THE EXACTNESS OF ACOUSTIC WAVE FRONTS CALCULATED VIA FRESNEL-KIRCHHOFF INTEGRALS

By

WILLIAM G. HEDGPETH

Bachelor of Science
Central Missouri State College
Warrensburg, Missouri
1965

Master of Science
Central Missouri State College
Warrensburg, Missouri
1966

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF EDUCATION
May 16, 1971

THE EXACTNESS OF ACOUSTIC WAVE FRONTS CALCULATED

VIA FRESNEL-KIRCHHOFF INTEGRALS

Thesis Approved:

Thomas S. Winter
Thesis Adviser

D. L. Rutledge

Arlo B. Schmidt

Kenneth E. Wiggins

D. D. Durham
Dean of the Graduate College

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to Dr. Thomas G. Winter for his supervision and guidance during the course of this study. He is especially indebted to Mike Reisinger for his assistance in the laboratory for helping with the "photographing" of the wave front patterns. And finally he wishes to express his deep gratitude to his wife, Linda, for her assistance and understanding during the course of this investigation. Also, her help in drawing the figures for this paper were invaluable.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF FIGURES (Continued)

CHAPTER I

INTRODUCTION

Wave propagation is fundamental to many branches of Physics but a detailed knowledge of the field near an object which is reflecting or diffracting the waves is difficult to attain. The best and most common approach is to calculate the field near such an object by Fresnel-Kirchhoff theory, but this approach usually incorporates some approximations which are not valid within a few wavelengths of the object. In this work, we have photographed the wavefront structure in front of and behind a 30 cm by 30 cm board in an acoustic field. We have calculated what the wave front structure should be by Fresnel-Kirchoff theory and compared the two. And it should be noted that such wave front patterns have never been calculated before.

The Fresnel-Kirchhoff diffraction theory should apply quite well to the study of acoustic waves. In fact, it should be more applicable to the study of sound than to the study of light. The Fresnel-Kirchhoff theory is a scalar wave theory, and sound is a scalar wave phenomenon. Light must be explained vectorally.

Several methods, in addition to the Fresnel-Kirchhoff theory, are available for the solution of diffraction fields about objects. These are, generally, direct solutions to Poisson's equation subject to the appropriate boundary conditions. At times, Poisson's equation can be solved in a closed form with the appropriate transformations. But most physical situations are too complex for this to be done and numerical analysis techniques must be used. A large variety of methods for solving differential equations numerically exist (4). Two of the most commonly used techniques for the numerical solution of Poisson's equation are the method of finite differences and the method of moments (2,10).

In the method of finite differences, the differential equation is reduced to a series of simple algebraic equations which are subject to the given boundary equations. From these equations, the field at points throughout the region is evaluated.

The method of moments is an operator technique of solution. A series solution $\Sigma\alpha_n f_n$ is established. The α_n's are evaluated through the use of inner products and inverse operators.

History of the Fresnel-Kirchhoff Diffraction Theory

The Fresnel-Kirchhoff diffraction theory and its origins were initially developed in the field of optics. But, as all ready mentioned, the theory is very applicable to the study of sound. In 1690 Huygens published his Traite de la Lumiere. It was in this treatise that he expounded upon the idea which has since become known as Huygens' Principle (1). According to this principle,

> "...every point of a wave-front may be considered as a centre of a secondary disturbance which gives rise to spherical wavelets, and the wave-front at any later instant may be regarded as the envelope of these wavelets" (7).

But Huygens had to make certain ad hoc assumptions in order to make his principle work. He had to assume that the secondary envelope directed back toward the source had no effect. Also, he had to assume that the wavelets had no effect except where they touched the envelope. With this wave theory Huygens was able to explain the laws of reflection and refraction.

In 1818, Fresnel published a paper on diffraction. In this paper Fresnel replaced Huygens' isolated spherical waves with periodic trains of spherical waves and employed the principle of interference. This refinement of Huygens' basic idea is known as the Huygens-Fresnel principle. With this refinement diffraction phenomena could finally be explained. But even this was not based upon a firm mathematical foundation. Kirchhoff was finally able to put the Huygens-Fresnel theory upon a solid mathematical basis. He showed that the theory could be derived from Greens theorem. This derivation is examined in the following chapter.

Evaluation of the Fresnel-Kirchhoff Integrals

The Fresnel-Kirchhoff integral is of the form

$$\mathcal{U}(P) = \iint g(x,y)\, e^{ikf(x,y)}\, dx\, dy \qquad (1-1)$$

In general this equation cannot be solved exactly. Hence techniques of calculating approximate solutions must be used. One technique which is found in most elementary textbooks on optics involves substitution of g(x,y) and f(x,y) by approximately equivalent functions which allow the integral (1-1) to be integrated in a closed form. For example, see Jenkins and White's Optics (12). But this type of solution is usable only for very simplified problems.

Generally, with an integral of the form of (1-1), one must resort to numerical techniques. One possibility would be to expand Simpson's rule to two dimensions. Monte Carlo techniques can also be used to solve multiple integrals. This latter technique is a statistical method of integrating. One obtains a random sample of points lying in the surface of integration. With this random sample one obtains an estimate to the true value of the integral. The larger the sample size, the more accurate the estimate will become (4).

Another method of evaluating a multiple integral is to generate a series expansion for the integrand and integrate term by term. One technique of series solution is known as the method of asymptotic expansions (24). This method was originally developed for integrals of one variable of the form of Equation (1-2).

$$I_2 = \int_a^b g(x)\, e^{ik\, f(x)}\, dx \qquad (1-2)$$

The major contributions to this integral are those points for which the phase kx is stationary. These points are known as "critical points." An asymptotic solution to (1-2) can be found by expanding a power series about these points and integrating the first few terms. The result is a series involving terms of $k^{-1/m}$, $k^{-2/m}$, etc., where m is an integer.

A formal and rigorous application of asymptotic expansions to one dimensional integrals has been developed by A. Erdelyi (6). Jones and Kline (13) have extended Erdelyi's treatment of single integrals to the treatment of multiple integrals.

Van Kampen (25, 26, 27) has developed an asymptotic technique for integrals of the form of (1-1) by applying a two-dimensional analogue of the method of stationary phase. Certain objections have been raised against his methodology, but, in practice, his results work very well (28). Van Kampen's method of solution of Kirchhoff's integral, i.e., Equation (1-1), is the one used in this paper.

Van Kampen's technique of solution is fully examined and applied to the Kirchhoff integral in the second chapter. Series solutions of the Kirchhoff integral for diffraction behind a 30 cm by 30 cm square and for reflection above a 30 cm by 30 cm square are obtained. In the third chapter these series solutions are used to calculate lines of constant phase (wave-fronts) for diffraction and reflection. A computerized search routine was developed to map the wave-fronts. Chapter IV contains information on the experimental mapping of acoustic wave-fronts. The method of experimentally mapping wave-fronts is explained. And "pictures" of the wave-fronts for the cases of diffraction and reflection about a square plane are examined. The fifth (and concluding) chapter compares the theoretically calculated wave-fronts with the experimentally determined wave-fronts.

CHAPTER II

THEORY

Kirchhoff's Diffraction Integral

The Fresnel-Kirchhoff diffraction integral is developed in many advanced texts upon optics or wave theory. But as this whole paper is based upon this type of integral, it is deemed appropriate to present a development of it.

As was previously mentioned in the last chapter, the Fresnel-Kirchhoff theory is based upon a certain integral theorem. This is Green's theorem. Stated without proof it is as follows: Let U(P) and G(P) be any two complex-valued functions of position, and let S be a closed surface surrounding a volume V. If U, G, and their first and second partial derivatives are single valued and continuous within and on S, then, we have

$$\iiint_V (\mathcal{G}\,\nabla^2 \mathcal{U} - \mathcal{U}\,\nabla^2 \mathcal{G})\, dv = \iint_S \left(\mathcal{G}\frac{\partial \mathcal{U}}{\partial n} - \mathcal{U}\frac{\partial \mathcal{G}}{\partial n}\right) d\sigma \qquad (2\text{-}1)$$

where $\partial/\partial n$ signifies a partial derivative in the outward normal direction at each point on S (1, 3, 7). It should be noted that in this paper U and G represent pressures created by two different point sources.

Kirchhoff's integral is developed by applying Green's theorem to the geometry in Figure 1. Figure 1 consists of a closed surface consisting of two parts, Σ and Σ', and an enclosed volume V. The inner surface Σ' is a diminishingly small spherical shell used to keep the

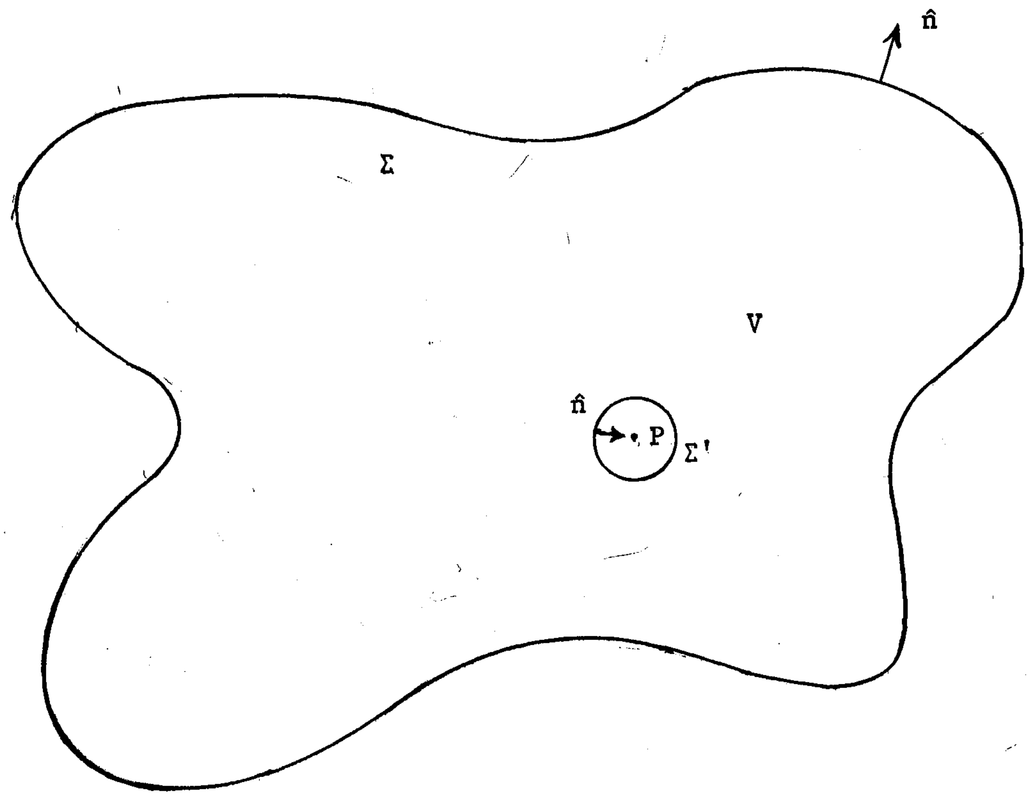

Figure 1. Geometry for Development of Kirchhoff Integral

point P, at which the field is to be found, out of the volume V.

As the volume V contains no sources, both G and U satisfy Poisson's equation. That is,

$$\begin{aligned} \nabla^2 U + k^2 U &= 0 \\ \nabla^2 G + k^2 G &= 0 \end{aligned} \qquad (2-2)$$

Applying Equations (2-2) to Equation (2-1), one obtains the result that the left hand side of (2-1) is identically equal to zero. Hence,

$$\iint_S \left(G \frac{\partial U}{\partial n} - U \frac{\partial G}{\partial n} \right) d\sigma \equiv 0 \qquad (2-3)$$

Assuming that G is a unit-amplitude acoustic wave expanding spherically about the point P, it takes on the following form

$$G = \frac{e^{iks}}{s} \qquad (2-4)$$

where s denotes the distance from the point P to a general point (x,y,z) on the surface Σ. Since s = 0 is a singularity for the function G, the point P must be excluded from the domain of integration.

Substituting (2-4) into (2-3) gives

$$\iint_\Sigma + \iint_{\Sigma'} \left\{ \frac{e^{iks}}{s} \frac{\partial U}{\partial n} - U \frac{\partial}{\partial n} \left(\frac{e^{iks}}{s} \right) \right\} d\sigma = 0 \qquad (2-5)$$

For integration over Σ', the small spherical shell with radius ε about point P, the normal derivative ∂/∂n becomes −∂/∂s. Hence, the integral over Σ' becomes

$$\iint_{\Sigma'} \left(\frac{U}{\varepsilon^2} - \frac{ikU}{\varepsilon} + \frac{1}{\varepsilon} \frac{\partial U}{\partial n} \right) e^{ik\varepsilon} d\Sigma' \qquad (2-6)$$

$$\int_0^{2\pi} \int_0^{\pi} \left(\frac{U}{\varepsilon^2} - \frac{ikU}{\varepsilon} + \frac{1}{\varepsilon} \frac{\partial U}{\partial n} \right) e^{ik\varepsilon} \sin\theta \, \varepsilon \, d\theta \, d\phi \qquad (2-7)$$

The final result for the integral over Σ' approaches 4π U(P). Hence, Equation (2-5) becomes

$$\mathcal{U}(P) = \frac{1}{4\pi} \iint_{\Sigma} \left[\mathcal{U} \frac{\partial}{\partial n} \left(\frac{1}{s} e^{iks} \right) - \frac{1}{s} e^{iks} \frac{\partial \mathcal{U}}{\partial n} \right] d\Sigma \qquad (2-8)$$

The above equation is known as the Fresnel-Kirchhoff diffraction integral. U is the pressure of some sound source which lies outside the volume V (note Figure 1). The sound pressure which one wants to determine at the point P within the closed surface is represented by U(P). And the distance s is the distance between the point P and points on the surface Σ. The physical significance of (2-8) is perhaps more readily understood in the following statement: If one knows the functional dependence of pressure U from some sound source upon any arbitrary closed surface, he can then calculate the pressure U(P) at any point P within the surface by means of the integral (2-8).

The Kirchhoff Integral for the Case of Reflection

Stone (23) mentions the fact that the Kirchhoff integral can be applied to reflection from plane surfaces. Thus if one wants to see what the total field looks like, he must add the result from the Kirchhoff integral and a contribution directly from the source. It seems reasonable that this can be done merely by referral to the principle of superposition. But this idea can be explicitly proved in a manner quite analogous to that of the proof of the Fresnel-Kirchhoff diffraction integral.

Consider a point source $\frac{Ae^{ikr}}{r}$ at point Q within the volume V in Figure 2. The surface Ω is a reflecting surface. The field U is to be

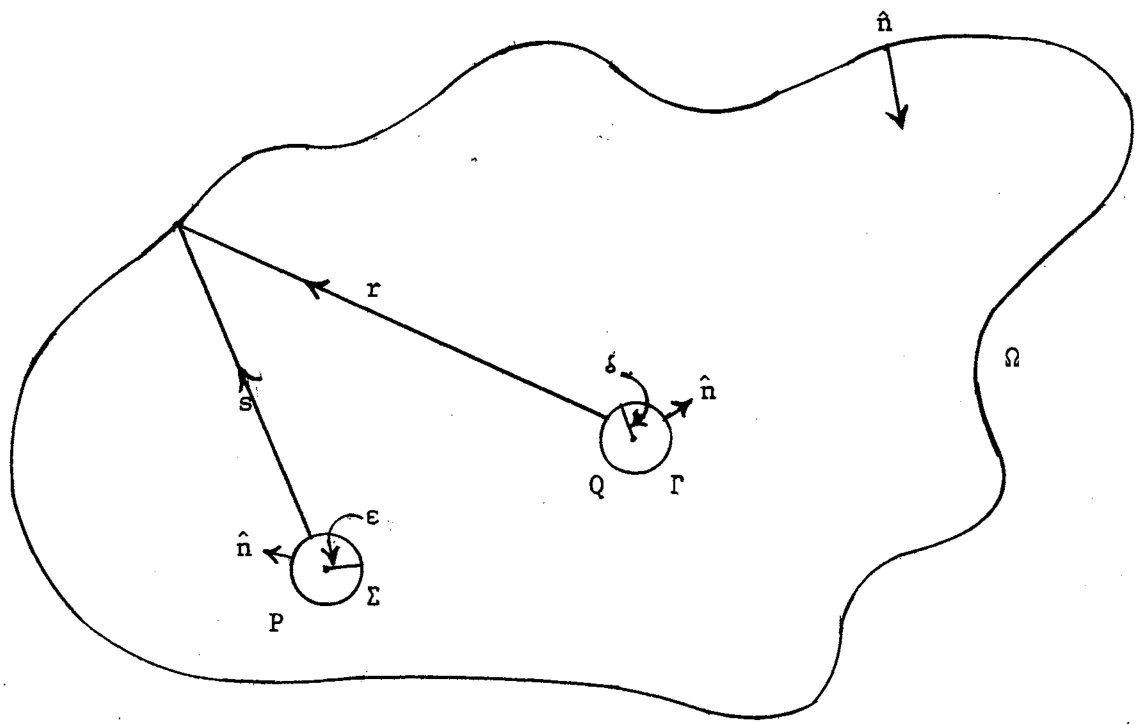

Figure 2. Geometry for Development of Kirchhoff's Integral for Reflection

found at the point P. Note that the volume V excludes both points P and Q as they are surrounded by spherical shells with diminishingly small radii ε and δ, respectively. The points P and Q must be excluded from the enclosed volume in order that Green's theorem can be satisfied.

Green's theorem states that

$$\int_V (\mathcal{G}\nabla^2 \mathcal{U} - \mathcal{U}\nabla^2 \mathcal{G})\, dv = \int_S (\mathcal{G}\frac{\partial \mathcal{U}}{\partial n} - \mathcal{U}\frac{\partial \mathcal{G}}{\partial n})\, d\sigma \qquad (2-9)$$

But as there are no sources in the volume V, the left hand member of (2-9) is identically zero. Therefore,

$$\iint_S (\mathcal{G}\frac{\partial \mathcal{U}}{\partial n} - \mathcal{U}\frac{\partial \mathcal{G}}{\partial n})\, d\sigma = 0 \qquad (2-10)$$

Letting $G = \frac{e^{iks}}{s}$, Equation (2-10) becomes

$$\iint_{\Sigma+\Gamma+\Omega} \left[\mathcal{U}\frac{\partial}{\partial n}\left(\frac{e^{iks}}{s}\right) - \frac{e^{iks}}{s}\frac{\partial \mathcal{U}}{\partial n}\right] d\sigma = 0 \qquad (2-11)$$

For integration over Σ and Γ, $\frac{\partial}{\partial n}$ becomes $\frac{\partial}{\partial s}$. As a result,

$$\left.\frac{\partial \mathcal{G}}{\partial n}\right|_{s=\varepsilon} = \frac{e^{ik\varepsilon}}{\varepsilon}\left(ik - \frac{1}{\varepsilon}\right) \qquad (2-12)$$

From Equations (2-11) and (2-12), one finds that the integral over Σ is

$$\iint_\Sigma \left(-\frac{\mathcal{U}}{\varepsilon^2} + \frac{ik\mathcal{U}}{\varepsilon} - \frac{1}{\varepsilon}\frac{\partial \mathcal{U}}{\partial n}\right) e^{ik\varepsilon}\, d\Sigma \qquad (2-13)$$

which becomes

$$\int_0^{2\pi}\int_0^{\pi} \left(-\frac{\mathcal{U}}{\varepsilon^2} + \frac{ik\mathcal{U}}{\varepsilon} - \frac{1}{\varepsilon}\frac{\partial \mathcal{U}}{\partial n}\right) e^{ik\varepsilon} \sin\theta\, \varepsilon^2\, d\theta\, d\phi$$

The resulting integration gives an answer of

$$\left(-\mathcal{U} + ik\mathcal{U}\varepsilon - \varepsilon\frac{\partial \mathcal{U}}{\partial n}\right) e^{ik\varepsilon} 4\pi \qquad (2-14)$$

And, as ε shrinks to zero, the integral over Σ approaches $-4\pi\, U(P)$ in the limit. Hence, Equation (2-11) can be replaced by

$$\mathcal{U}(P) = \frac{1}{4\pi}\iint_{\Omega}\left[\mathcal{U}\frac{\partial}{\partial n}\left(\frac{e^{iks}}{s}\right) - \frac{e^{iks}}{s}\frac{\partial \mathcal{U}}{\partial n}\right]d\Omega$$

$$+\frac{1}{4\pi}\iint_{\Gamma}\left[\mathcal{U}\frac{\partial}{\partial n}\left(\frac{e^{iks}}{s}\right) - \frac{e^{iks}}{s}\frac{\partial \mathcal{U}}{\partial n}\right]d\Gamma \qquad (2-15)$$

Next, the summation over the surface Γ needs to be examined. Γ is the surface of a small sphere of radius δ centered at Q and excluding Q from the volume V. An acoustic point source is located at Q. Hence, on Γ it can be seen that

$$\mathcal{U} = \frac{A e^{ik\delta}}{\delta}$$
$$\left.\frac{\partial \mathcal{U}}{\partial n}\right|_{\delta} = \frac{ikA e^{ik\delta}}{\delta}\left(ik - \frac{1}{\delta}\right) \qquad (2-16)$$

Substituting Equations (2-16) into the integral over Γ, i.e., Equation (2-11), one obtains

$$A\iint\left[\frac{e^{ik\delta}\, ik e^{iks}}{\delta s}\left(ik - \frac{1}{s}\right)\cos(\hat{n},\vec{s}) - \frac{e^{iks}}{s}\frac{e^{ik\delta}}{\delta}\left(ik - \frac{1}{\delta}\right)\varepsilon^2 d\omega \qquad (2-17)$$

where ω is the solid angle. As the sphere of radius δ shrinks to the point Q, the integral approaches a value

$$\frac{4\pi A e^{ikr}}{r} \qquad (2-18)$$

where r is the distance from point P to point Q.

Thus, the general expression (2-15) for the field U(P) becomes

$$\mathcal{U}(P) = \frac{1}{4\pi}\iint\left(\mathcal{U}\frac{\partial}{\partial n}\left(\frac{e^{iks}}{s}\right) - \frac{e^{iks}}{s}\frac{\partial \mathcal{U}}{\partial n}\right)d\sigma + \frac{A e^{ikr}}{r} \qquad (2-19)$$

The above equation is a general expression for the field U at point P inside an arbitrary, homogeneous volume with one point source. The first term in Equation (2-19) is for reflection from the arbitrary reflecting surface. The second term is a point source contribution. Therefore, the total field due to reflection from a surface is a superposition of a component due to a Fresnel-Kirchhoff integral and a component due to the point source which lies in the field.

Application of the Fresnel-Kirchhoff Diffraction Integral to a Plane Surface

In Figure 3 a point source exists at P_o, and one wants to observe the field at P due to diffraction about the screen B. The closed surface consists of three regions: A, B, and C. B is the screen or plane surface about which the wave is diffracted. C is part of a spherical shell of radius R. A is that part of an infinite plane surface which does not include the diffraction screen B. It should be pointed out that the surface A and C are arbitrarily chosen.

When R becomes sufficiently large, the values of the acoustic field U and of the normal derivative of the acoustic field $\frac{\partial U}{\partial n}$ become very small. Hence, one should be able to neglect the contribution of the surface C to the total integral. The summation over C can also be forced to zero by a much more accurate strategum. One can definitely state that the acoustic radiation field came into existence at some particular time t in the past. And, as one moves away from the source, one can find a region in which the field does not exist yet. Thus, the radius R can be chosen large enough such that the surface C lies outside the region in which the point source radiation field exists.

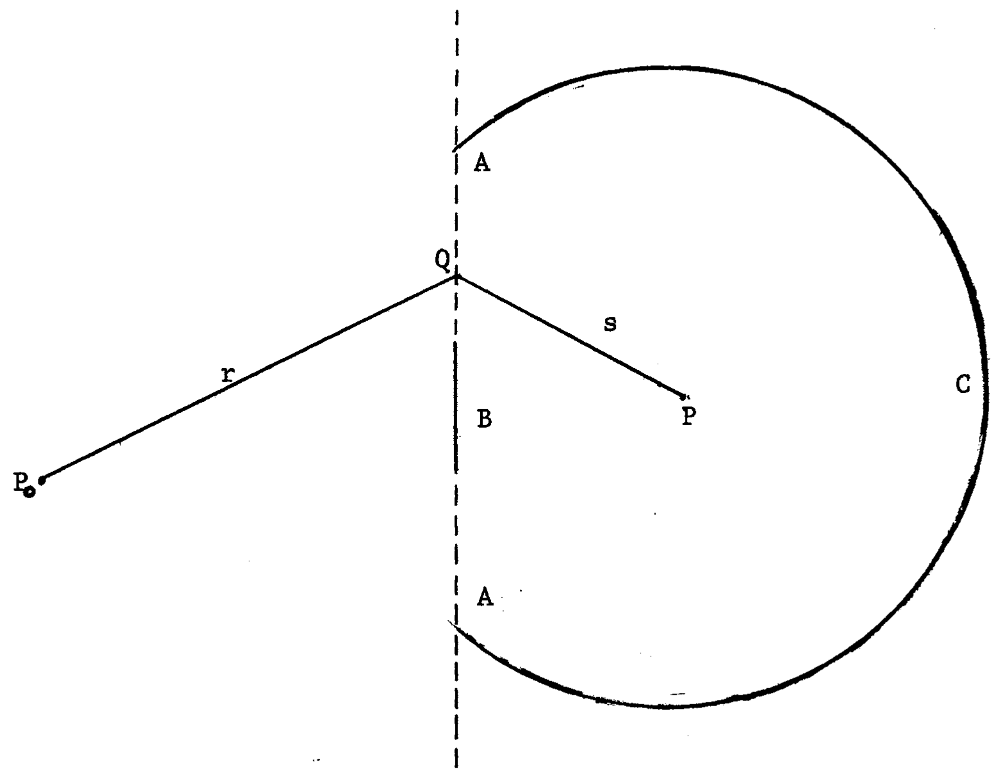

Figure 3. Geometry for Diffraction About a Screen

Kirchhoff made two postulates about the surfaces A and B: On A he assumed that U and $\frac{\partial U}{\partial n}$ were the same as if no screen existed. On B he assumed that U and $\frac{\partial U}{\partial n}$ were zero. That is,

$$\text{on A:} \quad \mathcal{U} = \mathcal{U}^{(i)} \qquad \frac{\partial \mathcal{U}}{\partial n} = \frac{\partial \mathcal{U}^{(i)}}{\partial n}$$

$$\text{on B:} \quad \mathcal{U} = 0 \qquad \frac{\partial \mathcal{U}}{\partial n} = 0 \tag{2-20}$$

where $$\mathcal{U}^{(i)} = \frac{A e^{ikr}}{r} \qquad \frac{\partial \mathcal{U}^{(i)}}{\partial n} = \frac{A e^{ikr}}{r}\left(ik - \tfrac{1}{r}\right)\cos(\hat{n}, \vec{r}) \tag{2-21}$$

These two approximations are known as the Kirchhoff boundary conditions. With the use of the Kirchhoff boundary conditions, Equation (2-19) becomes

$$\mathcal{U}(P) = \frac{ikA}{4\pi}\iint \frac{e^{ik(r+s)}}{rs}\left[\cos(\hat{n}, \vec{r}) - \cos(\hat{n}, \vec{s})\right] dA \tag{2-22}$$

if one assumes r and s are large compared to the wavelength.

Considering a screen 30 cm by 30 cm in size in Figure 4, Equation (2-22) becomes explicitly

$$\mathcal{U}(P) = \iint \frac{iA}{4\pi} \, \frac{e^{ik\left(\sqrt{\xi^2+\eta^2+b^2} + \sqrt{(\xi-x)^2+(\eta-y)^2+z^2}\right)}}{\sqrt{\xi^2+\eta^2+b^2}\,\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}} \times$$

$$\left\{\frac{-b}{\sqrt{\xi^2+\eta^2+b^2}} - \frac{z}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}}\right\} d\xi\, d\eta \tag{2-23a}$$

Physically it seems reasonable that only the values of ξ and ζ near the diffraction screen give any significant contribution to the above integral. In this region ξ and ζ are much smaller than b. Hence,

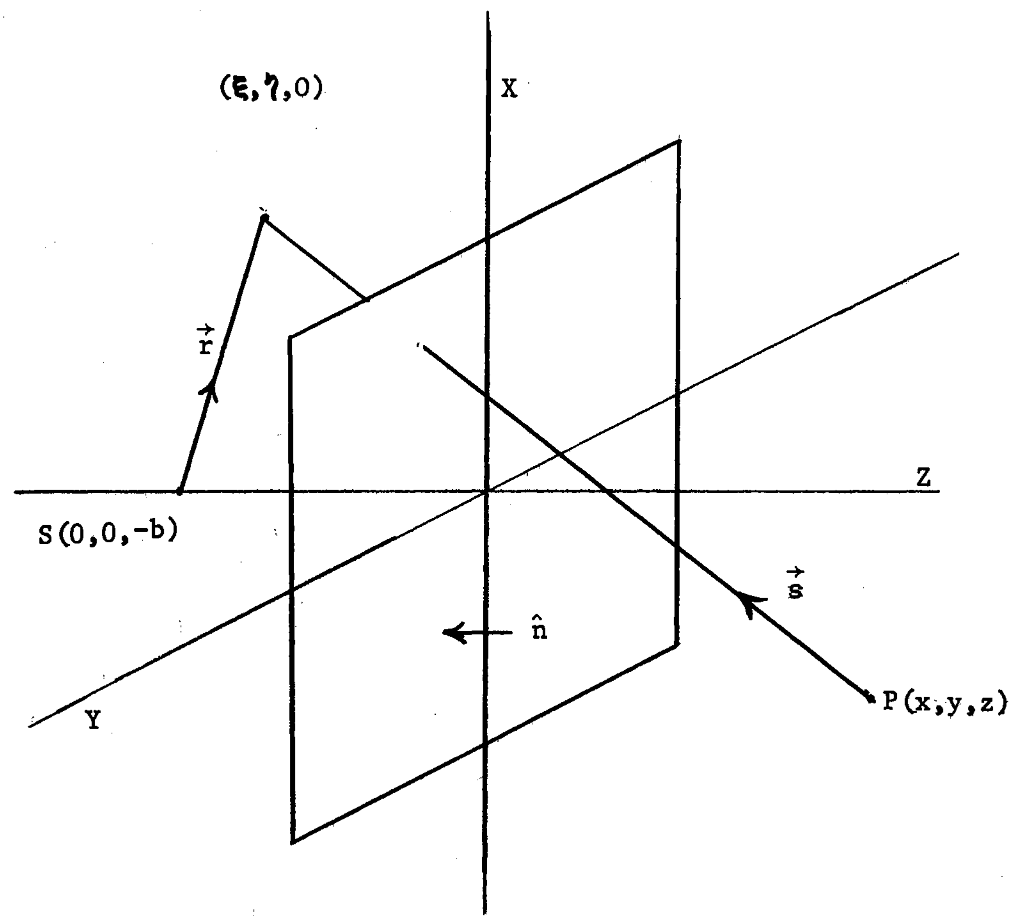

Figure 4. Diffraction About a Square Screen

$$\mathcal{U}(P) = \iint \frac{iAk}{4\pi b} \frac{e^{ik\left(\sqrt{\xi^2+\eta^2+b^2} + \sqrt{(\xi-x)^2+(\eta-y)^2+z^2}\right)}}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}} \times$$

$$\left\{-1 - \frac{z}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}}\right\} d\xi\, d\eta \qquad (2\text{-}23b)$$

If one allows the following substitutions to be made in (2-23b),

$$f(\xi,\eta) = \sqrt{\xi^2+\eta^2+b^2} + \sqrt{(\xi-x)^2+(\eta-y)^2+z^2}$$

$$g(\xi,\eta) = \frac{iA}{4\pi b}\left(-1 - \frac{z}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}}\right)\frac{1}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}}$$

it is of the form

$$\mathcal{U}(P) = k \iint g(\xi,\eta)\, e^{ik f(\xi,\eta)}\, d\xi\, d\eta$$

This type of integral can be solved asymptotically by a technique developed by Van Kampen (25, 26, 27) and Van Der Corput (24).

Application of the Fresnel-Kirchhoff Integral for the Case of Reflection From a Plane Surface

Equation (2-19) is the general form of Kirchhoff's integral for the case of reflection. The next task is to apply this equation to the specific case of reflection from a 30 cm by 30 cm square reflector. Figure 5 indicates the geometry of the situation. S is the point where the source is located. P is the point at which the field is to be calculated. The 30 cm by 30 cm square is symmetrically located about the origin in the z = 0 plane.

One has to be quite careful in determining the direction of the

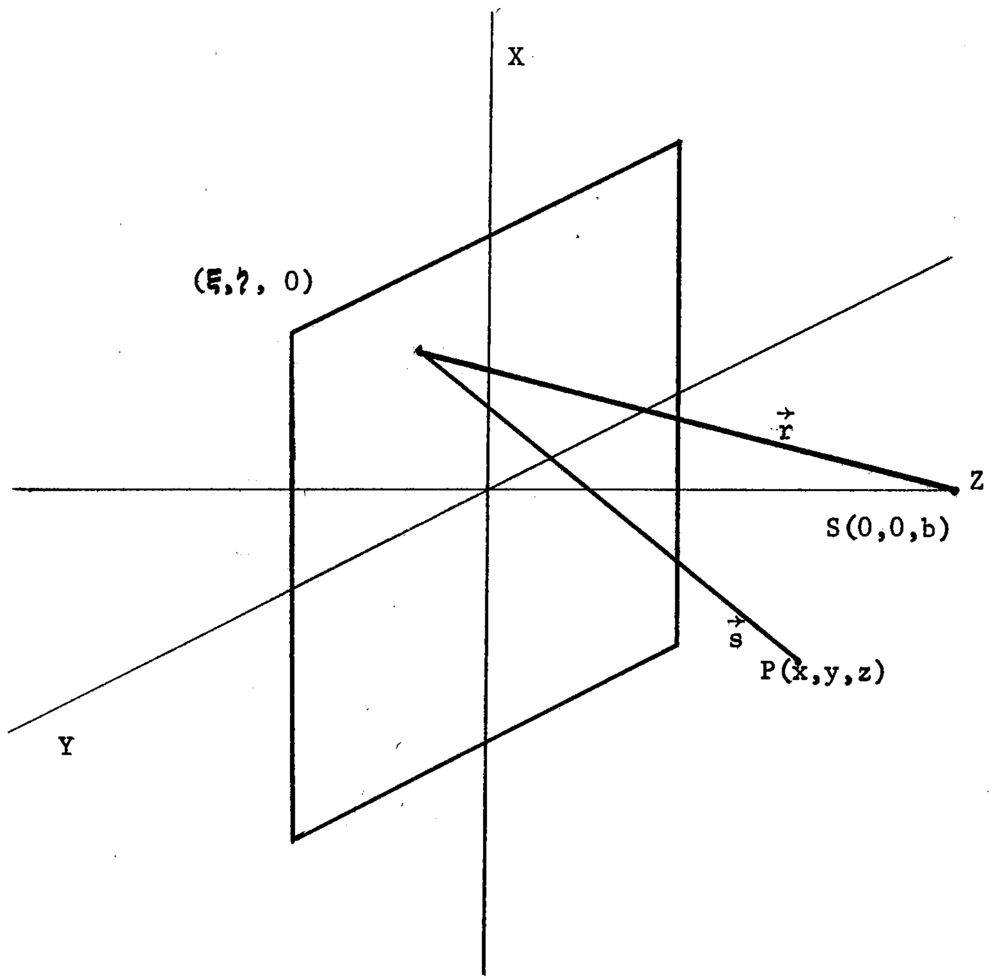

Figure 5. Reflection From a Finite Plane

normal. Equation (2-19) was derived by assuming the normal direction is out of the object space and into the image space. In terms of object and image spaces, this normal is indicated in Figure 6. But this makes for a rather unique situation physically. In Figure 4 the normal associated with $\vec{r}$ is in the negative z direction; the normal associated with $\vec{s}$ is in the positive z direction.

There is a contribution to the integral in (2-19) only over the surface of the plane as there is no reflection from any other part of the arbitrarily chosen closed surface. I.e., U and $\frac{\partial U}{\partial n}$ are zero except on the surface of the reflection plane itself. On the reflection plane,

$$\mathcal{U} = \frac{A e^{ikr}}{r}$$
$$\frac{\partial \mathcal{U}}{\partial n} = \frac{A e^{ikr}}{r}\left(ik - \frac{1}{r}\right)\frac{\partial r}{\partial n} \tag{2-24}$$

And expression (2-19) becomes (i.e., the value of the field at the point P):

$$\mathcal{U}(P) = \frac{1}{4\pi}\iint \frac{A e^{ik(s+r)}}{sr}\left[ik\left(\frac{\partial s}{\partial r} + \frac{\partial r}{\partial s}\right) - \left(\frac{1}{s}\frac{\partial s}{\partial n} + \frac{1}{r}\frac{\partial r}{\partial n}\right)\right] d\xi\, d\eta$$
$$+ \frac{A e^{ikd}}{d} \tag{2-25}$$

From Figure 6 one can see that the normal derivatives take on the following values

$$\frac{\partial s}{\partial n} = \cos(\hat{n}, \vec{s})$$
$$\frac{\partial r}{\partial n} = \cos(\hat{n}, \vec{r}) \tag{2-26}$$

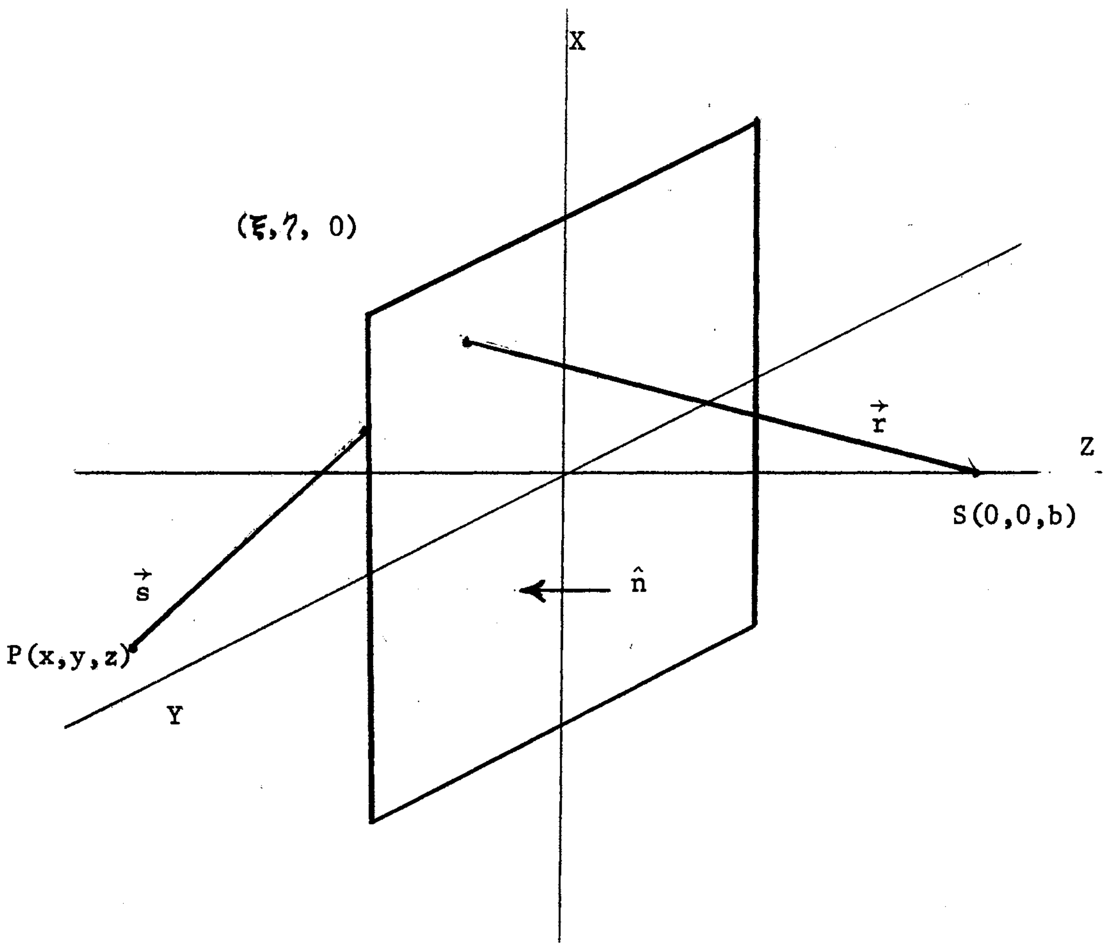

Figure 6. Separation of Object Space and Image Space for Reflection From a Finite Plane

It should be noted that cos(n,s) is an obtuse angle and is therefore inherently negative. From (2-26)

$$\left(\frac{1}{s}\frac{\partial s}{\partial n} + \frac{1}{r}\frac{\partial r}{\partial n}\right)\frac{1}{rs} \qquad (2\text{-}27)$$

offers a very small contribution to the integration over the surface of the plane. But if r and s become quite small, neglecting this term can introduce considerable error. In other words if either the acoustic source or the point at which one wants to find the pressure lies close to the reflecting surface, neglecting (2-27) can introduce significant error. Hence, in order to neglect (2-27) r and s should be several wavelengths long. By applying (2-26) and (2-27) to (2-25) one obtains the generally seen form of the Fresnel-Kirchhoff integral. That is,

$$\mathcal{U}(P) = \frac{1}{4\pi}\iint \frac{ikA}{sr}\, e^{ik(s+r)}\left[\cos(\hat{n},\vec{s}) - \cos(\hat{n},\vec{r})\right] d\xi\, d\eta + \frac{A\, e^{ikd}}{d} \qquad (2\text{-}28)$$

The preceding expression can be clarified by referring to Figure 4 and Figure 5.

$$s = \sqrt{(\xi - x)^2 + (\eta - y)^2 + z^2}$$

$$\cos(\hat{n},\vec{s}) = \frac{-z}{\sqrt{(\xi - x)^2 + (\eta - y)^2 + z^2}} \qquad (2\text{-}29)$$

$$r = \sqrt{\xi^2 + \eta^2 + b^2}$$

$$\cos(\hat{n},\vec{r}) = \frac{b}{\sqrt{\xi^2 + \eta^2 + b^2}} \qquad (2\text{-}30)$$

But ξ and ζ are always much less than b; and, for all practical purposes,

$$r \approx b$$
$$\cos(\hat{n}, \vec{r}) \approx 1 \qquad (2-31)$$

Substituting the above values into the Fresnel Kirchhoff integral (2-28), one obtains

$$\mathcal{U}(P) = \frac{-ikA}{4\pi b} \iint \frac{e^{ik\left(b+\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}\right)}}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}} \left[1 + \frac{z}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}}\right] d\xi\, d\eta$$

$$+ \frac{A e^{ikd}}{d} \qquad (2-32)$$

By allowing the substitutions

$$f(\xi,\eta) = b + \sqrt{(\xi-x)^2+(\eta-y)^2+z^2}$$

$$g(\xi,\eta) = \frac{-iA}{4\pi b} \frac{1}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}} \left(1 + \frac{z}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}}\right)$$

one has an integral of the form

$$k \iint_{\Omega} g(\xi,\eta)\, e^{ikf(\xi,\eta)}\, d\Omega \qquad (2-33)$$

This is the type of integral Van Kampen evaluated in his paper by the technique of asymptotic expansions.

An Asymptotic Treatment of the Fresnel-Kirchhoff Integral

In 1948 Van der Corput (24) devised a technique by which an integral of the form

$$I = \int_a^b g(x)\, e^{i\omega f(x)}\, dx \qquad (2-34)$$

could be evaluated asymptotically. This technique is known as the method of critical points or the method of stationary phase (MSP). It should be noted that a, b, g(x), and f(x) are independent of ω. When ω

is relatively large, the integral (2-34) has a rapidly oscillating part and a slowly changing amplitude part. Due to the rapidly oscillating phase aspect of the integrand, the contributions to the integral of most points x in the interval [a,b] are cancelled. The only points which significantly contribute to the integral (2-34) are those points at which the phase is stationary and the end points of the interval of integration. The phase is stationary at points which satisfy the following expression:

$$\frac{\partial f}{\partial x} = 0 \qquad (2-35)$$

In 1949 Van Kampen (25) extended this method of stationary phase to double integrals of the form of (2-33). Van Kampen found that, on account of the rapidity of fluctuation of the exponential term kge^{ikf}, the value of the Fresnel-Kirchhoff integral depends substantially upon the behavior of the integrand near a limited number of critical points. These critical points are of three kinds: internal points of the domain of integration at which the function $f(\xi,n)$ is stationary, i.e., where

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial \eta} = 0 \qquad (2-36)$$

boundary points of the domain of integration at which the function $f(\xi,n)$ is stationary, i.e., where

$$\frac{\partial f}{\partial s} = 0 \qquad (2-37)$$

s denoting the derivative along the boundary curve of the region of integration; corner points or boundary points where two analytic curves join, i.e., points of discontinuity or singular points lying upon the boundary of the region of integration.

MSP Solution to the Kirchhoff Integral for Diffraction About a Square Plane

For the case of diffraction about a 30 cm by 30 cm square plane, the Fresnel-Kirchhoff integral to be evaluated has the form of Equation (2-23b), i.e.,

$$\mathcal{U}(P)=\iint \frac{iAk}{4\pi b}\,\frac{e^{ik\left(b+\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}\right)}}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}}\left(-1-\frac{z}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}}\right)d\xi\, d\eta$$

The above expression is to be integrated over all space excluding the diffracting screen itself. As was mentioned earlier, the above integral is of the form

$$\mathcal{U}(P)=k\iint g(\xi,\eta)\,e^{ikf(\xi,\eta)}\,d\xi\, d\eta \qquad (2\text{-}38)$$

$$g(\xi,\eta)=\frac{-iA}{4\pi b}\,\frac{1}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}}\left(1+\frac{z}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}}\right) \qquad (2\text{-}39)$$

$$f(\xi,\eta)=b+\sqrt{(\xi-x)^2+(\eta-y)^2+z^2} \qquad (2\text{-}40)$$

Van Kampen's technique of solution can be applied to the expression (2-38) to solve for the field behind a diffracting plane.

<u>Contribution of Critical Points of the First Kind</u>

By applying the condition (2-36) to the rapidly oscillating function (2-40) the points of stationary phase within the interior of the region of integration can be obtained. These are found to be

$$(\xi,\eta)=(x,y) \qquad (2\text{-}41)$$

for all x and y such that $|x| > 15$ and $|y| > 15$ where (x,y,z) is the coordinate of the point in space at which one is trying to find the acoustic intensity and phase.

To find the value of the contribution $U_M(P)$ of a critical point of the first kind to the integral (2-38), the functions $f(\xi,\zeta)$ and $g(\xi,\zeta)$ must be expanded in powers of ξ and ζ about the critical point (after the origin of ξ and ζ has been moved to the critical point).

$$f(\xi,\eta) = a_{00} + a_{10}\xi + a_{01}\eta + a_{20}\xi^2 + a_{11}\xi\eta + \cdots$$
$$g(\xi,\eta) = b_{00} + b_{10}\xi + b_{01}\eta + b_{20}\xi^2 + b_{11}\xi\eta + \cdots \qquad (2\text{-}42)$$

where $a_{10} = a_{01} = a_{11} = 0$ since $\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial \zeta} = 0.$

The integrand of (2-38) can be written as

$$kge^{ikf} = ke^{ika_{00}}e^{ik(a_{20}\xi^2 + a_{02}\eta^2)}(b_{00} + b_{10}\xi + b_{01}\eta + \cdots)\cdot\{1 + ik(a_{30}\xi^3 + \cdots) + \cdots\} \qquad (2\text{-}43)$$

If one expands the product of the two series and then integrates term by term over all space ($-\infty$ to ∞), the following integral results:

$$\mathcal{U}_M(P) = ke^{ika_{00}}\iint_{-\infty}^{+\infty} e^{ik(a_{20}\xi^2 + a_{02}\eta^2)}[b_{00} + b_{10}\xi + \cdots + ikb_{00} + a_{30}\xi^3 + \cdots]\,d\xi\,d\eta \qquad (2\text{-}44)$$

Only very small values of $|\xi|$ and $|\zeta|$ make any significant contribution to the above integral. Hence, the integral (2-44) is a very good approximation to the contribution of a critical point of the first kind to the Fresnel-Kirchhoff integral (2-23b).

The expression (2-44) can be integrated by the definite integrals found in Appendix A. The result of the integration is found to be

$$\mathcal{U}_M(P) = \frac{\pi e^{ika_{00}} i}{\sqrt{|a_{20}\, a_{02}|}} \left[b_{00} + \frac{i\, b_{20}}{2ka_{20}} + \frac{i\, b_{02}}{2ka_{02}} - \frac{3\, b_{10}\, a_{30}}{4k\, a_{20}^2} + \cdots \right] \qquad (2-45)$$

It can be noted that as the wave number k increases without bound (the wavelength approaches zero), the first term of (2-45) does not vanish. This term is due to geometrical optics. All the other terms are due to diffraction effects.

By evaluating the partial derivatives for the two functions $f(\xi,\zeta)$ and $g(\xi,\zeta)$ at the critical points $(\xi,\zeta) = (x,y)$, the expression (2-45) becomes

$$\mathcal{U}_M(P) = \frac{A\, e^{ik(b+z)}}{b} \left[\frac{1}{2} + \frac{13i}{8kz} + \cdots \right] \qquad (2-46)$$

It must be remembered that a critical point of the first kind exists if and only if $|x| > 15$ and $|y| > 15$. Thus (2-46) contributes to the Fresnel-Kirchhoff integral only if the x and y coordinates of the point at which one is trying to calculate the field lies outside the boundaries of the diffracting screen.

Contribution of Critical Points of the Second Kind

By applying the condition (2-37) to the rapidly oscillating function (2-40), the points of stationary phase lying along the edge of the surface of integration can be obtained. The boundary curves to be examined are:

$$\begin{aligned} \eta &= \pm 15 \qquad |\xi| < 15 \\ \xi &= \pm 15 \qquad |\eta| < 15 \end{aligned} \qquad (2-47)$$

The critical point associated with the boundary $\eta = 15$ is found to be

$$(\xi, \eta) = (x, +15) \qquad (2-48)$$

for all x such that $|x| < 15$ and for all y.

The critical point associated with the boundary $\eta = -15$ is found to be

$$(\xi, \eta) = (x, -15) \qquad (2-49)$$

for all x such that $|x| < 15$ and for all y.

The critical point associated with the boundary $\xi = 15$ is

$$(\xi, \eta) = (15, y) \qquad (2-50)$$

for all y such that $|y| < 15$ and for all x.

And the critical point of the second kind associated with the boundary $\xi = -15$ is

$$(\xi, \eta) = (-15, y) \qquad (2-51)$$

for all y such that $|y| < 15$ and for all x. Again, as in the preceding section, the reader must remember that the coordinates x and y are two of the position coordinates associated with the point (x,y,z) at which one wants to find the acoustic field.

In order to find the contribution $U_N(P)$ of a critical point of the second kind N, the origin of ξ and η must be shifted to N and then f and g must be expanded in powers of ξ and η. The resulting power series have the form of

$$\begin{aligned} f(\xi, \eta) &= a_{00} + a_{10}\xi + a_{01}\eta + a_{20}\xi^2 + \cdots \\ g(\xi, \eta) &= b_{00} + b_{10}\xi + b_{01}\eta + b_{20}\xi^2 + \cdots \end{aligned} \qquad (2-52)$$

The next task is to examine each critical point of the second kind individually. The first one to be considered is $(\xi,\eta) = (x,15)$ on the boundary $\eta = 15$. From the definition of the critical point of the second kind, it can be noted that $a_{10} = a_{11} = 0$. As a result,

$$k g e^{ikf} = k e^{ika_{00}} e^{ik(a_{01}\eta + a_{20}\xi^2)} (b_{00} + b_{10}\xi + \cdots)\cdot$$
$$\left\{1 + ik(a_{02}\eta^2 + a_{30}\xi^3 + \cdots) + \cdots\right\} \qquad (2-53)$$

The surface over which the above function is integrated is the half plane for which ξ goes from $-\infty$ to $+\infty$ and η goes from 0 to ∞. Stated in terms of an integral,

$$\mathcal{U}_N(P) = \int_0^\infty \int_{-\infty}^\infty k g e^{ikf} d\xi d\eta$$

The definite integrals necessary to evaluate the above can be found in Appendix A. The series resulting from the integration is

$$\mathcal{U}_N(P) = e^{ika_{00}} \left(\frac{\pi}{k|a_{20}|}\right)^{1/2} \frac{i e^{\pi i/4}}{a_{01}} \left\{ b_{00} - \frac{b_{01}}{ika_{01}} - \frac{b_{20}}{2ika_{20}} \right.$$
$$\left. + \frac{2a_{02}b_{00}}{ik a_{01}^2} + \frac{a_{21}b_{00}}{2k a_{20} a_{01} i} + \frac{3a_{40}b_{00}}{4ik a_{20}^2} + \cdots \right\} \qquad (2-54)$$

Evaluating the partial derivatives for $f(\xi,\eta)$ and $g(\xi,\eta)$ at the critical point $\xi = x$, $\eta = 15$ and substituting them into (2-54) gives the following series.

$$\mathcal{U}_N(P) = e^{ik(b+R)} \left(\frac{\pi R}{k}\right)^{1/2} \frac{e^{\pi i/4}}{4\pi b(15-y)} \left\{ -(1 + z/R) \right.$$
$$\left. + \frac{i}{k}\left[-\frac{13}{4R} - \frac{7z}{4R^2} + \frac{2(R+z)}{(15-y)^2}\right] + \cdots \right\} \qquad (2-55)$$

where $R = \sqrt{(15 - y)^2 + z^2}$.

The series (2-56) is not valid for $14 < y < 16$ as the series will then increase without bound. A series which is valid in the boundary edge region will be determined later.

The next critical point to be considered is $\xi = x$, $\eta = -15$. In this case the integrand kge^{ikf} is integrated over the limits $-\infty \leq \xi \leq \infty$ and $-\infty \leq \eta \leq 0$. The resulting series is

$$\mathcal{U}_N(P) = e^{ika_{00}}\left(\frac{\pi}{k|a_{20}|}\right)^{1/2}\frac{i e^{\pi i/4}}{a_{01}}\left\{-b_{00} + \frac{b_{01}}{ika_{01}} + \frac{b_{20}}{2ka_{20}i}\right.$$

$$\left. - \frac{2a_{02}b_{00}}{ka_{01}^2 i} - \frac{a_{21}b_{00}}{2ka_{20}a_{01}i} - \frac{3a_{40}b_{00}}{4ka_{20}^2 i} + \cdots\right\} \qquad (2-56)$$

Evaluating the necessary partial derivatives at the critical point $\xi = x$, $\eta = -15$, one obtains the series contribution

$$\mathcal{U}_N(P) = e^{ik(b+R)}\left(\frac{\pi R}{k}\right)^{1/2}\frac{e^{\pi i/4}}{4\pi b(15+z)}\left\{-\left(1 + \frac{z}{R}\right)\right.$$

$$\left. + \frac{i}{k}\left[-\frac{13}{4R} - \frac{7z}{4R^2} + \frac{2R}{(15+z)^2}\left(1 + \frac{z}{R}\right)\right] + \cdots\right\} \qquad (2-57)$$

where $R = \sqrt{(15 + z)^2 + z^2}$.

For the critical point of the second kind $\xi = 15$, $\eta = y$ the integrand kge^{ikf} can be expanded as below after moving the origin to the critical point. It can be noted that a_{01} and a_{11} are zero due to the nature of the critical point.

$$kge^{ikf} = ke^{ika_{00}}e^{ik(a_{10}\xi + a_{02}\eta^2)}(b_{00} + b_{10}\xi + \cdots).$$

$$\left\{1 + ik(a_{20}\xi^2 + \cdots) + \cdots\right\} \qquad (2-58)$$

To obtain the contribution of this critical point to the Fresnel-Kirchhoff integral, the surface of integration is the half plane $\xi = 0$ to $\xi = \infty$ and $\eta = -\infty$ to $\eta = +\infty$. That is,

$$\mathcal{U}_N(P) = \int_{-\infty}^{\infty}\int_{0}^{\infty} kg\, e^{ikf}\, d\xi\, d\eta \qquad (2-59)$$

The integration of (2-59) gives

$$\mathcal{U}_N(P) = e^{ika_{00}}\left(\frac{\pi}{ka_{02}}\right)^{1/2}\frac{i e^{\pi i/4}}{a_{10}}\left\{b_{00} - \frac{b_{10}}{ika_{10}} - \frac{b_{02}}{2ika_{02}} + \frac{2b_{00}a_{20}}{ika_{10}^2} + \frac{a_{12}b_{00}}{2a_{10}ka_{02}i} + \cdots\right\} \qquad (2-60)$$

Evaluating the necessary partial derivatives at the critical point $\xi = 15$, $\eta = y$ and substituting these into (2-60) gives the series contribution of the critical point to the integral.

$$\mathcal{U}_N(P) = e^{ik(b+R)}\left(\frac{\pi R}{k}\right)^{1/2}\frac{e^{\pi i/4}}{4\pi b(15-x)}\left\{-(1+z/R) + \frac{i}{k}\left[-\frac{13}{4R} - \frac{7z}{4R^2} + \frac{2(R+z)}{(15-x)^2}\right] + \cdots\right\} \qquad (2-61)$$

where $R = \sqrt{(15-x)^2 + z^2}$.

The series (2-61) is not valid for $14 < x < 16$ as $U_N(P)$ will increase without bound as x approaches the boundary edge 15.

The integrand of the remaining critical point $\xi = -15$, $\eta = y$ can also be expanded as in (2-58) since $a_{01} = a_{11} = 0$. In this case the Kirchhoff integral has the limts

$$\mathcal{U}_N(P) = \int_{-\infty}^{\infty}\int_{-\infty}^{0} kg\, e^{ikf}\, d\xi\, d\eta \qquad (2-62)$$

The resulting series in terms of partial derivatives is

$$\mathcal{U}_N(P) = e^{ika_{00}}\left(\frac{\pi}{ka_{02}}\right)^{1/2}\frac{i e^{\pi i/4}}{a_{10}}\left\{-b_{00} + \frac{b_{10}}{ika_{10}} + \frac{b_{02}}{2ika_{02}}\right.$$

$$\left. -\frac{2a_{20}b_{00}}{ka_{10}^2 i} - \frac{a_{12}b_{00}}{2a_{10}ka_{02}i} - \frac{3a_{04}b_{00}}{4k a_{02}^2 i} + \cdots\right\} \qquad (2-63)$$

When the partial derivatives are evaluated at the critical point about which the series is expanded, (2-63) becomes

$$\mathcal{U}_N(P) = e^{ika_{00}}\left(\frac{\pi}{ka_{02}}\right)^{1/2}\frac{i e^{\pi i/4}}{4\pi b(15+x)}\left\{-(1 + z/R)\right.$$

$$\left. + \frac{i}{k}\left[-\frac{13}{4R} - \frac{7z}{4R^2} + \frac{2(R+z)}{(15+x)^2} + \cdots\right.\right\} \qquad (2-64)$$

where $R = \sqrt{(15 + x)^2 + z^2}$.

One can observe that (2-64) is not valid for $-16 < x < -14$ as in this region it is possible for the series $U_N(P)$ to increase without bound as x approaches the boundary edge -15.

Each of the four series contributions for the critical points of the second kind presented above has a region for which it cannot be used as the series goes to infinity within this range. These regions of singularity involve the boundary edges of the 30 cm by 30 cm diffraction plane itself. That is, when the (x,y,z) coordinates of the point at which one wishes to calculate the field approach within one unit of being directly above a boundary edge, the series expansion associated with the critical point on that boundary cannot be used.

The next task is to develop the series which are valid when one wishes to calculate the phase perpendicularly above a boundary edge. But as the phase is going to be calculated only along the eta coordinate in the $\xi = 0$ plane, the only singularity expansions which need to be made are those at the boundaries $\eta = 15$ and $\eta = -15$.

Noting that $\frac{\partial f}{\partial \xi}$ and several other partial derivatives with respect to ξ are zero, the integrand kge^{ikf} of the Fresnel-Kirchhoff integral can be expanded in the following power series expansion about the critical point $\xi = x$, $\eta = 15$ or $\xi = x$, $\eta = -15$ (after the origin of ξ and η has been moved to the critical point in question).

$$kge^{ikf} = ke^{ika_{00}} e^{ik(a_{01}\eta + a_{02}\eta^2 + a_{20}\xi^2)} (b_{00} + b_{01}\eta + \cdots) \left\{ 1 + ik(a_{21}\xi^2\eta + a_{03}\eta^3 + \cdots) + \cdots \right\} \qquad (2\text{-}65)$$

By applying the same limits of integration to the integrand (2-65) as were used for the nonsingular expansions about the critical points (x,15) and (x,-15), one obtains the following series (Appendix A and Appendix B list the definite integrals which are necessary to evaluate the two integrals).

The series contribution $U_N(P)$ due to the critical point $\xi = x$, $\eta = 15$ is

$$\mathcal{U}_{N_1}(P) = e^{ika_{00}} \left(\frac{\pi}{ka_{20}}\right)^{1/2} \frac{e^{\pi i/4}}{2a_{02}} \left\{ \left(\frac{b_{21}}{2a_{20}k} - \frac{b_{03}}{ka_{02}} - \frac{a_{03}b_{00}a_{01}^2 k}{4a_{02}^2} + \frac{a_{03}b_{01}a_{01}^3 k}{8a_{02}^3} \right) + i\left(-b_{01} - \frac{b_{02}a_{01}}{2a_{02}} + \frac{b_{03}a_{01}^2}{4a_{02}^2} + \frac{a_{21}b_{00}}{2a_{20}} - \frac{a_{03}b_{00}}{a_{02}} + \frac{5a_{03}b_{01}a_{01}}{4a_{02}^2} \right) + \cdots \right\} \qquad (2\text{-}66)$$

$$U_{N_2}(P) = e^{ika_{00}} e^{-\frac{ka_{01}^2 i}{4a_{02}}} \frac{\pi e^{\frac{\pi i}{4}}}{\sqrt{a_{20}a_{02}}} \Phi^{-} \left\{ \frac{b_{00}}{2} - \frac{b_{01}a_{01}}{4a_{02}} \right.$$

$$+ \frac{b_{02}a_{01}^2}{8a_{02}^2} - \frac{b_{03}a_{01}^2}{16a_{02}^3} + \frac{a_{21}b_{00}a_{01}}{8a_{20}a_{02}} + \frac{3a_{03}b_{00}a_{01}}{8a_{02}^2}$$

$$\left. - \frac{3a_{03}b_{01}a_{01}^2}{8a_{02}^3}\right) + i\left(\frac{b_{20}}{4ka_{20}} + \frac{b_{02}}{4ka_{02}} - \frac{b_{21}a_{01}}{8ka_{20}a_{02}} - \frac{3b_{03}a_{01}}{8ka_{02}^2}\right.$$

$$\left.\left. - \frac{ka_{03}b_{00}a_{01}^3}{16a_{02}^3} - \frac{3a_{03}b_{01}}{8ka_{02}^2} + \frac{ka_{03}b_{01}a_{01}^4}{32a_{02}^4}\right) + \cdots \right\} \quad (2\text{-}67)$$

where $\Phi^{-} = e^{\pi i/4} - \Phi$ and $U_N(P) = U_{N_1}(P) + U_{N_2}(P)$.

The reader should note that (2-66) and (2-67) are valid only within one unit of the boundary $\eta = 15$. That is, the y coordinate of the point P(x,y,z) at which one wishes to calculate the phase must lie within the interval (-16, - 14).

The series contribution $U_N(P)$ due to the critical point $\xi = x$, $\eta = -15$ is the negative of (2-66) and (2-67) due to the fact that one of the limits of integration has been changed. Also, this series is valid only within one unit of the boundary in question--notably, $\eta = -15$.

Contribution of Critical Points of ~~the Third~~ Kind

Critical points of the third kind are those points along the boundary at which the function $f(\xi,\eta)$ is discontinuous. For the 30 cm by 30 cm square plane surface, these are the four corner points (15,15), (15,-15), (-15,-15), and (-15,15).

Each of these four corner points gives a series contribution to the Kirchhoff integral. As with the two preceding kinds of critical points, the functions $f(\xi,\eta)$ and $g(\xi,\eta)$ are expanded into power series.

$$f(\xi,\eta) = a_{00} + a_{10}\xi + a_{01}\eta + \cdots$$
$$g(\xi,\eta) = b_{00} + b_{10}\xi + b_{01}\eta + \cdots$$

The integrand kge^{ikf} of the Fresnel-Kirchhoff integral becomes

$$kge^{ikf} = ke^{ika_{00}}e^{ik(a_{10}\xi + a_{01}\eta)}(b_{00} + b_{10}\xi + b_{01}\eta + \cdots)\cdot\left\{1 + ik(a_{20}\xi^2 + a_{11}\xi\eta + a_{02}\eta^2) + \cdots\right\} \quad (2\text{-}68)$$

Integration of (2-68) over the appropriate limits for each of the four critical points of the third kind will give the series contribution of these points to the Kirchhoff integral.

By remembering that the Kirchhoff integral is integrated over all ξ and η, excluding those points lying within the boundaries of the diffraction screen and by referring to Figure 7, one can easily determine the limits of integration on the integrand (2-68) for the four critical points.

The surface of integration for the contribution of the critical point (15,15) is the area including all of the first, second, and fourth quadrants of two space (these are the quadrants associated with the origin lying at the critical point itself). One integrates over the first, second, and third quadrants for the critical point (-15,15). For the critical point (-15,-15), the areas to be integrated over are the second, third, and fourth quadrants. And for the last critical point (15,-15) one integrates over the first, third and fourth quadrants.

The definite integrals needed to perform the above integrations are listed in Appendix A. And after performing the integration, one obtains

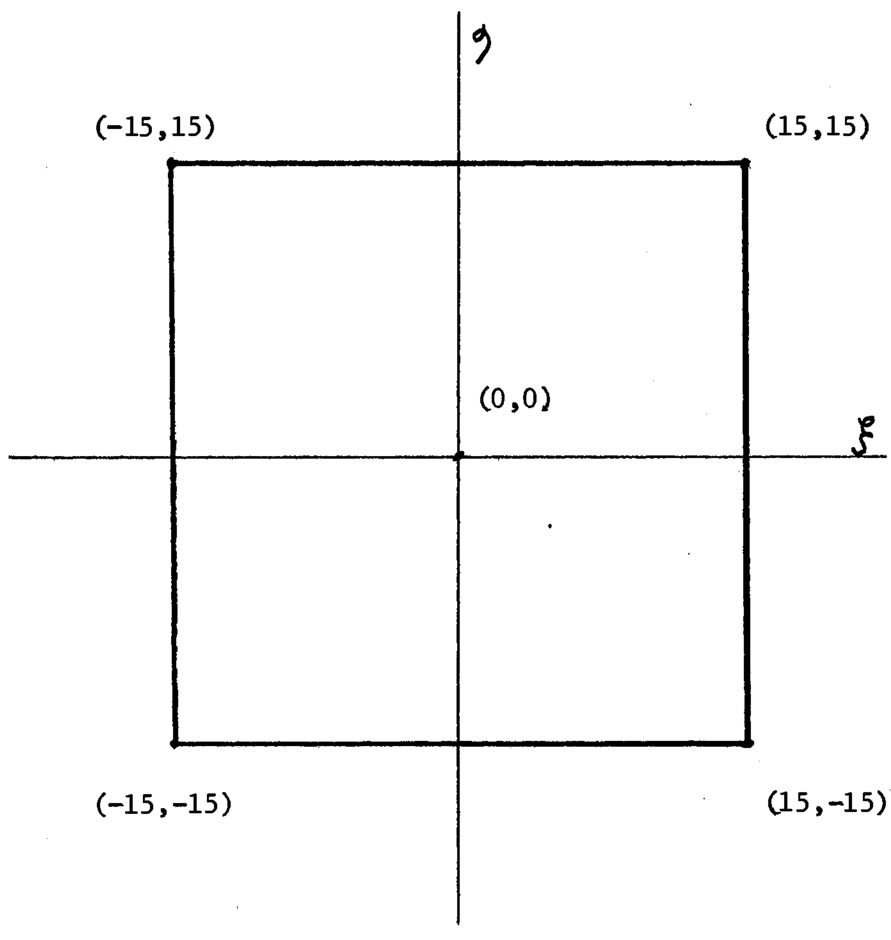

Figure 7. The Diffraction Screen

the series contributions to the Fresnel-Kirchhoff integral.

For the critical point of the third kind (15,15) the series contribution $U_L(P)$ of the point is

$$\mathcal{U}_L(P) = \frac{e^{ik(b+R)}}{(15-x)(15-y)4\pi bk}\Big\{-i(R+z) + \tfrac{1}{k}\Big(3 + z/R$$

$$-2(R^2+Rz)\left[\frac{1}{(15-x)^2} + \frac{1}{(15-y)^2}\right] + \cdots\Big\} \qquad (2-69)$$

where $R = \sqrt{(15-x)^2 + (15-y)^2 + z^2}$.

The series contribution for the critical point (-15,15) is

$$\mathcal{U}_L(P) = \frac{e^{ik(b+R)}}{(15+x)(15-y)4\pi bk}\Big\{-i(R+z) + \tfrac{1}{k}\Big(3 + z/R$$

$$-2(R^2+Rz)\left[\frac{1}{(15+x)^2} + \frac{1}{(15+y)^2}\right]\Big) + \cdots\Big\} \qquad (2-70)$$

where $R = \sqrt{(15+x)^2 + (15-y)^2 + z^2}$.

The contribution $U_L(P)$ for the critical point of the third kind (-15,-15) is

$$\mathcal{U}_L(P) = \frac{e^{ik(b+R)}}{(15+x)(15+y)4\pi bk}\Big\{-i(R+z) + \tfrac{1}{k}\Big(3 + z/R$$

$$2(R^2+Rz)\left[\frac{1}{(15+x)^2} + \frac{1}{(15+y)^2}\right] + \cdots\Big\} \qquad (2-71)$$

where $R = \sqrt{(15+x)^2 + (15+y)^2 + z^2}$.

And the contribution $U_L(P)$ of the fourth critical point (15,-15) is

$$u_L(P) = \frac{e^{ik(b+R)}}{(15-x)(15+z)4\pi bk}\left\{-i(R+z) + \frac{1}{k}\left(3 + \frac{z}{R} - 2(R^2+Rz)\left[\frac{1}{(15-x)^2} + \frac{1}{(15+z)^2}\right]\right) + \cdots\right\} \qquad (2\text{-}72)$$

where $R = \sqrt{(15-x)^2 + (15+z)^2 + z^2}$.

But if one examines the four preceding series contributions, he can see that they are not valid if the point P(x,y,z) at which one wishes to find the phase lies within one unit of being directly above a boundary edge.

Hence new series expansions will have to be made which will give valid results across the boundary edges of the diffraction screen. Since the only points at which the acoustic field is going to be examined is in the plane, $\xi = 0$, it is possible to keep the number of new expansions down to four (otherwise, one would need three different expansions for each critical point: one expansion as the critical point itself is approached, a second as one of the straight line boundaries intersecting the critical point is approached, and a third as the other adjoining straight line boundary is approached). In order to be able to calculate the phase of the acoustic field above the $\eta = -15$ boundary edge of the diffraction screen in the $\xi = 0$ plane, a special series for each of the critical points (-15,-15) and (-15,15) is needed. And to be able to calculate the field at a point P(x,y,z) above the $\eta = 15$ boundary of the screen, special series are needed for the critical points (15,-15) and (15,15).

The series expansion (of the integrand of the Fresnel-Kirchhoff integral) needed to find the contributions of the critical points of the third kind in the boundary edge regions is

$$k g e^{ikf} = k e^{ika_{00}} e^{ika_{10}\xi} e^{ik(a_{01}\eta + a_{02}\eta^2)} (b_{00} + b_{10}\xi + b_{01}\eta + \cdots) \cdot \{1 + ik(a_{20}\xi^2 + a_{11}\xi\eta + a_{30}\xi^3 + \cdots) + \cdots\} \quad (2\text{-}73)$$

The same limits of integration apply over (2-73) for each of the critical points as applied over (2-68). The results of the integrations give the following series (which are valid only within one unit of a boundary edge).

The contribution $U_L(P)$ of the critical point of the third kind (-15,-15) in the boundary region $\eta = -15$ to the Fresnel-Kirchhoff integral is

$$\mathcal{U}_{L_1}(P) = \frac{e^{ika_{00}}}{2a_{10}a_{02}} \left\{ \left(-\frac{b_{01}}{k} - \frac{b_{02}a_{01}}{2ka_{02}} + \frac{a_{11}b_{00}}{ka_{10}} + \frac{b_{00}a_{12}a_{01}}{2ka_{10}a_{02}} - \frac{b_{00}a_{03}}{ka_{02}} + \frac{b_{01}a_{11}a_{01}}{2ka_{10}a_{02}} \right) + i\left(-\frac{b_{11}}{k^2 a_{10}} + \frac{b_{00}a_{03}a_{01}^2}{4a_{02}^2} \right) + \cdots \right\} \quad (2\text{-}74)$$

$$\mathcal{U}_{L_2}(P) = \frac{e^{ika_{00}} e^{-\frac{k a_{01}^2 i}{4a_{02}}}}{2a_{10}} \left(\frac{\pi}{k a_{02}} \right)^{1/2} \Phi \left\{ \left(\frac{b_{10}}{ka_{10}} - \frac{b_{11}a_{01}}{2ka_{10}a_{02}} + \frac{b_{02}}{2ka_{02}} - \frac{2b_{00}a_{20}}{ka_{10}^2} - \frac{b_{00}a_{12}}{2ka_{10}a_{02}} - \frac{b_{00}a_{03}a_{01}^3 k}{8a_{02}^3} \right. \right.$$

$$\left.\frac{-b_{01}a_{11}}{2ka_{10}a_{02}}\right) + i\left(-b_{00} + \frac{b_{01}a_{01}}{2a_{02}} + \frac{2b_{20}}{k^2a_{10}^2} - \frac{b_{02}a_{01}^2}{4a_{02}^2}\right.$$

$$\left.\left. - \frac{a_{11}b_{00}a_{01}}{2a_{10}a_{02}} + \frac{b_{00}a_{12}a_{01}^2}{4a_{10}a_{02}^2} - \frac{3b_{00}a_{03}a_{01}}{4a_{02}^2} + \frac{b_{01}a_{11}a_{01}^2}{4a_{10}a_{02}^2}\right) + \cdots\right\} \quad (2\text{-}75)$$

where $\Phi^{-} = e^{\pi i/4} - \Phi$ and $U_L(P) = U_{L_1}(P) + U_{L_2}(P)$. The definite integrals used in arriving at (2-74) and (2-75) are listed in Appendix A and Appendix B.

The contribution $U_L(P)$ of the critical point of the third kind (15,-15) in the boundary region $\eta = -15$ to the Fresnel-Kirchhoff integral is

$$\mathcal{U}_{L_1}(P) = \frac{e^{ika_{00}}}{2a_{10}a_{02}}\left\{\left(\frac{b_{01}}{k} + \frac{b_{02}a_{01}}{2a_{02}k} - \frac{b_{00}a_{11}}{a_{10}k} - \frac{b_{00}a_{12}a_{01}}{2a_{10}a_{02}k}\right.\right.$$

$$\left.\left. + \frac{b_{00}a_{03}}{a_{02}k} - \frac{b_{01}a_{11}a_{01}}{2ka_{10}a_{02}}\right) + i\left(\frac{b_{11}}{k^2a_{10}} - \frac{b_{00}a_{03}a_{01}^2}{4a_{02}^2}\right) + \cdots\right\} \quad (2\text{-}76)$$

$$\mathcal{U}_{L_2}(P) = \frac{e^{ika_{00}}e^{-\frac{ika_{01}^2}{4a_{02}}}}{2a_{10}}\left(\frac{\pi}{ka_{02}}\right)^{1/2}\Phi^{-}\left\{\left(-\frac{b_{10}}{ka_{10}} + \frac{b_{11}a_{01}}{2ka_{10}a_{02}} - \frac{b_{02}}{2ka_{02}}\right.\right.$$

$$\left. + \frac{2b_{00}a_{20}}{ka_{10}^2} + \frac{b_{00}a_{12}}{2ka_{10}a_{02}} + \frac{b_{00}a_{03}a_{01}^3k}{8a_{02}^3} + \frac{b_{01}a_{11}}{2ka_{10}a_{02}}\right) + i\left(b_{00}\right.$$

$$- \frac{b_{01}a_{01}}{2a_{02}} - \frac{2b_{20}}{k^2a_{10}^2} + \frac{b_{02}a_{01}^2}{4a_{02}^2} + \frac{b_{00}a_{11}a_{01}}{2a_{10}a_{02}} - \frac{b_{00}a_{12}a_{01}^2}{4a_{10}a_{02}^2}$$

$$\left.\left. - \frac{3b_{00}a_{03}a_{01}}{4a_{02}^2} + \frac{b_{01}a_{11}a_{01}^2}{4a_{10}a_{02}^2}\right) + \cdots\right\} \quad (2\text{-}77)$$

where $\Phi^{-} = e^{\pi i/4} - \Phi$ and $U_L(P) = U_{L_1}(P) + U_{L_2}(P)$.

The contribution $U_L(P)$ of the critical point of the third kind (-15,15) in the boundary region $\eta = 15$ to the Kirchhoff integral is the negative of (2-74) and (2-75). But naturally, it must be remembered that the partial derivatives will take on different values.

And the contribution $U_L(P)$ of the critical point of the third kind (15,15) in the boundary region $\eta = 15$ is the negative of (2-76) and (2-77). Again one must remember that the partial derivatives take on different values in the two cases.

By adding the contributions of the various critical points $U_M(P)$, $U_N(P)$, and $U_L(P)$ one obtains a solution to the Fresnel-Kirchhoff integral for diffraction about a 30 cm by 30 cm plane. And from this solution, the phase at various points in space can be calculated.

MSP Solution to the Kirchhoff Integral for

Reflection About a Square Plane

For the case of reflection from a square 30 cm by 30 cm plane the solution via the Kirchhoff theory takes the form (2-32). I.e.,

$$\mathcal{U}(P) = \frac{-iAk}{4\pi b}\int_{-15}^{15}\int_{-15}^{15}\frac{e^{ik\left(b+\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}\right)}}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}}\left(1+\frac{z}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}}\right)d\xi\, d\eta + A\frac{e^{ikd}}{d}$$

For the above expression, the origin of the coordinate system lies in the center of the reflection plane. Figure 4 indicates the geometry of the situation. The non-integral term $\frac{A\, e^{ikd}}{d}$ is the point source contribution to the field at the position (x,y,z).

The integral part of (2-32) is of the form

$$k \iint_{\Omega} g(\xi,\eta)\, e^{ik f(\xi,\eta)}\, d\Omega$$

if one allows the following substitutions.

$$f(\xi,\eta) = b + \sqrt{(\xi-x)^2 + (\eta-y)^2 + z^2} \qquad (2-78)$$

$$g(\xi,\eta) = \frac{-iA}{4\pi i\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}}\left(1 + \frac{z}{\sqrt{(\xi-x)^2+(\eta-y)^2+z^2}}\right) \qquad (2-79)$$

And it can be seen that the integrands of the Fresnel-Kirchhoff integral for both diffraction and reflection are identical. Hence, the same critical points exist. The only differences in the two integrals are the limits of integration. For the case of reflection the surface of integration is the 30 cm by 30 cm screen. For the case of diffraction the surface of integration is all two space excluding the 30 cm by 30 cm screen.

The contribution of the critical points of the first kind $\xi = x$, $\eta = y$ for the case of reflection is identical to that of diffraction. I.e.,

$$\mathcal{U}_M(P) = \frac{e^{ik(b+z)}}{b}\left[\frac{1}{2} + \frac{13i}{8kz} + \cdots\right] \qquad (2-80)$$

The region of integration for the expression kge^{ikf} about the critical point of the second kind $\xi = x$, $\eta = 15$ is the square plane itself. But, since only points near the critical point itself give any significant contributions to the Fresnel-Kirchhoff integral, the limits of integration can be, for all practical purposes, changed to the following: $-\infty \leq \xi + \infty$ and $-\infty \leq \eta \leq 0$. That is,

$$\mathcal{U}_N(P) = \int_{-\infty}^{0}\int_{-\infty}^{\infty} kg\, e^{ikf}\, d\xi\, d\eta \qquad (2\text{-}81)$$

This approximation must be employed in the development of the series contributions of all the critical points in the case of reflection. The general series resulting from the integration (2-81) is

$$\mathcal{U}_N(P) = e^{ika_{00}}\left(\frac{\pi}{ka_{20}}\right)^{1/2} \frac{i e^{\pi i/4}}{a_{01}} \left\{ -b_{00} + \frac{b_{01}}{ika_{01}} + \frac{b_{20}}{2ika_{20}} \right.$$

$$\left. - \frac{2a_{02}b_{00}}{ika_{01}^2} - \frac{a_{21}b_{00}}{2ka_{20}a_{01}i} - \frac{3a_{40}b_{00}}{4kia_{20}^2} + \cdots \right\} \qquad (2\text{-}82)$$

Evaluating the partial derivatives at the critical point $\xi = x$, $\eta = 15$ gives the following contribution of the point to the Kirchhoff integral.

$$\mathcal{U}_N(P) = e^{ik(b+R)}\left(\frac{\pi R}{k}\right)^{1/2} \frac{e^{\pi i/4}}{4\pi b(15-y)} \left\{ (1 + z/R) \right.$$

$$\left. + i/k\left(\frac{13}{4R} + \frac{7z}{4R^2} - \frac{2(R+z)}{(15-y)^2}\right) + \cdots \right\} \qquad (2\text{-}83)$$

where $R = \sqrt{(15 - y)^2 + z^2}$.

The series (2-82) and (2-83) are the negative of the series derived for diffraction about a plane. Also, it should be noted that in the integral (2-81), the coordinate η is integrated from $\eta = -\infty$ to 0. In the integral (2-54) for the case of diffraction η is integrated from

0 to ∞.

By following the techniques outlined in the above and in the section on diffraction, the remaining series contributions of the various critical points can be developed. Performing the necessary calculations, one finds that the series contributions of the various critical points for the case of reflection are the negative of those for the case of diffraction about a 30 cm by 30 cm plane. Consequently the reader can obtain the value of the Fresnel-Kirchhoff integral for reflection by taking the negative of the series for diffraction about the square plane. And in summary, the material presented in this chapter allows one to obtain values of the acoustic field both for diffraction behind a plane and for reflection above a plane.

CHAPTER III

THEORETICAL MAPPING OF CONSTANT PHASE SURFACES

The Computer Technique of Mapping

The evaluation of the Fresnel-Kirchhoff integrals by the method of asymptotic expansions gives the amplitude and phase only at one particular point at a time. Hence, the integrals have to be evaluated at a lattice of points. And, then, those points in the lattice which have nearly the same phase will constitute the loci of the wave-fronts (a wave-front is a surface of constant phase).

The calculation of the wave-fronts is a three dimensional problem. But, in practice, it is possible to keep the lattice to two dimensions if one chooses lattice planes which are symmetrically located with respect to the diffracting and reflecting surfaces.

The major problem in the calculation of the wave-fronts is the locating of those points in the two dimensional lattice which are the constant phase points for a particular wave-front. A computerized cyclic search technique is used. Consider the lattice in Figure 8. A point (y,z) is picked in the lattice. It is the first point of the constant phase curve being mapped. Then, compare the phase of this point (y,z) with the four nearest neighbor points (y + a,z), (y,z + a), (y - a,z), and (y, z - a). The "a" is the lattice spacing. The nearest neighbor point with phase nearest that of the originally chosen point becomes the second point of the constant phase curve (wave-front). Then, the

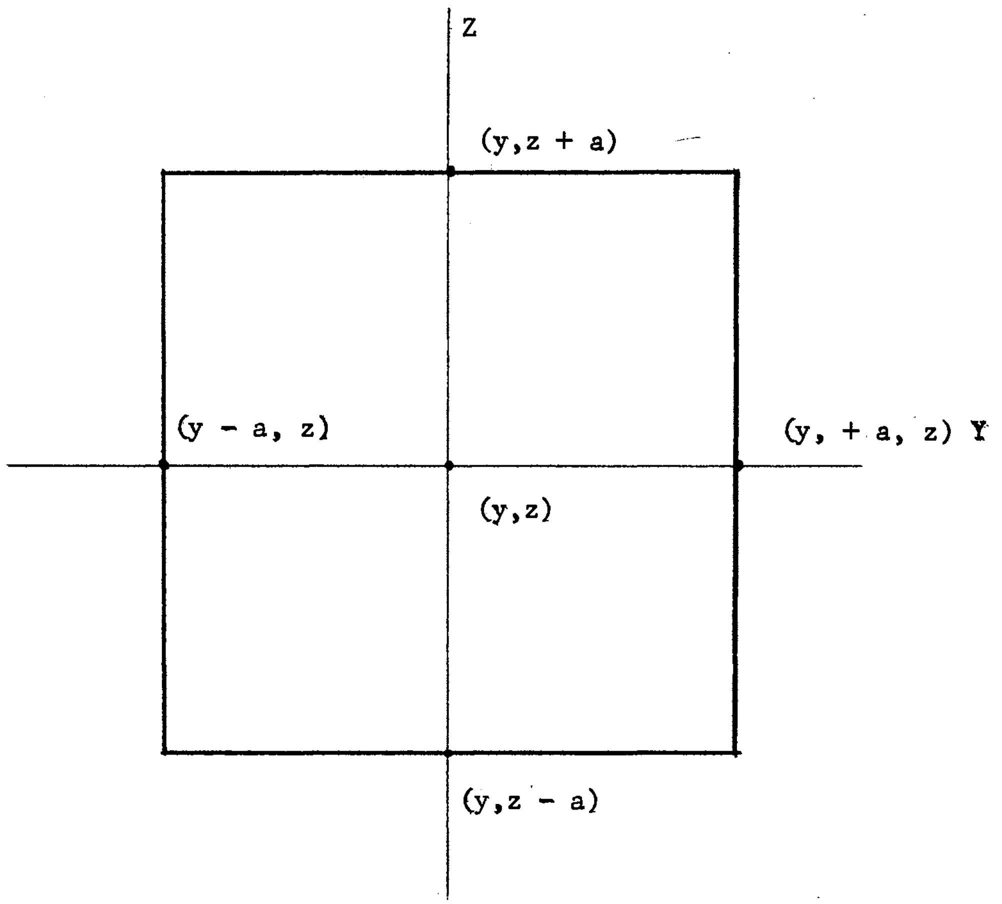

Figure 8. Lattice for Selection of Constant Phase Points

phases of the nearest neighbors of this second point (excluding the point known to lie upon the constant phase curve) are compared with the phase of the first point upon the curve. The nearest neighbor point with phase nearest that of the originally chosen point (point #1) becomes the third point upon the constant phase curve. With a continuation of this technique, a two dimensional slice of a wave-front can be mapped. Of course, one must be very careful with the boundary conditions which must be placed into the computer program to perform this operation.

The computer program which does the above is listed in Appendix C. The acoustic field (and hence, the phase) is calculated via the asymptotic solutions developed in the previous chapter in a subroutine which must accompany this search program. This subroutine is called PHASE(Y,Z). Any time this statement occurs in the main program, the value of the phase at the point (y,z) is calculated.

The points constituting the wave-fronts for diffraction and reflection about a 30 cm by 30 cm square plane were calculated on an IBM 360 computer and punched onto cards. These cards were fed into an IBM 1620 computer and plotted on a Calcomp plotter which was linked with the computer. The resulting graphs of the wave-fronts are presented below.

Figure 9 and Figure 10 indicate the theoretical wave-front patterns for the case of diffraction when the speaker is 150 cm (approximately 87 wavelengths for 20,000 hertz sound) from the diffracting surface. It should be noted that the wave-fronts depicted in these graphs are not one wavelength apart. In fact the various wave-fronts do not even have the same value of phase. The wave-fronts shown in Figure 9 are approximately 5 cm apart (which is several wavelengths) so as to give a good overall view of the wave-front pattern near the surface of the square.

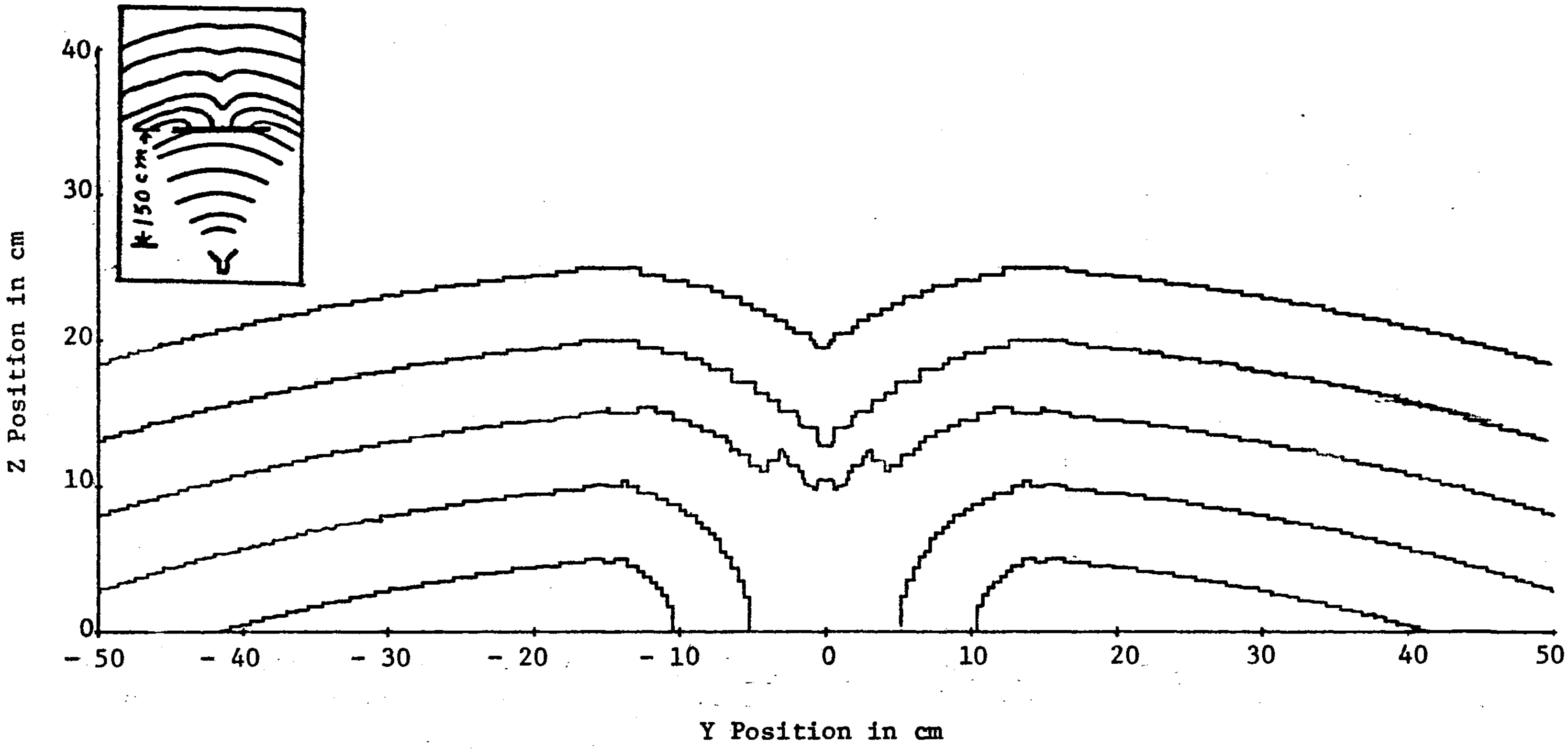

Figure 9. Wave Fronts for Diffraction: Speaker Distance Equals 150 cm

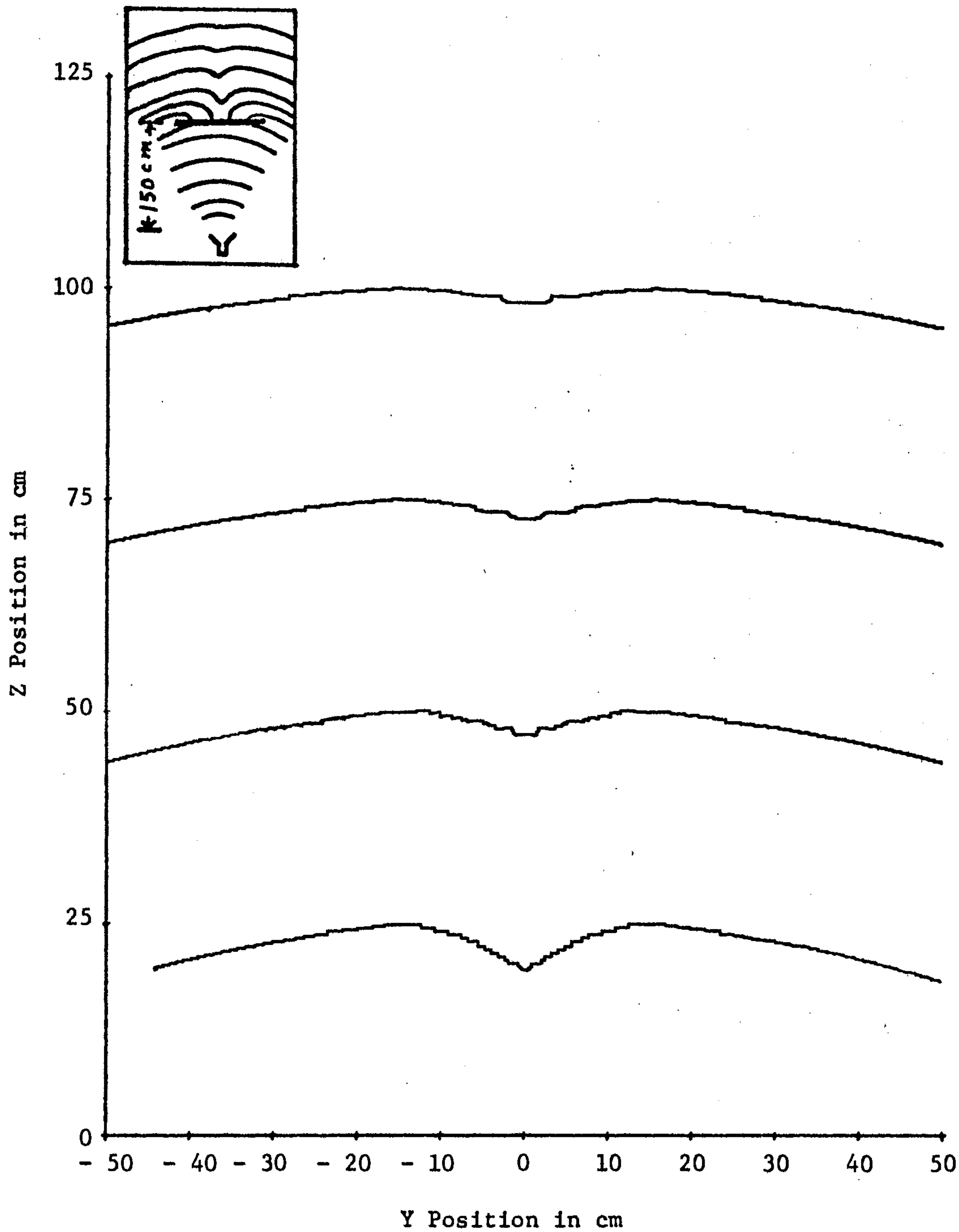

Figure 10. Wave Fronts for Diffraction: Speaker Distance Equals 150 cm

And in Figure 10 the lines of constant phase are roughly 25 cm apart so as to give one a general view of the wave-front patterns to a considerable distance from the surface of the square. Furthermore, the rest of the figures depicting wave-front patterns within this chapter are presented in the same manner as Figure 9 and Figure 10. Referring to the coordinate system in the two figures, the position of the speaker is $x = 0$, $y = 0$, $z = -150$ and the square is located from $y = -15$ to $y = 15$. Near the surface of the square plane, the wave-fronts bend toward and actually touch the surface. The acoustic field appears to be behaving as if the boundaries $y = -15$ and $y = +15$ were acoustic line sources. As one goes farther away from the 30 cm by 30 cm diffraction plane (in the positive z direction), the wave-fronts no longer come down and touch its surface. And as one goes even farther away, the wave-fronts are bent even less and less. And at very large distances, the wave-fronts start approximating spherical wave-fronts.

Figure 11 and Figure 12 indicate the theoretical wave-front patterns for the case of diffraction when the speaker is 250 cm (approximately 145 wavelengths for 20,000 hertz sound) from the diffracting 30 cm by 30 cm square plane. The exact location of the speaker is $x = 0$, $y = 0$, $z = -250$. And, as before, the location of the square is $y = -15$ to $y = +15$. The statements which were made for Figure 2 and Figure 3 also hold for these two figures.

A comparison of the diffraction patterns for the speaker distances 150 cm and 250 cm indicates that in the region behind the square ($y = -15$ to $y = +15$) the wave-fronts have essentially identical shapes. But outside the boundary edges of the square ($y < -15$, $y > 15$), the wave-fronts associated with the speaker distance 150 cm are more curved.

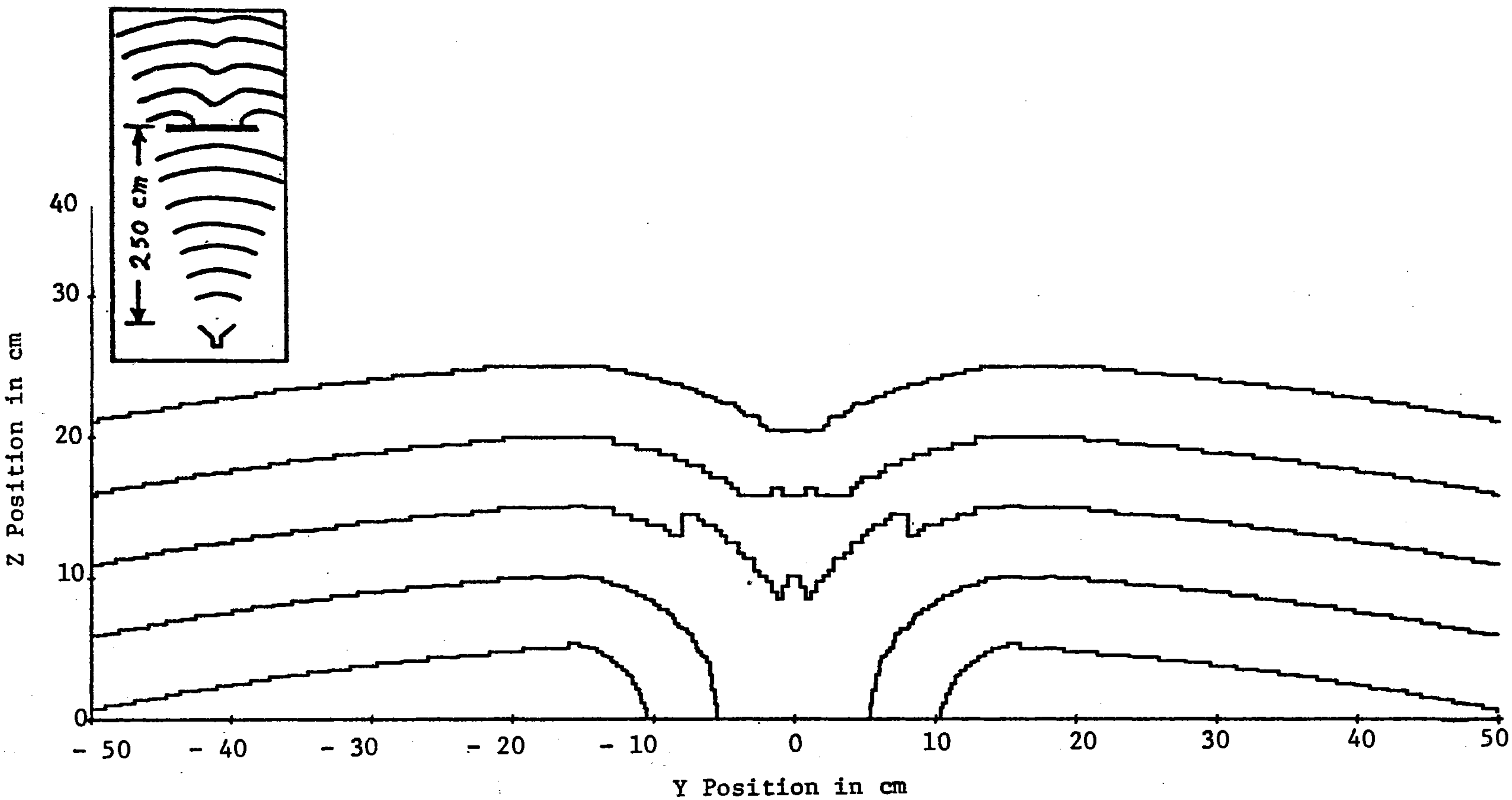

Figure 11. Wave Fronts for Diffraction: Speaker Distance Equals 250 cm

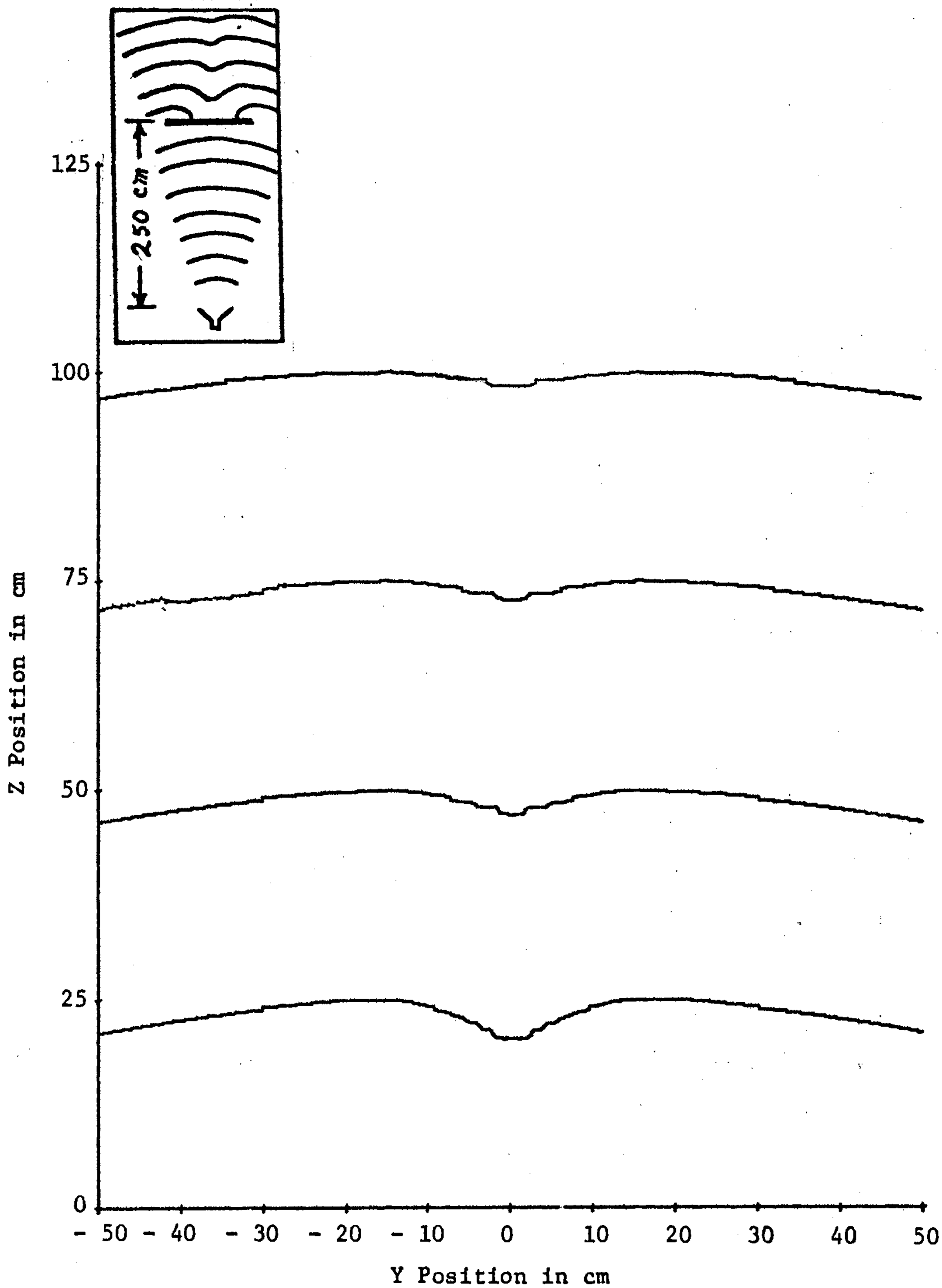

Figure 12. Wave Fronts for Diffraction: Speaker Distance Equals 250 cm

This last result one would naturally expect as the speaker is 100 cm closer to the area being examined.

Figure 13 and Figure 14 indicate that the theoretical wave-front patterns for the case of reflection when the speaker is 150 cm (approximately 87 wavelengths for 20,000 hertz sound) above the 30 cm by 30 cm square. In both pictures, the reflecting square is located from $y = -15$ to $y = +15$, and the position of the speaker is at $x = 0$, $y = 0$, $z = 150$. Just above the surface of the square, the wave-fronts are very flat--essentially plane. Although there are some variations above the edges $y = -15$ and $y = +15$. And as one proceeds away from the surface of the plane in the positive z direction, the areas of the wave-fronts perpendicularly above the reflecting square still remain very flat.

Figure 15 and Figure 16 indicate the theoretical wave-front patterns for the case of reflection when the speaker is 250 cm (approximately 145 wavelengths for 20,000 hertz sound) above the 30 cm by 30 cm square. The location of the square is from $y = -15$ to $y = +15$, and the position of the speaker is $x = 0$, $y = 0$, $z = 250$. The same general statements which were made for the wave-fronts in Figure 13 and Figure 14 are also true for these two figures.

A comparison of the wave-fronts in the region above the square ($y = -15$ to $y = +15$) shows very little difference. But the wave-fronts for the larger speaker distance do seem to be slightly flatter. The major difference between the two sets of wave-fronts is to be found in the regions of space lying outside the boundaries of the square reflector, i.e., for $y < -15$ and $y > 15$. When the speaker or acoustic source is only 150 cm above the square reflector, the wave-fronts are

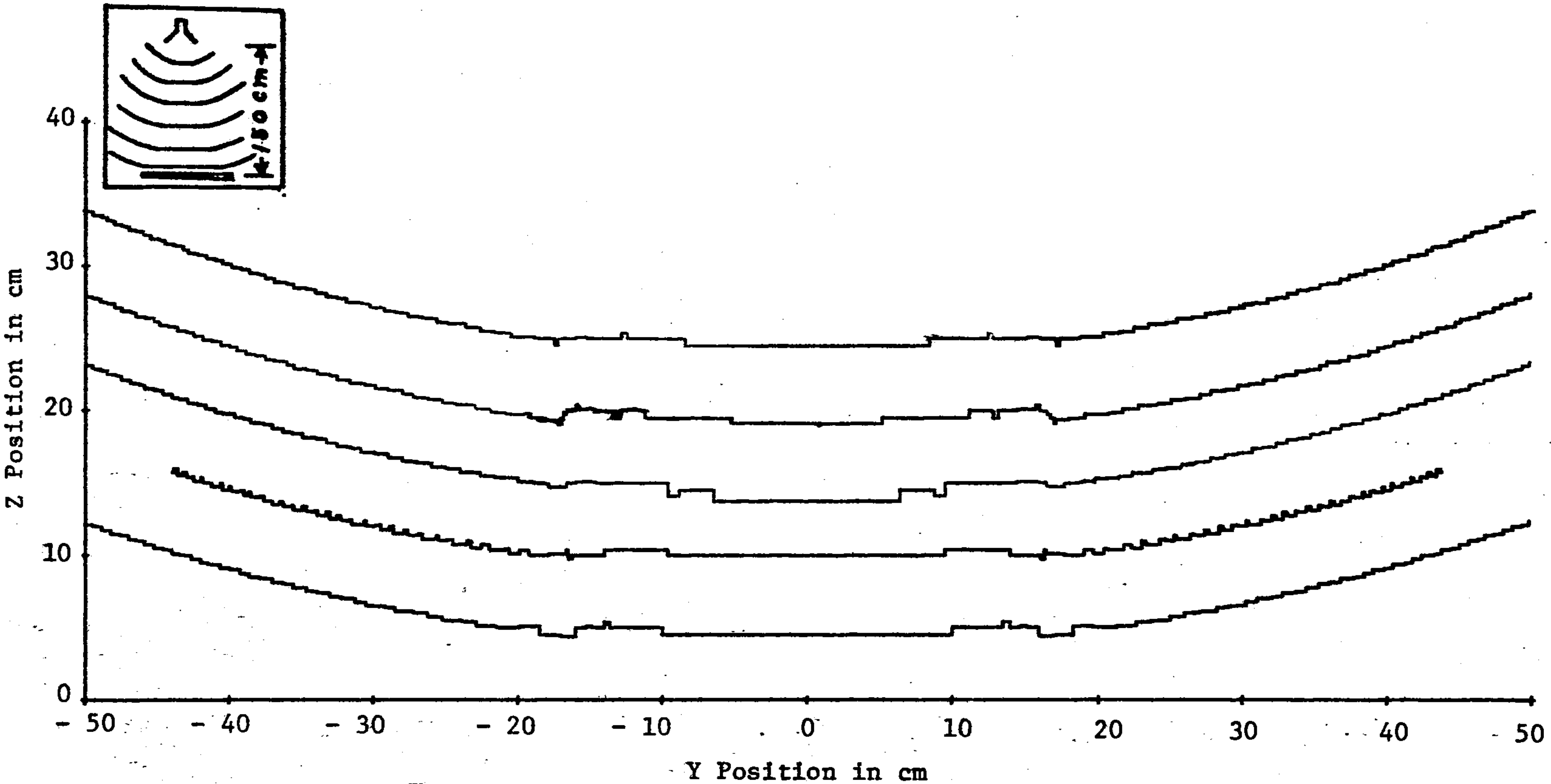

Figure 13. Wave Fronts for Reflection: Speaker Distance Equals 150 cm

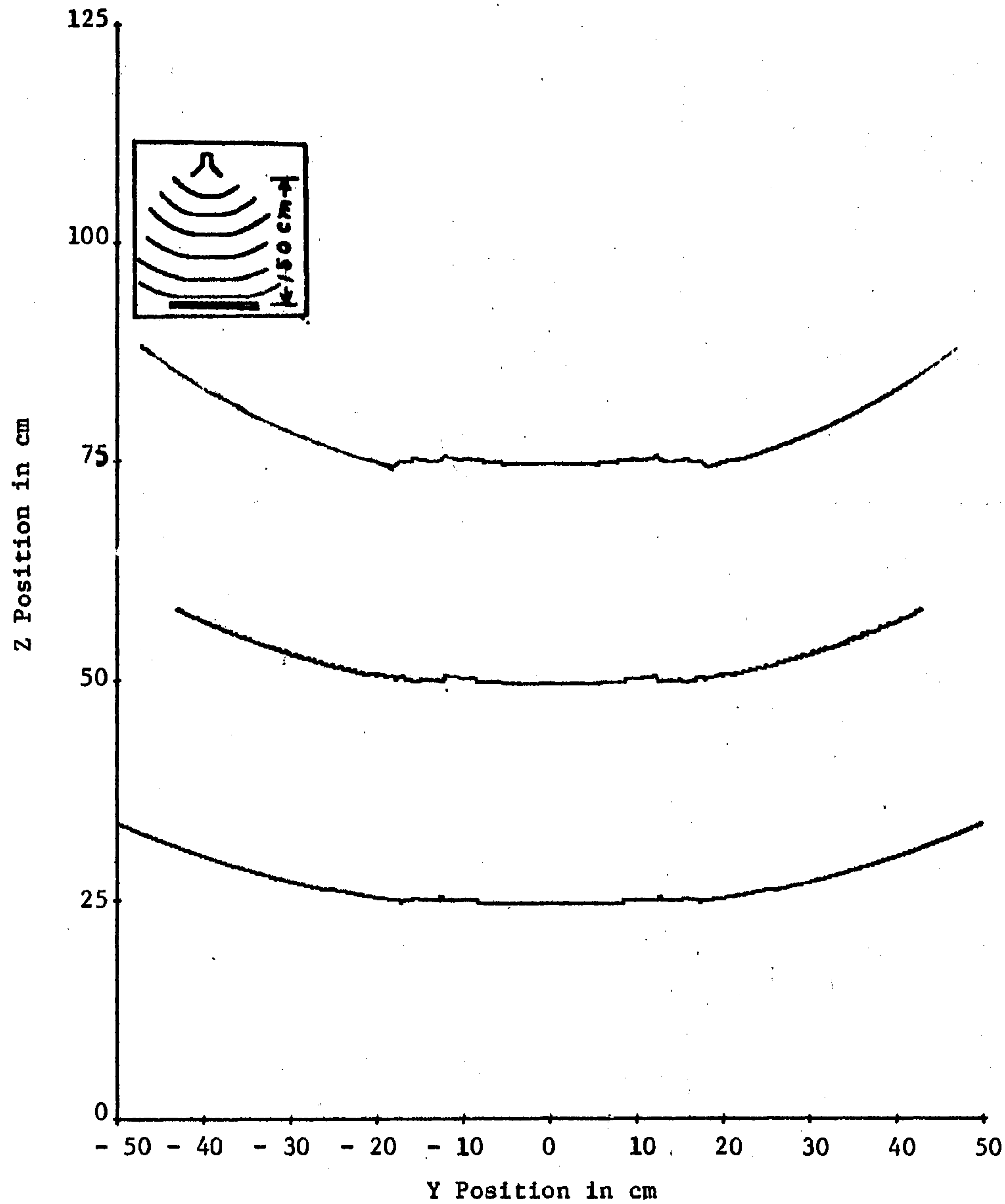

Figure 14. Wave Fronts for Reflection: Speaker Distance Equals 150 cm

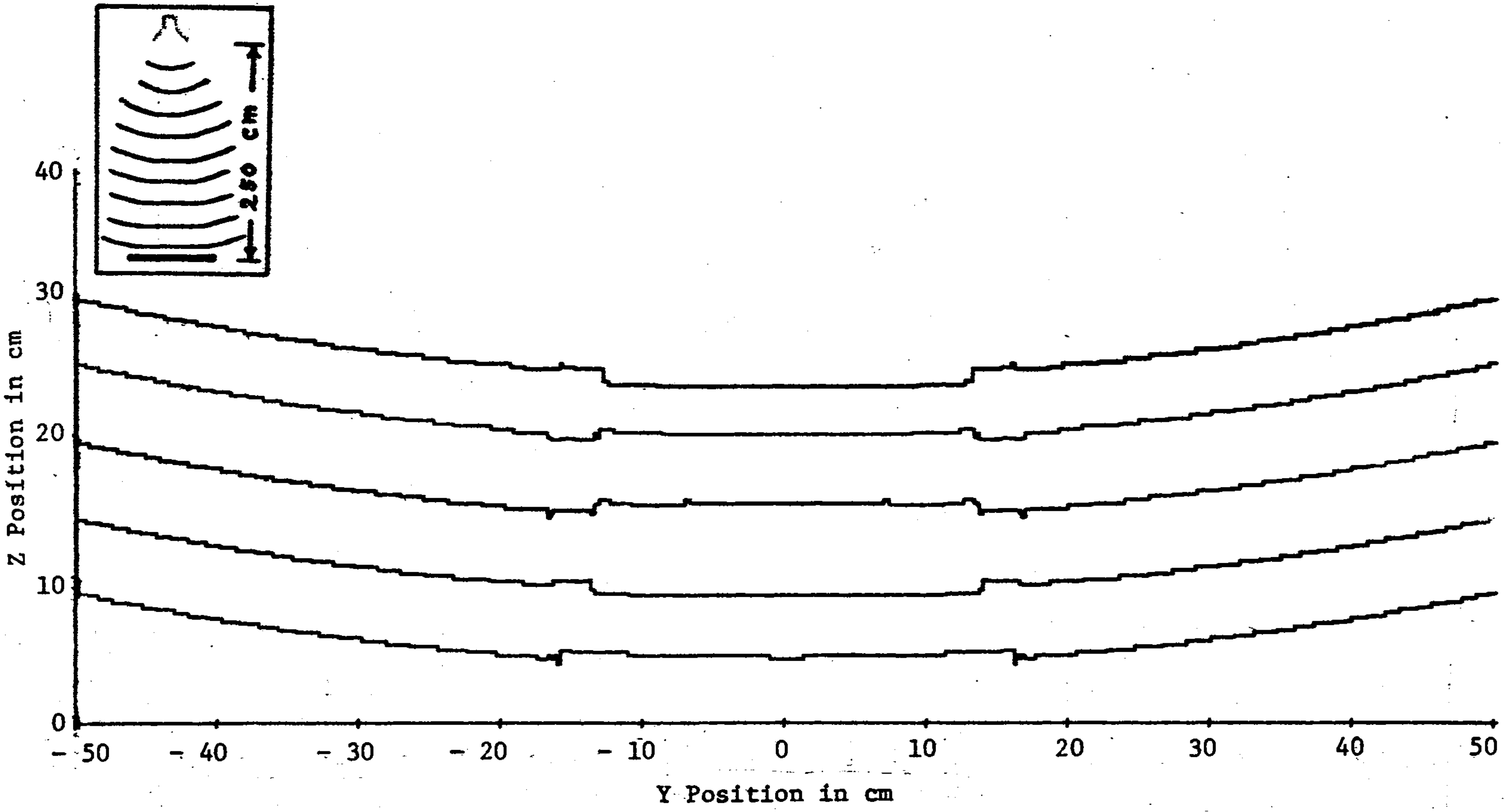

Figure 15. Wave Fronts for Reflection: Speaker Distance Equals 250 cm

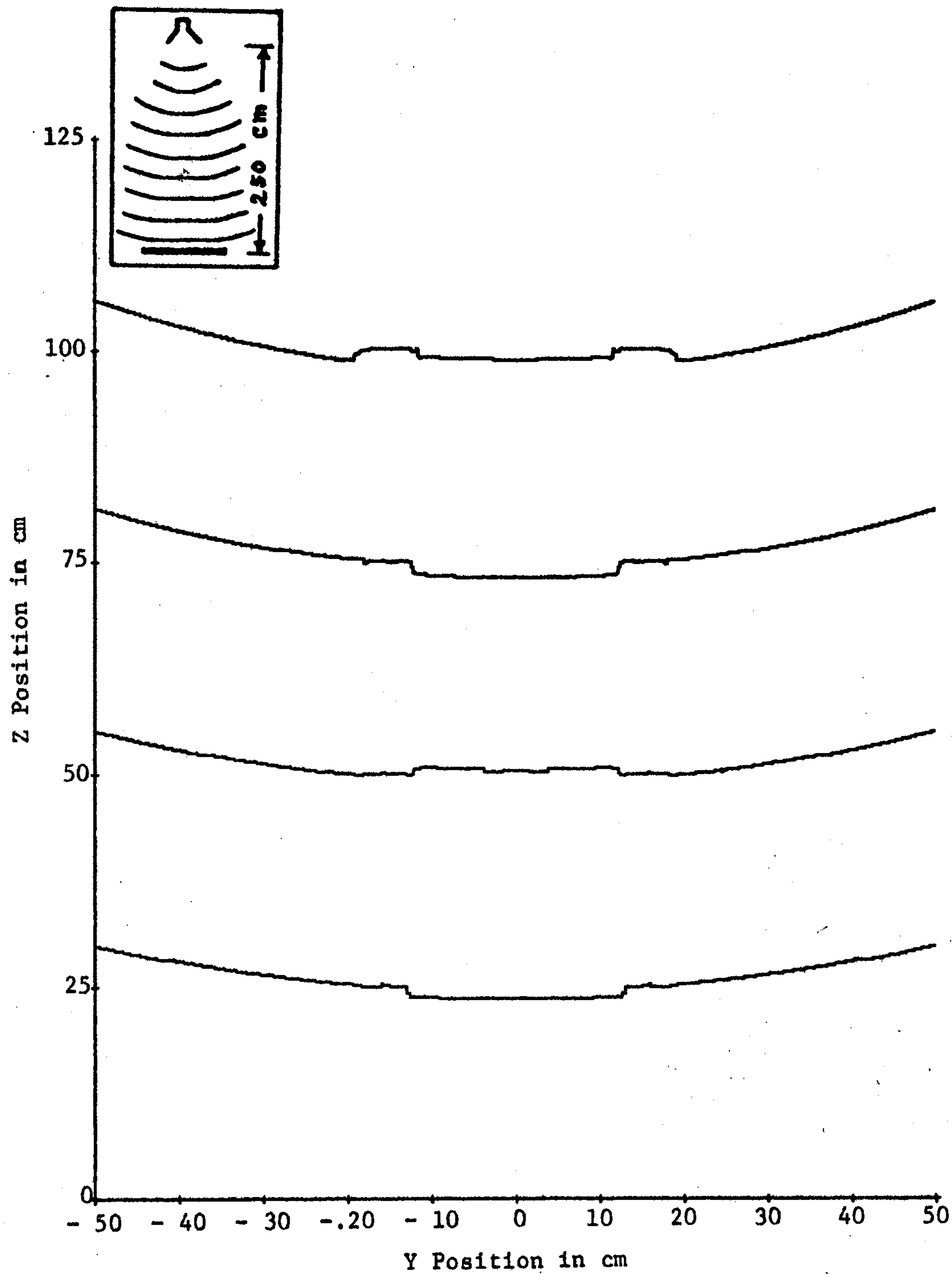

Figure 16. Wave Fronts for Reflection: Speaker Distance Equals 250 cm

much more curved than when the speaker is 250 cm above the reflector. Again, this seems to be a naturally logical occurrence. As one approaches a source, the wave-fronts will definitely become more curved.

CHAPTER IV

EXPERIMENTAL APPARATUS AND RESULTS

Technique of mapping wave-fronts of electromagnetic and acoustic waves have existed for several years. These were initiated by Harley Iams (11) and his colleagues of RCA laboratories in the late forties. By using a dipole receiver and current sensitive paper he was able to map wave-fronts of centimeter waves around various types of radio wave antennas. Basically this was done by mixing a reference signal and the dipole receiver signal together and feeding the sum of the two to a stylus in contact with current sensitive paper. If the two signals were in phase, the summed signal would be a maximum and the current sensitive paper would be considerably darkened. On the other hand, if the two signals were out of phase, the summed signal would be very small and the current sensitive paper would be affected very little. And, as the dipole probe was moved throughout the region under investigation, the result was a series of light and dark lines on the paper representing the wave-fronts (i.e., a series of points of constant phase).

This method of mapping wave-fronts does have one distinct disadvantage. The stylus is physically attached to the dipole probe and, hence, the conducting paper (and its associated conducting backing plate) has to be placed within the region of space which is being mapped. As a result, only a very small region of space can be examined. Furthermore, the resolution of the wave-fronts by this method is not too good.

A much better method of mapping wave-fronts using photography has since been developed (16). In this method the stylus and current sensitive paper of Iam's device are replaced with a camera and a small light bulb. When the probe signal and the reference signal are in phase, the bulb will glow brightly. When they are out of phase, very little light will be emitted by the bulb. If the room in which the experiment is being conducted is darkened and the shutter to a camera is opened, a "photograph" of the wave-fronts can be taken.

The above mentioned photographic technique was the one used by the author in experimentally plotting the wave-fronts of acoustic waves. But instead of the small dipole probe and a radiating antenna used in plotting wave-fronts of microwaves, a speaker and a microphone were used. A block diagram of the experimental set-up can be seen in Figure 17. An audio oscillator drives the speaker. A microphone picked up the generated sound wave. The microphone and reference (audio oscillator) signals were added together electronically in a summing circuit (labeled "mixer"). The resultant signal was fed to a small light bulb. And, as was previously mentioned, if the reference and microphone signals were in phase, i.e., if the sound wave at the point being probed was in phase with the audio oscillator, the bulb glowed brightly. If the acoustic field at a point in space and the reference signal were out of phase, the bulb glowed weakly or not at all.

For some of the "photographs" of wave-fronts the simple electronic summing circuit was replaced by a phasemeter. Instead of adding the two signals, the device compared the two signals. If the microphone signal and the reference signal were in phase, the phasemeter would give an output signal of constant voltage. If the microphone and reference

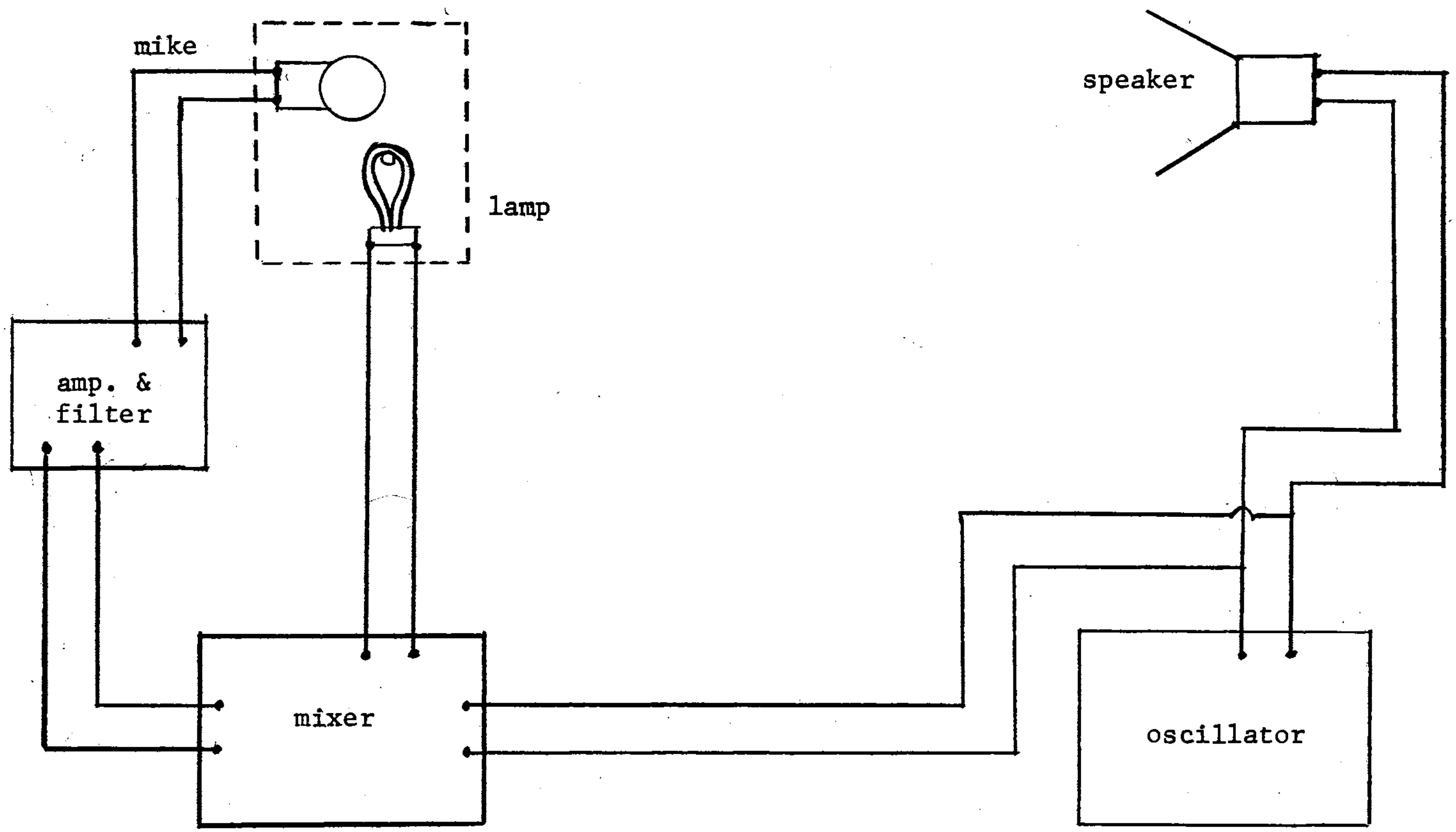

Figure 17. Diagram of Circuitry for Wave Front Plotter

signal were out of phase, there was no output signal. As the lamp was connected to the phasemeter output, the lamp was either lighted or not lighted.

There was one major difference between the use of the simple summing circuit and the phasemeter. While the phasemeter gives only phase information in the wave-front plots, the summing circuit gives both phase and amplitude information in the "photograph" of the wave-fronts. The microphone signal amplitude depended upon the location of the microphone in space. The amplitude variation of this signal was retained when it was summed with the reference signal. Consequently, the summed output to the bulb contained a variable maximum output. In other words the acoustic field at two different points in space which had the same phase could have quite different output from the summing circuit. Hence the brightness of the bulb depended both upon the amplitude and phase of the microphone signal, i.e., the amplitude and phase of the acoustic field.

The mechanical part of the scanning apparatus consisted of a movable cart upon a horizontal track. When the cart moved to the end of the track, a switch was tripped. Tripping this switch allows two things to occur. The horizontal track itself dropped a small distance. And the cart started moving in the opposite direction, i.e., away from the end of the track. The switch reversed the direction of the motor which drove the cart, and it activated a solenoid which, by means of a gravity driven gear assembly, allowed the track to fall a small distance. The microphone and small lamp were attached to the motor-driven cart. Thus a two dimensional slice of space could be scanned and the wave-fronts "photographed."

The above apparatus was used to map wave front patterns for both reflection and diffraction about a 30 cm by 30 cm plane. The physical geometry for obtaining these wave-front pictures is indicated in Figure 18. The microphone probe extended to the middle of the 30 cm by 30 cm square plane so as to sample the acoustic field along a center line of the plane. By appropriately placing a reversing switch upon the cart track, the microphone probe could be brought very close to the surface of the square plane. The speaker was placed at two convenient distances from the surface of the plane: 150 cm and 250 cm.

The frequency used in developing the acoustic field was 20,000 hertz. One reason for choosing this particular frequency was that the Altec Lansing microphone used as the probe would not respond at higher frequencies. Also, at the lower frequencies (in the neighborhood of 10,000 hertz) natural noises in the building tended to pass through the amplifier and filter of Figure 18. As a result the wave-front pictures would have been distorted.

Figure 19 and Figure 20 are the diffraction patterns behind the 30 cm by 30 cm square plane with the speaker distance from the plane set at 150 cm (approximately 87 wavelengths) and 250 cm (approximately 145 wavelengths) respectively. One can note in the top and bottom parts of the two pictures (the region which essentially obeys geometrical optics) that the wave-fronts have different curvatures. Furthermore the wave-fronts in the picture in which the speaker distance is 250 cm are curved less than in the picture in which the speaker distance is 150 cm. This is to be expected as the wave-fronts of a source become more flat if the distance from the source is increased.

In the diffracting region to the right of the square in both pic-

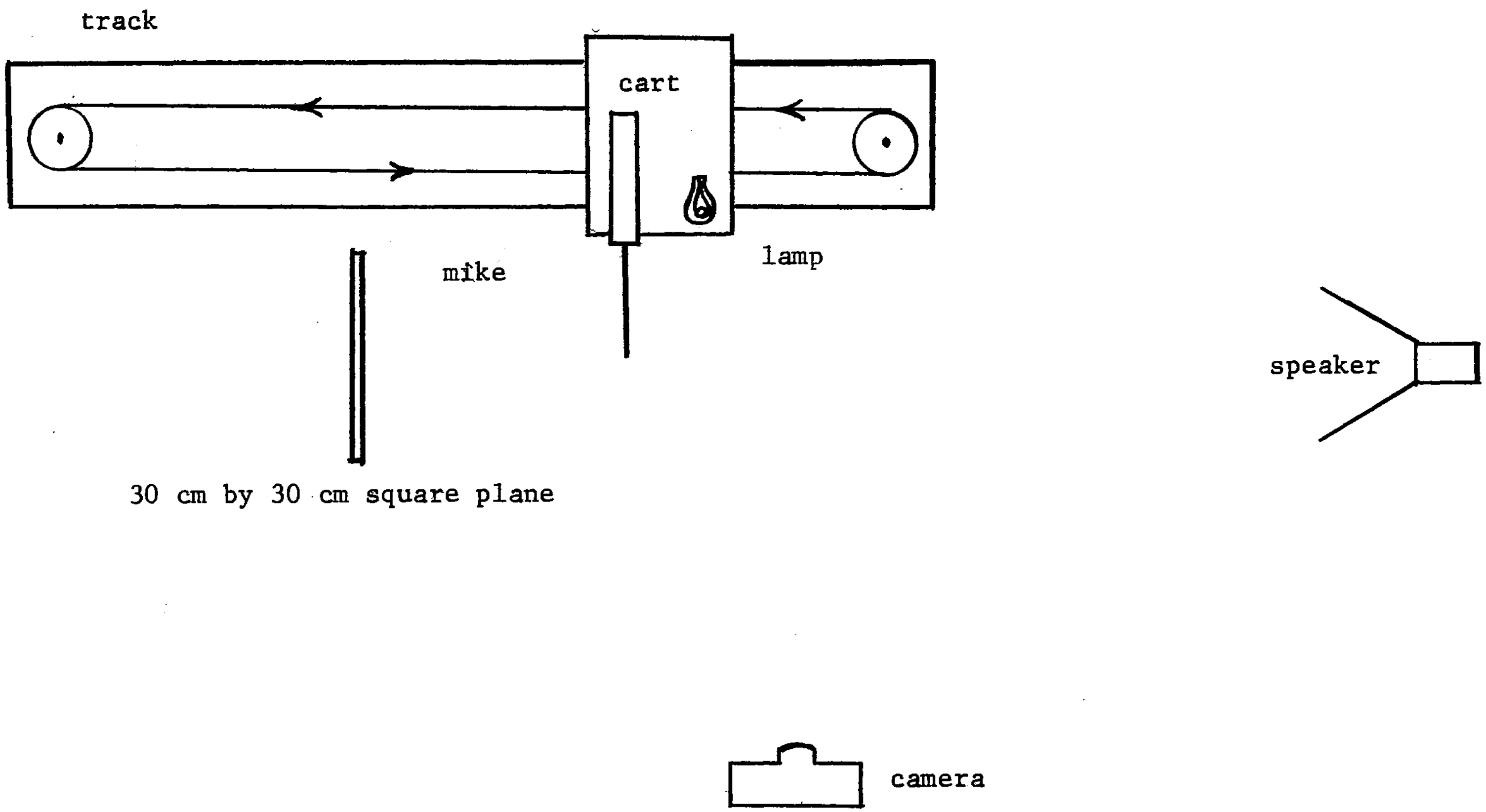

Figure 18. Top View of Scanning Geometry

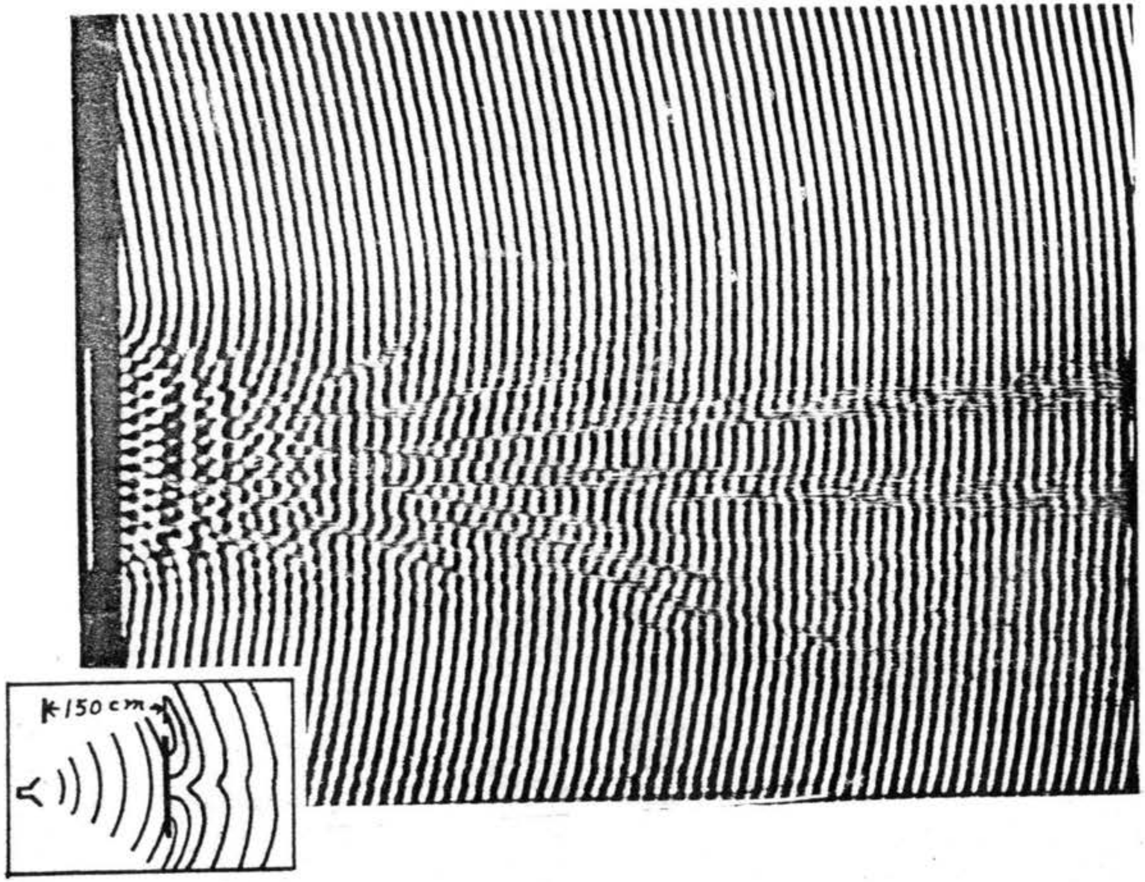

Figure 19. Diffraction Behind a Square Plane: Speaker Located 150 cm Above Plane

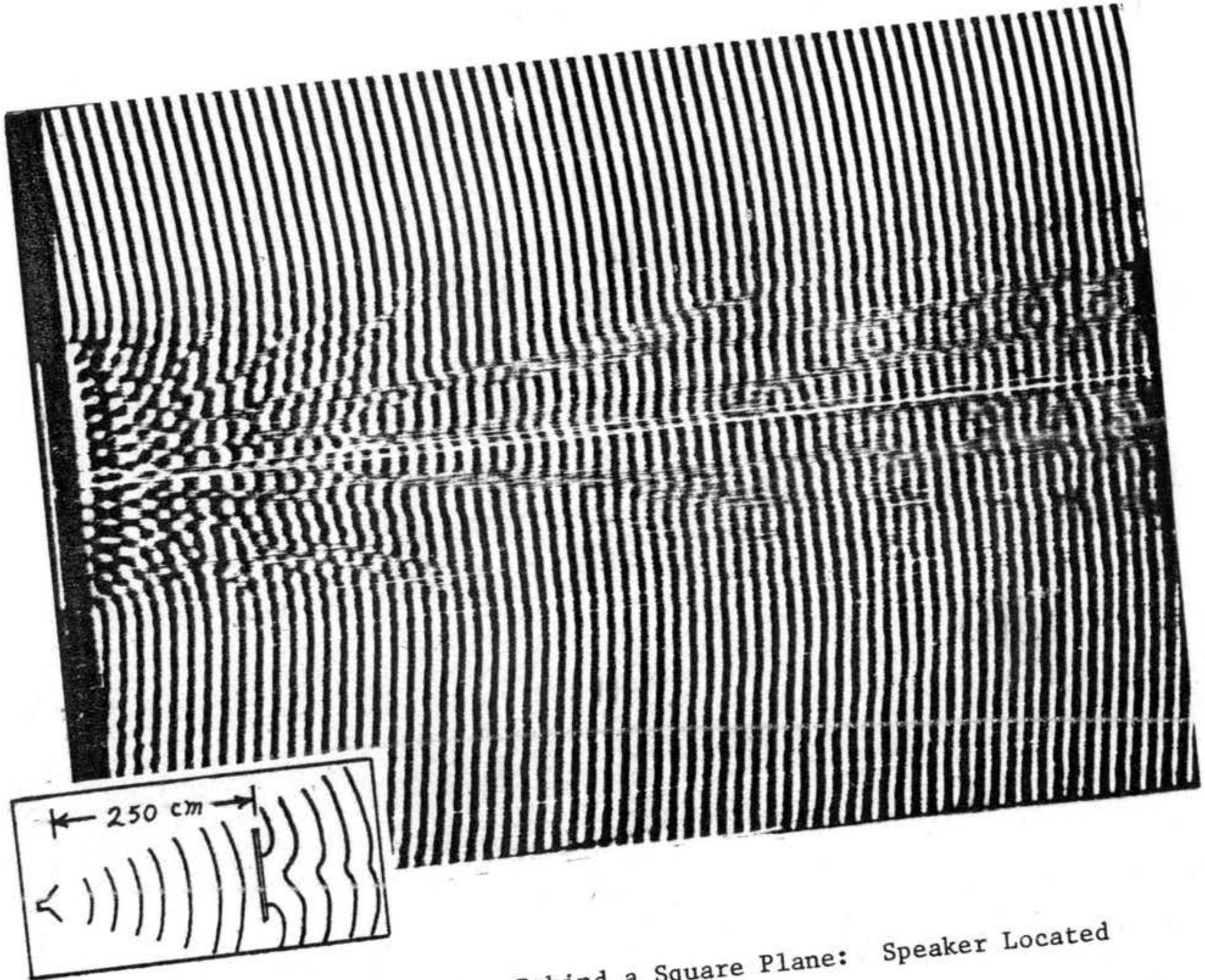

Figure 20. Diffraction Behind a Square Plane: Speaker Located 250 cm Above Plane

tures (the speaker is located to the left), the wave-fronts curve in toward the surface of the square. But as one goes further from the diffraction screen itself one finally reaches a region in which the wave-fronts dip down toward the surface of the square and then go back away. And in between these two regions there lies a rather extensive area in which one cannot distinctly determine if the wave-fronts bend down to the surface of the square or turn down and then sweep back away from the surface.

There is one further comment which should be added about the diffraction patterns. There is a possibility of a considerable amount of transmission of sound through the diffraction screen as it was made of fiberboard. This possibility was checked. And it was found that the amount of sound transmitted through the fiberboard square could not be detected above the natural noise level of the microphone. Consequently, there was very little possibility of the wave-front patterns being distorted by transmission through the square.

Figure 21 and Figure 22 are the wave-front patterns for the case of reflection in front of a 30 cm by 30 cm square plane with the speaker located to the right at the distance of 150 cm (approximately 87 wavelengths) and 250 cm (approximately 145 wavelengths) respectively. For Figure 21 the wave-front pattern approaches within two and one fourth wavelengths of the square. And for Figure 22, the wave-front pattern approaches within two and one fourth wavelengths of the surface of the square. One can easily see the distance of the speaker from the reflection plane is greater for Figure 22 than for Figure 21 as the curvature of the wave-fronts is much less in Figure 22 than in Figure 21.

The interesting thing about the reflection wave-front patterns is

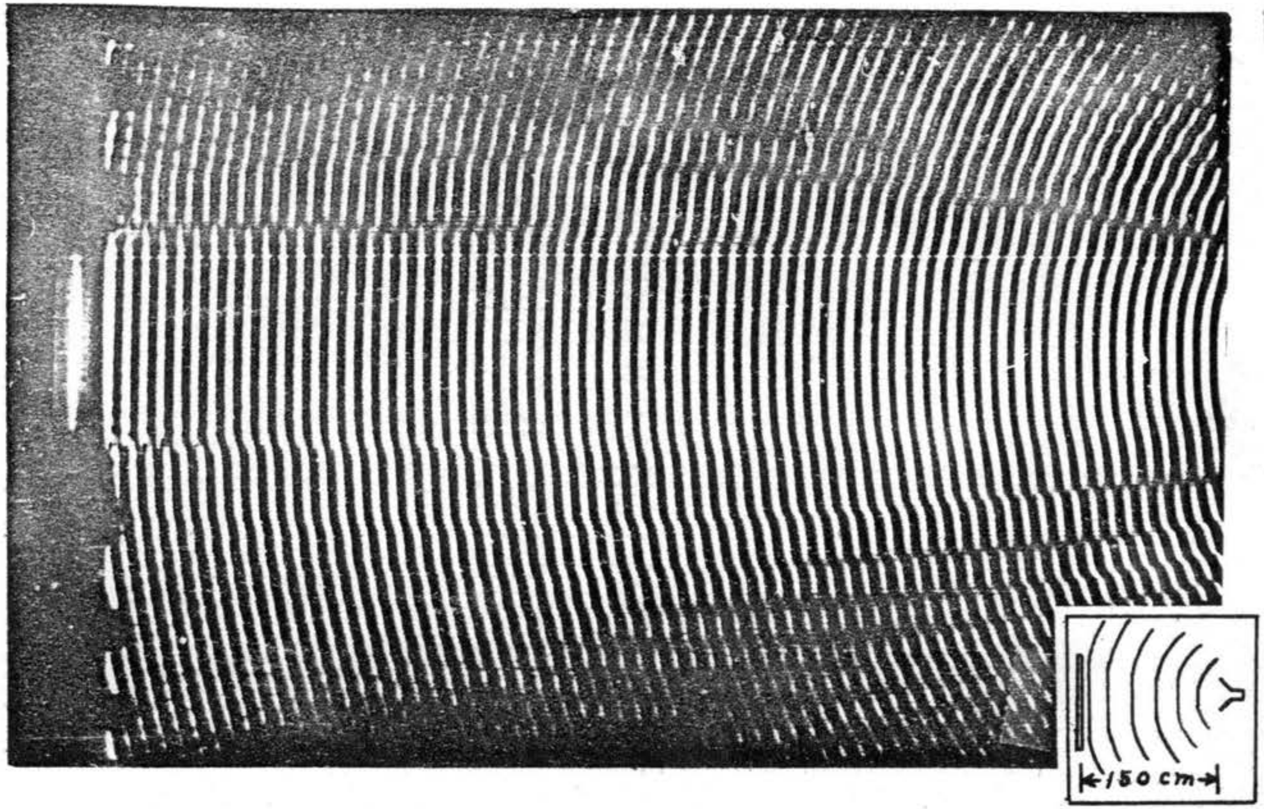

Figure 21. Reflection Above Square Plane: Speaker Located 150 cm Above Plane

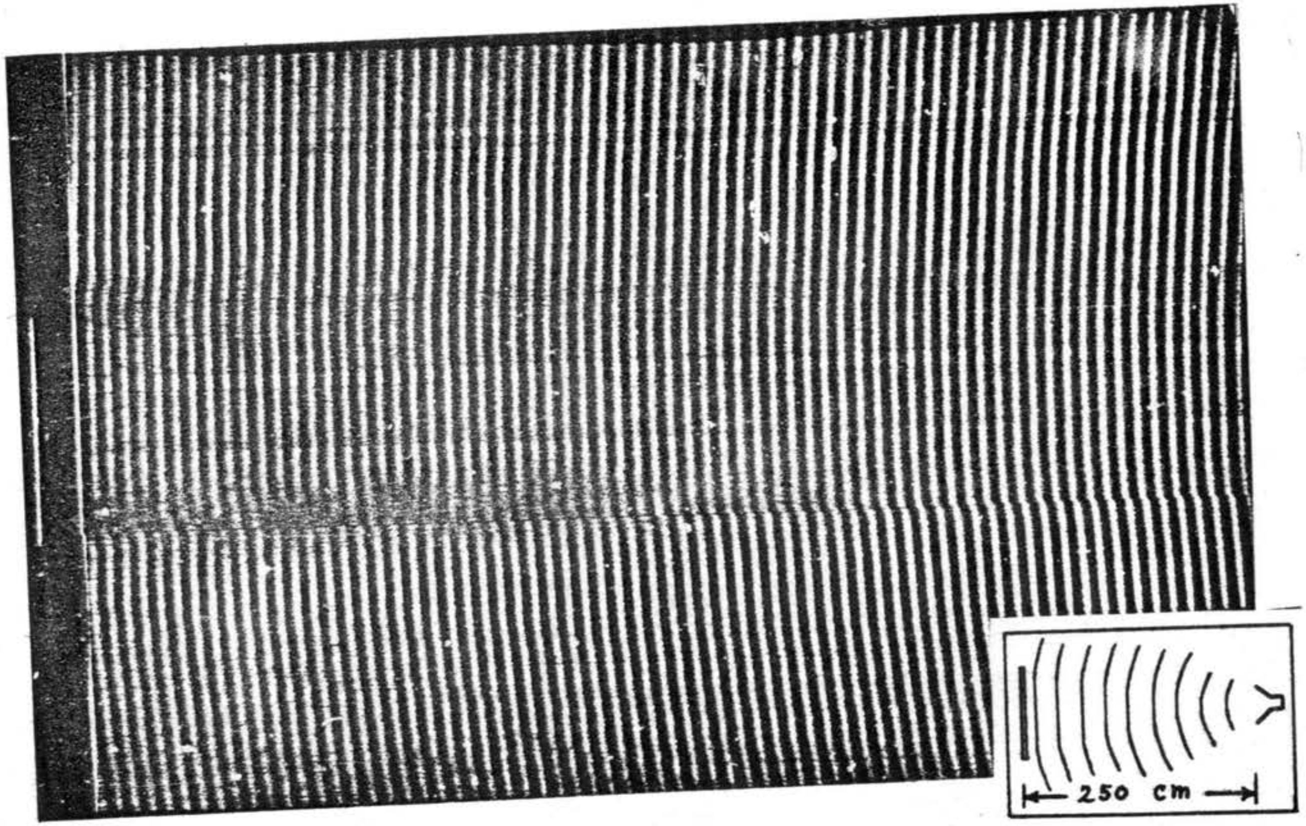

Figure 22. Reflection Above the Square Plane: Speaker Located 250 cm Above Plane

that in the region to the right of the square reflector, i.e., in the region above the reflection surface, the wave-fronts become very flat. In fact near the surface of the reflector the wave-fronts are almost plane. But as one proceeds in the direction of the speaker (to the right) the wave-fronts become slightly curved. And in the picture where the speaker distance is 150 cm the wave-fronts perpendicularly above the reflection surface the wave-fronts start curving slightly more rapidly than in the picture where the speaker distance is 250 cm as one proceeds toward the speaker. This result is as one would logically expect.

CHAPTER V

CONCLUSION

The pictures and graphs of wave-fronts for diffraction behind a square screen look very much the same. In the graphs near the surface of the square the wave-fronts bend toward and touch the surface. In the experimentally taken pictures the wave-fronts have been mapped to within approximately two wavelengths of the surface of the square. And the wave-front patterns near the surface look the same as the theoretically calculated ones. In the theoretical wave-front pictures at larger distances, the wave-fronts bend down toward the surface and then sweep back away. And at even larger distances, the diffracted wave-fronts begin to approximate a spherical wave. The same type of phenomenon occurs in the experimental wave-front pictures. And, as a result, the Fresnel-Kirchhoff theory does predict the nature of wave-fronts for diffraction behind a square plane to within at least two wavelengths of the diffracting surface.

The pictures and graphs of wave-fronts for the case of reflection in front of a 30 cm by 30 cm plane reflector also look quite similar. In the theoretical graphs of the wave-fronts, the wave-fronts are very flat near the surface of the square reflector. In fact the wave-fronts remain very flat even at large distances from the reflecting surface. The same type of occurrence is to be found in the pictures of the wave-fronts mapped in the laboratory. Again the experimental mappings of the wave-fronts approach to within two wavelengths of the reflecting surface. Hence, one can definitely say that the Fresnel-Kirchhoff theory does indeed correctly predict the general structure of the wave-front patterns due to reflection from a 30 cm by 30 cm plane surface to within at least two wavelengths of the surface.

There are two things in regard to this research which might be interesting to examine in the future. Due to the size of the microphone probe there was a limit to the resolution of the wave-fronts. The pressure of a finite area of space instead of a point was always examined. If one could decrease the size of the probe relative to the wavelength, much better resolution should occur. This should eliminate some of the haziness which occurs in the experimental wave-front picture. Also, with a decreased probe size, one should be able to obtain wave-front patterns much closer to the diffracting and reflecting surfaces than was done in this work.

A SELECTED BIBLIOGRAPHY

1. Baker, B. B. and E. T. Copson. The Mathematical Theory of Huygens' Principle, 2nd ed. Oxford: Clarendon Press, 1950.

2. Binns, K. J. and P. J. Lawrenson. Analysis and Computation of Electric and Magnetic Field Problems. New York: MacMillan Co., 1963.

3. Born, Max and E. Wolf. Principles of Optics, 2nd ed. New York: MacMillan Co., 1964.

4. Davis, Philip J. and Philip Rabinowitz. Numerical Integration. Waltham, Massachusetts: Blaisdell Publishing Co., 1967.

5. de Haan, D. Bierens. Nouvelles Tables D'Integrales Definies. New York: G. E. Stechert and Co., 1939.

6. Erdelyi, A. Asymptotic Expansions. New York: Dover Publications, Inc., 1956.

7. Goodman, Joseph W. Introduction to Fourier Optics. New York: McGraw-Hill Book Company, Inc., 1968.

8. Gradshteyn, I. S. and I. M. Ryzhik. Table of Integrals, Series, and Products, trans. Scripta Technica, Inc. New York: Academic Press, 1965.

9. Grobner, Wolfgang and Nikolaus Hofreiter. Integraltafel Vol. I and II. Vienna: Springer-Verlag, 1961.

10. Harrington, Roger F., ed. Field Computation by Moment Methods. New York: MacMillan Co., 1968.

11. Iams, Harley. "Phase-front Plotter for Centimeter Waves." RCA Review, Vol. 8 (June, 1947), 270.

12. Jenkins, Francis A. and Harvey E. White. Principles of Optics, 3rd ed. New York: McGraw-Hill Book Company, Inc., 1957.

13. Jones, Douglas and Morris Kline. "Asymptotic Expansion of Multiple Integrals and Method of Stationary Phase." Journal of Mathematical Physics, Vol. 37 (1958), 1-28.

14. King, R. Electromagnetic Engineering, Vol. I. New York: McGraw-Hill Book Company, Inc., 1945.

15. Kinsler, Lawrence and Austin R. Frey. Fundamentals of Acoustics. New York: John Wiley and Sons, 1950.

16. Koch, W. Sound Waves and Light Waves, 1st ed. Garden City, New York: Anchor Books, 1965.

17. Kottler, Fredrick. "Diffraction at a Black Screen, Part I: Kirchoff's Theory." Progress in Optics, Vol. 4 (1965), 283-313.

18. Lamb, Horace. Dynamical Theory of Sound, 2nd. ed. New York: Dover Publications, Inc., 1925.

19. Levine, H. and J. Schwinger. "On the Theory of Diffraction by an Aperture in an Infinite Plane Screen, Part I." Physical Review, Vol. 74 (October 15, 1948), 958-973.

20. Levine, H. and J. Schwinger. "On the Theory of Diffraction by an Aperture in an Infinite Plane Screen, Part II." Physical Review, Vol. 75 (May 1, 1945), 1423-1432.

21. Morse, Philip M. and K. Uno Ingard. Theoretical Acoustics. New York: McGraw-Hill Book Company, Inc., 1968.

22. Rossi, Bruno. Optics. Reading, Massachusetts: Addison-Wesley, 1965.

23. Stone, John M. Radiation and Optics. New York: McGraw-Hill Book Company, Inc., 1963.

24. van der Corput, J. G. "On the Method of Critical Points." Koninklijke Nederlandsche Akademie van Wettenschappen. Proceedings, Vol. 51 (1948), 650-658.

25. van Kampen, N. G. "An Asymptotic Treatment of Diffraction Problems." Physica, Vol. 14 (1949), 575-589.

26. van Kampen, N. G. "An Asymptotic Treatment of Diffraction Problems, Part II." Physica, Vol. 16 (1950), 817-821.

27. van Kampen, N. G. "Method of Stationary Phase and Method of Fresnel Zones." Physica, Vol. 24 (1958), 437-444.

28. Wolf, E. "The Diffraction Theory of Aberrations." Report of the Progress in Physics, Vol. 14 (1951), 97-108.

APPENDIX A

INTEGRALS NEEDED FOR NON-BOUNDARY REGION EXPANSIONS

$$\int_{-\infty}^{\infty} y^n e^{ikay^2} dy = 0 \qquad (n \text{ odd})$$

$$= \Gamma\left(\frac{n+1}{2}\right)\left(k|a|\right)^{-\frac{n+1}{2}} \varepsilon^{n+1} \quad (n \text{ even})$$

$$\varepsilon = e^{\pi i/4}$$

$$\int_{0}^{\infty} y^n e^{ikay} dy = \frac{n!}{(-ika)^{n+1}}$$

$$\int_{-\infty}^{0} y^n e^{ikay} dy = (-1)^n \frac{n!}{(ika)^{n+1}}$$

$$\int_{-\infty}^{\infty} y^n e^{ikay} dy = 0$$

APPENDIX B

INTEGRALS NEEDED FOR BOUNDARY REGION EXPANSIONS

$$\int_0^\infty e^{(ax^2+bx)i} f(x)\, dx = \frac{1}{\sqrt{a}}\, e^{-\frac{b^2 i}{4a}} \int_{\frac{b}{2\sqrt{a}}}^{\infty} e^{y^2 i} f\left(\frac{y}{\sqrt{a}} - \frac{b}{2a}\right) dy$$

$$\int_0^\infty e^{(ax^2+bx)} dx = \frac{e^{-\frac{b^2 i}{4a}}}{2} \sqrt{\frac{\pi}{a}} \left\{ e^{\pi i/4} - \Phi\left(\frac{b}{2\sqrt{a}}\right) \right\}$$

$$\Phi\left(\frac{b}{2\sqrt{a}}\right) = \frac{2}{\sqrt{\pi}} \int_0^{\frac{b}{2\sqrt{a}}} e^{y^2 i}\, dy$$

$$\int_0^\infty e^{(ax^2+bx)i} x\, dx = - e^{-\frac{b^2 i}{4a}} \frac{b}{4a} \sqrt{\frac{\pi}{a}}\, \bar{\Phi} + \frac{i}{2a}$$

$$\bar{\Phi} = e^{\pi i/4} - \Phi\left(\frac{b}{2\sqrt{a}}\right)$$

$$\int_0^\infty e^{(ax^2+bx)i} x^2 dx = \frac{e^{-\frac{b^2 i}{4a}}}{4a}\, \bar{\Phi} \sqrt{\frac{\pi}{a}} \left(i + \frac{b^2}{2a}\right) - \frac{bi}{4a^2}$$

$$\int_0^\infty e^{(ax^2+bx)i} x^3 dx = \frac{e^{-\frac{b^2 i}{4a}}\, b}{8a^2} \sqrt{\frac{\pi}{a}}\, \bar{\Phi} \left(-3i - \frac{b^2}{2a}\right) - \frac{1}{2a^2} - \frac{b^2}{8a^3 i}$$

APPENDIX C

WAVE FRONT PLOTTING PROGRAM

```
      DIMENSION YDISP(4),ZDISP(4),NAVOID(4)
      DELT=0.4
      Z=20.0
      Y=-14.0
      YDISP(1)=DELT
      YDISP(2)=0.0-DELT
      YDISP(3)=0.0
      YDISP(4)=0.0
      ZDISP(1)=0.0
      ZDISP(2)=0.0
      ZDISP(3)=DELT
      ZDISP(4)=0.0-DELT
      NAVOID(1)=2
      NAVOID(2)=1
      NAVOID(3)=4
      NAVOID(4)=3
      I=1
      PSTRT=PHASE(Y,Z)
      JAVOID=2
      WRITE (6,3)
      WRITE (6,4) I,Y,Z,PSTRT,JAVOID
      DO 2 I=2,100
      PBEST=1.0E+60
      DO 1 J=1,4
      IF(J.EQ.JAVOID) GO TO 1
      IF(J.EQ.2) GO TO 1
      TEST=PHASE(Y+YDISP(J),Z+ZDISP(J))
      ANGLE1=ABS(TEST-PSTRT)
      IF(ANGLE1-3.14159) 10,10,5
    5 ANGLE1=2.0*3.14159-ANGLE1
   10 ANGLE2=ABS(PBEST-PSTRT)
      IF(ANGLE2-3.14159) 20,20,15
   15 ANGLE2=ABS(2.0*3.14159-ANGLE2)
   20 IF(ANGLE1.GE.ANGLE2) GO TO 1
      PBEST=TEST
      JBEST=J
    1 CONTINUE
      Y=Y+YDISP(JBEST)
      Z=Z+ZDISP(JBEST)
      P=PBEST
      JAVOID=NAVOID(JBEST)
    2 WRITE (6,4) I,Y,Z,P,JAVOID
    3 FORMAT (' ',T3,'NO',T11,'Y',T20,'Z',T26,'PHASE',
     1T38,'AVOID'/)
    4 FORMAT (' ',I4,2(2XF8.4),2X1PE12.4,2XI2)
      CALL EXIT
      END
```

VITA

3

William G. Hedgpeth

Candidate for the Degree of

Doctor of Education

Thesis: THE EXACTNESS OF ACOUSTIC WAVE FRONTS CALCULATED VIA FRESNEL-KIRCHHOFF INTEGRALS

Major Field: Higher Education

Biographical:

Personal Data: Born near Osceola, Missouri, January 31, 1944, the son of Mr. and Mrs. Lester Hedgpeth.

Education: Graduated from Lowry City High School, Lowry City, Missouri, in May, 1962; received the Bachelor of Science degree from Central Missouri State College in 1965, with a major in physics and mathematics; received the Master of Science degree from Central Missouri State College in 1966, with a major in physics; completed requirements for the Doctor of Education degree at Oklahoma State University in May, 1971 with a major in higher education.

Professional Experience: Graduate Teaching Assistant, Central Missouri State College, 1965-66; Graduate Teaching Assistant, Oklahoma State University, 1966-1970.