SUPERADDITIVE n-HOMOGENEOUS FUNCTIONS AND n-CONVEX SETS

By

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PREFACE

This paper contains a study of n-convex sets and the nonnegative, continuous, superadditive, n-homogeneous, real valued functions defined on the nonnegative orthant of euclidean n^2-space. One such function is the well-known, permanent function. An interesting feature of such a function is that each above set, that is, the set of points for which the function is greater than or equal to a nonnegative constant, has the property that if x and y belongs to the above set then $\alpha x + \beta y$ also belongs to the above set, where $\alpha \geq 0$, $\beta \geq 0$ and $\alpha^n + \beta^n = 1$. Sets with this property are called n-convex. Moreover, the collection of all such functions, $\mathcal{P}_n$, forms a convex cone, whose is the zero function, in the real linear space of real functions defined on the positive orthant of euclidean n^2-space.

Following the introductory chapter, Chapter II contains a study of n-convex sets and some examples of the 2-convex hull of pairs of points in the plane. A characterization of the n-convex hull of a set, similar to that of Caratheodory for the convex hull, is given. Also characterized is the n-convex hull of a convex set.

In the following chapter the functions of $\mathcal{P}_n$, whose above sets are the n-convex hulls of finite subsets of the above sets, are studied. Chapter V contains a study of the extremal elements of this convex cone of functions. In particular, the functions $p_a(x) = \sup\{\lambda^n : x \geq \lambda a\}$, where a is a nonzero point of euclidean n^2-space, are shown to be extremal elements of $\mathcal{P}_n$.

A study of the infimum of a function of $\mathcal{P}_n$ over an n-convex subset of the nonnegative orthant of euclidean n^2-space is found in Chapter V. The last chapter is a summary and includes several questions that are open for further study.

To avoid confusion, the collection of all elements that belong to set A but not set B will be denoted by $A \setminus B$.

I would like to express my appreciation to all those who assisted me in the completion of this degree. Special thanks go to Professor E. K. McLachlan for his thoughtful guidance and encouragement throughout my graduate studies at Oklahoma State University and to Professors John Jewett, Forrest Whitfield and Harold Fristoe for serving on my advisory committee.

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CHAPTER I

INTRODUCTION

The Permanent Function

Let M_n denote the class of all $n \times n$ matrices with real entries. Then, with addition and scalar multiplication defined in the usual way, M_n is a real vector space isomorphic to E_{n^2}, where E_{n^2} denotes euclidean n^2-space. In particular, consider the n-square matrices with nonnegative entries. The <u>permanent</u> of an n-square matrix $A = (a_{ij})$, written $\mathrm{per}(A)$, is defined as

$$\mathrm{per}(A) = \sum_{\sigma} \prod_{1}^{n} a_{i\sigma(i)},$$

where the summation extends over all $n!$ permutations σ of the numbers $1, \ldots, n$ and $\sigma(i)$ denotes the i-th number in the permutation σ.

For example, if $n = 2$ the permanent of

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is

$$\mathrm{per}(A) = a_{11}\, a_{22} + a_{12}\, a_{21}.$$

When $n = 3$, the permanent of

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is

$$\mathrm{per}(A) = a_{11}\, a_{22}\, a_{33} + a_{11}\, a_{23}\, a_{32} + a_{12}\, a_{21}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} + a_{13}\, a_{22}\, a_{31}.$$

An $n \times n$ matrix $A = (a_{ij})$ with elements in the real field is said to be <u>doubly stochastic</u> if and only if $a_{ij} \geq 0$ for all i and j and each of the row sums and each of the column sums are 1. An example is

$$J_n = \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}.$$

Let K_n denote the set of all $n \times n$ doubly stochastic matrices. Then K_n is a compact convex subset of $E^+_{n^2}$, where $E^+_{n^2}$ is the positive orthant of E_{n^2} space [6]. The following unsolved conjecture was made by B.L. Van der Waerden [10] in 1926: If A is an n-square doubly stochastic matrix, then

$$\mathrm{per}(A) \geq \frac{n!}{n^n},$$

with equality if and only if $A = J_n$.

The permanent function is continuous everywhere and

$$\operatorname{per}(\alpha A) = \alpha^n \operatorname{per}(A),$$

where α is real. Furthermore, if A and B are $n \times n$ real matrices with nonnegative entries, then

$$\operatorname{per}(A + B) \geq \operatorname{per}(A) + \operatorname{per}(B).$$

The purpose of this thesis is to study a family of functions with the above properties. A more detailed discussion of the permanent function may be found in Marcos and Minx [5].

The Convex Cone of $\mathcal{P}_n$ Functions

Recall $E^+_{n^2} = \{x: x = (x_1, \ldots, x_{n^2}),\ x_i \geq 0,\ 1 \leq i \leq n^2\}$, where $n = 1, 2, \ldots$. Let $\mathcal{P}_n$ be the collection of finite-valued functions such that for $p \in \mathcal{P}_n$ it follows that

$$p: E^+_{n^2} \to E^+_1$$

and in addition

a. p is continuous,

b. $p(\alpha x) = \alpha^n p(x), \quad \alpha \geq 0,$

c. $p(x + y) \geq p(x) + p(y).$

That is, the functions of $\mathcal{P}_n$ are nonnegative, continuous, n-homogeneous and superadditive.

It follows that the permanent function belongs to $\mathcal{P}_n$. Also, note that since the functions in $\mathcal{P}_n$ are n-homogeneous, they are zero at the origin.

Definition 1.1: A set C in a linear space L is a cone with vertex x_0 if $x \in C$ and $\lambda > 0$ imply $\lambda x + (1 - \lambda)x_0 \in C$. C is called a convex cone if it is also convex.

It can be shown that C is a convex cone in L with vertex the origin if and only if

$$\text{a. } C + C \subset C$$
$$\text{b. } \lambda C \subset C, \text{ for } \lambda > 0$$

(cf [7], p. 14).

If $p,q \in \mathcal{P}_n$, define $(p + q)(x) = p(x) + q(x)$. Also, if $\alpha \geq 0$, define $(\alpha p)(x) = \alpha[p(x)]$. With addition and scalar multiplication defined as above, $\mathcal{P}_n$ is a convex cone, whose vertex is the zero function, in the linear space of real functions defined on $E^+_{n^2}$.

Proposition 1.1: The set $\mathcal{P}_n$ is a convex cone whose vertex is the zero function.

Proof: Let $p,q \in \mathcal{P}_n$. Then $p + q$ is continuous and if $\lambda \geq 0$, then

$$\begin{aligned}(p + q)(\lambda x) &= p(\lambda x) + q(\lambda x) = \lambda^n\, p(x) + \lambda^n\, q(x) \\ &= \lambda^n(p(x) + q(x)) = \lambda^n[(p + q)(x)].\end{aligned}$$

Also,

$$(p + q)(x + y) = p(x + y) + q(x + y) \geq p(x) + p(y) + q(x) + q(y)$$
$$= (p(x) + q(x)) + (p(y) + q(y))$$
$$= (p + q)(x) + (p + q)(y).$$

Therefore, $p + q \in \mathcal{P}_n$. Now let $\alpha > 0$. Then αp is continuous and if $\lambda \geq 0$, then

$$(\alpha p)(\lambda x) = \alpha[p(\lambda x)] = \alpha[\lambda^n p(x)] = (\alpha\lambda^n)p(x)$$
$$= (\lambda^n \alpha)p(x) = \lambda^n[\alpha \cdot p(x)] = \lambda^n(\alpha p)(x).$$

Also,

$$(\alpha p)(x + y) = \alpha[p(x + y)] \geq \alpha[p(x) + p(y)]$$
$$= \alpha[p(x)] + \alpha[p(y)] = (\alpha p)(x) + (\alpha p)(y).$$

Hence, $\alpha p \in \mathcal{P}_n$ and $\mathcal{P}_n$ is a convex cone with vertex the zero function.

Consider the convex cone C in Figure 1. Notice that if $x_i = y + z$, where $i = 1, 2$ and $y,z \in C$, then both y and z are proportional to x_i. However, if $x \in C$ such that x does not lie on either ray extending from 0 through x_1 or x_2, then there exists points $y,z \in C$, neither of which are proportional to x, but such that $x = y + z$. The vectors having the properties of x_1 and x_2 are given a special name in the following definition.

<u>Definition 1.2</u>: The <u>extremal elements</u> of a convex cone C in a real linear space are those $x \neq 0$ such that if $x = y + z$, then there exists $\alpha \geq 0$ and $\beta \geq 0$ such that $y = \alpha x$ and $z = \beta x$.

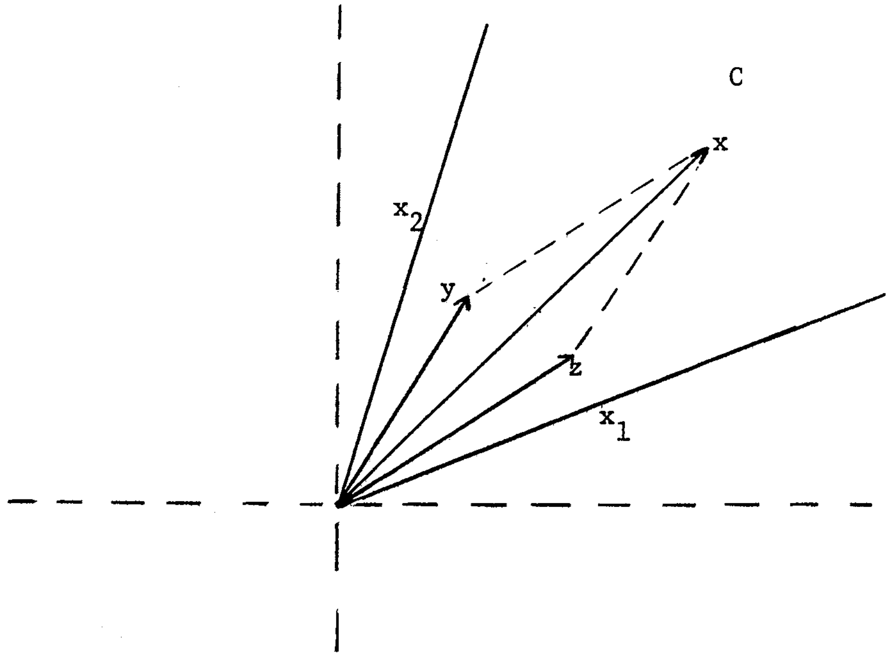


Figure 1.

Consider the convex cone in Figure 2. The points x_i, $i = 1,2,3$ are all extremal elements of the convex cone C. If x is a nonzero element of C, then there exists $\lambda > 0$ such that λx belongs to the smallest convex set containing x_1, x_2 and x_3, the convex hull of $\{x_1, x_2, x_3\}$. There exists nonnegative real numbers α_1, α_2, α_3 such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $\lambda x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$. Therefore,

$$x = \frac{\alpha_1}{\lambda} x_1 + \frac{\alpha_2}{\lambda} x_2 + \frac{\alpha_3}{\lambda} x_3.$$

Hence, for every $x \varepsilon C$, $x \neq 0$, x can be expressed as a finite sum of extremal elements of C.

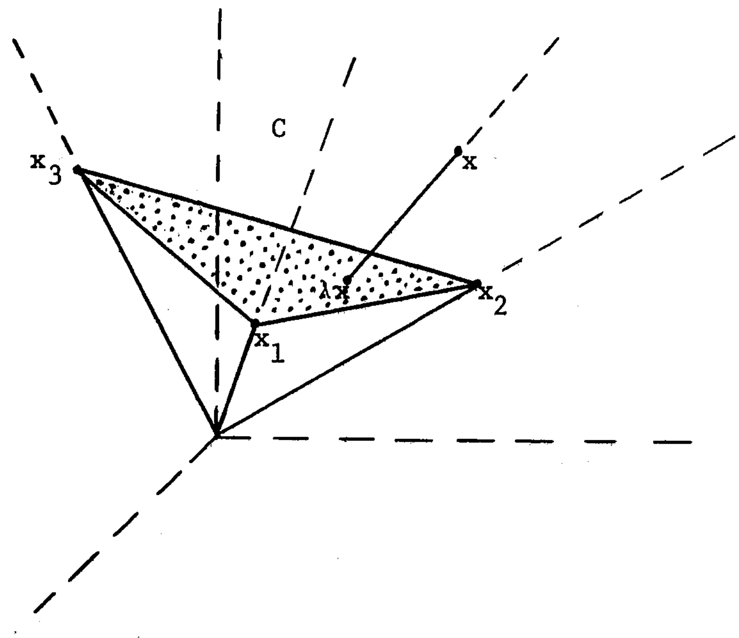


Figure 2.

However, this is not true in general. In more complicated situations analogous results are possible, by using a theorem of Choquet, to give an integral representation of any element of the cone in terms of the extremal elements of the cone. An excellent reference is _Lectures on Analysis_ by Choquet [2].

Part of Chapter III and all of Chapter IV of this thesis are devoted to a study of the extremal elements of $\mathcal{P}_n$.

Monotone Concave Gauges

Let $x = (x_1, \ldots, x_{n^2})$ and $y = (y_1, \ldots, y_{n^2})$ belong to $E^+_{n^2}$. Then define $x \geq y$ $(x > y)$ if and only if $x_i \geq y_i$ $(x_i > y_i)$ for all i.

The following family of functions have been studied by R. T. Rockafellar [8].

<u>Definition 1.3</u>: A <u>monotone concave gauge</u> on E_n^+ is a continuous real-valued function on E_n^+ such that

$$\text{a.}\quad f(\alpha x) = \alpha f(x),\quad \alpha \geq 0,$$

$$\text{b.}\quad f(x + y) \geq f(x) + f(y),$$

$$\text{c.}\quad f(x) \geq f(y) \quad \text{when} \quad x \geq y.$$

For brevity mc-gauge means monotone convex gauge.

<u>Definition 1.4</u>: A <u>monotone set of convex type</u> is a nonempty closed convex set $C \subset E_n^+$ such that $x \geq y$ and $y \in C$ implies $x \in C$.

If C is a monotone set of convex type, Rockefellar defines the <u>monotone support function</u> of C by

$$\langle C,x\rangle = \inf\{x\cdot y\colon y \in C\},$$

where $x\cdot y$ denotes inner product, and proves the following proposition.

<u>Proposition 1.2</u>: If C is a monotone set of convex type in E_n^+, then $\langle C,x\rangle$ is a monotone concave gauge. Conversely, each monotone concave gauge f on E_n^+ is of the form

$$f(x) = \langle C,x\rangle,$$

where

$$C = \{y \in E_n^+ : y \cdot z \geq f(z) \quad \text{for all} \quad z \in E_n^+\}.$$

The set C is a monotone set of convex type.

The collection of all monotone concave gauges defined on $E_{n^2}^+$ will be denoted as $\wp_n'$. The following theorem shows that the product of n mc-gauges belongs to $\wp_n$.

Theorem 1.1: If for all $i = 1, \ldots, n$, $A_i \in \wp_n'$, then the function A defined as

$$A(x) = \prod_1^n A_i(x)$$

is an element of $\wp_n$.

Proof: Clearly, A is continuous. Also,

$$A(\alpha x) = \prod_1^n A_i(\alpha x) = \prod_1^n \alpha A_i(x) = \alpha^n \prod_1^n A_i(x) = \alpha^n A(x),$$

and

$$A(x + y) = \prod_1^n A_i(x + y) \geq \prod_1^n (A_i(x) + A_i(y)) \geq \prod_1^n A_i(x) + \prod_1^n A_i(y)$$

$$= A(x) + A(y).$$

Hence, $A \in \wp_n$.

Theorem 1.1 raises the question of whether all functions of $\mathcal{P}_n$ are the products of n monotone concave guages. However, this is not the case for consider the following example:

Example 1.1: For every $x = (x_1, \ldots, x_4) \varepsilon E_4^+$ define $p(x)$ as follows,

$$p(x) = x_1^2 + x_2^2.$$

It is easy to see that $p \varepsilon \mathcal{P}_2$. Now suppose there exists monotone concave functions A_1 and A_2 such that

$$p(x) = A_1(x)\, A_2(x).$$

Notice that if $x_1 > 0$ or $x_2 > 0$, then $p(x) > 0$, which implies $A_1(x) > 0$ and $A_2(x) > 0$. In this case

$$A_1(x) = \frac{x_1^2 + x_2^2}{A_2(x)} .$$

Consider $(1,1,0,0) = (1,0,0,0) + (0,1,0,0)$. Since A_1 is superadditive, then

$$A_1((1,1,0,0)) \geq A_1((1,0,0,0)) + A_1((0,1,0,0)).$$

Hence,

$$\frac{2}{A_2((1,1,0,0))} \geq \frac{1}{A_2((1,0,0,0))} + \frac{1}{A_2((0,1,0,0))} ,$$

which implies that

$$
\begin{aligned}
2A_2((1,0,0,0))A_2((0,1,0,0)) &\geq A_2((1,1,0,0))A_2((0,1,0,0)) \\
&\quad + A_2((1,1,0,0))A_2((1,0,0,0)) \\
&\geq [A_2((1,0,0,0)) \\
&\quad + A_2((0,1,0,0))]A_2((0,1,0,0)) \\
&\quad + [A_2((1,0,0,0)) \\
&\quad + A_2((0,1,0,0))]A_2((1,0,0,0)) \\
&= 2A_2((1,0,0,0))A_2((0,1,0,0)) \\
&\quad + A_2^2((0,1,0,0)) + A_2^2((1,0,0,0)) \\
&> 2A_2((1,0,0,0))A_2((0,1,0,0)),
\end{aligned}
$$

since $A_2((0,1,0,0)) > 0$ and $A_2((1,0,0,0)) > 0$. This is a contradiction. Therefore, there does not exist two mc-gauges A_1 and A_2 such that $p(x) = A_1(x)\ A_2(x)$. However, if $f(x) = x_1$ and $g(x) = x_2$, then $f,g \in \mathcal{P}'_n$ and p is the finite sum of products of functions belonging to $\mathcal{P}'_n$.

Let $\mathcal{S}_n$ denote all those $p \in \mathcal{P}_n$ which are finite linear combinations of functions of the type

$$A(x) = \prod_{1}^{n} A_i(x),$$

where $A_i \in \mathcal{P}'_n$. Thus, $\mathcal{S}_n$ is clearly a subcone of $\mathcal{P}_n$. Also, the permanent function belongs to $\mathcal{S}_n$. It would be of interest to know if $\mathcal{S}_n$ is indeed $\mathcal{P}_n$.

n-Convex Sets

A convex functional p is a mapping from a convex set K in a linear space L into the real numbers such that if $x,y \in K$ and

$\alpha \in [0,1]$, then

$$p(\alpha x + (1 - \alpha)y) \leq \alpha p(x) + (1 - \alpha)p(y).$$

For a convex functional f and real number λ the below set $\{z: f(z) \leq \lambda\}$ and the strictly below set $\{z: f(z) < \lambda\}$, relative to λ, are convex.

For $p \in \mathcal{P}_n$ the below sets and the strictly below sets are not necessarily convex. For example, if $x = (x_1, \ldots, x_4) \in E_4^+$, define

$$p(x) = x_1 \, x_2.$$

Then $p \in \mathcal{P}_2$. Notice that $(4,0,0,0)$ and $(0,4,0,0)$ belong to $\{x: p(x) \leq 1\}$. Let $\alpha = \frac{1}{2}$. Then $\frac{1}{2}(4,0,0,0) + \frac{1}{2}(0,4,0,0) = (2,2,0,0)$ and $p((2,2,0,0)) = 4 > 1$. Hence, $\{z: p(z) \leq 1\}$ is not convex.

However, some interesting results are obtained when the above sets $\{z: p(z) \geq \lambda\}$ are studied. To get these results the following definition is given.

<u>Definition 1.5</u>: In a real linear space a set S is <u>n-convex</u> if for all $x,y \in S$, $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha^n + \beta^n = 1$, then $\alpha x + \beta y \in S$.

Notice that a set is convex if and only if it is 1-convex.

<u>Proposition 1.3</u>: For all $p \in \mathcal{P}_n$ and all $\lambda \geq 0$, the sets $\{z: p(z) \geq \lambda\}$ and $\{z: p(z) > \lambda\}$ are n-convex.

Proof: Let $x,y \in \{x: p(z) \geq \lambda\}$. Let $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha^n + \beta^n = 1$. Then

$$p(\alpha x + \beta y) \geq p(\alpha x) + p(\beta y) = \alpha^n p(x) + \beta^n p(y) \geq \alpha^n \lambda + \beta^n \lambda$$
$$= (\alpha^n + \beta^n)\lambda = \lambda.$$

Hence, $\alpha x + \beta y \in \{z: p(z) \geq \lambda\}$ and thus $\{z: p(z) \geq \lambda\}$ is n-convex. Likewise, $\{z: p(z) > \lambda\}$ is n-convex.

The sets $\{x: p(x) \geq \lambda\}$ will be denoted $\mathrm{Lev}_\lambda p$, i.e.,

$$\mathrm{Lev}_\lambda p = \{x: p(x) \geq \lambda\}.$$

A somewhat similar concept has been investigated by M. Landsberg [4]. He defines a set to be p-convex, where $0 < p \leq 1$, if $x,y \in S$, $\alpha \geq 0$, $\beta \geq 0$ and $\alpha^p + \beta^p = 1$ implies $\alpha x + \beta y \in S$.

In the following proposition, each member of a family of sets, which have been of some interest in recent literature, is shown to be n-convex [8].

Proposition 1.4: A monotone set of convex type in E_n^+ is n-convex.

Proof: Let $x,y \in S$ and $\alpha \geq 0$, $\beta \geq 0$ such that $\alpha^n + \beta^n = 1$. Notice that $\alpha \leq 1$ and $\beta \leq 1$. Since S is convex, then

$$\alpha^n x + \beta^n y \in S.$$

Since S is a monotone set of convex type, then $\alpha x + \beta y \geq \alpha^n x + \beta^n y$ implies $\alpha x + \beta y \in S$. Hence, S is n-convex.

The converse is not true since, as will be seen in later examples, an n-convex set is not always convex.

Particular use is made of the following concepts in studying n-convex sets.

Definition 1.6: If x and y belong to a real linear space L, then

$$n\text{-cr}[x,y] = \{\alpha x + \beta y \colon \alpha \geq 0,\ \beta \geq 0 \ \text{ and } \ \alpha^n + \beta^n = 1\}$$

and

$$n\text{-cr}(x,y) = \{\alpha x + \beta y \colon \alpha > 0,\ \beta > 0 \ \text{ and } \ \alpha^n + \beta^n = 1\}.$$

The sets $n\text{-cr}[x,y]$ and $n\text{-cr}(x,y)$ will be called n-curves and open n-curves, respectively.

In a linear space L, the line segment, xy, joining $x,y \in L$ is the set of all points $\alpha x + \beta y$, where $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta = 1$. The set of all such points where both $\alpha > 0$ and $\beta > 0$ is denoted by intv xy. Notice furthermore that if $z \in$ intv xy, then

$$\text{intv } xz \subset \text{intv } xy.$$

Clearly, the concept of an n-curve is a generalization of the concept of a line segment. However, the following example shows that an n-curve does not necessarily have the simple property noticed above for line segments. The absence of this property somewhat complicates the work in Chapter V.

Example 1.2: In E_2 let $x = (1,0)$ and $y = (0,1)$. Notice that if $(\alpha,\beta) = \alpha x + \beta y \in 2-\text{cr}(x,y)$, then $\alpha^2 + \beta^2 = 1$. Suppose

$$z = \frac{1}{2}x + \frac{\sqrt{3}}{2}y = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

Then $z \in 2\text{-cr}(x,y)$. Let $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. If $w = \frac{1}{2}z + \frac{\sqrt{3}}{2}x$, then $w \in 2 - \mathrm{cr}(x,z)$ and

$$w = \frac{1}{2} z + \frac{\sqrt{3}}{2} x = \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right) + \left(\frac{\sqrt{3}}{2}, 0\right) = \left(\frac{1 + 2\sqrt{3}}{4}, \frac{\sqrt{3}}{4}\right).$$

However, since

$$\left(\frac{1 + 2\sqrt{3}}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2 = \frac{1 + 4\sqrt{3} + 12}{16} + \frac{3}{16} = \frac{16 + 4\sqrt{3}}{16} > 1,$$

then $w \notin 2 - \mathrm{cr}(x,y)$ (cf. Figure 3.).

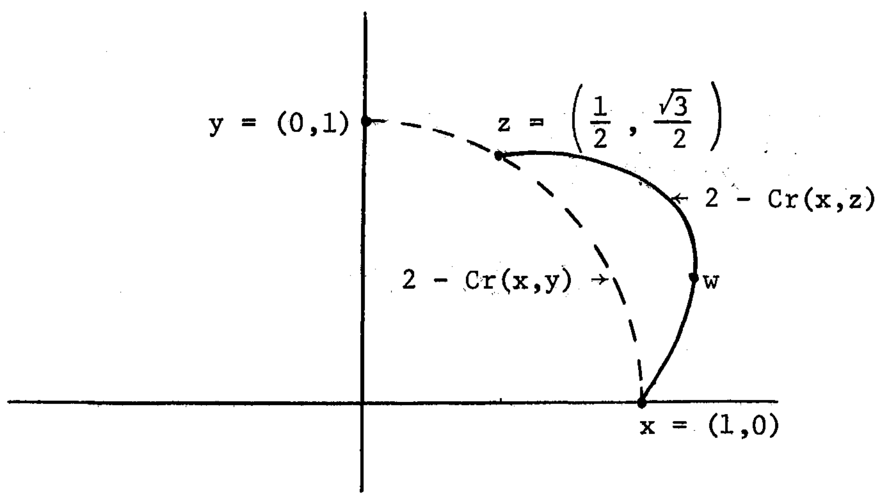


Figure 3.

Let x, y and z be three distinct points in a linear space L. If $u \in \mathrm{intv}\ xy$ and $v \in \mathrm{intv}\ yz$, then there exists a point w at which uz and xv intersect (cf. Figure 4).

The following result is a striking analogue with n-curves.

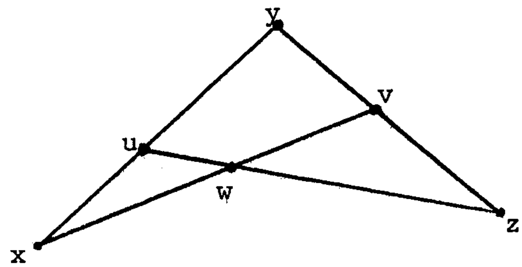


Figure 4.

Proposition 1.5: Suppose x, y, z are three distinct points of a real linear space L. If $u \in n - cr(x,y)$ and $v \in n - cr(y,z)$, then $(n - cr[u,z]) \cap (n - cr[x,v]) \neq \emptyset$.

Proof: Since $u \in n - cr(x,y)$, $u = \alpha x + \beta y$, where $\alpha > 0$, $\beta > 0$ and $\alpha^n + \beta^n = 1$. Also, $v = \sigma z + \gamma y$, where $\sigma > 0$, $\gamma > 0$ and $\sigma^n + \gamma^n = 1$. Let

$$\lambda = \frac{\beta}{\sqrt[n]{\gamma^n\alpha^n + \beta^n}} \quad \text{and} \quad \omega = \frac{\sigma\beta}{\sqrt[n]{\gamma^n\alpha^n + \beta^n}} .$$

Clearly, $\lambda > 0$ and $\omega > 0$. Also,

$$\sqrt[n]{1 - \lambda^n} = \sqrt[n]{1 - \frac{\beta^n}{\gamma^n\alpha^n + \beta^n}} = \sqrt[n]{\frac{\gamma^n\alpha^n + \beta^n - \beta^n}{\gamma^n\alpha^n + \beta^n}} = \frac{\gamma\alpha}{\sqrt[n]{\gamma^n\alpha^n + \beta^n}} > 0$$

and

$$\sqrt[n]{1 - \omega^n} = \sqrt[n]{1 - \frac{\sigma^n\beta^n}{\gamma^n\alpha^n + \beta^n}} = \sqrt[n]{\frac{\gamma^n\alpha^n + \beta^n - \sigma^n\beta^n}{\gamma^n\alpha^n + \beta^n}} = \sqrt[n]{\frac{\gamma^n\alpha^n + \beta^n(1 - \sigma^n)}{\gamma^n\alpha^n + \beta^n}}$$

$$= \sqrt[n]{\frac{\gamma^n\alpha^n + \beta^n\gamma^n}{\gamma^n\alpha^n + \beta^n}} = \frac{\gamma}{\sqrt[n]{\gamma^n\alpha^n + \beta^n}} > 0.$$

Since

$$\lambda^n + \left(\sqrt[n]{1 - \lambda^n}\right)^n = 1 \quad \text{and} \quad \omega^n + \left(\sqrt[n]{1 - \omega^n}\right)^n = 1,$$

then

$$\lambda(\sigma z + \gamma y) + \sqrt[n]{1 - \lambda^n}\, x = \lambda v + \sqrt[n]{1 - \lambda^n}\, x \in n - cr[x,v]$$

and

$$\sqrt[n]{1 - \omega^n}\,(\alpha x + \beta y) + \omega z = \sqrt[n]{1 - \omega^n}\, u + \omega z \in n - cr[u,z].$$

Now

$$\lambda(\sigma z + \gamma y) + \sqrt[n]{1 - \lambda^n}\, x = \frac{\beta}{\sqrt[n]{\gamma^n\alpha^n + \beta^n}}(\sigma z + \gamma y) + \frac{\gamma\alpha}{\sqrt[n]{\gamma^n\alpha^n + \beta^n}}\, x$$

and

$$\sqrt[n]{1 - \omega^n}\,(\alpha x + \beta y) + \omega z = \frac{\gamma}{\sqrt[n]{\gamma^n\alpha^n + \beta^n}}(\alpha x + \beta y) + \frac{\sigma\beta}{\sqrt[n]{\gamma^n\alpha^n + \beta^n}}\, z.$$

Hence,

$$\lambda(\sigma z + \gamma y) + \sqrt[n]{1 - \lambda^n}\, x = \sqrt[n]{1 - \omega^n}\,(\alpha x + \beta y) + \omega z$$

implies $(n - cr[u,z]) \bigcap (n - cr[x,v]) \neq \emptyset$.

As the next example shows, $n - cr[u,z]$ and $n - cr[x,v]$ may intersect at more than one point.

Example 1.3: In E_2, let $x = \left(\frac{1}{2}, \frac{1}{2}\right)$, $y = \left(-\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{2}\right)$ and $z = (1,0)$ (cf. Figure 5). Let $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$, then

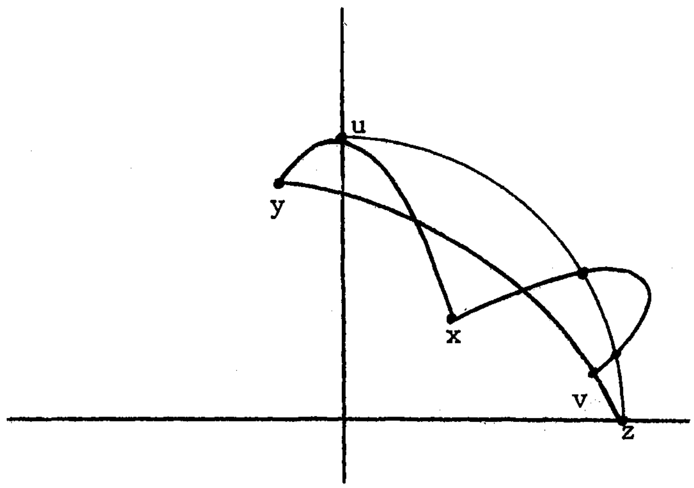


Figure 5.

$\alpha x + \beta y \in 2 - cr(x,y)$. Now

$$\alpha x + \beta y = \frac{1}{2}\left(\frac{1}{2}, \frac{1}{2}\right) + \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{2}\right) = \left(\frac{1}{4}, \frac{1}{4}\right) + \left(-\frac{1}{4}, \frac{3}{4}\right)$$
$$= (0,1).$$

Let $u = (0,1)$. Now let $\sigma = \frac{\sqrt{3}}{2}$ and $\gamma = \frac{1}{2}$. Then

$$\sigma z + \gamma y \in 2 - cr(y,z).$$

Further,

$$\sigma z + \gamma y = \frac{\sqrt{3}}{2}(1,0) + \frac{1}{2}\left(-\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{2}\right) = \left(\frac{\sqrt{3}}{2}, 0\right) + \left(-\frac{\sqrt{3}}{12}, \frac{\sqrt{3}}{4}\right)$$
$$= \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{12}, \frac{\sqrt{3}}{4}\right) = \left(\frac{5\sqrt{3}}{12}, \frac{\sqrt{3}}{4}\right).$$

Let

$$v = \left(\frac{5\sqrt{3}}{12}, \frac{\sqrt{3}}{4}\right).$$

Now

$$2 - \mathrm{cr}[u,z] = \{\lambda(1,0) + \omega(0,1): \lambda \geq 0,\ \omega \geq 0,\ \lambda^2 + \omega^2 = 1\}$$
$$= \{(\sqrt{1-\omega^2}, \omega): 0 \leq \omega \leq 1\},$$

i.e., that portion of the unit circle that is contained in the first quadrant. Also,

$$2 - \mathrm{cr}[x,v] = \{\lambda x + \omega v: \lambda \geq 0,\ \omega \geq 0,\ \lambda^2 + \omega^2 = 1\}$$
$$= \left\{\lambda\left(\frac{1}{2}, \frac{1}{2}\right) + \sqrt{1-\lambda^2}\left(\frac{5\sqrt{3}}{12}, \frac{\sqrt{3}}{4}\right): 0 \leq \lambda \leq 1\right\}$$
$$= \left\{\left(\frac{\lambda}{2} + \frac{5\sqrt{3}\sqrt{1-\lambda^2}}{12}, \frac{\lambda}{2} + \frac{3\sqrt{3}\sqrt{1-\lambda^2}}{12}\right): 0 \leq \lambda \leq 1\right\}.$$

Let

$$f(\lambda) = \left(\left(\frac{\lambda}{2} + \frac{5\sqrt{3}\sqrt{1-\lambda^2}}{12}\right)^2 + \left(\frac{\lambda}{2} + \frac{3\sqrt{3}\sqrt{1-\lambda^2}}{12}\right)^2\right)^{\frac{1}{2}},$$

where $\lambda \in [0,1]$. This $f(\lambda)$ gives the distance from the origin of the points on $2 - \mathrm{cr}[x,v]$. Moreover, f is a continuous real function. Also,

$$f(0) = \left(\left(\frac{5\sqrt{3}}{12}\right)^2 + \left(\frac{3\sqrt{3}}{12}\right)^2\right)^{\frac{1}{2}} = \left(\frac{75}{144} + \frac{27}{144}\right)^{\frac{1}{2}} = \left(\frac{102}{144}\right)^{\frac{1}{2}} < 1,$$

$$f\left(\frac{1}{2}\right) = \left(\left(\frac{1}{4} + \frac{15}{24}\right)^2 + \left(\frac{1}{4} + \frac{9}{24}\right)^2\right)^{\frac{1}{2}}$$
$$= \left(\left(\frac{21}{24}\right)^2 + \left(\frac{15}{24}\right)^2\right)^{\frac{1}{2}} = \left(\frac{441}{576} + \frac{225}{576}\right)^{\frac{1}{2}}$$
$$= \left(\frac{666}{576}\right)^{\frac{1}{2}} > 1,$$

and

$$f(1) = \left(\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right)^{\frac{1}{2}} = \left(\frac{1}{2}\right)^{\frac{1}{2}} < 1.$$

The Intermediate Value Theorem from calculus implies there exists $\lambda \varepsilon (0, \frac{1}{2})$ such that $f(\lambda) = 1$ and there exists $\lambda' \varepsilon (\frac{1}{2}, 1)$ such that $f(\lambda') = 1$. Therefore, since x and $v \varepsilon E_2^+$, then both

$$\lambda x + \sqrt{1 - \lambda^2}\, v \quad \text{and} \quad \lambda' x + \sqrt{1 - \lambda'^2}\, v$$

belong to $(2 - cr[u,z]) \cap (2 - cr[x,v])$.

Pressing further the analogy between n-curves and line segments, the n-curves are used to define a type of extreme point.

Definition 1.7: Let K be a n-convex subset of a linear space L. A point $z \varepsilon K$ is an n-extreme point of K if there does not exist $x,y \varepsilon K$ such that $z \varepsilon n - cr(x,y)$.

Just as for convex sets, a point x of an n-convex set K is an n-extreme point of K if and only if $K\setminus\{x\}$ is n-convex.

Some Basic Properties of $\wp_n$ Functions

In the remainder of this chapter several basic properties of $\wp_n$ functions are developed.

Proposition 1.6: Let $p \varepsilon \wp_n$. If $x,y \varepsilon E^+_{n^2}$ such that $x \geq y$, then $p(x) \geq p(y)$.

Proof: Since $x \geq y$, then $x_i \geq y_i$ for all i. Let

$$z_i = x_i - y_i \geq 0.$$

Then $z \in E^+_{n^2}$ and $x = y + z$. Therefore,

$$p(x) = p(y + z) \geq p(y) + p(z) \geq p(y).$$

Thus, $p(x)$ is nondecreasing on $E^+_{n^2}$. The converse is not true. Let $p(x) = x_1 x_2$, where $x = (x_1, \ldots, x_4) \in E^+_4$. Then

$$p((2,1,1,1)) = 2 > 1 = p((1,1,4,5)).$$

However, $(2,1,1,1) \not\geq (1,1,4,5)$.

If $p \neq 0$, the next proposition shows that $p(x) > 0$ at each interior point of $E^+_{n^2}$.

<u>Proposition 1.7</u>: Let $p \in \mathcal{P}_n$. If there exists $a \in \text{int } E^+_{n^2}$ such that $p(a) = 0$, then $p = 0$.

Proof: Let $a = (a_1, \ldots, a_{n^2})$. Since $a \in \text{int } E^+_{n^2}$, each $a_i > 0$. Let $x = (x_1, \ldots, x_{n^2}) \in E^+_{n^2}$. For each i there exists $\lambda_i \geq 0$ such that $x_i = \lambda_i a_i$. Let $\lambda = \max\{\lambda_i : i = 1, \ldots, n^2\}$. Then $x_i = \lambda_i a_i \leq \lambda a_i$, implies $\lambda a \geq x$. By Proposition 1.6

$$0 \leq p(x) \leq p(\lambda a) = \lambda^n p(a) = 0,$$

implies $p(x) = 0$, for all $x \in E^+_{n^2}$. Hence, $p = 0$.

With the aid of the following comment, $p(x)$ is shown to be strictly increasing.

<u>Comment 1.1</u>: If $p \in \mathcal{P}_n$ and $x,y \in E^+_{n^2}$ such that $x \geq y$, then $p(x) - p(y) \geq p(x - y)$.

Proof: Since $p(x) = p(x - y + y) \geq p(x - y) + p(y)$, then $p(x) - p(y) \geq p(x - y)$.

Proposition 1.8: If $x,y \in E^+_{n^2}$ such that $x > y$, then $p(x) > p(y)$.

Proof: Since $x > y$, then $x - y > 0$. Using Comment 1.1 and Proposition 1.7, it follows that

$$p(x) - p(y) \geq p(x - y) > 0.$$

Hence, $p(x) > p(y)$.

For $n = 1$, very special results hold for $p \in \mathcal{P}_n$.

Comment 1.2: If $n = 1$ and $p \in \mathcal{P}_1$, then p is linear.

Proof: If $x \in E^+_1$, then $x = x\cdot 1$ and $p(x) = p(x\cdot 1) = x\cdot p(1)$. In particular,

$$\begin{aligned} p(x + y) &= p((x + y)\cdot 1) = (x + y)p(1) = xp(1) + yp(1) \\ &= p(x\cdot 1) + p(y\cdot 1) = p(x) + p(y). \end{aligned}$$

Actually, p is not linear for $n > 1$, but the following proposition gives necessary and sufficient conditions for p being additive.

Proposition 1.9: Let $n > 1$ and $p \in \mathcal{P}_n$. Then p is additive if and only if $p = 0$.

Proof: Clearly, if $p = 0$, then p is additive. Suppose p is additive. Then

$$p(x) = p\left(\frac{1}{2}x + \frac{1}{2}x\right) = p\left(\frac{1}{2}x\right) + p\left(\frac{1}{2}x\right)$$
$$= \left(\frac{1}{2}\right)^n p(x) + \left(\frac{1}{2}\right)^n p(x) = \frac{1}{2^{n-1}} p(x),$$

which implies $p(x) = 0$, for all $x \in E^+_{n2}$.

Note that if $p,q \in \mathcal{P}_n$ and $p(x) = q(x)$ for all $x \in \text{int } E^+_{n2}$, then $p(x) = q(x)$ for all $x \in E^+_{n2}$. This follows from the continuity of p and q.

If p is a nonnegative, superadditive and n-homogeneous function on E^+_{n2} and p is continuous at one nonzero point of E^+_{n2}, then p is continuous at 0.

<u>Proposition 1.10</u>: Suppose $p\colon E^+_{n2} \to E^+_1$ is superadditive and $p(\alpha x) = \alpha^n p(x)$, for all $\alpha \geq 0$ and $x \in E^+_{n2}$. If p is continuous at any $x \in E^+_{n2}$, where $x \neq 0$, then p is continuous at 0.

Proof: Let $\varepsilon > 0$. If p is continuous at $x \neq 0$, then there exists $\delta > 0$ such that $h \in E^+_{n2}$ and $||h|| = ||(x + h) - x|| < \delta$ implies $|p(x + h) - p(x)| < \varepsilon$. It follows from Comment 1.1 that

$$|p(h) - p(0)| = |p(h)| = |p(x + h - x)|$$
$$\leq |p(x + h) - p(x)| < \varepsilon.$$

Hence, p is continuous at 0.

CHAPTER II

n-CONVEX SETS

The n-Convex Hull

For any set S, let $n(S)$ be the intersection of all n-convex sets containing S. Then $n(S)$ is called the <u>n-convex hull</u> of S.

If $\text{conv}(S)$ denotes the intersection of all convex sets containing the set S, then it is known that

$$\text{conv}(S) = \left\{ \sum_{1}^{k} \alpha_i x_i : x_i \in S,\ \alpha_i \geq 0,\ \sum_{1}^{k} \alpha_i = 1,\ k \text{ an integer} \right\}$$

[9]. In this section a similar characterization will be developed for $n(S)$. Consider the following:

<u>Proposition 2.1</u>: Let $\mathfrak{C}$ be a collection of n-convex sets. Then

a. if $\{A_\lambda : \lambda \in \Omega\} \subset \mathfrak{C}$, then $\bigcap A_\lambda \in \mathfrak{C}$,

b. if $A, B \in \mathfrak{C}$, then $A + B \in \mathfrak{C}$,

c. if $A \in \mathfrak{C}$ and σ is real, then $\sigma A \in \mathfrak{C}$.

The proofs are simple and straightforward. This proposition means that the collection of n-convex sets is closed under arbitrary intersections, finite vector sums and under scalar multiplication.

Thus, the n-convex hull of a set S is n-convex, in fact, the smallest n-convex set containing S. The following theorem gives a characterization of $n(S)$.

Theorem 2.1: For any set S,

$$n(S) = \left\{ \sum_{1}^{k} \alpha_i x_i : x_i \in S,\ \alpha_i \geq 0,\ \sum_{1}^{k} \alpha_i^n = 1 \right\}$$

where k is an integer and the x_i are not necessarily distinct.

Proof: Let

$$B(S) = \left\{ \sum_{1}^{k} \alpha_i x_i : x_i \in S,\ \alpha_i \geq 0,\ \sum_{1}^{k} \alpha_i^n = 1 \right\},$$

where k is an integer and the x_i are not necessarily distinct. Let $x \in S$. Then $x = 1 \cdot x \in B(S)$, implies that $S \subset B(S)$. Now show that $B(S)$ is n-convex. Let $x, y \in B(S)$. Then

$$x = \sum_{1}^{k} \alpha_i x_i \quad \text{and} \quad y = \sum_{1}^{\ell} \beta_i y_i$$

where $\alpha_i \geq 0$, $\beta_i \geq 0$ and

$$\sum_{1}^{k} \alpha_i^n = \sum_{1}^{\ell} \beta_i^n = 1.$$

Let $\gamma \geq 0$ and $\sigma \geq 0$ such that $\gamma^n + \sigma^n = 1$. Now

$$\sigma x + \gamma y = \sigma \sum_{1}^{k} \alpha_i x_i + \gamma \sum_{1}^{\ell} \beta_i y_i = \sum_{1}^{k} \sigma\alpha_i x_i + \sum_{1}^{\ell} \gamma\beta_i y_i,$$

where $x_i, y_i \in S$, $\sigma\alpha_i \geq 0$ and $\gamma\beta_i \geq 0$. Also,

$$\sum_{1}^{k} (\sigma\alpha_i)^n + \sum_{1}^{\ell} (\gamma\beta_i)^n = \sigma^n \sum_{1}^{k} \alpha_i^n + \gamma^n \sum_{1}^{\ell} \beta_i^n = \sigma^n + \gamma^n = 1,$$

implies that $\sigma x + \gamma y \in B(S)$. Hence, $B(S)$ is n-convex and contains S. Therefore, $n(S) \subset B(S)$.

Now show $B(S) \subset n(S)$ by inducting on k. Suppose $x \in B(S)$ such that $x = \alpha y$, then $\alpha = 1$ and $y \in S$. Hence, $x \in S \subset n(S)$.

Suppose that for each $x \in B(S)$ such that

$$x = \sum_{1}^{k-1} \alpha_i x_i,$$

where $x_i \in S$, $\alpha_i > 0$,

$$\sum_{1}^{k-1} \alpha_i^n = 1,$$

then $x \in n(S)$. Let

$$x = \sum_{1}^{k} \alpha_i x_i$$

where $x_i \in S$, $\alpha_i > 0$ and

$$\sum_{1}^{k} \alpha_i^n = 1.$$

Since

$$\sum_{1}^{k} \alpha_i^n = 1,$$

then

$$\alpha_k = \sqrt[n]{1 - \sum_{1}^{k-1} \alpha_i^n}.$$

Hence,

$$x = \sum_{1}^{k-1} \alpha_i x_i + \alpha_k x = \sqrt[n]{\sum_{1}^{k-1} \alpha_j^n} \sum_{1}^{k-1} \left(\frac{\alpha_i}{\sqrt[n]{\sum_{1}^{k-1} \alpha_j^n}} \right) x_i + \left(\sqrt[n]{1 - \sum_{1}^{k-1} \alpha_i^n} \right) x_m.$$

Since

$$\sum_{1}^{k-1} \left(\frac{\alpha_i}{\sqrt[n]{\sum_{1}^{k-1} \alpha_j^n}} \right)^n = \sum_{1}^{k-1} \left(\frac{\alpha_i^n}{\sum_{1}^{k-1} \alpha_j^n} \right) = \frac{\sum_{1}^{k-1} \alpha_i^n}{\sum_{1}^{k-1} \alpha_j^n} = 1,$$

then

$$\sum_{1}^{k-1} \left(\frac{\alpha_i}{\sqrt[n]{\sum_{1}^{k-1} \alpha_j^n}} \right) x_i \in n(S).$$

Let

$$y = \sum_{1}^{k-1} \left(\frac{\alpha_i}{\sqrt[n]{\sum_{1}^{k-1} \alpha_j^n}} \right) x_i .$$

Then

$$x = \left(\sqrt[n]{\sum_{1}^{k-1} \alpha_j^n} \right) y + \left(\sqrt[n]{1 - \sum_{1}^{k-1} \alpha_i^n} \right) x_m .$$

Since n(S) is n-convex and

$$\left(\sqrt[n]{\sum_{1}^{k-1} \alpha_j^n} \right)^n + \left(\sqrt[n]{1 - \sum_{1}^{k-1} \alpha_i^n} \right)^n = \sum_{1}^{k-1} \alpha_j^n + 1 - \sum_{1}^{k-1} \alpha_i^m = 1,$$

then $x \varepsilon n(S)$. Hence, $B(S) \subset n(S)$ which implies $n(S) = B(S)$.

Notice that $B(S) = n(S) = n(n(S)) = B(B(S))$. The following theorem shows that $B(S)$ is <u>inverse starlike</u> from the origin, that is, $x \varepsilon B(S)$ implies $\lambda x \varepsilon B(S)$ for all $\lambda \geq 1$.

<u>Theorem 2.2</u>: If $x = \lambda y$ where $\lambda \geq 1$ and $y \varepsilon B(S)$, then $x \varepsilon B(S)$.

Proof: Consider

$$k \cdot \frac{1}{\sqrt[n]{k}} = k^{\frac{n-1}{n}},$$

where k is a positive integer. There exists a k such that

$$k^{\frac{n-1}{n}} \geq \lambda .$$

Also,

$$k\left(\frac{1}{\sqrt[n]{k}}\right)^n = 1.$$

Therefore,

$$k^{\frac{n-1}{n}} y \in B(S) \quad \text{and} \quad k^{\frac{n-1}{n}} y \geq \lambda y.$$

Hence,

$$y \leq \lambda y \leq k^{\frac{n-1}{n}} y.$$

Let $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha^n + \beta^n = 1$. Then

$$\alpha y + \beta k^{\frac{n-1}{n}} y \in B(S), \quad \text{i.e.,} \quad \left(\sqrt[n]{1 - \beta^n} + \beta k^{\frac{n-1}{n}}\right) y \in B(S).$$

It remains to be shown that there exists a real number β such that

$$\lambda = \sqrt[n]{1 - \beta^n} + \beta k^{\frac{n-1}{n}}.$$

For $\beta \in [0,1]$, let

$$f(\beta) = \sqrt[n]{1 - \beta^n} + \beta k^{\frac{n-1}{n}}.$$

Notice that f is continuous on $[0,1]$,

$$f(0) = 1 \quad \text{and} \quad f(1) = k^{\frac{n-1}{n}}.$$

Since

$$1 \leq \lambda \leq k^{\frac{n-1}{n}},$$

then there exists a $\beta \in [0,1]$ such that $f(\beta) = \lambda$. Hence,

$$x = \lambda y \in B(S).$$

Now for any set S, let

$$B'(S) = \left\{ \sum_1^k \alpha_i x_i : x_i \in S,\ \alpha_i \geq 0,\ \sum_1^k \alpha_i^n \geq 1 \right\},$$

where k is an integer and the x_i are distinct.

It is easy to see that $B(S) \subset B'(S)$. For example, if

$$x = \alpha_1 x_1 + \alpha_2 x_1 + \alpha_3 x_3 \in B(S),$$

then $\alpha_1^n + \alpha_2^n + \alpha_3^n = 1$. However, $x = (\alpha_1 + \alpha_2)x_1 + \alpha_3 x_3$, where $(\alpha_1 + \alpha_2)^n + \alpha_3^n \geq 1$. Hence, $x \in B'(S)$. The following proposition shows these sets are in fact equal.

<u>Proposition 2.2</u>: $B(S) = B'(S)$.

Proof: Let $x \in B'(S)$. Then

$$x = \sum_1^k \alpha_i x_i,$$

where $x_i \in S$, $\alpha_i \geq 0$ and

$$\sum_1^k \alpha_i^n \geq 1.$$

Let

$$\alpha = \left(\sum_{1}^{k} \alpha_i^n \right)^{\frac{1}{n}}.$$

Then

$$\sum_{1}^{k} \left(\frac{\alpha_i}{\alpha} \right)^n = 1,$$

implies that $\frac{1}{\alpha} x \in B(S)$. Since $\alpha \geq 1$, Theorem 2.2 implies that $x = \alpha \frac{1}{\alpha} x \in B(S)$.

Proposition 2.2 and Theorem 2.1 together give another characterization of $n(S)$.

Theorem 2.3: For any set S,

$$n(S) = \left\{ \sum_{1}^{k} \alpha_i x_i : x_i \in S, \alpha_i \geq 0, \sum_{1}^{k} \alpha_i^n \geq 1 \right\},$$

where k is an integer and the x_i are distinct.

Actually, the following characterization of $n(S)$ will prove to be the most useful.

Theorem 2.4: For any set S,

$$n(S) = \left\{ \lambda \sum_{1}^{k} \alpha_i x_i : \lambda \geq 1, x_i \in S, \alpha_i \geq 0, \sum_{1}^{k} \alpha_i^n = 1 \right\},$$

where k is an integer and the x_i are distinct.

Proof: Let

$$A = \left\{ \lambda \sum_{1}^{k} \alpha_i x_i : \lambda \geq 1,\ x_i \varepsilon S,\ \alpha_i \geq 0,\ \sum_{1}^{k} \alpha_i^n = 1 \right\},$$

where k is an integer and the x_i are distinct. Clearly, $A \subset n(S)$. Let $y \varepsilon n(S)$. Then

$$y = \sum_{1}^{m} \alpha_i x_i,$$

where $\alpha_i \geq 0$,

$$\sum_{1}^{m} \alpha_i^n \geq 1$$

and $x_i \varepsilon S$. Let

$$\lambda = \sqrt[n]{\sum_{1}^{m} \alpha_i^n},$$

then

$$y = \lambda \sum_{1}^{m} \left(\frac{\alpha_i}{\lambda} \right) x_i \quad \text{and} \quad \sum_{1}^{m} \left(\frac{\alpha_i}{\lambda} \right)^n = 1.$$

Therefore, $y \varepsilon A$ and $n(S) = A$.

Theorem 2.2 implies that an n-convex set is inverse starlike from 0. The following example shows that the converse is not true.

Example 2.1: Let S be the shaded area in Figure 6. S is inverse starlike from 0.

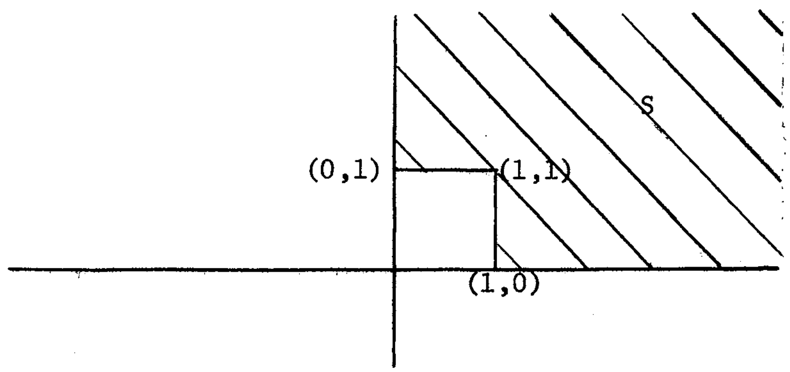


Figure 6.

Let $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Then $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{1}{2}(1,0) + \frac{\sqrt{3}}{2}(0,1) \in 2(S)$. Since $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \notin S$, S is not 2-convex.

Examples of 2-Convex Sets

In this section, several examples of the 2-convex hull of sets of two points in the plane are given.

Example 2.2: In E_2 let $S = \{(0,y_1),(1,0)\}$, where $y_1 > 0$. Find $2(S)$.

Solution: Let $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha^2 + \beta^2 = 1$. Then

$$(\beta,\alpha y_1) = \alpha(0,y_1) + \beta(1,0) \in 2(S).$$

Let $x = \beta$ and $y = \alpha y_1$. Then

$$\alpha = \frac{y}{y_1}$$

and

$$x = \beta = \sqrt{1 - \alpha^2} = \sqrt{1 - \left(\frac{y}{y_1}\right)^2} = \frac{\sqrt{y_1^2 - y^2}}{y_1} .$$

Hence, $xy_1 = \sqrt{y_1^2 - y^2}$ implies $x^2 y_1^2 = y_1^2 - y^2$. Thus,

$$x^2 + \frac{y^2}{y_1^2} = 1,$$

where $x \geq 0$, $y \geq 0$ and $y_1 > 0$. When $y_1 = 1$, then Theorem 2.4 implies $2(S)$ is the region indicated in Figure 7(a). Likewise, if $y_1 > 1$, then $2(S)$ is the region indicated in Figure 7(b). Finally, if $y_1 < 1$, then $2(S)$ is the region indicated in Figure 7(c).

Example 2.3: In E_2 let $S = \{(x_1,y_1),(1,0)\}$, where $x_1 > 0$ and $y_1 > 0$. Find $2(S)$.

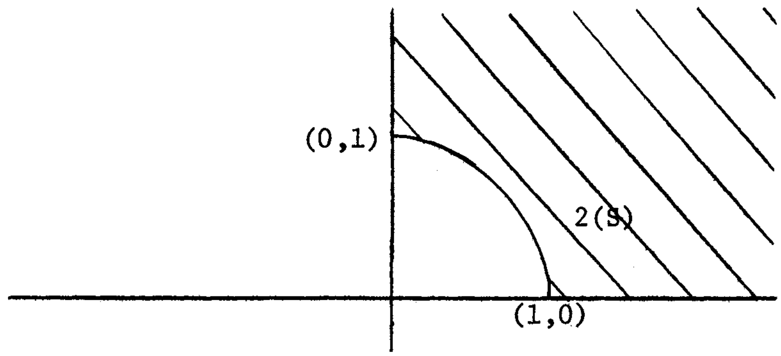
Solution: Let $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha^2 + \beta^2 = 1$. Then

$$(\alpha x_1 + \beta,\ \alpha y_1) = \alpha(x_1,y_1) + \beta(1,0) \in 2(S).$$

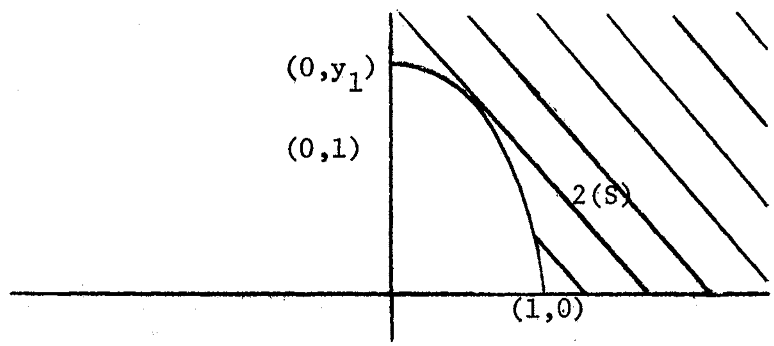
Let $x = \alpha x_1 + \beta$ and $y = \alpha y_1$. Thus, x and y are nonnegative. Substituting,

$$x = x_1 \frac{y}{y_1} + \sqrt{1 - \frac{y^2}{y_1^2}} = x_1 \frac{y}{y_1} + \frac{\sqrt{y_1^2 - y^2}}{y_1} .$$

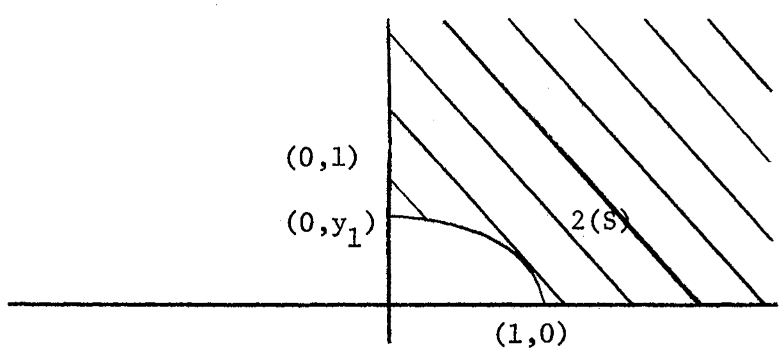
Hence,



(a)



(b)



(c)

Figure 7.

$$xy_1 - x_1y = \sqrt{y_1^2 - y^2} \ ,$$

$$y_1^2x^2 - 2x_1y_1xy + x_1^2y^2 = y_1^2 - y^2,$$

$$y_1^2x^2 - 2x_1y_1xy + (x_1^2 + 1)y^2 = y_1^2.$$

This can be shown to be the following ellipse

$$(y_1^2\cos^2\phi - 2x_1y_1\sin\phi\cos\phi + (x_1^2 + 1)\sin^2\phi)x'^2$$

$$+ (y_1^2\sin^2\phi + 2x_1y_1\sin\phi\cos\phi + (x_1^2 + 1)\cos^2\phi)y'^2 = y_1^2$$

by making the following change of variables

$$x = x'\cos\phi - y'\sin\phi,$$

$$y = x'\sin\phi + y'\cos\phi,$$

where $0 \leq \phi \leq \frac{\pi}{2}$ such that

$$\cot 2\phi = \frac{x_1^2 - y_1^2 + 1}{2x_1y_1} .$$

Notice that the coefficients of x'^2 and y'^2 are positive.

Let γ be the angle between the vectors (x_1,y_1) and $(1,0)$ (cf. Figure 8), then

$$\cot \gamma = \frac{x_1}{y_1} .$$

Therefore,

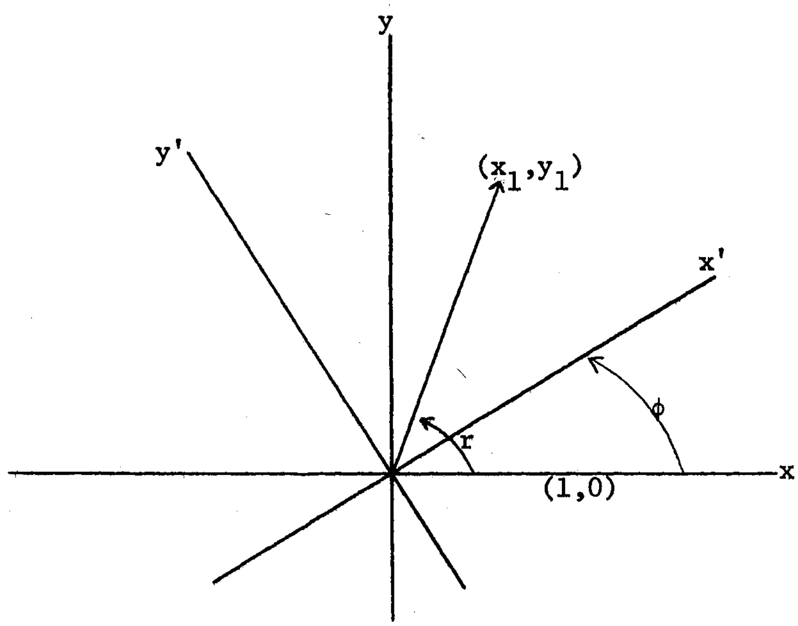


Figure 8.

$$\cot 2\gamma = \frac{\cot^2\gamma - 1}{2\cot\gamma} = \frac{\left(\frac{x_1}{y_1}\right)^2 - 1}{2\,\frac{x_1}{y_1}} = \frac{x_1^2 - y_1^2}{y_1^2} \cdot \frac{y_1}{2x_1} = \frac{x_1^2 - y_1^2}{2x_1y_1}.$$

Hence,

$$\cot 2\phi = \frac{x_1^2 - y_1^2}{2x_1y_1} + \frac{1}{2x_1y_1} = \cot 2\gamma + \frac{1}{2x_1y_1} > \cot 2\gamma.$$

Also, $\gamma, \phi \in [0, \frac{\pi}{2}]$, implies that $2\gamma > 2\phi$. Hence, $\gamma > \phi$.

Let

$$r = \sqrt{(x_1^2 + 1 - y_1^2)^2 + (2x_1y_1)^2}.$$

If $y_1^2 \leq x_1^2 + 1$, then $\cot 2\phi \geq 0$. Hence, $0 \leq 2\phi \leq \frac{\pi}{2}$ implies $\cos 2\phi \geq 0$ and $\sin 2\phi \geq 0$. Therefore,

$$\cos 2\phi = \frac{x_1^2 + 1 - y_1^2}{r}$$

and

$$\sin 2\phi = \frac{2x_1y_1}{r} .$$

If $y_1^2 > x_1^2 + 1$, then $\cot 2\phi < 0$. Hence, $\frac{\pi}{2} \leq 2\phi \leq \pi$ implies $\sin 2\phi \geq 0$ and $\cos 2\phi \leq 0$. Hence,

$$\sin 2\phi = \frac{2x_1y_1}{r} \quad \text{and} \quad \cos 2\phi = \frac{x_1^2 + 1 - y_1^2}{r}.$$

In either case $2x_1y_1 = r \sin 2\phi$ and $x_1^2 + 1 - y_1^2 = r \cos 2\phi$. Calculating the difference of the y^{1^2} and x^{1^2} coefficients gives

$$\begin{aligned}
&y_1^2\sin^2\phi + 2x_1y_1\sin\phi\cos\phi + (x_1^2 + 1)\cos^2\phi - (y_1^2\cos^2\phi - 2x_1y_1\sin\phi\cos\phi \\
&\qquad + (x_1^2 + 1)\sin^2\phi) \\
&\qquad\quad = -y_1^2(\cos^2\phi - \sin^2\phi) + (x_1^2 + 1)(\cos^2\phi - \sin^2\phi) + 2x_1y_1\sin 2\phi \\
&\qquad\quad = (x_1^2 + 1 - y_1^2)\cos 2\phi + 2x_1y_1\sin 2\phi \\
&\qquad\quad = r \cos^2 2\phi + r \sin^2 2\phi = r > 0.
\end{aligned}$$

The coefficient of y^{1^2} is greater than the coefficient of x^{1^2}. This implies the major axis of the ellipse is the x'-axis. Therefore, using Theorem 2.4, it follows that $2(S)$ is the indicated region in Figure 9.

The next example considers the case of two vectors with an obtuse angle between them.

<u>Example 2.3</u>: In E_2 let $S = \{(x_1,y_1),(1,0)\}$ where $x_1 < 0$ and $y_1 > 0$. Find $2(S)$.

Solution: Let $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha^2 + \beta^2 = 1$. Then $(\alpha x_1 + \beta,\ \alpha y_1) \in B(S)$. Let $x = \alpha x_1 + \beta$ and $y = \beta y_1$. As in the previous example

$$y_1^2x^2 - 2x_1y_1xy + (x_1^2 + 1)y^2 = y_1^2.$$

Now let $0 \leq \phi \leq \frac{\pi}{2}$ such that

$$\cot 2\phi = \frac{x_1^2 + 1 - y_1^2}{2x_1y_1}.$$

Making the change of variables

$$x = x'\cos\phi - y'\sin\phi,$$

$$y = x'\sin\phi + y'\cos\phi,$$

then as before

$$(y_1^2\cos^2\phi - 2x_1y_1\sin\phi\cos\phi + (x_1^2 + 1)\sin^2\phi)x^{1^2} + (y_1^2\sin^2\phi$$

$$+ 2x_1y_1\sin\phi\cos\phi + (x_1^2 + 1)\cos^2\phi)y^{1^2} = y_1^2.$$

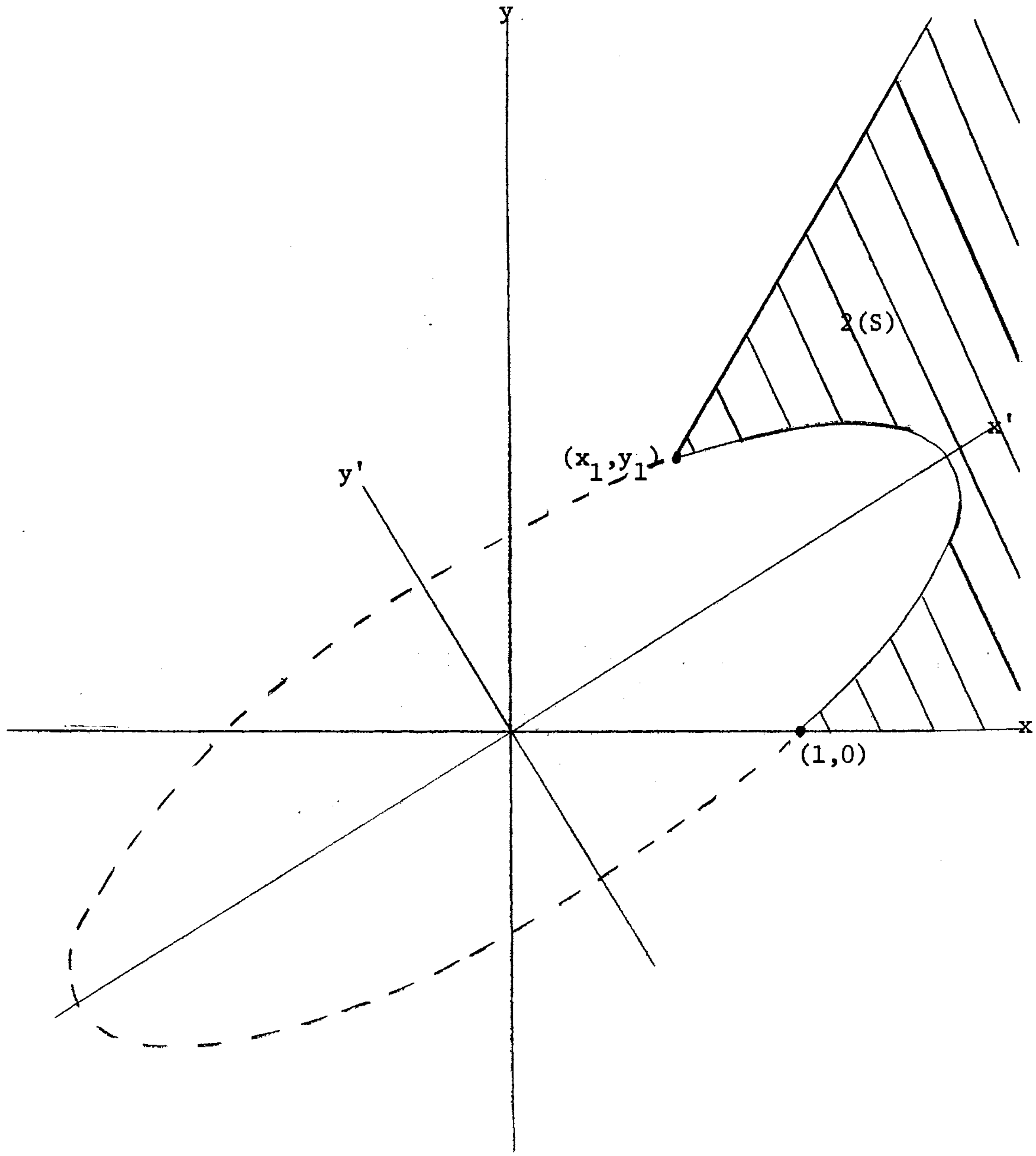


Figure 9.

If γ is the angle between (x_1,y_1) and $(1,0)$, then

$$\cot \gamma = \frac{x_1}{y_1}$$

implies that

$$\cot 2\gamma = \frac{x_1^2 - y_1^2}{2x_1y_1} .$$

Since $x_1 < 0$, then

$$\cot 2\phi = \cot 2\gamma + \frac{1}{2x_1y_1} < \cot 2\gamma .$$

Also, $\pi \leq 2\gamma \leq 2\pi$. Hence, $\pi < 2\gamma < \pi + 2\phi$, which gives

$$\frac{\pi}{2} < \gamma < \frac{\pi}{2} + \phi$$

(cf. Figure 10).

Again let

$$r = \sqrt{(x_1^2 + 1 - y_1^2)^2 + (2x_1y_1)^2} .$$

Recall that

$$\cot 2\phi = \frac{x_1^2 + 1 - y_1^2}{2x_1y_1}$$

and $2x_1y_1 < 0$. Suppose $y_1^2 \geq x_1^2 + 1$. Then $\cot 2\phi \geq 0$, implies that $0 \leq 2\phi \leq \frac{\pi}{2}$. Hence, $\sin 2\phi \geq 0$ and $\cos 2\phi \geq 0$. Therefore,

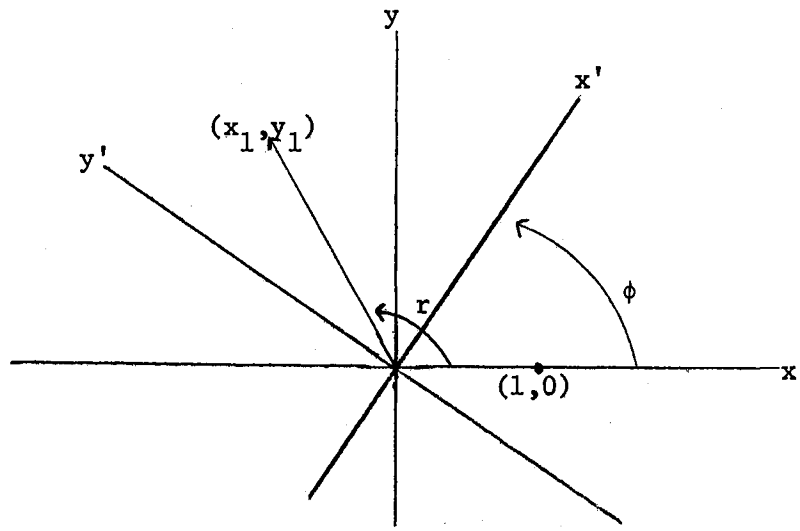


Figure 10.

$$\sin 2\phi = -\frac{2x_1y_1}{r} \quad \text{and} \quad \cos 2\phi = -\frac{x_1^2 + 1 - y_1^2}{r}\,.$$

If $y_1^2 < x_1^2 + 1$, then $\cot 2\phi < 0$. Hence, $\frac{\pi}{2} < 2\phi < \pi$, implies that $\sin 2\phi \geq 0$ and $\cos 2\phi \leq 0$. Therefore,

$$\sin 2\phi = -\frac{2x_1y_1}{r} \quad \text{and} \quad \cos 2\phi = -\frac{x_1^2 + 1 - y_1^2}{r}\,.$$

In either case, $2x_1y_1 = -r \sin 2\phi$ and $x_1^2 + 1 - y_1^2 = -r \cos 2\phi$.

Therefore, the difference of the y^{1^2} coefficient and the x^{1^2} coefficient is

$$y_1^2\sin^2\phi + 2x_1y_1\sin\phi\cos\phi + (x_1^2 + 1)\cos^2\phi - (y_1^2\cos^2\phi - 2x_1y_1\sin\phi\cos\phi$$
$$+ (x_1^2 + 1)\sin^2\phi$$

$$= (x_1^2 + 1 - y_1^2)\cos 2\phi + 2x_1y_1\sin 2\phi = -r\cos^2 2\phi - r\sin^2 2\phi$$

$$= -r < 0.$$

Thus, the major axis of the ellipse is the y'-axis. Again using Theorem 2.4, it follows that $2(S)$ is the indicated region in Figure 11.

It remains to consider the cases of the two points on a line through the origin. In the following example the two points are on opposite sides of the origin.

Example 2.4: In E_2 let $S = \{(1,0),(x_1,0)\}$ where $x_1 < 0$. Find $n(S)$.

Solution: Let $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha^n + \beta^n = 1$. Then $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$. Also,

$$(\alpha + \beta x_1, 0) = \alpha(1,0) + \beta(x_1,0) \in n(S).$$

Let

$$f(\lambda) = \lambda + \sqrt[n]{1 - \lambda^n}\, x_1,$$

where $\lambda \in [0,1]$. Then f is continuous, $f(0) = x_1$ and $f(1) = 1$. Let $a \in (x_1,1)$, then by the Intermediate Value Theorem there exists $\lambda \in (0,1)$ such that $f(\lambda) = a$. Therefore,

$$\{(a,0): a \in [x_1,1]\} \subset n(S).$$

Since Theorem 2.4 implies $n(S)$ is inverse starlike from the origin, then $n(S)$ is the x-axis (cf. Figure 12).

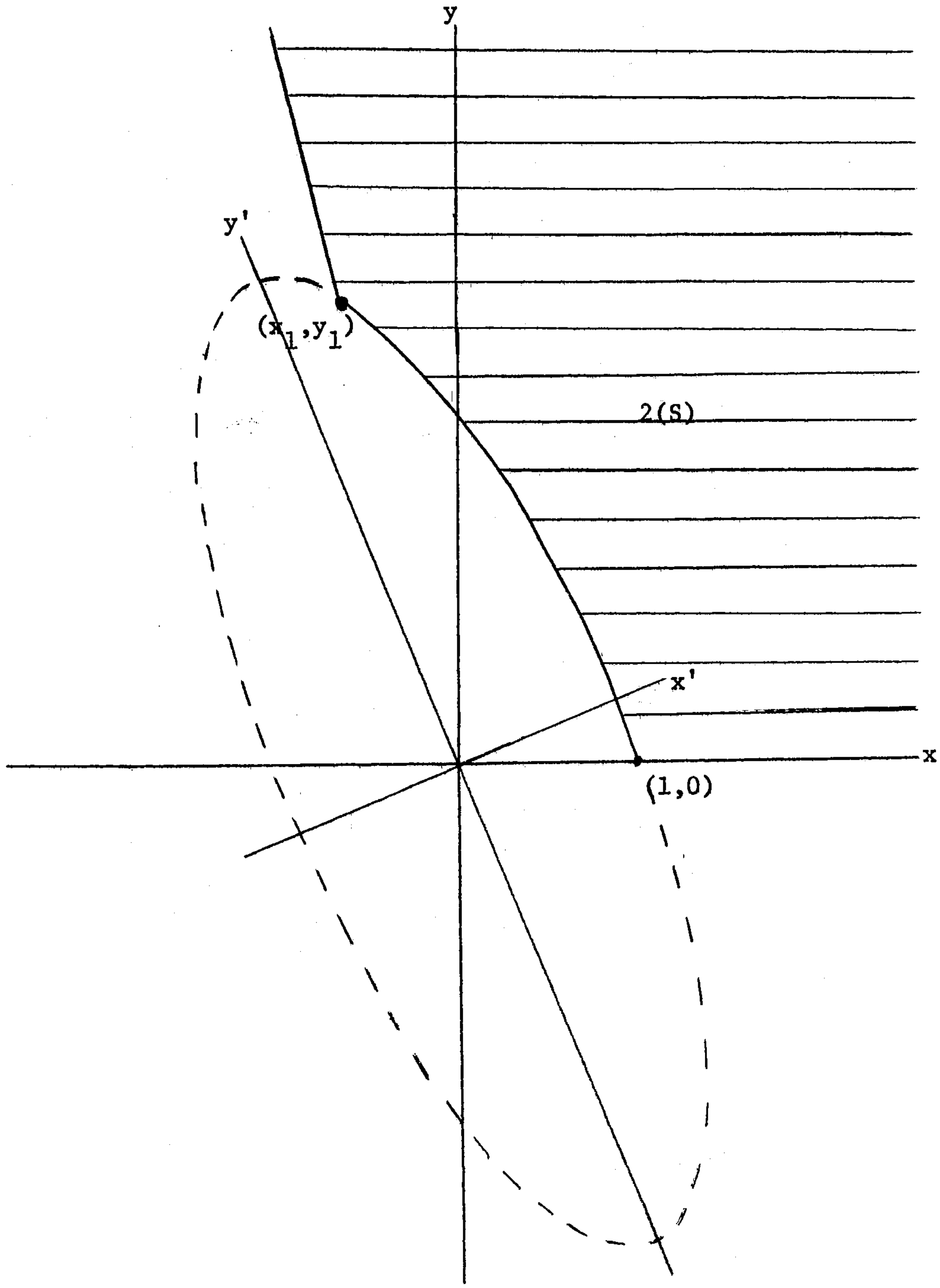


Figure 11.

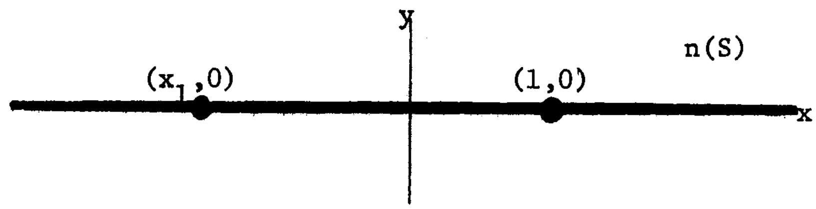


Figure 12.

Example 2.5: In E_2 let $S = \{(0,0),(1,0)\}$. Find $n(S)$.

Solution: Let $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha^n + \beta^n = 1$. Then $(\beta,0) = \alpha(0,0) + \beta(1,0) \in n(S)$, where $0 \leq \beta \leq 1$. As in Example 2.4, $n(S) = \{(\beta,0)\colon \beta \geq 0\}$ (cf. Figure 13).

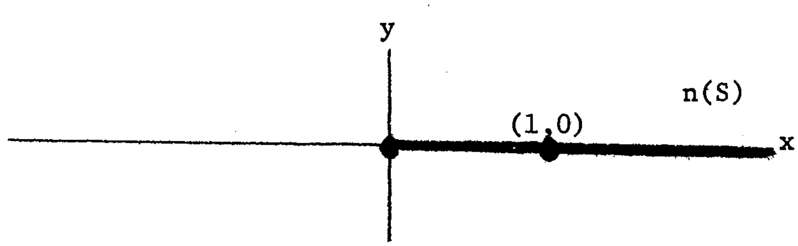


Figure 13.

Additional Properties

This section contains several additional properties of n-convex sets and the n-convex hull of a set. The first proposition characterizes the n-convex hull of the union of two n-convex sets.

Proposition 2.3: Let C and K be two nonempty n-convex sets, then

$$n(C \cup K) = \bigcup\{n - cr[x,y]: x \in C \text{ and } y \in K\}.$$

Proof: Clearly $\bigcup\{n - cr[x,y]: x \in C \text{ and } y \in K\}$ is contained in $n(C \cup K)$. Let $z \in n(C \cup K)$. By Theorem 2.1,

$$z = \sum \alpha_i x_i + \sum \beta_i y_i,$$

where $\alpha_i \geq 0$, $\beta_i \geq 0$,

$$\sum \alpha_i^n + \sum \beta_i^n = 1,$$

$x_i \in C$ and $y_i \in K$. If $\alpha_i = 0$ for all i, then $z \in K$. Hence, for any $v \in C$, $z \in n - cr[v,z]$. Therefore, suppose not all $\alpha_i = 0$. Likewise, assume not all $\beta_i = 0$. Let

$$\alpha = \sqrt[n]{\sum \alpha_i^n} \quad \text{and} \quad \beta = \sqrt[n]{\sum \beta_i^n}.$$

Then

$$z = \alpha \sum \left(\frac{\alpha_i}{\alpha}\right) x_i + \beta \sum \left(\frac{\beta_i}{\beta}\right) y_i.$$

Since

$$\frac{\alpha_i}{\alpha} \geq 0 \quad \text{and} \quad \sum \left(\frac{\alpha_i}{\alpha}\right)^n = 1,$$

then

$$x = \sum \left(\frac{\alpha_i}{\alpha} \right) x_i \; \varepsilon \; C.$$

Likewise,

$$y = \sum \left(\frac{\beta_i}{\beta} \right) y_i \; \varepsilon \; K.$$

Also, $\alpha^n + \beta^n = 1$ implies $z \; \varepsilon \; n - cr[x,y]$ and the proof is complete.

Note this analogous to the result that the convex hull of the union of two convex is the union of the line segments whose endpoints are such that one is in one of the sets and the other endpoint is in the other set.

In the next proposition the n-convex hull of a convex set is shown to be convex.

Proposition 2.4: If C is a convex set, then

$$n(C) = \{\alpha x\colon \alpha \geq 1, \; x \; \varepsilon \; C\},$$

which is a convex set.

Proof: First it will be shown that $\{\alpha x\colon \alpha \geq 1, \; x \; \varepsilon \; C\}$ is convex. Let $\alpha x, \beta y \; \varepsilon \; \{\alpha x\colon \alpha \geq 1, \; x \; \varepsilon \; C\}$ and $\lambda \; \varepsilon \; [0,1]$. Let

$$\sigma = \frac{\lambda\alpha}{\lambda\alpha + \beta(1 - \lambda)} \quad \text{and} \quad \gamma = \lambda\alpha + \beta(1 - \lambda).$$

Then $0 \leq \sigma \leq 1$ and $\gamma \geq 1$. Hence, $\sigma x + (1 - \sigma)y \; \varepsilon \; C$ and $\gamma(\sigma x + (1 - \sigma)y) \; \varepsilon \; \{\alpha x\colon \alpha \geq 1, \; x \; \varepsilon \; C\}$. Since

$$1 - \sigma = \frac{\lambda\alpha + \beta(1 - \lambda) - \lambda\alpha}{\lambda\alpha + \beta(1 - \lambda)} = \frac{\beta(1 - \lambda)}{\lambda\alpha + \beta(1 - \lambda)},$$

then

$$\gamma(\sigma x + (1 - \sigma)y) = \gamma\sigma x + \gamma(1 - \sigma)y = \lambda\alpha x + (1 - \lambda)\beta y,$$

which implies that $\{\alpha x\colon \alpha \geq 1,\ x \in C\}$ is convex.

By Theorem 2.4 $\{\alpha x\colon \alpha \geq 1,\ x \in C\} \subset n(C)$. By induction on m inclusion in the other direction will be shown. Suppose $m = 1$, then $y = \alpha x \in B(C) = n(C)$. Thus, $\alpha = 1$ and $x \in C$, which implies that $y \in \{\alpha x\colon \alpha \geq 1,\ x \in C\}$. Assume the induction hypothesis, that is, for every $y \in B(C)$ such that

$$y = \sum^{m-1} \alpha_i x_i,$$

where $\alpha_i > 0$,

$$\sum^{m-1} \alpha_i^n = 1$$

and $x_i \in C$ then $y \in \{\alpha x\colon \alpha \geq 1,\ x \in C\}$. Let

$$y = \sum^{m} \alpha_i x_i,$$

where $\alpha_i > 0$,

$$\sum^{m} \alpha_i^n = 1$$

and $x_i \in C$. Let

$$\alpha = \sqrt[n]{\sum^{m-1} \alpha_i^n},$$

then

$$y = \alpha \sum^{m-1} \left(\frac{\alpha_i}{\alpha}\right) x_i + (1 - \alpha) \frac{\alpha_m}{1 - \alpha} x_m.$$

Since

$$\sum^{m-1} \left(\frac{\alpha_i}{\alpha}\right)^n = 1,$$

then

$$\sum^{m-1} \left(\frac{\alpha_i}{\alpha}\right) x_i \in \{\alpha x: \alpha \geq 1, x \in C\}.$$

Also, $0 < \alpha < 1$ implies $0 \leq \alpha^n \leq \alpha$. Hence, $1 - \alpha^n \geq 1 - \alpha > 0$ which implies that

$$\frac{1}{1 - \alpha^n} \leq \frac{1}{1 - \alpha}.$$

Therefore,

$$\frac{\alpha_m}{1 - \alpha} \geq \frac{\alpha_m}{1 - \alpha^n} = \frac{\alpha_m}{\sum^{m} \alpha_i^n - \sum^{m-1} \alpha_i^n} = \frac{\alpha_m}{\alpha_m^n} = (\alpha_m)^{1-n} \geq 1.$$

As a result,

$$\frac{\alpha_m}{1 - \alpha} x_m \in \{\alpha x: \alpha \geq 1, x \in C\}.$$

Then $y \in \{\alpha x: \alpha \geq 1, x \in C\}$ which implies $n(C) = \{\alpha x: \alpha \geq 1, x \in C\}$.

Figure 14 illustrates the converse is not true. That is, C a set in the xy-plane is not convex but 2(C) is convex. Recall that 2(C) is inverse starlike from the origin. The examples of the last section show that 2(C) is as indicated.

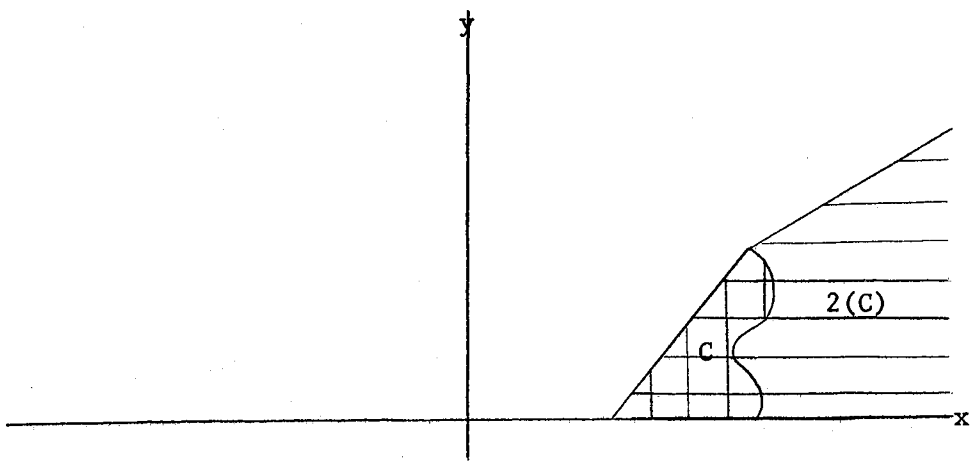


Figure 14.

Next the convex hull of an n-convex set is shown to be n-convex.

Proposition 2.5: If K is a n-convex set, then conv(K) is n-convex.

Proof: The conv(K) will be n-convex if and only if

$$n(\text{conv}(K)) = \text{conv}(K).$$

Clearly, $\text{conv}(K) \subset n(\text{conv}(K))$. Let $x \in n(\text{conv}(K))$. By Proposition 2.4, there exists $a \in \text{conv}(K)$ and $\alpha \geq 1$ such that $x = \alpha a$. Since

$a \in \mathrm{conv}(K)$,

$$a = \sum \alpha_i a_i,$$

where $a_i \in K$, $\alpha_i \geq 0$ and

$$\sum \alpha_i = 1.$$

Therefore,

$$x = \alpha a = \alpha \sum \alpha_i a_i = \sum \alpha_i (\alpha a_i).$$

Since K is n-convex, Theorem 2.2 implies $\alpha a_i \in K$. Hence,

$$x \in \mathrm{conv}(K) \text{ and } \mathrm{conv}(K) = n(\mathrm{conv}(K)).$$

Propositions 2.4 and 2.5 imply the following:

Proposition 2.6: For any set K, $n(\mathrm{conv}(K)) = \mathrm{conv}(n(K))$.

Proof: Proposition 2.4 implies $\mathrm{conv}(n(D)) = n(D)$ when D is convex. Clearly, $K \subset \mathrm{conv}(K)$. Hence, $n(K) \subset n(\mathrm{conv}(K))$. Thus,

$$\mathrm{conv}(n(K)) \subset \mathrm{conv}(n(\mathrm{conv}(K)) = n(\mathrm{conv}(K))$$

by the above statement.

Also, Proposition 2.5 implies $n(\mathrm{conv}(D)) = \mathrm{conv}(D)$ when D is n-convex. Now $K \subset n(K)$ and $\mathrm{conv}(K) \subset \mathrm{conv}(n(K))$. Hence,

$$n(\mathrm{conv}(K)) \subset n(\mathrm{conv}(n(K))) = \mathrm{conv}(n(K)).$$

Therefore, $n(\mathrm{conv}(K)) = \mathrm{conv}(n(K))$.

It was noticed earlier that an n-convex set is inverse starlike from the origin. Example 2.1 showed that the converse was not true. However, if a set S is inverse starlike from the origin and convex, then Proposition 2.4 implies the set S is n-convex. Clearly, the converse of this statement is not true. The next two results are similar to a couple of results obtained by Allen (cf. [1], pp. 16-17).

Comment 2.1: If K is n-convex, then $K + K \subset K$.

Proof: If $x + y \in K + K$, then

$$x + y = \left(\frac{1}{2}\right)^{-\frac{1}{n}}\left(\left(\frac{1}{2}\right)^{\frac{1}{n}} x + \left(\frac{1}{2}\right)^{\frac{1}{n}} y\right).$$

Since

$$\left(\left(\frac{1}{2}\right)^{\frac{1}{n}}\right)^{n} + \left(\left(\frac{1}{2}\right)^{\frac{1}{n}}\right)^{n} = \frac{1}{2} + \frac{1}{2} = 1,$$

then

$$\left(\frac{1}{2}\right)^{\frac{1}{n}} x + \left(\frac{1}{2}\right)^{\frac{1}{n}} y \in K.$$

Also,

$$\left(\frac{1}{2}\right)^{-\frac{1}{n}} > 1.$$

Hence, $x + y \in K$.

Proposition 2.7: Let $0 \in K$. Then K is a convex cone whose vertex is 0 if and only if K is n-convex.

Proof: Suppose K is a convex cone whose vertex is 0. Since K is convex, $n(K) = \{\alpha x\colon x \in K, \alpha \geq 1\} = K$. Hence, K is n-convex.

Now suppose K is n-convex. By the previous comment, $K + K \subset K$. Let $\alpha > 0$ and $x \in K$. If $\alpha \geq 1$, then $\alpha x \in K$. Suppose $0 < \alpha < 1$, then $0 < \alpha^n < 1$. Let $\beta^n = 1 - \alpha^n$. Then $\alpha x = \alpha x + \beta \cdot 0 \in K$. Hence, $\alpha K \subset K$ and K is a convex cone with vertex 0.

A Separation Theorem

Let S be a subset of a linear space L. The <u>core</u> of S, denoted by $Cr(S)$ is the set of all $x \in S$, such that for all $y \in L$, $y \neq x$, there exists $z \in \text{intv } xy$ such that $xz \subset S$.

A standard separation theorem is as follows: If A and B are two convex subsets of a linear space L, where $Cr(B) \neq \emptyset$, $A \neq \emptyset$ and $A \cap Cr(B) = \emptyset$, then there exists a hyperplane which separates A and B [10]. This section contains a similar result for n-convex sets. Consider first the following lemma.

<u>Lemma 2.1</u>: Let K and C be two n-convex sets in a linear space L such that $K \cap C = \emptyset$, then $\text{conv}(K) \cap \text{conv}(C) = \emptyset$.

Proof: Suppose $x \in \text{conv}(K) \cap \text{conv}(C)$. Then

$$x = \sum \alpha_i x_i,$$

where $\alpha_i \geq 0$, $x_i \in K$ and

$$\sum \alpha_i = 1.$$

Also,

$$x = \sum \beta_i y_i,$$

where $\beta_i \geq 0$, $y_i \varepsilon C$ and

$$\sum \beta_i = 1.$$

Suppose $\alpha_\ell > 0$ and $\beta_m > 0$, then there exists $\sigma \geq 1$ such that $\sigma\alpha_\ell \geq 1$ and $\sigma\beta_m \geq 1$. Therefore,

$$\sigma x = \sum (\sigma\alpha_i) x_i,$$

where $\sigma\alpha_i \geq 0$, $x_i \varepsilon K$ and

$$\sum (\sigma\alpha_i)^n \geq 1.$$

Also,

$$\sigma x = \sum (\sigma\beta_i) y_i,$$

where $\sigma\beta_i \geq 0$, $y_i \varepsilon C$ and

$$\sum (\sigma\beta_i)^n \geq 1.$$

This implies $\sigma x \varepsilon K \cap C$, a contradiction. Hence,

$$\text{conv}(K) \cap \text{conv}(C) = \emptyset.$$

With the aid of Lemma 2.1, the following separation theorem is possible.

Theorem 2.5: Let K and C be two n-convex sets in linear space L such that $Cr(C) \neq \emptyset$, $K \neq \emptyset$ and $K \cap C = \emptyset$, then there exists a hyperplane that separates K and C.

Proof. Since $Cr(C) \subset C \subset conv(C)$, then $cr(conv(C)) \neq \emptyset$. Also, $K \neq \emptyset$ implies $conv(K) \neq \emptyset$. Further, $K \cap C \neq \emptyset$ implies that $conv(K) \cap conv(C) = \emptyset$; hence, $conv(K) \cap Cr(conv(C)) = \emptyset$. Therefore, there exists a hyperplane that separates $conv(K)$ and $conv(C)$; hence, K and C.

Complementary n-Convex Sets

Let C and D be subsets of a linear space L. Then C and D are complementary if and only if $L = C \cup D$ and $C \cap D = \emptyset$.

In this section two disjoint n-convex subsets of a linear space L are shown to be contained in two complementary n-convex subsets of L. The following lemma is similar to Proposition 1.5.

Lemma 2.2: Suppose x, y, z are three distinct points of a linear space L. If $u \in n(\{x,y\})$ and $v \in n(\{y,z\})$, then

$$n(\{u,z\}) \cap n(\{x,v\}) \neq \emptyset.$$

Proof: Since $u \in n(\{x,y\})$, then $u = \lambda(\alpha x + \beta y)$, where $\lambda \geq 1$, $\alpha \geq 0$, $\beta \geq 0$ and $\alpha^n + \beta^n = 1$. Also, $v \in n(\{y,z\})$ implies that $v = \omega(\sigma z + \gamma y)$, where $\omega \geq 1$, $\sigma \geq 0$, $\gamma \geq 0$ and $\sigma^n + \gamma^n = 1$. If $\beta = 0$, then $u = \lambda\alpha x = \lambda(\alpha x + \beta v) \in n(\{x,v\})$ and

$$n(\{u,z\}) \cap n(\{x,v\}) \neq \emptyset.$$

Suppose $\beta > 0$. Then

$$\sqrt[n]{\gamma^n\alpha^n + \beta^n} > 0.$$

Let

$$\theta = \frac{\beta}{\sqrt[n]{\gamma^n\alpha^n + \beta^n}} \quad \text{and} \quad \phi = \frac{\sigma\beta}{\sqrt[n]{\gamma^n\alpha^n + \beta^n}}.$$

Then $\theta \geq 0$ and $\phi \geq 0$. As in Proposition 1.5

$$\sqrt[n]{1 - \theta^n} = \frac{\gamma\alpha}{\sqrt[n]{\gamma^n\alpha^n + \beta^n}} \geq 0,$$

$$\sqrt[n]{1 - \phi^n} = \frac{\gamma}{\sqrt[n]{\gamma^n\alpha^n + \beta^n}} \geq 0,$$

and

$$\sqrt[n]{1 - \phi^n}\,(\alpha x + \beta y) + \phi z = \theta(\sigma z + \gamma y) + \sqrt[n]{1 - \theta^n}\,x.$$

Let

$$t = \sqrt[n]{1 - \phi^n}\,(\alpha x + \beta y) + \phi z = \theta(\sigma z + \gamma y) + \sqrt[n]{1 - \theta^n}\,x.$$

Let $\xi = \max\{2\omega^2, 2\lambda^2\}$. Consider

$$\xi t = \sqrt[n]{1 - \phi^n}\,\xi(\alpha x + \beta y) + \phi\xi z.$$

Since $\xi > \lambda^2 \geq 1$, there eixsts $\xi' \geq 1$ such that $\xi = \xi'\lambda^2$. Hence,

$$\xi t = \sqrt[n]{1 - \phi^n}\,\xi'\lambda^2(\alpha x + \beta y) + \phi\xi z = \left(\sqrt[n]{1 - \phi^n}\,\xi'\lambda\right)u + \phi\xi z,$$

where

$$\sqrt[n]{1 - \phi^n}\ \xi'\lambda \geq 0,$$

$\phi\xi \geq 0$ and

$$\left(\sqrt[n]{1 - \phi^n}\ \xi'\lambda\right)^n + (\phi\xi)^n = (1 - \phi^n)\xi'^n\lambda^n + \phi^n\xi^n \geq (1 - \phi^n) + \phi^n = 1.$$

Hence, $\xi t \in n(\{u,z\})$. Likewise, $\xi t \in n(\{x,v\})$. Therefore,

$$n(\{u,z\}) \bigcap n(\{x,v\}) \neq \emptyset.$$

The proof of the next lemma is similar to that of Proposition 2.3. The basic difference is that a single point is not an n-convex set.

Lemma 2.3: If K is a n-convex subset of a linear space L and $x \in L$, then

$$n(K \bigcup \{x\}) = \{\alpha x + \beta y\colon \alpha \geq 0,\ \beta \geq 0,\ \alpha^n + \beta^n \geq 1,\ y \in K\}.$$

Proof: Clearly,

$$\{\alpha x + \beta y\colon \alpha \geq 0,\ \beta \geq 0,\ \alpha^n + \beta^n \geq 1,\ y \in K\} \subset n(K \bigcup \{x\}).$$

Let $z \in n(K \bigcup \{x\})$. By Theorem 2.3

$$z = \sum_1^m \alpha_i x_i + \alpha x,$$

where $\alpha_i \geq 0$, $\alpha \geq 0$,

$$\sum_1^m \alpha_i^n + \alpha^n \geq 1$$

and $x_i \in K$. The result clearly holds if all the $\alpha_i = 0$. Suppose not all the $\alpha_i = 0$. Let

$$\lambda = \sqrt[n]{\sum_1^m \alpha_i^n}\ .$$

Then

$$z = \lambda \sum_1^m \left(\frac{\alpha_i}{\lambda}\right) x_i + \alpha x.$$

As in Proposition 2.3,

$$\sum_1^m \left(\frac{\alpha_i}{\lambda}\right) x_i \in K.$$

Since $\lambda^n + \alpha^n \geq 1$, $z \in \{\alpha x + \beta y\colon \alpha \geq 0,\ \beta \geq 0,\ \alpha^n + \beta^n \geq 1,\ y \in K\}$.

The following result is analogous to a result for convex sets found in Valentine [10]. With the use of the two previous lemmas, the proposition is proved as in Valentine.

Proposition 2.8: If A and B are disjoint n-convex sets in a linear space L, then there exists complementary n-convex sets C and D of L such that $A \subset C$ and $B \subset D$.

Proof: Let $\mathfrak{R} = \{(A_i,B_i)\colon A_i$ and B_i are n-convex, $A \subset A_i$, $B \subset B_i$ and $A_i \cap B_i = \emptyset\}$. Partially order $\mathfrak{R}$ by $(A_i,B_i) < (A_j,B_j)$ if and only if $A_i \subset A_j$ and $B_i \subset B_j$. Now let

$$\mathfrak{R}' = \{(A_\sigma, B_\sigma) : \sigma \in \Omega\}$$

be a linearly ordered subset of $\mathfrak{R}$. Consider $(\bigcup A_\sigma, \bigcup B_\sigma)$, where $\sigma \in \Omega$. Clearly, for every $(A_\alpha, B_\alpha) \in \mathfrak{R}'$, $(A_\alpha, B_\alpha) < (\bigcup A_\sigma, \bigcup B_\sigma)$. Also, $A \subset \bigcup A_\sigma$ and $B \subset \bigcup B_\sigma$. Consider $\bigcup A_\sigma$. Let $x,y \in \bigcup A_\sigma$. Since $\mathfrak{R}'$ is linearly ordered, there exists an $\sigma' \in \Omega$ such that $x,y \in A_{\sigma'}$. Hence, $\alpha x + \beta y \in A_{\sigma'} \subset \bigcup A_\sigma$, for $\alpha \geq 0$, $\beta \geq 0$ and $\alpha^n + \beta^n = 1$. Therefore, $\bigcup A_\sigma$ is n-convex. Likewise, $\bigcup B_\sigma$ is n-convex. Suppose $x \in [\bigcup A_\sigma] \cap [\bigcup B_\sigma]$. Then there exists $\alpha, \beta \in \Omega$ such that $x \in A_\alpha$ and $x \in B_\beta$. Without loss of generality, assume $(A_\alpha, B_\alpha) < (A_\beta, B_\beta)$. Then $x \in A_\beta \cap B_\beta$, a contradiction. Hence, $[\bigcup A_\sigma] \cap [\bigcup B_\sigma] = \emptyset$, implies that $(\bigcup A_\sigma, \bigcup B_\sigma) \in \mathfrak{R}$. Therefore, every linearly ordered subset $\mathfrak{R}'$ of $\mathfrak{R}$ has an upper bound in $\mathfrak{R}$. As a result, there exists a maximal element (C,D) in $\mathfrak{R}$. Since $(C,D) \in \mathfrak{R}$, then $A \subset C$, $B \subset D$, both C and D are n-convex and $C \cap D = \emptyset$. It remains to be shown that $C \cup D = L$.

Suppose $x \in L \setminus (C \cup D)$. Consider the n-convex sets $n(C \cup \{x\})$ and $n(D \cup \{x\})$. Notice that $A \subset C \cup \{x\}$ and $B \subset D \cup \{x\}$. Consider $(n(C \cup \{x\}), D)$. Since (C,D) is maximal in $\mathfrak{R}$, then $(n(C \cup \{x\}), D) \notin \mathfrak{R}$. This means $n(C \cup \{x\}) \cap D \neq \emptyset$. Therefore, there exists a $d_1 \in n(C \cup \{x\}) \cap D$. Likewise, there exists a $c_1 \in n(D \cup \{x\}) \cap C$. By the last lemma $d_1 = \alpha c + \beta x$, where $c \in C$, $\alpha \geq 0$, $\beta \geq 0$ and $\alpha^n + \beta^n \geq 1$. Also, $c_1 = \sigma d + \gamma x$, where $d \in D$, $\sigma \geq 0$, $\gamma \geq 0$ and $\sigma^n + \gamma^n \geq 1$. Since $x \notin C \cup D$, then $x \neq c$ and $x \neq d$. Also, $C \cap D = \emptyset$ implies $c \neq d$. Therefore, x, c, d are three distinct points of L, $d_1 \in n(\{x,c\})$ and $c_1 \in n(\{x,d\})$. By Lemma 2.2, $n(\{d_1,d\}) \cap n(\{c_1,c\}) \neq \emptyset$. However, $n(\{d_1,d\}) \subset D$ and

$n(\{c_1, c\}) \subset C$. This implies that $C \cap D \neq \emptyset$, a contradiction. Hence, $L = C \cup D$.

The following proposition shows that the complementary n-convex sets in Proposition 2.8 are actually complementary convex sets.

Proposition 2.9: If C and D are nonempty complementary n-convex subsets of a linear space L, then both C and D are convex.

Proof: Consider C. If $C = \{0\}$, then for any $x \in L$, $x \neq 0$, both x and $-x$ belong to D. However, as in Example 2.4, this implies $0 \in D$, a contradiction. If $\lambda \geq 1$, then $\lambda v \in C$. Suppose $0 < \lambda < 1$. If $\lambda v \notin C$, then $\lambda v \in D$. Hence, $v = \frac{1}{\lambda}(\lambda v) \in D$, a contradiction. Therefore, $\lambda v \in C$ for all $\lambda > 0$.

Now let $x, y \in C$ and $\alpha \in [0,1]$. Let $z = \alpha x + (1 - \alpha)y$. Since $\alpha^n + (1 - \alpha)^n > 0$, then

$$z = \sqrt[n]{\alpha^n + (1 - \alpha)^n} \left(\frac{\alpha}{\sqrt[n]{\alpha^n + (1 - \alpha)^n}} x + \frac{1 - \alpha}{\sqrt[n]{\alpha^n + (1 - \alpha)^n}} y \right).$$

Now

$$\frac{\alpha}{\sqrt[n]{\alpha^n + (1 - \alpha)^n}} x + \frac{1 - \alpha}{\sqrt[n]{\alpha^n + (1 - \alpha)^n}} y \in C.$$

Hence, $z \in C$ and C is convex. Likewise, D is convex.

CHAPTER III

APPLICATIONS OF n-CONVEX SETS

For every $x = (x_1,x_2,x_3,x_4) \in E_4^+$, define

$$p(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Then $p \in \mathcal{P}_2$, the nonnegative, continuous, 2-homogeneous, superadditive functions defined on E_4^+. Let $f_i(x) = x_i^2$, where $i = 1, 2, 3, 4$. Suppose there exists $\alpha \geq 0$ such that $f_1 = \alpha p$. Then

$$1 = f_1((1,0,0,0)) = \alpha p((1,0,0,0)) = \alpha$$

and

$$1 = f_1((1,1,0,0)) = \alpha p(1,1,0,0) = 2\alpha,$$

which is a contradiction. Therefore, there does not exists $\alpha \geq 0$ such that $f_1 = \alpha p$. Likewise, there does not exists $\alpha \geq 0$ such that $f_i = \alpha p$ for any i. Hence, p is not an extremal element of the convex cone $\mathcal{P}_2$.

Also, notice that

$$p(e_1) = p(e_2) = p(e_3) = p(e_4) = 1,$$

where e_k is that vector having all zero coordinates except the k-th coordinate which is 1.

If $p \in \mathcal{P}_n$, let

$$[p:\alpha] = \{x: p(x) = \alpha\}.$$

Note that $[p:\alpha]$ would be a hyperplane if p were a linear functional. It would be desirable to know more about $[p:\alpha]$. For example, the n-convex hull of $[p:\alpha]$ is given in the following proposition.

Proposition 3.1: If $p \in \mathcal{P}_n$, then $n([p:1]) = \mathrm{Lev}_1 p$.

Proof: Clearly $n([p:1]) \subset \mathrm{Lev}_1 p$. Let $x \in \mathrm{Lev}_1 p$. Then by Theorem 2.2,

$$x = \sqrt[n]{p(x)}\,\frac{x}{\sqrt[n]{p(x)}} \in n([p:1]).$$

Hence, $n([p:1]) = \mathrm{Lev}_1 p$.

Now for the function $p(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2$,

$$\{e_1,e_2,e_3,e_4\} \subset [p:1].$$

Let $x = (x_1,x_2,x_3,x_4) \in \mathrm{Lev}_1 p$. Notice

$$x = \sum_1^4 x_i e_i.$$

Also,

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = p(x) \geq 1.$$

Hence, $x \in 2(\{e_1,e_2,e_3,e_4\})$. Therefore, $\mathrm{Lev}_1 p = 2(\{e_1,e_2,e_3,e_4\})$, a finite subset of $[p:1]$. The following question arises: Is there any relation between the fact that $\mathrm{Lev}_1 p$ is the n-hull of a finite subset of $[p:1]$ and the fact that p is not an extremal element of the cone $\mathcal{P}_n$?

The following concept will be needed.

<u>Definition 3.1</u>: If A is a subset of a linear space L, the <u>conical hull</u> of A, denoted by $\mathrm{coni}(A)$, is the intersection of all convex cones containing A.

The conical hull of A may be characterized as

$$\mathrm{coni}(A) = \left\{ \sum_1^m \lambda_i x_i : x_i \in A, \lambda_i > 0 \right\}.$$

[7]. This characterization is used in the following:

<u>Theorem 3.1</u>: If $p \in \mathcal{P}_n$ such that $p \neq 0$ and

$$S = \{a_1, \ldots, a_m\} \subset [p:1]$$

such that $n(S) = \mathrm{Lev}_1 p$, then for every $x \in E^+_{n^2}$ such that $x \neq 0$, $p(x) > 0$.

Proof: Let $x \in E^+_{n^2}$ such that $x \neq 0$. Suppose $p(x) = 0$. Let

$$A = \left\{ \lambda \sum_1^m \alpha_i a_i : \alpha_i \geq 0, \sum \alpha_i^n = 1, \lambda > 0 \right\} \cup \{0\}.$$

Notice that $n(S) \subset A$. Clearly, $A \subset \mathrm{coni}(S) \cup \{0\}$. Let

$$y \in \mathrm{coni}(S) \cup \{0\}.$$

If $y = 0$, then $y \in A$. Suppose $y \neq 0$, then

$$y = \sum_{1}^{m} \alpha_i a_i,$$

where $\alpha_i \geq 0$. Notice that not all of the α_i are zero. If

$$\sum_{1}^{m} \alpha_i^n = 1,$$

then $y \in A$. Suppose

$$\sum_{1}^{m} \alpha_i^n \neq 1.$$

Clearly,

$$\sum_{1}^{m} \alpha_i^n \neq 0.$$

Let

$$\alpha = \sqrt[n]{\sum_{1}^{m} \alpha_i^n}.$$

Then

$$y = \alpha \sum_{1}^{m} \left(\frac{\alpha_i}{\alpha}\right) a_i \quad \text{and} \quad \sum_{1}^{m} \left(\frac{\alpha_i}{\alpha}\right)^n = 1.$$

Hence, $y \in A$. Therefore, $A = \text{coni}(S) \cup \{0\}$. Now let $y \in A \setminus \{0\}$. Then

$$y = \lambda \sum_{1}^{m} \alpha_i a_i,$$

$\lambda > 0$, $\alpha_i \geq 0$,

$$\sum_{1}^{m} \alpha_i^n = 1.$$

Again notice that not all the α_i are zero. Thus,

$$p(y) = p\left(\lambda \sum_{1}^{m} \alpha_i a_i\right) = \lambda^n p\left(\sum_{1}^{m} \alpha_i a_i\right) \geq \lambda^n \sum_{1}^{m} \alpha_i^n \, p(a_i)$$

$$= \lambda^n \sum_{1}^{m} \alpha_i^n = \lambda^n > 0.$$

Hence, $y \neq x$ since $p(x) = 0$. The set A is a closed set since S is finite and hence there exists an open neighborhood N of x such that $N \cap A = \emptyset$. Let $z \in N \cap \text{int } E^+_{n^2}$. Since $p \neq 0$, $p(z) > 0$. Hence,

$$p\left(\frac{z}{\sqrt[n]{p(z)}}\right) = 1,$$

which implies that

$$\frac{z}{\sqrt[n]{p(z)}} \in \text{Lev}_1 p = n(S) \subset A.$$

Therefore, $z \in A$. This is a contradiction. Hence, $p(x) > 0$.

The next theorem shows that each positive axis of $E^+_{n^2}$ contains an element of the set S of Theorem 3.1.

Theorem 3.2: Let $p \in \mathcal{P}_n$ such that $p \neq 0$. Suppose there exists a set

$$S = \{a_1, \ldots, a_m : a_i = (a_{i1}, \ldots, a_{in^2})\} \subset [p:1]$$

such that $n(S) = \mathrm{Lev}_1 p$. Then for every $i \in \{1, \ldots, n^2\}$

$$\mathrm{Ray}(e_i) \cap S \neq \emptyset,$$

where $\mathrm{Ray}(e_i) = \{\alpha e_i : \alpha \geq 0\}$.

Proof: First notice $0 \notin S$ since $S \subset [p:1]$. Suppose there exists k such that $\mathrm{Ray}(e_k) \cap S = \emptyset$. Let $x \in \mathrm{Ray}(e_k)$ such that $x \neq 0$. By the previous theorem $p(x) > 0$. Hence,

$$\frac{x}{\sqrt[n]{p(x)}} \in [p:1].$$

Thus, without loss of generality assume $p(x) = 1$. Since $x \in \mathrm{Lev}_1 p = n(S)$,

$$x = \sum_1^m \alpha_i a_i$$

where $\alpha_i \geq 0$, not all the $\alpha_i = 0$ and

$$\sum_{1}^{m} \alpha_i^n = 1.$$

Therefore,

$$x = \sum_{1}^{m} \alpha_i a_i = \sum_{1}^{m} \alpha_i (a_{i1}, \ldots, a_{in^2}) = \left(\sum_{1}^{m} \alpha_i a_{i1}, \ldots, \sum_{1}^{m} \alpha_i a_{in^2}\right).$$

Also, $x \in \mathrm{Ray}(e_k)\setminus\{0\}$ implies there exists $\lambda > 0$ such that $x = \lambda e_k$. Hence,

$$\sum_{1}^{m} \alpha_i a_{i1} = 0, \ldots, \sum_{1}^{m} \alpha_i a_{ik} = \lambda, \ldots, \sum_{1}^{m} \alpha_i a_{in^2} = 0.$$

Since $\lambda > 0$, there exists $j \in \{1, \ldots, n^2\}$ such that $a_{jk} > 0$ and $\alpha_j > 0$. Consider

$$\begin{array}{ccccccccc}
\alpha_1 a_{11} & + & \cdots & + & \alpha_j a_{j1} & + & \cdots & + & \alpha_m a_{m1} & = 0 \\
\vdots & & & & & & & & & \vdots \\
\alpha_1 a_{1k} & + & \cdots & + & \alpha_j a_{jk} & + & \cdots & + & \alpha_m a_{mk} & = \lambda \\
\vdots & & & & & & & & & \vdots \\
\alpha_1 a_{1\ell} & + & \cdots & + & \alpha_j a_{j\ell} & + & \cdots & + & \alpha_m a_{m\ell} & = 0 \\
\vdots & & & & & & & & & \vdots \\
\alpha_1 a_{1n^2} & + & \cdots & + & \alpha_j a_{jn^2} & + & \cdots & + & \alpha_m a_{mn^2} & = 0.
\end{array}$$

Notice there exists $\ell \neq k$ such that $a_{j\ell} > 0$, since otherwise $a_j = (a_{j1}, \ldots, a_{jk}, \ldots, a_{jn^2}) = (0, \ldots, a_{jk}, \ldots, 0) = a_{jk} e_k \in \mathrm{Ray}(e_k)$. Hence, $\alpha_j a_{j\ell} > 0$. This implies that

$$\sum_{1}^{m} \alpha_i a_{i\ell} \neq 0,$$

a contradiction. Therefore, $\mathrm{Ray}(e_k) \cap S \neq \emptyset$ for every $k \in \{1, \ldots, n^2\}$.

Notice that S contains at least n^2 elements since the sets $\mathrm{Ray}(e_i)$ have only the origin as a common point and S does not meet the origin.

The next theorem attempts to answer the question raised at the first part of this chapter.

Theorem 3.3: Let $p \in \mathcal{P}_n$ such that $p \neq 0$. If there exists a set $S = \{a_1, \ldots, a_{n^2}\} \subset [p:1]$ such that $n(S) = \mathrm{Lev}_1 p$, then p is not an extremal element of $\mathcal{P}_n$.

Proof: By the previous theorem, it can be assumed with loss of generality that $a_i = (0, \ldots, a_{ii}, \ldots 0) = a_{ii}e_i$, where $a_{ii} > 0$. Let $x \in E^+_{n^2}\backslash\{0\}$. Then by Theorem 3.1 $p(x) > 0$. Since

$$\frac{x}{\sqrt[n]{p(x)}} \in \mathrm{Lev}_1 p = n(S),$$

it follows from Theorem 2.4 that

$$\frac{x}{\sqrt[n]{p(x)}} = \lambda \sum_{1}^{n^2} \alpha_i a_i,$$

where $\lambda \geq 1$, $\alpha_i \geq 0$ and

$$\sum_{1}^{n^2} \alpha_i^n = 1.$$

Now

$$1 = p\left(\frac{x}{\sqrt[n]{p(x)}}\right) = p\left(\lambda \sum_{1}^{n^2} \alpha_i a_i\right) \geq \lambda^n \sum_{1}^{n^2} \alpha_i^n\, p(a_i) = \lambda^n \sum_{1}^{n^2} \alpha_i^n$$

$$= \lambda^n \geq 1.$$

Therefore, $\lambda^n = 1$ which implies $\lambda = 1$. Hence,

$$x = \sqrt[n]{p(x)} \sum_{1}^{n^2} \alpha_i a_i.$$

Also, suppose

$$x = \sqrt[n]{p(x)} \sum_{1}^{n^2} \alpha_i' a_i.$$

That is, suppose the representation is not unique. Clearly,

$$\sum_{1}^{n^2} \alpha_i a_i = \sum_{1}^{n^2} \alpha_i' a_i.$$

Therefore,

$$0 = \sum_{1}^{n^2} (\alpha_i - \alpha_i') a_i = \sum_{1}^{n^2} (\alpha_i - \alpha_i') a_{ii} e_i$$

$$= ((\alpha_1 - \alpha_1')a_{11}, \ldots, (\alpha_{n^2} - \alpha'_{n^2})a_{n^2n^2}).$$

Thus, for all i, $(\alpha_i - \alpha_i')a_{ii} = 0$ and $a_{ii} > 0$, which implies that

$\alpha_i = \alpha_i'$. Hence, the representation for

$$x = \sqrt[n]{p(x)} \sum_{1}^{n^2} \alpha_i a_i$$

is unique. To illustrate the relationship between the α_i and x, denote

$$x = \sqrt[n]{p(x)} \sum_{1}^{n^2} \alpha_{i(x)} a_i.$$

In order to show that p is not an extremal element of $\mathcal{P}_n$, it will be necessary to write p as a sum of elements in $p \in \mathcal{P}_n$ which are not proportional to p.

Now for every $i \in \{1, \ldots, n^2\}$ define $f_i\colon E^+_{n^2} \to E^+_1$ as follows:

$$f_i(x) = \begin{cases} \alpha^n_{i(x)} p(x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

If $x = 0$, then

$$p(x) = \sum_{1}^{n^2} f_i(x).$$

If $x \in E^+_{n^2} \backslash \{0\}$, then

$$p(x) = p(x) \sum_{1}^{n^2} \alpha^n_{i(x)} = \sum_{1}^{n^2} \alpha^n_{i(x)} p(x) = \sum_{1}^{n^2} f_i(x).$$

In either case,

$$p = \sum f_i .$$

It remains to be shown that each f_i belongs to $\mathcal{P}_n$. First show n-homogenity. Let $\alpha \geq 0$ and $x \in E^+_{n^2}$. If $x = 0$, then $\alpha x = 0$. Hence, $f_i(\alpha x) = 0 = \alpha^n f_i(x)$. Suppose $x \neq 0$. If $\alpha = 0$, then $\alpha x = 0$. Thus, $f_i(\alpha x) = 0 = \alpha^n f_i(x)$. Suppose $x \neq 0$ and $\alpha > 0$. Then

$$x = \sqrt[n]{p(x)} \sum_1^{n^2} \alpha_{i(x)} a_i .$$

This implies that

$$\alpha x = \alpha \left(\sqrt[n]{p(x)} \sum_1^{n^2} \alpha_{i(x)} a_i \right) = \sqrt[n]{\alpha^n p(x)} \sum_1^{n^2} \alpha_{i(x)} a_i$$

$$= \sqrt[n]{p(\alpha x)} \sum_1^{n^2} \alpha_{i(x)} a_i .$$

Therefore, $\alpha_{i(\alpha x)} = \alpha_{i(x)}$. Hence, for each i

$$f_i(\alpha x) = \alpha^n_{i(x)} p(\alpha x) = \alpha^n (\alpha^n_{i(x)} p(x)) = \alpha^n f_i(x).$$

Thus, for all $\alpha \geq 0$ and $x \in E^+_{n^2}$, $f_i(\alpha x) = \alpha^n f_i(x)$.

Now show superadditivity. Let $x,y \in E^+_{n^2}$. If $x = 0$ and $y = 0$, then $x + y = 0$. As a result, $f_i(x + y) = 0 = f_i(x) + f_i(y)$. Suppose $x = 0$ and $y \neq 0$, then $f_i(y) = \alpha^n_{i(y)} p(y)$. Also, $x + y = y$. Hence,

$f_i(x+y) = f_i(y) = f_i(y) + f_i(x)$. Now suppose both $x \neq 0$ and $y \neq 0$. Then

$$x = \sqrt[n]{p(x)} \sum_1^{n^2} \alpha_{i(x)} a_i \quad \text{and} \quad y = \sqrt[n]{p(y)} \sum_1^{n^2} \alpha_{i(y)} a_i .$$

Hence,

$$x + y = \sqrt[n]{p(x+y)} \sum_1^{n^2} \left(\sqrt[n]{\frac{p(x)}{p(x+y)}}\, \alpha_{i(x)} + \sqrt[n]{\frac{p(y)}{p(x+y)}}\, \alpha_{i(y)} \right) a_i .$$

Then

$$\begin{aligned} f_i(x+y) &= \left(\sqrt[n]{\frac{p(x)}{p(x+y)}}\, \alpha_{i(x)} + \sqrt[n]{\frac{p(y)}{p(x+y)}}\, \alpha_{i(y)} \right)^n p(x+y) \\ &\geq \left[\left(\sqrt[n]{\frac{p(x)}{p(x+y)}}\, \alpha_{i(x)} \right)^n + \left(\sqrt[n]{\frac{p(y)}{p(x+y)}}\, \alpha_{i(y)} \right)^n \right] p(x+y) \\ &= \left[\frac{p(x)}{p(x+y)}\, \alpha^n_{i(x)} + \frac{p(y)}{p(x+y)}\, \alpha^n_{i(y)} \right] p(x+y) \\ &= \alpha^n_{i(x)}\, p(x) + \alpha^n_{i(y)}\, p(y) = f_i(x) + f_i(y). \end{aligned}$$

Therefore, $f_i(x+y) \geq f_i(x) + f_i(y)$ for all $x,y \in E^+_{n^2}$.

To show $f_i \in \mathcal{P}_n$, it only remains to be shown that f_i is continuous. In fact from Proposition 1.10, it is sufficient to show that f_i is continuous for each $x \neq 0$. Suppose $x \in E^+_{n^2}$ such that $x \neq 0$. Let $\{x_k\} \subset E^+_{n^2}$ such that $x_k \to x$. Without loss of generality, assume $x_k \neq 0$ for each k. Then

$$x = \sqrt[n]{p(x)} \sum_{1}^{n^2} \alpha_{i(x)} a_i \quad \text{and} \quad x_k = \sqrt[n]{p(x_k)} \sum_{1}^{n^2} \alpha_{i(x_k)} a_i .$$

Also,

$$\frac{x_k}{\sqrt[n]{p(x_k)}} \to \frac{x}{\sqrt[n]{p(x)}} ,$$

which implies that

$$\sum_{1}^{n^2} \alpha_{i(x_k)} a_i \to \sum_{1}^{n^2} \alpha_{i(x)} a_i .$$

However,

$$((\alpha_{1(x_k)} - \alpha_{1(x)}) a_{11}, \ldots, (\alpha_{n^2(x_k)} - \alpha_{n^2(x)}) a_{n^2 n^2})$$

$$= \sum_{1}^{n^2} (\alpha_{i(x_k)} - \alpha_{i(x)}) a_i$$

$$= \sum_{1}^{n^2} \alpha_{i(x_k)} a_i - \sum_{1}^{n^2} \alpha_{i(x)} a_i \to 0.$$

Therefore, for each i, $(\alpha_{i(x_k)} - \alpha_{i(x)}) a_{ii} \to 0$, which implies that $\alpha_{i(x_k)} \to \alpha_{i(x)}$. As a result,

$$f_i(x_k) = \alpha^n_{i(x_k)} p(x_k) \to \alpha^n_{i(x)} p(x) = f_i(x).$$

Hence, f_i is continuous. Thus, each $f_i \in \mathcal{P}_n$.

Lastly, show the f_i are not proportional to p. Suppose $f_i = \sigma p$, $\sigma \geq 0$. Since $p(a_i) = 1$ and

$$a_i = \sqrt[n]{p(a_i)}\,\frac{a_i}{\sqrt[n]{p(a_i)}}\ ,$$

then

$$\alpha_{i(a_i)} = \frac{1}{\sqrt[n]{p(a_i)}}\ .$$

Hence,

$$\sigma = \sigma p(a_i) = f_i(a_i) = \left(\frac{1}{\sqrt[n]{p(a_i)}}\right)^n p(a_i) = 1.$$

Now consider a_j for some $j \neq i$. Since $p(a_j) = 1$ and

$$a_j = \sqrt[n]{p(a_j)}\,\frac{a_j}{\sqrt[n]{p(a_j)}}\ ,$$

then $\alpha_{i(a_j)} = 0$. Thus,

$$\sigma = \sigma p(a_j) = f_i(a_j) = 0p(a_j) = 0,$$

a contradiction. Therefore, there does not exist a real number $\sigma \geq 0$ such that $f_i = \sigma p$. Hence, p is not an extremal element of $\mathcal{P}_n$.

The technique used in this proof provided the motivation from which arose Theorem 4.2 in the next chapter. Theorem 3.3 answers the question raised at the first of this chapter for a finite set S that contains exactly n^2 elements of $[p:1]$ and has the property that $n(S) = \mathrm{Lev}_1 p$. The question is still open if S contains more than n^2 elements.

In trying to determine whether or not $p \in \mathcal{P}_n$ is an extremal element of $\mathcal{P}_n$, it is important to determine how f and g are related to p when $p = f + g$. One immediate question is the following: If $p = f + g$, then how are the sets $\mathrm{Lev}_1 p$, $\mathrm{Lev}_1 f$ and $\mathrm{Lev}_1 g$ related? The following proposition is an attempt to shed some light on this question. First consider the following lemma:

Lemma 3.1: Let K be a n-convex set. If $0 < \alpha \leq 1$, then $K \subset \alpha K$.

Proof: Let $x \in K$, then $x = \frac{1}{\alpha}(\alpha x)$. Since αK is n-convex and $\frac{1}{\alpha} \geq 1$, then $x \in \alpha K$. Hence, $K \subset \alpha K$.

Proposition 3.2: Let p, f and g be nonzero elements of $\mathcal{P}_n$. If $p = f + g$, then

$$n(\mathrm{Lev}_1 f \cup \mathrm{Lev}_1 g) \subset \mathrm{Lev}_1 p \subset \frac{1}{2}\, n(\mathrm{Lev}_1 f \cup \mathrm{Lev}_1 g).$$

Proof: If $x \in \mathrm{Lev}_1 f$, then $p(x) = f(x) + g(x) \geq 1$, which implies that $x \in \mathrm{Lev}_1 p$. Hence, $\mathrm{Lev}_1 f \subset \mathrm{Lev}_1 p$. Likewise, $\mathrm{Lev}_1 g \subset \mathrm{Lev}_1 p$. Therefore, $\mathrm{Lev}_1 f \cup \mathrm{Lev}_1 g \subset \mathrm{Lev}_1 p$, and hence $n(\mathrm{Lev}_1 f \cup \mathrm{Lev}_1 g) \subset \mathrm{Lev}_1 p$.

Now let $x \in \mathrm{Lev}_1 p$. If $f(x) \geq 1$ or $g(x) \geq 1$, then

$$x \in n(\mathrm{Lev}_1 f \cup \mathrm{Lev}_1 g) \subset \frac{1}{2}\, n(\mathrm{Lev}_1 f \cup \mathrm{Lev}_1 g)$$

by Lemma 3.1. Suppose $f(x) < 1$ and $g(x) < 1$. If $f(x) = 0$, then $1 \leq p(x) = f(x) + g(x) = g(x)$, a contradiction. Hence, $0 < f(x) < 1$. Likewise, $0 < g(x) < 1$. Then

$$x = \frac{1}{2}\left[\sqrt[n]{f(x)}\,\frac{x}{\sqrt[n]{f(x)}} + \sqrt[n]{g(x)}\,\frac{x}{\sqrt[n]{g(x)}}\right],$$

where

$$\frac{x}{\sqrt[n]{f(x)}} \in \mathrm{Lev}_1 f \quad \text{and} \quad \frac{x}{\sqrt[n]{g(x)}} \in \mathrm{Lev}_1 g.$$

Also,

$$\left(\sqrt[n]{f(x)}\right)^n + \left(\sqrt[n]{g(x)}\right)^n = f(x) + g(x) = p(x) \geq 1.$$

Therefore, $x \in \frac{1}{2}\, n(\mathrm{Lev}_1 f \cup \mathrm{Lev}_1 g)$.

The following are three basic properties of the sets $\mathrm{Lev}_1 p$ and $[p:1]$.

Proposition 3.3: If $p, q \in \mathcal{P}_n$, then $p \geq q$ if and only if $\mathrm{Lev}_1 q \subset \mathrm{Lev}_1 p$.

Proof: Suppose $p \geq q$. Let $x \in \mathrm{Lev}_1 q$, then $p(x) \geq q(x) \geq 1$. Then $x \in \mathrm{Lev}_1 p$, which implies that $\mathrm{Lev}_1 q \subset \mathrm{Lev}_1 p$.

Now suppose $\mathrm{Lev}_1 q \subset \mathrm{Lev}_1 p$. Let $x \in E^+_{n^2}$. Clearly, if $q(x) = 0$ then $p(x) \geq q(x)$. Therefore, suppose $q(x) > 0$. Then

$$\frac{x}{\sqrt[n]{q(x)}} \in \mathrm{Lev}_1 q.$$

Hence,

$$\frac{p(x)}{q(x)} = p\left(\frac{x}{\sqrt[n]{q(x)}}\right) \geq 1,$$

since $Lev_1 q \subset Lev_1 p$. This implies that $p \geq q$.

Proposition 3.4: If $\alpha > 0$, then

$$Lev_\lambda \alpha p = Lev_{\frac{\lambda}{\alpha}} p \quad \text{and} \quad \alpha Lev_\lambda p = Lev_{\alpha^n \lambda} p.$$

Proof: Let $x \in Lev_\lambda \alpha p$. Then $\alpha p(x) \geq \lambda$, which implies that $p(x) \geq \frac{\lambda}{\alpha}$. Hence,

$$x \in Lev_{\frac{\lambda}{\alpha}} p.$$

Let

$$x \in Lev_{\frac{\lambda}{\alpha}} p.$$

Then $p(x) \geq \frac{\lambda}{\alpha}$, which implies that $\alpha p(x) \geq \lambda$. Therefore, $x \in Lev_\lambda \alpha p$ and

$$Lev_\lambda \alpha p = Lev_{\frac{\lambda}{\alpha}} p.$$

Next let $\alpha x \in \alpha Lev_\lambda p$. Then $p(\alpha x) = \alpha^n p(x) \geq \alpha^n \lambda$, and hence $\alpha x \in Lev_{\alpha^n \lambda} p$. Now let $x \in Lev_{\alpha^n \lambda} p$. Then $p(x) \geq \alpha^n \lambda$. Hence,

$$\frac{1}{\alpha^n} p(x) \geq \lambda.$$

Thus, $p\left(\frac{1}{\alpha} x\right) \geq \lambda$, which implies that $\frac{1}{\alpha} x \in Lev_\lambda p$. Therefore,

$x \in \alpha\mathrm{Lev}_\lambda p$ and $\alpha\mathrm{Lev}_\lambda p = \mathrm{Lev}_{\alpha^n\lambda} p$.

Proposition 3.5: If $p \in \mathcal{P}_n$ and $\alpha \geq 0$, then

$$\alpha[p:1] = [p:\alpha^n].$$

Proof: The proof is similar to the preceding proof.

CHAPTER IV

EXTREMAL ELEMENTS OF $\mathcal{P}_n$

The first theorem gives some of the extremal elements of $\mathcal{P}_n$. It is conjectured that this set includes all of the extremal elements of $\mathcal{P}_n$. The following lemmas will be needed.

Lemma 4.1: Let $p,q \in \mathcal{P}_n$. Define

$$p \wedge q(x) = \min\{p(x),q(x)\}.$$

Then $p \wedge q \in \mathcal{P}_n$.

Proof: First notice that $p \wedge q$ is n-homogeneous, that is,

$$\begin{aligned} p \wedge q(\alpha x) &= \min\{p(\alpha x),q(\alpha x)\} = \min\{\alpha^n p(x),\alpha^n q(x)\} \\ &= \alpha^n \min\{p(x),q(x)\} = \alpha^n p \wedge q(x). \end{aligned}$$

Also,

$$\begin{aligned} p \wedge q(x + y) &= \min\{p(x + y),\ q(x + y)\} \\ &\geq \min\{p(x) + p(y),\ q(x) + q(y)\} \\ &\geq \min\{p(x),q(x)\} + \min\{p(y),q(y)\} \\ &= p \wedge q(x) + p \wedge q(y). \end{aligned}$$

Finally, $p \wedge q$ is continuous since

$$p \wedge q = \frac{1}{2}(p + q - |p - q|)$$

[3]. Therefore, $p \wedge q \in \mathcal{P}_n$.

For all $k = 1, \ldots, n^2$, let $p_k(x) = x_k^n$, where

$$x = (x_1, \ldots, x_{n^2}) \in E^+_{n^2}.$$

Then $p_k \in \mathcal{P}_n$. With this in mind, consider the following:

<u>Lemma 4.2</u>: Let $a = (a_1, \ldots, a_{n^2}) \in E^+_{n^2}\backslash\{0\}$. Define p_a as follows:

$$p_a(x) = \sup\{\lambda^n: x \geq \lambda a, \lambda \geq 0\}.$$

Then $p_a \in \mathcal{P}_n$.

Proof: Without loss of generality, assume the nonzero coordinates of a are $a_1, \ldots, a_k$, $k \leq n^2$. Let

$$p(x) = \left(\frac{1}{a_1^n} p_1 \wedge \ldots \wedge \frac{1}{a_k^n} p_k\right)(x).$$

Lemma 4.1 implies $p \in \mathcal{P}_n$. Now for x given suppose

$$p(x) = \frac{1}{a_\ell^n} x_\ell^n ,$$

$1 \leq \ell \leq k$. Then for every $i \in \{1, \ldots, k\}$

$$\frac{1}{a_i^n} x_i^n \geq \frac{1}{a_\ell^n} x_\ell^n .$$

Therefore,

$$x_i^n \geq \frac{a_i^n}{a_\ell^n} x_\ell^n .$$

This implies

$$x_i \geq \frac{a_i}{a_\ell} x_\ell = \frac{x_\ell}{a_\ell} a_i,$$

with equality when $i = \ell$. If $i \in \{1, \ldots, n^2\} \backslash \{1, \ldots, k\}$, then $a_i = 0$ in which case

$$x_i \geq \frac{x_\ell}{a_\ell} a_i .$$

Thus,

$$x \geq \frac{x_\ell}{a_\ell} a.$$

Notice that there does not exist

$$\lambda > \frac{x_\ell}{a_\ell}$$

such that $x \geq \lambda a$, since otherwise

$$x_\ell \geq \lambda a_\ell > \frac{x_\ell}{a_\ell} a_\ell = x_\ell .$$

Hence,

$$p_a(x) = p(x) = \left(\frac{1}{a_1^n} p_1 \wedge \cdots \wedge \frac{1}{a_k^n} p_k \right)(x) \tag{4-1}$$

for every $x \in E^+_{n^2}$, which implies that $p_a \in \mathcal{P}_n$.

Notice that if a_i is a nonzero coordinate of a and

$$x = (x_1, \ldots, x_{n^2}) \in E^+_{n^2}$$

such that $x_i = 0$, then $x \geq \lambda a$ implies $\lambda = 0$. Thus, $p_a(x) = 0$.

Also, if $a = e_k$, then $p_a = p_k$.

In general, if $p \in \mathcal{P}_n$, the set $[p:1]$ is difficult to characterize. However, a characterization is possible when $p = p_a$ for some $a \neq 0$.

To do this, let $a = (a_1, \ldots, a_{n^2}) \in E^+_{n^2}$, $a \neq 0$. For every $i \in \{1, \ldots, n^2\}$, define

$$R(a_i) = \{(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_{n^2}) : x_j \geq a_j \text{ for } j \neq i\}.$$

Lemma 4.3: If $a \in E^+_{n^2} \backslash \{0\}$, then

$$[p_a:1] = \bigcup_1^{n^2} R(a_i).$$

Proof: Let $y \in R(a_i)$. Clearly, $y \geq a$. Notice there does not exist $\lambda > 1$ such that $y \geq \lambda a$, for otherwise $a_i \geq \lambda a_i > a_i$. Hence, by definition $p_a(y) = 1$. This implies that

$$\bigcup_1^{n^2} R(a_i) \subset [p_a:1].$$

Now suppose $y \in [p_a:1]$. Considering equation (4-1), there exists $k \in \{1, \ldots, n^2\}$ such that $a_k > 0$ and

$$\frac{y_k^n}{a_k^n} = 1.$$

This implies $y_k^n = a_k^n$, which in turn implies $y_k = a_k$. For all other $i \in \{1, \ldots, n^2\}$ such that $a_i > 0$,

$$\frac{y_i^n}{a_i^n} \geq 1.$$

Hence, $y_i^n \geq a_i^n$ which implies $y_i \geq a_i$. If $i \in \{1, \ldots, n^2\}$ such that $a_i = 0$, then $y_i \geq a_i$. Therefore, $y \in R(a_k)$. Hence,

$$[p_a:1] = \bigcup_{1}^{n^2} R(a_i).$$

Using this result, it is possible to show that $p_a = p_b$ if and only if $a = b$.

Comment 4.1: $p_a = p_b$ if and only if $a = b$.

Proof: Clearly, $a = b$ implies $p_a = p_b$. Suppose $p_a = p_b$. Then

$$\bigcup_{1}^{n^2} R(a_i) = [p_a:1] = [p_b:1] = \bigcup_{1}^{n^2} R(b_i).$$

If $a \neq b$, then, without loss of generality, there exists k such that $a_k > b_k$. Let $x \in R(b_k)$. Then $x_k = b_k < a_k$. Hence, $x \notin R(a_k)$. Therefore, $x \in R(a_i)$ for some $i \neq k$. Then $x_k \geq a_k > x_k$, a contradiction. Hence, $a = b$.

Next p_a is shown to be an extremal element of $\mathcal{P}_n$.

Theorem 4.1: The function p_a is an extremal of $\mathcal{P}_n$.

Proof: Suppose $p_a = f + g$. Let $y \in R(a_i)$, where $i \in \{1, \ldots, n^2\}$, then

$$p_a(a) = p_a(y) = f(y) + g(y) \geq f(a) + g(a) = p_a(a).$$

This implies $f(y) = f(a)$ and $g(y) = g(a)$, since $f(y) \geq f(a)$ and $g(y) \geq g(a)$. Also, $p_a(a) = f(a) + g(a)$ implies $p_a(a) \geq f(a)$ and $p_a(a) \geq g(a)$. Therefore, there exists $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha p_a(a) = f(a)$ and $\beta p_a(a) = g(a)$.

Again, without loss of generality, suppose the nonzero coordinates of a are $a_1, \ldots, a_k$. Let $x \in E^+_{n^2}$ such that $x_1 > 0, \ldots, x_k > 0$. Then for every $i \in \{1, \ldots, k\}$ there exists $\lambda_i > 0$ such that $a_i = \lambda_i x_i$. Let $\lambda = \max\{\lambda_i : i \in \{1, \ldots, k\}\}$. Notice there exists a $j \in \{1, \ldots, k\}$ such that $\lambda = \lambda_j$. Hence, $\lambda x_i \geq a_i$ with equality when $i = j$. Clearly, if $i \in \{1, \ldots, n^2\} \setminus \{1, \ldots, k\}$, then $\lambda x_i \geq a_i$. Therefore, $\lambda x \in R(a_j)$. Let $y = \lambda x$. Then $x = \frac{1}{\lambda} y$, where $y \in R(a_j)$. Therefore,

$$f(x) = f\left(\frac{1}{\lambda} y\right) = \frac{1}{\lambda^n} f(y)$$

$$= \frac{1}{\lambda^n} f(a) = \frac{1}{\lambda^n} \alpha p_a(a)$$

$$= \frac{1}{\lambda^n} \alpha p_a(y) = \alpha p_a\left(\frac{1}{\lambda} y\right)$$

$$= \alpha p_a(x).$$

Clearly, if $x \in E^+_{n^2}$ such that $x_i = 0$ for some $i \in \{1, \ldots, k\}$, then $0 = p_a(x)$. This implies $f(x) = 0$, which in turn implies that $f(x) = \alpha p_a(x)$. In either case $f(x) = \alpha p_a(x)$. Hence, $f = \alpha p_a$. Likewise, $g = \beta p_a$. Therefore, p_a is an extremal element of $\mathcal{P}_n$.

Another property of the functions p_a is of interest.

<u>Proposition 4.1</u>: The function p_a is minimal in the set of all elements of $\mathcal{P}_n$ which agree with p at a.

Proof: Let $g \in \mathcal{P}_n$ such that $g(a) = 1 = p_a(a)$. Without loss of generality, assume the nonzero coordinates of a are $a_1, \ldots, a_k$. Let $x \in E^+_{n^2}$. If $x_1 > 0, \ldots, x_k > 0$, then, as in the previous theorem, there exists $i \in \{1, \ldots, k\}$ such that $x = \alpha y$, where $y \in R(a_i)$. Hence,

$$g(x) = g(\alpha y) = \alpha^n g(y) \geq \alpha^n g(a) = \alpha^n p_a(a)$$

$$= \alpha^n p_a(y) = p_a(\alpha y) = p_a(x).$$

If there exists $j \in \{1, \ldots, k\}$ such that $x_j = 0$, then $p_a(x) = 0$. Hence, $g(x) \geq p_a(x)$. Therefore, for all $x \in E^+_{n^2}$, $g(x) \geq p_a(x)$.

For these functions p_a the sets $[p_a:1]$ have the following properties.

Proposition 4.2: If $\alpha > 0$, then

$$\alpha[p_a:1] = [p_a:\alpha^n] = [p_{\alpha a}:1].$$

Proof: The first equality follows from Proposition 3.4 and holds in general. It remains to be shown that $[p_a:\alpha^n] = [p_{\alpha a}:1]$. Let $x \in [p_a:\alpha^n]$. Then $p_a(x) = \sup\{\lambda^n: x \geq \lambda_a\} = \alpha^n$. As in the proof of Lemma 4.2, if $p_a(x) = \alpha^n$, then $x \geq \alpha a$ and there exists $i \in \{1, \ldots, n^2\}$ such that $x_i = \alpha a_i$. Consider $p_{\alpha a}(x)$. Since $x \geq \alpha a$, then $p_{\alpha a}(x) \geq 1$. If $p_{\alpha a}(x) = \lambda^n > 1$, then $x \geq \lambda(\alpha a) > \alpha a$, a contradiction since $x_i = \alpha a_i$. Hence, $p_{\alpha a}(x) = 1$ which implies that $x \in [p_{\alpha a}:1]$. Therefore, $[p_a:\alpha^n] \subset [p_{\alpha a}:1]$.

Now suppose $x \in [p_{\alpha a}:1]$, then $p_{\alpha a}(x) = 1$. This implies $x \geq \alpha a$ and there exists $i \in \{1, \ldots, n^2\}$ such that $x_i = \alpha a_i$. Consider $p_a(x)$. Since $x \geq \alpha a$, $p_a(x) \geq \alpha^n$. Suppose $p_a(x) = \lambda^n > \alpha^n$, then $x \geq \lambda a > \alpha a$, a contradiction. Hence, $p_a(x) = \alpha^n$ which implies that $x \in [p_a:\alpha^n]$. Therefore, $[p_{\alpha a}:1] \subset [p_a:\alpha^n]$.

Recall that $\mathcal{S}_n$ is the set of linear combinations of products of n functions of $\mathcal{P}'_n$. For $a \in E^+_{n^2}\setminus\{0\}$, let $q_a(x) = \sup\{\lambda: x \geq \lambda a\}$. Then, as in the case for $\mathcal{P}_n$, q_a is an extremal element of $\mathcal{P}'_n$. Also, $p_a(x) = [q_a(x)]^n$, which implies that $p_a \in \mathcal{S}_n$. Since $\mathcal{S}_n$ is a subcone of $\mathcal{P}_n$, then p_a is an extremal element of $\mathcal{S}_n$. It is conjectured that $\{p_a: a \in E^+_{n^2}\setminus\{0\}\}$ represents all the extremal elements of $\mathcal{S}_n$.

Lemma 4.4: If

$$p(x) = \prod_{1}^{n} A_i(x),$$

where $A_i \in \mathcal{P}_n'$, is an extremal element of $\mathcal{S}_n$, then each A_i is an extremal element of $\mathcal{P}_n'$.

Proof: Suppose there exists a $k = 1, \ldots, n^2$ such that A_k is not extremal in $\mathcal{P}_n'$. Then there exists $f, g \in \mathcal{P}_n'$ such that $A_k = f + g$ and neither f or g is proportional to A_k. Hence,

$$p(x) = \prod_{1}^{n} A_i(x) = A_1(x) \cdots (f(x) + g(x)) \cdots A_n(x)$$

$$= A_1(x) \cdots f(x) \cdots A_n(x) + A_1(x) \cdots g(x) \cdots A_n(x).$$

Since p is extremal in $\mathcal{S}_n$, there exists $\alpha \geq 0$ and $\beta \geq 0$ such that

$$A_1(x) \cdots f(x) \cdots A_n(x) = \alpha p(x) \quad \text{and} \quad A_1(x) \cdots g(x) \cdots A_n(x) = \beta p(x).$$

Let $x \in \operatorname{int} E^+_{n^2}$. Then $p(x) > 0$. Also, as in Proposition 1.3, each $A_i(x) > 0$, $f(x) > 0$ and $g(x) > 0$. Therefore,

$$\alpha A_1(x) \cdots A_k(x) \cdots A_n(x) = \alpha p(x) = A_1(x) \cdots f(x) \cdots A_n(x),$$

which implies that $\alpha A_k(x) = f(x)$, for all $x \in \operatorname{int} E^+_{n^2}$. It follows, as in Comment 1.3, that $\alpha A_k(x) = f(x)$ for all $x \in E^+_{n^2}$. This is a contradiction. Therefore, A_k is extremal in $\mathcal{P}_n'$ for each k.

In any convex cone, if the sum of two nonzero elements is an extremal element, then the two elements are proportional. Hence, the only possible extremal elements of $\mathcal{S}_n$ are those elements of the form

$$p(x) = \prod_{1}^{n^2} A_i^{\ell(i)}(x), \tag{4-2}$$

where $\ell(i)$ is a nonnegative integer and

$$\sum_{1}^{n^2} \ell(i) = n.$$

Notice that $\ell(i) > 0$ for at most n values of $i = 1, \ldots, n^2$. Moreover, Lemma 4.4 implies the $A_i(x)$ must be extremal elements of $\mathcal{P}_n'$. The Lemma 4.4 and these comments give conditions that are necessary when p is an extremal element in $\mathcal{S}_n$. These conditions are not sufficient as will be seen in Proposition 4.3.

Attention will now be given to considering the extremal elements of $\mathcal{P}_n$.

Theorem 4.2: Let p be defined as in (4-2). Let k be the number of $i = 1, \ldots, n^2$ for which $\ell(i) > 0$. If $k > 1$, then p is not an extremal element of $\mathcal{P}_n$.

Proof: Without loss of generality, assume

$$p(x) = \prod_{1}^{k} A_i^{\ell(i)}(x),$$

where $\ell(i)$ is a positive integer,

$$\sum_{1}^{k} \ell(i) = n$$

and the A_i are distinct (pairwise nonproportional) extremal elements in $\mathcal{P}_n'$. Define

$$f_i(x) = \begin{cases} \dfrac{A_i(x)}{A_1(x) + \cdots + A_k(x)}\, p(x), & A_1(x) + \cdots + A_k(x) > 0 \\ 0 & , \quad A_1(x) + \cdots + A_k(x) = 0. \end{cases}$$

If $A_1(x) + \cdots + A_k(x) = 0$, then $A_i(x) = 0$, for $i = 1, \ldots, k$. Hence,

$$p(x) = 0 = \sum_{1}^{k} f_i(x).$$

If $A_1(x) + \cdots + A_k(x) > 0$, then

$$p(x) = \sum_{1}^{k} f_i(x).$$

In either case

$$p(x) = \sum_{1}^{k} f_i(x).$$

It will now be shown that $f_i \in \mathcal{P}_n$, for each i.

n-Homogenity: Let $\alpha \geq 0$ and $x \in E^+_{n^2}$. If $\alpha = 0$, then $\alpha x = 0$. Hence, $A_1(\alpha x) = \ldots = A_k(\alpha x) = 0$. Therefore, $f_i(\alpha x) = 0 = \alpha^n f_i(x)$.

Suppose $\alpha > 0$. If $0 = A_1(\alpha x) + \cdots + A_k(\alpha x) = \alpha(A_1(x) + \cdots + A_k(x))$, then $A_1(x) + \cdots + A_k(x) = 0$. Hence, $f_i(\alpha x) = \alpha^n f_i(x)$. Suppose $\alpha > 0$ and $\alpha(A_1(x) + \cdots + A_k(x)) = A_1(\alpha x) + \cdots + A_k(\alpha x) > 0$, then $A_1(x) + \cdots + A_k(x) > 0$. Therefore,

$$f_i(\alpha x) = \frac{A_i(\alpha x)}{A_1(\alpha x) + \cdots + A_k(\alpha x)} p(\alpha x)$$

$$= \alpha^n \frac{A_i(x)}{A_1(x) + \cdots + A_k(x)} p(x) = \alpha^n f_i(x).$$

So for all $\alpha \geq 0$ and $x \in E^+_{n^2}$, $f_i(\alpha x) = \alpha^n f_i(x)$.

Superadditivity: Let $x, y \in E^+_{n^2}$.

Case I: If $A_1(x + y) + \cdots + A_k(x + y) = 0$, then

$$0 = A_1(x + y) + \cdots + A_k(x + y)$$
$$\geq A_1(x) + \cdots + A_k(x) + A_1(y) + \cdots + A_k(y) \geq 0,$$

which implies $A_1(x) + \cdots + A_k(x) = 0$ and $A_1(y) + \cdots + A_k(y) = 0$. Therefore, $f_i(x + y) = 0 = f_i(x) + f_i(y)$.

Case II: Suppose $A_1(x + y) + \cdots + A_k(x + y) > 0$, $A_1(x) + \cdots + A_k(x) = 0$ and $A_1(y) + \cdots + A_k(y) = 0$. Clearly, $f_i(x + y) \geq f_i(x) + f_i(y)$.

Case III: Suppose $A_1(x + y) + \cdots + A_k(x + y) > 0$, $A_1(x) + \cdots + A_k(x) > 0$ and $A_1(y) + \cdots + A_k(y) = 0$. Then

$$f_i(x + y) = \frac{A_i(x + y)}{A_1(x + y) + \cdots + A_k(x + y)} p(x + y),$$

$$f_i(x) = \frac{A_i(x)}{A_1(x) + \cdots + A_k(x)} p(x)$$

and $f_i(y) = 0$. It must be shown that

$$\frac{A_i(x + y)}{A_1(x + y) + \cdots + A_k(x + y)} p(x + y) \geq \frac{A_i(x)}{A_1(x) + \cdots + A_k(x)} p(x).$$

This is true if and only if

$$[A_1(x) + \cdots + A_k(x)]A_i(x + y) \prod_1^k A_j^{\ell(j)}(x + y)$$

$$\geq [A_1(x + y) + \cdots + A_k(x + y)]A_i(x) \prod_1^k A_j^{\ell(j)}(x). \quad (4\text{-}3)$$

Consider a term on the right side of this last inequality. Without loss of generality, consider

$$A_k(x + y)A_i(x) \prod_1^k A_j^{\ell(j)}(x).$$

Now consider the term

$$A_k(x)A_i(x + y) \prod_1^k A_j^{\ell(j)}(x + y)$$

on the left side of the inequality. Now

$$A_k(x)A_i(x+y)\prod_{1}^{k} A_j^{\ell(j)}(x+y)$$

$$= A_k(x)A_i(x+y)A_1^{\ell(1)}(x+y)\cdots A_k^{\ell(k)-1}(x+y)A_k(x+y)$$

$$= A_k(x+y)A_i(x+y)A_1^{\ell(1)}(x+y)\cdots A_k^{\ell(k)-1}(x+y)A_k(x)$$

$$\geq A_k(x+y)A_i(x)\prod_{1}^{k} A_j^{\ell(j)}(x),$$

which implies that (4-3) is true. Therefore, $f_i(x+y) \geq f_i(x) + f_i(y)$.

Case IV: Suppose $A_1(x+y) + \cdots + A_k(x+y) > 0$, $A_1(x) + \cdots + A_k(x) > 0$ and $A_1(y) + \cdots + A_k(y) > 0$. Then

$$f_i(x+y) = \frac{A_i(x+y)}{A_1(x+y) + \cdots + A_k(x+y)}\, p(x+y),$$

$$f_i(x) = \frac{A_i(x)}{A_1(x) + \cdots + A_k(x)}\, p(x)$$

and

$$f_i(y) = \frac{A_i(y)}{A_1(y) + \cdots + A_k(y)}\, p(y).$$

It must be shown that

$$\frac{A_i(x+y)}{A_1(x+y) + \cdots + A_k(x+y)}\prod_{1}^{k} A_j^{\ell(j)}(x+y)$$

$$\geq \frac{A_i(x)}{A_1(x) + \cdots + A_k(x)}\prod_{1}^{k} A_j^{\ell(j)}(x) + \frac{A_i(y)}{A_1(y) + \cdots + A_k(y)}\prod_{1}^{k} A_j^{\ell(j)}(y).$$

This will be true if and only if

$$A_i(x+y)[A_1(x)+\cdots+A_k(x)][A_1(y)+\cdots+A_k(y)]\prod_1^k A_j^{\ell(j)}(x+y)$$

$$\geq A_i(x)[A_1(y)+\cdots+A_k(y)][A_1(x+y)+\cdots+A_k(x+y)]\prod_1^k A_j^{\ell(j)}(x)$$

$$+A_i(y)[A_1(x)+\cdots+A_k(x)][A_1(x+y)+\cdots+A_k(x+y)]\prod_1^k A_j^{\ell(j)}(y).$$

Since

$$A_i(x+y)[A_1(x)+\cdots+A_k(x)][A_1(y)+\cdots+A_k(y)]\prod_1^k A_j^{\ell(j)}(x+y)$$

$$\geq A_i(x)[A_1(x)+\cdots+A_k(x)][A_1(y)+\cdots+A_k(y)]\prod_1^k A_j^{\ell(j)}(x+y)$$

$$+A_i(y)[A_1(x)+\cdots+A_k(x)][A_1(y)+\cdots+A_k(y)]\prod_1^k A_j^{\ell(j)}(x+y),$$

it is sufficient to show

$$A_i(x)[A_1(x)+\cdots+A_k(x)][A_1(y)+\cdots+A_k(y)]\prod_1^k A_j^{\ell(j)}(x+y)$$

$$\geq A_i(x)[A_1(y)+\cdots+A_k(y)][A_1(x+y)+\cdots+A_k(x+y)]\prod_1^k A_j^{\ell(j)}(x)$$

(4-4)

and

$$A_i(y)[A_1(x) + \cdots + A_k(x)][A_1(y) + \cdots + A_k(y)] \prod_1^k A_j^{\ell(j)}(x + y)$$

$$\geq A_i(y)[A_1(x) + \cdots + A_k(x)][A_1(x + y) + \cdots + A_k(x + y)] \prod_1^k A_j^{\ell(j)}(y).$$

(4-5)

Consider (4-4). Clearly, (4-4) holds when $A_i(x) = 0$. Suppose $A_i(x) > 0$, then (4-4) holds if and only if

$$[A_1(x) + \cdots + A_k(x)] \prod_1^k A_j^{\ell(j)}(x + y)$$

$$\geq [A_1(x + y) + \cdots + A_k(x + y)] \prod_1^k A_j^{\ell(j)}(x). \quad (4\text{-}6)$$

Consider a term on the right side of (4-6). In fact, without loss of generality, consider

$$A_k(x + y) \prod_1^k A_j^{\ell(j)}(x).$$

Consider the term

$$A_k(x) \prod_1^k A_j^{\ell(j)}(x + y)$$

on the left side of (4-6). Since

$$A_k(x) \prod_1^k A_j^{\ell(j)}(x+y) = A_k(x) A_1^{\ell(1)}(x+y) \cdots A_k^{\ell(k)-1}(x+y) A_k(x+y)$$

$$= A_k(x+y) A_1^{\ell(1)}(x+y) \cdots A_k^{\ell(k)-1}(x+y) A_k(x)$$

$$\geq A_k(x+y) \prod_1^k A_j^{\ell(j)}(x),$$

inequality (4-6) is true, which implies (4-4) holds. Likewise, (4-5) holds. Therefore, each f_i is superadditive.

Continuity of f_i: Let $x \in E_{n2}^{+}$ and $\{y_j\} \subset E_{n2}^{+}$ such that $y_j \to x$. Suppose $A_1(x) + \cdots + A_k(x) > 0$, then without loss of generality, it may be assumed that $A_1(y_j) + \cdots + A_k(y_j) > 0$ for each j. In this case

$$f_i(y_j) = \frac{A_i(y_j)}{A_1(y_j) + \cdots + A_k(y_j)} p(y_j) \to \frac{A(x)}{A_1(x) + \cdots + A_k(x)} p(x) = f_i(x).$$

Now suppose $A_1(x) + \cdots + A_k(x) = 0$, then $p(x) = f_i(x) = 0$. If there exists $m \in \{1, \ldots, k\}$ such that $A_m(y_j) = 0$, then $f_i(y_j) = 0 = f_i(x)$. Suppose $A_m(y_j) > 0$, for $m = 1, \ldots, k$. Let

$$f_i(y_j) = \frac{A_i(y_j)}{A_1(y_j) + \cdots + A_k(y_j)} \prod_1^k A_m^{\ell(m)}(y_j)$$

$$= \Bigg(\frac{1}{A_1^{\ell(1)-1}(y_j) \cdots A_i^{\ell(i)+1}(y_j) \cdots A_k^{\ell(k)}(y_j)}$$

$$+ \frac{1}{A_1^{\ell(1)}(y_j) \cdots A_i^{\ell(i)}(y_j) \cdots A_k^{\ell(k)}(y_j)}$$

$$\left. + \frac{1}{A_1^{\ell(1)}(y_j) \cdots A_i^{\ell(i)+1}(y_j) \cdots A_k^{\ell(k)-1}(y_j)} \right)^{-1}. \qquad (4\text{-}7)$$

As $y_j \to x$, $A_m(y_j) \to A_m(x) = 0$, for each m. If there is a subsequence of $\{y_j\}$ such that each $A_m(y_j) > 0$, then (4-7) implies that $f_i(y_j) \to 0$ as the subsequence approaches x. Therefore, given $\varepsilon > 0$, there exists an integer N such that $j \geq N$ implies that

$$|f_i(x) - f_i(y_j)| < \varepsilon,$$

i.e., f is continuous. Hence, each $f_i \in \mathcal{P}_n$.

It remains to be shown that the functions f_i form a nonproportional decomposition of p. Suppose $f_i(x) = \alpha p(x)$, for all $x \in E^+_{n^2}$. Let $x \in E^+_{n^2}$. There exists a sequence $\{y_j\} \subset \operatorname{int} E^+_{n^2}$ such that $y_j \to x$. Since $y_j \in \operatorname{int} E^+_{n^2}$, $A_1(y_j) > 0, \ldots, A_k(y_j) > 0$ and $p(y_j) > 0$. Hence,

$$\alpha p(y_j) = f_i(y_j) = \frac{A_i(y_j)}{A_1(y_j) + \cdots + A_k(y_j)} p(y_j),$$

which implies that

$$\alpha = \frac{A_i(y_j)}{A_1(y_j) + \cdots + A_k(y_j)} .$$

or

$$A_i(y_j) = \alpha(A_1(y_j) + \cdots + A_k(y_j)).$$

Also,

$A_i(y_j) \to A_i(x)$ and $\alpha(A_1(y_j) + \cdots + A_k(y_j)) \to \alpha(A_1(x) + \cdots + A_k(x))$.

Hence,

$$A_i(x) = \alpha(A_1(x) + \cdots + A_k(x)).$$

Since $A_1, \ldots, A_i, \ldots, A_k$ are pairwise nonproportional extremal elements in $\mathcal{P}_n'$, this is a contradiction. Therefore, there does not exist $\alpha \geq 0$ such that $f_i = \alpha p$. Hence, the decomposition is non-proportional, which implies that p is not an extremal element of $\mathcal{P}_n$.

Two questions immediately arise. First, is $f_i \in \mathcal{S}_n$? Secondly, is every extremal element of $\mathcal{P}_n'$ of the form q_a, where $a \in E^+_{n^2}\setminus\{0\}$? If both answers are affirmative, then every extremal element of $\mathcal{S}_n$ is of the form p_a, where $a \in E^+_{n^2}\setminus\{0\}$. It is entirely possible that the functions f_i do not belong to $\mathcal{S}_n$.

The following is an example of a subcone of $\mathcal{P}_n$ that has as extremal elements some functions that are not extremal in $\mathcal{P}_n$.

<u>Example 4.1</u>: Let Q_n be the set of all $p\colon E^+_{n^2} \to E^+_1$ such that

$$p(x) = \sum_{i1,\ldots,in=1}^{n^2} \alpha_{i1,\ldots,in} x_{i1} \cdots x_{in},$$

where $i1 \leq \cdots \leq in$, $\alpha_{i1,\ldots,in} \geq 0$ and $x = (x_1, \ldots, x_{n^2})$. Thus, Q_n is the set of nonnegative, superadditive n-forms. Clearly, Q_n is a subcone of $\mathcal{S}_n \subset \mathcal{P}_n$. Therefore, the functions $p_1, \ldots, p_{n^2}$ are extremal elements of Q. However, these are not all of the extremal elements of Q_n. In fact, the extremal elements of Q_n are the positive scalar multiples of functions of the form

$$p(x) = x_{k1} \cdots x_{kn},$$

where $k_j \in \{1, \ldots, n^2\}$, for $j = 1, \ldots, n$, and $k1 \leq \cdots \leq kn$.

Proof: Let p be a function of the above form. Suppose $p = f + g$, where $f,g \in Q_n$. Suppose

$$f(x) = \sum_{i1,\ldots,in=1}^{n^2} \alpha_{i1,\ldots,in} x_{i1} \cdots x_{in}$$

and

$$g(x) = \sum_{i1,\ldots,in=1}^{n^2} \beta_{i1,\ldots,in} x_{i1} \cdots x_{in},$$

where $i1 \leq \cdots \leq in$. Then

$$\begin{aligned} x_{k1} \cdots x_{kn} &= f(x) + g(x) \\ &= \sum \alpha_{i1,\ldots,in} x_{i1} \cdots x_{in} + \sum \beta_{i1,\ldots,in} x_{i1} \cdots x_{in} \\ &= \sum (\alpha_{i1,\ldots,in} + \beta_{i1,\ldots,in}) x_{i1} \cdots x_{in}, \end{aligned}$$

which implies that

$$\alpha_{i1,\ldots,in} + \beta_{i1,\ldots,in} = \begin{cases} 1, & i1 = k1,\ldots,in = kn, \\ 0, & \text{elsewhere.} \end{cases}$$

Therefore,

$$f(x) = \sum \alpha_{i1,\ldots,in} x_{i1} \cdots x_{in}$$

$$= \alpha_{k1,\ldots,kn} x_{k1} \cdots x_{kn} = \alpha_{k1,\ldots,kn}\, p(x)$$

and

$$g(x) = \sum \beta_{i1,\ldots,in} x_{i1} \cdots x_{in}$$

$$= \beta_{k1,\ldots,kn} x_{k1} \cdots x_{kn} = \beta_{k1,\ldots,kn}\, p(x).$$

Hence, p is an extremal element of Q_n.

Now for every $x = (x_1, \ldots, x_{n^2}) \in E^+_{n^2}$ define $p(x)$ as

$$p(x) = x_1^{\ell(1)} \cdots x_{n^2}^{\ell(n^2)}, \tag{4-7}$$

where $\ell(i)$ is a nonnegative integer and

$$\sum_{1}^{n^2} \ell(i) = n.$$

Notice that $\ell(i) > 0$ for at most n values of $i = 1, \ldots, n$. Clearly, $p \in Q_n$. In fact, the preceding example shows that p is an extremal element of Q_n. If k is the number of $i \in \{1, \ldots, n^2\}$ for which $\ell(i) > 0$ and $k > 1$, Theorem 4.2 says that p is not an extremal element of $\mathcal{P}_n$. The following proposition shows that it is possible that p is not an extremal element of $\mathcal{S}_n$.

<u>Proposition 4.3</u>: Let p be defined as in (4-7). If $k = 2$, then p is not an extremal element of $\mathcal{S}_n$.

Proof: Without loss of generality, assume

$$p(x) = x_1^{\ell(1)} x_2^{\ell(2)},$$

where $\ell(1) > 0$ and $\ell(2) > 0$. As seen in the proof of Theorem 4.2,

$$p(x) = \sum_{1}^{2} f_i(x)$$

where

$$f_i(x) = \begin{cases} \dfrac{x_i}{x_1 + x_2} x_1^{\ell(1)} x_2^{\ell(2)}, & x_1 + x_2 > 0, \\ 0, & x_1 + x_2 = 0. \end{cases}$$

Consider $f_1(x)$. Notice

$$f_1(x) = \begin{cases} \dfrac{x_1 x_2}{x_1 + x_2} x_1^{\ell(1)} x_2^{\ell(2)-1}, & x_1 + x_2 > 0, \\ 0, & x_1 + x_2 = 0. \end{cases}$$

Let

$$g(x) = \begin{cases} \dfrac{x_1 x_2}{x_1 + x_2}, & x_1 + x_2 > 0, \\ \\ 0 \quad, & x_1 + x_2 = 0. \end{cases}$$

Then $f_1(x) = g(x) x_1^{\ell(1)} x_2^{\ell(2)-1}$. Since the objective is to show that $f_1 \in \mathcal{S}_n$, it remains to be shown that $g \in \mathcal{P}'_n$. As in Theorem 4.2, g is continuous and homogeneous of degree 1. It remains to prove the superadditivity.

Let $x,y \in E^+_{n^2}$. Suppose $x_1 + x_2 = 0$ and $y_1 + y_2 = 0$, then $x_1 + y_1 + x_2 + y_2 = 0$. Hence, $g(x + y) = 0 = g(x) + g(y)$. Suppose $y_1 + y_2 > 0$, then $x_1 + y_1 + x_2 + y_2 = y_1 + y_2 > 0$. Hence,

$$g(x + y) = \frac{(x_1 + y_1)(x_2 + y_2)}{x_1 + y_1 + x_2 + y_2} = \frac{y_1 y_2}{y_1 + y_2} = g(y) + g(x).$$

Finally, suppose $x_1 + x_2 > 0$ and $y_1 + y_2 > 0$. In this case it must be shown that

$$\frac{(x_1 + y_1)(x_2 + y_2)}{x_1 + y_1 + x_2 + y_2} \geq \frac{x_1 x_2}{x_1 + x_2} + \frac{y_1 y_2}{y_1 + y_2}$$

which is equivalent to proving

$$(x_1 + y_1)(x_2 + y_2)(x_1 + x_2)(y_1 + y_2) - [x_1 x_2 (y_1 + y_2)(x_1 + y_1 + x_2 + y_2) + y_1 y_2 (x_1 + x_2)(x_1 + y_1 + x_2 + y_2)] \geq 0.$$

By direct calculation

$$
\begin{aligned}
(x_1 + y_1)(x_2 + y_2)(x_1 + x_2)(y_1 + y_2) &- [x_1x_2(y_1 + y_2)(x_1 + y_1 + x_2 + y_2) \\
&+ y_1y_2(x_1 + x_2)(x_1 + y_1 + x_2 + y_2)] \\
&= (x_1x_2 + x_1y_2 + x_2y_1 + y_1y_2)(x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2) \\
&\quad - [(x_1x_2y_1 + x_1x_2y_2)(x_1 + y_1 + x_2 + y_2) \\
&\quad + (x_1y_1y_2 + x_2y_1y_2)(x_1 + y_1 + x_2 + y_2)] \\
&= x_1^2x_2y_1 + x_1^2x_2y_2 + x_1x_2^2y_1 + x_1x_2^2y_2 + x_1^2y_1y_2 + x_1^2y_2^2 \\
&\quad + x_1x_2y_1y_2 + x_1x_2y_2^2 + x_1x_2y_1^2 + x_1x_2y_1y_2 \\
&\quad + x_2^2y_1^2 + x_2^2y_1y_2 + x_1y_1^2y_2 + x_1y_1y_2^2 \\
&\quad + x_2y_1^2y_2 + x_2y_1y_2^2 - [x_1^2x_2y_1 + x_1x_2y_1^2 \\
&\quad + x_1x_2^2y_1 + x_1x_2y_1y_2 + x_1^2x_2y_2 + x_1x_2y_1y_2 \\
&\quad + x_1x_2^2y_2 + x_1x_2y_2^2 + x_1^2y_1y_2 + x_1y_1^2y_2 \\
&\quad + x_1x_2y_1y_2 + x_1y_1y_2^2 + x_1x_2y_1y_2 \\
&\quad + x_2y_1^2y_2 + x_2^2y_1y_2 + x_2y_1y_2^2] \\
&= x_1^2y_2^2 - 2x_1x_2y_1y_2 + x_2^2y_1^2 \\
&= (x_1y_2 - x_2y_1)^2 \geq 0.
\end{aligned}
$$

Hence, g is superadditive and $f_1 \in \mathcal{S}_n$. Likewise, $f_2 \in \mathcal{S}_n$. Hence, p is not an extremal element of $\mathcal{S}_n$.

CHAPTER V

THE INFIMUM OF A $\mathcal{P}_n$ FUNCTION

The supremum of $f(x)$ where f is a convex function defined on a convex set has been studied by many persons. For example, Rockafellar (cf. [7], pp. 342-349). The discussion here will consider the infimum $p(x)$, for $p \in \mathcal{P}_n$, where x is restricted on an n-convex set. Recall that if $z \in n - cr(x,y)$, then $n - cr(x,z)$ is not always contained in $n - cr(x,y)$ (cf. Example 1.2).

The first two results are analogous to results involving the supremum of a convex function over a convex set.

<u>Proposition 5.1</u>: If C is a nonempty subset of $E^+_{n^2}$ and $p \in \mathcal{P}_n$, then $\inf\{p(x) : x \in C\} = \inf\{p(x) : x \in n(C)\}$.

Proof: Since $C \subset n(C)$, then

$$\inf\{p(x) : x \in n(C)\} \leq \inf\{p(x) : x \in C\}.$$

Let $x \in n(C)$. Then

$$x = \lambda \sum_1^k \alpha_i x_i,$$

where $\lambda \geq 1$, $\alpha_i \geq 0$,

$$\sum_{1}^{k} \alpha_i^n = 1$$

and $x_i \in C$. Hence,

$$p(x) = p\left(\lambda \sum_{1}^{k} \alpha_i x_i\right) \geq \lambda^n \sum_{1}^{k} \alpha_i^n p(x_i) \geq \sum_{1}^{k} \alpha_i^n p(x_i)$$

$$\geq \inf\{p(x): x \in C\} \sum_{1}^{k} \alpha_i^n = \inf\{p(x): x \in C\}.$$

Therefore, $\inf\{p(x): x \in n(C)\} \geq \inf\{p(x): x \in C\}$. This implies that $\inf\{p(x): x \in C\} = \inf\{p(x): x \in n(C)\}$.

If C, in Proposition 5.1, is compact, then there exists $x \in C \subset K$ such that $p(x) = \inf\{p(y): y \in n(C)\}$. The following proposition is analogous to the maximum principle of harmonic functions.

Proposition 5.2: Let $p \in \mathcal{P}_n$ and K be a n-convex subset of $E^+_{n^2}$. If there exists $z \in \text{rel int } (K)$ such that

$$p(z) = \inf\{p(x): x \in K\},$$

then p is zero over K.

Proof: Let x be any other point of K. Let

$$f(\beta) = \beta z - \sqrt[n]{\beta^n - 1}\, x,$$

where $\beta \geq 1$. Then f is continuous and $f(1) = z$. Since $z \in \text{rel int } (K)$, there exists $\gamma > 1$ such that $f(\gamma) \in \text{rel int } (K)$.

Let

$$y = f(\gamma) = \gamma z - \sqrt[n]{\gamma^n - 1}\, x.$$

Then

$$\gamma z = y + \sqrt[n]{\gamma^n - 1}\, x,$$

$$z = \frac{1}{\gamma} y + \frac{\sqrt[n]{\gamma^n - 1}}{\gamma} x = \frac{1}{\gamma} y + \sqrt[n]{\frac{\gamma^n - 1}{\gamma^n}}\, x = \frac{1}{\gamma} y + \sqrt[n]{1 - \left(\frac{1}{\gamma}\right)^n}\, x.$$

Let $\alpha = \frac{1}{\gamma}$, then $\alpha \in (0,1)$ and

$$z = \alpha y + \sqrt[n]{1 - \alpha^n}\, x.$$

Also,

$$p(z) \geq \alpha^n p(y) + (1 - \alpha^n) p(x) \geq \alpha^n p(z) + (1 - \alpha^n) p(z) = p(z).$$

Hence, $p(x) \nmid p(z)$, which implies that $p(x) = p(z)$. Now $z \in K$ implies that $2z \in K$. Therefore,

$$2^n p(z) = p(2z) = p(z),$$

which implies that $p(z) = 0$.

The following proposition is used several times in the remainder of this chapter.

<u>Proposition 5.3</u>: Let $p \in \mathcal{P}_n$. If there exists $z \in n - cr(x,y)$ such that $p(z) = \inf\{p(w) : w \in n - cr[x,y]\}$, then $p(x) = p(z) = p(y)$.

Proof: Since $z \in n - cr(x,y)$, then $z = \alpha x + \beta y$, where $\alpha > 0$, $\beta > 0$ and $\alpha^n + \beta^n = 1$. Hence,

$$p(z) \geq \alpha^n p(x) + \beta^n p(y) \geq \alpha^n p(z) + \beta^n p(z) = p(z).$$

Thus, $p(z) \nless p(x)$ and $p(z) \nless p(y)$, which implies that

$$p(x) = p(z) = p(y).$$

In the results to follow n-extreme points are used to characterize those points of certain sets for which the infimum of a function $p \in \mathcal{P}_n$ is obtained.

Proposition 5.4: If $K = n(C)$, where C is a convex subset of $E^+_{n^2}$, $p \in \mathcal{P}_n$ and there exists $z \in K$ such that

$$p(z) = \inf\{p(x): x \in K\},$$

then z is a n-extreme point of K and $z \in C$.

Proof: First notice that $z \in C$, since otherwise by Proposition 2.4, $z = \alpha x$ where $\alpha > 1$ and $x \in C$. Hence,

$$p(z) = \alpha^n p(x) > p(x) \geq p(z),$$

a contradiction. Suppose z is not an n-extreme point of K. Then there exists $x,y \in K$ such that $z = \alpha x + \beta y$, where $\alpha > 0$, $\beta > 0$ and $\alpha^n + \beta^n = 1$. By Proposition 5.3, $p(x) = p(z) = p(y)$. As above, $x,y \in C$. Since $\alpha + \beta > 1$,

$$z = (\alpha + \beta)\left(\frac{\alpha}{\alpha + \beta}\, x + \frac{\beta}{\alpha + \beta}\, y\right) > \frac{\alpha}{\alpha + \beta}\, x + \frac{\beta}{\alpha + \beta}\, y.$$

The set C is convex implies that

$$\frac{\alpha}{\alpha + \beta} x + \frac{\beta}{\alpha + \beta} y \in C.$$

Hence,

$$p(z) > p\left(\frac{\alpha}{\alpha + \beta} x + \frac{\beta}{\alpha + \beta} y\right),$$

a contradiction. Therefore, z is an n-extreme point of K.

In the following the set K is allowed to be more general but p is restricted to belonging to $\mathcal{S}_n$.

<u>Theorem 5.1</u>: If K is a n-convex subset of $\operatorname{int} E^+_{n^2}$, $p \in \mathcal{S}_n$ and there exists $z \in K$ such that $p(z) = \inf\{p(v): v \in K\} > 0$, then z is a n-extreme point of K.

Proof: Suppose z is not a n-extreme point of K. Then there exists $x,y \in K$ such that $z = \alpha x + \beta y$, where $\alpha > 0$, $\beta > 0$ and $\alpha^n + \beta^n = 1$. Since $p \in \mathcal{S}_n$, let

$$p(x) = \sum_{j=1}^{m} \left(\prod_{i=1}^{n} A_{ji}(x) \right).$$

As in Proposition 1.7, $x \in \operatorname{int} E^+_{n^2}$ implies that $A_{ji}(x) > 0$, for each i and j. Likewise, $A_{ji}(y) > 0$ for each i and j. By Proposition 5.3 $p(x) = p(z) = p(y)$. Hence,

$$p(y) = p(z) = \sum_1^m \left(\prod_1^n A_{ji}(\alpha x + \beta y) \right)$$

$$\geq \sum_1^m \left(\prod_1^n (\alpha A_{ji}(x) + \beta A_{ji}(y)) \right)$$

$$\geq \sum_1^m \left(\prod_1^n \alpha A_{ji}(x) + \prod_1^n \beta A_{ji}(y) \right)$$

$$= \sum_1^m \left(\prod_1^n \alpha A_{ji}(x) \right) + \sum_1^m \left(\prod_1^n \beta A_{ji}(y) \right)$$

$$= \alpha^n \sum_1^m \left(\prod_1^n A_{ji}(x) \right) + \beta^n \sum_1^m \left(\prod_1^n A_{ji}(y) \right)$$

$$= \alpha^n p(x) + \beta^n p(y) = (\alpha^n + \beta^n) p(y) = p(y).$$

This implies that

$$\prod_1^n (\alpha A_{ji}(x) + \beta A_{ji}(y)) = \prod_1^n \alpha A_{ji}(x) + \prod_1^n \beta A_{ji}(y)$$

for each j, a contradiction, since $\alpha > 0$, $\beta > 0$, $A_{ji}(x) > 0$ and $A_{ji}(y) > 0$. Therefore, z is a n-extreme point of K.

The following example shows that if the condition $K \subset \text{int } E^+_{n^2}$ is removed, then the theorem is no longer true.

Example 5.1: For $x = (x_1, x_2, x_3, x_4) \in E^+_4$, let

$$p(x) = x_1 x_2 + x_3 x_4.$$

Let $K = 2(\{(2,3,0,0),(0,0,2,3)\})$. Notice that

$$p((2,3,0,0)) = p((0,0,2,3)) = 6.$$

Let

$$z = \frac{1}{2}(2,3,0,0) + \frac{\sqrt{3}}{2}(0,0,2,3) = \left(1, \frac{3}{2}, \sqrt{3}, \frac{3\sqrt{3}}{2}\right).$$

Then $z \varepsilon K$. Also, $p(z) = \frac{3}{2} + \frac{9}{2} = 6$. If $x \varepsilon K$, then

$$x = \alpha(2,3,0,0) + \beta(0,0,2,3),$$

where $\alpha \geq 0$, $\beta \geq 0$ and $\alpha^2 + \beta^2 \geq 1$. Hence,

$$p(x) \geq \alpha^2 p((2,3,0,0)) + \beta^2 p((0,0,2,3)) \geq 6.$$

Hence, $p(z) = \inf\{p(x): x \varepsilon K\}$ and z is not a 2-extreme point of K.

It should be noted, however, that in Theorem 5.1 the condition that $K \subset \text{int } E^+_{n^2}$ is stronger than necessary. Hence, it should be possible to obtain results analogous to those of Theorem 5.1 for more general n-convex sets K.

The analogous question for $p \varepsilon \mathcal{P}_n$ and $K \subset \text{int } E^+_{n^2}$ is more difficult to answer. Considering Example 2.3, it would perhaps seem likely that if $z \varepsilon n - cr(x,y)$ where $x,y \varepsilon K$, then either $z > x$ or $z > y$. This would imply that $p(z) > p(x)$ or $p(z) > p(y)$, which in turn would imply that the $\inf\{p(x): x \varepsilon K\}$ is obtained at an n-extreme point of K if it is obtained at all. However, it is not true that $x,y \varepsilon K \subset \text{int } E^+_{n^2}$ implies that $z > x$ or $z > y$, for all $z \varepsilon n - cr(x,y)$. For example, consider $2 - cr(\{(4,1)(1,4)\})$. Let $\alpha = \frac{10}{17}$. Then

$$\sqrt{1 - \alpha^2} = \sqrt{1 - \frac{100}{289}} = \frac{\sqrt{189}}{17}\,.$$

Therefore,

$$\left(\frac{40 + \sqrt{189}}{17}, \frac{10 + 4\sqrt{189}}{17}\right)$$

$$= \alpha(4,1) + \sqrt{1 - \alpha^2}\,(1,4) \in 2 - cr(\{(4,1),(1,4)\}).$$

However,

$$\frac{40 + \sqrt{189}}{17} < \frac{40 + 20}{17} < 4.$$

It can be shown, for $p \in \mathcal{P}_n$ and K an n-convex subset of int $E^+_{n^2}$, that if there exists $z \in K$ such that

$$p(z) = \inf\{p(x): x \in K\},$$

then either z is an n-extreme point of K or there exists

$$x = (x_1, \ldots, x_{\ 2}) \quad \text{and} \quad y = (y_1, \ldots\ y_{\ 2})$$

such that

$$z = \alpha x + \sqrt[n]{1 - \alpha^n}\; y,$$

where

$$0 < \min\left\{\frac{2x_i y_i}{x_i^2 + y_i^2}: x_i \geq y_i\right\} \leq \alpha \leq \max\left\{\frac{y_i^2 - x_i^2}{y_i^2 + x_i^2}; x_i < y_i\right\} < 1.$$

CHAPTER VI

SUMMARY

The basic purpose of this research has been to study the functions that belong to $\mathcal{P}_n$ and to study n-convex sets. It was found that the product of n monotone concave gauges was a function in $\mathcal{P}_n$. In fact, the collection $\mathcal{S}_n$ of all linear combinations of such products is a subcone of $\mathcal{P}_n$.

The sets $\text{Lev}_\alpha p$, where $p \in \mathcal{P}_n$ and $\alpha \geq 0$ are n-convex. Also, for any set S the n-convex hull of S is given by

$$n(S) = \left\{ \lambda \sum_{1}^{k} \alpha_i x_i : \lambda \geq 1, \quad x_i \in S, \ \alpha_i \geq 0, \ \sum_{1}^{k} \alpha_i^n = 1 \right\}.$$

In particular, $n(S)$ is inverse starlike from the origin. Several examples of the 2-convex hull of points in the plane were given. The n-convex hull of a convex set C was shown to be $\{\alpha x: \alpha \geq 1, x \in C\}$. Moreover, if K and C are two n-convex sets, where $Cr(C) \neq \emptyset$, $K \neq \emptyset$ and $K \cap C = \emptyset$, then there exists a hyperplane that separates K and C. Also, if K and C are disjoint n-convex sets in a linear space L, then there exists complementary n-convex sets A and B of L such that $K \subset A$ and $C \subset B$. Moreover, A and B are convex sets. Further, if x, y, z are three distinct points of a linear space, $u \in n - cr(x,y)$, $v \in n - cr(y,z)$, then

$$(n - cr[z,u]) \cap (n - cr[x,v]) \neq \emptyset.$$

In fact, the point of intersection is not always unique.

If $p \in \mathcal{P}_n$ such that $p \neq 0$ and $S = \{a_1, \ldots, a_m\} \subset [p:1]$ such that $n(S) = \mathrm{Lev}_1 p$, then it was shown that for all $x \in E^+_{n^2}$, where $x \neq 0$, then $p(x) > 0$. In particular, there exists a point of S on each positive axis. Moreover, if $m = n^2$, then p is not an extremal element of $\mathcal{P}_n$.

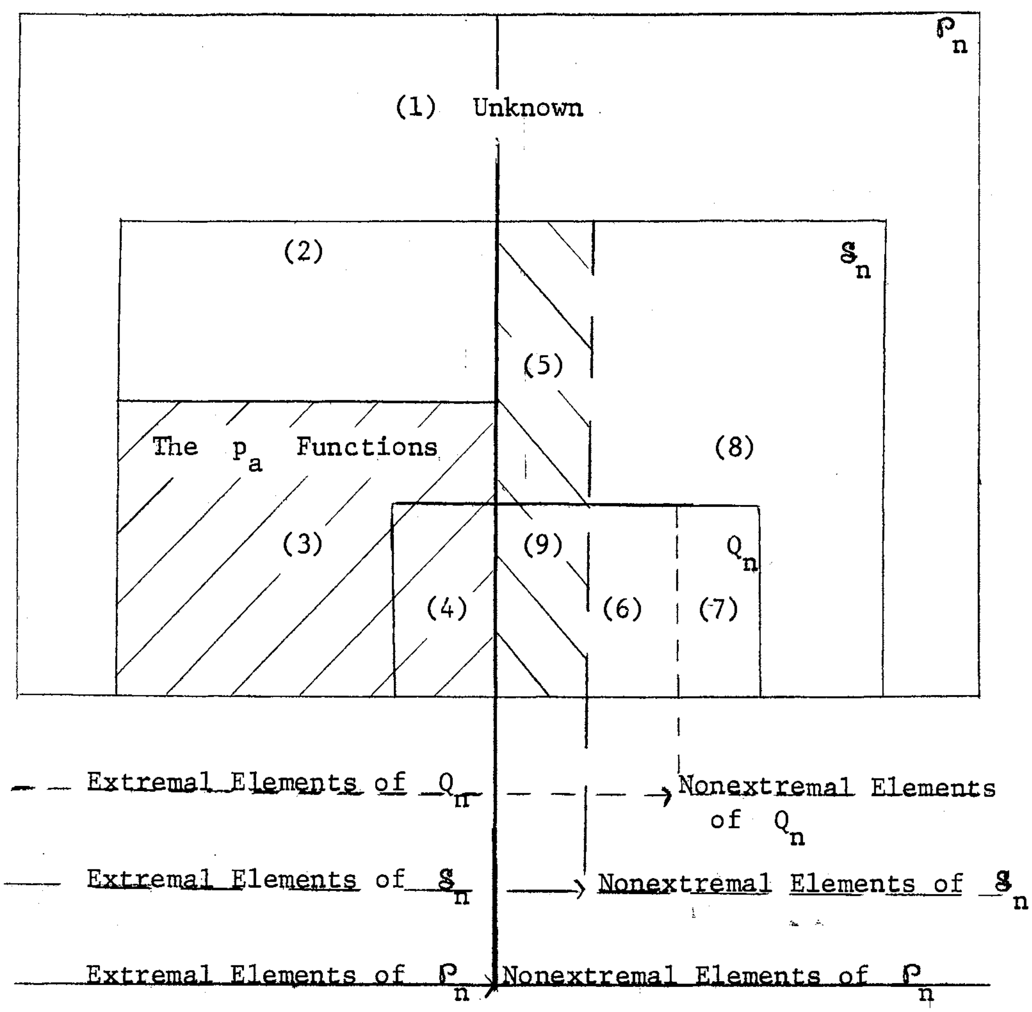
For each $a \in E^+_{n^2} \setminus \{0\}$, the functions p_a, where

$$p_a(x) = \sup\{\lambda^n : x \geq \lambda a\},$$

are all extremal elements of $\mathcal{P}_n$. Since $\mathcal{S}_n$ is a subcone of $\mathcal{P}_n$ and since each $p_a(x) = [q_a(x)]^n$, where $q_a(x) = \sup\{\lambda : x \geq \lambda a\} \in \mathcal{P}'_n$, then each p_a function is an extremal element of $\mathcal{S}_n$ (cf (3), Figure 15). Notice that the functions that belong to (4) of Figure 15 are the functions p_k, where $k = 1, \ldots, n^2$. Also, if

$$p(x) = \prod_1^n A_i(x),$$

where $A_i \in \mathcal{P}'_n$, is an extremal element of $\mathcal{S}_n$, then each A_i is an extremal element in $\mathcal{P}'_n$. One problem for further study would be to determine what are the extremal elements of $\mathcal{P}'_n$. Are they just those functions q_a, where $a \in E^+_{n^2} \setminus \{0\}$? If so, then (2) and (5) in Figure 15 are empty. Does there exist an extremal element A, which is not a q_a function, in $\mathcal{P}'_n$ such that $p(x) = [A(x)]^n$ is an extremal element



$\mathcal{P}_n$
(1) Unknown
(2)
$\mathcal{S}_n$
(5)
The p_a Functions
(8)
(3)
(9)
Q_n
(4)
(6)
(7)
Extremal Elements of Q_n
Nonextremal Elements of Q_n
Extremal Elements of $\mathcal{S}_n$
Nonextremal Elements of $\mathcal{S}_n$
Extremal Elements of $\mathcal{P}_n$
Nonextremal Elements of $\mathcal{P}_n$

Figure 15.

of $\mathcal{P}_n$ (cf (2), Figure 15)? Moreover, does there exist an extremal element A, again not a q_a function, in $\mathcal{P}'_n$ such that $p(x) = [A(x)]^n$ is an extremal element of $\mathcal{S}_n$ but not $\mathcal{P}_n$ (cf (5), Figure 15)? It was shown that if

$$p(x) = \sum_{1}^{n^2} A_i^{\ell(i)}(x),$$

where $\ell(i)$ is a nonnegative integer and

$$\sum_{1}^{n^2} \ell(i) = n,$$

belongs to $\mathcal{S}_n$ and if k is the number of $i = 1, \ldots, n^2$ for which $\ell(i) > 0$, then $k > 1$ implies that p is not an extremal element of $\mathcal{P}_n$. Some immediate problems for further study would be: Determine what functions, if any, belong to $\mathcal{P}_n \setminus \mathcal{S}_n$ (cf. (1), Figure 15). In particular, do the functions f_i in Theorem 4.2 belong to $\mathcal{S}_n$? Notice that together Proposition 4.3 and Example 4.1 give examples of functions that belong to (6) of Figure 15. Is it possible to find a topology for $\mathcal{P}_n$ in which the closure of $\mathcal{S}_n$ would be $\mathcal{P}_n$?

Proposition 4.3 implies the functions p that belong to (9), if any, in Figure 15 are of the form

$$p(x) = x_1^{\ell(1)} \cdots x_{n^2}^{\ell(n^2)},$$

where $\ell(i)$ is a nonnegative integer,

$$\sum_{1}^{n^2} \ell(i) = n$$

and there exists at least three $i = 1, \ldots, n^2$ for which $\ell(i) > 0$. Also, the functions in (7) of Figure 15 are those $p \in Q_n$ that can be expressed as the sum of two or more functions that also belong to Q_n. The functions in (8) of Figure 15 are for the most part unknown.

It was found that for $p \in \mathcal{P}_n$ and K a n-convex subset of $E^+_{n^2}$, the existence of $z \in \text{rel int } K$ such that $p(z) = \inf\{p(x) \colon x \in K\}$ implies p is zero over K. Further, if $K = n(C)$, where C is a convex subset of $E^+_{n^2}$, $p \in \mathcal{P}_n$ and there exists $z \in K$ such that $p(z) = \inf\{p(x) \colon x \in K\}$, then z is a n-extreme point of K and $z \in C$. Also, if K is a n-convex subset of $\text{int } E^+_{n^2}$, $p \in \mathcal{S}_n$ and there exists $z \in K$ such that $p(z) = \inf\{p(v) \colon v \in K\} > 0$, then z is a n-extreme point of K.

Numerous questions arise which would be of interest for further research. For example, if $p \in \mathcal{P}_n$ and K is a n-convex subset of $E^+_{n^2}$, then does there exist a n-extreme point of K at which p assumes its minimum value over K? If not, what modifications are necessary for the result to hold? Perhaps K might be chosen to be the n-convex hull of a compact set. Is it possible to prove a Krein-Milman type theorem for n-convex sets and n-extreme points? Another possibility would be to assume the functions in $\mathcal{P}_n$ are differentiable and study the cone $\mathcal{P}_n$. A further study of n-convex sets analogous to that for convex sets might prove profitable. Also, a further study of the topological properties of n-convex sets might prove interesting. Certainly, results analogous to Helly's Theorem and Blaschke's Theorem for convex sets would be of interest. Finally, let p be a superadditive, nonnegative, homogeneous function of degree n defined on some subset of $E^+_{n^2}$, find an extension of p to all of $E^+_{n^2}$. For example, let $J^+_{n^2} \subset E^+_{n^2}$ where $J^+_{n^2} = \{(x_1, \ldots, x_{n^2}): x_i \text{ is an integer}\}$. If $p: J^+_{n^2} \to E^+_1$ such that $p(x + y) \geq p(x) + p(y)$ and $p(\alpha x) = \alpha^n p(x)$ for every nonnegative integer α, extend p to an element of $\mathcal{P}_n$.

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