

DYNAMICS OF BALLISTIC BODIES WITH
VARIABLE ASYMMETRIES

By

CHARLES FREMONT HUFF

Bachelor of Science
The University of Texas
Austin, Texas
1961

Master of Science
The University of Texas
Austin, Texas
1963

Submitted to the Faculty of the
Graduate College of the
Oklahoma State University
in partial fulfillment of
the requirements for
the Degree of
DOCTOR OF PHILOSOPHY
May, 1971

OKLAHOMA
STATE UNIVERSITY
LIBRARY
AUG 11 1971

DYNAMICS OF BALLISTIC BODIES WITH
VARIABLE ASYMMETRIES

Thesis Approved:

James B. Partin
Thesis Adviser
W. C. Brooke
R. L. Lowery
J. L. Norton
Henry R. Seibert
D. D. Durham
Dean of the Graduate College

788339

PREFACE

The information in the following work is presented in a manner which makes the problem easy to comprehend. The motion of a slender symmetric missile is described to give the reader an understanding of the basic problem. The effect of variable asymmetries on the motion is then developed to demonstrate their ability to alter the motion pattern of the original vehicle. This is specifically done to allow anyone familiar with dynamics and vibrations, but not the specific problem, to understand the basic problem and then follow the development of the effect of a variable asymmetry.

I am deeply indebted to Dr. James R. Partin, my thesis adviser; Dr. W. Bryan Brooks, and Dr. Henry R. Sebesta for their assistance in conducting the research. The many long hours that each of these gentlemen spent discussing the problem and contributing ideas have been invaluable in the completion and presentation of this investigation. I would like to thank Dr. R. L. Lowery for serving as chairman of my committee and Dr. J. R. Norton for being on the committee. In addition I would like to thank the many members of the faculty who have so graciously given their time.

I am grateful to the National Science Foundation for providing partial financial support for this work.

Without the support of my family, my research could not have been a success. In addition my wife, Mary, has provided immeasurable help in the preparation of this document.

TABLE OF CONTENTS

Chapter	Page
I. INTRODUCTION	1
II. DEVELOPMENT OF THE EQUATIONS OF MOTION FOR RESIDENT ASYMMETRIES	11
III. ANALYSIS OF THE GENERAL EFFECT OF STATIONARY ASYMMETRIES	23
IV. EFFECTS OF A MASS IN A RACE	34
V. ANALYSIS	53
VI. EXAMPLES	60
VII. CONCLUSIONS AND RECOMMENDATIONS	75
SELECTED BIBLIOGRAPHY	78
APPENDIX A - MOTION OF A MASS IN A MOVING COORDINATE SYSTEM	80
APPENDIX B - FINITE DIFFERENCE SIMULATION	84
APPENDIX C - EFFECTS OF A VARIABLE TRIM MOMENT	86
APPENDIX D - NOMENCLATURE	90

LIST OF TABLES

Table	Page
I. Example Parameters	61
II. Parameters for Example Trajectories	66

LIST OF FIGURES

Figure	Page
1. Typical Motion Behavior for a Ballistic Missile with Small Asymmetries	2
2. Basic Coordinate Systems	14
3. Trim Angles Versus Roll-Rate Ratio for an Asymmetry About the z-Axis	29
4. Variable Asymmetry Positional Coordinates	37
5. Effect of a Variable Asymmetry on the Angle of Attack as a Function of Roll-Rate Ratio with $m_b/m_v = .01$	64
6. Effect of a Variable Asymmetry on the Angle of Attack as a Function of Roll-Rate Ratio with $m_b/m_v = .03$	65
7. Roll-Rate Ratio and Damping Parameter as a Function of Time for Selected Examples	69
8. Roll Rate as a Function of Time for Selected Examples	70
9. Roll Rate as a Function of Time for Selected Examples	73
10. General Coordinates of a Moving Point in a Body-Fixed Coordinate System	81

CHAPTER I

INTRODUCTION

Small symmetric ballistic missiles flying through an atmosphere often exhibit anomalous motion behavior that can cause severe degradation of accuracy or total destruction of the vehicle. The undesirable motion can be attributed to certain small mass and aerodynamic asymmetries that arise in manufacturing, handling, or flight. Typical behavior of the roll rate and angle of attack is shown in Figure 1 for vehicles with and without asymmetries. Often a need arises to alter the resident motion pattern to reduce undesirable effects. Efforts to control the motion through close manufacturing tolerances have met with only limited success partly because the magnitude of asymmetries that can be troublesome may arise from physical changes that can take place in handling and flight. Active control devices with sensing elements can eliminate the anomalous motion but at a great weight, reliability, and cost penalty.

The major contribution of this work is in the description of the effect of variable asymmetries on ballistic missiles with small resident asymmetries. In addition, the effects of the resident asymmetries are described more completely than in previous works. This work will investigate

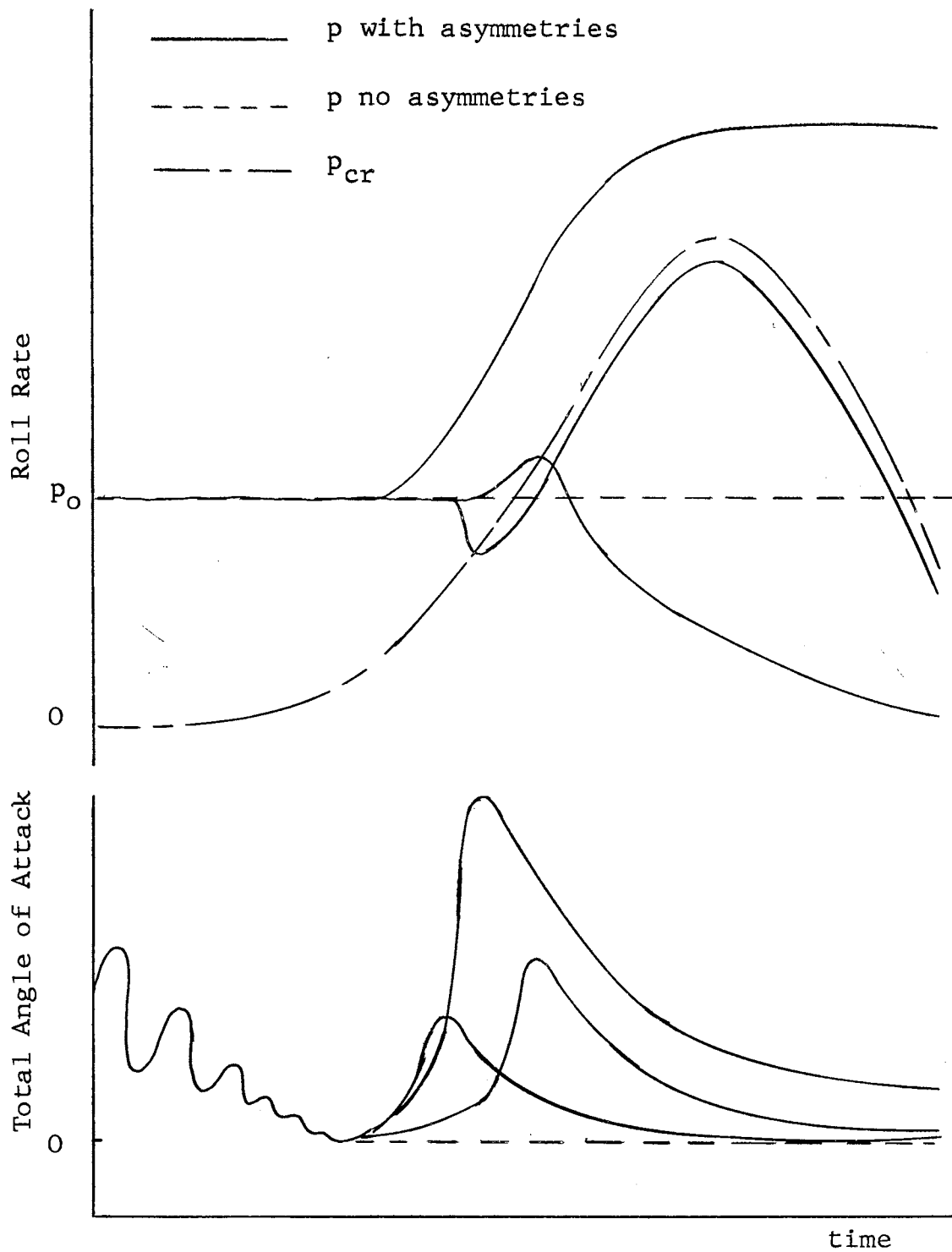


Figure 1. Typical Motion Behavior for a Ballistic Missile with Small Asymmetries

some of the effects of two types of variable asymmetries that can be used to disrupt a particular motion pattern. The first asymmetry consists of a ball in a circular race mounted perpendicularly to the axis of symmetry. This ball may be either forced or free depending on the desired effect. The "free" ball is actually constrained by the race and the damping in the race. The effect of such a ball is to add a moving trim force and moment to the vehicle. The actual configuration need not be a ball in a race but any mass asymmetry capable of moving about the axis of symmetry. Another possible asymmetry that will affect the motion of the vehicle is a rotating flap that can be free or forced. The effect of such a device will also be described.

An analytical approach is presented that is capable of predicting the effect of the variable asymmetries on the quasi-steady state motion of a missile. The analytical theory that is developed is quite an accurate method for predicting the influence of the asymmetries. The asymmetries are shown to be very effective in altering the characteristics of the motion of a vehicle. Numerical examples are given to demonstrate the effectiveness of the asymmetries in changing the motion pattern. These examples are compared with finite-difference simulations of the full equations of motion. The exact effect is controllable through the geometric parameters of the asymmetry producing device which are alterable, subject to design constraints.

Equations of motion are developed for resident asymmetries before the addition of a variable asymmetry is considered. This is done because the basic question to be answered in this work is; what is the effect of the variable asymmetry upon the motion of a vehicle with resident asymmetries. A variable asymmetry is then added to the system and its effect on the motion of the vehicle is determined. The analysis of the effect of the variable mass asymmetry is begun with the basic equations of motion of a two body system to insure that all effects of the variable asymmetry are considered. The resulting equations are combined with the resident asymmetry equations. The equations that result when the variable asymmetries are stationary are special cases of the equations for resident asymmetries. The general situation with resident asymmetries can be developed in the same manner as that used in describing the motion with a variable asymmetry by considering asymmetry positions to be stationary. However, this would be cumbersome because multiple masses and flaps must be used to completely represent the general cases of fixed asymmetries.

The investigation is limited to slender axisymmetric ballistic missiles traveling at high velocities. The vehicle is assumed to have constant aerodynamic coefficients over the small angles of attack to which the motion is limited.

The basic description of the motion of ballistic missiles with small asymmetries has been developed by

Nelson (1), Nicolaides (2), and Murphy (3). Their work developed suitable coordinate systems and simplified equations of motion but failed to provide adequate explanation of the anomalous motion behavior that has been observed for vehicles without roll controlling fins. More recently work by Pettus (4), Barbera (5), and Hodapp and Clark (6) have given suitable asymmetric conditions to account for most of the observed motion patterns on finless vehicles.

Pettus defined the asymmetries responsible for the anomalous behavior and gave the basic simplified equations of motion but did little analysis in this paper. Barbera analyzed Pettus's angle of attack equations for steady state motion but only treated the roll equation by finite difference techniques. Hodapp and Clark analyzed similar equations of motion assuming that part of the moment asymmetries are caused by products of inertia instead of geometric trim moments. In the following work it is shown that these situations are actually the same. Care must be taken when comparing the two methods to insure that the same reference axes are used in describing the aerodynamic forces and moments to avoid comparing cases that appear to be the same but are actually quite different.

The general development used is as follows. The equations of motion completely describing the motion of a ballistic vehicle in body-fixed axes are:

$$\bar{\mathbf{F}} = m\dot{\bar{\mathbf{V}}} + \bar{\boldsymbol{\omega}} \times m\bar{\mathbf{V}} \quad (1-1)$$

$$\text{and } \bar{\mathbf{M}} = \dot{\bar{\mathbf{H}}} + \bar{\boldsymbol{\omega}} \times \bar{\mathbf{H}} \quad (1-2)$$

where $\bar{\mathbf{F}}$ is the external force,

$\bar{\boldsymbol{\omega}}$ inertial angular velocity,

m the vehicle mass,

$\bar{\mathbf{V}}$ is the velocity of the center of mass,

$\bar{\mathbf{M}}$ is the external moment about the center of mass,

and $\bar{\mathbf{H}}$ is the angular momentum of the vehicle about the center of mass.

Coordinate systems are chosen to relate the body position to a reference coordinate system arbitrarily defined as an inertial system. The coordinate systems used are either a body-fixed coordinate system (used in this work) or a non-rolling precessing aeroballistic coordinate system that is referenced to a flight-path-fixed coordinate system. Generally the change in flight path angle is small enough to allow it to be considered one axis of the inertial set. The second system requires an additional coordinate to define the body angular position since the precessing aeroballistic axes have one coordinate fixed to the vehicle axis of symmetry and the other two axes not rolling with the body. Care must be taken when comparing the two systems as the angular rates in the precessing aeroballistic system are slightly different than those of the body

fixed system.

The next step in the analysis is the simplification of the equations of motion so that analytical information can be obtained. Because of aerodynamic loading, the most important parameters are the wind-body angles (see Figure 2) which describe angle of attack. For this reason the velocities are expressed in terms of wind-body relative velocities (u, v, w). The angular velocities are expressed in inertial-body-angular rates (p, q, r). The angles α and β may be approximated by

$$\alpha \cong w/V$$

and $\beta \cong v/V$

for small angles of attack (α) and sideslip (β). The aerodynamic forces and moments are then approximated as functions of $Q, \alpha, \beta, p, q,$ and r and their derivatives. The effect of gravity on the angular motion of missiles in high velocity flight is considered negligible.

The equations orthogonal to the axis of symmetry are combined using the complex transformation $\theta = \beta + i\alpha$ and $\xi = q + ir$. The translational equation is used to eliminate ξ from the rotational equation. The other remaining equations are a translational and a rotational equation along and about the x-axes (axis of symmetry), respectively. The axial motion equation is generally assumed to have little effect on the angular motion. This is true when the velocity change does not affect the $\dot{\alpha}$ and $\dot{\beta}$ equations.

When terms with small coefficients are neglected, the angular motion is now described by a second-order ordinary differential equation in a complex angle of attack, θ , and a first order ordinary differential equation describing the roll acceleration \dot{p} . These equations are coupled and have time-varying coefficients. Generally the assumption of a slowly varying dynamic pressure and roll rate allows the use of the method of variation of parameters described by Cunningham (7) to solve the now linearized second order equations for θ as a function of p . The solution $\theta(p)$ can be substituted into the roll rate equation resulting in a first-order nonlinear differential equation in p alone. Analytical treatment of this equation has only recently been attempted by Pettus (8) and independently by the author.

The physical mechanism that gives rise to the anomalous motion is a resonance phenomena which is common to all driven oscillatory systems. However, in this system it is possible for the roll rate which is equivalent to the driving frequency of the oscillator to be excited by the vehicle mass and aerodynamic asymmetries. The angle of attack is equivalent to the magnitude and phase response of the oscillator. The forces generated by the small mass and aerodynamic asymmetries are equivalent to the driving force of an oscillator. If a body-fixed coordinate system is used the equations do not appear similar to those of an oscillator but in precessing aerobalistic coordinates

the equations are comparable directly. This relationship is most clearly seen in the reference by Vaughn (9). It is possible with small asymmetries for an equilibrium condition (called "persistent roll resonance") to be established. This condition is characterized by a constant roll rate to resonant frequency ratio. The natural or resonant frequency of the system varies with the dynamic pressure. Persistent roll resonance motion occurs in the steady-state portion of the angle of attack motion. In the steady-state part of the motion, because of asymmetries, the vehicle has one side preferentially oriented toward the wind vector (hence, the term "lunar motion"). The result of this motion is an amplified angle of attack and large roll rate change which causes dispersion, uneven ablation, and vehicle destruction. Figure 1 shows typical effects of asymmetries on vehicle roll rate and angle of attack for the flight of a vehicle on a reentry flight. In addition, large roll rates cause excessive centrifugal loadings.

In Chapter II the full set of equations of motion for a rigid body are given. The bodies are specified to be symmetric but can possibly have small resident distortions in this symmetry that are accounted for by small perturbations on the symmetric-case equations. From these equations of motion, simplified equations are developed for analysis. In Chapter III the approximate equations developed in Chapter II are analyzed to show the effect of stationary asymmetries on the quasi-steady-state motion of

high velocity ballistic missiles. The effects of variable asymmetries on the quasi-steady-state motion of vehicles with fixed asymmetries are described in Chapters IV and V. These chapters consider a moving internal mass while Appendix C considers a moving flap. The flap is shown to have a basic effect very similar to the mass. The analytical expressions are compared with finite difference solutions of the basic equations of motion in Chapter VI. Chapter VII concludes the presentation and recommends additional investigations that can be conducted.

CHAPTER II

DEVELOPMENT OF THE EQUATIONS OF MOTION FOR RESIDENT ASYMMETRIES

The following is a development of the equations of motion for a basically symmetric ballistic missile flying through an atmosphere at high velocity and small angles of attack. The vehicle is assumed to have small fixed distortions in its symmetry which cause deviations in the applied aerodynamic forces and moments. The complete equations of motion are presented in Equations (2-1) through (2-6) with the dominant forces and moments approximated by commonly accepted functions in (2-7) through (2-12). The wind relative velocities are expressed by the linear equivalent angles of attack and sideslip.

The equations about the axes orthogonal to the axis of symmetry are combined for simplicity to give an angle of attack equation. The translational equations are not solved for displacement since the angular motion is of primary interest. The only use made of the translational equations is to relate the inertial roll rates q and r to the angles of attack α and β .

The motion of a rigid body in free flight in an atmosphere can be completely described by six equations of

motion in cartesian coordinates. The equations define the six degrees of positional freedom available to the body axis system referred to the inertial axis system. The equations of motion, in terms of wind-referenced velocities and inertially referenced angular rates in a coordinate system fixed to the body at the center of mass as given by Thomson (10), are:

$$F_x = m(\dot{u} - vr + wq), \quad (2-1)$$

$$F_y = m(\dot{v} - wp + ur), \quad (2-2)$$

$$F_z = m(\dot{w} - uq + vp), \quad (2-3)$$

$$\begin{aligned} M_x = & \dot{p}I_{xx} - qr(I_{yy} - I_{zz}) + (r^2 - q^2)I_{yz} \\ & - (\dot{r} + pq)I_{xz} + (pr - \dot{q})I_{xy}, \end{aligned} \quad (2-4)$$

$$\begin{aligned} M_y = & \dot{q}I_{yy} - pr(I_{zz} - I_{xx}) + (p^2 - r^2)I_{zx}, \\ & - (\dot{p} + qr)I_{xy} + (pq - \dot{r})I_{zy}, \end{aligned} \quad (2-5)$$

$$\begin{aligned} \text{and } M_z = & \dot{r}I_{zz} - pq(I_{xx} - I_{yy}) + (q^2 - p^2)I_{xy} \\ & - (\dot{q} + pr)I_{yz} + (qr - \dot{p})I_{xz}. \end{aligned} \quad (2-6)$$

These equations are the expanded form of Equations (1-1) and (1-2) where

p, q, r are inertial angular rates,

u, v, w are wind-relative velocities in the atmosphere along the x, y , and z axes,

m is vehicle mass,

I 's are components of the inertia tensor,

F 's are external forces,

and M 's are external moments.

The coordinate systems are shown in Figure 2. The external forces arise from aerodynamic forces and gravitational effects. This set of equations represents a set of six nonlinear first order differential equations for which no closed form solution is available. Approximations must be made in order to obtain information about a vehicle's motion. The most common approaches are equation simplification and finite difference simulation with complete equations. The former may be used to gain insight about general motion properties while the latter is an excellent means of simulating explicit individual flights. Both methods require explicit definitions for the aerodynamic forces and moments.

The aerodynamic forces and moments are generally expressed as functions of wind-related kinematic functions and flow conditions. In this work they will be assumed to be functions of α , β , p , q , r , and Q . Variations due to Mach and Reynolds number changes will not be considered. The forces and moments are generally considered to be linear functions of the dynamic pressure, Q , so that dividing F by Q and a reference area, S , and M by Q , S ,

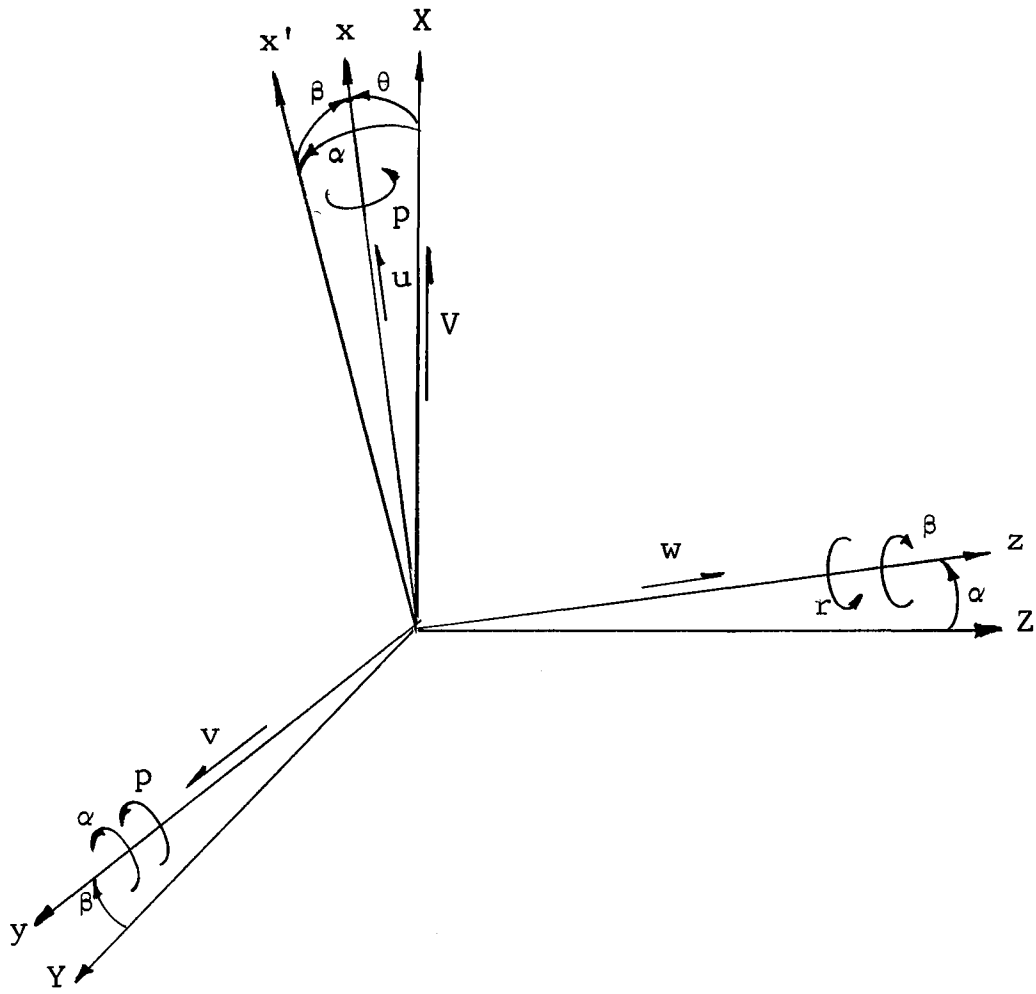


Figure 2. Basic Coordinate Systems

and a reference length, d , results in non-dimensional forces and moments that are not functions of Q . The assumption of linearity in Q is quite good in the situation considered since Q varies very slowly for slender vehicles at small angles of attack. The dimensionless quantities can be expanded in a Taylor's series about the zero values of α , β , p , q , and r . The resulting series consists of constant differential terms that are often called "stability derivatives" or "aerodynamic coefficients" multiplied by the various combinations of the variables. A complete listing of the stability derivatives and a discussion of them is given by Nielsen (11).

Only certain terms of the series contribute significant forces and moments. The terms used in this analysis will include those retained in References 1 through 6. Only zero and first order terms are retained since α , β , q , and r are small. Exceptions to this are several second order terms containing p since p is not restricted to small angular rates.

The general scheme of identification consists of subscripting the force coefficients with x , y , z and the moment coefficients with l , m , n to identify the direction or axis on which they act. A moment about the x -axis is signified by l , a moment about the y -axis by m , and a moment about the z -axis by n . The additional subscripts indicate the variables of differentiation in the Taylor's series. For instance

$$C_{m\alpha} = \frac{\partial}{\partial \alpha} (M/QdS) \text{ about the } y \text{ axis,}$$

$$\text{and } C_{mp\beta} = \frac{\partial^2}{\partial p \partial \beta} (M/QdS) \cdot 2V/d \text{ also about the } y \text{ axis.}$$

The coefficients used are described as

C_{x0} - axial force coefficient,

$C_{y\beta}$, $C_{z\alpha}$, C_{y0} , C_{z0} - normal force coefficients,

$C_{m\alpha}$, $C_{n\beta}$, C_{m0} , C_{n0} - restoring moment coefficients,

C_{lp} , C_{lo} - roll moment coefficients,

and C_{mq} , C_{nr} , $C_{mp\beta}$, $C_{np\alpha}$ - damping coefficients.

The signs associated with the coefficients are the standard for missile aerodynamics which is used in the first six references. The constant terms C_{y0} , C_{z0} , C_{m0} , and C_{n0} are the results of deviations of the vehicle from perfect symmetry. A lower case (o) indicates the constant terms from the series. The coefficients differentiated with respect to p , q , or r have been multiplied by $2V/d$ to make them dimensionless.

The forces and moments in terms of retained coefficients are given in Equations (2-7) through (2-9). The basic moments are derived about an axis system through the axis of symmetry. The moments given in (2-10) through (2-12) have been transferred to an axis system located at the center of mass. This accounts for the terms containing

F_x , F_y , and F_z . The resulting expressions for the forces and moments used in the analysis are:

$$F_x = -(C_{x0})QS, \quad (2-7)$$

$$F_y = (C_{y\beta}\beta + C_{y0})QS, \quad (2-8)$$

$$F_z = (C_{z\alpha}\alpha + C_{z0})QS, \quad (2-9)$$

$$M_x = (C_{lp}pd/2V + C_{l0})QSd + F_z\delta Y - F_y\delta Z, \quad (2-10)$$

$$M_y = (C_{m\alpha}\alpha + C_{mp\beta}p\beta d/2V + C_{mq}qd/2V + C_{m0})QSd + F_x\delta Z, \quad (2-11)$$

$$\text{and } M_z = (C_{n\beta}\beta + C_{np\alpha}p\alpha d/2V + C_{nr}rd/2V + C_{n0})QSd - F_x\delta Y, \quad (2-12)$$

where δY and δZ are the distances from the center of mass to the axis of geometric symmetry. All other external forces have been neglected.

The systems considered will be limited to flights with small angles of attack. Small angular motions can be treated as vectors which makes the order of angular rotation unimportant. A proof of the vector properties of infinitesimal rotations is given by Goldstein (12). If the angles are limited to small values then $\alpha \cong w/V$ and $\beta \cong v/V$ and if \dot{V} is of the order of v and w then $\dot{\alpha} \cong \dot{w}/V$ and $\dot{\beta} \cong \dot{v}/V$. The translational equations become

$$F_x = mV(\dot{u}/V - \beta r + \alpha q), \quad (2-13)$$

$$F_y = mV(\dot{\beta} - \alpha p + r), \quad (2-14)$$

$$\text{and } F_z = mV(\dot{\alpha} - q + \beta p). \quad (2-15)$$

Since the body is symmetric except for the small asymmetries, an orthogonal transformation such as the complex number system is convenient to simplify writing the equations. Multiplication of equations (2-6) and (2-15) by i and adding to (2-5) and (2-14), respectively, gives

$$F_y + iF_z = mV(\dot{\beta} + i\dot{\alpha} + p(i\beta - \alpha) + r - iq) \quad (2-16)$$

$$\text{and } M_y + iM_z = (\dot{q} + i\dot{r})I + p(I - I_x)(-r + iq) \quad (2-17)$$

where the coordinate axes are the principal axes with the assumed condition $I_{yy} \cong I$ and $I_{zz} \cong I$.

These equations represent a pair of complex first-order differential equations. It is convenient to let $\theta = \beta + i\alpha$ and $\xi = q + ir$. In addition the basic symmetry of the vehicles that are considered allows the use of a single coefficient for the orthogonal force and moment coefficients. If these coefficients are defined as $C_{N\theta} = -C_{y\beta} = -C_{z\alpha}$, $C_{M\theta} = C_{m\alpha} = -C_{n\beta}$, $C_{M\dot{\theta}} = C_{mq} = C_{nr}$, and $C_{Mp\theta} = C_{mp\alpha} = C_{np\beta}$; then equations (2-16) and (2-17) after use of (2-7) through (2-12) become

$$(-C_{N\theta}\dot{\theta} + C_{y\dot{\theta}} + iC_{z\dot{\theta}})QS = mV(\dot{\theta} + ip\theta - i\dot{\xi}) \quad (2-18)$$

$$\begin{aligned} \text{and } & (-iC_{M\theta}\dot{\theta} + (C_{Mp\theta}p\dot{\theta} + C_{M\theta}\dot{\xi})d/2V + C_{mo} + iC_{no})QSd \\ & - (+C_{xo})(\delta Z - i\delta Y)QS = \ddot{\xi}I + ip\dot{\xi}(I - I_x). \end{aligned} \quad (2-19)$$

Substitution of (2-18) into (2-19) and rearrangement of the results gives

$$\ddot{\theta}_P + E_P\dot{\theta}_P + F_P\theta_P = G_P \quad (2-20)$$

where $E_P = -C_{M\theta}\dot{d}/2VI' + C_{N\theta}/M' + ip(2 - I_x/I)$,

$$\begin{aligned} F_P = & - (C_{M\theta} + (ip(C_{Mp\theta} + C_{M\theta}\dot{\xi}) + C_{M\theta}C_{N\theta}/M')d/2V)/I' \\ & - p^2(1 - I_x/I) + ip(1 - I_x/I)C_{N\theta}/M' + ip, \end{aligned}$$

$$\begin{aligned} G_P = & C_{M\theta}(C_{yo} + iC_{zo})d/2VM'I' + i(C_{mo} + iC_{no})/I' \\ & - i(+C_{xo})(\delta Z - i\delta Y)/I'd \\ & + ip(1 - I_x/I)(C_{yo} + iC_{zo})/M', \end{aligned}$$

$$I' = I/QSd,$$

$$M' = mV/QS,$$

$$\text{and } Q = \frac{1}{2}\rho^*V^2S.$$

It should be noted that Equation (2-18) must be differentiated to obtain $\ddot{\xi}$. The subscript P indicates principal-axis-referenced quantities. The roll equation is

$$\begin{aligned} \dot{p} = & - (\delta Y(C_{N\theta}\alpha_P - C_{zo}) - \delta Z(C_{N\theta}\beta_P + C_{yo}))/I'_x \\ & + (C_{lp}pd/2V + C_{lo})/I'_x \end{aligned} \quad (2-21)$$

where $I'_x = I_x/QSd$.

Since the above is developed for systems with V large and ρ^* small, terms with equal powers of ρ^* and V will be negligible compared to terms with higher orders of V . Thus, terms divided by $I'M'V$ fall into this class and are deleted from the analysis.

The terms C_{y0} , C_{z0} , C_{m0} , and C_{n0} are asymmetry terms arising from forces and moments on the body when the principal x-axis or the wind vector are aligned. Possible causes are geometrical distortions of the body and mass asymmetries which cause a shift of the principal x-axis from the axis of geometric symmetry. If the equations of motion are written about the geometric axes or any other axes instead of principal axes, the parts of the trim coefficients of (2-20) due to axis misalignment are replaced by terms containing trim angles that arise from the rotation of the axes. The angles of rotation between the principal and geometric axes given by Vaughn (13) are

$$\alpha_J = I_{xz}/(I - I_x),$$

$$\beta_J = I_{xy}/(I - I_x),$$

$$\text{and } \theta_J = \beta_J + i\alpha_J \quad (2-22)$$

where the inertia terms are about the geometric axes. The angles are measured from the geometrical to the principal axes and represent rotations about the y and z axes. The

orthogonal axes y and z are virtually identical in either system for small θ_J . It is now possible to present Equation (2-20) written about the geometric axes by use of a simple transformation.

If one defines a $\theta = \theta_P + \theta_J$ and deletes the small terms from (2-20) one obtains

$$\ddot{\theta} + E\dot{\theta} + F\theta = G \quad (2-23)$$

where $E = -dC_{M\theta}/2VI' + C_{N\theta}/M' + ip(2 - I_x/I)$,

$$F = - (C_{M\theta} + ip(C_{Mp\theta} + C_{M\theta}^{\cdot})d/2V)/I' + ip^2(1 - I_x/I) + ip(1 - I_x/I)(C_{N\theta}/M'),$$

$$\begin{aligned} \text{and } G = & i(C_{mo} + iC_{no})/I' - C_{xo}(\delta Z - i\delta Y)/I'd \\ & + ip(1 - I_x/I)(C_{yo} + iC_{zo})M' - C_{M\theta}\theta_J/I' \\ & + ip(C_{Mp\theta} + C_{M\theta}^{\cdot})\theta_J d/2VI' - p^2(1 - I_x/I)\theta_J \\ & + ip(1 - I_x/I)C_{N\theta}\theta_J/M' + ip\theta_J^{\cdot}. \end{aligned}$$

The equation for roll is

$$\begin{aligned} I_{xp}^{\cdot\cdot} = & (\delta Z(C_{N\theta}(\beta - \beta_J) - C_{zo}) - \delta Y(C_{N\theta}(\alpha - \alpha_J) - C_{yo})) \\ & + (C_{1p}pd/2V + C_{1o}) \end{aligned} \quad (2-24)$$

It should be noted that the value of p in the two sets of equations is the same. Equations (2-23) and (2-24) are approximations to Equations (2-1) through (2-6) under the stated conditions. These equations are similar to those

obtained by Hodapp and Clark (6). Equations (2-20) and (2-21) express the same information with the terms containing θ_J in Equation (2-23) appearing in (2-20) in the C_{y0} , C_{z0} , C_{m0} , and C_{n0} terms. If information from the two sets of equations are compared, one should be certain that comparison is made in the forms given above. It is quite easy to compare different cases of trim conditions because the terms containing θ_J could be included in C_{y0} , C_{z0} , C_{m0} , and C_{n0} in (2-23) and (2-24).

The equations are approximations of the equations of motion for a vehicle with fixed asymmetries. They are nonlinear and a closed form solution is not known. Some type of nonlinear analysis must be applied to obtain information analytically. The following chapter gives an analysis of these equations.

CHAPTER III

ANALYSIS OF THE GENERAL EFFECT OF STATIONARY ASYMMETRIES

In this section the approximate equations of motion developed in the previous chapter will be analyzed. The Equations (2-20) and (2-21) approximately describe the angular motion of a ballistic missile. These two equations actually represent a fifth-order set of differential equations. Complete solution of the simplified equations is not feasible because of nonlinearities and time varying coefficients. Several physical considerations make the analysis easier. This analysis is developed for well designed vehicles. As such the aerodynamic restoring force is positive and the effective damping is great enough to insure that the transient oscillations decay over a relative short period of the trajectory. The dynamic pressure, Q , and the roll rate, p , vary slowly compared to the natural frequency of the system. Because of this the equations can be analyzed satisfactorily using a decoupling technique such as variation of parameters. This allows the angle of attack equation to be solved with p and Q as noninfluencing variables.

Equations (2-20) and (2-21) are of the form $\ddot{\theta} = F(\theta, \dot{\theta}, p)$ and $\dot{p} = f(\theta)$. The solution of these can be accomplished by using the variation of parameters technique described by Cunningham (7) by approximating p and Q in the $\ddot{\theta}$ equation by $p = \bar{p} + \dot{\bar{p}}\delta t$ and $Q = \bar{Q} + \dot{\bar{Q}}\delta t$ where $\bar{p} = p$ and $\bar{Q} = Q$ at time t . This approach is acceptable because the analysis can be conducted over short time periods. The $\ddot{\theta}$ equation is solved by retaining only the constant portion of p and Q . The resulting θ is used in the \dot{p} equation. This \dot{p} can then be used in the $\ddot{\theta}$ equation to include the time varying terms $\dot{\bar{p}}\delta t$ and $\dot{\bar{Q}}\delta t$ and an improved θ obtained. If the value of $\dot{\bar{p}}$ and $\dot{\bar{Q}}$ are small as in the type problems considered here, the improvement of the second step is minor and the first θ and \dot{p} generated are used allowing the \bar{p} and \bar{Q} used to be the values at time t . The resulting function $\theta(p, Q)$ is valid for arbitrary values of p and Q . The equation for angle of attack is then substituted into the roll equation. This results in a differential equation in p and \dot{p} that is nonlinear and can be studied using stability analysis about the equilibrium points.

It is actually more meaningful to transform the roll equation into function of frequency ratio, λ . The frequency ratio is defined as p/p_{cr}

$$\text{with } p_{cr} = \pm (-C_{M\theta} Q S d / (I - I_x))^{1/2}.$$

The importance of λ will become apparent in the discussion of the stability of the vehicle about constant roll-rate

ratios. The value of \dot{p}_{cr} , though small, is left in the roll ratio equation because the torque required to maintain equilibrium is a function of \dot{p}_{cr} .

The solution of Equations (2-20) or (2-23) can be divided into two parts; a transient or homogenous solution, and a steady-state or particular solution. The transient solution has an incoherent effect on the roll equation. The reason for this is the homogenous part of the solution for α and β are high frequency sinusoidal functions of small amplitude that enter directly into the roll torque. Therefore, the roll torque will also be sinusoidal with relatively high frequency and low amplitude. The integral of the torque will be a small number. The asymmetries which are responsible for the anomalous motion give rise to the particular solution of the angle of attack equation. The particular solution is non-oscillatory so that the torque produced by this portion of the solution will have a coherent effect on the roll torque, and consequently an effect on the overall vehicle behavior. Because of this the analysis will be limited to the steady-state portion of the solution.

The particular solution for Equation (2-23), under the assumptions made, may be written as

$$\beta = (AC + BD) / (C^2 + D^2) \quad (3-1)$$

and $\alpha = (BC - AD) / (C^2 + D^2)$

$$\text{where } A = (C_{no} + C_{xo} \delta Y/d)/C_{M\theta} - \lambda^2 \beta_J + pR_1 + \beta_J,$$

$$B = (-C_{mo} + C_{xo} \delta Z/d)/C_{M\theta} - \lambda^2 \alpha_J - pR_2 + \alpha_J,$$

$$C = 1 - \lambda^2,$$

$$D = \mu \lambda + R_3,$$

$$\lambda = p/p_{cr},$$

$$p_{cr} = \pm (-C_{M\theta}/(1 - I_x/I)I')^{\frac{1}{2}},$$

$$\mu = p_{cr} \left[+ \frac{d(C_{Mp\theta} + C_{M\theta}^{\cdot})}{2VC_{M\theta}} - \frac{(I - I_x)(C_{N\theta}/C_{M\theta})}{(mdV)} \right],$$

$$R_1 = (I - I_x) \frac{(C_{zo} + C_{N\theta} \alpha_J)}{mdVC_{M\theta}} + \frac{(C_{Mp\theta} + C_{M\theta}^{\cdot})}{2VC_{M\theta}} \alpha_J d \\ + p \alpha_J I' / C_{M\theta},$$

$$R_2 = (I - I_x) \frac{(C_{yo} + C_{N\theta} \beta_J)}{mdVC_{M\theta}} + \frac{(C_{Mp\theta} + C_{M\theta}^{\cdot})}{2VC_{M\theta}} \beta_J d \\ - p \beta_J I' / C_{M\theta},$$

$$\text{and } R_3 = p I' / C_{M\theta}.$$

The R terms are generally small and therefore will be deleted from this analysis. This equation as written is also the particular solution of (2-20) if one lets $\theta_J = 0$. The values of the trim coefficients are for the principal axes. If the trim coefficients for the geometric axes are desired they are obtained by combining the β_J and α_J terms into the trim terms in A and B of (3-1).

Equation (2-24) may be written in terms of roll-rate ratio as

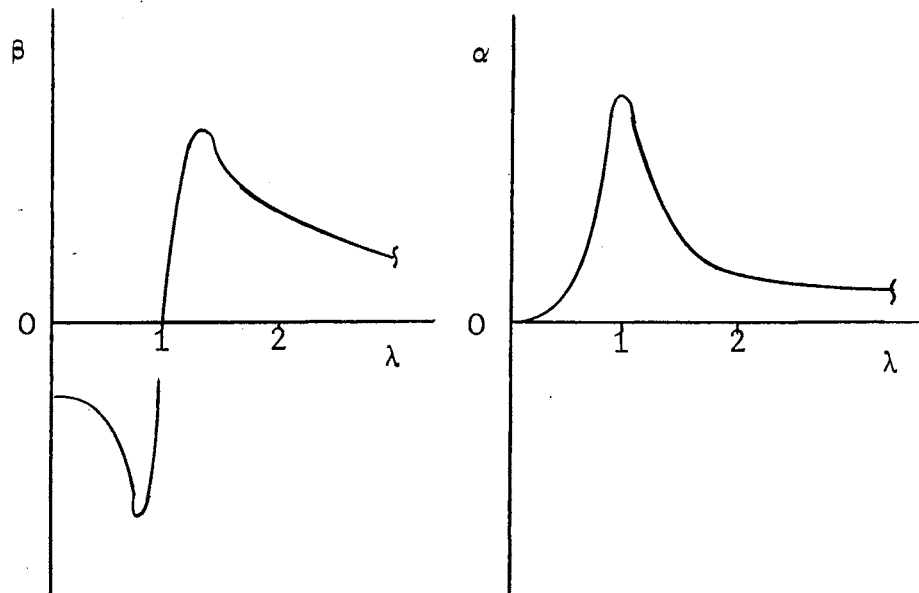
$$\begin{aligned} \dot{\lambda} = & - (\delta Y(C_{N\theta}\alpha - C_{zo}) - \delta Z(C_{N\theta}\beta - C_{yo}))/I_x' p_{cr} \\ & + (\lambda C_{lp}d/2V + C_{lo}/p_{cr})/I_x' - \lambda \dot{p}_{cr}/p_{cr} \end{aligned} \quad (3-2)$$

with the trim coefficients for the same axis system as α and β . The variations in p_{cr} and \dot{p}_{cr} are assumed to have no effect on the solution form of λ in the range of analysis under the assumptions of slowly varying parameters.

On ballistic missile flights the dynamic pressure Q will change causing p_{cr} to vary. Since the resonance region is avoided when possible, only those vehicles whose trajectories have a wide variation in Q will be designed to encounter resonance. The problems are encountered when p is near p_{cr} where because of asymmetries the angle of attack increases. With a center of mass offset, this can cause a large roll rate change. Either condition, large angle of attack or roll rate change, can be considered degrading to vehicle performance.

The increase in angle of attack near $\lambda = 1$ corresponds to the resonant amplification of the displacement magnitude of an oscillator. The self-driven change in roll rate (forcing frequency) is a phenomena not normally encountered in oscillators. The roll rate is changed by the torque producing asymmetries predominantly near $\lambda = 1$. The change in p as it passes through the resonant region in most cases would not be significant except that an equilibrium

condition can be established such that λ asymptotically approaches a constant value of λ which means that p will follow p_{cr} . The phenomena is called "roll lock-in" or "persistent roll resonance". Torque is produced by an asymmetry when the resulting angle of angle of attack is about the same axis as the center of mass offset from the axis of symmetry as seen in Equation (3-2). The total angle of attack θ from Equation (3-1) can be broken down into two components; one component which is a function of C and an orthogonal component which is a function of D . This may be thought of as rotating the coordinates of (3-1) and (3-2) to a location where β is a function of C and α is a function of D . This is the case in the example shown in Figure 3. In Equation (3-1), the quantity $C = 1 - \lambda^2$ is negative for $\lambda > 1$, positive for $\lambda < 1$, and zero at $\lambda = 1$ while the quantity $D = \mu\lambda$ does not change signs. Therefore, the resulting angle of attack for any single asymmetry will be composed of one component that starts with one sign, goes through zero, and ends with the opposite sign as λ varies through $\lambda = 1$. The orthogonal component will start small, maximize, and get small again as λ varies through unity. For any center of mass offset and any asymmetry, a torque that is a function of λ will be produced. Since the torques vary with λ , the possibility of a null torque condition for Equation (3-2) exists if the asymmetries are sufficiently large to produce the necessary torque to equal the $\lambda \dot{p}_{cr}/p_{cr}$ term from (3-2). For an additional discussion



$$C_{no} = \text{constant}$$

$$C_{mo} = 0$$

$$\delta Y = \text{constant}$$

$$\delta Z = 0$$

Figure 3. Trim Angles Versus Roll-Rate Ratio for an Asymmetry About the z -Axis

of this, the reader is referred to the work by Pettus (8).

The situation can be seen from the following analysis. The roll-rate ratio change is given by Equation (3-2). The value of $\dot{\lambda}$ is influenced by the variables α , β , and Q in combination with vehicle parameters and asymmetries. The condition $\dot{\lambda} = 0$ is an equilibrium condition which defines a constant roll-rate ratio λ . The stability of this condition determines whether persistent roll resonance can occur.

The equilibrium conditions are determined when $\dot{\lambda} = 0$ in Equation (3-2) with α and β given by (3-1). The stability of these singular points may be determined by expanding $\dot{\lambda} = f(\lambda)$ in a Taylor's series about λ_{eq} as

$$\begin{aligned} \dot{\lambda} = & \left[\partial f(\lambda_{eq}) / \partial \lambda \right] (\lambda - \lambda_{eq}) + \left[\partial^2 f(\lambda_{eq}) / \partial \lambda^2 \right] (\lambda - \lambda_{eq})^2 \\ & + O[(\lambda - \lambda_{eq})^3]. \end{aligned} \quad (3-3)$$

For small variations only the first term need be retained and the equation becomes

$$\begin{aligned} \dot{\lambda} = & \left(\left(- \frac{\delta Y \partial \alpha(\lambda_{eq})}{\partial \lambda} + \frac{\delta Z \partial \beta(\lambda_{eq})}{\partial \lambda} \right) C_{N\theta} / I_{x'} p_{cr} \right) \delta \lambda \\ & + (C_{lp} d / 2 V I_{x'} - \dot{p}_{cr} / p_{cr}) \delta \lambda \end{aligned} \quad (3-4)$$

where $\frac{\partial \alpha}{\partial \lambda} = - (2\lambda A - \mu B + 2\lambda \beta_J C + 2\lambda \alpha_J D) / (C^2 + D^2)$

$$+ (AC + BD) (4\lambda C - 2\mu D) / (C^2 + D^2)^2,$$

$$\frac{\partial \beta}{\partial \lambda} = - (2\lambda B + \mu A + 2\lambda \alpha_J C + 2\lambda \beta_J D) / (C^2 + D^2) \\ + (BC - AD)(4\lambda C - 2\mu D) / (C^2 + D^2)^2,$$

and $\delta\lambda = \lambda - \lambda_{eq}$.

The Equation (3-4) is a first order linear equation in λ that is applicable in the region about λ_{eq} . The solution for λ will asymptotically converge to λ_{eq} for negative coefficients on λ and diverge for positive coefficients. When these conditions are met, the system is said to be stable or unstable in the Liapounov sense as described by Minorsky (14).

The existence of persistent roll lock-in is determined by the zeroes of (3-2) and the sign on the coefficient of $\delta\lambda$ in (3-4). The value of Q is changing throughout the trajectory and the last term of (3-2) is divided by Q while the others are multiplied by $Q^{\frac{1}{2}}$. This makes it impossible for $\dot{\lambda} = 0$ over any time period when Q is changing. Therefore, persistent roll lock-in cannot be maintained. A condition can occur where λ is almost constant and varies only enough to maintain a condition as close to $\dot{\lambda} = 0$ as possible. The true condition can be stated as a minimization of $\dot{\lambda}$ in (3-2). The actual value of $\dot{\lambda}$ will be very near zero for Q slowly varying so that for practical purposes lock-in occurs.

In Figure 3 the values of α and β versus λ for trim moments about the z-axis are shown. Other cases can be

derived from this example by a simple axis rotation. The major portion of the roll torque results from these trim angles multiplied by center of mass offsets. Therefore, these curves with an appropriate magnitude change can also represent roll torques. Equilibrium conditions for $\dot{\lambda} = \min$ occur at some value of roll torque other than zero because of changing p_{cr} (the dimensionless roll-ratio torque is zero for equilibrium). The total asymmetry torque will be a linear combination of the torques from each asymmetry.

An additional observation on equilibrium can be made with the aid of Figure 3. A λ equilibrium condition occurs at some value of α and β where the roll-ratio torque is zero. Any transient variation in α and β will change the torque. If the change is within the region where the vehicle is stable, it will return to the stable λ equilibrium condition. However, if the transient perturbation drives the torque across a local maximum and through an unstable equilibrium point, the vehicle will not return to the original equilibrium condition.

The equations of motion for a vehicle with fixed asymmetries have been analyzed in the region where the quasi-steady-state solution dominates the motion. The steady-state solution of the angle of attack equation for slowly varying roll and dynamic pressure is given. The roll-rate ratio equation is analyzed about equilibrium conditions to determine the conditions for quasi-steady roll-rate ratio.

In the next chapter a mass in a race will be added to the system to determine its effect on motion of the vehicle with resident asymmetries. Again the region of interest will be where the steady-state solution predominates.

CHAPTER IV

EFFECTS OF A MASS IN A RACE

The addition of a small mass in a race will alter the motion of a symmetric missile with small asymmetries. The overall effect on a trajectory depends on initial conditions to a great extent. The various parameters of the ball in the race have an effect on the motion of a vehicle that can be described by differential equations. These equations can be approximated by the same techniques used for vehicles with fixed asymmetries. The resulting equations are complicated enough to preclude general analysis of the total system motion, but the equilibrium positions of the ball in the vehicle can be analyzed with success. The following analysis will determine the effect of a mass in race on the steady-state solution of a missile flying at high speed through the atmosphere.

If the ball position is varying slowly, the angle of attack can be determined from the steady-state solution and the instantaneous ball position. The ball position can be determined most feasibly from difference techniques unless it is at an equilibrium condition. The ball equilibrium condition is defined as that position under steady trajectory conditions where there is no force on the ball in the

tangential direction. This position does not imply an equilibrium condition for the roll-rate ratio. When both ball and roll-rate ratio are in equilibrium the situation will be called trajectory equilibrium. The equilibrium ball position varies with roll rate and trajectory conditions. For certain trajectory conditions no equilibrium position exists and the ball will be driven around in the body. When an equilibrium condition exists the ball will be driven toward it, but since the condition varies with roll rate and trajectory parameters, the ball may not be able to respond rapidly enough to reach and remain at the equilibrium position. In addition any transient motion may move the ball away from equilibrium and cause a change in roll-rate ratio which can significantly alter ball equilibrium conditions. For these reasons only the effect of the variable asymmetry at equilibrium will be described. Ball motion between equilibrium positions will not be determined.

The effects of small masses on a ballistic vehicle can be determined by several methods. It is most convenient to express the situation as a system of masses and consider the effects of the small mass on the main vehicle to be perturbation torques and moments. The resulting equation can then be solved in a form identical to a single-body problem. In the following analysis it is assumed that $m_v \gg m_b$ where m_v is the vehicle mass and m_b is the ball mass. The equations of motion may be written about the

center of mass of the main vehicle or the center of mass of the system. The equations of motion (see Thomson (10)) of the ball-vehicle system about the system center of mass written in vector form are:

$$\begin{aligned}
 \bar{M} = & \bar{I} \cdot \dot{\bar{\omega}} + \bar{\omega} \times \bar{I} \cdot \bar{\omega} + m_v \bar{s}_v \times \dot{\bar{\omega}} \times \bar{s}_v + m_b \bar{s}_b \times \dot{\bar{\omega}} \times \bar{s}_b \\
 & + m_v \bar{\omega} \times \bar{s}_v \times \bar{\omega} \times \bar{s}_v + m_b \bar{\omega} \times \bar{s}_b \times \bar{\omega} \times \bar{s}_b \\
 & + 2m_v \bar{s}_v \times \bar{\omega} \times \dot{\bar{s}}_{vr} + 2m_b \bar{s}_b \times \bar{\omega} \times \dot{\bar{s}}_{br} \\
 & + m_v \bar{s}_v \times \ddot{\bar{s}}_{vr} + m_b \bar{s}_b \times \ddot{\bar{s}}_{br}
 \end{aligned} \tag{4-1}$$

$$\text{and } \bar{F} = (m_b + m_v) \ddot{\bar{s}} \tag{4-2}$$

where $\ddot{\bar{s}}$ is the inertial acceleration of the system center of mass with the coordinate systems shown in Figure 4. The term \bar{I} is the principal inertia tensor of the main vehicle and the subscripts v and b represent vehicle and ball, respectively. The subscript r refers to quantities relative to the body fixed coordinate systems. Several auxiliary equations can be developed to allow the equations to be transferred to the center of mass of the main vehicle (point 0) from the system center of mass. From the expressions $\bar{s} = \bar{s}_b - \bar{s}_v$ and $m_v \bar{s}_v = -m_b \bar{s}_b$, the expressions for \bar{s}_b and \bar{s}_v in terms of \bar{s} are $\bar{s}_b = \bar{s} m_v / m_t$ and $\bar{s}_v = -\bar{s} m_b / m_t$.

The moment transfer equation is $\bar{M} = \bar{M}_0 - \bar{F} \times \bar{s}_v$ where \bar{M}_0 is the moment about the center of mass of the vehicle. Equation (4-1) becomes

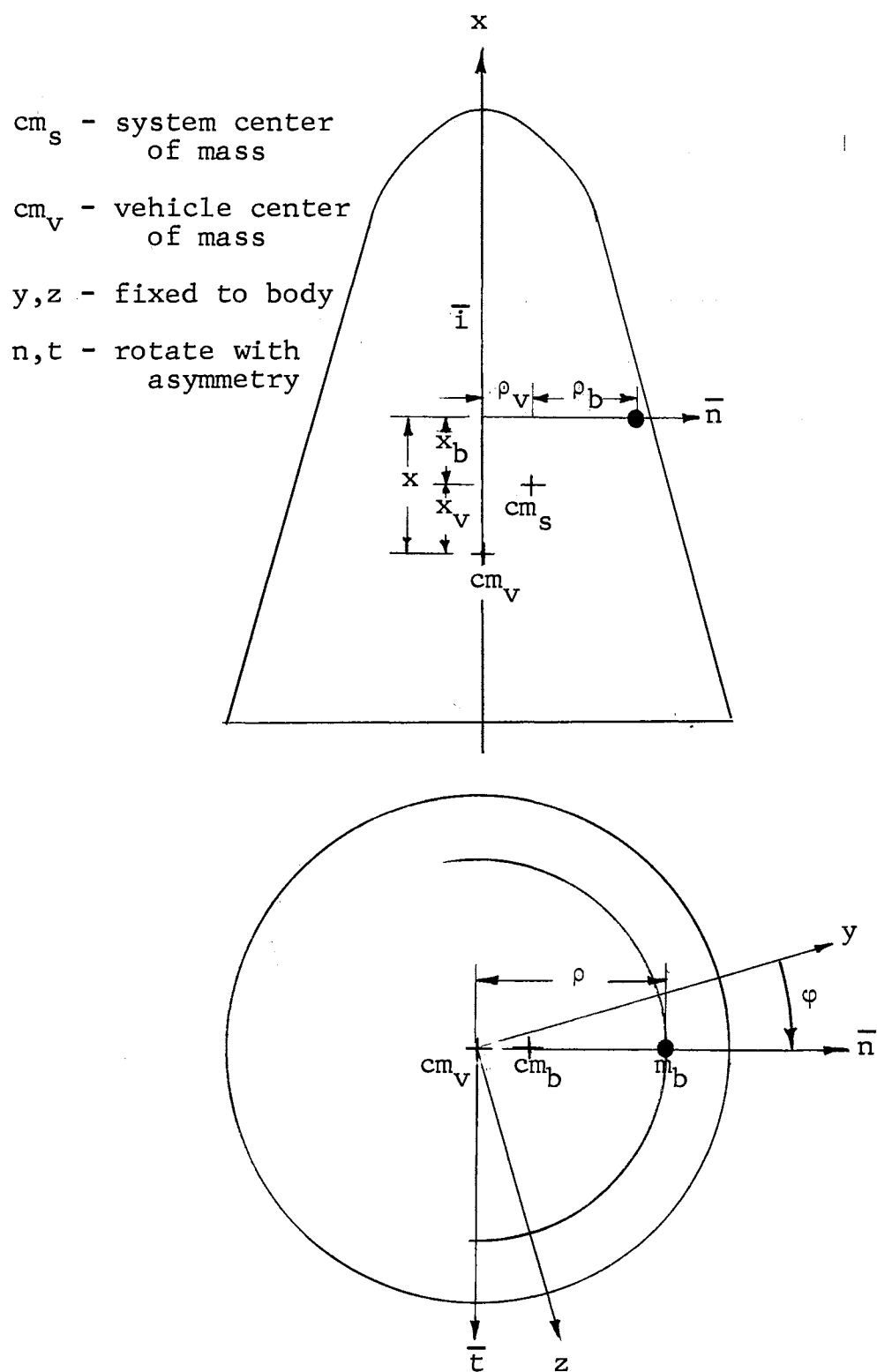


Figure 4. Variable Asymmetry Positional Coordinates

$$\begin{aligned}\bar{M}_O = \bar{I} \cdot \dot{\bar{\omega}} + \bar{\omega} \times \bar{I} \cdot \bar{\omega} + m\bar{s} \times \dot{\bar{\omega}} \times \bar{s} + m\bar{\omega} \times \bar{s} \times \bar{\omega} \times \bar{s} \\ + m\bar{s} \times \ddot{\bar{s}}_r + 2m\bar{s} \times \bar{\omega} \times \dot{\bar{s}}_r - (m_b/m_t)\bar{F} \times \bar{s}\end{aligned}\quad (4-3)$$

with $m = m_v m_b / m_t$ and $m_t = m_v + m_b$. With some algebraic rearrangement and the information in Appendix A, Equation (4-3) may be reduced to

$$\bar{M}_O = \bar{I} \cdot \dot{\bar{\omega}} + \bar{\omega} \times \bar{I} \cdot \bar{\omega} + \bar{s} \times \bar{F}_b. \quad (4-4)$$

The small mass, m_b , will be considered to be constrained to move in a circular race mounted perpendicular to the x-axis of the geometric symmetry. The expressions describing the relative position, velocity, and acceleration of the mass in the race are:

$$\bar{s} = \rho \bar{n} + x \bar{i},$$

$$\dot{\bar{s}}_r = \rho \dot{\varphi} \bar{t},$$

$$\text{and } \ddot{\bar{s}}_r = -\rho \dot{\varphi}^2 \bar{n} + \rho \ddot{\varphi} \bar{t}$$

where \bar{i} , \bar{n} , and \bar{t} are unit vectors. Two axis systems are used; an x, y, z and an x, n, t. These systems are shown in Figure 4.

For analysis Equation (4-4) may be expanded in Cartesian form in the x, y, z, axes to

$$M_O^x = \dot{p} I_x + \rho F_b^t, \quad (4-5)$$

$$M_0^y = \dot{q}I - pr(I - I_x) + xF_b^t \cos\varphi - (xF_b^n - \rho F_b^x) \sin\varphi, \quad (4-6)$$

$$\text{and } M_0^z = \dot{r}I + pq(I - I_x) + xF_b^t \sin\varphi + (xF_b^n - \rho F_b^x) \cos\varphi \quad (4-7)$$

where F_b^x , F_b^n , and F_b^t are given by Equations (4-12), (4-13), and (4-14) and the M 's are aerodynamic moments about the vehicle center of mass. For a constant φ these equations reduce to Equations (2-4), (2-5), and (2-6) and can be analyzed by the methods of Chapters II and III.

The equations of motion can be written about the x , n , t axes in order to facilitate analysis. The equations are:

$$M_0^x = I_x \dot{p} + \rho F_b^t \quad (4-8)$$

$$M^n = M_0^y \cos\varphi + M_0^z \sin\varphi = I(\dot{q} \cos\varphi + \dot{r} \sin\varphi) - xF_b^t, \\ - p(r(I - I_x) \cos\varphi - q(I - I_x) \sin\varphi) \quad (4-9)$$

$$\text{and } M_0^t = -M_0^y \sin\varphi + M_0^z \cos\varphi = I(-\dot{q} \sin\varphi + \dot{r} \cos\varphi) \\ + p(r(I - I_x) \sin\varphi + q(I - I_x) \cos\varphi) \\ + xF_b^n - \rho F_b^x \quad (4-10)$$

$$\text{where } F_b^x/m = F^x/m_v + 2\rho\dot{\varphi}(r \sin\varphi + q \cos\varphi) - x(r^2 + q^2) \\ + \rho(\dot{q} \sin\varphi - \dot{r} \cos\varphi) + \rho p(q \cos\varphi + r \sin\varphi) \quad (4-11)$$

$$\begin{aligned}
F_b^n/m &= F^n/m_v + 2\rho q r \sin\varphi \cos\varphi + x(\dot{r}\cos\varphi + \dot{q}\sin\varphi) \\
&+ p x(q\cos\varphi + r\sin\varphi) - \rho \dot{\varphi}^2 - 2\rho p \dot{\varphi} \\
&- \rho(p^2 + q^2 \sin^2\varphi + r^2 \cos^2\varphi)
\end{aligned} \tag{4-12}$$

$$\begin{aligned}
\text{and } F_b^t/m &= F^t/m_v + \rho \ddot{\varphi} + \rho \dot{p} - x(\dot{q}\cos\varphi + \dot{r}\sin\varphi) \\
&- \rho(q^2 - r^2)\sin\varphi\cos\varphi - \rho q r(\sin^2\varphi - \cos^2\varphi) \\
&- x p(q\sin\varphi - r\cos\varphi)
\end{aligned} \tag{4-13}$$

where the unsubscripted F 's are aerodynamic forces on the system and $m = m_b m_v / m_t$. The tangential force can also be expressed by a function

$$F_b^t = f(\varphi, \dot{\varphi}, \ddot{\varphi}). \tag{4-14}$$

The above equations describe the motion of a ballistic vehicle containing a small mass that is either stationary in the vehicle or moving relative to the vehicle in a circular race under viscous forces. The case of the condition φ equal to a constant will be considered first, to determine the effect of a ball that is fixed or in equilibrium. An analysis similar to that used in Chapter II and III will be used to determine an approximate particular solution to the equations. The \bar{M} and \bar{F} terms are due to aerodynamic forces and moments. The terms of F_b containing F/m_v 's are eliminated by transferring the moments to the system center of mass. In doing this the value of $C_{m\theta}$ for the system is

changed by $-x_{N\theta} m_b / m_t$ and the effective center of mass offset from the geometric center by $(-m_b \rho / m_t)$. The remaining terms of the F_b expressions are perturbation torques due to axis misalignment. In the particular solution perturbation terms containing \dot{p} , \dot{q} , \dot{r} , qr , q^2 , and r^2 are small.

Equations (4-9), (4-10) or (4-6), and (4-7) can be combined by the orthogonal complex transformation into an equation similar to Equation (2-20). After manipulations similar to those in Chapter II, the resulting angle of attack equation is

$$\ddot{\theta}_P + E_P \dot{\theta}_P + F_P \theta_P = G_P + G_{Pb} \quad (4-15)$$

where E_P , F_P , and G_P are defined in Equation (2-20)

and $G_{Pb} = m p x^2 (q + i r) / I - m x p p^2 (\cos \varphi + i \sin \varphi) / I$

$$- m p^2 p (q \cos \varphi + r \sin \varphi) (\cos \varphi + i \sin \varphi) / I.$$

This equation describes the motion of a vehicle with small resident asymmetries acted upon by a stationary mass in a race where G_{Pb} represents the effect of the mass on the basic vehicle. The first term in G_{Pb} can be entered directly into E_P and F_P with the aid of Equation (2-18). It adds directly into the terms $(I - I_x) / I$. The second term in G_{Pb} contains the product of inertia terms I_{xy} and I_{xz} and will be handled as in (2-23). The final term contains the product of inertia I_{yz} and the nonsymmetric

portion added to I due to the small mass m_b . Their effect will be included in correction terms α_c and β_c using $r \cong \alpha p$ and $q \cong \beta p$ in the third term of G_{pb} . The particular solution to (4-15) becomes with $\theta = \beta + i\alpha$

$$\beta = (AC + BD)/(C^2 + D^2) + \beta_c \quad (4-16)$$

$$\alpha = (BC - AD)/(C^2 + D^2) + \alpha_c \quad (4-17)$$

where μ , λ , A , B , C , and D are defined in Equation (3-1) but now

$$\delta Y = Y_M - m\rho \cos\varphi/m_t,$$

$$\delta Z = Z_M - m\rho \sin\varphi/m_t,$$

$$p_{cr} = ((-C_{M\theta} + x m_b C_{N\theta}/m_t d) Q S d / (I_A - I_x))^{1/2}$$

$$\beta_J = m x \rho \cos\varphi / (I_A - I_x),$$

$$\alpha_J = m x \rho \sin\varphi / (I_A - I_x),$$

$$\text{and } I_A = I + m x^2$$

where I_A is used for I in all $(I - I_x)$ terms

$$\text{and } \beta_c = -\lambda^2 m \rho^2 \left[(AC^2 + BCD) \cos^2\varphi + (BCD - AD^2) \sin^2\varphi + B(C^2 + D^2) \sin\varphi \cos\varphi \right] / ((I_A - I_x)(C^2 + D^2)^2)$$

$$\text{and } \alpha_c = -\lambda^2 m \rho^2 \left[(-ACD - D^2) \cos^2\varphi + (BC - AD) C \sin^2\varphi + A(C^2 + D^2) \sin\varphi \cos\varphi \right] / ((I_A - I_x)(C^2 + D^2)^2).$$

The equation for roll motion becomes

$$M^x = (I_x + \rho^2 m) \ddot{\phi} - m x \rho p^2 (\beta \sin \phi - \alpha \cos \phi) \quad (4-18)$$

where $M^x = M_0^x - F_b^t \rho m_b / m_t$.

Equations (4-16), (4-17), and (4-18) describe the effect of a stationary mass on the motion of the vehicle. By proper selection of equivalent masses these equations can describe the motion of a vehicle whose principal and geometric axes do not coincide (a vehicle with "products of inertia").

When the ball is moving it is constrained to move on a circular race about the axis of symmetry. It is therefore constrained in the x and n directions by the race, and restrained in the t direction by damping forces. For the special case of viscous damping (h is the damping coefficient)

$$F_b^t = - h \dot{\theta}. \quad (4-19)$$

The equation for roll motion (4-5) becomes

$$M_0^x = p I_x \ddot{\phi} - \rho h \dot{\phi}. \quad (4-20)$$

This equation shows the total direct effect of the ball on the roll motion of the basic vehicle to be the viscous damping force. This fact is extremely important to the overall effect of the ball on the vehicle. The portion of the center of mass offset from the center of aerodynamic pressure does not directly contribute any roll torque as does a center of mass offset due to a rigidly attached mass. This center of mass offset does alter α and β as seen in

(4-16) and (4-17). It should be noted that the roll moment and roll moment of inertia in Equation (4-21) is that of the original vehicle about its axis of symmetry.

The effect of a ball in a stationary position has been described. Next the motion of the ball must be described so that positions of ball equilibrium can be determined. For viscous damping in the race, the ball motion is described from the tangential ball force equation as

$$\begin{aligned} \rho \ddot{\varphi} = & -h\dot{\varphi}/m_b - \rho \dot{p} + x(\dot{r} \sin\varphi + \dot{q} \cos\varphi) \\ & + xp(q \sin\varphi - r \cos\varphi) - \rho r^2 \cos\varphi \sin\varphi \\ & + \rho q^2 \sin\varphi \cos\varphi + \rho qr(\sin^2\varphi - \cos^2\varphi) \\ & + F^y \sin\varphi/m_t - F^z \cos\varphi/m_t. \end{aligned} \quad (4-21)$$

For the steady-state solution at small angles of attack, $q \cong \beta p$ and $r \cong \alpha p$ and \dot{q} , \dot{r} , α^2 , and β^2 are small so that when the ball is near equilibrium and $\dot{\varphi}$ is small, Equation (4-21) becomes

$$\begin{aligned} \rho \ddot{\varphi} + h\dot{\varphi}/m_b - ((xp^2 - C_{N\theta}QS/m_t)\beta + C_{y0}QS/m_t) \sin\varphi \\ + ((xp^2 - C_{N\theta}QS/m_t)\alpha + C_{z0}QS/m_t) \cos\varphi = -\rho \dot{p} \end{aligned} \quad (4-22)$$

where the terms replacing F_y and F_z are from Equations (2-8) and (2-9), and C_{lp} and C_{lo} are small. For slow ball motion this equation approximately describes the motion of the ball in the race. Force equilibrium is obtained when the last three terms sum to zero and the ball is stationary.

Stability of these equilibrium points is easily determined through a small angle perturbation analysis. The important considerations are the terms containing $(xp^2 - C_{N\theta}QS/m_t)$ which are composed of a gyroscopic term and an aerodynamic term and the trim forces (terms containing C_{y0} and C_{z0}). The sign of this term is extremely important in the stability analysis of the ball equilibrium position.

It is important to note that both the gyroscopic effect and the aerodynamic effect are linear functions of the angle of attack. The significance of this is that the gyroscopic forces on the ball are not necessarily dominant at small angles of attack but decrease directly with the aerodynamic forces. Since α and β are functions of ball position and fixed asymmetries, substitution of (4-16), (4-17), and (4-20) into Equation (4-22) will describe the equilibrium positions of the ball. The resulting equation is

$$\rho\ddot{\varphi} + h\dot{\varphi}/m_b + R_M = 0 \quad (4-23)$$

with $R_M = (K\alpha + C_{z0}QS/m_t)\cos\varphi - (K\beta + C_{y0}QS/m_t)\sin\varphi$

$$+ \rho(\delta Z(C_{N\theta}\beta + C_{y0}) - \delta Y(C_{N\theta}\alpha + C_{z0}))/I'_x$$

where $K = xp^2 - C_{N\theta}QS/m_t$

and R_M is a remainder term that represents the position sensitive force on the ball. Solution of the equation for φ when $R_M = 0$ is cumbersome for the general case. Some simplification is obtained considering cases when C_{y0} and

and C_{zo} are small or can be incorporated in α and β . The \dot{p} term will be small and in general only cause a slight phase shift in φ_{eq} . With these considerations (4-23) reduces to

$$\tan \varphi_{eq} = \alpha/\beta, \quad (4-24)$$

$$\text{with } \sin \varphi_{eq} = \alpha/\theta,$$

$$\cos \varphi_{eq} = \beta/\theta,$$

$$\text{and } |\theta| = (\alpha^2 + \beta^2)^{\frac{1}{2}}.$$

These relationships can be used to find approximate equilibrium conditions. Substitution of (4-24) into (4-16) and (4-17) gives

$$\beta = \beta_T + H(C\beta + D\alpha)/\theta \quad (4-25)$$

$$\alpha = \alpha_T + H(C\alpha - D\beta)/\theta \quad (4-26)$$

$$\text{where } \beta_T = (A_1C + B_1D)/(C^2 + D^2),$$

$$\alpha_T = (B_1C - A_1D)/(C^2 + D^2),$$

$$A_1 = (C_{no} + C_{xo}Y_M/d)/C_{M\theta},$$

$$B_1 = - (C_{mo} - C_{xo}Z_M/d)/C_{M\theta},$$

$$H = - (m_p C_{xo} / (m_t C_{M\theta} d) + \lambda^2 m_{x\rho} / (I_A - I_x)) / (C^2 + D^2),$$

$$C = 1 - \lambda^2,$$

$$\text{and } D = \mu\lambda$$

where α_c and β_c are deleted from the analysis for simplicity. Solving for α and β yields

$$\beta_{eq} = (\beta_T N + \alpha_T HD) \theta' / (\alpha_T^2 + \beta_T^2) \quad (4-27)$$

$$\alpha_{eq} = (-\beta_T HD + \alpha_T N) \theta' / (\alpha_T^2 + \beta_T^2) \quad (4-28)$$

where $\theta' = HC + N$,

and $N = \pm (\alpha_T^2 + \beta_T^2 - H^2 D^2)^{1/2}$.

The stability of the ball position about the equilibrium conditions must be determined by some means. Two values of α and β are possible for N real. The stability of each root may be determined from the simplified ball position equation

$$\ddot{\rho\varphi} + h\dot{\varphi}/m_b + R_M = 0 \quad (4-29)$$

with $R_M = K(\alpha \cos\varphi - \beta \sin\varphi)$

and the angle of attack Equation (4-15). The equations represent three coupled second-order nonlinear differential equations. A practical first approach is to linearize the equations about the equilibrium conditions. The equations with δ indicating deviation from equilibrium are:

$$\ddot{\delta\theta} + E_P \dot{\delta\theta} + F_P \delta\theta - iH' \exp(i\varphi_{eq}) \delta\varphi = 0$$

$$\ddot{\rho\delta\varphi} + h/m_b \dot{\delta\varphi} - KN\delta\varphi + K(\delta\alpha \cos\varphi_{eq} - \delta\beta \sin\varphi_{eq}) = 0$$

where the θ equation can be broken into two second-order

equations. The characteristic equation of the system

$$\begin{aligned}
 & (s^2 + E_R s + F_R)^2 (\rho s^2 + h/m_b s - KN) \\
 & + (E_I s - F_I)^2 (\rho s^2 + h/m_b s - KN) \\
 & + KH' (s^2 + E_R s + F_R) = 0
 \end{aligned} \tag{4-30}$$

where the subscript R indicates the real part of the coefficient and I the imaginary part of E_p and F_p defined in (4-15) and

$$H' = -m_p(x_p^2/I + C_{x0}/I'm_{td}).$$

If any root of the characteristic equation has a positive real part the equilibrium position is unstable. The general expressions for the roots of this polynomial are not tractable. The roots must be numerically determined by standard numerical polynomial root finders. This method gives the roots for discrete values of the parameters but does not add enough insight into the problem because of the wide range in values of the variables E_p , F_p , K , and N over any particular trajectory and the various values of vehicle parameters. Further analysis will be conducted to aid insight into the vehicle response.

Equation (4-30) may be rewritten in the form

$$(\rho s^2 + h/m_b s - KN) + M(s) = 0$$

$$\text{with } M(s) = \frac{KH' (s^2 + E_R s + F_R)}{(s^2 + E_R s + F_R)^2 + (E_I s - F_I)^2}.$$

The roots of $(\rho s^2 + h/m_b s - KN)$ will give valuable insight into stability under certain conditions. If substitution of these roots into $M(s)$ yields a number small compared to $-KN$, then the roots are good approximations to the polynomial (4-30). A necessary condition for stability when this approximation holds is for all coefficients of the reduced polynomial to be positive. Since there are two values of N of opposite sign for each condition, the vehicle has one position where $-KN$ is positive. It should be remembered that this does not insure stability but is only a necessary condition. Sufficient conditions for stability must be established from (4-30).

Under the restrictions imposed on the vehicles considered in this work, the condition that $-KN$ must be positive can be determined from the requirement that R_M be of the opposite sign of $\delta\varphi$ when $\dot{\theta}$ and $\ddot{\theta}$ terms are negligible compared to θ terms. This implies the general condition that the vehicle moves instantaneously to any ball movement. In the region of interest this is a good assumption. From Equations (4-16) and (4-17) neglecting α_c and β_c written in the format of (4-25) and (4-26), one gets

$$R_M = -K(\beta_T + H(C \cos\varphi + D \sin\varphi)) \sin\varphi \\ + K(\alpha_T + H(C \sin\varphi - D \cos\varphi)) \cos\varphi$$

or
$$R_M = -K(\beta_T \sin\varphi - \alpha_T \cos\varphi + HD)$$

where $dR_M/d\varphi = -K(\beta_T \cos\varphi + \alpha_T \sin\varphi)$

which yields with (4-24), (4-27), and (4-28)

$$dR_M/d\varphi)_{eq} = -KN \quad (4-31)$$

where $K = xp^2 - C_{N\theta}QS/m_t$.

Therefore the stable condition for $K < 0$ is positive N and for $K > 0$ negative N . It should be remembered this is a necessary but not a sufficient condition for stability.

When N is imaginary no equilibrium condition exists for the mass. The term R_M always has the sign of $-KH\mu$ since μ and p_{cr} are of the same sign and by the Schwarz inequality

$$|HD| > |\beta_T \sin\varphi - \alpha_T \cos\varphi| \quad (4-32)$$

from N being imaginary. This fact indicates the ball position, φ , to be driven in the direction of $-KH\lambda$ regardless of the value of φ . The signs of K and H are dependent on the dominant terms in each expression.

Although the ball may have a stable position this position will vary with λ . For the system to be in equilibrium $\dot{\lambda}$ must be zero in addition to the requirement that the ball be in equilibrium. The equation for $\dot{\lambda}$ as derived from Equation (4-20) is

$$\begin{aligned} \dot{\lambda} = & (Z_M(C_{N\theta}\beta - C_{y0}) - Y_M(C_{N\theta}\alpha - C_{z0}))/I'_x p_{cr} - \rho h \dot{\varphi} \\ & + (\lambda C_{1p} d/2V + C_{10}/p_{cr})/I'_x - \lambda \dot{p}_{cr}/p_{cr}. \end{aligned} \quad (4-33)$$

This equation is the same as (3-2) except for the $\dot{\varphi}$ term. The values α and β are determined differently because the mass creates a trim angle of attack. When α and β as defined by Equations (4-27) and (4-28) represent a stable ball position with $\dot{\varphi} \cong 0$ and when $\dot{\lambda} \cong 0$, then the vehicle with ball can be considered "locked-in" an equilibrium roll condition. This condition will persist as long as roll rate variations can adjust α and β to maintain a zero roll-rate-ratio torque or until a discontinuity in the solutions for α_{eq} and β_{eq} occurs. The vehicle will then go into a transient motion until the trajectory is terminated or another equilibrium condition is reached. If the ball moves slowly Equations (4-16), (4-17), and (4-32) adequately describe the vehicle motion. If $\ddot{\varphi}$ and $\dot{\varphi}$ become significant the angle of attack equations and roll equation must be altered to include these effects primarily because the rapid change in ball position can excite transient response in the motion.

The simplified equations developed in this chapter can be utilized in an analysis that will approximately describe the motion of a vehicle with resident asymmetries containing a moving mass. The computational effort will be much less than that required using the complete equations of motion (4-1) and (4-2). The discussion of the effect of a moving flap is given in Appendix C. The effect of the flap is the same as that of the ball in the race; therefore,

only a cursory analysis is given. Additional flap aerodynamic information is necessary for a more complete analysis.

CHAPTER V

ANALYSIS

In this chapter a discussion of the equations developed in Chapter IV is given. The material is presented to give insight into vehicle motion when a variable asymmetry is present. The discussion is also intended to relate the response of the vehicle with a ball to the response characteristics of vehicles with resident asymmetries.

The effect of the stationary or slowly moving mass in a race is given by Equations (4-16), (4-17), and (4-20). The stability of the ball equilibrium conditions given in (4-24), (4-27), and (4-28) is determined under the assumption of slowly varying trajectory and roll-rate ratio conditions. Trajectory equilibrium conditions are established when the ball is in an equilibrium position in the body and the roll-rate ratio is in equilibrium. Total system stability cannot be determined simultaneously by normal linearization techniques because of the varying coefficients. The situation is limited to a quasi-equilibrium condition as discussed in Chapter III.

The position where the ball is in equilibrium with angle of attack is given by Equations (4-27) and (4-28). These equations describe the resulting angle of attack only

where $\ddot{\varphi} = 0$ when $\dot{\varphi} = 0$ and do not contradict the angle of attack given by (4-16) and (4-17). The two solutions are the same only when $\varphi = \varphi_{eq}$. When $\varphi \neq \varphi_{eq}$ the steady state condition $\ddot{\varphi} = 0$ does not exist and the ball will move toward or away from a φ_{eq} . When the Equations (4-27) and (4-28) have no solution the ball will be driven continuously as discussed with Equation (4-32). If the value of $\dot{\varphi}$ is small then the angle of attack is adequately defined by (4-16) and (4-17) for arbitrary φ .

The ball being in equilibrium with the angle of attack is a necessary but not sufficient condition for the angle of attack to be in equilibrium with the roll rate. For the roll-rate ratio to be in equilibrium, $\dot{\lambda}$ as given by (4-32) must be approximately zero. If $\dot{\lambda} \cong 0$, and α and β equal those defined by (4-27) and (4-28) with the ball stationary, the vehicle can be said to be in "roll lock-in". This implies a condition similar to that developed in Chapter III but it is more restrictive since the asymmetry must remain stationary. Conditions may exist where the asymmetry moves slowly and the roll-rate ratio changes gradually but this cannot be classed as a pseudo-equilibrium condition because the system will drift away from the equilibrium trajectory.

Examination of the angle of attack equations and the roll rate equations for either fixed or free asymmetries indicates that at each point in time for any particular

trajectory condition several values of p can possibly give the pseudo-equilibrium conditions. Further, if plots of p and p_{cr} versus time for vehicles with roll torque producing asymmetries are studied for various cases, it can be deduced that for any p at any time in the trajectory the roll rate is driven toward or away from an equilibrium condition provided an equilibrium condition exists. The dimensionless roll ratio (λ) torques for stationary asymmetries are functions of α and β and Y_M and Z_M as given in Equation (3-2) or (4-33). Each moment-producing asymmetry produces an angle of attack about the axis on which it acts and one about the orthogonal axis as discussed in Chapter III. The basic shape of each as a function of λ is shown in Figure 3 on page 29. The magnitude of the force caused by the angles of attack, α and β , producted with Y_M and Z_M give the roll torque. The roll torque must be sufficiently large to balance the \dot{p}_{cr} term in Equation (3-2) or (4-33). The resulting roll torque curve due to all asymmetries has an absolute maximum in the region of $\lambda = 1$ because the trim angles of attack maximize in this area. When the maximum roll torque term is larger than the p_{cr} term and they are of opposite sign, an equilibrium condition exists at some point away from its maximum. If the torque is not large enough or of the wrong sign then no equilibrium condition exists.

If a transient perturbation occurs the possibility exists for driving the roll rate across the maximum torque

through an unstable equilibrium. When this happens the roll rate change will continue in the direction of the roll torque until another stable equilibrium condition is reached or the flight terminates. A similar transient condition occurs with a free asymmetry when it is moving. The relative motion between the asymmetry and the vehicle provides a roll torque as shown by Equation (4-20). Of course a changing asymmetry position also alters the roll torque because the angle of attack varies with asymmetry position. Analysis of this transient effect will not be investigated here except to note that in the region where α_{eq} and β_{eq} for a vehicle with a variable asymmetry have no solution (no ball equilibrium position) a positive $\dot{\varphi}$ will increase the roll rate while a negative $\dot{\varphi}$ will spin it down. Thus when $\dot{\varphi}$ is of the sign of \dot{p}_{cr} it will aid in maintaining the roll-rate ratio equilibrium while it helps drive the roll-rate ratio away from equilibrium, if of the opposite sign. It should be noted from Equation (4-33) that the size of the torque of a moving ball is a function of ρ and h as well as of $\dot{\varphi}$. This torque can be greater than the torque from the angle of attack and center of mass offset. If this is the case, then the torque due to $\dot{\varphi}$ will control the roll rate change.

Examination of Equations (4-27) and (4-28) indicates that there are two basic effects of variable asymmetries that are independent of initial conditions. When N is real, a stable asymmetry equilibrium position may exist and

the stability is determined by the characteristic equations of the linearized angle of attack and ball position equation. If N is imaginary no equilibrium position exists and the ball runs continuously as defined by Equation (4-32) and supporting comments. These conditions cover the two basic steady-state situations. The conditions are covered in the analysis. It is possible that the ball may not be at the equilibrium location or its velocity may be other than zero at the equilibrium position. In addition since ball equilibrium does not imply roll-rate ratio equilibrium, the ball equilibrium position may change quite rapidly in certain regions due to changes in λ . The above conditions make it possible for a vehicle to pass through a trajectory equilibrium condition if the ball is running too fast (low race damping) or unable to move to the equilibrium condition (high race damping). These transient conditions are functions of system parameters and initial conditions and are not included in the analysis. Because no previous history of the system is assumed no specific initial conditions on the ball will be dictated.

In addition to the two possible conditions for the variable asymmetry (stationary or moving), both fixed resident and variable asymmetries have different effects on the vehicle motion above and below resonance (see Figure 3, page 29). This results from the sign reversal on the trim angle component about the axis on which the asymmetry moment acts due to the term $1 - \lambda^2$. The region of λ near

unity is also important since the effect of the terms multiplied by D is maximized. This also increases the possibility that no variable asymmetry equilibrium position exists since N has no solution for large enough HD (see Equation (4-28)). Therefore, the discussion of the effects of variable asymmetries will be divided into three regions: the first with $\lambda > 1$, then near $\lambda = 1$, and finally $\lambda < 1$. Before turning to specific examples the general effects in each of the regions will be given. The discussion and examples will center on reentry vehicle trajectories since they offer the widest variety of conditions in a single flight.

The condition of no ball equilibrium position for the variable asymmetry can occur in all three regions but is most likely to occur near $\lambda = 1$ since the condition $N^2 < 0$ where $N^2 = \alpha_T^2 + \beta_T^2 - (HD)^2$ occurs when $(HD)^2 > (\alpha_T^2 + \beta_T^2)$.

When $N^2 > 0$, equilibrium conditions exist and are given by Equations (4-27) and (4-28) with total amplitude of the angle of attack given by

$$\theta' = HC + N \quad (5-1)$$

where the sign on N is positive or negative. A negative θ' indicates a sign reversal on Equations (4-27) and (4-28).

The magnitude of N is always less than θ_T , the trim due to resident asymmetries, so that at $\lambda = 1$ where $C = 0$,

$|\theta| < \theta_T$. Above and below $\lambda = 1$ the magnitude of θ depends upon HC and N . If HC is of the opposite sign of N then

$|\theta| < \theta_T$. Since C changes sign and HD gets small away from $\lambda = 1$ the variable asymmetry will generally amplify total angle of attack at some value of λ either above or below resonance. The variable asymmetry also causes a phase shift in θ . The phase shift is a function of HD and N so will be most pronounced near $\lambda = 1$ as seen from Equations (4-27) and (4-28).

With this analysis of the equations of motion the discussion proceeds to specific examples in Chapter VI.

CHAPTER VI

EXAMPLES

In this chapter the theory developed in the previous chapters is compared with finite difference simulations of the six-degree of freedom equations of motion to substantiate the theory. The finite difference program is described in Appendix B. A small reentry vehicle is used as a model in the simulations so that the results can be associated with those of References (5), (6), and (8). The vehicle parameters are arbitrarily chosen and may not represent physically realizable situations. The parameters used are listed in Table 1. The net static margin of the system is held constant for all examples. The examples are confined to two variable asymmetries on a basic vehicle with four resident asymmetry combinations. This is done to present the greatest number of equilibrium cases with reasonable computational effort.

In this chapter the angle of attack information is presented in terms of $|\theta|$ and φ

$$\text{where } |\theta| = (\alpha^2 + \beta^2)^{\frac{1}{2}} \quad (6-1)$$

$$\text{and } \varphi = \arctan \alpha/\beta. \quad (6-2)$$

The term φ represents phase position of either wind vector

TABLE I
EXAMPLE PARAMETERS

$$m_b/m_v = .01, .02, \text{ or } .03$$

$$I/I_x = 10$$

$$\text{Initial } V = 25000.$$

$$\text{Flight path angle} = - 20.^{\circ}$$

$$\text{Altitude} = 175000. \text{ at } t_o$$

$$\text{Vehicle static margin} = .03d$$

$$C_{x0} = .07$$

$$C_{N\theta} = 0.7/\text{rad}$$

$$C_{M\dot{\theta}} = - 1.8/\text{rad}$$

$$Y_M/d = 0.0$$

$$Z_M/d = 0.00043$$

$$C_{M0} = 0.001/\text{rad}$$

$$S = \text{maximum vehicle area}$$

$$d = \text{vehicle length}$$

or ball position from the y-axis.

The variables ρ^* and V in the analytical expressions must be supplied from some external source since no expressions are included in the analysis to generate them. Inclusion of a specific atmosphere or trajectory theory has no influence on the results of this work since the assumption is made that ρ^* and V are slowly varying parameters. In the comparisons between analytical predictions and finite difference simulations the density and velocity generated by the difference simulations are used. In addition the analytical solutions use the same value of p as the simulation at each point of comparison. Thus the comparisons are point by point estimations of ability of the theory to predict the angle of attack of the vehicle. This is done since trajectory equilibrium conditions are of primary interest. If the angle of attack estimation is good, equilibrium conditions are easy to determine. When the equilibrium conditions of the methods are similar, integration of the roll equation from any point away from the trajectory equilibrium conditions will follow close to the finite difference simulation if the instantaneous value of the torques do not differ radically from each other because both are asymptotically approaching the same boundary. Since the angle of attack given by the analysis is accurate, the roll torque will be about the same as that of the finite difference equations and the slight changes in trajectory between the two will not be consequential.

The angle of attack effects of a variable asymmetry on a vehicle with fixed resident asymmetries can be shown to be a function of roll-rate ratio, resident trim amplitude, and variable asymmetry parameters. Figures 5 and 6 are plots of amplitude ratios, θ_{eq}/θ_T , and phase angle shift, $\varphi_T - \varphi_{eq}$, versus λ where θ_{eq} signifies angle of attack when the ball is at its equilibrium position and θ_T the angle of attack for the same vehicle with only resident asymmetries. The values for θ_T are given by Equation (3-1) and the values for θ_{eq} by (4-27) and (4-28). The plots are independent of angular position of the asymmetries and rely only on total magnitude of asymmetries. They are obtained by plotting the values obtained from the theoretical equations. The general resident asymmetry moment coefficient $C_{Mo} = .001$ and two m_b/m_v values are used. The fixed ball position parameters are $x = -2.5$ and $\rho = 1.0$. Appearing on the plots are individual data points generated by finite difference simulations listed in Table 2. These simulations had C_{Mo} values slightly different than .001 due to the trim moment caused by the fixed center of mass offset. The data points represent points from the simulations where the ball is at the equilibrium conditions. The asterisks on the curves indicate end-point values of λ regions where the theoretical equations have solutions due to an imaginary value for N . The agreement is quite good over a wide range of values of λ . The only limitation of these plots is that no indication is given on how a

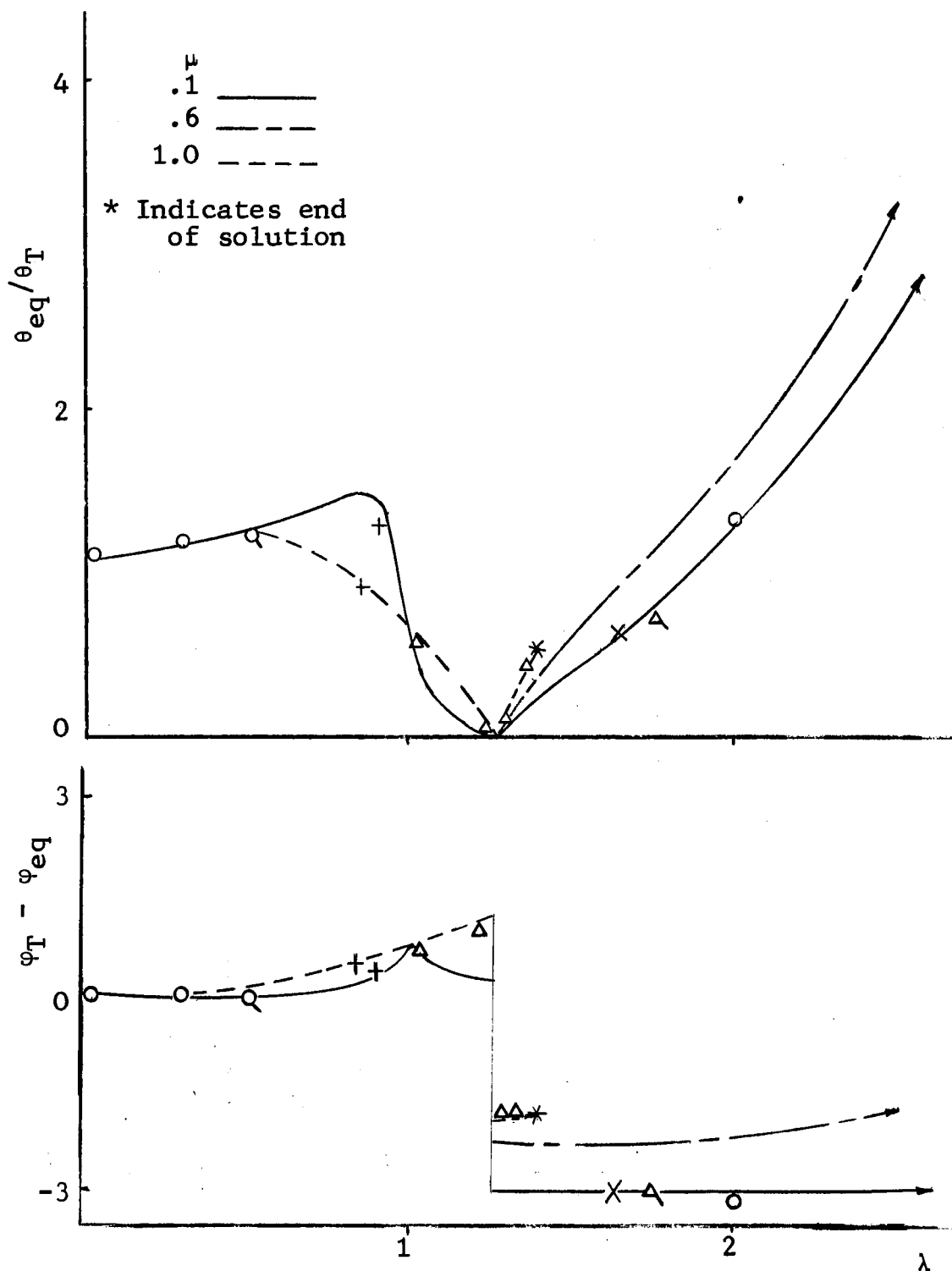


Figure 5. Effect of a Variable Asymmetry on the Angle of Attack as a Function of Roll-Rate Ratio with $m_b/m_v = .01$

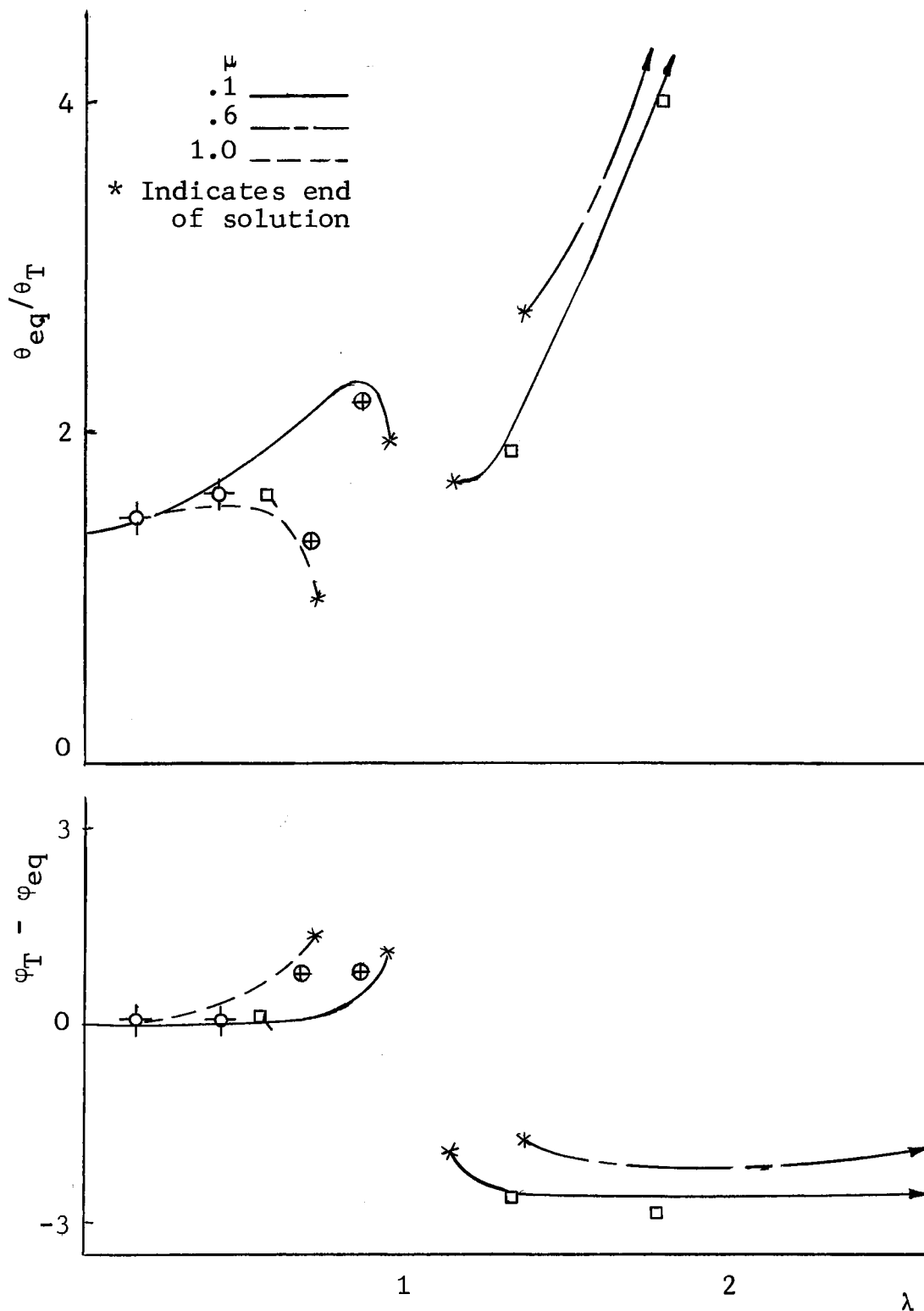


Figure 6. Effect of a Variable Asymmetry on the Angle of Attack as a Function of Roll-Rate Ratio with $m_b/m_v = .03$

TABLE II
PARAMETERS FOR EXAMPLE TRAJECTORIES

Run	Symbol	λ	μ	C_{mo}	C_{no}	m_b/m_v	h
1	○	.02	1.0	-.001	0	.01	.1
	○	.3	.3				
	○	2.0	.07				
2	Q	.5	.2	-.001	0	.01	1.0
3	△	1.02	.25	.001	0	.01	.1
	△	1.21	.79				
	△	1.26	.84				
	△	1.35	.90				
4	△	1.75	.08	.001	0	.01	.3
5	+	.86	.98	0	-.001	.01	1.0
	+	.92	.20				
6	×	1.62	.1	0	.001	.01	1.0
7	□	1.29	.09	.001	0	.03	3.0
	□	1.75	.08				
8	□	.53	.30	.001	0	.03	.9
9	⊕	.67	.91	0	-.001	.03	.3
	⊕	.85	.20				
10	⊖	.40	.18	-.001	0	.03	.3
11	○	-	-	.001	0	.01	1.0
12	△	-	-	0	-.001	0	-
13	◇	-	-	.001	0	0	-

vehicle will get to the stated conditions. Additionally, roll equilibrium conditions are not directly given and must be approximated using additional information.

The λ values for roll-rate ratio equilibrium can be determined with the aid of Equation (3-2) or (4-33) knowing the values of the resident center of mass offset with its relative position to the resident trim asymmetry and p_{cr} and \dot{p}_{cr} along with the information given in Figures 5 and 6 for vehicles with the stated asymmetry and ball parameters. Additional insight can be gained by further considerations of the plots. At large values of λ the ball produces a trim of θ_J because of a principal axis shift given by Equation (2-22) while a trim moment gives a very small trim angle. For this reason θ_{eq}/θ_T becomes large but actually indicates only that θ_T is small. If θ_T was referenced to a non-principal axis, the value of θ_{eq}/θ_T would approach a constant value. For large enough vehicle damping, μ , no equilibrium conditions exist when $\lambda \gg 1$ because the term $HD > \theta_T$ in Equations (4-27) and (4-28). If HC and N are of opposite sign at $\lambda \gg 1$ then at some λ the value of $\theta_{eq} = 0$ since $HC = 0$ at $\lambda = 1$. When $\theta_{eq} = 0$, the value of φ_{eq} for the ball changes by 180° . An example of this can be seen in Figure 5. Another condition when the value of φ_{eq} will shift abruptly is when K changes signs. This causes a change in the sign of the value N which describes the stable equilibrium ball position.

For the conditions with the variable asymmetry

$m_b/m_v = .01$ shown in Figure 5 the largest value of damping that gives a real N at $\lambda = 2.5$ is $\mu = .6$. The larger asymmetry $m_b/m_v = .03$ shown in Figure 6 has the value $\mu = .2$ the largest with a real N at $\lambda = 2.5$. Since $\mu(t)$ is relatively unaffected by the asymmetries the larger the value of m_b/m_v the smaller the regions of possible equilibrium (real N). The close agreement between the analytical approximations of Chapter IV and the finite difference simulations for ball equilibrium conditions is apparent in Figures 5 and 6. It is unfortunate that the location of trajectory equilibrium points cannot be as easily illustrated. The point by point comparisons between the analysis and finite difference simulations give values of λ for trajectory equilibrium that are in good agreement. This is expected since the angle of attack values are so close.

It is worthwhile to examine some specific example trajectories to follow the motion of a vehicle through time with actual variations in λ to illustrate trajectory equilibrium conditions. Figure 7 gives the value of $|\lambda|$ versus time for several trajectories along with the approximate values of $|\mu|$ for all of the trajectories in this and succeeding figures. Figure 8 shows the actual roll rate along with p_{cr} as a function of time. The vehicle parameters are given in Tables 1 and 2. Run 13 has only the fixed or resident asymmetry and no moving mass. The vehicle of Run 4 which has $m_b/m_v = .01$ has a decreasing λ value as time increases until $\lambda = 1.03$ at which point the

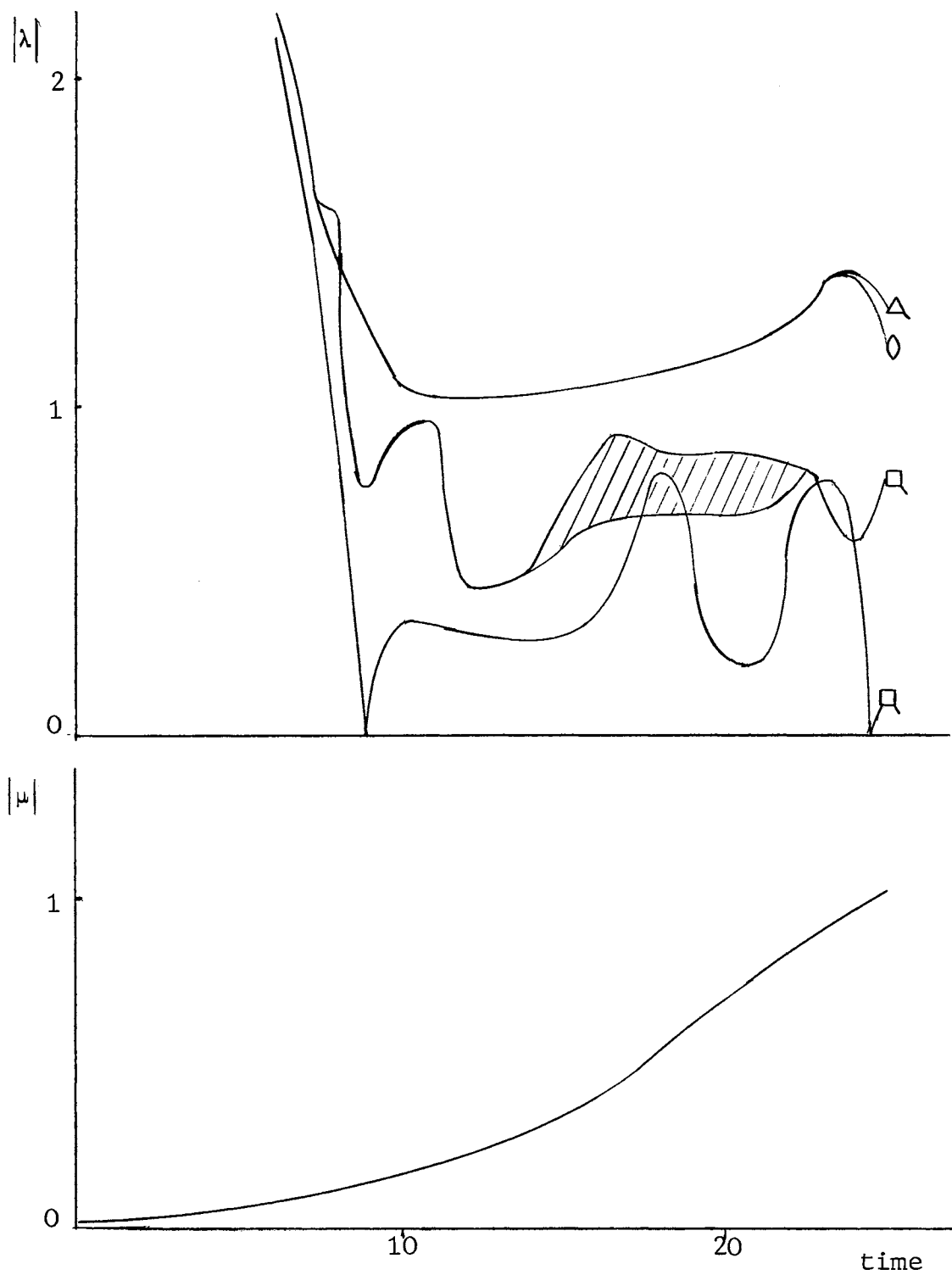


Figure 7. Roll-Rate Ratio and Damping Parameter as a Function of Time for Selected Examples

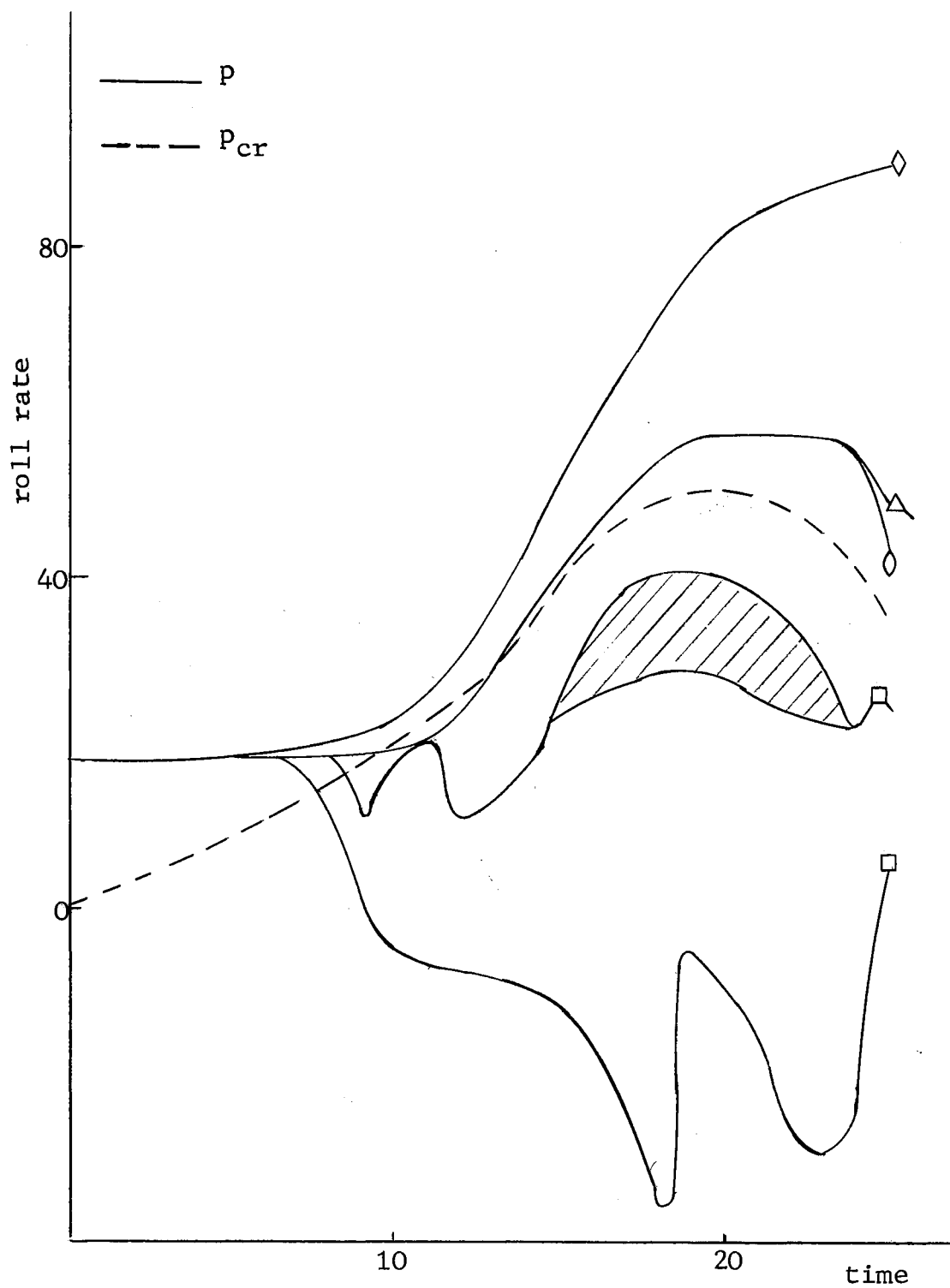


Figure 8. Roll Rate as a Function of Time for Selected Examples

roll ratio is in equilibrium. This condition persists with a slow increase in λ . When \dot{p}_{cr} goes negative the increase in λ becomes more apparent because the available roll torque is positive (see Equations (4-27), (4-28), and (4-33)).

From Figure 5 it is seen that no equilibrium condition exists after $\lambda = 1.4$ if $\mu = 1$ and the ball will drive the roll ratio down as given by Equation (4-32). This occurs on both Runs 4 and 11 at about $t = 24$. Run 11 is identical to Run 4 except that $h = 1.0$ instead of 0.3. Equation (4-33) shows that with a driven ball, the torque produced is a function of h so that a larger h provides a larger torque when the ball is moving, thus driving the roll rate down more rapidly in 11 than 4.

The other two runs shown on Figure 7 are for vehicles with the heavier asymmetry mass. Runs 7 and 8 are for identical vehicles except for the h value shown in Table 2. For both runs possible equilibrium conditions end at $\lambda = 1.1$. Run 7 has larger ball damping so that its roll rate is driven down further than Run 8. In fact its roll rate goes negative as seen in Figure 8. The possibility of equilibrium solutions is established at $\lambda = .9$ but the combination of the roll torque due to resident asymmetries which is negative and the torque from the moving ball combine to drive the roll rate lower. In Run 7 the viscosity in the race is so high that the ball moves very slowly. The equilibrium position at $\lambda = 1.1$ is about 180° from the equilibrium position at $\lambda = .9$ and the ball is unable to

respond quickly enough to move to the new position. The angle of attack contribution of the variable asymmetry is effective in the torque Equation (4-33) and is responsible for the negative torque. The lower race viscosity for Run 8 allows the ball in this case to run to the ball equilibrium condition for $\lambda = .81$ at $t = 9$. The vehicle has a $\dot{\lambda} > 0$ under these conditions and spins up until the ball has no equilibrium ($t = 10.5$) and then the vehicle is spun down where equilibrium is reestablished ($t = 12.5$). This situation is repeated throughout the remainder of the trajectory. The envelopes of extremities of roll values are plotted on Figures 7 and 8 for Run 8.

A similar situation occurs for Run 7 after it spins through zero and finally obtains an equilibrium condition in both ball position and λ for $\lambda = - .3$. This condition persists although $|\lambda|$ begins to increase after $t = 15$ due to a decrease in \dot{p}_{cr} (a loss of trajectory equilibrium). Ball equilibrium solutions are lost again at $\lambda = - .8$ and $\mu = .5$ and the vehicle spins up again (λ decreases) with the process repeated to termination of the trajectory.

Figure 9 shows the effect of various ball sizes on another fixed asymmetry condition. Run 12 is for the resident asymmetry alone while Run 5 is for $m_b/m_v = .01$ and Run 9 for $m_b/m_v = .03$. The resident asymmetry acts on the same axis as the center of mass is located. The torque produced by the resident asymmetry is negative for $\lambda > 1$ and becomes positive just below $\lambda = 1$. This creates a

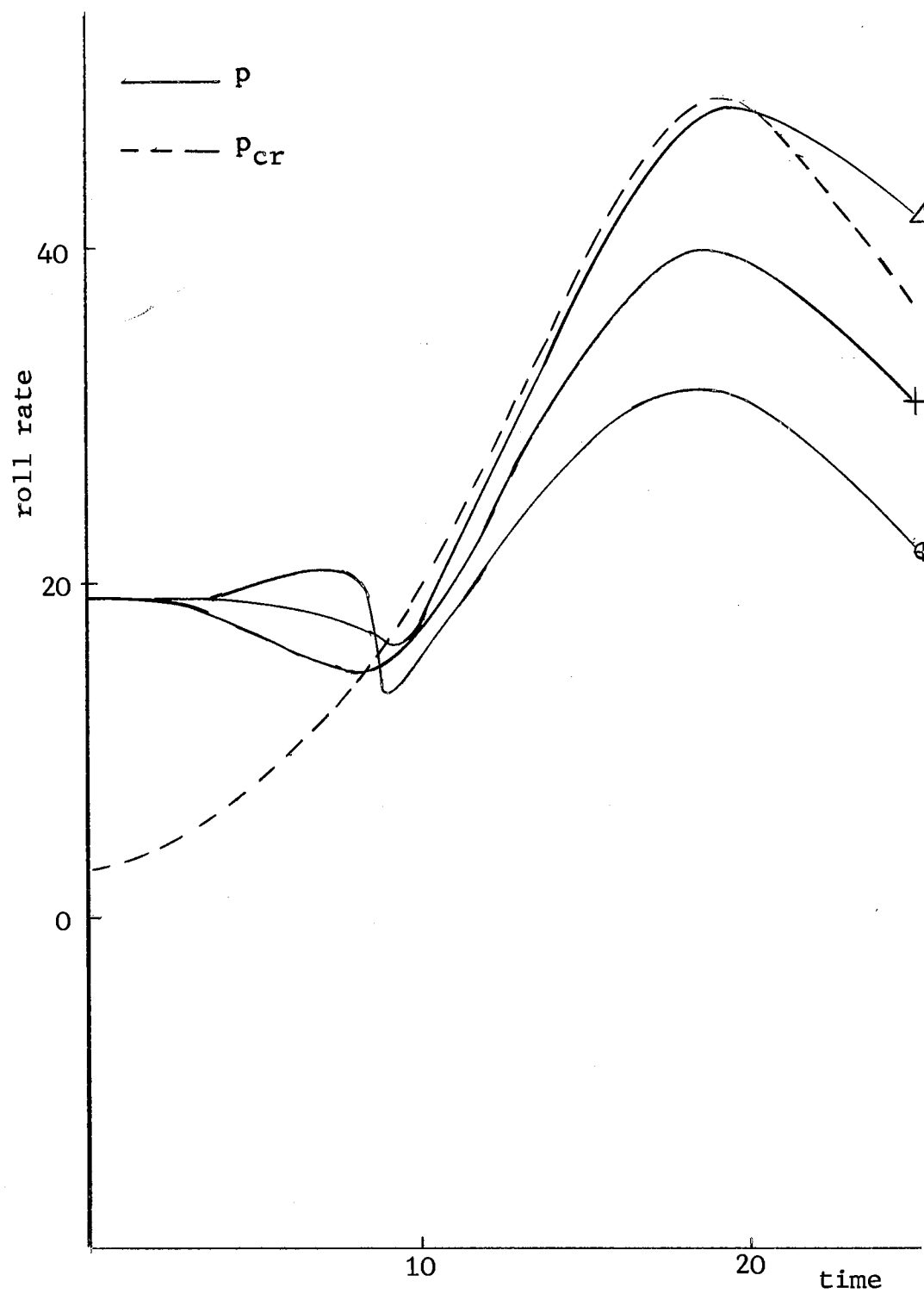


Figure 9. Roll Rate as a Function of Time for Selected Examples

stable equilibrium point below $\lambda = 1$ for \dot{p}_{cr} increasing. The effect of a variable mass is to significantly lower the λ for trajectory equilibrium. This acts to reduce the angle of attack in the region of large Q . Run 9 has a stable ball equilibrium position that gives a positive roll torque until $Q = 7000$ when the position becomes unstable. This instability is only detected through consideration of the stability of the full angle of attack and ball position equations combined. The stability condition determined by Equation (4-31) is insufficient and the complete characteristic equation must be considered. In this case no stable equilibrium ball position exists until λ decreases to less than unity.

The results of this chapter show the basic effects of variable asymmetries through comparison of the equations developed in the previous chapters with specific examples. Because of the innumerable possible combinations of vehicle and trajectory parameters all types of situations cannot be described and demonstrated. The examples presented do show typical responses for ballistic vehicles.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

It has been shown that a variable asymmetry can be used to alter the motion of a slender symmetric missile with small configurational and mass asymmetries. A vehicle with resident asymmetries will in general experience an amplification of the angle of attack near resonance and a large change in roll rate as a result of the roll torque produced by the asymmetries. It is shown that mass and moment asymmetries are identical whether fixed or variable.

The angular motion of the vehicle is described by equations derived from the complete equations of motion through appropriate simplifications. The significant changes in roll rate are attributed to the non-oscillatory portion of the angle of attack which is the particular solution of the angle of attack equation. At certain values of the roll rate to natural frequency ratio the roll torque will be equal to that required to maintain a constant value of the roll-rate ratio. If this condition is stable the vehicle is said to be in "roll lock-in". This condition is objectionable because the angle of attack will remain large if the stable equilibrium conditions occur near a roll ratio of unity. This condition approximates a

condition of resonance and in fact is often called "persistent roll resonance". In addition a large roll torque can cause a significant change in roll rate. Both effects can lead to structural problems and are to be avoided.

The addition of a variable asymmetry has been shown to be capable of producing a significant change in the conditions of equilibrium. An equilibrium condition with a variable asymmetry requires that the variable asymmetry itself be stationary. This additional requirement reduces the range of trajectory equilibrium conditions because both roll-rate ratios and asymmetry position must be in equilibrium simultaneously.

Over certain regions of a trajectory no equilibrium ball position exists and the ball is driven around in its race. This causes a roll torque due to race friction which spins the body up or down. The direction of ball motion is given in Chapter IV.

When a stable equilibrium position for the asymmetry exists, the ball will move towards this point. This position is a function of vehicle parameters and trajectory conditions and consequently varies during a flight. If the variable asymmetry is not at an equilibrium position it will take a finite amount of time to reach such a position. In fact the equilibrium position may change faster than the ball is able to respond. This results in the possibility that although conditions exist for both

asymmetry and roll-rate ratio to be in equilibrium simultaneously, the asymmetry may not be at its equilibrium position and the vehicle will not "lock-in".

No attempt has been made in this analysis to specify a variable asymmetry that will keep the angle of attack and roll rate parameters within certain bounds for a particular vehicle and trajectory. This problem will be left to the designer of the specific vehicle.

The criteria for determining the equilibrium conditions and their effect on vehicle motion have been developed. The effect of a slowly moving asymmetry has been given. Further work into the dynamic effects of a moving asymmetry presents an interesting but complex area for future investigation. The investigation of the effect of a variable flap asymmetry can be expanded for flaps with specific aerodynamics. In addition other race configurations for internal mass asymmetries can be considered. The techniques developed in this work are adequate to handle these problems.

SELECTED BIBLIOGRAPHY

1. Nelson, R. L., "The Motions of Rolling Symmetrical Missiles Referred to a Body Axis System," NACA Report TN 3737, Nov. 1956.
2. Nicolaides, J. D., "On the Free Flight Motion of Missiles Having Slight Configurational Asymmetries," BRL Report 858, 1953.
3. Murphy, C. H., "Free Flight of Symmetric Missiles," BRL Report 1216, July 1963.
4. Pettus, J. J., "Persistent Re-Entry Vehicle Roll Resonance," AIAA Paper 66-49, New York, 1966.
5. Barbera, F. J., "An Analytical Technique for Studying the Anomalous Roll Behavior of Ballistic Re-Entry Vehicles," AIAA Paper 69-103, New York, 1969.
6. Hodapp, A. E. and Clark, E. L., "The Effects of Products of Inertia on the Roll Behavior of Ballistic Re-Entry Vehicles," AIAA Paper 70-204, New York, 1970.
7. Cunningham, W. J., Nonlinear Analysis, McGraw-Hill, New York, 1958, pp. 257-259.
8. Pettus, J. J., "Slender Entry Vehicle Roll Dynamics," AIAA Paper 70-560, Tullahoma, Tenn., 1970.
9. Vaughn, H. R., "A Detailed Development of the Tricyclic Theory," Sandia Laboratories SC-M-67-2933, Feb. 1968.
10. Thomson, W. T., Introduction to Space Dynamics, John Wiley and Sons, New York, 1961, pp. 194-195.
11. Nielson, J. N., Missile Aerodynamics, McGraw-Hill, New York, 1960, pp. 349-402.
12. Goldstein, H., Classical Mechanics, Addison-Wesley, Reading, Mass., 1959, pp. 124-132.

13. Vaughn, H. R., "Boundary Conditions for Persistent Roll Resonance on Re-Entry Vehicles," Sandia Corporation SC-RR-67-287, May 1967.
14. Minorsky, N., Nonlinear Oscillations, D. Van Nostrand Co., New York, 1962, pp. 118-162.
15. Nayfeh, A. H., "A Multiple Time Scaling Analysis of Re-Entry Roll Dynamics," AIAA Journal, Vol. 7, No. 11, Nov. 1969, pp. 2155-2157.

APPENDIX A

MOTION OF A MASS IN A MOVING

COORDINATE SYSTEM

The following is a kinematic description of the motion of a small fixed or moving mass in a body fixed to an intermediate coordinate system moving in space relative to an inertial coordinate system. The general coordinate system is shown in Figure 10 where the intermediate x, y, z axis system is fixed to the main body and angle φ and distances x and ρ describe the position of the satellite mass with respect to the body-fixed coordinates. The situations are limited to ρ and x being constants with φ fixed or varying. The bodies are coupled through forces given by Newton's third law of motion. The absolute acceleration of a mass b referring to Figure 10 is:

$$\bar{a}_b = \bar{a}_0 + \bar{a}_{b/0}, \quad (A-1)$$

$$\bar{a}_b = \bar{a}_0 + \bar{a}_{b'/0} + \bar{a}_{b/b'},$$

$$\text{or} \quad \bar{a}_b = \bar{a}_0 + \bar{a}_{b'/0} + (\bar{a}_{b/0})_{\text{rel}} + 2\bar{\omega} \times (\bar{v}_{b/0})_{\text{rel}}$$

$$\text{where } \bar{a}_{b'/0} = \dot{\bar{\omega}} \times \bar{\rho} + \bar{\omega} \times \bar{\omega} \times \bar{\rho},$$

and b is fixed to the coordinate system. The terms

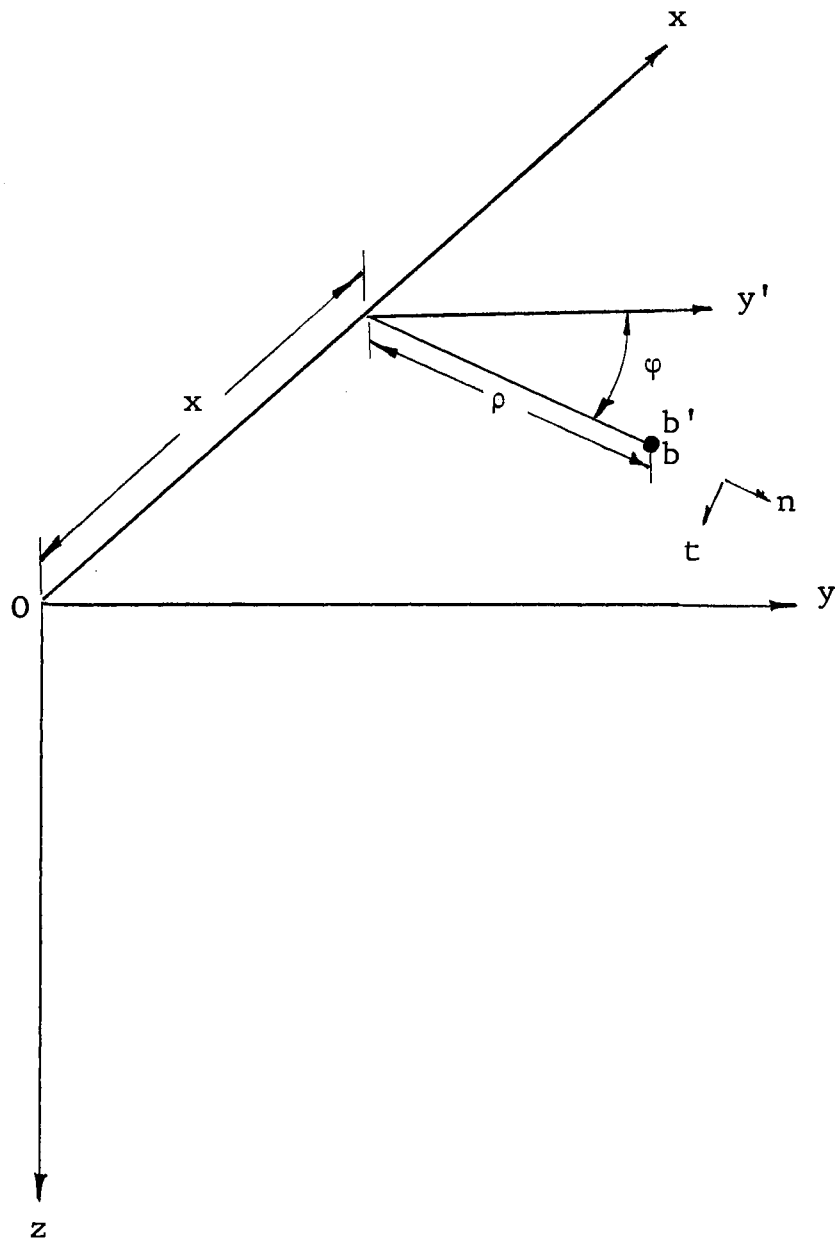


Figure 10. General Coordinates of a Moving Point in a Body-Fixed Coordinate System

subscripted with (rel) are quantities relative to the intermediate or body-fixed coordinate system.

The acceleration a_0 is the acceleration of the point 0 which is the origin of the intermediate coordinate system. The acceleration $a_{b/0}$ will be written in cylindrical coordinates since this system is most applicable to the problem to be considered. Under the constraints $\ddot{x} = \dot{x} = 0$ and $\ddot{\rho} = \dot{\rho} = 0$, the acceleration of b relative to 0 is

$$\begin{aligned} a_{b/0}^x = & 2\rho\dot{\varphi}\dot{r}\sin\varphi - \dot{r}\dot{\rho}\cos\varphi + 2\rho q\dot{\varphi}\cos\varphi + \dot{q}\dot{\rho}\sin\varphi \\ & + p\dot{r}\dot{\rho}\sin\varphi - (r^2 + q^2)x + pq\dot{\rho}\cos\varphi, \end{aligned} \quad (A-2)$$

$$\begin{aligned} a_{b/0}^n = & -\rho\dot{\varphi}^2 - 2p\rho\dot{\varphi} + (\dot{r}\cos\varphi + \dot{q}\sin\varphi)x - p^2\rho \\ & + p(q\cos\varphi + r\sin\varphi)x - r^2\rho\cos^2\varphi \\ & - q^2\rho\sin^2\varphi + 2qr\rho\sin\varphi\cos\varphi, \end{aligned} \quad (A-3)$$

$$\begin{aligned} \text{and } a_{b/0}^t = & \rho\ddot{\varphi} + \dot{p}\dot{\rho} - (\dot{r}\sin\varphi + \dot{q}\cos\varphi)x \\ & - p(q\sin\varphi - r\cos\varphi)x - qr(\sin^2\varphi - \cos^2\varphi)\rho \\ & + (r^2 - q^2)\rho\cos\varphi\sin\varphi. \end{aligned} \quad (A-4)$$

From Newton's second and third laws

$$\bar{F}_b = m_b \bar{a}_b \quad (A-5)$$

$$\text{and } \bar{F}_v = -\bar{F}_b. \quad (A-6)$$

where \bar{F}_b is the force on the mass and \bar{F}_v is the force on the vehicle. The forces in the x and normal (n) directions

are dictated by the holonomic constraints. The force in the tangential (t) direction arises solely from the relative resistance to motion between the small mass and the constraining race. The equation of motion for the ball in the t direction becomes

$$F^t = m_b(a_0^t + a_{b/0}^t). \quad (A-7)$$

For viscous damping $F^t = -h\dot{\varphi}$ the equation of motion of the ball position φ becomes

$$\begin{aligned} \ddot{\rho}\varphi = & -h\dot{\varphi}/m_b - \dot{p}\rho + (\dot{r}\sin\varphi + \dot{q}\cos\varphi)x + \ddot{y}\sin\varphi \\ & + p(q\sin\varphi - r\cos\varphi)x - r^2\rho\cos\varphi\sin\varphi - \ddot{z}\cos\varphi \\ & + q^2\rho\sin\varphi\cos\varphi + qrp(\sin^2\varphi - \cos^2\varphi). \end{aligned} \quad (A-8)$$

The acceleration of 0 may be expressed as

$$a_0^t = -\ddot{y}\sin\varphi + \ddot{z}\cos\varphi \quad (A-9)$$

$$\text{or } a_0^t = -F^y\sin\varphi/m_t + F^z\cos\varphi/m_t. \quad (A-10)$$

The motion of the system is described by the above equations and the equations of motion of the main vehicle with the interacting forces F_v included.

APPENDIX B

FINITE DIFFERENCE SIMULATION

Finite difference simulation of the equations of motion of the mathematical model is used to substantiate the theoretical approximations of this work. The solution of the finite difference equations was accomplished through use of a high speed digital computer. This technique is commonly used in flight simulations.

The equations of motion for the basic vehicle are given in Equations (2-1) through (2-12). The forces and moments of the equations are augmented with ball forces using equations (A-1) and (A-6). The relative motion of the ball is restricted to a circular path in the body-fixed coordinate system. Therefore, the relative motion of the ball is described by Newton's second law and the tangential acceleration given by Equation (A-3). The tangential force on the ball is a linear viscous damping force proportional to a constant and the relative angular velocity.

The equations of motion of the body are integrated once to give velocities while the ball motion equation is integrated twice to give relative velocity and position. The total velocity vector is held at a constant flight path angle with the horizontal. This velocity multiplied by the

sine of the flight path angle is integrated to get an altitude variation for use in the expression for density

$$\rho^* = .0027 \exp(-.000045 Z). \quad (B-1)$$

The resulting density is used with the total velocity vector to determine the dynamic pressure

$$Q = \frac{1}{2} \rho^* V^2. \quad (B-2)$$

No curvature or rotational effects of the atmosphere are included. Gravity effects are neglected since kinetic energy of the missile is large compared with geopotential energy of the vehicle. The aerodynamic coefficients in the equations are held constant. The above assumptions are not severely limiting since the relative position coordinates are not calculated and the motion characteristics are short time duration functions that are secondarily effected by the above considerations. An analysis of this is given by Nayfeh (15).

The solutions are accomplished through use of a basic program called "DYSIMP" that integrates simultaneous first-order ordinary differential equations. This program uses a fourth-order Runge-Kutta integration scheme with a fixed step size. This method proves exceptionally stable when the step size is chosen to be much less than the period of the highest frequency oscillation. The particular differential and supporting equations are supplied as a subroutine to DYSIMP.

APPENDIX C

EFFECTS OF A VARIABLE TRIM MOMENT

Another type of variable asymmetry that can be considered using the analysis of the previous chapters is a variable trim moment. This variable moment will be considered to be a configurational asymmetry attached to the body and rotating about the main vehicle axis of symmetry. This "flap" could be driven or free to assume a stable condition. If unforced its position must be determined by the relative wind velocity. The stable position will be a strong function of the relative wind velocity and a secondary function of other considerations such as angular velocity and accelerations. The asymmetry can be balanced or unbalanced without increasing the difficulty of solution.

The analysis of the rotating flap is limited because the aerodynamics of flaps are strong functions of geometry and related flow conditions. Since only general assumptions can be made, the methods for analysis are briefly outlined and are noted to be the same as those for a rotating mass. The results for the two cases are so similar that a more extensive analysis without specific flap aerodynamics would not add significant information.

The analysis parallels that for a ball in a race

through Equation (4-15) with flap forces appearing in the same manner as ball forces. The term G_p should now be augmented with a G_{pf} which represents the moment due to the flap. Only the moments due to the flap are considered since the normal force terms on the body due to the flap will have a secondary effect on the angular motion as shown in Equation (3-1). The term G_{pf} contains terms similar to those for a fixed moment asymmetry except that they vary with relative position. The augmentation is

$$G_{pf} = i(C_{Mf}(\cos(\varphi + \varphi_f) + i \sin(\varphi + \varphi_f)))/I' \quad (C-1)$$

where C_{Mf} is the flap trim moment coefficient and φ is the relative wind position and φ_f is the phase of C_{Mf} from this position. Equations for the particular solution similar to (4-25) and (4-26) result and are given as

$$\beta = \beta_T + C_{Tf}(C \sin(\varphi + \varphi_f) - D \cos(\varphi + \varphi_f)), \quad (C-2)$$

$$\alpha = \alpha_T - C_{Tf}(C \cos(\varphi + \varphi_f) + D \sin(\varphi + \varphi_f)), \quad (C-3)$$

and $C_{Tf} = C_{Mf}/(C_{M\theta}(C^2 + D^2))$

where $\cos\varphi = \beta/\theta$ and $\sin\varphi = \alpha/\theta$.

These equations can be solved for α and β and yield

$$\beta_{eq} = (\beta_T N_f + \alpha_T C_{Tf}(D \cos\varphi_f + D \sin\varphi_f))\theta_f \quad (C-4)$$

and $\alpha_{eq} = (\alpha_T N_f - \beta_T C_{Tf}(C \cos\varphi_f + D \sin\varphi_f))\theta_f \quad (C-5)$

where $\theta_f = C_{Tf}(C \sin \varphi_f - D \cos \varphi_f + N_f)/(\alpha_T^2 + \beta_T^2)$

$$N_f = \pm \left[(D \cos \varphi_f - C \sin \varphi_f)^2 - (D^2 + C^2) + (\alpha_T^2 + \beta_T^2)/C_{Tf}^2 \right]^{1/2}.$$

In general $\varphi_f = 90^\circ$ since the asymmetry will align itself into or away from the wind vector. If the asymmetry is not balanced this may be included in the equations and treated as a moving mass by the methods of Chapter IV.

It is interesting to note that the Equations (C-4) and (C-5) are similar to Equations (4-27) and (4-28). This is expected since it was shown in Chapter II that a principal axis shift creates a trim moment exactly like a geometric trim. If φ_f is other than 90° , the equilibrium condition will be altered but the form of the equations will be the same. This phase angle will appear also if the flap force and center of mass are not on the same radius from the center of rotation of the flap.

The stability of the flap equilibrium position is similar to that of the mass in the race if the consideration for windward or leeward stability is included in the sign of C_{Nf} . The equation for ball motion will be augmented with terms to account for relative rotational torque on the flap. The remainder is the same as given in (4-23) except that now

$$K = x p^2 - C_{N\theta} Q S / m_t - \rho_f C_{Nf} Q S / \rho m_f \quad (C-6)$$

where ρ_f is the radius to the aerodynamic surface,

ρ is the radius of gyration of the flap,

and m_f is the flap mass.

A positive C_{Nf} indicates the flap pointing into the wind and a negative C_{Nf} indicates a leeward position. The conditions for stability are the same as given with Equations (4-30) and (4-31).

The effect of a control flap on vehicle motion is the same as that of a ball in a race except that it does not necessarily apply a moment only orthogonal to the relative wind vector. The effect on the roll equation will be the same as for the ball in the race. The influence is entirely through a change in angle of attack. Illustrative examples are omitted because of the similarity to the variable mass case.

APPENDIX D

NOMENCLATURE

- A - total trim coefficient about the z-axis
- B - total trim coefficient about the y-axis
- C - trim amplification factor in plane of the trim coefficient, $1 - \lambda^2$
- C_{ijk} - aerodynamic coefficient where subscripts give particular definition
- D - trim amplification factor in plane orthogonal to trim coefficient, $\mu\lambda$
- d - reference length
- E - coefficient on $\dot{\theta}$
- F - coefficient on θ
- \bar{F} - force
- F_i, F^i - force in specific direction
- G - forcing function on angle of attack equation
- H - trim amplification factor caused by variable asymmetry
- \bar{H} - angular momentum
- h - damping coefficient on variable asymmetry
- I - vehicle inertia orthogonal to axis of symmetry
- I_x - vehicle inertia about the axis of symmetry
- I_{ij} - vehicle inertia tensor component
- I' - I/QSd
- I'_x - I_x/QSd

- K - coefficient in asymmetry position equation,
 $x p^2 - C_{N\theta} Q S / m_t$
- M - moment
- M' - mV/QS
- m - mass
- N - solution parameter, $\pm (\alpha_T^2 + \beta_T^2 - H^2 D^2)^{\frac{1}{2}}$
- p, q, r - inertial roll rates about the x, y, z axes
- p_{cr} - critical or natural frequency, $\pm (-C_{M\theta} Q S d / (I - I_x))^{\frac{1}{2}}$
- Q - dynamic pressure, $\frac{1}{2} \rho^* V^2$
- R - remainder coefficient
- R_M - remainder in asymmetry position equation
- S - reference area
- \bar{s} - distance from vehicle center to asymmetry mass
- u, v, w - wind referenced velocities along x, y, z axes
- V - total velocity
- X, Y, Z - inertial axes
- x, y, z - body-fixed axes
- x - x distance from coordinate center
- Y_M, Z_M - resident geometric center offset from center of mass
- $\delta Y, \delta Z$ - total geometric center offset from center of mass
- α - angle of attack
- β - angle of sideslip
- δ - small perturbation on a parameter
- θ - total angle of attack
- λ - frequency ratio, p/p_{cr}
- μ - damping coefficient

ξ - $q + ir$
 ρ - radius of race
 ρ^* - air density
 φ - angular position of asymmetry
 $\bar{\omega}$ - angular velocity

Subscripts

A - augmented value
 b - asymmetry mass
 cm - center of mass
 eq - equilibrium condition
 f - flap
 J - trim due to mass asymmetry
 l,m,n - moments about x, y, z axes
 M - moment
 N - force
 Nf - normal force due to flap
 o - trim value for coefficients
 O - vehicle center referenced
 p,q,r - partial derivatives with respect to p, q, r
 P - principal axes referenced
 T - trim due to resident asymmetries
 v - vehicle
 x,y,z - coordinate directions
 α,β,θ - partial derivative with respect to $\alpha, \beta, \text{ or } \theta$

Superscripts

x,n,t - directions on ball coordinates
 x,y,z - directions on body-fixed coordinates

VITA

Charles Fremont Huff

Candidate for the Degree of

Doctor of Philosophy

Thesis: DYNAMICS OF BALLISTIC BODIES WITH VARIABLE
ASYMMETRIES

Major Field: Mechanical Engineering

Biographical:

Personal Data: Born in Alice, Texas, July 21, 1939,
the son of John Arleigh and Nellmarie Huff.

Education: Graduated from Boerne High School, Boerne, Texas, in May, 1957; entered Trinity University in San Antonio, Texas; in 1958 entered The University of Texas in Austin and received a Bachelor of Science degree in Mechanical Engineering in August, 1961; while an undergraduate elected to membership in Tau Beta Pi and Pi Tau Sigma; during the summers of 1958 and 1959 attended San Antonio College and Colorado University respectively; received the Master of Science degree from The University of Texas in Austin in 1963, with a major in Mechanical Engineering; did graduate work at the University of New Mexico, Albuquerque, New Mexico; completed the requirements for the Doctor of Philosophy degree at Oklahoma State University in May, 1971, as a National Science Foundation Fellow.

Professional Experience: Worked as a graduate teaching assistant at The University of Texas, 1961-63; as Staff Member Technical, Sandia Laboratories, Albuquerque, New Mexico, 1963-68; and as a research assistant, Oklahoma State University, Basic Fluid Power Study, 1969-70.

Professional Organizations: Member of the American Society of Mechanical Engineers, and the American Institute of Aeronautics and Astronautics.