ already, it suffices to show that G is the group generated by H and C. Let $R = \langle H,C \rangle$ denote this group, and suppose indeed that there is a nonzero element $g' \in G$ such that $g' \notin R$. Then $\langle H,g' \rangle$ properly contains H, and $\langle H,g' \rangle \supseteq nG$ since $H \supseteq nG$. Thus $\langle H,g' \rangle \cap C \neq (0)$, and so there are elements $h' \in H$ and $c' \in C$ such that $c' = h' + kg'$ for some integer k. Thus $kg' \in R$, and so $k(g' + R) = 0$. This proves that G/R is a torsion group. Since every torsion group is a direct sum of p-groups (3.3.4.), there must exist a nonzero element $g + R \in G/R$ such that $p(g + R) = 0$ for some prime p. This means that $pg \in R$, and so there exist $h \in H$ and $t \in Z$ such that $h + pg = ta$. Since $nG \subseteq R$, p must divide n; write $n = pm$. Thus $mh + ng = mta$; the left hand side is an element of H, while the right hand side is in C. Hence $mta \in C \cap H = (0)$, and so $n|mt$; i.e., $p|t$. Set $t = ps$ and consider the element $z = g - sa$.

First, $z \notin H$, for if it is, then $g \in R$, a contradiction to the choice of g. Hence $\langle H,z \rangle \cap C \neq (0)$ by the choice of H, and so there is a nonzero element $c \in C$ such that $c = h + rz$. This implies that $rz \in R$, so $r(z + R) = 0$. Since $z + R = g + R$ has order p in G/R, we have $p|r$; write $r = pr'$. Finally, then, $c = h + rz = h + r'pz = h + r'(pg - psa) = h + r'(ta - h - ta) = h - r'h \in C \cap H$. Since $c \neq 0$, this contradicts the fact that $C \cap H = (0)$. Hence $G = R$. ||

3.4.3. <u>Proposition</u> [Kaplansky, 18] Let G be a group. Then the following are equivalent:

(1) G is algebraically compact.

(2) Suppose B and A are any groups and B is a pure subgroup of A. Let $f:B \to G$ be any group homomorphism. Then f can be extended to a homomorphism from A to G. Pictorially,

$$\begin{array}{ccccc} 0 & \to & B & \xrightarrow{\iota} & A \\ & & f\downarrow & \swarrow g & \\ & & G & & \end{array}$$

We can find a homomorphism $g:A \to G$ such that $f = g\iota$, where $\iota:B \to A$ is the usual injection of a subgroup into the containing group.

<u>Proof</u>. (2) implies (1). Assume the conditions of (2) and let G be a pure subgroup of B. By assumption, there is a homomorphism $g:B \to G$ such that $1_G = g\iota$. Hence G is a direct summand of B, by 2.5.1.

(1) implies (2). Suppose G is AC, B is a pure subgroup of A and $f:B \to G$. We follow a construction known as a "pushout"; we will use it again in 5.2.4. In this method, we "push out" to $B \oplus G$ and extract a convenient subgroup, which, when factored out, makes everything work!

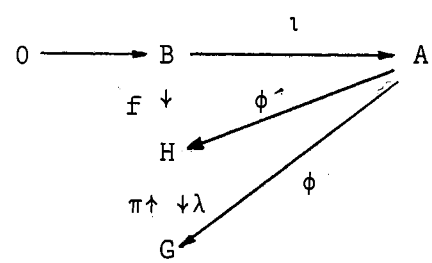
Let $E = \{(b,-f(b)) \mid b \in B\}$. Then E is a subgroup of $A \oplus G$. Take $H = (A \oplus G)/E$ and consider $\eta:G \to H$ defined by $\eta(g) = (0,g)+E$. Now $\text{Ker}\,\eta = \{g \in G \mid (0,g) \in E\} = \{g \in G \mid g = -f(b)$ for some $b \in B$ and $b = 0\} = (0)$. Hence η is monic and affords us with an imbedding of G in H. Hence G may be considered a subgroup of H. We wish to show that $\eta(G)$ is a pure subgroup of H, and so thereby prove that G is a pure subgroup of H.

Suppose $(0,g) + E = n((a,g') + E)$ for $n \in Z$, $a \in A$, $g \in G$ and $g' \in G$. Then $(-na, g - ng') \in E$, and so there exists $b \in B$ such that $b = -na$ and $g - ng' = -f(b)$. Since B is a pure subgroup of A, the first equation allows us to conclude that $b = nb''$ for some b'' in B. Set $g'' = f(b'')$, so $g'' \in G$. Then $b = nb''$ implies $g - ng' = -f(b) = -nf(b'') = -ng''$. This means that $g + n(g'' - g') = 0$. Hence $(0,g) + E = n((0,g''-g') + E)$, and so $\eta(G)$ is a pure subgroup of H.

By assumption, then, G is a direct summand of H, and so the map $\sigma((a,g) + E) = g$ is a homomorphism from H to G. Define $\phi:A \to G$ by $\phi(a) = \sigma((a,0) + E)$. Let $b \in B$, so $\iota(b) = b \in A$. Then $\phi(\iota(b)) = \sigma((b,0) + E) = \sigma((0,f(b)) + E) = f(b)$, and so ϕ is the desired extension of f. ||

3.4.4. <u>Corollary</u> Any direct summand of an AC group is itself AC.

<u>Proof.</u> Let G be AC and H a direct summand of G. Suppose B is a pure subgroup of A. Let $\pi:G \to H$ and $\lambda:H \to G$ be the projection out of and the injection into G, respectively. Then:

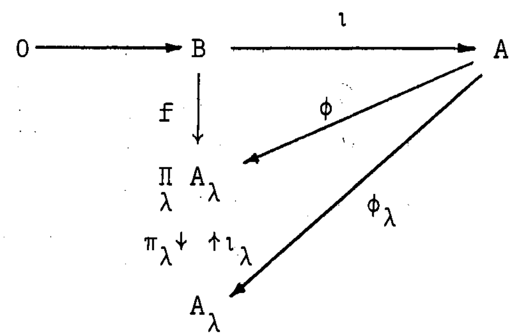


Here $f:B \to H$ is an arbitrary homomorphism. By the assumption that G is AC, λf can be extended to all of A, so there is a homomorphism $\phi:A \to G$ such that $\lambda f = \phi\iota$. Define $\phi':A \to H$ by $\phi' = \pi\phi$. Then $\phi'\iota = f$, and so ϕ' is the desired extension of f to all of A. ||

3.4.5. <u>Corollary</u> Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of AC groups. Then $\prod_\lambda A_\lambda$ is also algebraically compact.

<u>Proof.</u> Let π_λ and ι_λ be the injections and projections as we previously defined them, and let $f:B \to \prod_\lambda A_\lambda$ be an arbitrary homomor-

phism. Since each A_λ is AC, there are homomorphisms $\phi_\lambda : A \to A_\lambda$ such that for all $\lambda \varepsilon \Lambda$, $\phi_\lambda \iota = \pi_\lambda f$. The situation is as follows:



We seek $\phi : A \to \prod_\lambda A_\lambda$ such that $f = \phi\iota$. Set $\phi(a)(\lambda) = \phi_\lambda(a)$ for $a \varepsilon A$ and $\lambda \varepsilon \Lambda$. Now let $b \varepsilon B$ and $\lambda \varepsilon \Lambda$. Then $\phi(\iota(b))(\lambda) = \phi_\lambda(\iota(b)) = \pi_\lambda(f(b)) = f(b)(\lambda)$. Hence $\phi(\iota(b)) = f(b)$ for all $b \varepsilon B$. ||

3.4.6. <u>Corollary</u> A direct summand of a direct product of cyclic torsion groups is algebraically compact.

<u>Proof.</u> Put the four previous results together.

3.5. Basic Subgroups

The notion of a "basic" subgroup of a group is due to Kulikov [22] who first introduced them and proved that they formed a fundamental tool for the study of p-groups. We will define basic subgroups and prove that torsion groups indeed have basic subgroups; this is a simpler version of a more general result due to Kulikov [22] and Fuchs [9]. We will then use basic subgroups to give an elegant proof for a generalization of 3.3.5. Basic subgroups will also be used in our characterization of torsion groups (see 8.2).

3.5.1. <u>Definition</u> Let G be a group. We say B is a basic subgroup of G if

(1) B is a pure subgroup of G.

(2) B is a direct sum of cyclic groups.

(3) G/B is divisible.

To show that torsion groups have basic subgroups, we claim that it suffices to show that all p-groups have basic subgroups. Indeed, suppose we have show this to be the case, and let T be any torsion group. Since $T \simeq \sum_p G_p$ (3.3.4.), we can construct a basic subgroup of T from those of each G_p: let B_p be a basic subgroup of G_p, and take $B = \sum_p B_p$. Certainly, B is a direct sum of cyclic groups because each is such a sum. By (11) of 3.3.7., we see B is a pure subgroup of T. Finally, $T/B \simeq \sum_p G_p / \sum_p B_p \simeq \sum_p (G_p/B_p)$, and since each G_p/B_p is divisible, so is the group T/B. This means B is a basic subgroup of T.

From now on in <u>this</u> section, then, every group G wil be assumed to be a p-group. Our work will be simplified by recalling that in any p-group, divisibility is equivalent to p-divisibility.

3.5.2. <u>Lemma</u>. If m and n are relatively prime nonzero integers such that $mg' = ng$ for $g \neq 0$ in G and $g' \in G$, then there exists $g'' \neq 0$ in G such that $mg'' = g$.

<u>Proof</u>. Write $1 = mr + ns$ for integers r and s. Then $g = (rm + ns)g = m(rg) + m(sg') = m(rg + sg')$. Set $g'' = rg + sg' \neq 0$. ||

3.5.3. <u>Definition</u>. Let $G[p^n] = \{x \in G \mid p^n x = 0\}$. For a p-group G, $G[p^n]$ is called the <u>p^n-socle</u> of G. If $n = 1$, $G[p]$ is called the <u>socle</u> of G.

The use of the word "socle" (base or pedistal in German) is due to Remak. Since every element of G has order a power of p, we have

$$G = \bigcup_{n=0}^{\infty} G[p^n]$$

The remarkable fact is how the behavior of $G[p]$ can effect all of G. This is the content of the next lemma.

3.5.3. <u>Lemma</u> If p^n divides every element of $G[p]$ (in G) for every $n \geq 0$, then G is divisible.

<u>Proof</u> It is enough to show that $G = pG$ (3.2.6.). If $x \varepsilon G[p]$, then, by assumption, $p|x$, so therefore $x \varepsilon pG$. Thus $G[p] \subseteq pG$. We induct to show $G[p^n] \subseteq pG$ for all $n \geq 1$. Indeed, suppose $G[p^k] \subseteq pG$ for $k \geq 1$. Let $g \varepsilon G[p^{k+1}]$. Then $p^k g \varepsilon G[p]$, so, by assumption, $p^{k+1}|p^k g$; write $p^k g = p^{k+1} g'$ for some $g' \varepsilon G$. Thus $p^k(g - pg') = 0$, and so $g - pg' \varepsilon G[p^k] \subseteq pG$. Hence $g - pg' = pg''$ for some $g'' \varepsilon G$. But then $g = p(g' + g'') \varepsilon pG$, and so $G[p^{k+1}] \subseteq pG$. Finally, then, $G = \bigcup_{n=0}^{\infty} G[p^n] \subseteq pG$; i.e., $G \subseteq pG$. ||

3.5.4. <u>Lemma</u> Suppose G is not divisible. Then G contains a nonzero pure cyclic subgroup.

<u>Proof.</u> Since G is not divisible, we may conclude the following from the previous lemma: there must exist an element $a \varepsilon G[p]$ and a largest integer $n \geq 0$ for which $p^n x = a$ has a nonzero solution in G. Let $b \neq 0$ be such a solution and set $B = \langle b \rangle$. Then remember that $p^n b = a$ and $p^{n+1} b = pa = 0$.

If B is not a pure subgroup of G, then there exist integers m and k and an element $g \varepsilon G$ such that $p^k g = mb$ but for no $sb \varepsilon B$ is it true that $p^k(sb) = mb$. This forces certain restrictions on k and m: first, writing $m = p^j t$ with $(p,t) = 1$, we must have $k > j$. For if $k \leq j$, then $p^k(p^{j-k} tb) = (p^j t)b = mb$, a contradiction to the choice of m, k and g. Also $j < n + 1$; for if $j \geq n + 1$, we have $mb = t(p^j b) = 0$, and so $p^k x = mb$ has a solution in B (namely 0), which is again a contradiction.

Thus, let $s = n+1-j > 0$. Then $p^{k+s-1}g = (p^n t)b = ta$. Since $(p,t) = 1$, 3.5.2. asserts the existence of an element $g'' \in G$ such that $p^{k+s-1}g'' = a$. Since $k + s - 1 = k + n - j > n$, this contradicts the choice of n. It must be, after all, that B is pure in G. ||

3.5.5. Definition A set $A = \{a_\lambda\} \subseteq G$ is independent if for all $a_{\lambda_1}, \cdots, a_{\lambda_k} \in A$ and integers $n_{\lambda_1}, \cdots, n_{\lambda_k}$,

$$n_{\lambda_1}a_{\lambda_1} + \cdots + n_{\lambda_k}a_{\lambda_k} = 0 \text{ implies } n_{\lambda_1} = \cdots = n_{\lambda_k} = 0.$$

A is called pure independent if A is independent and $\sum_\lambda \langle a_\lambda \rangle$ is a pure subgroup of G. A is called a maximal independent set if A is an independent set and no set properly containing A is independent.

Note that if B is a direct sum of cyclic groups, $B \simeq \sum_\lambda \langle a_\lambda \rangle$, then $\{a_\lambda\}$ is an independent set, and conversely.

3.5.6. Lemma Let $B \simeq \sum_\lambda \langle a_\lambda \rangle$ be a subgroup of G. Then the following are equivalent:

(1) $A = \{a_\lambda\}$ is a maximal pure independent set.

(2) B is a basic subgroup of G.

Proof. (1) implies (2). Already, by assumption, B is a pure subgroup of G and is a direct sum of cyclic groups. It remains to show that G/B is divisible; i.e., $p(G/B) = G/B$. If this is not the case, then, 3.5.4. allows us to assert that G/B contains a nonzero pure cyclic subgroup $C/B = \langle x + B \rangle$ (with $x \notin B$). If this cyclic subgroup is finite, then part (12) of 3.3.7. allows us to assume that $x + B$ and x have the same order.

By (9) of 3.3.7., we have that C is pure in G. Also, $C \simeq B \oplus \langle x \rangle$. The directness follows if $nx \notin B$ for all integers n, which is just what will happen if $\langle x + B \rangle$ above is an infinite cyclic group. If it is a finite group, set $H = \langle x \rangle$, and select $t \in H \cap B$. Then $t = sx$, and so $B = t + B = s(x + B)$. This means that $s | o(x + B) = o(x)$, and so $sx = 0$. Thus $t = 0$, and the directness of the sum follows.

Thus $\{a_\lambda\} \cup \{x\}$ is a pure independent set: it generates the pure subgroup C, and because $B \cap \langle x \rangle = (0)$, we get the necessary independence relation. Since $\{x\} \cup \{a_\lambda\}$ properly contains $\{a_\lambda\}$, this contradicts the maximality of $\{a_\lambda\}$. Hence G/B is divisible.

(2) implies (1). Suppose B is a basic subgroup of G. Then B is pure in G, so $\{a_\lambda\}$ is a pure independent set. We must show it is maximal. Take $x \notin B$ such that $\{a_\lambda\} \cup \{x\}$ is a pure independent set. Then $\langle x \rangle \simeq [B \oplus \langle x \rangle]/B$ is a pure subgroup of the divisible group G/B. By (2) of 3.3.7. $\langle x \rangle$ is a divisible subgroup of G/B. But no nonzero cyclic group is divisible! Hence $\{a_\lambda\}$ is a maximal pure independent set. ||

3.5.7. <u>Lemma</u>. Every p-group has a basic subgroup.

<u>Proof</u>. If G is divisible, then (0) is a basic subgroup of G. Otherwise, we proceed via a Zorn's Lemma argument. Let $P = \{X \subseteq G \mid X$ is a pure independent set$\}$. By 3.5.4., $P \neq \emptyset$, since G contains a nonzero pure cyclic subgroup $B = \langle x \rangle$. Then $\{x\} \in P$. Let C be any chain in P, say $C = \{X_\lambda\}_{\lambda \in \Lambda}$. Set $T = \bigcup_\lambda X_\lambda$. We must show that $T \in P$.

Select $x_1, \cdots, x_k \in T$. Since C is a chain, there exists $X_\rho \in C$ such that all $x_i \in X_\rho$ for $i = 1, \cdots, k$. Hence if $n_1, \cdots, n_k$ are any integers whatsoever,

$$n_1x_1 + \cdots + n_kx_k = 0 \text{ implies } n_1 = \cdots = n_k = 0$$

because of the independence of X_ρ. Hence T is an independent set.

A similar argument shows that T is a pure independent set. By Zorn's Lemma, then, G contains a maximal pure independent set $M = \{t_\mu\}$. By 3.5.6. G must contain a basic subgroup, namely $\sum_\mu <t_\mu>$. ||

Incidentally, the reader may have noticed that this proof proceeds just like the one used to show that every vector space has a basis. We are now in a position to give the promised generalization of 3.3.4. But first, we prove one more lemma we will use later on.

3.5.8. Lemma Let H be a pure subgroup of G such that G/H is a direct sum of cyclic groups. Then H is a direct summand of G.

Proof. Let $G/H = \Sigma<y_\lambda>$. There must exist $g_\lambda \varepsilon G$ such that $y_\lambda = g_\lambda + G$ for each λ. Among all y_λ such that $o(y_\lambda) = n_\lambda < \infty$, select g_λ such that $o(g_\lambda) = n_\lambda$. This is possible by (12) of 3.3.7. Let K be the subgroup generated by the $\{g_\lambda\}$ above. We show that $G \simeq K \oplus H$.

First, let $t \varepsilon G$; as usual, let t^* be the image of t in G/H. Then $t^* = \sum_i a_i y_i$, so $(t - \sum_i a_i g_i)^* = 0$. Hence $t - \sum_i a_i g_i = h$ for some $h \varepsilon H$, so $t = h + \sum_i a_i g_i$. Thus G is generated by $K \bigcup H$.

Now we show that $K \bigcap H = (0)$. Let $w \varepsilon K \bigcap H$, so that $w = \sum_i a_i g_i$. Since $w \varepsilon H$, we have $0 = w^* = \sum_i a_i y_i$. Thus $a_j = 0$ for all indices j such that $o(y_j) = \infty$. Among the remaining indices i such that $o(y_i) = n_i < \infty$, we have $n_i | a_i$. Thus $a_i = k_i n_i$ for integers k_i. But the g_i have been chosen such that $n_i g_i = 0$. Hence $w = \sum_i a_i g_i = \sum_i k_i n_i g_i = 0$. ||

3.5.9. Definition A group G is bounded if $nG = (0)$ for some integer $n > 0$.

Obviously, bounded groups are torsion groups, but not conversely.

Also, every finite group is bounded. An infinite group can be bounded if the order of its elements has a finite upper bound. If G is any group whatsoever, a homomorphic image of it is always bounded: G/nG is bounded for all n.

The following theorem, first proved by Baer [2] and Prüfer [31], is perhaps as satisfying a generalization of 3.3.4. as we might expect. With basic subgroups at hand, the proof seems trivial.

3.5.10. <u>Theorem</u> Every bounded group is a direct sum of cyclics.

<u>Proof.</u> Let G be given. Since G is torsion, we may assume that G is a p-group, by 3.3.4. Hence $p^nG = (0)$ for some integer n. Let B be a basic subgroup of G. Then $G/B = p^n(G/B) = 0$ because G/B is divisible and $p^nG = 0$. Thus $G = B$. ||

3.6. The group $Z(p^\infty)$

We close this chapter with a short discussion of a very interesting infinite Abelian group. This group, which we denote by $Z(p^\infty)$, was first introduced by Prüfer [31], and it has come to be quite important in the study of Abelian groups.

Let p be a fixed prime. Let $P = \{r/s \in Q \mid s = p^n$ for some $n \geq 0\}$. Hence P is a subgroup of Q and consists of all those (reduced) fractions whose denominators are powers of p.

We define $Z(p^\infty) = P/Z$, so $Z(p^\infty)$ is a subgroup of Q/Z. By the way $Z(p^\infty)$ is defined, addition is performed modulo 1; for example, let $x \in P$ and x^* denote $x + Z \in Z(p^\infty)$. For convenience, select $p = 2$. Then $(\frac{1}{2})^* + (\frac{1}{2})^* = 0$ and $(1/4)^* + (7/8)^* = (1/8)^*$.

3.6.1. <u>Proposition</u> Let $Z(p^\infty)$ be as defined above, with p a fixed prime.

(1) $Z(p^\infty)$ is generated by a set of its elements $\{c_1, c_2, \cdots\}$ such that $c_1 \neq 0$, $pc_1 = 0$, $pc_2 = c_1$, and $pc_{n+1} = c_n$ for all $n \geq 1$. In fact, every $x \in Z(p^\infty)$ is a multiple of some c_n. If $0 < i \leq j$, then $p^{j-i}c_j = c_i$. Every c_n has the property that $p^n c_n = 0$ for $n \geq 1$, and $\langle c_n \rangle \simeq Z(p^n)$.

(2) Every nonzero proper subgroup of $Z(p^\infty)$ is isomorphic to $Z(p^n)$ for some $n \geq 1$. Every $Z(p^n)$ may be imbedded in $Z(p^\infty)$.

(3) $Z(p^\infty)$ is a p-group.

(4) $Q/Z \simeq \sum_p Z(p^\infty)$ for various primes p.

(5) $Z(p^\infty)$ is divisible.

Proof. (1) Let $c_n = (1/p^n)^*$ for $n = 1,2,\cdots$. Then $c_1 = 1/p + Z \neq 0$, but $pc_1 = 1 + Z = 0$. Clearly, $pc_{n+1} = c_1$ for $n \geq 1$. A straightforward induction on n shows $p^n c_n = 0$ for all $n \geq 1$. Hence $o(c_n)$ divides p^n. But if $o(c_n) = p^j$ with $j < n$, we are led to the conclusion that $c_1 = 0$. Hence $o(c_n) = p^n$, so $\langle c_n \rangle \simeq Z(p^n)$.

Another induction shows that $c_i = p^{j-i}c_j$ if $0 < i \leq j$. If x is any nonzero element of $Z(p^\infty)$ which is not equal to any c_n, then $x = r/p^n + Z$, where $(r,p) = 1$ and $0 < r < p^n$. Thus $x = rc_n$.

(2) Let H be a nonzero proper subgroup of $Z(p^\infty)$. Then H does not contain all of the generators c_n, and so there must exist c_{n+1} with the smallest possible index such that $c_{n+1} \notin H$; i.e., c_{n+1} is the "first" of the c_n's not in H. What we claim is that H is isomorphic to $\langle c_n \rangle$, which is is also isomorphic to $Z(p^n)$. Clearly, $c_n \in H$ by the choice of c_{n+1}. But any element h of H can be written as $h = r(1/p^k + Z)$ for integers r and k. Also $(r,p) = 1$,

so there are integers s and t such that $rs + p^k t = 1$. Then $c_k = rs(1/p^k + Z) + tp^k(1/p^k + Z) = sh \in H$. Now either $k \leq n$ or $k > n$. If the latter holds, then $k \geq n+1$, and so $c_{n+1} = p^{k-(n+1)}c_k \in H$. This is a contradiction to the choice of c_{n+1}. Hence $k \leq n$, and so $c_k = p^{n-k}c_n \in \langle c_n \rangle$. Thus $h = rc_k \in \langle c_n \rangle$.

The only proper subgroups of $Z(p^\infty)$ have therefore been shown to be the cyclic groups $Z(p^n)$ for all $n = 1,2,\cdots$; they form a chain

$$0 \subset Z(p) \subset Z(p^2) \subset Z(p^3) \subset \cdots \subset Z(p^n) \subset \cdots.$$

Thus $Z(p^\infty)$ is neither finitely generated nor cyclic, but every proper subgroup is! Thus $Z(p^\infty)$ is an example which shows that the converse of the well known result "Every subgroup of a cyclic group is cyclic," is false (although the Klein Four Group or S_3 will also show this).

(3) Clear, since $p^n c_n = 0$ for all n.

(4) We know Q/Z is a torsion group. From (3), the groups $Z(p^\infty)$ for various primes are precisely its p-primary subgroups. By the uniqueness assertion of 3.3.4., we must have $Q/Z \approx \sum_p Z(p^\infty)$.

(5) Since Q/Z is divisible (3.2.3.), we must have that every $Z(p^\infty)$ is also divisible, from (4) above and (4) of 3.2.2. ||

From the above result, we see that any direct sum of various $Z(p^\infty)$ is a divisible group. It is a remarkable fact that the only divisible groups are essentially Q, $Z(p^\infty)$ and their direct sums.

3.6.2. <u>Theorem</u> [Fuchs, 8] A divisible group is a direct sum of $Z(p^\infty)$ for various primes p and groups isomorphic to Q. Any two such decompositions are isomorphic.

3.6.3. <u>Corollary</u> A group G is torsion-free and divisible if and only if G is isomorphic to a direct sum of copies of Q.

CHAPTER IV

GROUPS OF HOMOMORPHISMS

We introduce in this chapter the first of the five derived groups we will examine in the homological method: the group of homomorphisms from one group to another. Fundamental in the method is the construction of an induced exact sequence of these groups given an exact sequence of Abelian groups. The difficulties that naturally arise in this project will lead to the important homological concepts of projectivity and injectivity of groups. Examples of groups of homomorphisms are given along with a characterization of precisely which groups are projective.

4.1. The "functor" Hom

We begin with a slight generalization, but later we will specialize only to groups. Let A and B be R-modules, where R is a fixed commutative ring with unity. We define

$$\mathrm{Hom}_R(A,B) = \{f:A \to B \mid f \text{ is an } R\text{-homomorphism}\}.$$

Under the definitions $(f + g)(a) = f(a) + g(a)$ and $r(f(a)) = (rf)(a)$, $\mathrm{Hom}_R(A,B)$ is itself an R-module. Note that when one shows that $(tf)(ra) = r[(tf)(a)]$ for all $r \in R$ and $t \in R$, the commutativity of R is strongly used. When R is a field, $\mathrm{Hom}_R(A,B)$ is the vector space of all linear transformations from the vector space A to the vector space B. We only need A and B to be groups; i.e., Z-modules. We will supress the Z when writing $\mathrm{Hom}_Z(A,B)$. Thus Hom(A,B) is the collection of all Abelian group homomorphisms from A to B, and is itself an Abelian group when A and B are Abelian groups.

Now that we have the groups Hom(A,B) the problem is how to paste them together. The technique that works nicely here is the concept of an <u>induced homomorphism</u>.

Let B and B' be groups. Given a fixed group homomorphism $\beta:B \to B'$, β induces a homomorphism $\beta_*:\mathrm{Hom}(A,B) \to \mathrm{Hom}(A,B')$ defined by $\beta_*(f) = \beta f$. This follows immediately from the fact that $(\beta_*(f + g))(a) = (\beta(f + g))(a) = \beta(f(a) + g(a)) = \beta(f(a)) + \beta(g(a)) = (\beta_*(f))(a) + (\beta_*(g))(a) = (\beta_*(f) + \beta_*(g))(a)$ for all $a \in A$. Note also that $(1_B)_*$ is equal to the identity homomorphism on Hom(A,B), since $\beta_*(f) = f$ if β is the identity map on B.

Now let $\beta':B' \to B''$ be another fixed homomorphism of groups and consider the composition $\gamma = \beta'\beta$. Then $\gamma_*:\mathrm{Hom}(A,B) \to \mathrm{Hom}(A,B'')$, $\beta'_*:\mathrm{Hom}(A,B') \to \mathrm{Hom}(A,B'')$ and $\beta_*:\mathrm{Hom}(A,B) \to \mathrm{Hom}(A,B')$. We show that $\gamma_* = (\beta')_*\beta_*$. Let $f \in \mathrm{Hom}(A,B)$ and $a \in A$. Then $(\gamma_*(f))(a) = (\gamma f)(a) = \gamma(f(a)) = (\beta'\beta)(f(a)) = \beta'(\beta(f(a))) = \beta'(\beta_*(a)) = ((\beta')_*\beta_*)(a)$. Thus $(\beta'\beta)_* = (\beta')_*\beta_*$.

What we have shown can be succinctly stated using category terminology: "Hom(A,B) is a covariant functor in the category of groups, for a fixed A." We will not have occasion to pursue this idea further, but it is hoped that the reader will find it helpful.

What happens when the second variable is fixed? Now everything is "turned around." If $\alpha:A \to A'$ is a fixed group homomorphism, then α induces $\alpha^*:\mathrm{Hom}(A',B) \to \mathrm{Hom}(A,B)$ for any group B, where α^* is defined by $\alpha^*(f) = f\alpha$. As before, α^* is a homomorphism of groups, and $(1_A)^*$ is the identity homomorphism on $Hom(A',B)$; but now if $\alpha':A' \to A''$, then $(\alpha'\alpha)^* = \alpha^*(\alpha')^*$. The short form for these facts is "For a fixed B, Hom(A,B) is a contravariant functor on the category of Abelian groups."

For reference, we bring this all together in the next proposition.

4.1.1. <u>Proposition</u> Let A and B be groups. Then Hom(A,B) is an Abelian group. If $\alpha:A \to A'$ is a fixed group homomorphism, define $\alpha^*:Hom(A',B) \to Hom(A,B)$ by $\alpha^*(f) = f\alpha$, and if $\beta:B \to B'$ is a fixed homomorphism, define $\beta_*:Hom(A,B) \to Hom(A,B')$ by $\beta_*(f) = \beta f$. The maps α^* and β_* are homomorphisms and if $\alpha':A' \to A''$ and $\beta':B' \to B''$ are fixed group homomorphisms, then

$$(1_B)_* = 1_{Hom(A,B)}\ , \quad (1_A)^* = 1_{Hom(A,B)}$$

$$(\beta'\beta)_* = (\beta')_*\beta_*\ , \text{ and } (\alpha'\alpha)^* = \alpha^*(\alpha')^*.$$

4.2. Examples of Homomorphism Groups

In this short section, we list many examples of general homomorphism groups. Some proofs are not given, but reference is made for the reader so he may pursue those facts he finds interesting.

4.2.1. <u>Proposition</u> Let A, B and G be groups; further, let 0 denote the trivial group.

(1) $Hom(A,0) = Hom(0,A) = 0$ for all groups A.

(2) $Hom(Z(n),Z) = 0$ for all $n > 1$.

(3) $Hom(A,B) = 0$ if A is cyclic and B is torsion-free.

(4) $Hom(Z,G) \simeq G$ for all groups G.

(5) $Hom(Z(n),A) \simeq A[n]$.

(6) $Hom(Z(p),Z(q)) = 0$ for all distinct primes p and q.

(7) $\text{Hom}(Z(p^n),Z(p^m)) \simeq Z(p^{\min\{m,n\}})$ for integers m and n.

(8) G is reduced if and only if $\text{Hom}(Q,G) = 0$.

(9) If G is divisible, then $\text{Hom}(G,A)$ is torsion-free for all A.

(10) G is torsion iff $\text{Hom}(G,A) = 0$ for all torsion-free A.

(11) [Harrison, 13] If A is torsion, then $\text{Hom}(A,C)$ is reduced and algebraically compact.

Proof. (1) Trivial

(2) Let $f \in \text{Hom}(Z(n),Z)$, and let $x \in Z(n)$. Then $nx = 0$, so $0 = f(nx) = nf(x) \in Z$. Hence $f(x) = 0$, since $n \neq 0$. Thus $f = 0$.

(3) Exactly the same proof given for (2) works here also.

(4) This is an important fact we will use many times. Let f be in $\text{Hom}(Z,G)$ and define $\psi:\text{Hom}(Z,G) \to G$ by $\psi(f) = f(1)$. Then ψ is a homomorphism; it is also onto, for if $g \in G$ is given, the function f such that $f(1) = g$ is completely determined by this assignment, it is a homomorphism from Z to G and it maps onto g under ψ. A short computation shows that $\text{Ker}\ \psi = (0)$.

(5) The same assignment $f \to f(1)$, where now $1 \in Z(n)$, easily is shown to define an isomorphism between $\text{Hom}(Z(n),A)$ and $A[n]$.

(6) From (5), $\text{Hom}(Z(p),Z(q)) \simeq Z(q)[p] = 0$.

(7) Again from (5), $\text{Hom}(Z(p^n),Z(p^m)) \simeq Z(p^m)[p^n] \simeq Z(p^{\min\{m,n\}})$.

(8) Suppose G is reduced and $f \in \text{Hom}(Q,G)$. If $f \neq 0$, then $f(Q)$ is a nonzero divisible subgroup of G, which is a contradiction. Conversely, suppose $\text{Hom}(Q,G) = 0$; write $G = D \oplus R$, where D is divisible and R is reduced (3.2.4.). Since D must be the image of Q under the zero homomorphism (we are using the fact that the homomorphic image of a divisible must be divisible), we have $D = 0$, so $G = R$.

(9) and (10) Consult Fuchs [8]. ||

4.3. Hom and Direct Sums

We now investigate the obvious question: does Hom preserve direct sums? A partial answer follows.

4.3.1. Proposition Let A, B and C be groups. Then

(1) $\mathrm{Hom}(A \oplus B, C) \simeq \mathrm{Hom}(A,C) \oplus \mathrm{Hom}(B,C)$

(2) $\mathrm{Hom}(A, B \oplus C) \simeq \mathrm{Hom}(A,B) \oplus \mathrm{Hom}(A,C)$

Proof. The idea is to somehow use 2.3.5. From the universal diagram

$$B \underset{\pi_1}{\overset{\iota_1}{\rightleftarrows}} B \oplus C \underset{\pi_2}{\overset{\iota_2}{\leftrightarrows}} C$$

we obtain the following diagram, after applying 4.1.1.:

$$\mathrm{Hom}(A,B) \underset{\pi_{1*}}{\overset{\iota_{1*}}{\rightleftarrows}} \mathrm{Hom}(A, B \oplus C) \underset{\pi_{2*}}{\overset{\iota_{2*}}{\leftrightarrows}} \mathrm{Hom}(A,C).$$

Note that the following hold:

$$\pi_{1*}\iota_{1*} = (\pi_1\iota_1)_* = (1_B)_* = 1_{\mathrm{Hom}(A,B)},$$

$$\pi_{2*}\iota_{2*} = (\pi_2\iota_2)_* = (1_C)_* = 1_{\mathrm{Hom}(A,C)}, \quad \text{and}$$

$$\iota_{1*}\pi_{1*} + \iota_{2*}\pi_{2*} = 1_{\mathrm{Hom}(A,B\oplus C)}.$$

Thus, by 2.3.5, $\mathrm{Hom}(A, B \oplus C) \simeq \mathrm{Hom}(A,B) \oplus \mathrm{Hom}(A,C)$. A similar argument gives the first assertion, except this time the diagram used is

$$A \underset{\iota_1}{\overset{\pi_1}{\leftrightarrows}} A \oplus B \underset{\iota_2}{\overset{\pi_2}{\rightleftarrows}} B.$$

We then get the following relations, again using 4.1.1.:

$$\iota_1^*\pi_1^* = (\pi_1\iota_1)^* = (1_A)^* = 1_{\mathrm{Hom}(A,C)},$$

$$\iota_2{}^*\pi_2{}^* = (\pi_2\iota_2)^* = 1_B{}^* = 1_{Hom(B,C)},$$

$$\pi_1{}^*\iota_1{}^* + \pi_2{}^*\iota_2{}^* = 1_{Hom(A\oplus B,C)}.$$

Thus $Hom(A \oplus B, C) \simeq Hom(A,C) \oplus Hom(B,C)$. ||

4.3.2. Corollary For any groups A, B and C, the following sequence is always exact:

$$0 \rightarrow Hom(A,B) \xrightarrow{\iota_{1*}} Hom(A, B\oplus C) \xrightarrow{\pi_{2*}} Hom(A,C) \rightarrow 0$$

4.3.3. Proposition Let $A_1, \cdots, A_n$ be any finite collection of groups. Then for all groups B

$$(1) \quad Hom(\sum_{i=1}^{n} A_i, B) \simeq \sum_{i=1}^{n} Hom(A_i, B)$$

$$(2) \quad Hom(B, \sum_{i=1}^{n} A_i) \simeq \sum_{i=1}^{n} Hom(B, A_i)$$

Proof. Induct on n and use 4.3.1. ||

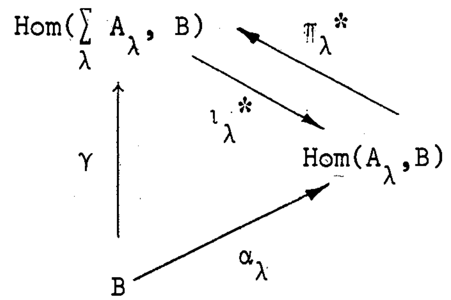
We are now going to extend the results of the previous propositions to the general case.

4.3.4. Proposition Let $\{A_\lambda\}_{\lambda\varepsilon\Lambda}$ be an arbitrary collection of groups. Then for all groups B

$$(1) \quad Hom(\sum_\lambda A_\lambda, B) \simeq \prod_\lambda Hom(A_\lambda, B)$$

$$(2) \quad Hom(B, \prod_\lambda A_\lambda) \simeq \prod_\lambda Hom(B, A_\lambda)$$

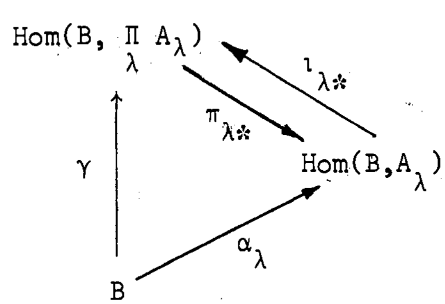
Proof. To show (1), we show that the diagram is couniversal: $Hom(\sum_\lambda A_\lambda, B) \xrightarrow{\iota_\lambda{}^*} Hom(A_\lambda, B)$. To this end, let B be an Abelian group and $\alpha_\lambda : B \rightarrow Hom(A_\lambda, B)$ a family of group homomorphisms. We must define a unique $\gamma : B \rightarrow Hom(\Sigma_\lambda A_\lambda, B)$ such that $\iota_\lambda{}^*\gamma = \alpha_\lambda$ for all $\lambda \varepsilon \Lambda$. Diagramatically, the situation is as follows:



Note that $\gamma(b)$ will be an element of $\mathrm{Hom}(\Sigma_\lambda A_\lambda, B)$ and so $\gamma(b)(f) \varepsilon B$ for all $f \varepsilon \sum_\lambda A_\lambda$. Define γ by $\gamma(b)(f) = \sum_\lambda [\alpha_\lambda(b)](f(\lambda))$. Since $f(\lambda) \varepsilon A_\lambda$ and $\alpha_\lambda(b) \varepsilon \mathrm{Hom}(A_\lambda, B)$, the sum on the right is an element of B: note also that the sum is finite since $f(\lambda) \neq 0$ for only finitely many $\lambda \varepsilon \Lambda$. Now let $\lambda \varepsilon \Lambda$ and $b \varepsilon B$ be arbitrary but fixed. Then if $a_\lambda \varepsilon A_\lambda$ is given, we have $(\iota_\lambda^* \gamma(b))(a_\lambda) = \gamma(b)(\iota_\lambda(a_\lambda)) = \alpha_\lambda(b)(a_\lambda)$ by the way γ is defined. Hence we do have $\iota_\lambda^* \gamma = \alpha$ for all $\lambda \varepsilon \Lambda$.

Now if γ has been defined in some way so that $\iota_\lambda^* \gamma = \alpha_\lambda$ for all $\lambda \varepsilon \Lambda$, write $f = \Sigma_\lambda h_\lambda$ where $h_\lambda = \iota_\lambda(f(\lambda))$ (see page 21). Then $\gamma(b)(f) = \gamma(b)(\Sigma_\lambda h_\lambda) = \Sigma_\lambda \gamma(b)(h_\lambda) = \Sigma_\lambda \gamma(b)(\iota_\lambda(f(\lambda)) = \Sigma_\lambda [\alpha_\lambda(b)](f(\lambda))$, and we have the definition of γ.

The situation for (2) is similar, except now the diagram is:



To define $\gamma: B \to \mathrm{Hom}(B, \Pi_\lambda A_\lambda)$, let $b \varepsilon B$, $b' \varepsilon B$, and $\lambda \varepsilon \Lambda$. Then set $\gamma(b)(b')(\lambda) = \alpha_\lambda(b)(b')$. One easily shows that this is the only way that γ can be defined so that $\pi_{\lambda *}\gamma = \alpha_\lambda$ for all $\lambda \in \Lambda$. ||

We record one more result in this vein before introducing the exact sequence for Hom. Most of the details are left to the reader.

4.3.5. <u>Proposition</u> For any groups A, B and C there is an isomorphism ψ:Hom(A,Hom(B,C)) $\to$ Hom(B,Hom(A,C)) defined, for a given f:A $\to$ Hom(B,C), by $\psi(f) = g$, where $g(b)(a) = f(a)(b)$.

<u>Proof.</u> One easily shows that all the functions involved make sense and are homomorphisms. Given g:B $\to$ Hom(A,C), then when we define f by $f(a)(b) = g(b)(a)$, we get a homomorphism from A to Hom(B,C). A short argument shows that the kernel of ψ is trivial, so ψ is an isomorphism. ||

Remember we mentioned that when U and V are vector spaces over a field F, then $\text{Hom}_F(U,V)$ is the Abelian group of all linear transformations from U to V. The above proposition is a generalization of the following result: $\text{Hom}_F(U,\hat{V}) \simeq \text{Hom}_F(V,\hat{U})$, where $\hat{W} = \text{Hom}_F(W,F)$ is the dual space of the vector space W.

4.4. Hom and Exact Sequences

One of the basic questions in our method is the following: if $A \to B \to C$ is exact, is an associated sequence of Hom groups also exact? The answer is typical: "yes and no." The attempts to resolve the difficulties therein give rise to many important related concepts.

We have already hinted at a partial result in 4.3.2. Now we record the general result.

4.4.1. <u>Proposition</u> Let $0 \to A \underset{\tau}{\overset{\kappa}{\rightleftarrows}} B \underset{\omega}{\overset{\sigma}{\rightleftarrows}} C \to 0$ be split exact and D any group. Then

$$0 \to \text{Hom}(D,A) \xrightarrow{\kappa_*} \text{Hom}(D,B) \xrightarrow{\sigma_*} \text{Hom}(D,C) \to 0$$

is also split exact.

Proof. These essence of the proposition is often expressed by: Hom(D,-) takes splits exact sequences of Abelian groups to split exact sequences of Abelian groups. First, $\text{Ker}\ \kappa_* = \{f \in \text{Hom}(D,A) \mid \kappa_*(f) = \kappa f = 0\} = \{f \in \text{Hom}(D,A) \mid f = 0\}$ since κ is monic. Thus κ_* is monic. We have not used the splitting assumption here, only the fact that κ is monic. But to show that σ_* is epic does require the splitting hypothesis.

Let $g \in \text{Hom}(D,C)$. Set $\phi = \omega g$. Then $\phi{:}D \to B$ and $\sigma_*(\phi) = \sigma\phi = (\sigma\omega)g = g$ since $\sigma\omega = 1_C$. Thus σ_* is an epimorphism.

Since $(\sigma\kappa)_* = \sigma_*\kappa_*$, we have $\sigma_*\kappa_* = 0$ because $\sigma\kappa$ is zero. Thus $\text{Im}\ \kappa_* \subseteq \text{Ker}\ \sigma_*$.

To show the other containment, and thus finish the argument, let $f \in \text{Ker}\ \sigma_* \cap \text{Hom}(D,B)$. Then $\sigma f = 0$ and $f{:}D \to B$. Thus (1) of 2.2.5. applies, and we can find $\rho{:}D \to A$ such that $f = \kappa\rho$; i.e., $f = \kappa_*(\rho)$. Hence the sequence is exact; that it splits follows from the fact that $B \simeq A \oplus C$. Then apply 4.3.1. to finish the proof. ||

We learned this much from the proof: If $0 \to A \to B$ is exact, then so is $0 \to \text{Hom}(D,A) \to \text{Hom}(D,B)$. But we cannot be so lucky for exact sequences $A \to B \to 0$. Before trying to remedy the situation, we record what is true without further hypotheses.

4.4.2. Theorem [Cartan and Eilenburg, 4] Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} L$ and $M \xrightarrow{\chi} B \xrightarrow{\xi} C \to 0$ be exact sequences of Abelian groups. Then for any group D, the following are also exact:

$$(1) \quad 0 \to \text{Hom}(D,A) \xrightarrow{\alpha_*} \text{Hom}(D,B) \xrightarrow{\beta_*} \text{Hom}(D,L)$$

$$(2) \quad 0 \to \text{Hom}(C,D) \xrightarrow{\xi^*} \text{Hom}(B,D) \xrightarrow{\chi^*} \text{Hom}(M,D)$$

Proof. The proof of (1) follows precisely the reasoning of the proof of proposition 4.4.1. For (2), note that $\text{Ker } \xi^* = \{f \in \text{Hom}(C,D) \mid \xi^*(f) = 0\} = \{f \in \text{Hom}(C,D) \mid f\xi = 0\} \supseteq \{f \mid f = 0\}$. But if $f\xi = 0$ for $f \in \text{Hom}(C,D)$, then $f = 0$, because ξ is epic. Hence ξ^* is monic. As before, $\chi^*\xi^* = (\xi\chi)^*$ is zero, so $\text{Im } \xi^* \subseteq \text{Ker } \chi^*$. Now let $f \in \text{Ker } \chi^* \cap \text{Hom}(B,D)$. Then $f\chi = 0$, and we have

$$\begin{array}{ccccccc} M & \xrightarrow{\chi} & B & \xrightarrow{\xi} & C & \rightarrow & 0 \\ & & \downarrow f & & & & \\ & & D & & & & \end{array}$$

By (2) of 2.2.5., there is a homomorphism $\phi:C \rightarrow D$ such that $f = \phi\xi$, and so $f = \xi^*(\phi)$; i.e., $f \in \text{Im } \xi^*$, and (2) is therefore exact. ||

One of the fundamental problems of Homological Algebra is trying to restore exactness on the right in (1) and (2). All of the next chapter will be devoted to a construction which restores this exactness.

But we can give a partial answer to the problem by specifying certain properties of D. Suppose $0 \rightarrow A \xrightarrow{\kappa} B$ is exact. When is $\text{Hom}(B,D) \xrightarrow{\kappa^*} \text{Hom}(A,D) \rightarrow 0$ also exact? Let $f:A \rightarrow D$. In pictures, we have:

$$\begin{array}{ccccc} 0 & \rightarrow & A & \xrightarrow{\kappa} & B \\ & & \downarrow f & & \\ & & D & & \end{array}$$

For κ^* to be epic, we need $\phi:B \rightarrow D$ such that $f = \kappa^*(\phi) = \phi\kappa$. Thus what is needed is an <u>extension of f</u>, namely ϕ, through κ. On the other hand, suppose $B \xrightarrow{\sigma} C \rightarrow 0$ is exact. Then when is $\sigma_*:\text{Hom}(D,B) \rightarrow \text{Hom}(D,C)$ also epic? Let $f:D \rightarrow C$. We have

$$\begin{array}{ccccc} & & D & & \\ & & \downarrow f & & \\ B & \xrightarrow{\sigma} & C & \rightarrow & 0 \end{array}$$

Thus σ_* epic means that we can find $\phi : D \to B$ such that the diagram commutes; i.e., $\sigma\phi = f$. We offer the following example to show that such a homomorphism ϕ need not exist.

4.4.3. Example Consider the exact sequence $0 \to Z \xrightarrow{\alpha} Z \xrightarrow{\eta} Z(m) \to 0$ where $\alpha(z) = mz$ (thus α is multiplication by m), η is the natural projection and $m > 1$. Let $D = Z(m)$. We then have

$$0 \to \text{Hom}(Z(m),Z) \xrightarrow{\alpha_*} \text{Hom}(Z(m),Z) \xrightarrow{\eta_*} \text{Hom}(Z(m),Z(m))$$

as the induced sequence of homomorphism groups. But the first two are zero (see (2) of 4.2.1.), while the last is not zero, for it at least contains the identity homomorphism.

Note that the above demands for homomorphisms which make the given diagrams commute are properties of the group D, and we single them out with the following definitions, due to Cartan and Eilenburg.

4.4.4. Definition Let D be a group. Then D is projective if for all exact sequences $B \xrightarrow{\sigma} C \to 0$ and all homomorphisms $f : D \to C$ there exists a homomorphism $\phi : D \to B$ such that $f = \sigma\phi$. Also D is injective if for all exact sequences $0 \to A \xrightarrow{\kappa} B$ and all homomorphisms $f : A \to D$ there exists $\psi : B \to D$ such that $f = \psi\kappa$.

One of our major problems will be to characterize these groups in terms of their internal structures and to investigate the roles they play in the entire theory. For now we content oursevles with a formal statement of what we have already said above.

4.4.5. Proposition Let $A \xrightarrow{\kappa} B \xrightarrow{\sigma} C$ be an arbitrary sequence of groups which is exact at B. Then

(1) $\text{Hom}(P,A) \xrightarrow{\kappa_*} \text{Hom}(P,B) \xrightarrow{\sigma_*} \text{Hom}(P,C)$ is exact at $\text{Hom}(P,B)$

if P is projective and σ is epic.

(2) $\mathrm{Hom}(C,I) \overset{\sigma^*}{\to} \mathrm{Hom}(B,I) \overset{\kappa^*}{\to} \mathrm{Hom}(A,I)$ is exact at $\mathrm{Hom}(B,I)$ if I injective and κ is monic.

4.4.6. Corollary No finite cyclic group is projective.

Proof. Suppose $Z(m)$ is projective; the sequence $Z \overset{\eta}{\to} Z(m) \to 0$ is exact, where η is the natural projection. Then, by the previous result, $0 = \mathrm{Hom}(Z(m),Z) \to \mathrm{Hom}(Z(m),Z(m)) \to 0$ is also exact, which it manifestly is not (see 4.4.3.). Hence $Z(m)$ is not projective. ||

4.5. Free Groups

Everything we say in this section can be generalized to arbitrary modules over rings. However, we do not need this generality; the notion of "free" modules solves our problem of the characterization of projective groups very nicely. It is due to MacLane [24].

4.5.1. Definition A group G is free on a subset T of its elements if $\{\mu_t : Z \to G\}$ forms a universal diagram with ends Z, one for each $t \in T$, where $\mu_t(n) = nt$ for all $n \in Z$. This means, simply, that $G \simeq \sum_{t \in T} Z$ is a direct sum of copies of Z and each element of G can therefore be uniquely represented as a sum $\sum_t n_t(t)$, where n_t are integers that are necessarily zero for all but a finite number of $t \in T$.

How does one construct groups free over given sets? Let T be an arbitrary set. Denote by Zt the group induced by Z and $t \in T$; i.e., $Zt = \{nt \mid n \in Z\}$, where $nt + mt$ means $(n + m)t$. Then Z is isomorphic to Zt. Set $G = \sum_{t \in T} Zt$. Then G is a free group, free on T, and T can be thought of as a subset of G by identifying t with the tuple that has t in the t^{th} coordinate and zeros elsewhere.

4.5.2. Proposition Every group is isomorphic to a quotient of a free group.

Proof. Let G be the given group and let F be a group free on G itself (as above). Define $\Theta: F \to G$ using precisely the same homomorphism that was used in the proof of 3.3.8. Just as we showed there, Θ is an epimorphism, and so $G \simeq F/\text{Ker }\Theta$. ||

4.5.3. Proposition Given any group G there are free groups F_0 and F such that $0 \to F_0 \to F \to G \to 0$ is exact.

Proof. We use the previous result, where $F_0 = \text{Ker }\Theta$. Then we have the exact sequence, and F_0 is free because it is a subgroup of a free group (see Kuroŝ [21], page 143 for a proof of this "obvious" fact). ||

What makes free groups so terribly interesting is the following fact.

4.5.4. Proposition Every free group is projective.

Proof. Assume we have the diagram

$$\begin{array}{ccccc} & & F & & \\ & & \downarrow \lambda & & \\ A & \overset{\sigma}{\to} & B & \to & 0 \end{array}$$

where F is free and the bottom row is exact. We define $\mu: F \to A$ as follows: let F be free on $T \subseteq F$. For each $t \in T$, there exists $a_t \in A$ such that $\sigma(a_t) = \lambda(t)$. Let $x \in F$, so $x = \sum_t n_t(t)$ in a unique manner and set $\mu(x) = \sum_t n_t a_t$. Then μ is a homomorphism, and

$(\sigma\mu)(x) = \sum_t n_t \sigma(a_t) = \sum_t n_t \lambda(t) = \lambda(\sum_t n_t t) = \lambda(x)$. ||

4.5.5. Corollary Given an Abelian group A, there exist projective groups P and P_0 such that

$$0 \to P_0 \to P \to A \to 0$$

is exact.

4.5.6. Corollary Z is a projective group.

4.5.7. Proposition [MacLane, 24] An Abelian group G is projective if and only if G is a direct summand of a free group.

Proof. Suppose $F = G \oplus H$, where F is free. Suppose further that $A \overset{\sigma}{\to} B \to 0$ is exact and $\gamma:G \to B$ is a homomorphism. Let $\pi:F \to G$ be the usual projection. Since F is free, there is a homomorphism $\beta:F \to A$ such that $\sigma\beta = \gamma\pi$. Let $\iota:G \to F$ be the usual injection and set $\zeta = \beta\iota$. Then $\sigma\zeta = \sigma\beta\iota = \gamma\pi\iota = \gamma$.

Conversely, suppose G is projective. As before, there is a free group F and an epimorphism Θ such that $\Theta:F \to G$. Since G is projective, we may "lift" the identity homomorphism on G to F; i.e., there is a homomorphism $\beta:G \to F$ such that $1_G = \Theta\beta$. By 2.5.1., $F \simeq \text{Ker}\,\Theta \oplus G$. ||

We can now piece together the previous propositions and results to give the first major theorem of our method.

4.5.8. Theorem [Cartan and Eilenburg, 4] Let G be an Abelian group. Then the following are equivalent.

(1) G is projective.

(2) $\sigma:B \to C$ epic implies $\sigma_*:\text{Hom}(G,B) \to \text{Hom}(G,C)$ is also epic.

(3) $0 \to A \to B \to C \to 0$ exact implies

$0 \to \text{Hom}(G,A) \to \text{Hom}(G,B) \to \text{Hom}(G,C) \to 0$ is also exact.

(4) Every exact sequence $0 \to A \to B \to G \to 0$ splits.

Proof. (1) implies (2). This follows immediately from the definition of projective.

(2) implies (3). From 4.4.5., we know that the sequence in (3) is exact up to Hom(G,C). Our assumption gives us that σ_* is epic, and so (3) is exact at Hom(G,C) also.

(3) implies (4). Let $0 \to A \to B \to G \to 0$ be an exact sequence. Now $\mathrm{Hom}(G,B) \xrightarrow{\sigma_*} \mathrm{Hom}(G,G)$ is epic, by assumption, and so there is a homomorphism $\phi \in \mathrm{Hom}(G,B)$ such that $\sigma_*(\phi) = 1_G$; i.e., $1_G = \sigma\phi$ for some $\phi: G \to B$. By 2.5.1., the given sequence splits.

(4) implies (1). Suppose every sequence $0 \to A \to B \to G \to 0$ splits. There is a free group F such that $0 \to F_o \to F \to G \to 0$ is exact (4.5.3.), and hence $F \simeq G \oplus F_o$ by the splitting hypothesis. Hence G is projective, by 4.5.7. ||

We intend to carry out a similar program for injective groups; it turns out that they are precisely the torsion-free groups! But the way will be much smoother if we wait for the machinery of the next chapter.

CHAPTER V

EXTENSIONS OF GROUPS

It is the purpose of this chapter to study a method, ultimately due to R. Baer, which restores exactness to the sequences (1) and (2) of 4.4.2. To do this, we construct new groups, denoted by Ext(A,B) for given groups A and B, and we identify certain elements of this group as "obstructions" to the possibility of extending group homomorphisms.

4.1. Definitions

Let A and C be Abelian groups. By an <u>extension of A by C</u> we mean a short exact sequence

$$E: \quad 0 \to A \xrightarrow{\kappa} B \xrightarrow{\sigma} C \to 0$$

of Abelian groups and group homomorphisms. It is our objective to make the collection of all such extensions into a group in some reasonable way. It is hoped that knowledge of this group (whose general structure is still being investigated) will reflect upon the structure of A and C.

Let $E': \quad 0 \to A' \xrightarrow{\kappa'} B' \xrightarrow{\sigma'} C' \to 0$ be another extension of groups. A <u>morphism of extensions</u> $\phi: E \to E'$ is a triple $\phi = (\alpha,\beta,\gamma)$ of group homomorphisms such that the following diagram commutes:

$$\begin{array}{lccccccccc} E: & 0 & \to & A & \overset{\kappa}{\to} & B & \overset{\sigma}{\to} & C & \to & 0 \\ & & & \alpha\downarrow & & \beta\downarrow & & \downarrow\gamma & & \\ E': & 0 & \to & A' & \overset{\kappa'}{\to} & B' & \overset{\sigma'}{\to} & C' & \to & 0 \end{array}$$

In particular, if $A = A'$ and $C = C'$, we say E and E' are <u>congruent</u> and write $E \equiv E'$ if and only if there is a morphism $\phi:E \to E'$ with $\phi = (1_A,\beta,1_C)$. When this is the case, the Three Lemma (1.2.3.) shows that β is an isomorphism. Thus $\equiv$ is an equivalence relation defined on the class of all extensions of A by C. We denote this collection of equivalence classes by <u>Ext(C,A)</u>. It is this collection that we will make into a group.

For example, one extension of A by C is always

$$E: \quad 0 \to A \to A \oplus C \to C \to 0.$$

Another extension of A by C, E', will be congruent to E precisely when E' splits. One condition which is equivalent to this splitting condition is that fact that C is projective (4.5.8.). Thus all extensions of A by C, when C is projective, are congruent to E, so somehow $\text{Ext}(C,A)$ is trivial. Thus we would expect E to be the zero element of $\text{Ext}(C,A)$. It will also turn out that if A is injective, then $\text{Ext}(C,A)$ is again trivial.

Without further delay, then, we define an addition of homomorphisms which will eventually lead to the addition in $\text{Ext}(C,A)$.

Let $\alpha:A \to A'$ and $\beta:B \to B'$ be homomorphisms of groups. Define $\alpha\oplus\beta$: $A \oplus B \to A' \oplus B'$ by $(\alpha\oplus\beta)(a,b) = (\alpha(a),\beta(b))$. Note that $\alpha\oplus\beta$ is epic (monic) if and only if both α and β are epic (monic). Also if $\delta:A' \to A''$ and $\gamma:B' \to B''$, then $(\delta\oplus\gamma)(\alpha\oplus\beta) = (\delta\alpha)\oplus(\gamma\beta)$. Note also that $1_A \oplus 1_B = 1_{A\oplus C}$.

The <u>diagonal map</u> $\Delta:C \to C \oplus C$ and the <u>co-diagonal map</u> $\nabla:C \oplus C \to C$

are defined by:

$$\Delta(c) = (c,c) \quad \text{and} \quad \nabla(b,c) = b + c.$$

Then if $f:C \to A$ and $g:C \to A$ are homomorphisms, we have $f + g = \nabla_A(f \oplus g)\Delta_C$. Our final definition of the addition of extensions will be reminiscent of this complicated way of adding homomorphisms.

A preliminary addition of extensions can be put forth immediately. Let $E: 0 \to A \overset{\kappa}{\to} B \overset{\sigma}{\to} C \to 0$ and $E': 0 \to A' \overset{\kappa'}{\to} B' \overset{\sigma'}{\to} C' \to 0$ be exact sequences of groups. Denote by $E \oplus E'$ the following exact sequence

$$E \oplus E': \quad 0 \to A \oplus A' \overset{\kappa\oplus\kappa'}{\to} B \oplus B' \overset{\sigma\oplus\sigma'}{\to} C \oplus C' \to 0.$$

The addition in $Ext(C,A)$ will be defined in terms of the above addition of extensions, but as in all arguments involving equivalence classes, we must worry about the uniqueness of the results regardless of the representatives used. The following sections help clear the way.

5.2. Composite Extensions

Let $E: 0 \to A \overset{\kappa}{\to} B \overset{\sigma}{\to} C \to 0$ be an extension of A by C, and $\gamma:C' \to C$ be a given group homomorphism. We are going to construct an extension $E': 0 \to A \overset{\kappa'}{\to} B' \overset{\sigma'}{\to} C' \to 0$ and a morphism $\phi = (1_A,\beta,\gamma)$: $E' \to E$; i.e., we will construct a homomorphism $\beta:B' \to B$ such that the following diagram commutes:

$$\begin{array}{lccccccccc} E': & 0 & \to & A & \overset{\kappa'}{\to} & B' & \overset{\sigma'}{\to} & C' & \to & 0 \\ & & & \downarrow 1 & & \downarrow \beta & & \downarrow \gamma & & \\ E: & 0 & \to & A & \overset{\kappa}{\to} & B & \overset{\sigma}{\to} & C & \to & 0 \end{array} \qquad (I)$$

It is understood that we are to construct a group B' and homomorphisms κ', σ' and β such that the top row is exact and both squares commute. The construction is somewhat common; categorically, it is known as a "pull-back" (a term due to S. Lang).

To get commutativity in the right hand square, we are tempted to define B' as that subgroup of $B \oplus C'$ which "forces" commutativity: $B' = \{(b,c') \mid \sigma(b) = \gamma(c')\}$. Now B' is indeed a subgroup of $B \oplus C'$ since σ and γ are homomorphisms. We define the following homomorphisms: $\sigma'(b,c') = c'$, $\beta(b,c') = b$ and $\kappa'(a) = (\kappa(a),0)$; they are merely the projections onto one of the coordinates or the injection induced by κ. Clearly, κ' is a monomorphism because κ is monic. If $c' \in C'$ is given, then $\gamma(c') \in C$, so $\sigma(b) = \gamma(c')$ for some $b \in B$, since σ is epic. Thus $(b,c') \in B'$, and so σ' is epic. The definition of the maps themselves give the commutativity of the squares.

The final thing to show in the exactness argument is $\text{Ker}\,\sigma' = \text{Im}\,\kappa'$. Note that if $c' = 0$ in C', then any $b \in B$ such that $\sigma(b) = \gamma(c') = 0$ is automatically in $\text{Ker}\,\sigma = \text{Im}\,\kappa$. Hence $b = \kappa(a)$ for some $a \in A$. Thus $\text{Ker}\,\sigma' = \{(b,c') \mid 0 = \sigma'(b,c') = c'\} = \{(b,c') \mid b = \kappa(a)$ for some $a \in A$ and $c' = 0\} = \text{Im}\,\kappa'$.

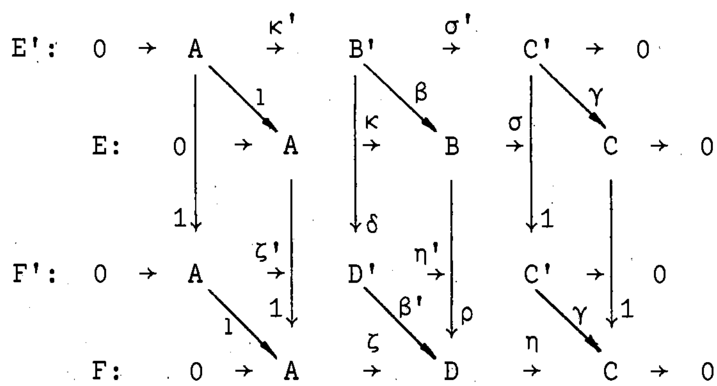
Hence we do indeed have an extension E' and a morphism of extensions. We call the extension E' a composite extension. Naturally, we would like to use the word "the",at least up to congruence; i.e., if E and F are congruent extensions of A by C, then any two composite extensions E' and F' are congruent. Indeed, suppose $E \equiv F$. Then the following diagram commutes, where ρ is an isomorphism.

$$\begin{array}{lccccccccc} E: & 0 & \to & A & \xrightarrow{\kappa} & B & \xrightarrow{\sigma} & C & \to & 0 \\ & & & 1\downarrow & & \downarrow\rho & & \downarrow 1 & & \\ F: & 0 & \to & A & \xrightarrow{\zeta} & D & \xrightarrow{\eta} & C & \to & 0 \end{array}$$

We wish to show that the following diagram of composite extensions also commutes, where δ is some isomorphism:

$$\begin{array}{lccccccccc}
E': & 0 & \to & A & \xrightarrow{\kappa'} & B' & \xrightarrow{\sigma'} & C' & \to & 0 \\
 & & & \downarrow 1 & & \downarrow \delta & & \downarrow 1 & & \\
F': & 0 & \to & A & \xrightarrow{\zeta'} & D' & \xrightarrow{\eta'} & C' & \to & 0
\end{array}$$

We will actually undertake to show a little more. Diagrammatically, the situation is as follows:



(II)

We are assuming that (1) all rows of the prism are exact, (2) $(1,\rho,1):E \to F$ is a morphism, (3) the map β' is that given by the pull-back construction and (4) F' is a composite extension of F (and so the bottom face commutes) and (the weaker hypothesis) E' is any sequence which makes the top face commute (in particular, a composite extension of E will do the job). The task we have set for ourselves is to define $\delta:B' \to D'$ so that the back face of the prism commutes. This will show $E' \equiv F'$.

Define $\delta:B' \to D'$ by $\delta(b') = (\rho(\beta(b')), \sigma'(b'))$. Then δ is a group homomorphism; since $\gamma(\sigma'(b')) = \sigma(\beta(b')) = \eta(\rho(\beta(b')))$, we have that $\delta(b') \in D'$. The desired commutativity relations follow directly: $1_{C'}\sigma' = \eta'\delta$ and $\delta\kappa' = \zeta' 1_A$.

We have shown, then, that $E \equiv F$ implies $E' \equiv F'$; in particular,

$E \equiv E$, and so any two composite extensions of E are congruent. In this sense, then, E' is unique; i.e., E' is unique up to equivalence, and we denote the dependence of E upon γ by using the notation $E\gamma$ to denote the composite extension E'.

Hence for each extension of A by C and each homomorphism $\gamma:C' \to C$ there is a well-defined class $E\gamma$ in $\mathrm{Ext}(C',A)$ such that (I) holds for any representative of $E\gamma$. If $\gamma':C'' \to C'$, then $\gamma\gamma':C'' \to C$, so $E(\gamma\gamma')$ is that extension, unique up to equivalence, such that the following diagram commutes:

$$\begin{array}{rccccccccc}
E(\gamma\gamma'): & 0 & \to & A & \xrightarrow{\kappa''} & B'' & \xrightarrow{\sigma''} & C'' & \to & 0 \\
 & & & 1\downarrow & & \downarrow\beta_1 & & \downarrow\gamma\gamma' & & \\
E: & 0 & \to & A & \xrightarrow{\kappa} & B & \xrightarrow{\sigma} & C & \to & 0
\end{array}$$

But in two steps $(E\gamma)\gamma'$ also has that property:

$$\begin{array}{rccccccccc}
(E\gamma)\gamma': & 0 & \to & A & \xrightarrow{\kappa''} & B'' & \xrightarrow{\sigma''} & C'' & \to & 0 \\
 & & & 1\downarrow & & \downarrow\beta' & & \downarrow\gamma' & & \\
E\gamma: & 0 & \to & A & \xrightarrow{\kappa'} & B' & \xrightarrow{\sigma'} & C' & \to & 0 \\
 & & & 1\downarrow & & \downarrow\beta & & \downarrow\gamma & & \\
E: & 0 & \to & A & \xrightarrow{\kappa} & B & \xrightarrow{\sigma} & C & \to & 0
\end{array}$$

because each small square commutes. Hence $(E\gamma)\gamma' \equiv E(\gamma\gamma')$.

We put all of these facts together in one place for reference.

5.2.1. <u>Lemma</u>. Let $E: \ 0 \to A \to B \to C \to 0$ be an extension of A by C and $\gamma:C' \to C$ a group homomorphism. Then there is an extension E' of A by C' and a morphism $(1_A,\beta,\gamma):E' \to E$ such that E' is unique with respect to the property that (I) commutes; i.e., $E \equiv F$ implies $E' \equiv F'$. Denote the well-defined class determined by E' in $\mathrm{Ext}(C',A)$ by $E\gamma$. Then $E(1_C) \equiv E$ and $E(\gamma\gamma') \equiv (E\gamma)\gamma'$, where $\gamma':C'' \to C$.

5.2.2. <u>Corollary</u> If $E: \; 0 \to A \overset{\kappa}{\to} B \overset{\sigma}{\to} C \to 0$ is exact, then $E\sigma$ splits.

<u>Proof.</u> Let $G: \; 0 \to A \overset{\iota}{\to} A \oplus B \overset{\pi}{\to} B \to 0$ be the canonical direct sum exact sequence. Now $E\sigma: \; 0 \to A \to B' \to B \to 0$, and we exhibit a morphism $(1,\delta,\sigma):G \to E$. By the previous lemma this will show $G \equiv E\sigma$. Then we only have to note that splitting of sequences is preserved between congruent classes in Ext(B,A). Referring to (II), we have $E = F$ in the diagram, $E' = E\sigma$ and G plays the role of F'. Define $\beta:A \oplus B \to B$ by $\beta(a,b) = \kappa(a) + b$. Then β is a group homomorphism, and a short calculation shows that the diagram below commutes:

$$\begin{array}{llllllllll} G: & 0 & \to & A & \overset{\iota}{\to} & A \oplus B & \overset{\pi}{\to} & B & \to & 0 \\ & & & 1\downarrow & & \downarrow\beta & & \downarrow\sigma & & \\ E: & 0 & \to & A & \overset{\kappa}{\to} & B & \overset{\sigma}{\to} & C & \to & 0 \end{array}$$

Thus $E\sigma \equiv G$, and so $E\sigma$ splits. ||

The fact that Lemma 5.2.1. was proven without explicit use of E as in (I) should make us suspect that it is a factorization property that really is at work here. That is the essence of the next lemma.

5.2.3. <u>Lemma</u> (Uniqueness of Factorization) Let

$E_1: \; 0 \to A_1 \overset{\kappa_1}{\to} B_1 \overset{\sigma_1}{\to} C' \to 0$ and $E: \; 0 \to A \overset{\kappa}{\to} B \overset{\sigma}{\to} C \to 0$. Given $\gamma:C' \to C$, we have the extension $E\gamma: \; 0 \to A \overset{\kappa'}{\to} B' \overset{\sigma'}{\to} C' \to 0$. Then any morphism $(\alpha_1, \beta_1,\gamma):E_1 \to E$ can be uniquely factored through $E\gamma$; i.e., there is a unique homomorphism $\beta':B_1 \to B'$ such that we get the following composition of morphisms: $(\alpha_1,\beta_1,\gamma) = (1,\beta,\gamma)(\alpha_1,\beta',1)$, where β is the map of 5.2.1., $(\alpha_1,\beta',1):E_1 \to E\gamma$, and $(1,\beta,\gamma):E\gamma \to E$.

<u>Proof.</u> Our task is to show that there is only one way β' can be

defined so that the following diagram commutes:

$$\begin{array}{lccccccccc}
E_1: & 0 & \to & A_1 & \xrightarrow{\kappa_1} & B_1 & \xrightarrow{\sigma_1} & C' & \to & 0 \\
 & & & \alpha_1\downarrow & \kappa' & \downarrow\beta' & \sigma' & \downarrow 1 & & \\
E\gamma: & 0 & \to & A & \to & B' & \to & C' & \to & 0
\end{array}$$

Let $a_1 \varepsilon A_1$ and $b_1 \varepsilon B_1$. A short calculation shows that the two squares above commute if and only if $\beta'(b_1) = (\beta_1(b_1), \sigma_1(b_1))$. Then note also that $\beta\beta' = \beta_1$. The claimed composition immediately holds. ||

The real usefulness of the lemma is: suppose we have some morphism $(1,\beta_1,\gamma):F \to E$, where $F: 0 \to A \to B_1 \to C' \to 0$ is exact. Then $(1,\beta_1,\gamma) = (1,\beta',1)(1,\beta,\gamma)$, where the first morphism is such that $(1, \beta',1):F \to E\gamma$. This means, then, that $F \equiv E\gamma$.

With this behind us, we proceed directly to the solution of a similar problem: if $\alpha:A \to A'$ is a homomorphism of groups, define the class αE. After all, what will hopefully serve as the additive inverse of $E \varepsilon \text{Ext}(C,A)$? A candidate is $(-1_A)E$, where $(-1_A)(a) = -a$ for all $a \varepsilon A$. All this work is aimed at 5.2.6. and 5.2.7.

5.2.4. <u>Lemma</u> For each extension $E: 0 \to A \to B \to C \to 0$ and homomorphism $\alpha:A \to A'$ there is an extension $E': 0 \to A' \to B' \to C \to 0$ and a morphism $(\alpha,\beta,1):E \to E'$. The extension E' is unique up to equivalence, and so depends upon E and α only; the well-defined class E' determines in $\text{Ext}(C,A')$ is denoted by αE. Any morphism $(\alpha,\beta_1,\gamma_1):E \to E_1$, where $E_1: 0 \to A' \to B_1 \to C_1 \to 0$ is exact, can be uniquely factored through αE in this fashion: $(a,\beta_1,\gamma_1) = (1,\beta',\gamma_1)(\alpha,\beta,1)$, where $(\alpha,\beta,1):E \to \alpha E$ and $(1,\beta',\gamma_1):\alpha E \to E_1$.

<u>Proof</u>. The situation can be pictured in diagram (III), where κ', β and σ' are to be constructed along with the group B' so that the

bottom row is exact and the diagram commutes.

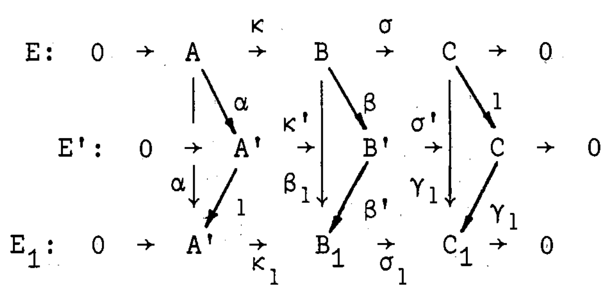
$$\begin{array}{lccccccccc}
E: & 0 & \to & A & \xrightarrow{\kappa} & B & \xrightarrow{\sigma} & C & \to & 0 \\
 & & & \alpha\downarrow & & \downarrow\beta & & \downarrow 1 & & \\
E': & 0 & \to & A' & \xrightarrow{\kappa'} & B' & \xrightarrow{\sigma'} & C & \to & 0
\end{array} \qquad \text{(III)}$$

The construction, again, is a standard one; it is called a "push-out" (the same one we used in 3.4.3.), a term due to P. Freyd.

Let $N \subseteq A' \oplus B$ be that <u>subgroup</u> defined as follows: $N = \{(-\alpha(a), \kappa(a)) \mid a \in A\}$. Define $B' = (A' \oplus B)/N$ and the following homomorphisms: $\kappa'(a') = (a',0) + N$ for all $a' \in A'$, $\sigma'((a',b) + N) = \sigma(b)$ for all $(a',b) + N$ in B', and finally $\beta(b) = (0,b) + N$. Then all of these maps are group homomorphisms. The proof that σ' is well-defined shows somewhat the reason for the choice of N: if $(a',b) - (a_1',b_1)$ is an element of N, then $(a' - a_1', b - b_1) = (-\alpha(a), \kappa(a))$ for some $a \in A$. Hence $\sigma(b - b_1) = \sigma(\kappa(a)) = 0$ implies $\sigma(b) = \sigma(b_1)$, and so $\sigma'((a',b) + N) = \sigma((a_1',b_1) + N)$.

One easily shows that (III) commutes when κ', σ', and β are defined in the above manner. Similarly, the exactness of the bottom row is easily established from the exactness of the top row and the definitions of the given homomorphisms.

Now let $E_1\colon\ 0 \to A' \xrightarrow{\kappa_1} B_1 \xrightarrow{\sigma_1} C_1 \to 0$ be an arbitrary extension of A' and $(\alpha, \beta_1, \gamma_1)\colon E \to E_1$ a morphism of extensions.



Our task is to show there is only one way to define $\beta':B' \to B_1$ such that $\beta_1 = \beta'\beta$, $\gamma_1\sigma' = \sigma_1\beta'$ and $\beta'\kappa' = \kappa_1 1_{A'}$; i.e., the lower faces of the prism commute.

We claim that all this is accomplished if and only if $\beta'((a',b) + N) = \kappa_1(a') + \beta_1(b)$. A straightforward, but tedious, computation shows that when β' is defined in this manner, it is a well-defined homomorphism and the claimed squares do commute. Conversely, if the squares commute and β' is some homomorphism which makes them so, then we arrive at the above formula for β . Examining the diagram, we have shown that $(\alpha,\beta_1,\gamma_1) = (1,\beta',\gamma_1)(\alpha,\beta,1)$.

We are now ready to show that E' is unique up to congruence. Let $(\alpha,\beta_1,1):E \to F'$ be a morphism, where F': $0 \to A' \to B' \to C \to 0$ is any extension of A'. Then $(\alpha,\beta_1,1) = (\alpha,\beta,1_C)(1_{A'},\beta',1_C)$, where $(1,\beta',1):E' \to F'$ is a morphism; i.e., $E' \equiv F'$. ||

The lemma allows us to speak of the well-defined class, αE, in $\mathrm{Ext}(C,A')$ determined by E, where $\alpha:A \to A'$. We have, as in the previous result, $(\alpha'\alpha)E \equiv \alpha'(\alpha E)$, where $\alpha':A' \to A''$; also $1_A E \equiv E$. Again, the usefulness of the uniqueness assertion comes to light when we have a morphism $(\alpha,\beta,1):E \to F$, where F: $0 \to A' \to B_1 \to C \to 0$ is an exact sequence and $\alpha:A \to A'$. Then $F \equiv \alpha E$.

Using a slight modification of the proof of 5.2.2., we have the following result.

5.2.5. <u>Corollary</u> Let E: $0 \to A \overset{\kappa}{\to} B \to C \to 0$ be an exact sequence of groups. Then κE splits.

We are now able to state what we have been after all along!

5.2.6. <u>Lemma</u> Let E: $0 \to A \to B \to C \to 0$ be an element representing a typical class in $\mathrm{Ext}(C,A)$. Let $\gamma:C' \to C$ and $\alpha:A \to A'$

be group homomorphisms. Then $\alpha(E\gamma) \equiv (\alpha E)\gamma$.

Proof. Note $\alpha(E\gamma)$ and $(\alpha E)\gamma$ are both elements of $\mathrm{Ext}(A',C')$. We have, by the definitions of $E\gamma$ and αE, morphisms $(1,\beta_1,\gamma):E\gamma \to E$ and $(\alpha,\beta_2,1):E \to \alpha E$. The composite morphism $(\alpha,\beta_2\beta_1,\gamma):E\gamma \to \alpha E$ factors uniquely through $(\alpha E)\gamma$, by 5.2.3., as $(\alpha,\beta_2\beta_1,\gamma) = (1,\beta,\gamma)(\alpha,\beta',1)$. Now the morphism $(\alpha,\beta',1):E\gamma \to (\alpha E)\gamma$ just generated is exactly like the one used to define $\alpha(E\gamma)$ from $E\gamma$. By the uniqueness assertion just above, we must have $\alpha(E\gamma) \equiv (\alpha E)\gamma$. ||

5.2.7. Lemma Any morphism $\Gamma_1 = (\alpha,\beta,\gamma):E \to E'$ of extensions gives rise to the congruence $\alpha E \equiv E'\gamma$.

Proof. We have $E: 0 \to A \to B \to C \to 0$; to use the uniqueness assertion of 5.2.4., we are required to exhibit a morphism $(\alpha,\beta,1):E \to E'\gamma$. By 5.2.3., we may factor $\Gamma_1:E \to E'$ through $\Gamma:E'\gamma \to E'$. When we do, we get $\Gamma_1 = \Gamma\Gamma_2$, where $\Gamma_2:E \to E'\gamma$ is a morphism of the form $(\alpha,\beta,1)$. ||

5.3. The Group Ext(C,A)

We have already shown that the elements of Ext(C,A) are equivalence classes of extensions of A by C, where $E \equiv F$ if and only if there is a morphism $(1,\beta,1):E \to F$. It was R. Baer who first noted that these classes formed a group, and the addition in this group is named for him.

5.3.1. Definition For E and F in Ext(C,A), their Baer sum is defined by $E + F = \nabla_A(E \oplus F)\Delta_C$.

Importantly, this definition makes sense, by 5.2.6. Also, the reader may compute that $E + F \in \mathrm{Ext}(C,A)$, recalling the definition of $E \oplus F$ (see page 73).

To clear the way to the proof that Ext(C,A) is a group, we first prove the following computation lemma.

5.3.2. Lemma Let $\alpha_i : A \to A'$ and $\gamma_i : C' \to C$ for $i = 1,2$ be group homomorphisms and let $E_i \in \text{Ext}(C,A)$ for $i = 1,2$. Then

(1) $\alpha_1(E_1 + E_2) \equiv \alpha_1 E_1 + \alpha_1 E_2$

(2) $(E_1 + E_2)\gamma_1 \equiv E_1\gamma_1 + E_2\gamma_2$

(3) $\Delta E \equiv (E \oplus E)\Delta$ and $E\nabla \equiv \nabla(E \oplus E)$ for all $E \in \text{Ext}(C,A)$.

(4) $(\alpha_1 + \alpha_2)E_1 \equiv \alpha_1 E_1 + \alpha_2 E_1$

(5) $E_1(\gamma_1 + \gamma_2) \equiv E_1\gamma_1 + E_1\gamma_2$

Proof. Only (1), (3) and (4) will be proven since the approach for the others is similar.

(1) Note that $\alpha\nabla = \nabla(\alpha \oplus \alpha)$ and $\Delta\gamma = (\gamma \oplus \gamma)\Delta$. Then

$$\alpha_1(E_1 + E_2) = \alpha_1(\nabla(E_1 \oplus E_2)\Delta)$$

$$\equiv \nabla(\alpha_1 + \alpha_1)[(E_1 \oplus E_2)\Delta]$$

$$\equiv \nabla[\alpha_1 E_1 \oplus \alpha_1 E_2]\Delta$$

$$= \alpha_1 E_1 + \alpha_1 E_2.$$

(3) One must verify that $(\Delta_A, \Delta_B, \Delta_C)$ is a morphism of extensions E and $E \oplus E$, where E is any element of Ext(C,A). This follows directly from the definitions of the homomorphisms involved. Then, by 5.2.7., $\Delta_A E \equiv (E \oplus E)\Delta_C$. The remaining fact needs $(\nabla_A, \nabla_B, \nabla_C)$ is a morphism of extensions, $E \oplus E$ to E.

(4) $\alpha_1 E_1 + \alpha_2 E_1 = \nabla(\alpha_1 E_1 \oplus \alpha_2 E_1)\Delta$

$$\equiv \nabla[(\alpha_1 \oplus \alpha_2)(E_1 \oplus E_1)]\Delta$$

$$\equiv (\nabla[\alpha_1 \oplus \alpha_2])\Delta E_1 \quad \text{(by (3))}$$

$$\equiv (\nabla(\alpha_1 \oplus \alpha_2)\Delta)E_1$$

$$\equiv (\alpha_1 + \alpha_2)E_1 \qquad \text{(see page 73).} \;||$$

Let E_o: $0 \to A \to A \oplus C \to C \to 0$ denote the split exact extension of A by C. We show that for all $E \in Ext(C,A)$, $E + E_o = E_o + E = E$. Consider the following diagram:

$$\begin{array}{lccccccccc} E: & 0 \to A & \to & B & \overset{\sigma}{\to} & C \to 0 \\ & 0\downarrow & & \downarrow\gamma & & \downarrow 1 \\ E_o: & 0 \to A & \to & A\oplus C & \to & C \to 0 \end{array}$$

Here 0 denotes the zero homomorphism of A and $\gamma(b) = (0,\sigma(b))$ for all $b \in B$. Then γ is a homomorphism and the diagram is easily seen to commute. Thus $E_o \equiv 0_A E$. Similarly, $E_o \equiv E0_C$.

By the distributive laws just proven, we have: $E + E_o \equiv 1_A E + 0_A E \equiv (1_A + 0_A)E \equiv E$ and $E_o + E \equiv E0_C + E1_C \equiv E(0_C + 1_C) \equiv E$.

By the same token, $E + (-1_A)E \equiv (1_A + (-1)_A)E \equiv 0_A E \equiv E_o$ and $(-1_A)E + E \equiv E_o$, where $(-1_A):A \to A$ is defined by $(-1_A)(x) = -x$ for all $x \in A$. Hence $(-1)E$ is the additive inverse of E.

The only thing remaining to show is the associative and distributive laws. Let

$$E_i: \; 0 \to A \overset{\kappa_i}{\to} B_i \overset{\sigma_i}{\to} C \to 0$$

$(i = 1,2)$ be two extensions of A by C. Let $\tau_A:A_1 \oplus A_2 \to A_2 \oplus A_1$ by defined by $\tau_A(a_1,a_2) = (a_2,a_1)$, the so called "switching homomorphism." Then immediately we have $\nabla_A\tau_A = \nabla_A$ and $\Delta_C = \tau_C\Delta_C$, and drawing the diagrams involved shows that (τ_A,τ_B,τ_C) is a morphism of extensions, $E_1 \oplus E_2$ to $E_2 \oplus E_1$. This allows us to conclude that $\tau_A(E_1 \oplus E_2) \equiv (E_2 \oplus E_1)\tau_C$. Finally,

$$\begin{aligned} E_1 + E_2 &= \nabla(E_1 \oplus E_2)\Delta \equiv \nabla(\tau_A(E_1 \oplus E_2))\Delta \\ &\equiv \nabla((E_2 \oplus E_1)\tau_C)\Delta \equiv \nabla(E_2 \oplus E_1)(\tau_C\Delta) \\ &\equiv \nabla(E_2 \oplus E_1)\Delta = E_2 + E_1. \end{aligned}$$

Thus addition is commutative. A similar purely "computational" argument can be given for the associative law, if we note that the diagonal and codiagonal maps have the following properties:

$$\nabla(\nabla \oplus 1) = \nabla(1 \oplus \nabla) \quad \text{and} \quad (\Delta \oplus 1)\Delta = (1 \oplus \Delta)\Delta.$$

We must identify $(C \oplus C) \oplus C$ with $C \oplus (C \oplus C)$, however, through the obvious isomorphism, to complete the verifications.

Recall that $(\alpha\oplus\beta)(E \oplus F) = \alpha E \oplus \beta F$ for all groups E and F. Then:

$$\begin{aligned} (E_1 + E_2) + E_3 &= \nabla[(E_1 + E_2) \oplus E_3]\Delta \\ &\equiv \nabla[(\nabla(E_1 \oplus E_2)\Delta) \oplus E_3]\Delta \\ &\equiv \nabla[(\nabla \oplus 1)[((E_1 \oplus E_2)\Delta) \oplus E_3]]\Delta \\ &\equiv \nabla[(\nabla \oplus 1)[(E_1 \oplus E_2) \oplus E_3](\Delta \oplus 1)]\Delta \\ &\equiv [\nabla(\nabla \oplus 1)[E_1 \oplus (E_2 \oplus E_3)](\Delta \oplus 1)]\Delta \\ &\equiv \nabla(1 \oplus \nabla)[E_1 \oplus (E_2 \oplus E_3)(1 \oplus \Delta)\Delta] \\ &\equiv \nabla[(1 \oplus \nabla)(E_1 \oplus (E_2 \oplus E_3))(1 \oplus \Delta)]\Delta \\ &\equiv \nabla[E_1 \oplus [\nabla(E_2 \oplus E_3)\Delta]]\Delta \\ &\equiv \nabla[E_1 \oplus [E_2 + E_3]]\Delta \\ &= E_1 + (E_2 + E_3). \end{aligned}$$

This completes the proof that Ext(C,A) is an Abelian group.

5.3.3. Corollary Ext(G,A) = 0 for all projective groups G.

Proof. Every sequence $0 \to A \to B \to G \to 0$ splits when G is projective (4.5.8.). ||

5.4. The Exact Sequence for Ext

As we have seen, the functor Hom does not preserve exact sequences of groups, because, for one thing, a homomorphism $\alpha:A \to G$ defined on a subgroup A of a group B may not, in general, extend to the entire group. It is fundamental in our method that Ext comes to the rescue to restore the exactness of the Hom sequence.

Before examining this prospect, we characterize those group homomorphism which do extend in terms of a split exact sequence.

5.4.1. Lemma Let A be a subgroup of B and E: $0 \to A \to B \to B/A \to 0$ the corresponding exact sequence. Then a homomorphism $\alpha:A \to G$ can be extended to all of B if and only if αE splits.

Proof. Suppose we have a homomorphism $\hat{\alpha}:B \to G$ such that $\hat{\alpha}|A = \alpha$. Let E: $0 \to G \to G\oplus C \to C \to 0$ be the usual split exact sequence, where $C = B/A$. Define a homomorphism $\beta:B \to G \oplus C$ by $\beta(b) = (\hat{\alpha}(b),\eta(b))$. Then the following diagram commutes:

$$\begin{array}{lccccccccc} E: & 0 & \to & A & \to & B & \overset{\eta}{\to} & C & \to & 0 \\ & & & \alpha\downarrow & & \downarrow\beta & & \downarrow 1 & & \\ E': & 0 & \to & G & \to & G\oplus C & \to & C & \to & 0 \end{array}$$

By 5.2.4., $E' \equiv \alpha E$; i.e., αE splits.

Conversely, suppose αE splits. Then the situation is as follows, where $\omega:B' \to G$ is a splitting homomorphism, so $\omega\zeta = 1_G$.

$$\begin{array}{ccccccccccc} E: & 0 & \to & A & \overset{\iota}{\to} & B & \overset{\sigma}{\to} & C & \to & 0 \\ & & & \alpha\downarrow & \zeta & \downarrow\beta & & \downarrow 1 & & \\ \alpha E: & 0 & \to & G & \underset{\omega}{\overset{\to}{\leftarrow}} & B' & \to & C & \to & 0 \end{array}$$

Set $\hat{\alpha} = \omega\beta : B \to G$. Then for all $a \in A$, $\hat{\alpha}(a) = \alpha(a)$; i.e., $\hat{\alpha}|A = \alpha$. ||

A more general result whose proof can be modeled on the one above is the following.

5.4.2. Proposition If $\kappa : A \to B$ is a monomorphism, any homomorphism $\alpha : A \to G$ can be extended through κ to all of G (that is, there exists $\beta : B \to G$ such that $\beta\kappa = \alpha$) if and only if αE splits, where $E: 0 \to A \to B \to C \to 0$ and $C = B/\kappa(A)$.

The extension αE is called the obstruction of each $\alpha : A \to G$, because if $\alpha E \equiv 0$ (so there is "no" obstruction), then α can be extended to all of B.

Viewed in another way, the problem of extending α to B is reflected in the splitting of αE. If we define a map $E^* : \mathrm{Hom}(A,G) \to \mathrm{Ext}(C,G)$ by $E^*(\alpha) = \alpha E$, then E^* is a homomorphism of groups by 5.3.2., and the problem of extending α to all of B is reflected in whether or not $\alpha \in \mathrm{Ker}\, E^*$. We call such a homomorphism E^* a connecting homomorphism.

The following is one of the major results of entire program.

5.4.3. Theorem [Cartan and Eilenburg, 4] Let $E: 0 \to A \overset{\kappa}{\to} B \overset{\sigma}{\to} C \to 0$ be an exact sequence of Abelian groups. Then the following sequences are also exact for all Abelian groups G:

$$(1) \quad 0 \to \mathrm{Hom}(C,G) \overset{\sigma^*}{\to} \mathrm{Hom}(B,G) \overset{\kappa^*}{\to} \mathrm{Hom}(A,G) \overset{E^*}{\to}$$

$$\mathrm{Ext}(C,G) \overset{\sigma^o}{\to} \mathrm{Ext}(B,G) \overset{\kappa^o}{\to} \mathrm{Ext}(A,G) \to 0$$

$$(2) \quad 0 \to \mathrm{Hom}(G,A) \xrightarrow{\kappa_*} \mathrm{Hom}(G,B) \xrightarrow{\sigma_*} \mathrm{Hom}(G,C) \xrightarrow{E_*}$$

$$\mathrm{Ext}(G,A) \xrightarrow{\kappa_o} \mathrm{Ext}(G,B) \xrightarrow{\sigma_o} \mathrm{Ext}(G,C) \to 0$$

The maps σ^*, κ^*, σ_*, and κ_* are those defined in 4.4.1. The other maps used above are defined in the following manner:

$E^*(\alpha) = \alpha E$, where $\alpha: A \to G$;

$E_*(\gamma) = E\gamma$, where $\gamma: G \to C$;

$\sigma^o(F) = F\sigma$ for $F \in \mathrm{Ext}(C,G)$;

$\kappa^o(F) = F\kappa$ for $F \in \mathrm{Ext}(B,G)$;

$\sigma_o(F) = \sigma F$ for $F \in \mathrm{Ext}(G,B)$;

$\kappa_o(F) = \kappa F$ for $F \in \mathrm{Ext}(G,A)$.

Proof. The six new maps above are all homomorphisms by 5.3.2. There is nothing new to show up to $\mathrm{Hom}(B,G)$: see 4.4.2. We show $\mathrm{Ker}\ E^* = \mathrm{Im}\ ^*$. By the previous result, we have: $\alpha \in \mathrm{Ker}\ E^*$ iff αE splits iff any homomorphism from A to G can be extended to all of B iff for all $\alpha: A \to G$ there exists $\beta: B \to G$ such that $\beta(\kappa(a)) = \alpha(a)$ for all $a \in A$ iff $\alpha \in \mathrm{Im}\ \kappa^*$.

To show exactness at $\mathrm{Ext}(C,G)$, we must show $\mathrm{Ker}\ \sigma^o = \mathrm{Im}\ E^*$. As in most arguments of this nature, it is usually easy to verify a zero sequence. Let $\alpha \in \mathrm{Hom}(A,G)$. Now $\sigma^o(E^*(\alpha)) = (\alpha E)\sigma$, which splits in $\mathrm{Ext}(B,G)$: see 5.2.2. Thus $\mathrm{Im}\ E^* \subseteq \mathrm{Ker}\ \sigma^o$.

For the reverse inclusion, let $F \in \mathrm{Ext}(C,G)$ such that $F\sigma$ splits. We must find a homomorphism $\alpha_1: A \to G$ such that F is an obstruction for α_1; i.e., $F \equiv \alpha_1 E$. Our assumptions yield the following commutative diagram (see following page). Let $\beta_1 = \beta\mu: B \to B_1$, so that $\sigma_1\beta_1 = \sigma$. Then $\sigma_1\beta_1\kappa = \sigma\kappa = 0$, and so applying 2.2.5. to the sequence $G \to B_1 \to C$ (the role of G in 2.2.5. is played by A, and that of ρ by $\beta_1\kappa$), we have $\beta_1\kappa = \kappa_1\alpha_1$ for some $\alpha_1 : A \to G$.

$$
\begin{array}{lccccccccc}
 & & & & & & & A & & \\
 & & & & & & & \downarrow\kappa & & \\
F\sigma: & 0 & \to & G & \to & G \oplus B & \overset{\rho}{\underset{\mu}{\rightleftarrows}} & B & \to & 0 \\
 & & & 1\downarrow & & \beta\downarrow \quad \beta_1\swarrow & & \downarrow\sigma & & \\
F: & 0 & \to & G & \xrightarrow{\kappa_1} & B_1 & \xrightarrow{\sigma_1} & C & \to & 0
\end{array}
$$

Hence the following is also a commutative diagram:

$$
\begin{array}{lccccccccc}
E: & 0 & \to & A & \xrightarrow{\kappa} & B & \xrightarrow{\sigma} & C & \to & 0 \\
 & & & \alpha_1\downarrow & & \beta_1\downarrow & & \downarrow 1 & & \\
E_1: & 0 & \to & G & \xrightarrow{\kappa_1} & B_1 & \xrightarrow{\sigma_1} & C & \to & 0
\end{array}
$$

By the uniqueness assertion of 5.2.4., we have $F \equiv \alpha_1 E$. This verifies exactness at $\mathrm{Ext}(C,G)$.

<u>Exactness at $\mathrm{Ext}(B,G)$.</u> Since $\kappa^o\sigma^o = (\sigma\kappa)^o = 0$, we have immediately that $\mathrm{Im}\ \sigma^o \subseteq \mathrm{Ker}\ \kappa^o$.

Now let $F \in \mathrm{Ext}(B,G) \cap \mathrm{Ker}\ \kappa^o$; then $F\kappa$ splits. We must show $F \equiv \Gamma\sigma$ for some $\Gamma \in \mathrm{Ext}(C,G)$. We set up a diagram like the one above:

$$
\begin{array}{lccccccccc}
F\kappa: & 0 & \to & G & \xrightarrow{\xi} & G \oplus A & \overset{\nu}{\underset{\mu}{\rightleftarrows}} & A & \to & 0 \\
 & & & 1\downarrow & & \beta\downarrow & & \downarrow\kappa & & \\
F: & 0 & \to & G & \xrightarrow{\tau} & B_1 & \xrightarrow{\psi} & B & \to & 0 \\
 & & & 1\downarrow & & \eta\downarrow & & \downarrow\sigma & & \\
\Gamma: & 0 & \to & G & \xrightarrow{\zeta} & B_1/\mathrm{Im}\beta_1 & \xrightarrow{\rho} & C & \to & 0
\end{array}
$$

We are going to show that Γ as we have drawn it here does the job. By assumption, we have a splitting homomorphism $\mu : A \to G \oplus A$. As before, define $\beta_1 = \beta\mu : A \to B_1$, so, in particular, $\kappa = \psi\beta_1$. The map η is the usual projection. Define $\zeta : G \to B_1/\mathrm{Im}\ \beta_1$ by

$\zeta(g) = \eta(\tau(g))$ and $\rho: B_1/\operatorname{Im}\beta_1 \to C$ by $\rho(b_1 + \operatorname{Im}\beta_1) = \sigma(\psi(b_1))$. The idea is to show that Γ is exact and $(1,\eta,\sigma): F \to \Gamma$ is a morphism.

This much is easily verified: ρ and ζ are well-defined, ρ is onto, and the two lower squares commute with these definitions of the maps involved. We show ζ is a monomorphism: if $\eta(\tau(g)) = \eta(\tau(g'))$, then $\tau(g - g') = \beta_1(a)$ for some $a \in A$. Since $\beta\xi = \tau 1_G$, and $\beta_1 = \beta\mu$, we have that $\beta\xi(g - g') = \tau(g - g') = \beta_1(a) = \beta\mu(a)$. Now β is a monomorphism because 1 and κ are monic (see 1.2.3.) and so $\xi(g - g') = \mu(a)$; i.e., $a = \nu(\mu(a)) = \nu(\xi(g - g')) = 0$. Thus $\tau(g-g') = \beta_1(a) = 0$ and so $g - g' \in \operatorname{Ker}\tau$. Since τ is monic, we finally have that $g = g'$, and so ζ is monic.

The last thing to show is exactness at $B_1/\operatorname{Im}\beta_1$. We have $\operatorname{Im}\zeta \subseteq \operatorname{Ker}\rho$ because $\psi\tau = 0$. Now let $b_1 + \operatorname{Im}\beta_1 \in \operatorname{Ker}\rho$; i.e., $\sigma\psi(b_1) = 0$ with $b_1 \in B_1$. This immediately gives $\psi(b_1) \in \operatorname{Ker}\sigma = \operatorname{Im}\kappa$, and so $\psi(b_1) = \kappa(a)$ for some $a \in A$. But $\kappa(a) = \psi(\beta_1(a))$, and so $\psi(\beta_1(a)) = \psi(b_1)$; i.e., $\beta_1(a) = b_1 - r$, where $r \in \operatorname{Ker}\psi = \operatorname{Im}\tau$. Thus $r = \tau(g_1)$ for some $g_1 \in G$, and finally, $\beta_1(a) = b_1 - \tau(g_1)$. This in turn implies that $b_1 - \tau(g_1) \in \operatorname{Im}\beta_1$, and so $\zeta(g_1) = b_1 + \operatorname{Im}\beta_1$; i.e., $\operatorname{Ker}\rho \subseteq \operatorname{Im}\zeta$.

The last fact to show is that <u>κ^0 is an epimorphism</u>; i.e., $\kappa: A \to B$ monic implies $\kappa^0: \operatorname{Ext}(B,G) \to \operatorname{Ext}(A,G)$ is epic.

To this end, there is a free Abelian group F and an epimorphism $\Theta: F \to B$ (4.5.2.). Let $K = \operatorname{Ker}\Theta$ and $L = \Theta^{-1}(\kappa(A))$. Now $\Theta: L \to \kappa(A)$ is an epimorphism with kernel K (for $\Theta(f) = 0 = \kappa(0)$ implies that $\{f \in F \mid \Theta(f) = 0\} \subseteq \{f \in F \mid \Theta(f) = \kappa(a)$ for some $a \in A\}$). We have, then, the following commutative diagram (see following page), where all the maps ι are the usual injections.

$$\begin{array}{lccccccccc} E_1: & 0 & \to & K & \overset{\iota}{\to} & L & \overset{\rho}{\to} & A & \to & 0 \\ & & & 1\downarrow & & \downarrow\iota & & \downarrow\kappa & & \\ E_2: & 0 & \to & K & \overset{\iota}{\to} & F & \overset{\Theta}{\to} & B & \to & 0 \end{array}$$

Define ρ by: let $\ell \in L$ and take for $\rho(\ell)$ an element of A such that $\kappa(a) = \Theta(\ell)$. Then ρ is a well-defined homomorphism (see the proof of 2.2.5.). Since Θ is epic, so is ρ. Finally, $\operatorname{Ker} \rho = \operatorname{Im} \iota|_L$. This follows because if $\rho(\ell) = 0$, then $\Theta(\ell) = \kappa(a) = 0$, and so $a = 0$ because κ is monic. All this shows that E_1 is exact, and so $E_1 \equiv E_2\kappa$ by 5.2.4. Hence the following diagram commutes:

$$\begin{array}{ccccc} \operatorname{Hom}(K,G) & \overset{E_1^*}{\to} & \operatorname{Ext}(B,G) & & \\ 1\downarrow & & \downarrow\kappa^o & & \\ \operatorname{Hom}(K,G) & \overset{E_1^*}{\to} & \operatorname{Ext}(A,G) & \overset{\rho^o}{\to} & \operatorname{Ext}(L,G); \end{array}$$

we have already shown that the bottom row is exact. But L is free, and hence projective (4.5.4.); thus $\operatorname{Ext}(L,G) = 0$ (5.3.3.). Thus E_1^* is epic, and so, therefore, is κ^o.

This completes the proof of the exactness of (1). The arguments given for the proof of the exactness of (1) are easily modified to give exactness of (2). The only real difficulty is trying to show that if $E\colon\ 0 \to A \to B \overset{\sigma}{\to} C \to 0$ is exact, then $\sigma_o\colon \operatorname{Ext}(G,B) \to \operatorname{Ext}(G,C)$ is epic.

Let $S\colon\ 0 \to C \overset{\mu}{\to} D \overset{\lambda}{\to} G \to 0$ be an arbitrary element of $\operatorname{Ext}(G,C)$. Since $\mu\colon C \to D$ is monic, the result proven just above says that $\mu^o\colon \operatorname{Ext}(D,A) \to \operatorname{Ext}(C,A)$ is epic; i.e., there is a sequence E': $0 \to A \overset{\delta}{\to} M \overset{\omega}{\to} D \to 0$ such that $\mu^o(E') = E$. This gives rise to the following commutative diagram:

$$\begin{array}{lccccccccc} E: & 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & & 1\downarrow & & \downarrow\beta & & \downarrow\mu & & \\ E': & 0 & \to & A & \to & M & \to & D & \to & 0 \end{array}$$

This can be extended to the following commutative diagram:

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
E: & 0 & \to & A & \to & B & \xrightarrow{\sigma} & C & \to & 0 \\
 & & & 1\downarrow & & \downarrow\beta & & \downarrow\mu & & \\
E': & 0 & \to & A & \xrightarrow{\delta} & M & \xrightarrow{\omega} & D & \to & 0 \\
 & & & & & \tau\downarrow & & \downarrow\lambda & & \\
 & & & & & G & \xrightarrow{1} & G & & \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & 0 & & 0 & &
\end{array}$$

Now β is monic because 1 and μ are monic, and τ, by a chase argument, can be defined so that the bottom square commutes and τ is epic. Thus we have the morphism $(\sigma,\omega,1):E'' \to S$, where E'' is the exact sequence in the middle column. Thus $S = \sigma E''$, and the proof of the theorem is complete. ||

Sequences (1) and (2) are the key to the homological method. We will use them to characterize injective groups. But first, we give some calculations for certain examples of $\text{Ext}(C,A)$.

5.4.4. Corollary Let C and G be given Abelian groups and F: $0 \to K \xrightarrow{\kappa} P \xrightarrow{\sigma} C \to 0$ an exact sequence with P projective (see 4.5.5.) Then

$$\text{Ext}(C,G) \simeq \text{Hom}(K,G)/\kappa^*\text{Hom}(P,G).$$

Proof. Note that $\text{Ext}(P,G) = 0$ since P is projective. By 5.4.3., we have the exact sequence

$$\text{Hom}(P,G) \xrightarrow{\kappa^*} \text{Hom}(K,G) \xrightarrow{E^*} \text{Ext}(C,G) \xrightarrow{\sigma^0} \text{Ext}(P,G) = 0.$$

Hence E^* is epic. By the Fundamental Theorem of Homomorphisms and the fact that $\text{Ker } E^* = \text{Im } \kappa^*$, we have the desired result. ||

5.4.5. Corollary Let G be a given group and $Z(m)$ the cyclic group of order $m > 1$. Then $Ext(Z(m),G) \simeq G/mG$.

Proof. We can use the previous corollary with the exact sequence $0 \to Z \overset{\kappa}{\to} Z \overset{\eta}{\to} Z(m) \to 0$, where κ is multiplication by $m > 1$ and η is the natural projection. Since Z is projective, we have $Ext(Z(m),G) \simeq Hom(Z,G)/\kappa^*Hom(Z,G)$. Refer to the proof that $Hom(Z,G) \simeq G$ (4.2.1.). The isomorphism involved takes $\kappa^*Hom(Z,G)$ isomorphically onto mG. Thus $Ext(Z(m),G) \simeq G/mG$. ||

5.4.6. Proposition $Ext(Z(m),Z) \simeq Z/mZ \simeq Z(m)$.

5.4.7. Proposition $Z(m)/nZ(m) \simeq Z(m,n)$ for all positive integers m and n.

Proof. Let 1_m and $1_{(m,n)}$ denote the classes represented by 1 in $Z(m)$ and $Z(m,n)$ respectively. These groups are generated as additive Abelian groups by these classes. The assignment $\psi(1_m) = 1_{(m,n)}$ defines a well-defined epimorphism from $Z(m)$ to $Z(m,n)$. We investigate the kernel of ψ. The claim is that $\text{Ker } \psi = nZ(m)$. It is clear that $nZ(m) \subseteq \text{Ker } \psi$. Let $\overline{s} \in \text{Ker } \psi \cap Z(m)$. Then $0 = \psi(\overline{s}) = s\,\psi(1_m) = s1_{(m,n)}$; hence $(m,n)|s$. Write $(m,n) = am + bn$ for some integers a and b. Then $s = k(am + bn) = (ka)m + (kb)n$. This means that $\overline{s} = n(\overline{k}\,\overline{b})$, so $\overline{s} \in nZ(m)$. We have verified our claim, and so $Z(m,n) \simeq Z(m)/nZ(m)$. ||

5.4.8. Corollary $Ext(Z(m),Z(n)) \simeq Z(m,n)$.

Proof. $Ext(Z(m),Z(n)) \simeq Z(n)/mZ(n) \simeq Z(m,n)$. ||

5.4.9. Corollary $Ext(Z(p),Z(q)) = 0$ for all primes $p \neq q$.

Proof. $Ext(Z(p),Z(q)) \simeq Z(p,q) \simeq Z(1) = 0$, since $Z(1) \simeq Z/(1) = 0$. ||

What we really used here was the fact that $(p,q) = 1$. We get the following corollary just as readily.

5.4.10. Proposition (Shur-Zassenhaus) Let $Z(m)$ and $Z(n)$ be given cyclic groups whose orders are relatively prime. Then any extension $0 \to Z(m) \to G \to Z(n) \to 0$ must split.

Proof. $Ext(Z(m),Z(n)) \simeq Z(m,n) = 0.$ ||

We close this section with a statement of how Ext operates through direct sums. The proof proceeds precisely along the lines of 4.3.4. To get the induced homomorphisms needed in the proof, refer to those defined in 5.4.3.

5.4.11. Proposition Let $\{G_\lambda\}_{\lambda\in\Lambda}$ be a given family of groups. Then:

$$(1) \quad Ext(\sum_\lambda G_\lambda, H) \simeq \prod_\lambda Ext(G_\lambda, H)$$

$$(2) \quad Ext(H, \prod_\lambda G_\lambda) \simeq \prod_\lambda Ext(H, G_\lambda).$$

Remember that when Λ is a finite index set, the direct product in the answer will be a direct sum. One may therefore compute the groups $Ext(G,H)$ completely using these propositions when G and H are finitely generated, and thus are direct sums of a finite number of cyclic groups.

5.5. Injective Groups

As innocent as this section might appear, it forms a keystone in our theory. We prove here the equivalence of injectivity and divisibility for Abelian groups and finish our development of divisible groups. With the group $Ext(C,A)$ we can give an elegant characterization of injective groups, again due to R. Baer [1].

Recall again that G is injective if for all exact sequences of groups $0 \to A \overset{\kappa}{\to} B$ with $A \subseteq B$ and all homomorphisms $\beta: A \to G$ there is a homomorphism $\alpha: B \to G$ such that $\beta\kappa = \alpha$. Our assumption that $A \subseteq B$ is not a serious drawback, since in any case $\kappa(A) \simeq A$ because κ is a monomorphism. It really is a simplification of the general case.

5.5.1. <u>Theorem</u> The following statements are equivalent for any group G:

(1) G is injective.

(2) For all monomorphisms $\kappa: A \to B$ with $A \subseteq B$, $\kappa^*: \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G)$ is an epimorphism.

(3) Every exact sequence $0 \to G \to A \to B \to 0$ of Abelian groups splits (i.e., $\mathrm{Ext}(B,G) = 0$ for all groups B).

(4) $\mathrm{Ext}(Z(m),G) = 0$ for all finite cyclic groups $Z(m)$.

<u>Proof</u>. (1) implies (2). Let $\kappa: A \to B$ be the prescribed monomorphism. Let $\rho: A \to G$ be any homomorphism. From the definition of injectivity, there exists $\tau: B \to G$ such that $\tau\kappa = \rho$: i.e., $\rho = \kappa^*(\tau)$, and so κ^* is an epimorphism.

(2) implies (3). Let $0 \to G \xrightarrow{\kappa} A \xrightarrow{\sigma} B \to 0$ be an exact sequence of Abelian groups. Then $\kappa^*: \mathrm{Hom}(A,G) \to \mathrm{Hom}(G,G)$ is an epimorphism, so $1_G = \rho\kappa$ for some $\rho: A \to G$. The sequence therefore splits (2.5.7.).

(3) implies (4). Trivial

(4) implies (1). Suppose every exact sequence $0 \to G \to D \to Z(m) \to 0$ splits. Let $0 \to A \xrightarrow{\kappa} B$ be exact (with $A \subseteq B$) and $\alpha: A \to G$ a homomorphism. We must show there exists $\beta: G \to B$ such that $\beta\kappa = \alpha$. The proof proceeds by a Zorn's Lemma argument. Let $P = \{(S,\gamma) \mid A \subseteq S \subseteq B$ and $\gamma: S \to G$ extends $\alpha\}$. Now $P \neq \emptyset$ because $(A,\alpha) \in P$. Let $\{(S_\lambda, \gamma_\lambda)\} \subseteq P$ be a chain. Let T be the union of the subgroups S_λ (and so T is a subgroup of B containing A) and let λ be that well-defined homomorphism $\lambda(t) = \gamma_\lambda(t)$ when $t \in S_\lambda$. Thus (T,λ) is

an upper bound of $\{(S_\lambda, \gamma_\lambda)\}$ in P. By Zorn's Lemma, there is a maximal pair (T,β) in P; we must show that $B = T$. Surely $T \subseteq B$. If the containment is proper, select $b \in B - T$ and consider $H = \langle T,b \rangle$. Define $\mu(z) = zb + T$ for $z \in Z$. Then μ is an epimorphism from Z to H/T and the kernel of μ is an ideal of Z. Since Z is a principal ideal domain, we may denote $\mathrm{Ker}\,\mu = (m)$; assume that $m > 1$. Then $H/T \cong Z/(m) \cong Z(m)$. Now $0 \to T \to H \to H/T \to 0$ is exact, and so is, therefore,

$$\mathrm{Hom}(H,G) \to \mathrm{Hom}(T,G) \to \mathrm{Ext}(H/T,G).$$

But $\mathrm{Ext}(H/T,G) = \mathrm{Ext}(Z(m),G) = 0$ by hypothesis, and so $\mathrm{Hom}(H,G) \to \mathrm{Hom}(T,G)$ is an epimorphism. This means that each homomorphism $T \to G$ can be extended to a map $H \to G$; in particular, β can be so extended, which contradicts the maximality of (T,β).

If $m = 1$ above, we then have $H/T = 0$, and so $\mathrm{Hom}(H,G) = \mathrm{Hom}(T,G)$. This certainly implies that every homomorphism $T \to G$ can be extended to a map from H to G, and the proof is complete. ||

5.5.2. Corollary An Abelian group G is injective if and only if G is divisible.

Proof. $0 = \mathrm{Ext}(Z(m),G) \cong G/mG$ if and only if G is divisible.

This extremely important cornerstone in the theory has been generalized to modules. Kaplansky [18] proved that divisibility implies injectivity for modules over Dedekind rings, and it was Cartan and Eilenburg [4] who proved that this characterized the Dedekind rings. For us, however, the following result is the most useful.

5.5.3. Proposition An Abelian group G is injective if and only if $\mathrm{Ext}(Q/Z,G) = 0$.

Proof. If G is injective, then every extension of G splits.

In particular, $Ext(Q/Z,G) = 0$. Conversely, suppose that $Ext(Q/Z,G) = 0$. We build up in pieces in the following way: every $Z(p^n)$ can be imbedded in $Z(p^\infty)$. For each integer n, $Z(n)$ is a direct sum of cyclic groups $Z(p_i^{n_i})$ for various primes p_i and integers n_i. Thus $Z(n)$ can be imbedded in a direct sum of $Z(p_i^\infty)$. Filling out the direct sum with all the primes p, we have that each $Z(n)$ can be imbedded in $\sum_p Z(p^\infty) = Q/Z$. By 5.4.3., $Ext(Q/Z,G) \to Ext(Z(n),G) \to 0$ is exact. But the group on the left is zero. Hence $Ext(Z(n),G) = 0$, and so G is injective, by 5.5.1. ||

5.5.4. Proposition Every Abelian group can be imbedded as a subgroup in some injective group.

Proof. We have just outlined above how each finite cyclic group can be imbedded in Q/Z, which is divisible, and hence injective. Also Z can be imbedded in Q which is also injective. Now any free group F is a direct sum of Z's, and so can be imbedded in that many copies of Q; this direct sum is divisible, and hence injective. Now let A be an arbitrary Abelian group. Then $A \simeq F/F_o$ where F is free. Imbed F in a divisible group D, so then A is imbedded in D/F_o. This last group is a quotient of a divisible group, and is thus divisible, and so finally injective. ||

5.5.5 Corollary For any Abelian group A, there are injective groups I and I_o such that the following is exact:

$$0 \to A \to I \to I_o \to 0.$$

Proof. There is an injective group I in which A can be imbedded, by the above result. Take $I_o = I/A$. As was pointed out, I/A is divisible, and so injective. ||

We are now in a position to continue the study of divisible groups begun in Chapter III.

5.5.6. Corollary Every group can be imbedded as a subgroup in some divisible group.

5.5.7. Proposition Let D be a group. Then the following are equivalent.

(1) D is a divisible group.

(2) Let G be any group and H a subgroup of G. Let f be a homomorphism of H into a group D. Then f can be extended to all of G.

(3) D is a direct summand of every group which contains it as a subgroup.

Proof. (1) implies (2). We are given f; let $\iota: H \to G$ be the identity map on H. We must find $g: G \to D$ such that $f = g\iota$. This means that $\iota^*: \text{Hom}(G,D) \to \text{Hom}(H,D)$ is an epimorphism, which by 5.5.1. is equivalent to the fact that D is injective. Since D is divisible, we have the desired result.

(2) implies (3). Suppose D is given and H is a group such that H contains D as a subgroup. The assumption allows us to assert the existence of a homomorphism $g: H \to D$ such that $1_D = g\iota$. By 2.5.1. $H \simeq D \oplus \text{Ker } g$, and so D is a direct summand of H.

(3) implies (1). Let D be the group in question. By 5.5.6. there is a divisible group H such that $D \subseteq H$. By assumption, then, $H = D \oplus R$. But this implies that D is divisible. ||

5.5.8. Corollary If G is a group and H is a divisible subgroup of G, then $G \simeq H \oplus M$ for some subgroup M of G.

5.5.9. Corollary Let G be a group. Then G contains a unique maximal divisible subgroup D such that $G = D \oplus G_o$, where G_o is reduced; i.e., the only divisible subgroup of G_o is (0).

Proof. Let D be the group generated by the family S of all divisible subgroups of G; $S \neq \emptyset$ because $(0) \in S$. We show that D is divisible. Let $x \in D$; then $x = x_1 + \cdots + x_k$, where each x_i lies in a divisible subgroup D_i of G. Select $n \in Z$; then $x_i = ny_i$ for $y_i \in D_i$, $i = 1,\cdots,k$. Hence $x = n(\sum_{i=1}^{k} y_i)$ with the summation lying in D. Hence D is divisible, and by the above corollary, $G = D \oplus G_o$. By the very definition of D, G_o has no nonzero divisible subgroups. ||

5.6. Injective Envelopes

We know that every group has a unique maximal divisible direct summand: that is the essence of 5.5.9. Now that we have the equivalence of injectivity and divisibility for groups, we start to work on the problem from the other direction: we show that given any group G there is a minimal divisible group D which contains it; D is unique in a very natural way. Such a group D is called the injective envelope of G, or the divisible hull of G.

As we mentioned, Kulikov [22] and Baer [1] showed the existence of injective envelopes. Our presentation, however, is based on the work of Eckmann and Schopf [7] who generalized the concept to modules.

To this end, we make the following definition.

5.6.1. Definition Let G and H be groups with G a subgroup of H. We say G is an essential subgroup of H (or H is an essential extension of G) if every nonzero subgroup of H has nonzero intersection with G.

Immediately, any group is an essential subgroup of itself. The value of the concept seems to lie in the fact that if G is an essential subgroup of H and H' is some subgroup of H such that $H' \cap G = (0)$, then $H' = (0)$. Note also that if G is an essential extension of H and H is an essential extension of K, then G is an essential extension of K. Similarly, if $G \supseteq H \supseteq K$ and G is an essential extension of K, then it is an essential extension of H.

5.6.2. <u>Proposition</u> Let G be an essential subgroup of H and I an injective group such that $G \subseteq I$ (see 5.5.4.) Then the identity map of G can be extended to a <u>monomorphism</u> of H into I.

<u>Proof</u>. Since I is injective, the identity map $\iota: G \to H$ induces an epimorphism $\iota^*: \mathrm{Hom}(H,I) \to \mathrm{Hom}(G,I)$ (see 5.5.1.). One of the homomorphisms in Hom(G,I) is the identity map 1_G, since $G \subseteq I$. Hence $1_G = \phi\iota$ for some $\phi \in \mathrm{Hom}(H,I)$. Consider the subgroup $\mathrm{Ker}\ \phi \subseteq H$, and let $x \in \mathrm{Ker}\ \phi \cap G$. Then $x = 1_G(x) = \phi(\iota(x)) = \phi(x) = 0$, and so $\mathrm{Ker}\ \phi \cap G = (0)$. Since G is an essential subgroup of H, $\mathrm{Ker}\ \phi = 0$.||

5.6.3. <u>Proposition</u> Let G be a group. Then G is injective if and only if G has no proper essential extensions.

<u>Proof</u>. Suppose G is injective and H is a group such that H is an essential extension of G. Since G is divisible, we have $H = G \oplus R$ with $G \cap R = (0)$. Thus $R = (0)$, since H is an essential extension of G. This means that $H = G$, and we do not have a proper essential extension of G.

Conversely, assume G has no proper essential extensions. Let $I \supseteq G$ be an injective group and select a subgroup I' of I which is maximal with respect to the property that $I' \cap G = (0)$. We show I/I' is an essential extension of $(G \oplus I')/I' \simeq G$.

To this end, let $I' \subset K \subset I$ and assume that K/I' has zero intersection with $(G \oplus I')/I'$; i.e., $K \cap (G \oplus I') \subset I'$. Thus $K \cap G \subset I' \cap G = (0)$, and so $K = I'$ by the maximality of I'.

Hence I/I' is an essential extension of G; since G has no proper essential extension, $I/I' \simeq G$, and so $I \simeq G \oplus I'$. Since I is divisible, so is G, and hence G is injective. ||

5.6.4. Theorem Every group G has a maximal essential extension H which is unique in the following sense: if H' is any other maximal essential extension of G, then the identity map of G can be extended to an isomorphism of H onto H'. In other words, there is an isomorphism from H onto H' leaving all the elements of G fixed.

Proof. Let I be an injective group containing G. Using a Zorn's Lemma argument, there is a maximal essential extension H of G inside of I. Ones uses $P = \{T \mid G \subseteq T \subseteq I$ and T is an essential extension of $G\}$ for the partially ordered set in the proof. Then $P \neq \emptyset$ since $G \varepsilon P$. The union of any chain of P is a group containing G as an essential subgroup. We must show that the maximality of H is independent of any I chosen.

Let $H' \supseteq H \supseteq G$ and H' be the supposed essential extension of G, where now H' may or may not be contained in I. Then H' is an essential extension of H, and so by 5.6.2. there is a monomorphism $\phi: H' \to I$ such that $\phi(H) = H$. Hence $H' \simeq \phi(H') \subseteq I$ is an essential extension of G in I. By the maximality of H, we have $H = \phi(H')$. Finally, then, $H = \phi^{-1}(H) = H'$, and so H is a maximal essential extension of G, not just in I, but absolutely.

Now any essential extension of H is an essential of G. This cannot happen, since H is maximal with respect to this property, so H has no proper essential extensions, and so by 5.6.3., is injective (and hence divisible).

If H' is any essential extension of G, the identity map of G can be extended to a monomorphism ϕ of H' into H. If H' itself is maximal, then $H \supseteq \phi(H') \simeq H'$, and so $H = \phi(H')$; i.e., ϕ is an isomorphism. ||

5.6.5. <u>Theorem</u> Let G and H be groups with G a subgroup of H. Then the following are equivalent:

(1) H is a maximal essential extension of G (every group has one of these by the previous result).

(2) H is an essential extension of G and H is injective.

(3) H is a minimal injective extension of G.

<u>Proof</u>. (1) implies (2). If (1) holds, then H has no proper essential extensions, and so H is injective (5.6.3.).

(2) implies (3). Suppose $G \subset I \subset H$ with I an injective group. Then $H = I \oplus R$, where R is a subgroup of H, since I is injective, and so divisible. But H is an essential extension of I; since $I \cap R = (0)$, we have $R = (0)$. Thus $I = H$.

(3) implies (1). Assume H is a minimal injective extension of G. Let $G \subseteq H' \subseteq H$, where H' is a maximal essential extension of G. Such an extension has no proper essential extension, and so H' is injective. Thus $H = H'$. ||

One should note that nowhere did we use any particular information peculiar to groups. Hence the notion of minimal injective extension generalizes immediately to modules over commutative rings with unity.

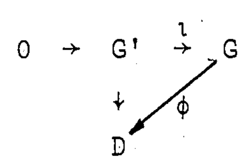
5.6.7. <u>Examples</u>.

(1) Considering $Z(p)$ as a subgroup of $Z(p^\infty)$, we have that $Z(p^\infty)$ is the injective envelope of $Z(p)$. For $Z(p^\infty)$ is injective since it is divisible; we need only show that $Z(p^\infty)$ is an essential extension of $Z(p)$. If H is any nonzero proper subgroup of $Z(p^\infty)$, then $H = Z(p^n)$ for some $n \geq 1$. Thus $H \cap Z(p) \neq 0$, and so $Z(p^\infty)$ is an essential extension of $Z(p)$. In fact $Z(p^\infty)$ is an essential extension of any $Z(p^n)$ for all $n \geq 1$.

(2) We now prove the following somewhat involved fact which needs the injective envelope for the proof.

5.6.8. <u>Proposition</u> Every group G can be imbedded as a pure subgroup in a direct sum $A \oplus D$, where D is a divisible group and A is a direct product of cyclic torsion groups.

<u>Proof</u>. Let $G' = \bigcap_{n=1}^{\infty} nG$ and set D equal to the injective envelope of G'. Denote the imbedding of G' in D by $G' \to D$ and let $\iota : G' \to G$ be the usual injection of a subgroup in the entire group. Since D is injective, let ϕ be the homomorphism which makes the following diagram commute:



Set $A = \prod_{n=1}^{\infty} G/nG$. Then A is a direct product of cyclic torsion groups: each G/nG is a bounded (and hence torsion) group. By 3.5.11. each G/nG is a direct sum of cyclic groups.

Define $\eta : G \to A$ by $\eta(g) = \langle g + nG \rangle$, and $\Theta : G \to A \oplus D$ by $\Theta(g) = \eta(g) + \phi(g)$. Then Θ is monic (and so affords the desired imbedding), and $\text{Im}\, \Theta$ is pure in $A \oplus D$; the details are left to the reader. ||

CHAPTER VI

TENSOR AND TORSION PRODUCTS OF GROUPS

It is the intent of this chapter to construct the second major exact sequence of our homological method. The program involves the tensor product $G \otimes H$ of two Abelian groups and the corresponding study of the effect of "tensoring" through an exact sequence. Tensor products have been with us for some time, but mostly in analysis, where people were interested in the tensor product of vector spaces. The central role of the tensor product in multilinear algebra was highlighted in Bourbaki's [3] treatise on this subject. However, the tensor product for Abelian groups was first defined by Whitney [33] in 1939. Our presentation follows that of MacLane and Cartan-Eilenburg.

6.1. Tensor Products

Let G and H be Abelian groups. We are going to define an Abelian group based on $G \times H$, and in a very natural way, this group will be unique.

6.1.1. Definition Let $f: G \times H \to J$ be an arbitrary function from the cross product of two groups G and H to a third group J. We say f is bi-additive if $f(g + g', h) = f(g,h) + f(g',h)$ and $f(g, h + h') = f(g,h) + f(g,h')$ for all elements g and g' in G and h and h' in H. We say f is balanced if f is bi-additive and $f(ng,h) = f(g,nh)$ for all integers n and elements $g \in G$ and

$h \in H$.

6.1.2. <u>Definition</u> Let J and J' be groups and $\phi: G \times H \to J'$ any balanced map. Let $f: G \times H \to J$ be a function. We say ϕ can be <u>factored through f</u> if there is an Abelian group <u>homomorphism</u> $\psi: J \to J'$ such that $\psi f = \phi$; i.e., the following diagram can be made to commute for some ψ:

$$\begin{array}{ccc} G \times H & \xrightarrow{f} & J \\ & \searrow^{\phi} & \downarrow \psi \\ & & J'. \end{array}$$

6.1.3. <u>Theorem</u> Let G and H be Abelian groups. Then there is an Abelian group T and a balanced map $t: G \times H \to T$ such that

(1) $t(G \times H)$ generates T; i.e., every element of T has the form $\sum_i n_i\, t(g_i, h_i)$, where $n_i \in Z$, $g_i \in G$, $h_i \in H$ and i runs over some finite indexing set.

(2) Every balanced map from $G \times H$ into an Abelian group can be uniquely factored through t.

(3) If T_1 is another Abelian group as in (1) and $t_1: G \times H \to T_1$ is some balanced map with property (2), then T and T_1 are isomorphic.

<u>Proof</u>. (1) Let F be the free Abelian group generated by $G \times H$ and S the <u>subgroup</u> of F generated by the collection of all elements of the following form:

$$(g' + g'', h) - (g', h) - (g'', h),$$

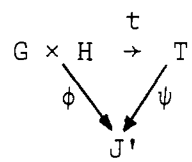
$$(g, h' + h'') - (g, h') - (g, h''),$$

$$\text{and } (ng, h) - (g, nh) \text{ for } n \quad Z.$$

Take $T = F/S$ and define $t: G \times H \to T$ by $t(g,h) = (g,h) + S$. Then by the way we sneaked up on T, the map t is automatically balanced, and every element of T is of the form

$$[\sum_i n_i\ (g_i,h_i)] + S = \sum_i n_i[(g_i,h_i) + S)] = \sum_i n_i t(g_i,h_i).$$

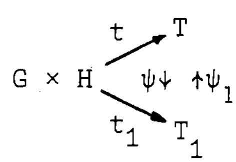
(2) Now let J' be any Abelian group and $\phi : G \times H \to J'$ a balanced map. Define $\psi' : F \to J'$ by $\psi'(\sum_i n_i(g_i,h_i)) = \sum_i n_i \phi(g_i,h_i)$. Since ϕ is balanced, we see that $\psi'(S)$ is zero. Thus ψ' induces a well-defined homomorphism ψ on T, given by $\psi((g,h) + S) = \psi'(g,h)$. Then $\psi(t(g,h)) = \psi((g,h) + S) = \psi'(g,h) = \phi(g,h)$; i.e., $\psi t = \phi$ for all $(g,h) \in G \times H$. Thus the following diagram commutes:



Now suppose $\rho : T \to J'$ also factors through t; i.e., $\rho t = \phi$. Then $\rho((g,h) + S) = \rho(t(g,h)) = \phi(g,h) = \psi(t(g,h)) = \psi((g,h) + S)$. This means that since ρ and ψ are homomorphisms, that $\psi = \rho$; i.e., ψ is unique with respect to this property.

We have the following special case: $J' = T$. Then there exits a unique homomorphism $\psi : T \to T$ such that $\psi t = t$. This means that $\psi = 1_T$, by the uniqueness of the map ψ.

(3) The situation is now



There is a homomorphism $\psi : T \to T_1$ such that $\psi t = t_1$, and there is a homomorphism $\psi_1 : T_1 \to T$ such that $\psi_1 t_1 = t$. Thus $(\psi\psi_1)t_1 = t_1$ and $(\psi_1\psi)t = t$. By the uniqueness assertion of (2), we have $\psi\psi_1 = 1_{T_1}$ and $\psi_1\psi = 1_T$. This means that ψ and ψ_1 are isomorphisms. ||

The theorem says many important things. The group T exists (1) and is unique up to isomorphism (3). In fact, (3) contains a method of showing how groups are isomorphic to T, as we show in the examples.

Since T is unique with respect to these properties, we may give it a name.

6.1.4. Notation The group T of the theorem will be denoted by $G \otimes H$, the tensor product of G and H. The image of (g,h) under t will be denoted by $g \otimes h$; this is an element of the generating set of T, and is called a simple tensor.

By the balanced property of t, simple tensors have the following properties:

(1) $(g' + g'') \otimes h = g' \otimes h + g'' \otimes h$

(2) $g \otimes (h' + h'') = g \otimes h' + g \otimes h''$

(3) $g \otimes nh = ng \otimes h = n(g \otimes h)$

Note that (3) follows from the fact that $(g \otimes h) + (g \otimes h) = g \otimes (h+h) = (g + g) \otimes h$. Thus a typical element of $G \otimes H$ is of the form

$$n_1(g_1 \otimes h_1) + \cdots + n_k(g_k \otimes h_k) = (n_1g_1) \otimes h_1 + \cdots + (n_kg_k) \otimes h_k.$$

The zero element of $G \otimes H$ is $0 \otimes 0$, since $(g \otimes h) + (0 \otimes 0) = (g \otimes h) + (0 \otimes h) = (g + 0) \otimes h = g \otimes h$. Here we have used the fact that zero comes in many guises: (let $\mathbf{0}$ denote the identity of Z) $0 \otimes h = \mathbf{0} \cdot 0 \otimes h = 0 \otimes h \cdot \mathbf{0} = 0 \otimes 0 = h \cdot \mathbf{0} \otimes 0 = h \otimes 0 \cdot \mathbf{0} = h \otimes 0$. The additive inverse of $g \otimes h$ is $(-g) \otimes h = g \otimes (-h)$.

An important fact that we will often use is that many algebraic properties of $G \otimes H$ need only be checked on the generating set, namely the set of simple tensors. Sometimes the entire structure of $G \otimes H$ is revealed by examining these simple tensors.

6.1.5. Proposition Let T be any torsion group. Then $T \otimes D = 0$ for any divisible group D.

Proof. Let $t \otimes d \in T \otimes D$. There is an integer m such that $mt = 0$, and since D is divisible, $d = md'$ for some $d' \in D$. Thus $t \otimes d = t \otimes md' = mt \otimes d' = 0 \otimes d' = 0$. ||

6.1.6. Corollary $Z(m) \otimes Q = 0$ for all $m > 1$.

The fact that zero comes in so many disguised forms makes the tensor product of groups a nonintuitive beast. For example, consider $Z(2) \otimes Z(3)$, where we consider both generated as Abelian groups by 1. Certainly $0 \otimes x$ is zero for all $x \in Z(3)$. The only other simple tensors left to worry about are of the form $1 \otimes x$. Now $1 \otimes 0$ is zero. What about $1 \otimes 1$? Certainly not zero, you say? Well, $1 \otimes 1 = 1 \otimes 4 = 1 \otimes 2 \cdot 2 = 2 \otimes 2 = 0 \otimes 2 = 0$ and $1 \otimes 2$ is immediately zero by the same argument. Hence: $Z(2) \otimes Z(3) = 0$. The trouble lies in the torsion property of $Z(2)$ and $Z(3)$.

The tensor product is better behaved if we give it more structure. There is a way of making $V \otimes W$ into a vector space over a field F if both V and W are vector spaces over F. Then $\dim(V \otimes W) = (\dim V)(\dim W)$, just what we would expect, and so $V \otimes W$ does not suddenly become zero when neither component is zero.

6.1.7. Proposition For all groups G and H, $G \otimes H \simeq H \otimes G$.

Proof. Consider the following diagram:

$$\begin{array}{ccc} G \times H & \xrightarrow{t_1} & G \otimes H \\ \iota\downarrow & & \psi\downarrow \ \uparrow\phi \\ H \times G & \xrightarrow{t_2} & H \otimes G \end{array}$$

Here $\iota(g,h) = (h,g)$ is a one-to-one function. The maps t_1 and t_2 are those of 6.1.3. The symmetric nature of the definition of balanced map makes $t_2\iota$ balanced from $G \times H$ to $H \otimes G$; thus there is a unique homomorphism $\psi: G \otimes H \to H \otimes G$ such that $\psi t_1 = t_2\iota$. Similarly, $t_1\iota^{-1}$ is a balanced map from $H \times G$ to $G \otimes H$, and so factors uniquely as $\phi t_2 = t_1\iota^{-1}$ for some homomorphism $\phi: H \otimes G \to G \otimes H$. Then $(\psi\phi)t_2 = t_2$ and $(\phi\psi)t_1 = t_1$, which as before implies that ϕ is an isomorphism. ||

6.1.8. Proposition $G \otimes (H \otimes K) \simeq (G \otimes H) \otimes K$ for all G, H, and K.

Proof. We must pick our way carefully in stages. For each fixed $k \in K$, define $\psi'_k(g,h) = g \otimes (h \otimes k)$; then $\psi'_k: G \times H \to G \otimes (H \otimes K)$. One easily shows that ψ'_k is bi-additive. Also $\psi'_k(ng,h) = ng \otimes (h \otimes k) = g \otimes n(h \otimes k) = g \otimes ((nh) \otimes k) = \psi'_k(g,nh)$. Hence ψ'_k is balanced, and so induces a unique homomorphism $\psi_k: G \otimes H \to G \otimes (H \otimes K)$ such that $\psi_k(g \otimes h) = g \otimes (h \otimes k)$. Now define $\phi': (G \otimes H) \times K \to G \otimes (H \otimes K)$ by $\phi'((g \otimes h),k) = \psi_k(g \otimes h)$.

Again, ϕ' is bi-additive. Also $\phi'(n(g \otimes h),k) = \psi_k(n(g \otimes h)) = \psi_k(g \otimes nh) = g \otimes (nh \otimes k) = g \otimes (h \otimes nk) = \psi_{nk}(g \otimes h) = \phi'(g \otimes h, nk)$. Then ϕ' is balanced, and so induces a homomorphism $\phi: (G \otimes H) \otimes K \to G \otimes (H \otimes K)$ such that $\phi((g \otimes h) \otimes k) = g \otimes (h \otimes k)$. Now if we repeat the process over again in the opposite order, i.e., for all fixed $g \in G$ define $\psi''_g(h,k) = (g \otimes h) \otimes k$, etc., we will eventually produce a homomorphism $\rho: G \otimes (H \otimes K) \to (G \otimes H) \otimes K$ such that $\rho(g \otimes (h \otimes k)) = (g \otimes h) \otimes k$. Then ρ and ϕ are, as before, isomorphisms. ||

6.2. Examples

We give now various examples of tensor products, along with how the tensor product behaves with direct sums and homomorphism groups.

6.2.1. Proposition $Z \otimes G \simeq G$ for all groups G.

Proof. Define $\psi'(n,g) = ng$ from $Z \times G$ to G. Then ψ' is balanced, and so ψ' induces a homomorphism $\psi: Z \otimes G \to G$ such that $\psi(n \otimes g) = ng$. Define $\phi: G \to Z \otimes G$ by $\phi(g) = 1 \otimes g$. Then ϕ is an Abelian group homomorphism, and $\phi\psi$ and $\psi\phi$ are the identities on the respective groups. ||

6.2.2. Corollary (1) $Z \otimes Z \simeq Z$ and $Z \otimes Q \simeq Q$

(2) $Z \otimes (\sum_\lambda G_\lambda) \simeq \sum_\lambda (Z \otimes G_\lambda)$

6.2.3. Proposition Let $Z(m)$ have order $m > 1$ and generator c. If A is any Abelian group, then $Z(m) \otimes A \simeq A/mA$.

Proof. Any $z \in Z(m)$ can be expressed as $z = kc$ with $k \in Z$. Define $\psi': Z(m) \times A \to A/mA$ by $\psi'(z,a) = ka + mA$. Then ψ' is well-defined and balanced. Hence it induces a homomorphism $\psi: Z(m) \otimes A \to A/mA$ such that $\psi(z \otimes a) = ka + mA$, where $z = kc$. Now define $\rho: A/mA \to Z(m) \otimes A$ by $\rho(a + mA) = c \otimes a$. We show that ρ is well-defined: if $a - a' \in mA$, then $a - a' = ma''$ for some $a'' \in A$. Thus $0 = mc \otimes a'' = c \otimes ma'' = (c \otimes a) - (c \otimes a') = \rho(a + mA) - \rho(a' + mA)$. Thus ρ is a well-defined group homomorphism. Now $\psi\rho(a + mA) = \psi(c \otimes a) = a + mA$ and $\rho\psi((kc) \otimes A) = \rho(ka + mA) = c \otimes ka = (kc) \otimes a$. Thus ψ is an isomorphism. ||

6.2.4. Corollary $Z(m) \otimes Z(n) \simeq Z(m,n)$ for all integers $m > 1$ and $n > 1$.

Proof. $Z(m) \otimes Z(n) \simeq Z(n)/mZ(n) \simeq Z(m,n)$ by 5.4.7. ||

6.2.5. Corollary $Z(m) \otimes Z(n) = 0$ for all relatively prime integers m and n.

We have already met a special case of this result on page 107.

6.2.6. Proposition For any groups A, B and C, we have the following isomorphism: $\mathrm{Hom}(A,\mathrm{Hom}(B,C)) \simeq \mathrm{Hom}(A \otimes B, C)$. The isomorphism is defined by $\zeta: \mathrm{Hom}(A \otimes B, C) \to \mathrm{Hom}(A,\mathrm{Hom}(B,C))$, where $\zeta(f) = g$,

$g(a)(b) = f(a \otimes b)$.

<u>Proof</u>. It is an easy exercise to see that g as we have defined it is indeed an element of $\text{Hom}(A,\text{Hom}(B,C))$ when $f \in \text{Hom}(A \otimes B,C)$, and conversely. We leave it to the reader to show ζ is a homomorphism.

Now let $g \in \text{Hom}(A,\text{Hom}(B,C))$ be given and define $f':A \times B \to C$ by $f'(a,b) = [g(a)](b)$. We show that f' is balanced. Clearly, the bi-additivity is no problem. Let $n \in Z$. Then $f'(na,b) = [g(na)](b) = [n(g(a))](b) = [g(a)](nb) = f'(a,nb)$; recall that g itself is a homomorphism, so $g(na) = ng(a)$, but $g(a)$ is also a homomorphism, so that $n(g(a))(b) = g(a)(nb)$. Hence by 6.1.3. there is a <u>unique</u> $f:A \times B \to C$ such that $f(a \otimes b) = g(a)(b)$. This means that ζ is an isomorphism.||

6.2.7. <u>Proposition</u> $A \otimes B$ is divisible when either A or B is divisible.

<u>Proof</u>. $A = nA$ implies $A \otimes B = (nA) \otimes B = n(A \otimes B)$. ||

6.2.8. <u>Proposition</u> If A and B are torsion-free, then so is $A \otimes B$.

<u>Proof</u>. We do not give the entire proof of this fact, but is rests upon $T(A \otimes B) \simeq \langle T(A) \otimes B,\ A \otimes T(B)\rangle$. Then $T(A) = T(B) = 0$ implies that $T(A \otimes B) = 0$. See Griffith [10] for a nice proof using only homological techniques. ||

6.3. Induced Homomorphisms and Exact Sequences

Let $\phi:G \to G'$ and $\psi:H \to H'$ be group homomorphisms. Define $\tau:G \times H \to G' \otimes H'$ by $\tau(g,h) = \phi(g) \otimes \psi(h)$. Then immediately τ is balanced, and so τ induces a unique homomorphism, which we will denote by $\phi \otimes \psi$, from $G \otimes H$ to $G' \otimes H'$. For all $g \otimes h \in G \otimes H$ we have that $(\phi \otimes \psi)(g \otimes h) = \phi(g) \otimes \psi(h)$, and we call $\phi \otimes \psi$ the induced homomorphism of ϕ and ψ. The problem is when is it that $\phi \otimes \psi$ is epic and monic?

6.3.1. Lemma Let $f':G \to G'$, $f'':G' \to G''$, $g':H \to H'$ and $g'':H' \to H''$ by group homomorphisms. Then

(1) $(f'' \otimes g'')(f' \otimes g') = (f''f') \otimes (g''g')$.

(2) $f' \otimes g'$ is zero if either f' or g' is zero.

(3) $1_G \otimes 1_H = 1_{G\otimes H}$.

Proof. The proof is immediate, since it suffices to check the formulas on the simple tensors only.

Using these induced homomorphisms, we can prove the following.

6.3.2. Proposition Let $\{G_\lambda\}_{\lambda\varepsilon\Lambda}$ and $\{H_\lambda\}_{\lambda\varepsilon\Lambda}$ be families of groups. Then $G \otimes (\sum_\lambda H_\lambda) \simeq \sum_\lambda (G \otimes H_\lambda)$ and $\sum_\lambda (G_\lambda \otimes H) \simeq (\sum_\lambda G_\lambda)\otimes H$.

Proof. We need prove only the first assertion, since the second is similarly proved. The diagram

$$\sum_\lambda H_\lambda \underset{\iota_\lambda}{\overset{\pi_\lambda}{\rightleftarrows}} H_\lambda$$

gives rise to

$$G \otimes (\sum_\lambda H_\lambda) \underset{1\otimes\iota_\lambda}{\overset{1\otimes\pi_\lambda}{\rightleftarrows}} G \otimes H_\lambda.$$

We show that the assumptions of 2.3.11. are satisfied. First,

$$(1 \otimes \pi_\lambda)(1 \otimes \iota_\lambda) = 1_G \otimes (\pi_\lambda \iota_\mu) = \begin{cases} 1_{G\otimes H} & \text{if } \mu = \lambda \\ 0 & \text{otherwise} \end{cases}$$

using the above facts and those of 2.3.7. Let $g \otimes f$ be a simple tensor in $G \otimes \sum H_\lambda$. Then $(1 \otimes \pi_\lambda)(g \otimes f) = g \otimes f(\lambda)$. Since $f(\lambda)$ is nonzero for only finitely many λ, we have that $(1 \otimes \pi_\lambda)(h)$ is nonzero for only finitely many λ for each fixed $h \varepsilon G \otimes \Sigma H_\lambda$.

Finally, let $f \in \sum_{\lambda} H_{\lambda}$. Then $\sum_{\lambda} (1 \otimes \iota_{\lambda})(1 \otimes \pi_{\lambda})(g \otimes f) =$

$\sum_{\lambda} (1 \otimes \iota_{\lambda})(g \otimes f(\lambda)) = \sum_{\lambda} (g \otimes \iota_{\lambda}(f(\lambda)) = g \otimes \sum_{\lambda} \iota_{\lambda}(f(\lambda)) = g \otimes f$. Hence

$\sum_{\lambda} (1 \otimes \iota_{\lambda})(1 \otimes \pi_{\lambda}) = 1_{G \otimes \Sigma H_{\lambda}}$, and so by 2.3.11. we have the desired isomorphism. ||

6.3.3. Theorem [Cartan and Eilenburg, 4] Let G be any group and $0 \to A \xrightarrow{\kappa} B \xrightarrow{\sigma} C \to 0$ an exact sequence of groups. Then the following are exact sequence of Abelian groups:

$$(1) \quad G \otimes A \xrightarrow{1 \otimes \kappa} G \otimes B \xrightarrow{1 \otimes \sigma} G \otimes C \longrightarrow 0$$

$$(2) \quad A \otimes G \xrightarrow{\kappa \otimes 1} B \otimes G \xrightarrow{\sigma \otimes 1} C \otimes G \longrightarrow 0$$

Proof. We prove the exactness of (1) only; the proof for (2) is exactly the same. First, $(1 \otimes \sigma)(1 \otimes \kappa) = 1 \otimes (\sigma\kappa) = 0$ since $\sigma\kappa = 0$. Hence $\mathrm{Im}(1 \otimes \kappa) \subseteq \mathrm{Ker}(1 \otimes \sigma)$. Also, σ epic implies that $1 \otimes \sigma$ is also epic, for if $g \otimes c \in G \otimes C$ is given, select $b \in B$ such that $\sigma(b) = c$; then $(1 \otimes \sigma)(g \otimes b) = g \otimes c$. Since every simple tensor in $G \otimes C$ has a preimage in $G \otimes B$, then so does every arbitrary element in $G \otimes C$.

We must now show that $\mathrm{Ker}(1 \otimes \sigma) \subseteq \mathrm{Im}(1 \otimes \kappa)$. We have a homomorphism $\rho: (G \otimes B)/\mathrm{Im}(1 \otimes \kappa) \to G \otimes C$ such that $1 \otimes \sigma = \rho\eta$, where $\eta: G \otimes B \to (G \otimes B)/\mathrm{Im}(1 \otimes \kappa)$ is the usual natural projection. We are using 2.2.6. Diagrammatically, we have:

$$\begin{array}{ccccccc} G \otimes A & \xrightarrow{1 \otimes \kappa} & G \otimes B & \xrightarrow{1 \otimes \sigma} & G \otimes C & \to & 0 \\ & & \eta \downarrow & \rho \nearrow \swarrow \psi & & & \\ & & \dfrac{G \otimes B}{\mathrm{Im}(1 \otimes \kappa)} & & & & \end{array}$$

Define $\psi':G \times C \to (G \otimes B)/\mathrm{Im}(1\otimes\kappa)$ by $\psi'(g,c) = (g \otimes b) + \mathrm{Im}(1\otimes\kappa)$, where $\sigma(b) = c$ for some $b \in B$. Since the given sequence is exact, a short argument shows that ψ' is independent of the choice of b. Since ψ' is also balanced, ψ' induces a unique homomorphism $\psi:G \otimes C \to (G \otimes B)/\mathrm{Im}(1\otimes\kappa)$. From their definitions, $\rho\psi$ and $\psi\rho$ are the identity maps on the respective groups, and so both are isomorphisms. Hence $\mathrm{Ker}(1\otimes\sigma) = \mathrm{Ker}(\rho\eta) = \mathrm{Ker}\ \eta = \mathrm{Im}(1\otimes\kappa)$. ||

Hence the induced sequence of tensor products is "almost" exact. But the kernel of $1 \otimes \kappa$ is quite a bottleneck. It can be very much nonzero even when $\kappa:A \to B$ is monic. For example, in 6.2.2. we showed that for $n > 1$, $Q \otimes Z(n) = 0$ and $Z \otimes Z(n) \simeq Z(n)$. Now the injection of Z into Q is monic, but the induced injection of $Z \otimes Z(n)$ into $Q \otimes Z(n)$ is very much not monic, because a nonzero group is being mapped to a zero group.

Another way of viewing this problem is that although A may be a subgroup of B, it does not follow that $G \otimes A$ will be a subgroup of $G \otimes B$, since some nonzero $g \otimes a$ in $G \otimes A$ may be zero in $G \otimes B$.

We can set out now in one of two paths: (1) declare that if $\kappa:A \to B$ is any monomorphism and G is a group such that $\kappa\otimes 1:A\otimes G \to B \otimes G$ is also monic, then G is, by definition, a flat group: discover the structure of all flat groups; (2) attempt to remedy the inexactitude in 6.3.3. in some manner similar to that of the last chapter.

Method (1) bear fruit; in fact, every free (and hence every projective) group is flat. But by taking method (2) we produce the second exact sequence of our method. We will see that $\kappa:A \to B$ monic implies $1\otimes\kappa:G \otimes A \to G \otimes B$ monic is tantamount to the fact that G is torsion-free.

As a first step in trying to remedy the situation, we now examine a typical tensor of $\mathrm{Ker}(1 \otimes \kappa)$. Let $g \in G$ be fixed. A tensor $g \otimes a \in G \otimes A$ arises in $\mathrm{Ker}(1 \otimes \kappa)$ whenever there is some integer m such that $mg = 0$ and $mb = \kappa(a)$ for some $b \in B$. Note that $g \otimes a$ is trivially in $\mathrm{Ker}(1 \otimes \kappa)$ if $m = 0$ or $a \in \mathrm{Ker}\ \kappa$. When the above holds, we have $(1 \otimes \kappa)(g \otimes a) = g \otimes \kappa(a) = mg \otimes b = 0$, and so $g \otimes a \in \mathrm{Ker}(1 \otimes \kappa)$. Certainly, $g \otimes a$ depends upon $m \in Z$ and $g \in G$. We show that $g \otimes a$ depends upon $c = \sigma(b) \in C$. For suppose $g \otimes a$ and $g \otimes a'$ have the properties that $mg = 0$, $mb = \kappa(a)$ and $mb' = \kappa(a')$. If $\sigma(b) = \sigma(b')$, then $b' = b + \kappa(a_o)$ for some $a_o \in A$. Thus $mb' = mb + \kappa(ma_o)$, and so $\kappa(a) = \kappa(a') + \kappa(ma_o)$. This means that $a - a' = ma_o$, since κ is monic. Thus $g \otimes (a - a') = mg \otimes a_o = 0$, and thus $g \otimes a = g \otimes a'$. ||

Let us denote this dependency by writing $[g,m,c] = g \otimes a$ for those tensors in $\mathrm{Ker}(1 \otimes \kappa)$. Then $[g,m,c]$ denotes an element of $G \otimes A$ such that (1) $mg = 0$ and $\kappa(a) = mb$ for $a \in A$ and $b \in B$, (2) $\sigma(b) = c$, so that $mc = 0$, and (3) $(1 \otimes \kappa)([g,m,c]) = 0$.

Note that the following properties hold, whenever both sides are defined:

(1) $[g,m,c + c'] = [g,m,c] + [g,m,c']$ $(mc = mc' = 0,\ mg = 0)$

(2) $[g + g',m,c] = [g,m,c] + [g',m,c]$ $(mg = mg' = 0,\ mc = 0)$

(3) $[g,mn,c] = [g,m,nc]$ $(mg = 0$ and $mnc = 0)$.

These properties follow directly from the definitions involved and the properties of the tensor product. For example, $[g,mn,c] = g \otimes a$ with $mng = 0$ in G, $(mn)c = 0$ in C, $\kappa(a) = mnb$ and $\sigma(b) = c$. Likewise $[g,m,nc] = g \otimes a'$ with $mg = 0$ in G, $m(nc) = 0$ in C, $\kappa(a') = mb'$

and $\sigma(b') = nc$. We see that these conditions are compatible, and guarantee the existence of the tensors involved when we assume at the outset that $mg = 0$ in G, $(mn)c = 0$ in G and take $b' = nb \in B$.

6.4. Torsion Products of Groups

With the discussion just ended for guidance, we may generate a group whose elements have the properties of those in $\mathrm{Ker}(1 \otimes \kappa)$. Our method is modeled on that of MacLane [25].

6.4.1. <u>Definition</u> Let G and H be group. Let $\mathrm{Tor}(G,H)$ denote the free Abelian group generated by all symbols $\langle g,m,h \rangle$ with $m \in Z$, $g \in G$, $h \in H$, $mg = 0$ in G, $mh = 0$ in H and subject to the following conditions:

(1) $\langle g + g',m,h \rangle = \langle g,m,h \rangle + \langle g',m,h \rangle$ $\quad (mg_i = 0 = mh)$

(2) $\langle g,m,h + h' \rangle = \langle g,m,h \rangle + \langle g,m,h' \rangle$ $\quad (mg = 0 = mh_i)$

(3) $\langle g,mn,h \rangle = \langle mg,n,h \rangle$ $\quad (mng = 0 = nh)$

(4) $\langle g,mn,h \rangle = \langle g,m,nh \rangle$ $\quad (mg = 0 = mnh)$.

6.4.2. <u>Proposition</u> Let G and H be groups.

(1) $\langle 0,m,h \rangle$ and $\langle g,m,0 \rangle$ are zero in $\mathrm{Tor}(G,H)$.

(2) If H or G is torsion-free, then $\mathrm{Tor}(G,H) = 0$.

(3) $\mathrm{Tor}(G,H) \simeq \mathrm{Tor}(H,G)$.

(4) $\langle g,m+n,h \rangle = \langle g,m,h \rangle + \langle g,n,h \rangle$ when both sides are defined.

(5) $\mathrm{Tor}(G,H)$ is a torsion group.

<u>Proof</u>. (1) Consider $\langle 0,m,h \rangle$ in $\mathrm{Tor}(G,H)$, and so $mh = 0$. As in the tensor case, this element comes in many forms; in fact, let $n \in Z$ and $h' \in H$ be given such that $nh' = 0$. Then $\langle 0,m,h \rangle = \langle 0,1,mh \rangle = \langle 0,1,0 \rangle = \langle 0,1,nh' \rangle = \langle 0,n,h' \rangle$, where we have used (4) of

the definition. Let $\langle g,n,h'\rangle$ be an arbitrary basis element of $\mathrm{Tor}(G,H)$. Then $nh' = 0$, and so $\langle g,n,h'\rangle + \langle 0,m,h\rangle = \langle g,n,h'\rangle + \langle 0,n,h'\rangle = \langle g+0,h,h'\rangle = \langle g,n,h'\rangle$. Since $\langle 0,m,h\rangle$ acts as a zero on each element of the generating set of $\mathrm{Tor}(G,H)$, it is zero on all of $\mathrm{Tor}(G,H)$.

(2) Let H be torsion-free and $\langle g,m,h\rangle$ an arbitrary element of $\mathrm{Tor}(G,H)$. Then $mh = 0$ in H, and so $h = 0$. By (1), $\langle g,m,h\rangle$ is zero, and so $\mathrm{Tor}(G,H) = 0$.

(3) $\mathrm{Tor}(G,H) \simeq \mathrm{Tor}(H,G)$ is clear from the symmetric nature of the defining relations.

(4) Let $g \in G$, $h \in H$ and choose integers m and n such that $\langle g,m,h\rangle$ and $\langle g,n,h\rangle$ are defined. Then $mg = 0$, $mg = 0$, $mh = 0$ and $nh = 0$. Therefore, $m|o(g)$ and $n|o(g)$. Write $o(g)m' = m$ and $o(g)n' = n$. Then $\langle g,m+n,h\rangle = \langle g,o(g)(m' + n'),h\rangle = \langle g,o(g),(m'+n')h\rangle = \langle g,o(g),m'h\rangle + \langle g,o(g),n'h\rangle = \langle g,o(g)m',h\rangle + \langle g,o(g)n',h\rangle$.

(5) Let $\langle g,m,h\rangle \in \mathrm{Tor}(G,H)$. Then $mg = 0$ and $mh = 0$. By (4), $m\langle g,m,h\rangle = \langle g,m^2,h\rangle = \langle mg,m,h\rangle = 0$; these elements make sense since $mg = 0$ implies $m(mg) = 0$. If $t = \sum_\lambda n_\lambda\langle g_\lambda,m_\lambda,h_\lambda\rangle$, where λ runs over a finite index set, then $m_\lambda\langle g_\lambda,m_\lambda,h_\lambda\rangle = 0$. Set s equal to the product of all the m_λ. Then $st = 0$, and so $\mathrm{Tor}(G,H)$ is a torsion group. ||

As we have done so many times before, we need induced homomorphisms between torsion products so we induce an exact sequence between them.

6.4.3. Definition Let $\rho:A \to A'$ and $\sigma:G \to G'$ be homomorphisms of groups. Define $\rho_\#:\mathrm{Tor}(G,A) \to \mathrm{Tor}(G,A')$ by $\rho_\#\langle g,m,a\rangle = \langle g,m,\rho(a)\rangle$ on the generating set of $\mathrm{Tor}(G,A)$. Similarly, define

$\sigma^{\#}: \text{Tor}(G,A) \to \text{Tor}(G',A)$ by $\sigma^{\#}\langle g,n,a\rangle = \langle\sigma(g),n,a\rangle$. Extend $\rho_{\#}$ and $\sigma^{\#}$ to all of $\text{Tor}(G,A)$ by linearity; i.e., $\rho_{\#}(\sum_\lambda m_\lambda\langle g_\lambda,n_\lambda,a_\lambda\rangle) = \sum_\lambda m_\lambda\rho_{\#}\langle g_\lambda,n_\lambda,a_\lambda\rangle$, and similarly for $\sigma^{\#}$.

6.4.4. Proposition. Let $\rho_{\#}$ and $\sigma^{\#}$ be as we have defined them above.

(1) If $\rho: A \to A'$ and $\rho': A' \to A''$ are group homomorphisms, then $(\rho'\rho)_{\#} = \rho'_{\#}\,\rho_{\#}$. Similarly, if $\sigma: G \to G'$ and $\sigma': G' \to G''$, then $(\sigma'\sigma)^{\#} = (\sigma')^{\#}\sigma^{\#}$.

(2) $(1_G)^{\#} = (1_A)_{\#} = 1_{\text{Tor}(G,A)}$ and $(0_G)^{\#} = (0_A)_{\#} = 0$.

(3) If $\rho_i: A \to A'$ for $i = 1,2$, then $(\rho_1 + \rho_2)_{\#} = \rho_{1\#} + \rho_{2\#}$. Similarly, if $\sigma_i: G \to G'$ for $i = 1,2$, then $(\sigma_1 + \sigma_2)^{\#} = \sigma_1^{\#} + \sigma_2^{\#}$.

Proof. It suffices to check the relation on the generating set of $\text{Tor}(G,A)$. We have $(\rho'\rho)_{\#}(\langle g,m,a\rangle) = \langle g,m,(\rho'\rho)(a)\rangle = \langle g,m,\rho'(\rho(a))\rangle = \rho'_{\#}(\rho_{\#}(\langle g,m,a\rangle))$. One checks the remaining assertion in the same way.

(2) Clear.

(3) On each generating element, the additivity relations clearly hold. Extending by linearity gives the assertion for every element in $\text{Tor}(G,A)$. ||

The following proposition describes how Tor acts with direct sums of groups. The proof uses the same approach as 6.3.2.

6.4.5. $\text{Tor}(G, \sum_\lambda A_\lambda) \simeq \sum_\lambda \text{Tor}(G, A_\lambda)$ and $\text{Tor}(\sum_\lambda A_\lambda, G) \simeq \sum_\lambda \text{Tor}(A_\lambda, G)$.

Proof. The diagram

$$\sum_\lambda A_\lambda \underset{\iota_\lambda}{\overset{\pi_\lambda}{\rightleftarrows}} A_\lambda$$

induces the following diagram, using the homomorphisms from 6.4.3.:

$$\mathrm{Tor}(\sum_{\lambda} A_{\lambda}, G) \overset{\pi_{\lambda}^{\#}}{\underset{\iota_{\lambda}^{\#}}{\rightleftarrows}} \mathrm{Tor}(A_{\lambda}, G)$$

Now finish the proof precisely the way 6.3.2. is done. ||

Thus to calculate Tor(G,A) for any finitely generated group G, it suffices to compute Tor(G,A) for finite cyclic groups G. This is accomplished in the following proposition.

6.4.6. Let $G = Z(m)$ be a given cyclic group of finite order m. Let $A[m] = \{a \in A \mid ma = 0\}$. Then $\mathrm{Tor}(G,A) \simeq A[m]$.

<u>Proof</u>. Define $\zeta: A[m] \to \mathrm{Tor}(G,A)$ by $\zeta(a) = \langle g,m,a \rangle$. This is a well-defined homomorphism, since $mg = 0$ in G, and $ma = 0$ by the choice of a. Clearly, $\zeta(a + a') = \zeta(a) + \zeta(a')$. Now let $\xi: \mathrm{Tor}(G,A) \to A[m]$ be defined as follows: let $\langle kg,n,a \rangle \in \mathrm{Tor}(G,A)$ be an arbitrary basis element; then $na = 0$ and $(nk)g = 0$. Thus $m|nk$; set $t = (nk)/m$, and define $\xi\langle kg,n,a \rangle = ta$. Now $ta \in A[m]$, since $m(ta) = (mt)a = k(na) = 0$. Note also that $\langle kg,n,a \rangle = \langle g,kn,a \rangle = \langle g,m,ta \rangle$, and that all the symbols are defined. A straightforward, but tedious, computation shows that ξ preserves (1)-(4) of definition 6.4.1. For example, let rg and sg be elements of G. Then consider the elements $\langle rg,n,a \rangle$ and $\langle sg,n,a \rangle$ in Tor(G,A); we have $na = 0$ in A and $(nr)g = (ns)g = 0$ in G. This means $\langle rg + sg,n,a \rangle \in \mathrm{Tor}(G,A)$. Let $t_1 = nr/m$ and $t_2 = ns/m$; then $t_1 + t_2 = (nr + ns)/m$ and so $\xi\langle (r+s)g,n,a \rangle = t_1 + t_2 = \xi\langle rg,n,a \rangle + \xi\langle sg,n,a \rangle$. One easily calculates that $\xi\zeta = 1$ on A[m] and $\zeta\xi = 1$ on Tor(G,A). ||

We are now ready to give the second major exact sequence of our method.

6.5. The Exact Sequence for $G \otimes A$

6.5.1. Theorem [Cartan and Eilenburg, 4] Let $0 \to A \overset{\kappa}{\to} B \overset{\sigma}{\to} C \to 0$ be an exact sequence of Abelian groups. Then for all groups G, we have the following exact sequences:

$$(1) \quad 0 \to \mathrm{Tor}(A,G) \overset{\kappa^\#}{\to} \mathrm{Tor}(B,G) \overset{\sigma^\#}{\to} \mathrm{Tor}(C,G) \overset{E^\#}{\to}$$

$$A \otimes G \overset{\kappa\otimes 1}{\to} B \otimes G \to C \otimes G \to 0$$

$$(2) \quad 0 \to \mathrm{Tor}(G,A) \overset{\kappa_\#}{\to} \mathrm{Tor}(G,B) \overset{\sigma_\#}{\to} \mathrm{Tor}(G,C) \overset{E_\#}{\to}$$

$$G \otimes A \overset{1\otimes\kappa}{\to} G \otimes B \to G \otimes C \to 0$$

The maps $\kappa^\#$, $\kappa_\#$, $\sigma^\#$, and $\sigma_\#$ are those of 6.4.3.; the map $E_\#$ is called a connecting homomorphism and is defined by $E_\#\langle g,m,c\rangle = [g,m,c]$, where $[g,m,c]$ appears in 6.3.

<u>Proof</u>. As we will mention at the end of the proof, it is sufficient to show that (2) is exact. That $E_\#$ is a homomorphism follows because $E_\#$ is defined on a generating set of Tor(G,C) whose defining relations match those shared by the tensors $[g,m,c]$ in $G \otimes A$ (see the discussion at the end of 6.3.). Since each $[g,m,c]$ lies in $\mathrm{Ker}(1 \otimes \kappa)$, we have that $(1 \otimes \kappa)E_\# = 0$. Since $\sigma_\#\kappa_\# = (\sigma\kappa)_\#$, we also have $\sigma_\#\kappa_\# = 0$, since $\sigma\kappa = 0$. To show $E_\#\sigma_\# = 0$, let $\langle g,m,b\rangle$ be an element in Tor(G,B). Then $mb = 0$ in B. Now $E_\#\sigma_\#\langle g,m,b\rangle = E_\#\langle g,m,\sigma(b)=c\rangle = g \otimes a$, where $a \in A$ is any element such that $\kappa(a) = mb$. But then $\kappa(a) = 0$, and so $a = 0$, since κ is monic. Hence $g \otimes a$ is also zero, and the result follows.

We have shown above that we have a zero sequence at Tor(G,B), Tor(G,C) and $G \otimes A$. We have already established exactness of the sequence at $G \otimes B$ and $G \otimes C$. The only thing left to show is exactness at Tor(G,A), Tor(G,B), Tor(G,C) and $G \otimes A$ by showing each kernel is in the respective image.

We are going to eventually reduce the problem by showing it is sufficient to prove the theorem for finite cyclic groups G. Let us accept this for the moment. Then using the isomorphisms of 6.2.3. and 6.4.6. we can replace (2) with another sequence whose exactness is easier to establish. Let $T = Z(m)$ be the cyclic group of order m and with generator g. We then have the following diagram:

$$\begin{array}{ccccccccccc} 0 & \to & Tor(T,A) & \overset{\kappa_\#}{\to} & Tor(T,B) & \overset{\sigma_\#}{\to} & Tor(T,C) & \overset{E_\#}{\to} & T \otimes A & \overset{1\otimes\kappa}{\to} & T \otimes B \\ & & \zeta\uparrow & & \zeta\uparrow & & \zeta\uparrow & & \rho\uparrow & & \rho\uparrow \\ 0 & \to & A[m] & \underset{\bar{\kappa}}{\to} & B[m] & \underset{\bar{\sigma}}{\to} & C[m] & \underset{E}{\to} & A/mA & \underset{\bar{\bar{\kappa}}}{\to} & B/mB \end{array}$$

Define $E(c) = a + mA$ for $c \in C[m]$, where $a \in A$ is such that $\kappa(a) = mb$ and $\sigma(b) = c$ for $b \in B$. Notice that $mc = 0$ implies $\sigma(mb) = 0$, so there exists $a \in A$ such that $\kappa(a) = mb$. Since σ and κ are homomorphisms, then so is E. This definition makes the third square commute: when $c \in C[m]$, $\rho(E(c)) = \rho(a + mA) = g_o \otimes a$, where $\kappa(a) = mb$, $\sigma(b) = c$ and $mc = 0$. But $E(\zeta(c)) = E(<g_o,m,c>) = g_o \otimes a'$, where $mc = 0$, $mg_o = 0$, and $a' \in A$ is any element such that $\kappa(a') = mb'$ and $\sigma(b') = c$. But then $\sigma(b) = \sigma(b')$, and as we have pointed out on page 114, $g_o \otimes a = g_o \otimes a'$. Thus $\rho E_\# = E\zeta$. The mappings $\bar{\sigma}$ and $\bar{\kappa}$ are the restrictions of σ and κ to the subgroups $A[m]$ and $B[m]$ respectively; likewise, $\bar{\bar{\kappa}}(a + mA) = \kappa(a) + mB$; i.e., $\bar{\bar{\kappa}}$ is induced by κ. A merry chase shows that all the other squares commute, too. Hence the exactness of the top row reduces to the exactness of the bottom row. From our previous work, it is only necessary to examine kernels contained in respective images.

Examine the following (to be shown exact) sequence:

$$0 \to A[m] \xrightarrow{\bar{\kappa}} B[m] \xrightarrow{\bar{\sigma}} C[m] \xrightarrow{E} A/mA \xrightarrow{\bar{\bar{\kappa}}} B/mB \xrightarrow{\bar{\bar{\sigma}}} C/mC$$

We get exactness at $A[m]$ and $B[m]$ because the original sequence is exact. Certainly $\bar{\kappa}$ is monic because κ is monic. If b is chosen in $B[m]$ and $\mathrm{Ker}\,\bar{\sigma}$, then $mb = 0$ and $\sigma(b) = 0$, so there is $a \in A$ such that $\kappa(a) = b$. But $\kappa(ma) = mb = 0$ implies $ma = 0$, and so $a \in A[m]$; thus $b \in \mathrm{Im}\,\bar{\kappa}$.

Now let $c \in C[m]$ and suppose $E(c) = 0$ in A/mA. Recall that $E(c) = a + mA$ where $\sigma(b) = c$ and $\kappa(a) = mb$ for some $b \in B$. Thus $a + mA = 0$ implies $a \in mA$, and so $a = ma'$; let $b' = \kappa(a')$ and $b'' = b - b'$. Then $mb'' = mb - mb' = \kappa(a) - \kappa(mb') = 0$, and so $b'' \in B[m]$. Likewise, $\sigma(b'') = \sigma(b) - \sigma(\kappa(a')) = c$, and so $c = \bar{\sigma}(b'')$; i.e., the sequence is exact at $C[m]$.

If $a + mA$ is in the kernel of $\bar{\bar{\kappa}}$, we have $\kappa(a) \in mB$; i.e., $\kappa(a) = mb$ for some $b \in B$. Thus $m\sigma(b) = \sigma(mb) = \sigma(\kappa(a)) = 0$, and so $\sigma(b) \in C[m]$. Setting $c = \sigma(b)$ gives $E(c) = a + mA$, and the exactness at A/mA follows.

Finally, if $\bar{\bar{\sigma}}(b + mB) = 0$, then $\sigma(b) = mc$ for some $c \in C$. Since σ is epic, $c = \sigma(b')$ for some $b' \in B$. Hence $\sigma(b - mb') = 0$, and so $b - mb' = \kappa(a)$ for some $a \in A$. Thus $b - \kappa(a) = mb'$ implies $\kappa(a) + mB = b + mB$; i.e., $\bar{\bar{\kappa}}(a + mB) = b + mB$ and the exactness of the bottom row is established.

We now turn to the reduction of the problem. Since we have just shown the sequence to be exact when G is any finite cyclic group, we show that it holds for any finitely generated group. Let G be finitely generated, so that $G \approx \sum_j Z(p_j^{k_j})$. Then

$$0 \to \mathrm{Tor}(A, Z(p_j^{k_j})) \to \cdots \to C \otimes Z(p_j^{k_j}) \to 0$$

is exact for all j. Then the following sequences are all exact:

$$0 \to \sum_j \mathrm{Tor}(A, Z(p_j^{k_j})) \to \cdots \to \sum_j C \otimes Z(p_j^{k_j}) \to 0$$

$$0 \to \mathrm{Tor}(A, \sum_j Z(p_j^{k_j})) \to \cdots \to C \otimes \sum_j Z(p_j^{k_j}) \to 0$$

$$0 \to \mathrm{Tor}(A,G) \to \mathrm{Tor}(B,G) \to \cdots\cdots \to B \otimes G \to C \otimes G \to 0$$

To complete the proof, we must show that if the result holds for all finitely generated groups, it holds for all Abelian groups. What makes the proof work is the fact that exactness is a "local" property: the proof involves at each stage only a finite number of elements from each group involved.

To this end, let G be arbitrary. As a sample of the argument, let us establish exactness at $G \otimes A$ under the assumption that the theorem holds for all finitely generated groups. A (fixed) element $u \in G \otimes A$ is of the form $u = \sum_i n_i(g_i \otimes a_i)$ and so involves only finitely many elements of G. If its image $(1 \otimes \kappa)(u) = \sum_i u_i(g_i \otimes \kappa(a_i))$ is zero in $G \otimes B$, it is so because of a finite number of defining relations in $G \otimes B$. These relations again involve only a finite list $h_1, \cdots, h_m$ of elements of G. Now take G_o to be the subgroup of G generated by all the elements $g_1, \cdots, g_n$, $h_1, \cdots, h_n$ which have appeared thus far in the argument, and let $\iota : G_o \to G$ be the usual inclusion map. Then $u_o = \sum_i n_i(g_i \otimes a_i)$ is an element of $G_o \otimes A$ such that $(\iota \otimes 1)(u_o) = u$. Thus the following squares are easily seen to commute:

$$\begin{array}{ccccc}
\mathrm{Tor}(G_o,C) & \xrightarrow{E} & G_o \otimes A & \xrightarrow{1\otimes\kappa} & G_o \otimes B \\
\iota_{\#}\downarrow & & \iota\otimes 1\downarrow & & \downarrow\iota\otimes 1 \\
\mathrm{Tor}(G,C) & \xrightarrow{E} & G \otimes A & \xrightarrow{1\otimes\kappa} & G \otimes B
\end{array}$$

We have that the top row is exact by our assumption. Now $(1 \otimes \kappa)(u) = 0$ implies $(1 \otimes \kappa)(u_o) = 0$ in $G_o \otimes A$, and so there is $t_o \in Tor(G_o,C)$ such that $E(t_o) = u_o$. But $E(\iota_{\#}(t_o)) = ((\iota \otimes 1)E)(t_o) = (\iota \otimes 1)(u_o) = u$, and so the bottom row is exact at $G \otimes A$. Repeating this type of argument at each position of the sequence of the theorem gives exactness at that point. Since all groups in (1) are isomorphic to the corresponding groups in (2), the exactness of the first sequence is implied by the exactness of the second sequence.

The proof is complete.

6.6. Torsion and Torsion-free Groups via Tor(A,B)

Let A be an arbitrary Abelian group with $T(A)$ its torsion subgroup. Then A is torsion-free iff $T(A) = 0$. The connection between $Tor(G,A)$ and $T(A)$ is the content of the following theorem. It also shows that Q/Z is a "universal" torsion group.

6.6.1. Theorem For any Abelian group A, $Tor(Q/Z,A) \simeq T(A)$.

Proof. First note that Q is torsion-free, and so $Tor(H,Q) = 0$ for all groups H. Similarly, $A/T(A)$ is torsion-free, so $Tor(A/T(A),H) = 0$ for all groups H. By 6.1.5. $T \otimes Q = 0$ for any torsion group T.

We take the following for exact sequences in 6.5.1.:

(1) $0 \to T(A) \to A \to A/T(A) \to 0$

(2) $0 \to Z \to Q \to Q/Z \to 0$.

Using (2) with $T(A)$ and (1) with A in 6.5.1., we have the following exact sequences, where the vertical exact sequence comes from using (1) with Q/Z in 6.5.1.

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ \mathrm{Tor}(T(A),Q) & \to & \mathrm{Tor}(T(A),Q/Z) & \overset{f}{\to} & T(A)\otimes Z & \to & T(A)\otimes Q \\ & & \downarrow g & & & & \\ \mathrm{Tor}(A,Q) & \to & \mathrm{Tor}(A,Q/Z) & \to & A\otimes Z & & \\ & & \downarrow & & & & \\ & & \mathrm{Tor}(A/T(A),Q/Z) & & & & \end{array}$$

where the vertical exact sequence comes from using (1) with Q/Z in 6.5.1.

Now $\mathrm{Tor}(T(A),Q)$, $\mathrm{Tor}(A/T(A),Q/Z)$, and $T(A) \otimes Q$ are all zero for the reasons pointed out above. Also $T(A) \otimes Z \simeq T(A)$. Putting these facts into the diagram and recalling that we have exact sequences, we have that f and g are isomorphisms, and so $T(A) \simeq \mathrm{Tor}(A,Q/Z)$. ||

6.6.2. <u>Corollary</u>. A is torsion-free if and only if $\mathrm{Tor}(Q/Z,A) \simeq \mathrm{Tor}(A,Q/Z) = 0$.

6.6.3. <u>Corollary</u> A group G is a torsion group if and only if $G \otimes Q = 0$.

<u>Proof</u>. Select the exact sequence $0 \to Z \to Q \to Q/Z \to 0$ and tensor with G. Then in 6.5.1. we have the following exact sequence: $\mathrm{Tor}(G,Q) \to \mathrm{Tor}(G,Q/Z) \to G \otimes Z \to G \otimes Q$. Now $\mathrm{Tor}(G,Q) = 0$ since Q is torsion-free, $G \otimes Z \simeq G$, and $G \otimes Q = 0$ by assumption. Thus $E_\#$ is an isomorphism, and so $G \simeq T(G)$. ||

6.6.4. <u>Corollary</u> Suppose $0 \to Z \to Q \to Q/Z \to 0$ remains exact upon tensor multiplication by an arbitrary group G. Then G is torsion-free.

<u>Proof</u>. The assumption implies that $1\otimes\kappa : G \otimes Z \to G \otimes Q$ is a monomorphism. From 6.5.1., we have the following exact sequence:

$$\mathrm{Tor}(G,Q) \overset{\sigma_\#}{\to} \mathrm{Tor}(G,Q/Z) \overset{E_\#}{\to} G \otimes Z \overset{1\otimes\kappa}{\to} G \otimes Q$$

But $\operatorname{Tor}(G,Q) = 0$ and $\operatorname{Im} E_{\#} = \operatorname{Ker}(1\otimes\kappa) = 0$; i.e., $E_{\#}$ is the zero homomorphism. Hence $\operatorname{Tor}(G,Q/Z) = 0$. ||

At the outset we hinted that through Tor we could characterize those Abelian groups G which preserve monomorphisms $\kappa: A \to B$ upon tensor multiplication. This is the content of the following theorem. The essential implication (5) implies (1) seems to have been known to Dieudonné [6].

6.6.5. Theorem Let G be any Abelian group. Then the following are equivalent:

(1) G is torsion-free.

(2) $\operatorname{Tor}(G,A) = 0$ for all Abelian groups A.

(3) If $\kappa: A \to B$ is monic, then so is $1\otimes\kappa: G \otimes A \to G \otimes B$.

(4) Any short exact sequence remains exact upon tensor multiplication by G.

(5) Any exact sequence remains exact when tensored by G.

Proof. (1) implies (2) See 6.4.2.

(2) implies (3) In 6.5.1., we have $\operatorname{Tor}(G,C) \to G \otimes A \to G \otimes B$ is exact, and so, if $\operatorname{Tor}(G,C) = 0$, then $1 \otimes \kappa$ is a monomorphism.

(3) implies (4) See 6.3.3.

(4) implies (5) Trivial.

(5) implies (1) The assumption allows us to use 6.6.4., and the result follows. ||

Before finishing the chapter, we record an interesting proposition which shows the strength of Q as a "universal" torsion-free group (compare with 6.6.1.). The proof is based on techniques of Harrison [11].

6.6.6 Proposition. Let X be any torsion-free group, and G any group. Then

(1) $\mathrm{Ext}(X,G) = 0$ iff $\mathrm{Ext}(Q,G) = 0$.

(2) $\mathrm{Hom}(G,X) = 0$ iff $\mathrm{Hom}(G,Q) = 0$.

<u>Proof.</u> (1) We need only prove ($\leftarrow$). For that purpose, the sequence $0 \to Z \to Q \to Q/Z \to 0$ gives rise to the exact sequence $0 \to \mathrm{Tor}(Q/Z,X) \to Z \otimes X \simeq X \to Q \otimes X \to Q/Z \otimes X \to 0$. Hence $0 \to X \to Q \otimes X \to Q/Z \otimes X \to 0$ is exact, and so $\mathrm{Ext}(Q\otimes X,G) \to \mathrm{Ext}(X,G) \to 0$ is exact.

Now $Q \otimes X$ is torsion-free and divisible by 6.2.7. and 6.2.8. Hence $Q \otimes X \simeq \Sigma\, Q$, by 3.5.3. Since $\mathrm{Ext}(Q \otimes X,G) = \mathrm{Ext}(\Sigma Q,G) \simeq \Pi\, \mathrm{Ext}(Q,G) = 0$ if $\mathrm{Ext}(Q,G) = 0$, we have $\mathrm{Ext}(X,G) = 0$.

(2) We proceed just like the above, except $0 \to \mathrm{Hom}(G,X) \to \mathrm{Hom}(G,Q \otimes X)$ is exact. Since $\mathrm{Hom}(G, \Sigma Q) \simeq \Pi \mathrm{Hom}(G,Q) = 0$ if $\mathrm{Hom}(G,Q) = 0$, we get $\mathrm{Hom}(G,X) = 0$. ||

The groups G such that $\mathrm{Ext}(X,G) = 0$ for all torsion-free groups X were called <u>cotorsion</u> groups in Harrison's article [11]. These groups yield a very elegant homological description. For example, the homomorphic image of an algebraically compact group is cotorsion, a direct product of cotorsion groups is again cotorsion and a reduced, torsion-free cotorsion group is algebraically compact. The reader is referred to Harrison's paper [11] for further details.

CHAPTER VII

PURE EXTENSIONS OF GROUPS

In this chapter we extend our homological method to the notion of purity. As we do so, we generate a useful subgroup of Ext(C,A), denoted by Pext(C,A), and the third major exact sequence of our method. These notions are due to Harrison [11].

7.1. The Subgroup Pext(C,A) of Ext(C,A)

Just as groups have pure subgroups, we would like to know when extensions of groups are "pure."

7.1.1. Definition Let E: $0 \to A \to B \to C \to 0$ be an arbitrary extension of A by C. We say E is pure if for all $n \in Z$, $nB \cap \kappa(A) \subseteq n\kappa(A)$. This means that for all $n \in Z$, the equation $nb = \kappa(a)$ with $a \in A$ and $b \in B$ implies $nb = \kappa(a')$ for some $a' \in A$.

The first proposition draws together some immediate conclusions.

7.1.2. Proposition Let E: $0 \to A \xrightarrow{\kappa} B \xrightarrow{\sigma} C \to 0$ be an extension of A by C.

(1) E is pure iff $\kappa(A)$ is a pure subgroup of B.

(2) B is a pure subgroup of A iff $0 \to B \xrightarrow{\iota} A \xrightarrow{\eta} A/B \to 0$ is pure.

(3) The sequence $0 \to A \to A \oplus B \to B \to 0$ is always pure.

(4) E is pure iff F: $0 \to A/nA \xrightarrow{\overline{\kappa}} B/nB \xrightarrow{\overline{\sigma}} C/nC \to 0$ is exact for every integer n; the maps $\overline{\kappa}$ and $\overline{\sigma}$ are the ones induced by κ and σ: $\overline{\kappa}(a + nA) = \kappa(a) + nB$ and $\overline{\sigma}(b + nB) = \sigma(b) + nC$.

Proof. (1) Suppose $\kappa(A)$ is a pure subgroup of B. Then $B \cap \kappa(A) \subseteq n\kappa(A)$, and conversely.

(2) This follows from (1) with $\kappa = \iota$.

(3) We know $\iota(a) = (a,0)$ for all $a \in A$. Thus $\iota(a) = n(a',b)$ implies $(a,0) = (na',nb)$, and so $a = na'$. Thus $n(a',b) = n\iota(a')$, and so $0 \to A \to A \oplus B \to B \to 0$ is pure.

(4) ($\to$) A straightforward argument shows that $\overline{\kappa}$ and $\overline{\sigma}$ are well-defined homomorphisms. Because E is pure, Ker $\overline{\kappa} = 0$. Automatically, $\overline{\sigma}$ is epic, and $\overline{\sigma\kappa}$ is zero. If $b + nB \in$ Ker $\overline{\sigma}$, then $\sigma(b) = nc$ for some $c \in C$. Let $b' \in B$ be such that $\sigma(b') = c$. Then $b - nb' = \kappa(a)$ for some $a \in A$ because $\sigma(b - nb') = 0$. Hence $\overline{\kappa}(a + nA) = b + nB$ and F is exact.

($\leftarrow$) If F is exact and n is given, suppose $nb = \kappa(a)$ for some $b \in B$ and $a \in A$. Then $\overline{\kappa}(a + nA) = \kappa(a) + nB = 0$. Because $\overline{\kappa}$ is given monic, this means $a \in nA$, so $a = na'$ for some $a' \in A$. Thus E is pure. ||

The following useful proposition says that for pure exact sequences the exact sequence of 6.5.1. splits into two pieces.

7.1.3. Proposition. Suppose $0 \to A \to B \to C \to 0$ is pure. Then for any group G the sequences E: $0 \to \text{Tor}(A,G) \to \text{Tor}(B,G) \to \text{Tor}(C,G) \to 0$ and F: $0 \to A \otimes G \to B \otimes G \to C \otimes G \to 0$ are exact. In fact, F is pure.

Proof. To show E and F are exact, we show that the connecting homomorphism $E_{\#}$ of 6.5.1. is the zero map. By definition, $E_{\#}\langle g,m,c\rangle = g \otimes a$, where $mg = 0$, $mc = 0$ and $a \in A$ is any element such that $\kappa(a) = mb$ and $\sigma(b) = c$. By the purity of A in B, $mb = m\kappa(a')$ for some $a' \in A$, and so $a = ma'$. Thus $g \otimes a = g \otimes ma' =$

$mg \otimes a' = 0$, and hence $E_{\#}$ is the zero map.

To show F is pure, we use a proof due to Harrison [11]. The above result allows us to assert that for any group G and integer n, the following sequence is exact:

$$0 \to A \otimes (G \otimes Z(n)) \to B \otimes (G \otimes Z(n)) \to C \otimes (G \otimes Z(n)) \to 0.$$

By 6.1.8., the following is also exact:

$$0 \to (A \otimes G) \otimes Z(n) \to (B \otimes G) \otimes Z(n) \to (C \otimes G) \otimes Z(n) \to 0.$$

Using 6.2.3., the following is also exact:

$$0 \to (A \otimes G)/n(A \otimes G) \to (B \otimes G)/n(B \otimes G) \to (C \otimes G)/n(C \otimes G) \to 0.$$

By the previous proposition, then, F is pure. ||

In fact, Fuchs [9] has even proved that if $0 \to A \to B \to C \to 0$ is pure, then the exact sequences of 3.3.2. are also pure exact.

7.1.4. Lemma Let E and F represent the same class in $Ext(C,A)$. Then E is pure iff F is pure.

Proof. The situation is as follows: there is a morphism $(1,\beta,1):E \to F$; recall that then β is an isomorphism. If E is pure we transfer the purity of E to that of F via the morphism. Conversely, $(1,\beta^{-1},1):F \to E$ is also a morphism; repeat the argument. ||

7.1.5. Lemma Let E: $0 \to A \to B \to C \to 0$ and $\gamma:C' \to C$ be a group homomorphism. Then E pure implies $E\gamma$ pure.

Proof.

$$\begin{array}{lccccccccc} E\gamma: & 0 & \to & A & \xrightarrow{\kappa'} & B' & \xrightarrow{\sigma'} & C' & \to & 0 \\ & & & \downarrow 1 & & \downarrow \beta & & \downarrow \gamma & & \\ E: & 0 & \to & A & \xrightarrow{\kappa} & B & \xrightarrow{\sigma} & C & \to & 0 \end{array}$$

Assume that E is pure, and suppose $x = nb' = \kappa'(a)$ for $a \in A$

and $b' \in B'$. Then $\beta(nb') = \beta(\kappa'(a)) = \kappa(a) \in \kappa(A)$, and $\beta(x) = n\beta(b')$ in nB. Thus $\beta(x) = n\kappa(a')$ for some $a' \in A$ by the purity of E. Hence $x = \kappa'(na') = n\kappa'(a')$; i.e., E is pure. ||

An attempt at a converse to the above result is repleat with difficulties without some added assumptions. But one converse which will be useful later is the following.

7.1.6. Lemma. Let E and γ be as in the previous lemma. If $E\gamma$ is pure, γ is an epimorphism and $\operatorname{Ker}\gamma$ is a pure subgroup of C', then E is pure.

Proof. The following diagram describes the situation.

$$\begin{array}{lccccccccc} E\gamma: & 0 & \to & A & \xrightarrow{\kappa'} & B' & \xrightarrow{\sigma'} & C' & \to & 0 \\ & & & 1\downarrow & & \downarrow\beta & & \downarrow\gamma & & \\ E: & 0 & \to & A & \xrightarrow{\kappa} & B & \xrightarrow{\sigma} & C & \to & 0 \end{array}$$

Recall that $B' = \{(b,c') \mid \sigma(b) = \gamma(c')\}$ and $\kappa'(a) = (\kappa(a),0)$. Suppose that $\kappa(a) = nb$ for $a \in A$ and $b \in B$. Then $\sigma(b) \in C$, so $\sigma(b) = \gamma(c')$ for some $c' \in C$, since γ is onto. This means $\gamma(nc') = \sigma(nb) = 0$, and so $nc' \in \operatorname{Ker}\gamma$. Let $nc' = t$, where $t \in \operatorname{Ker}\gamma$. Then $n|t$ in C', so $n|t$ in $\operatorname{Ker}\gamma$; i.e., $nc'' = t$ for some $c'' \in \operatorname{Ker}\gamma$. Thus $n(c' - c'') = 0$; note that $\gamma(c' - c'') = \sigma(b)$, so $b' = (b,c'-c'')$ in B'. But beyond this we get $\kappa'(a) = (\kappa(a),0) = (\kappa(a),n(c'-c'')) = (nb,n(c'-c'')) = nb'$. By the purity of $E\gamma$, we have $nb' = n\kappa'(a')$ for some $a' \in A$. Unraveling all this shows that $(n\kappa(a'),0) = (nb,n(c'-c''))$: thus $nb = \kappa(a')$. Thus, too, E is pure. ||

7.1.7. Example We do need that $\operatorname{Ker}\gamma$ is a pure subgroup of C'. Consider $0 \to Z \to Q \to Q/Z \to 0$. Now this sequence is not pure! Consider the following diagram.

$$\begin{array}{ccccccccccc} & & & & & & & & Z & & \\ & & & & & & & & \downarrow\iota & & \\ E: & 0 & \to & Z & \to & Q \oplus Z & \to & & Q & \to & 0 \\ & & & 1\downarrow & & \downarrow & & & \downarrow\eta & & \\ F: & 0 & \to & Z & \to & Q & \to & & Q/Z & \to & 0 \end{array}$$

We know that E is pure (7.1.2.), η is epic, but F is not pure: the problem is that $\text{Ker}\ \eta = \text{Im}\ \iota = Z$ is not a pure subgroup of Q.

Not content, we examine the similar question for αE.

7.1.8. <u>Lemma</u> Let $E:\ 0 \to A \to B \to C \to 0$ be pure exact and $\alpha: A \to A'$ a group homomorphism. Then αE is pure.

<u>Proof.</u> By 7.1.4., we may use the exact sequence which defines αE in $\text{Ext}(C,A')$. Then

$$\begin{array}{cccccccccc} E: & 0 & \to & A & \xrightarrow{\kappa} & B & \xrightarrow{\sigma} & C & \to & 0 \\ & & & \alpha\downarrow & & \downarrow\beta & & \downarrow 1 & & \\ \alpha E: & 0 & \to & A' & \xrightarrow{\kappa'} & \dfrac{A' \oplus B}{N} & \to & C & \to & 0, \end{array}$$

where $N = \{(-\alpha(a),\kappa(a)) \mid a \in A\}$ and $\kappa'(a') = (a',0) + N$ for all $a' \in A'$. Let $b' = (a',b) + N$ be an element of B' and suppose that $nb' = \kappa'(a_1')$, where $a_1' \in A'$. Then $na - a_1' = -\alpha(a)$ and $nb = \kappa(a)$ for some $a \in A$. But the last equation implies, along with the fact that E is pure that $nb = n\kappa(a_1) = \kappa(na_1)$. Let $a_2' = a' + \alpha(a_1) \in A'$. Then $na' - na_2' = -\alpha(na_1)$, and so $(na' - na_2', nb) \in N$. This means $nb' = n\kappa'(a_2')$, and so αE is pure. ||

7.1.9. <u>Corollary</u> If E is a pure sequence of groups, then so is $-E$.

<u>Proof.</u> By definition, $-E = (-1)E$. Apply 7.1.8. and 7.1.4. ||

7.1.10. <u>Lemma</u> Suppose E and α are as in 7.1.8. If αE is pure, α is a monomorphism and $\text{Im}\ \alpha$ is a pure subgroup of A', then E is pure.

<u>Proof.</u> Suppose $\kappa(a) = nb$ for $a \in A$ and $b \in B$. Let $a' = \alpha(a)$ and $b' = (0,b) + N$. Then $(-\alpha(a),\kappa(a)) \in N$ implies $nb' = (0,nb) + N = (\alpha(a),0) + N = \kappa'(a')$. Thus there is an element $a_1' \in A'$ (by the purity of αE) such that $nb' = n\kappa'(a_1')$. Unthreading the maze gives $(0,nb) + N = (na_1',0) + N$; i.e., $(-na_1',nb) \in N$. This means there is an element $a_1 \in A$ such that $na_1 = \alpha(a_1)$ and $nb = \kappa(a_1)$. Thus $n|\alpha(a_1)$ in A', and so $n|\alpha(a_1)$ in $\text{Im}\,\alpha$, by the purity of $\text{Im}\,\alpha$ in A'. Thus $n\alpha(a_2) = \alpha(a_1)$ for some $a_2 \in A$. Comparing equations and using the fact that α is monic, we get $a_1 = na_2$. Thus $nb = \kappa(a_1) = n\kappa(a_2)$; i.e., E is pure. ||

7.1.11. <u>Example</u> We need the purity of $\text{Im}\,\alpha$ in A': consider the sequence $0 \to Z \xrightarrow{\kappa} Z \to Z(2) \to 0$ where κ is multiplication by 2. Consider the diagram

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ E: & 0 \to & Z & \xrightarrow{\kappa} & Z & \to & Z(2) & \to & 0 \\ & & \kappa\downarrow & & & & & & \\ F: & 0 \to & Z & \to & Z \oplus Z(2) & \to & Z(2) & \to & 0 \end{array}$$

Now κ is monic, F is pure, but E is not pure, since Z is hardly pure in itself. The trouble: $\text{Im}\,\kappa = (2)$ is not a pure subgroup of Z.

7.1.12. <u>Lemma</u> Let E_i: $0 \to A \to B_i \to C \to 0$ be extensions of A by C, for $i = 1,2$. Then E_i pure imply $E_1 \oplus E_2$ is also pure.

<u>Proof.</u> Clear from the definition of $E_1 \oplus E_2$.

7.1.13. <u>Lemma</u> Let E_1 and E_2 be as above. Then $E_1 \oplus E_2$ is pure if both E_1 and E_2 are also pure.

<u>Proof.</u> $E_1 + E_2 = \nabla(E_1 \oplus E_2)\Delta$. Now apply 7.1.12., 7.1.8., and 7.1.5. ||

7.1.14. Theorem Let Pext(C,A) be the collection of all equivalence classes E represented by pure extensions of A by C in Ext(C,A). Then Pext(C,A) is a subgroup of Ext(C,A).

Proof. The result follows immediately from Lemmas 7.1.13., 7.1.9., and 7.1.2. ||

7.1.15. Corollary Let C be a torsion-free group. Then Ext(C,A) = Pext(C,A).

Proof. The result states that every extension of A by C is pure. Let E: $0 \to A \to B \to C \to 0$ be such a sequence. Then $C \simeq B/\kappa(A)$, and so $\kappa(A)$ is a pure subgroup of B (3.3.7.). Thus E is pure and $\text{Ext}(C,A) \subseteq \text{Pext}(C,A)$. ||

We intend to characterize precisely when Pext(C,A) and Pext(A,C) are trivial as one of the first problems in the chapter on Applications. For now we answer the perhaps more natural question: Does 5.4.3. remain exact when "Ext" is replaced by "Pext?" The pleasant answer is "yes."

7.2. The Exact Sequence for Pext

Refer back to 5.4.3. and examine the exact sequences (1) and (2). Let E: $0 \to A \to B \to C \to 0$ be a pure exact sequence. In (2) replace all the groups Ext(G,-) by Pext(G,-); do a similar replacement in (1) also. Are the resulting sequences still exact?

In (1), there is nothing to show up to Hom(A,G). Since $E^*(\gamma) = E\gamma$ is pure if E is pure, E^*: Hom(A,G) $\to$ Pext(C,G). Similarly, σ^0 and κ^0 take their values in Pext(-,G) when defined on the proper group Pext(-,G). Clearly, we have a zero sequence from Hom(A,G) on.

Let $F \in$ Pext(C,G) such that $\sigma^0(F) = F\sigma$ is zero. Then $F = E^*(\lambda)$ for some $\lambda \in$ Hom(A,G); hence we have exactness as Pext(C,G).

Now let $E \in \text{Pext}(B,G)$ such that $\kappa^0(E) = E\kappa = 0$. By 5.4.3., $E = \sigma^0(F) = F\sigma$ for some $F \in \text{Ext}(C,G)$. Thus we have that $F\sigma$ is pure, σ is epic and $\text{Ker}\ \sigma = \text{Im}\ \kappa$ is a pure subgroup of B. By 7.1.6., F is pure, and so we have exactness at $\text{Pext}(B,G)$.

The last thing to show is $\kappa^0:\text{Pext}(B,G) \to \text{Pext}(A,G)$ is epic. Refer to the proof that $\kappa^0:\text{Ext}(B,G) \to \text{Ext}(A,G)$ is epic. Instead of choosing F to be an arbitrary free Abelian group, let F be that direct sum of cyclics such that $0 \to K \to F \to B \to 0$ is pure exact (3.3.8.). Now proceed with the proof just as before. We produce an exact sequence $E_1 \equiv E_2\kappa$, and so E_1 is pure by 7.1.5. The rest of the argument, including the final diagram, goes through, except the groups Ext in that final diagram can be changed to Pext.

In (2), everything is exact up to $\text{Pext}(G,A)$, just like the above. Let $E \in \text{Pext}(G,B)$ such that $\sigma_0(E) = \sigma E$ is zero. Then there is an element $F \in \text{Ext}(G,A)$ such that $E = \kappa_0(F) = \kappa F$. Thus κF is pure, κ is monic and $\text{Im}\ \kappa$ is a pure subgroup of B. Thus by 7.1.10., F is pure, so $F \in \text{Pext}(G,A)$. To complete the argument, refer to the final diagram of the proof of 5.4.3. There the two rows are pure and the final column is pure, if we proceed through the proof making the appropriate changes to Pext groups. Then the middle column is also pure exact, and the equation $S = \sigma_0(E'')$ with both S and E'' pure states that σ_0 is epic, as we desired. We thus have the following result.

7.2.1. <u>Theorem</u> Let $E\colon\ 0 \to A \to B \to C \to 0$ be a pure exact sequence of Abelian groups. Then for all groups G, the following are also exact:

$$(1)\quad 0 \to \text{Hom}(C,G) \to \text{Hom}(B,G) \to \text{Hom}(A,G) \to$$

$$\text{Pext}(C,G) \to \text{Pext}(B,G) \to \text{Pext}(A,G) \to 0$$

$$(2) \quad 0 \to \mathrm{Hom}(G,A) \to \mathrm{Hom}(G,B) \to \mathrm{Hom}(G,C) \to$$

$$\mathrm{Pext}(G,A) \to \mathrm{Pext}(G,B) \to \mathrm{Pext}(G,C) \to 0$$

The homomorphisms involved are precisely those of 5.4.3.

Theorems 5.4.3., 6.5.1., and 7.2.1. form the core of what we have called a "homological method" in Abelian group theory. The next chapter gives illustrations of the use of this method.

CHAPTER VIII

APPLICATIONS

8.1. Pure-projective and pure-injective groups

Our first application of the homological method will be the characterization of the triviality of Pext(G,A) and Pext(A,G). We have already foreshadowed this work in our characterization of the algebraically compact groups in 3.4. These groups play a rôle in the final solution; in fact, the algebraically compact groups are the pure-injective groups. The explicit description of these classes of groups is due to J. Maranda [26].

8.1.1. <u>Definition</u> A group G is <u>pure-projective</u> if for all epimorphisms $\sigma:B \to C$ with $\text{Ker}\,\sigma$ pure in B and any homomorphism $f:P \to C$, there is a homomorphism $\phi:P \to B$ such that $\sigma\phi = f$; i.e., the following diagram commutes:

$$\begin{array}{ccccc} & & P & & \\ & {}^{\phi}\swarrow & \downarrow f & & \\ B & \xrightarrow{\sigma} & C & \to & 0 \end{array}$$

The only feature added to the concept of projectivity is the purity of $\text{Ker}\,\sigma$ in B. Clearly, every projective group is pure-projective.

8.1.2. <u>Definition</u> A group G is <u>pure-injective</u> if for all monomorphisms $\kappa:A \to B$ with $\text{Im}\,\kappa$ pure in B and any homomorphism $f:A \to G$, there is a homomorphism $\phi:B \to G$ such that $f = \phi\kappa$; i.e., the following diagram commutes:

$$\begin{array}{ccccc} 0 & \to & A & \overset{\kappa}{\to} & B \\ & & \downarrow f & \swarrow & \phi \\ & & G & & \end{array}$$

The added attraction beyond the usual notion of injectivity is the purity of $\text{Im}\ \kappa$ in B. Every injective group is pure-injective.

8.1.3. Theorem [J. M. Maranda, 26] Let G be a group. Then the following are equivalent.

(1) G is pure-projective.

(2) $\text{Pext}(G,A) = 0$ for all groups A.

(3) G is a direct sum of cyclic groups.

Proof. (1) implies (2). Suppose $0 \to A \overset{\kappa}{\to} B \overset{\sigma}{\to} C \to 0$ is a pure exact sequence. Then $\text{Ker}\ \sigma = \text{Im}\ \kappa$ is a pure subgroup of B, and so there exists a homomorphism $\rho : G \to B$ such that $\sigma\rho = 1_G$, since G is pure-projective. Hence the sequence splits, so $\text{Pext}(G,A) = 0$.

(2) implies (3). Let G be given such that $\text{Pext}(G,A) = 0$ for all groups A. By 3.3.8., G is the homomorphic image of a direct sum P of cyclic groups with pure kernel K; hence the sequence $0 \to K \to P \to G \to 0$ is pure exact, and splits by assumption. Thus G is a direct summand of P, and so it itself a direct sum of cyclic groups. (The seemingly "obvious" fact that a subgroup of a direct sum of cyclics is again a direct sum of cyclics is not so easy to prove; it involves techniques that do not beg the question in this proof, however. See Kuroŝ [21] or Griffith [10] for a proof of this fact. Compare this with the other "obvious" result that a subgroup of a free group is free, for which this author has yet to find an "understandable" proof with the scope, say, of a monograph like this one.)

(3) implies (1). Suppose $\sigma : B \to C$ is epic and $\text{Ker}\ \sigma$ is pure in

B. Let $f:G \to C$ be a homomorphism. By the "pullback" construction, we have

$$\begin{array}{ccccccccc} Ef: & 0 & \to & \text{Ker}\,\sigma & \to & M & \underset{\mu}{\overset{\rho}{\rightleftarrows}} & G & \to & 0 \\ & & & 1\downarrow & & \beta\downarrow & & \downarrow f & & \\ E: & 0 & \to & \text{Ker}\,\sigma & \to & B & \to & C & \to & 0. \end{array}$$

We know Ef is pure (7.1.5.). Now $G \simeq M/\text{Ker}\,\sigma$ is a direct sum of cyclic groups, and $\text{Ker}\,\sigma$ is pure in M. Thus $\text{Ker}\,\sigma$ is a direct summand of M (3.5.8.), and so Ef splits. Let $\mu:G \to M$ be the splitting homomorphism, and define $\phi:G \to B$ by $\phi = \beta\mu$. Then $f = \sigma\phi$ and so G is pure-projective. ||

8.1.4. Theorem Let G be a group. Then the following are equivalent.

(1) G is pure-injective.

(2) $\text{Pext}(C,G) = 0$ for all groups C.

(3) $G \simeq H \oplus D$, where H is a direct summand of a direct product of cyclic torsion groups and D is divisible.

Proof. (1) implies (2). Let $E: \ 0 \to G \overset{\kappa}{\to} B \overset{\sigma}{\to} C \to 0$ be a pure exact sequence. Then $\text{Im}\,\kappa$ is a pure subgroup of B, so there is a homomorphism $\rho:B \to G$ such that $\rho\kappa = 1_G$, since G is pure-injective. Thus E splits, so the class it determines in $\text{Pext}(C,G)$ is zero.

(2) implies (3). We know that G can be imbedded as a pure subgroup in a direct sum $A \oplus D$, where A is a direct product of cyclic torsion groups and D is divisible (5.6.8.). Thus $0 \to G \overset{\iota}{\to} A \oplus D \overset{\eta}{\to} (A \oplus D)/G \to 0$ is a pure exact sequence, and by our assumption, it must split. Hence $G \simeq \iota(G)$ is a direct summand of $A \oplus D$. Write $G \simeq D_1 \oplus R$, where D_1 is the maximal divisible subgroup of G and R is reduced. Then D_1 must be a subgroup of D, and

so D_1 is a summand of D (5.5.7.). This means R is a direct summand of A, which is a direct product of cyclic torsion groups.

(3) implies (1). Suppose $\kappa: A \to B$ is monic and $\kappa(A)$ is pure in B. Let $f: A \to G$ be a homomorphism. We link A and G together through a "pushout" diagram:

$$\begin{array}{lcccccccc} E: & 0 & \to & A & \xrightarrow{\kappa} & B & \xrightarrow{\eta} & B/\kappa(A) & \to & 0 \\ & & & f\downarrow & & \downarrow\beta & & \downarrow 1 & & \\ fE: & 0 & \to & G & \underset{\lambda}{\overset{\pi}{\rightleftarrows}} & M & \xrightarrow{\rho} & B/\kappa(A) & \to & 0 \end{array}$$

We know fE is pure because E is pure; we must show that fE splits. By assumption, $G \simeq H \oplus D$, where H is a direct summand of a direct product of cyclic torsion groups and D is divisible. Now $\pi(G) \simeq \pi(D) \oplus \pi(H)$ is contained in M as a pure subgroup, which means that $\pi(H)$ is a pure subgroup of M (3.3.7.), and so $\pi(H)$ is a direct summand of M, because $\pi(H)$ is algebraically compact (3.4.6.). Since $\pi(D)$ is divisible, it too is a direct summand of M (5.5.7.). Thus $\pi(G)$ is a direct summand of M, fE splits, and $\lambda: M \to G$ is a homomorphism such that $\lambda\pi = 1_G$. Set $\phi: B \to G$ equal to $\phi = \lambda\beta$. Then $f = \phi\kappa$ and so G is a pure-injective group. ||

8.1.5. <u>Corollary</u> $\mathrm{Pext}(G, Z(m)) = 0$ for all $m > 1$ and all groups G. Similarly, $\mathrm{Pext}(Z(m), G) = 0$ for all G and $m > 1$.

<u>Proof</u>. By the previous two theorems, every $Z(m)$ is trivially both pure-projective and pure-injective. Hence the corresponding Pext groups vanish. ||

The situation here is quite different from Ext. Although all $Z(m)$ are pure-projective and pure-injective, they are neither projective (since they are not free) nor injective (since they are not divisible).

8.2. $X \otimes Y$ and $\mathrm{Hom}(X,Y)$ for Torsion Groups X and Y

The propositions in this section bring together many of the diverse tools of the homological method. They show off the power of the techniques at their best.

Recall we showed that $Z(m) \otimes Z(n) \simeq Z(m,n)$. A generalization of this result follows, due to Harrison [11].

8.2.1. Let X and Y be torsion groups. Then $X \otimes Y$ is a direct sum of cyclic groups.

__Proof.__ Let X_b and Y_b be basic subgroups of X and Y, repectively. Select the pure exact sequences

$$0 \to X_b \to X \to X/X_b \to 0 \quad \text{and}$$

$$0 \to Y_b \to Y \to Y/Y_b \to 0.$$

We also have that X/X_b and Y/Y_b are divisible and both X_b and Y_b are direct sums of cyclic groups. By 7.1.3., the following are also exact:

$$0 \to X_b \otimes Y \to X \otimes Y \to X/X_b \otimes Y = 0 \qquad \text{and}$$

$$0 \to X_b \otimes Y_b \to X_b \otimes Y \to X_b \otimes Y/Y_b = 0.$$

The end groups are zero because X and Y are torsion groups, and X/X_b and Y/Y_b are divisible. Hence $X \otimes Y \simeq X_b \otimes Y \simeq X_b \otimes Y_b$. Since the tensor product preserves direct sums (6.3.2.), we have that $X \otimes Y$ is a direct sum of cyclic groups, because X_b and Y_b are also such a sum. ||

The following proposition somehow uses just about everything that we have developed so far, and then just a bit more! We will have to accept the following fact which lies deep in the subject (see Cartan and Eilenburg, 4): for any groups A, B and C,

$\mathrm{Ext}(A,\mathrm{Hom}(B,C)) \oplus \mathrm{Hom}(A,\mathrm{Ext}(B,C)) \simeq \mathrm{Ext}(A \otimes B,C) \oplus \mathrm{Hom}(\mathrm{Tor}(A,B),C)$.

In the next section we will develop one such "adjoint relation." The above is an example of what Cartan and Eilenburg call a Künneth relation.

8.2.2. <u>Theorem</u> [Harrison, 11] Let A be a torsion group. Then for any B, Hom(A,B) is reduced and is a direct summand of a direct product of finite cyclic groups.

<u>Proof</u>. We are set up to use the characterization of Pext(C,A). First, Hom(A,B) is reduced, because $\mathrm{Hom}(Q,\mathrm{Hom}(A,B)) \simeq \mathrm{Hom}(Q \otimes A,B) = 0$. We have used 6.1.5., 6.2.6. and (8) of 4.2.1.

To show the second assertion, we use 8.1.4. and show that $\mathrm{Pext}(G,\mathrm{Hom}(A,B)) = 0$ for all G. We build up in stages, starting at torsion groups G. By 3.3.8., we have a pure exact sequence $0 \to K \to C \to G \to 0$ such that C is a direct sum of cyclic groups. By 7.1.3., $0 \to K \otimes A \to C \otimes A \to G \otimes A \to 0$ is a pure exact sequence. Thus the following sequence is exact, by 7.2.1.:

$$0 \to \mathrm{Hom}(G \otimes A,B) \to \mathrm{Hom}(C \otimes A,B) \to \mathrm{Hom}(K \otimes A,B) \to \mathrm{Pext}(G \otimes A,B).$$

This last group is trivial because $G \otimes A$ is a direct sum of cyclic groups, by the previous result, and so is pure-projective (8.1.3.).

Using 6.2.6. we have the exact sequence

$$0 \to \mathrm{Hom}(G,\mathrm{Hom}(A,B)) \to \mathrm{Hom}(C,\mathrm{Hom}(A,B)) \to \mathrm{Hom}(K,\mathrm{Hom}(A,B)) \to 0.$$

But the sequence

$$0 \to \mathrm{Hom}(G,\mathrm{Hom}(A,B)) \to \mathrm{Hom}(C,\mathrm{Hom}(A,B)) \to \mathrm{Hom}(K,\mathrm{Hom}(A,B)) \to$$
$$\mathrm{Pext}(G,\mathrm{Hom}(A,B)) \to \mathrm{Pext}(C,\mathrm{Hom}(A,B)) = 0$$

is exact, where the last group is trivial because C is a direct sum of cyclic groups. This all means that $\mathrm{Pext}(G,\mathrm{Hom}(A,B)) = 0$.

If G is torsion-free, then $\mathrm{Pext}(G,\mathrm{Hom}(A,B)) = \mathrm{Ext}(G,\mathrm{Hom}(A,B))$

(7.1.15.). Now $\text{Ext}(G,\text{Hom}(A,B)) = 0$ if we show that (using 6.6.6.) $\text{Ext}(Q,\text{Hom}(A,B)) = 0$. But $\text{Ext}(Q,\text{Hom}(A,B)) \oplus \text{Hom}(\text{Tor}(Q,A),B) \simeq \text{Ext}(Q \otimes A,B) \oplus \text{Hom}(\text{Tor}(Q,A),B) \simeq \text{Ext}(0,B) \oplus \text{Hom}(0,B) = 0$. Here we have used the fact that $Q \otimes A = 0$ because Q is divisible and A is torsion-free, while $\text{Tor}(Q,A) = 0$ because Q is torsion-free. Hence $\text{Ext}(Q,\text{Hom}(A,B)) = 0$.

Finally, let G be arbitrary. Then link our previous two cases through the pure exact sequence $0 \to T(G) \to G \to G/T(G) \to 0$. Hence $0 = \text{Pext}(G/T(G),\text{Hom}(A,B)) \to \text{Pext}(G,\text{Hom}(A,B)) \to \text{Pext}(T(G),\text{Hom}(A,B)) = 0$. Thus $\text{Pext}(G,\text{Hom}(A,B)) = 0$. ||

The above proof also emphasizes the important role of the innocent exact sequence $0 \to T(G) \to G \to G/T(G) \to 0$. It serves as a link between the torsion and torsion-free cases. It is a pure exact sequence, and so is applicable to the exact sequence for Pext as well as Ext.

8.3. Adjoint Relations

The catchall title refers to interrelationships between the various groups involved in the homological method. The following proposition illustrates such a relationship; it was proven by Nunke [30] in a more general setting. Following the proposition is a list of more adjoint relations of interest; some have already appeared earlier in this paper.

8.3.1. Proposition Let A, B and C be groups with A projective. Then $\text{Ext}(A \otimes B,C) \simeq \text{Hom}(A,\text{Ext}(B,C))$.

Proof. The idea is to set up an appropriate diagram and use 2.2.7. Let I be an injective group such that $E\colon\ 0 \to C \xrightarrow{\kappa} I \xrightarrow{\sigma} N \to 0$ is exact (5.5.5.). Then $\text{Ext}(G,I) = 0$ for all G. From 5.4.3., the sequence

$$F: \ Hom(B,I) \xrightarrow{\sigma_*} Hom(B,N) \xrightarrow{E_*} Ext(B,C) \rightarrow Ext(B,I) = 0$$

is exact, along with the sequence

$$Hom(A \otimes B,I) \xrightarrow{\sigma_*} Hom(A \otimes B,N) \xrightarrow{E_*} Ext(A \otimes B,C) \rightarrow Ext(A \otimes B,I) = 0.$$

From F, $Hom(A,Hom(B,I)) \xrightarrow{\sigma_{**}} Hom(A,Hom(B,N)) \xrightarrow{E_{**}} Hom(A,Ext(B,C))$

is exact and $(E_*)_*$ is epic. Using the isomorphism ζ from 6.2.6., we have the following diagram:

$$\begin{array}{ccccc} Hom(A \otimes B,I) & \xrightarrow{\sigma_*} & Hom(A \otimes B,N) & \xrightarrow{E_*} & Ext(A \otimes B,C) \\ \downarrow\zeta & & \downarrow\zeta & & \downarrow\Psi \\ Hom(A,Hom(B,I)) & \xrightarrow{\sigma_{**}} & Hom(A,Hom(B,N)) & \xrightarrow{E_{**}} & Hom(A,Ext(B,C)). \end{array}$$

In order to use 2.2.7., we must show that the left-hand square commutes. First, $\zeta_*(f) = \zeta(\sigma f) = g$, where $g(a)(b) = f(a \otimes b)$, for all $a \in A$ and $b \in B$. But $\zeta(f) = g'$, where $g'(a)(b) = f(a \otimes b)$, and $\sigma_{**}(\zeta(f)) = \sigma_*(g')$. Now $\sigma_*(g')(a)(b) = \sigma(f(a \otimes b))$, and so $\zeta\sigma_* = \sigma_{**}\zeta$. By 2.2.7., there exists an isomorphism between $Ext(A \otimes B,C)$ and $Hom(A,Ext(B,C))$ such that the right hand square commutes. ||

A list of other intriguing adjoint relations follows, and concludes our work.

(1) $Ext(A,Ext(B,C)) \simeq Ext(Tor(A,B),C)$ [Nunke, 30]

(2) $Ext(A,Hom(B,C)) \simeq Hom(B,Ext(A,C))$ when B is projective [Nunke, 30]

(3) $Ext(A,Ext(B,C)) \simeq Ext(B,Ext(A,C))$ [Nunke, 30]

(4) $Tor(A,Tor(B,C)) \simeq Tor(Tor(A,B),C)$ [Nunke, 29]

CHAPTER IX

SUMMARY

This thesis has been concerned with homological methods and their role in Abelian group theory. In the first section, we give an outline for this method and the major results pertaining to it and its applications. In the second section, we indicate some areas for further consideration and investigation.

9.1. Outline of the Homological Method

Let A and B be groups. We obtain five derived Abelian groups: $\mathrm{Hom}(A,B)$ [4.1], $\mathrm{Ext}(A,B)$ [5.3], $\mathrm{Pext}(A,B)$ [7.1], $A \otimes B$ [6.1], and $\mathrm{Tor}(A,B)$ [6.4]. $\mathrm{Pext}(A,B)$ is a subgroup of $\mathrm{Ext}(A,B)$ [7.1.14.], with equality if A is torsion-free [7.1.15.].

Let $0 \to A \to B \to C \to 0$ be an exact sequence of Abelian groups, and let G be any group. We have the following long exact sequences:

(1) $0 \to \mathrm{Hom}(G,A) \to \mathrm{Hom}(G,B) \to \mathrm{Hom}(G,C) \to$
$\mathrm{Ext}(G,A) \to \mathrm{Ext}(G,B) \to \mathrm{Ext}(G,C) \to 0$ [5.4.3.]

(2) $0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G) \to$
$\mathrm{Ext}(C,G) \to \mathrm{Ext}(B,G) \to \mathrm{Ext}(A,G) \to 0$ [5.4.3.]

(3) $0 \to \mathrm{Tor}(G,A) \to \mathrm{Tor}(G,B) \to \mathrm{Tor}(G,C) \to$
$G \otimes A \to G \otimes B \to G \otimes C \to 0$ [6.5.1.]

If $0 \to A \to B \to C \to 0$ is a pure exact sequence, then the following are also exact, with (5) being pure exact:

(4) $0 \to \text{Tor}(G,A) \to \text{Tor}(G,B) \to \text{Tor}(G,C) \to 0$ [7.1.3.]

(5) $0 \to G \otimes A \to G \otimes B \to G \otimes C \to 0$ [7.1.3.]

(6) $0 \to \text{Hom}(G,A) \to \text{Hom}(G,B) \to \text{Hom}(G,C) \to$

$\text{Pext}(G,A) \to \text{Pext}(G,B) \to \text{Pext}(G,C) \to 0$ [7.2.1.]

(7) $0 \to \text{Hom}(C,G) \to \text{Hom}(B,G) \to \text{Hom}(A,G) \to$

$\text{Pext}(C,G) \to \text{Pext}(B,G) \to \text{Pext}(A,G) \to 0$ [7.2.1.]

Basic group theoretic notions are translated over to homological terms in the following manner:

(8) G is divisible iff $\text{Ext}(Q/Z, G) = 0$ [5.5.3.]

(9) G is reduced iff $\text{Hom}(Q,G) = 0$ [4.2.1.8.]

(10) G is torsion iff $Q \otimes G = 0$ [6.6.3.]

(11) G is torsion-free iff $\text{Tor}(Q/Z,G) = 0$ [6.6.2.]

(12) G is projective iff $\text{Ext}(G,A) = 0$ for all A. [4.5.8.]

(13) G is injective iff $\text{Ext}(Z(n),G) = 0$ for all n. [5.5.1.]

(14) G is a direct sum of cyclics iff

$\text{Pext}(G,A) = 0$ for all A. [8.1.3.]

(15) G is a direct summand of a direct product of

finite cyclic groups iff $\text{Pext}(A,G) = 0$ for all A. [8.1.4.]

We have the following reduction theorems:

(16) $\text{Hom}(\sum_\lambda G_\lambda, H) \simeq \prod_\lambda \text{Hom}(G_\lambda, H)$ [4.3.4.]

(17) $\text{Hom}(H, \prod_\lambda G_\lambda) \simeq \prod_\lambda \text{Hom}(H, G_\lambda)$ [4.3.4.]

(18) $\text{Ext}(\sum_\lambda G_\lambda, H) \simeq \prod_\lambda \text{Ext}(G_\lambda, H)$ [5.4.11.]

(19) $\text{Ext}(H, \prod_\lambda G_\lambda) \simeq \prod_\lambda \text{Ext}(H, G_\lambda)$ [5.4.11.]

(20) $\text{Pext}(\sum_\lambda G_\lambda, H) \simeq \prod_\lambda \text{Pext}(G_\lambda, H)$

(21) $\text{Pext}(H,\prod_\lambda G_\lambda) \simeq \prod_\lambda \text{Pext}(H,G_\lambda)$

(22) $(\sum_\lambda G_\lambda) \otimes H \simeq \sum_\lambda (G_\lambda \otimes H) \simeq H \otimes (\sum_\lambda G_\lambda)$ [6.3.2.]

(23) $\text{Tor}(\sum_\lambda G_\lambda, H) \simeq \sum_\lambda \text{Tor}(G_\lambda, H) \simeq \text{Tor}(H, \sum_\lambda G_\lambda)$ [6.4.5.]

Let $Z(n)$ be the finite cyclic group of order $n > 1$, and G any Abelian group.

(24) $Z(n) \otimes G \simeq G/nG \simeq \text{Ext}(Z(n),G)$ [5.4.5., 6.2.3.]

(25) $Z \otimes G \simeq G \simeq \text{Hom}(Z,G)$ [6.2.2., 4.2.1.4.]

(26) $\text{Tor}(Z(n),G) = G[n]$. [6.4.6.]

To apply the method, we have the following "standard" exact sequences; let G be any group.

(27) $0 \to T(G) \to G \to T/T(G) \to 0$ is pure exact [3.3.2., 3.3.7.]

(28) There are projective groups P and P_o such that $0 \to P_o \to P \to G \to 0$ is exact. [4.5.5.]

(29) There are injective groups I and I_o such that $0 \to G \to I \to I_o \to 0$ is exact. [5.5.5.]

(30) There is a direct sum of cyclic groups P and a subgroup K of P such that $0 \to K \to P \to G \to 0$ is pure exact. [3.3.8.]

(31) There is a divisible group D and a direct product of finite cyclic groups A such that $0 \to G \to A \oplus D \to (A \oplus D)/G \to 0$ is pure exact. [5.6.8.]

9.2. Suggestions for Further Work

One of the basic purposes of our presentation in the particularly simple context of Abelian groups was to make a little more meaningful the many results which generalize and extend those discussed here. There seems to be two major directions in which to proceed: (1) generalize to modules over rings and (2) introduce more general methods and more powerful tools to generalize the entire setting. Of course, these two directions sometimes overlap, but abstractions to within category theory, homology theory, or indeed homological algebra proper are examples of (2). The use of topological methods to find natural topologies for groups and completions of groups along with ordinal and cardinal arithmetic in discussing invariants for groups are further examples.

The first portion of D. K. Harrison's foundational paper on homological methods and Abelian groups [11] should now be readily transparent. His results on cotorsion groups use only the methods we have presented here along with a minimal amount of ordinal arithmetic. His chapter in compact groups presents the interesting result that an Abelian group G is algebraically isomorphic to a totally disconnected compact topological group if and only if G is isomorphic to a direct product of finite cyclic groups and p-adic integers, for various primes p. Incidentally, we have not mentioned before, since it was too far afield for our purposes, that $Hom(Z(p^\infty),Z(p^\infty))$ and the ring of p-adic integers are isomorphic as rings. See Kuroŝ [21] for more details on this interesting facet of $Z(p^\infty)$.

Readers interested in rings should consult Jans' book [16]. Jans develops Ext through homology and applies it to the structure of modules over rings. This method is not at all similar to ours; it involves what is called a projective resolution of a module, higher dimensional groups $Ext^n(C,A)$, and the concept of injective dimension. These all follow as natural extensions of the work done here, and so cannot seem so mysterious if studied in the order history made them! Incidentally, Jans [17] has an interesting paper on modules that are both projective and injective simultaneously. In particular, those rings R which have projective injective envelopes are characterized.

A very readable account of torsion theory for modules appears in Hattori's paper [14]. For example, he shows that in order to obtain a theorem like $Tor(Q/Z,G) \simeq T(G)$, it is necessary and sufficient that the module be taken over a ring in which every principal left ideal is projective.

Chase [5] has investigated those rings R which admit projective products of projective modules. What is surprising is that, in general, only direct <u>sums</u> of projectives are projective and direct products of injectives are injective. (The idea for the proof of the later is in 3.4.5.) He calls a module A over R finitely related if there is an exact sequence $0 \to K \to F \to A \to 0$ of R-modules such that F is free and F and K are finitely generated (compare with 4.5.3.). Then a direct product of any family of projective left R-modules is projective if and only if R satisfies the descending chain condition on principal right ideals and any finitely generated right ideal of R is finitely related.

Along this line, Matlis [27] shows that the sum of two injective submodules of a module over a ring R is injective iff R is left hereditary; i.e., every left ideal of R is a projective left module over R. His paper [28] on divisible modules extends much of our work on divisible groups to a module setting. As we mentioned, it was Ekmann and Schopf [7] who generalized the notion of injective envelope to modules.

We showed that free and projective for groups are the same concept. Kaplansky [19] and Cartan and Eilenburg [4] generalized the equivalence to modules, and in the process characterized the Dedekind rings.

C. P. Walker's [32] categorical presentation of purity showed that the notion of purity could be generalized to a categorical setting in essentially two different ways. Many properties of Pext and pure-projectives go over to this more general setting.

Our adjoint relations can be traced in a module setting to Nunke [30]. He also generalized Ext(A,B) when A and B are modules.

The general structure of Ext(A,B) and Tor(A,B) is still an open question. Nunke [29] and Harrison [12] have made some contributions to the problem. One easy result of Nunke's is that $Tor(A,B)[n] \simeq Tor(A[n],B[n])$.

For a readable account of topological methods in the study of Abelian groups, the reader is referred to Koyama and Irwin's excellent article [20].

A SELECTED BIBLIOGRAPHY

[1] Baer, R. "Abelian groups that are direct summands of every containing Abelian group," Bull. Amer. Math. Soc., Vol. 46 (1940), 800-806.

[2] Baer, R. "Der Kern, eine charackteristische Untergruppe," Composito Math., Vol. 1 (1934), 254-283.

[3] Bourbaki, N. Algèbra Multilinéare (Chapt. III of Book II in Eleménts de Mathématique), Paris: Herman, 1948; 2nd Edition, 1951. V. 11.

[4] Cartan, H. and S. Eilenburg. Homological Algebra, Princeton: Princeton University Press, 1956.

[5] Chase, S. "Direct products of modules," Trans. Amer. Math. Soc., Vol. 97 (1960), 457-473.

[6] Dieudonné, J. "Sur le produits tensoriels," Ann. Sci. École Norm. Sup., Vol. 64 (1948), 101-117.

[7] Eckmann, B. and A. Schopf, "Uber Injective Moduln," Archiv Math., Vol. 4 (1953), 75-78.

[8] Fuchs, L. Abelian Groups, Oxford: Pergamon Press, 1960

[9] Fuchs, L. "Notes on Abelian Groups, I," Ann. Univ. Sci. Budapest, Vol. 2 (1959), 5-23; II, Acta Math. Acad. Sci. Hungar., Vol. 11 (1960), 117-125.

[10] Griffith, P. Infinite Abelian Group Theory, Chicago: University of Chicago Press, 1970.

[11] Harrison, D. K. "Infinite Abelian groups and homological methods," Ann. of Math., Vol. 69 (1959), 366-391.

[12] Harrison, D. K. "On the Structure of Ext," in Topics in Abelian Groups, edited by J. M. Irwin and E. A. Walker, Proceedings of the Symposium of Abelian Groups, New Mexico State University, 1962, Chicago: Scott, Foresman and Co, 1963, pages 195-209.

[13] Harrison, D. K. "Two problems of L. Fuchs," Publ. Math. Debrecen, Vol. 7 (1960), 316-319.

[14] Hattori, A. "A foundation of torsion theory for modules over general rings," Nagoya Math. J. Vol. 17 (1960), 147-158.

[15] Herstein, I. N. Topics in Algebra, New York: Blaisdell Pub. Co., 1964.

[16] Jans, James. Rings and Homology, New York: Holt, Rinehart, 1964.

[17] Jans, J. "Projective injective modules," Pac. J. Math. Vol. 9 (1959), 1103-1108.

[18] Kaplansky, I. Infinite Abelian Groups, Ann Arbor: University of Michigan Press, 1954.

[19] Kaplansky, I. "Projective modules," Ann. of Math., Vol. 68 (1958), 372-377.

[20] Koyama, T. and J. Irwin, "On Topological methods in Abelian groups," in Studies in Abelian Groups edited by B. Charles, Paris: Springer-Verlag, Dunod, 1968, pages 207-222.

[21] Kuroŝ, A. G. The Theory of Groups, New York: Chelsea, 1955.

[22] Kulikov, L. Ya., "On the theory of Abelian groups of arbitrary cardinality," (Russian), Math. Sb., Vol. 16 (1945),129-162.

[23] Lambek, J. Lectures on Rings and Modules, Waltham, Mass.: Blaisdell Publishing Co., 1966.

[24] MacLane, S. "Duality for Groups," Bull. Amer. Math. Soc., Vol. 56 (1950), 485-516.

[25] MacLane, S. Homology, New York: Academic Press, 1963.

[26] Maranda, J. M. "On pure subgroups of Abelian groups," Arch. Math., Vol. 11 (1960), 1-13.

[27] Matlis, E. "Injective modules over Noetherian rings," Pac. J. Math., Vol. 8 (1958), 511-528.

[28] Matlis, E. "Divisible modules," Proc. Amer. Math. Soc., Vol. 11 (1960), 385-391.

[29] Nunke, R. J. "On the structure of Tor. I," Budapest: Proceedings of the Colloquium on Abelian Groups, 1964; II, Pacific J. Math., Vol. 22 (1967),453-464.

[30] Nunke, R. J. "Modules of extensions over Dedekind rings," Illinois J. Math., Vol. 3 (1959), 222-241.

[31] Prüfer, H. "Untersuchungen über die Zerlegbaikeit der abzählbaren primären abelschen Gruppen," Math. Z., Vol. 17 (1923), 35-61.

[32] Walker, C. P. "Relative homological algebra and Abelian groups," Ill. J. Math., Vol. 10 (1966), 196-209.

[33] Whitney, H. "Tensor products of Abelian groups," Duke Math. J., Vol. 4 (1939), 495-520.

[34] Wiener, H. "A simplification of the logic of relations," Proc. of the Cambridge Philosophical Society, Vol. 17, (1914), 387-390.

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