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CONVEXITY STRUCTURES AND THE THEOREMS OF

CARATHEODORY, RADON, AND HELLY

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CONVEXITY STRUCTURES AND THE THEOREMS OF
CARATHEODORY, RADON, AND HELLY

CHAPTER I

INTRODUCTION

There have been continuing efforts by various authors to generalize the concept of convexity and certain well known results in the theory of convexity by varying the underlying space, altering the definition of individual convex sets, or collectively defining a new class of "convex" sets. The latter technique is used in this paper to create abstract convexity structures in which to explore possible relationships between the classical Caratheodory, Radon, and Helly theorems on convexity and to discuss the properties of various convexity structures in E^n. Emphasis is placed on minimizing the structure imposed on the space in order to obtain the desired results. For the sake of completeness the well known theorems of Caratheodory, Radon, and Helly are stated below.

1.1. THEOREM. (Caratheodory [2]) If S is a subset of E^n and x lies in the convex hull of S, then x lies in the convex hull of n+1 (or fewer) members of S.

<u>1.2. THEOREM</u>. (Radon [16]) If S is a set of at least n+2 elements in E^n, then S can be expressed as the union of two disjoint subsets whose convex hulls have a non-empty intersection.

<u>1.3. THEOREM</u>. (Helly [12]) If $\mathcal{F}$ is a finite family of at least n+1 convex sets in E^n and each n+1 members of $\mathcal{F}$ have a non-empty intersection, then the intersection of all members of $\mathcal{F}$ is non-empty.

Although these results are intimately related to the dimension of the space, the properties involved therein can be abstracted and studied in axiomatic settings. F. W. Levi [15] takes a step in this direction by using an analogue of the Radon theorem to prove an analogue of the Helly theorem. M. Breen [1] constructs additional examples to shed light on the properties considered by Levi. In chapter two we define Caratheodory, Radon, and Helly properties, which the family of convex sets in E^n possesses by virtue of the classical theorems, and investigate the relationships among them in a general setting. Our definitions are basically different from those used by Levi and Breen; and except for a modification of Levi's main theorem, our results are independent of theirs.

In chapter three a characterization of the Caratheodory Radon, and Helly properties in terms of "half-spaces" appears, and this leads to an alternate proof of Levi's theorem in a slightly less general setting. Finally, chapter four contains results about convexity structures in E^n, the main theorem of which is a characterization of the collection of convex sets in E^n.

Although the author has attempted to make his notation consistent with that of Valentine [17], new symbols are defined as they are introduced, and a summary of notation appears in the appendix. The term linear space will be used to denote a real linear space.

CHAPTER II

RELATIONSHIPS AMONG THE CARATHEODORY, RADON, AND HELLY PROPERTIES IN AN ABSTRACT SETTING

The closure operator as studied in topology can be modified to give a weaker closure, or hull, operator that has been investigated and used in generalizing convexity by P. C. Hammer [8;9;11]. In any set X such an operator can be defined by using a collection of subsets of X. Consequently, we make the following definition.

2.1. DEFINITION. In a set X a collection of subsets $\mathcal{C}$ will be called a convexity structure if

(a) X and $\emptyset$ belong to $\mathcal{C}$, and

(b) $\mathcal{C}$ is closed under intersections; that is, if $\mathcal{F} \subset \mathcal{C}$, then $\cap \mathcal{F}$ belongs to $\mathcal{C}$.

The ordered pair (X, $\mathcal{C}$) will be called a closure space.

2.2. DEFINITION. If (X, $\mathcal{C}$) is a closure space and S is a subset of X, the $\mathcal{C}$-hull of S is denoted by $\mathcal{C}$(S) and is defined by

$$\mathcal{C}(S) = \cap \{C \in \mathcal{C} : S \subset C\}.$$

In order to distinguish the hull operator from the collection

$\mathcal{C}$, the operator will be referred to as $\mathcal{C}$-conv.

Closure spaces and the corresponding hull operators occur in all areas of mathematics, as illustrated by the following examples. In these examples X and $\emptyset$, in addition to the subsets listed, are understood to be members of $\mathcal{C}$.

(a) Let X be a topological space and let $\mathcal{C}$ be the family of all closed sets.

(b) Let X be a linear space and let $\mathcal{C}$ be the family of all convex sets.

(c) Let $X = E^n$ and let $\mathcal{C}$ be the collection of all convex sets with diameter not greater than one.

(d) Let $X = E^n$ and let $\mathcal{C}$ be the collection of all convex sets with dimension not greater than $k \leq n$.

(e) Let X be a group and let $\mathcal{C}$ be the collection of all subgroups with the identity element deleted.

(f) Let X be a ring and let $\mathcal{C}$ be the collection of all ideals with the additive identity deleted.

(g) Let X be the power set of some set M and let $\mathcal{C}$ be the collection of all σ-algebras in M with $\emptyset$ and M deleted.

(h) Let X be the power set of some set M and let $\mathcal{C}$ be the collection of all topologies on M with $\emptyset$ and M deleted.

The following theorem is an immediate result of the definitions and records some basic facts about the $\mathcal{C}$-hull of a set.

<u>2.3. THEOREM</u>. If $(X, \mathcal{C})$ is a closure space and S and T are

subsets of X, then

(a) $S \subset \mathcal{C}(S)$,

(b) if $S \subset T$, then $\mathcal{C}(S) \subset \mathcal{C}(T)$,

(c) $\mathcal{C}(\mathcal{C}(S)) = \mathcal{C}(S)$, and

(d) $\mathcal{C}(\emptyset) = \emptyset$.

Even though $\mathcal{C}$-conv is not in general a Kuratowski closure operator, it is interesting to note that Hammer's [8] generalization of Kuratowski's theorem shows that the family of sets generated from a single subset by $\mathcal{C}$-conv and complementation owns at most, and sometimes exactly, 14 sets. However, in special cases a better bound may be possible; in fact, W. Koenen [14] proves that if $X = E^n$ and $\mathcal{C}$ is the collection of convex sets, the best bound is 10.

2.4. DEFINITION. A set $S \subset X$ is *$\mathcal{C}$-convex* if $S = \mathcal{C}(S)$.

2.5. THEOREM. A set $S \subset X$ is $\mathcal{C}$-convex if and only if S belongs to $\mathcal{C}$.

Proof. If S is $\mathcal{C}$-convex then $S = \cap\{C \in \mathcal{C} : S \subset C\} \in \mathcal{C}$. Conversely, if $S \in \mathcal{C}$, then both $\cap\{C \in \mathcal{C} : S \subset C\} \subset S$ and $S \subset \cap\{C \in \mathcal{C} : S \subset C\}$. It follows that $S = \cap\{C \in \mathcal{C} : S \subset C\} = \mathcal{C}(S)$ and S is $\mathcal{C}$-convex by definition.

In a closure space $(X, \mathcal{C})$ we define the following properties.

2.6. DEFINITION. $\mathcal{C}$ is said to have the *Caratheodory property* if there exists a positive integer n such that for each subset $S \subset X$ and $x \in \mathcal{C}(S)$, there exist elements $x_1, x_2, x_3, \ldots, x_m$ in S, $m \leqslant n$, such that $x \in \mathcal{C}(\{x_1, x_2, \ldots, x_m\})$. The smallest such positive integer is called the Caratheodory number. (In the setting of logic, this might also be called the Tarski property.)

2.7. DEFINITION. $\mathcal{C}$ is said to have the Radon property if there exists an integer n such that for each finite subset $S \subset X$ with card $S \geqslant n$, there exist non-empty subsets S_1 and S_2 of S such that $S_1 \cap S_2 = \emptyset$, $S = S_1 \cup S_2$, and $\mathcal{C}(S_1) \cap \mathcal{C}(S_2) \neq \emptyset$. The smallest such positive integer is called the Radon number, and the pair (S_1, S_2) is called a Radon partition of S.

2.8. DEFINITION. $\mathcal{C}$ is said to have the Helly property if there exists a positive integer n with the property that for each finite subfamily $\mathcal{F} \subset \mathcal{C}$ such that card $\mathcal{F} \geqslant n$ and the intersection of every n members of $\mathcal{F}$ is non-empty, then $\cap \mathcal{F} \neq \emptyset$. The smallest such positive integer is called the Helly number.

From these definitions we see that the Radon number is never less than two and the Helly number is never less than one. It is easy to show that these smallest values are actually attained if and only if each finite subset $\mathcal{F}$ of non-empty members of $\mathcal{C}$ has a non-empty intersection.

The following theorem is a modification of a theorem due to Levi [15]; the proof is similar to that given by Levi but incorporates the above definitions.

2.9. THEOREM. If $(X, \mathcal{C})$ is a closure space and $\mathcal{C}$ has Radon number r, then $\mathcal{C}$ has Helly number $h \leq r-1$.

Proof. We shall use mathematical induction on card $\mathcal{F}$, where $\mathcal{F}$ is a finite subfamily of $\mathcal{C}$ and the intersection of every r-1 members of $\mathcal{F}$ is non-empty. If card $\mathcal{F} = r-1$, then $\cap\mathcal{F} \neq \emptyset$ by the choice of $\mathcal{F}$. Assume that if card $\mathcal{F} = k \geq r-1$, then $\cap\mathcal{F} \neq \emptyset$, for each such $\mathcal{F}$. Suppose, for some $\mathcal{F}$, card $\mathcal{F} = k+1$ and let $\mathcal{F} = \{F_1,\ldots,F_{k+1}\}$. By the induction assumption, the intersection of every k members of $\mathcal{F}$ is non-empty; so let $x_i \in \cap\{F_j: i \neq j\}$ and put $A = \{x_1,x_2,\ldots,x_{k+1}\}$. Since $k+1 \geq r$, the set A has a Radon partition (B, C) such that

$$D = \mathcal{C}(B) \cap \mathcal{C}(C) \neq \emptyset.$$

We prove that $D \subset \cap\mathcal{F}$. Let i be an integer such that $x_i \in C$; then $x_i \notin B$ and hence $x_j \in B$ implies $i \neq j$. Thus, by the choice of x_j, we have $x_j \in F_i$, which proves that B is a subset of F_i. It then follows from 2.3 (b) and 2.5 that

$$\mathcal{C}(B) \subset \mathcal{C}(\cap\{F_i: x_i \in C\}) = \cap\{F_i: x_i \in C\}.$$

Similarly,

$$\mathcal{C}(C) \subset \mathcal{C}(\cap\{F_i: x_i \in B\}) = \cap\{F_i: x_i \in B\}.$$

Consequently,

$$\emptyset \neq D \subset (\cap\{F_i: x_i \in C\}) \cap (\cap\{F_i: x_i \in B\}) = \cap\mathcal{F}.$$

Thus, $\mathcal{C}$ has the Helly property with respect to the number r-1 and hence has Helly number $h \leq r-1$.

The following examples show that if we assume only one of the three properties under consideration, then 2.9 is the only theorem possible.

<u>2.10. EXAMPLE.</u> (Hammer) Let $X = E^2$. We use the term <u>half open right sector at a point</u> p to denote the open portion of the plane between two perpendicular rays emanating from p together with exactly one of the closed bounding rays. Let $\mathcal{C}$ consist of X, $\emptyset$, complements of all half open right sectors at all points in E^2, and all possible intersections of such sets. Hammer [9] proves that $\mathcal{C}$ has Caratheodory number 7. However, using the fact that any collinear set of points in X is a member of $\mathcal{C}$, it is easy to show that the Helly property does not hold for $\mathcal{C}$. For, suppose $\mathcal{C}$ has Helly number h, take a set of h+1 collinear points $F = \{x_1, x_2, \ldots, x_{h+1}\}$, and put $F_i = F \sim \{x_i\}$. Then $\mathcal{F} = \{F_1, F_2, \ldots, F_{h+1}\}$ is obviously a family of h+1 sets which are members of $\mathcal{C}$ such that each h members have a non-empty intersection, but $\bigcap \mathcal{F} = \emptyset$. Thus, $\mathcal{C}$ does not have the Helly property, and by 2.9, cannot have the Radon property. This shows that neither the Radon property nor the Helly property is a consequence of the Caratheodory property.

<u>2.11. EXAMPLE.</u> Let $X = E^1$ and let $\mathcal{C}$ be the closed convex sets in X. Since the ordinary convex hull of a finite set of points in X is also the closed convex hull, or $\mathcal{C}$-hull, of that set, $\mathcal{C}$ has Radon number 3, and since all members of

$\mathcal{C}$ are convex, $\mathcal{C}$ has Helly number 2. But if

$$S = \{\tfrac{1}{n}: n = 1,2,3,\ldots\},$$

then 0 belongs to $\mathcal{C}(S)$ but does not belong to the $\mathcal{C}$-hull of any finite subset of S. Thus, neither the Radon property nor the Helly property implies the Caratheodory property.

The following class of examples leads to an example showing that the Radon property does not follow from the Helly property, and thus, the two properties are not equivalent in a closure space. In addition, it shows that even though the Helly number is less than the Radon number when both exist, the difference between the two numbers can be arbitrarily large.

2.12. EXAMPLE. Let $X = E^n$ and let $\mathcal{C}$ consist of X, $\emptyset$, and the collection of all closed rectangular solids with edges parallel to the coordinate axes; that is, C is a non-trivial member of $\mathcal{C}$ if and only if there exist numbers $a_1,\ldots,a_n$, $b_1,\ldots,b_n$, with $a_i \leq b_i$ for $i = 1,2,\ldots,n$ such that $C = \{x \in X: a_i \leq x_i \leq b_i\}$. Based on this definition of $\mathcal{C}$ we make the following observations, in general using the notation x_i to denote the i-th coordinate of a point x.

2.12.1. $\mathcal{C}$ is closed under intersections. In fact, if $\mathcal{F} = \{F_\alpha: \alpha \in A\}$ is a subset of $\mathcal{C}$ and $\cap\mathcal{F} \neq \emptyset$, where

$$F_\alpha = \{x \in X: a_{\alpha,i} \leq x_i \leq b_{\alpha,i}\}$$

then

$$\cap\mathcal{F} = \{x \in X: c_i \leq x_i \leq d_i\},$$

where $c_i = \sup \{a_{\alpha,i}: \alpha \in A\}$ and $d_i = \inf \{b_{\alpha,i}: \alpha \in A\}$.

<u>2.12.2</u>. If S is a bounded subset of X, then

$$\mathcal{C}(S) = \{x \in X: a_i \leqslant x_i \leqslant b_i\},$$

where $a_i = \inf \{x_i: x \in S\}$ and $b_i = \sup \{x_i: x \in S\}$.

<u>2.12.3</u>. If S and T are bounded subsets of X, then

$$\mathcal{C}(S) \cap \mathcal{C}(T) = \emptyset$$

if and only if there is an integer i such that either

$$\sup \{x_i: x \in S\} < \inf \{x_i: x \in T\}$$

or

$$\sup \{x_i: x \in T\} < \inf \{x_i: x \in S\}.$$

<u>2.12.4</u>. $\mathcal{C}$ has Helly number 2.

<u>Proof</u>. Let $\mathcal{F} = \{F_1,\ldots,F_k\}$ be a finite subset of $\mathcal{C}$ such that each two members of $\mathcal{F}$ have a non-empty intersection, where $F_j = \{x \in X: a_{j,i} \leqslant x_i \leqslant b_{j,i}\}$ for $j = 1,2,\ldots,k$. Let z be the point in X such that

$$z_i = \inf \{b_{j,i}: j = 1,2,\ldots,k\}$$

for $i = 1,2,\ldots,n$. Then $a_{j,i} \leqslant z_i \leqslant b_{j,i}$ for all j and i; for if not, there are two members of $\mathcal{F}$ which are disjoint, contrary to the choice of $\mathcal{F}$. Thus, z belongs to $\cap\mathcal{F}$, from which it follows that $\mathcal{C}$ has Helly number 2.

<u>2.12.5</u>. $\mathcal{C}$ has Radon number $r \leqslant 2n+1$.

<u>Proof</u>. From 2.12.2 it follows that if a point x_0 does not belong to $\mathcal{C}(\{x_1,\ldots,x_k\})$, then there is an integer i

such that $x_{0,i} < \inf \{x_{j,i}: j = 1,2,\ldots,k\}$ or $x_{0,i} > \sup \{x_{j,i}: j = 1,2,\ldots,k\}$. Thus, if $S = \{x_1,\ldots,x_{2n+1}\}$ is a subset of X and no member of S is in the $\mathcal{C}$-hull of the remaining 2n members of S, then for each point x_j of S there is an integer i such that

$$x_{j,i} < \inf \{x_{k,i}: k = 1,2,\ldots,2n+1,\ k \neq j\}$$

or

$$x_{j,i} > \sup \{x_{k,i}: k = 1,2,\ldots,2n+1,\ k \neq j\}.$$

This means that each of the 2n+1 members of S must have a coordinate which is an extreme value relative to the corresponding coordinates of the remaining members of S. But this is impossible since there are only 2n extreme values to be distributed among the 2n+1 points. Hence, for every 2n+1 points in X, one point must be in the $\mathcal{C}$-hull of the remaining 2n points. We conclude that $\mathcal{C}$ has Radon number not greater than 2n+1.

<u>2.12.6</u>. For every positive integer k, there is an integer n such that if $X = E^n$, then $\mathcal{C}$, as defined in 2.12, has Radon number greater than k.

<u>Proof</u>. Let k be a positive integer and let

$$n = \binom{k}{1} + \binom{k}{2} + \ldots + \binom{k}{[k/2]}.$$

Let M be the k by n matrix whose columns are given as follows: the first $\binom{k}{1}$ columns are the characteristic functions of all the one element subsets of $\{1,2,\ldots,k\}$, the next $\binom{k}{2}$ columns are the characteristic functions of all the two element subsets, and in general, the appropriate $\binom{k}{j}$ columns are the characteristic functions of all the j-element subsets of $\{1,2,\ldots,k\}$ for $j = 1,2,\ldots,[k/2]$. For $k = 6$, in which case $n = 41$, this process yields the matrix

$$M = \begin{pmatrix} 100000 & 111110000000000 & 11111111110000000000 \\ 010000 & 100001111000000 & 11110000001111110000 \\ 001000 & 010001000111000 & 10001110001110001110 \\ 000100 & 001000100100110 & 01001001101001101101 \\ 000010 & 000100010010101 & 00100101010101011011 \\ 000001 & 000010001001011 & 00010010110010110111 \end{pmatrix}.$$

Now let S be the k-element subset of X whose coordinates are given by the rows of M. If S_1 and S_2 are non-empty disjoint subsets of S such that $S = S_1 \cup S_2$, then either card $S_1 \leq [k/2]$ or card $S_2 \leq [k/2]$. Without loss of generality, suppose that card $S_1 \leq [k/2]$. There exists an integer i such that the i-th coordinate of each member of S_1 is 1 and the i-th coordinate of each member of S_2 is 0. Thus, the i-th coordinate of each member of $\mathcal{C}(S_1)$ is 1, and the i-th coordinate of each member of $\mathcal{C}(S_2)$ is 0, from which it follows that $\mathcal{C}(S_1) \cap \mathcal{C}(S_2) = \emptyset$. Thus, S is a k-element subset of X which has no Radon partition, so the Radon number of $\mathcal{C}$ must be greater than k.

Combining the results of 2.12.5 and 2.12.6, we know that the Radon number r for $\mathcal{C}$ in $X = E^n$, with $\mathcal{C}$ as defined in 2.12, satisfies the inequality $k < r \leq 2n+1$, where k is any integer such that $\sum_{j=1}^{[k/2]} \binom{k}{j} \leq n$.

<u>2.13. EXAMPLE</u>. Let $X = E^\infty$ and let $\mathcal{C}$ be defined as in 2.12,

with appropriate modifications being made to compensate for the fact that X is infinite dimensional. If k is a positive integer, the technique used in the proof of 2.12.6 can again be used to construct a set S of k points in X such that S has no Radon partition. Thus, $\mathcal{C}$ does not have the Radon property. However, the proof of 2.12.4 is still valid in $X = E^{\infty}$ to show that $\mathcal{C}$ has Helly number 2.

The above examples show that 2.9 is the only theorem possible if we assume only one of the properties under consideration; however, the next theorem shows that the Radon property does follow if both the Helly and Caratheodory properties are assumed.

2.14. THEOREM. If $(X, \mathcal{C})$ is a closure space such that $\mathcal{C}$ has Helly number h and Caratheodory number c $(c \geq 1)$, then $\mathcal{C}$ has Radon number $r \leq hc+1$.

Proof. Let S be an (hc+1)-element subset of X. Let $\mathcal{G}$ consist of all the subsets of S which contain at least (hc+1-c) elements, and let $\mathcal{F}$ be the set of $\mathcal{C}$-hulls of the members of $\mathcal{G}$. Thus, $\mathcal{F}$ is a finite subset of $\mathcal{C}$ containing at least h elements. In order to apply the Helly property and show that $\bigcap\mathcal{F} \neq \emptyset$, it is necessary to show that the intersection of any h-element subset of $\mathcal{F}$ is non-empty. Consequently, let $F_1, F_2, \ldots, F_h$ be members of $\mathcal{F}$, with $G_1, G_2, \ldots, G_h$ being the corresponding members of $\mathcal{G}$. Then,

$$\begin{aligned} \text{card}\ (\bigcap\{G_i: i = 1,2,\ldots,h\}) &= hc+1-\text{card}\ (S \sim (\bigcap\{G_i: i = 1,2,\ldots,h\})) \\ &= hc+1-\text{card}\ (\bigcup\{(S \sim G_i): i = 1,2,\ldots,h\}). \end{aligned}$$

But $hc+1-\text{card}\ (\bigcup\{(S \sim G_i): i = 1,2,\ldots,h\}) \geq hc+1-\sum_{i=1}^{h} \text{card}\ (S \sim G_i)$

$$\geq hc+1-hc$$
$$= 1.$$

Thus,

$$\bigcap\{G_i: i = 1,2,\ldots,h\} \neq \emptyset,$$

and since

$$\bigcap\{G_i: i = 1,2,\ldots,h\} \subset \bigcap\{F_i: i = 1,2,\ldots,h\},$$

then

$$\bigcap\{F_i: i = 1,2,\ldots,h\} \neq \emptyset.$$

Since $\mathcal{C}$ has Helly number h, then $\bigcap\mathcal{F} \neq \emptyset$. Let p be an element of $\bigcap\mathcal{F}$. Since S belongs to $\mathcal{S}$, then $\mathcal{C}(S)$ belongs to $\mathcal{F}$, and consequently, $p \in \mathcal{C}(S)$. Since $\mathcal{C}$ has Caratheodory number c, there is a subset $T = \{x_1, x_2, \ldots, x_n\}$ of S such that $n \leq c$ and p is in the $\mathcal{C}$-hull of T. But card $(S \sim T) \geq hc+1-c$, so $S \sim T$ belongs to $\mathcal{S}$ and $\mathcal{C}(S \sim T)$ belongs to $\mathcal{F}$. Since p is a member of $\bigcap\mathcal{F}$, then p is a member of $\mathcal{C}(S \sim T)$. Thus, $p \in [\mathcal{C}(T) \cap \mathcal{C}(S \sim T)]$, which shows that $(T, S \sim T)$ is a Radon partition of S. Hence, $\mathcal{C}$ has the Radon property with Radon number $r \leq hc+1$.

<u>2.15. COROLLARY</u>. If $(X, \mathcal{C})$ is a closure space and $\mathcal{C}$ has Caratheodory number c, then the Helly and Radon properties are equivalent, and the corresponding numbers satisfy the inequality $h+1 \leq r \leq hc+1$.

Another interesting family of examples is the following: Let $X = E^n$, and for some positive integer k, let

$C \in \mathcal{C}$ if and only if C is convex or card $C \leqslant k$. Then $\mathcal{C}$ has Caratheodory number $c = n+1$, and J. Eckoff [6] proves that if $2 \leqslant n \leqslant k+1$, then $\mathcal{C}$ has Radon number $2k+2$. By 2.9 $\mathcal{C}$ has Helly number $h \leqslant 2k+1$ and by 2.14 we have the inequality $2k+2 \leqslant h(n+1)+1$. If n remains constant and k is allowed to increase, then the Caratheodory number remains constant while the Helly and Radon numbers become large. Such examples also illustrate the fact that $\mathcal{C}$ can have the Caratheodory, Radon, and Helly properties simultaneously, even though some members of $\mathcal{C}$ are not convex in the usual sense.

CHAPTER III

HALF SPACES AND THEIR RELATIONSHIP TO THE CARATHEODORY, RADON, AND HELLY PROPERTIES

The importance of separation theorems in the study of convexity is indicated by the fact that Helly [12] used a separation theorem in the proof of his famous theorem (stated in 1.3) published in 1923. Separation theorems can also be used to prove the Caratheodory and Radon theorems, and in 1941 they were used by Dieudonné [3] to prove the Hahn-Banach theorem on the extension of linear functionals. Since separation theorems occupy a prominent position in the theory of convexity, it would be of interest to know how they are related to the Caratheodory, Radon, and Helly properties in a more general setting. In E^n, some separation theorems make use of hyperplanes while others make use of half-spaces. In this chapter a separation property using half-spaces will be defined and will lead to characterizations of the Caratheodory, Radon, and Helly properties.

3.1. DEFINITION. In a closure space $(X, \mathcal{C})$ a subset S of X is a $\mathcal{C}$-half-space if both S and $X \sim S$ belong to $\mathcal{C}$.

3.2. DEFINITION. Let $(X, \mathcal{C})$ be a closure space. $\mathcal{C}$ has the separation property if for every pair of disjoint members C and D of $\mathcal{C}$, there exist complementary $\mathcal{C}$-half-spaces A and B such that $C \subset A$ and $D \subset B$.

In a linear topological space with the usual definition of convexity, $\mathcal{C}$-half-spaces are convex sets containing an open half-space and contained in a closed half-space, and Kakutani [13] and others prove that the separation property as defined above is satisfied. In proving a generalized separation theorem J. W. Ellis [7] gives five properties which will insure the existence of $\mathcal{C}$-half-spaces and the separation property. Two of these are satisfied by all closure operators, while the other three can be used to impose new conditions on the hull operator $\mathcal{C}$-conv. These more primitive assumptions will be used in chapter four of this paper; however, in this chapter matters are simplified by assuming the existence of the separation property.

The following result (and its proof) is strictly classical when restricted to E^{n-1}. We include the proof for the sake of completeness in the more general setting.

3.3. LEMMA. If $(X, \mathcal{C})$ is a closure space and m is a positive integer, the following statements are equivalent.

(a) $\mathcal{C}$ has Helly number $h \leq m$.

(b) If $\mathcal{F}$ is a finite subfamily of $\mathcal{C}$ such that card $\mathcal{F} \geq m$ and any m members of $\mathcal{F}$ have a non-empty intersection, then any m+1 members of $\mathcal{F}$ have a non-empty intersection.

Proof. (a)$\longrightarrow$(b): Let $\mathcal{F}$ be a finite subfamily of $\mathcal{C}$ such that card $\mathcal{F} \geqslant m$ and any m members of $\mathcal{F}$ have a non-empty intersection. Since $\mathcal{C}$ has Helly number $h \leqslant m$, then $\bigcap \mathcal{F} \neq \emptyset$. Hence, any m+1 members of $\mathcal{F}$ have a non-empty intersection.

(b)$\longrightarrow$(a): Suppose that $\mathcal{C}$ satisfies (b), and let $\mathcal{F}$ be a finite subfamily of $\mathcal{C}$ such that card $\mathcal{F} \geqslant m$ with each intersection of m members of $\mathcal{F}$ non-empty. By (b) any m+1 members of $\mathcal{F}$ have a non-empty intersection. Proceeding by induction, assume that any m+k members of $\mathcal{F}$ have a non-empty intersection, and let $F_1, F_2, \ldots, F_{m+k+1}$ be any m+k+1 members of $\mathcal{F}$. Let $\mathcal{G} = \{G_1, G_2, \ldots, G_{m+1}\}$ where $G_i = F_i$ for $i = 1,2,\ldots,m$ and $G_{m+1} = \bigcap\{F_i: i = m+1,\ldots,m+k+1\}$. $\mathcal{G}$ is a subfamily of $\mathcal{C}$ and the intersection of any m members of $\mathcal{G}$ is identical with the intersection of either m or m+k members of $\mathcal{F}$. In either case the intersection is non-empty, and by applying (b) to $\mathcal{G}$, we have $\bigcap \mathcal{G} \neq \emptyset$. But $\bigcap\{F_i: i = 1,2,\ldots,m+k+1\} = \bigcap \mathcal{G} \neq \emptyset$. This shows that the intersection of any m+k+1 members of $\mathcal{F}$ is non-empty, and by mathematical induction it follows that $\bigcap \mathcal{F} \neq \emptyset$. Hence, the Helly number h of $\mathcal{C}$ exists and $h \leqslant m$.

The usual convex hull of a set S in a linear space is the intersection of all convex sets containing S. However, Hammer [10] proves that the convex hull of S may be obtained by intersecting a much smaller collection of convex sets, namely, the semispaces which contain S. (A semispace at a point p in a linear space is a maximal convex set which does not include p.) Since in linear spaces semispaces are special types of half-spaces, Hammer's theorem can be restated as follows: The convex hull of a set S in a real linear space is the intersection of all the half-spaces which contain S. The next theorem shows that the same is true in any closure space in which $\mathcal{C}$ has the separation property and contains all singletons.

3.4. THEOREM. Let $(X, \mathcal{C})$ be a closure space in which $\mathcal{C}$ has the separation property and contains all 1-element subsets of X. Then $\mathcal{C}(S)$ is the intersection of all $\mathcal{C}$-half-spaces containing S for all subsets S of X.

Proof. Let $S \subset X$ and suppose A is the intersection of all $\mathcal{C}$-half-spaces containing S. Since $\mathcal{C}$-half-spaces are members of $\mathcal{C}$, then for any $\mathcal{C}$-half-space $H \supset S$ we have $\mathcal{C}(S) \subset H$ and therefore $\mathcal{C}(S) \subset A$. Suppose $A \neq \mathcal{C}(S)$, and let x be a point in $A \sim \mathcal{C}(S)$. Since $\{x\}$ and $\mathcal{C}(S)$ are disjoint members of $\mathcal{C}$, there exist complementary $\mathcal{C}$-half-spaces B and C such that $\{x\} \subset B$ and $\mathcal{C}(S) \subset C$. Thus, C is a $\mathcal{C}$-half-space which contains $\mathcal{C}(S)$, and consequently S, but which does not contain x, a member of A. Hence, the contradiction proves $A = \mathcal{C}(S)$, and the proof is complete.

Since $\mathcal{C}$-hulls are intersections of members of $\mathcal{C}$, properties involving $\mathcal{C}$-hulls of sets can be restated in terms of members of $\mathcal{C}$, and with the use of 3.4 these properties can be given in terms of $\mathcal{C}$-half-spaces. The next two theorems restate the Radon and Helly properties in terms of finite point sets and $\mathcal{C}$-half-spaces, making it easier to see their similarities and differences; and after placing an additional restriction on the space $(X, \mathcal{C})$, the Caratheodory property is similarly characterized.

3.5. THEOREM. If $(X, \mathcal{C})$ is a closure space in which $\mathcal{C}$ has the separation property and contains all 1-element subsets of X, then the following statements are equivalent.

(a) $\mathcal{C}$ has Radon number $r \leqslant m$.

(b) If S is an m-element subset of X, there is a non-empty proper subset A of S and a point p in X such that every $\mathcal{C}$-half-space which contains either A or $S \sim A$ also contains p.

(c) If S is an m-element subset of X, there is a non-empty proper subset A of S such that every $\mathcal{C}$-half-space which contains A meets $S \sim A$.

Proof. (a)$\longrightarrow$(b): Assume $\mathcal{C}$ has Radon number $r \leqslant m$, S is an m-element subset of X, (A, B) is a Radon partition of S, and p belongs to the intersection $\mathcal{C}(A) \cap \mathcal{C}(B)$. Then each $\mathcal{C}$-half-space which contains either A or $B = S \sim A$ also contains p.

(b)$\longrightarrow$(c): Let S be an m-element subset of X and let A be the subset of S whose existence is asserted in (b). If there is a $\mathcal{C}$-half-space H which contains A and does not intersect $S \sim A$, then H and $X \sim H$ are disjoint $\mathcal{C}$-half-spaces which contain A and $S \sim A$, respectively. But this contradicts the existence of p in (b).

(c)$\rightarrow$(a): Let S be an m-element subset of X and let A be the non-empty proper subset of S whose existence is asserted in (c). If $\mathcal{C}(A) \cap \mathcal{C}(S \sim A)$ is empty, then by the separation property, there are $\mathcal{C}$-half-spaces H and $X \sim H$ such that $A \subset \mathcal{C}(A) \subset H$ and $(S \sim A) \subset \mathcal{C}(S \sim A) \subset (X \sim H)$. This shows that H is a $\mathcal{C}$-half-space which contains A but is disjoint from $S \sim A$, contradicting (c). Thus, $\mathcal{C}(A) \cap \mathcal{C}(S \sim A) \neq \emptyset$ and $(A, S \sim A)$ is a Radon partition of S. Hence, $\mathcal{C}$ has Radon number $r \leq m$.

<u>3.6. THEOREM</u>. If $(X, \mathcal{C})$ is a closure space in which $\mathcal{C}$ has the separation property and contains all 1-element subsets of X, then the following statements are equivalent.

(a) $\mathcal{C}$ has Helly number $h \leq m$.

(b) If S is an (m+1)-element subset of X, there exists an element p in X such that if C is a $\mathcal{C}$-half-space containing at least m members of S, then C contains p.

<u>Proof</u>. (a)$\rightarrow$(b): Suppose that $\mathcal{C}$ has Helly number $h \leq m$, and let $S = \{x_1, x_2, \ldots, x_{m+1}\}$. Suppose further that $S_i = S \sim \{x_i\}$ and $\mathcal{F} = \{F_i : F_i = \mathcal{C}(S_i)\} \subset \mathcal{C}$. $\mathcal{F}$ contains m+1 elements, and the intersection of any m-element subfamily of $\mathcal{F}$ contains a member of S. Since $\mathcal{C}$ has Helly number

$h \leq m$, then $\bigcap \mathcal{F} \neq \emptyset$. Let $p \in \bigcap \mathcal{F}$, in which case p belongs to the $\mathcal{C}$-hull of each m-element subset of S. Thus, p is in each $\mathcal{C}$-half-space which contains an m-element subset of S.

(b)$\rightarrow$(a): Let $\mathcal{F}$ be a finite subset of $\mathcal{C}$ such that card $\mathcal{F} \geq m$ and any m members of $\mathcal{F}$ have a non-empty intersection. In view of 3.3 it suffices to show that any m+1 members of $\mathcal{F}$ have a non-empty intersection. Let $\mathcal{F}_0 = \{F_1, F_2, \ldots, F_{m+1}\}$ be an (m+1)-element subset of $\mathcal{F}$. If we assume $\bigcap \mathcal{F}_0 = \emptyset$, there is a set of m+1 distinct points, $S = \{x_1, x_2, \ldots, x_{m+1}\}$, such that for each i, x_i belongs to all members of $\mathcal{F}_0$ except F_i, and thus if $i \neq j$, then $x_j \in F_i$. By (b) there is a point p such that any $\mathcal{C}$-half-space containing at least m elements of S also contains p. Since each member of $\mathcal{F}_0$ contains exactly m members of S, then each $\mathcal{C}$-half-space containing a member of $\mathcal{F}_0$ also contains p. Thus, by 3.4 and the fact that $\mathcal{F}_0 \subset \mathcal{C}$, we have $p \in \mathcal{C}(F_i) = F_i$ for $i = 1, 2, \ldots, m+1$, from which it follows that $p \in \bigcap \mathcal{F}_0$, a contradiction. Hence, any m+1 members of $\mathcal{F}$ have a non-empty intersection, from which it follows that $\mathcal{C}$ has Helly number $h \leq m$.

<u>3.7. DEFINITION</u>. The closure space $(X, \mathcal{C})$ is <u>domain</u> <u>finite</u> if for every subset S of X

$$\mathcal{C}(S) = \bigcup \{\mathcal{C}(T) : T \text{ is a finite subset of } S\}.$$

<u>3.8. THEOREM</u>. If m is a positive integer and $(X, \mathcal{C})$ is a

domain finite closure space in which $\mathcal{C}$ has the separation property and contains all 1-element subsets of X, then the following statements are equivalent.

(a) $\mathcal{C}$ has Caratheodory number $c \leq m$.

(b) If S is a subset of X which contains at least m+1 elements and $p \in \mathcal{C}(S)$, there is a proper subset S_1 of S such that every $\mathcal{C}$-half-space which contains S_1 also contains p.

<u>Proof</u>. (a)$\longrightarrow$(b): If $\mathcal{C}$ has Caratheodory number $c \leq m$ and p belongs to $\mathcal{C}(S)$, where S is a subset of X which contains at least m+1 elements, then there is a subset S_1 of S such that S_1 contains at most $c \leq m$ members and p belongs to $\mathcal{C}(S_1)$. Thus, S_1 is a proper subset of S, and every $\mathcal{C}$-half-space which contains S_1 also contains p.

(b)$\longrightarrow$(a): Let T be a subset of X containing at least m+1 elements and with $p \in \mathcal{C}(T)$. Since $(X, \mathcal{C})$ is domain finite, there is a finite subset S_0 of T such that p belongs to $\mathcal{C}(S_0)$. If card $S_0 > m$, by (b) there is a subset S_1 of S_0 such that card $S_1 <$ card S_0 and every $\mathcal{C}$-half-space containing S_1 also contains p. Thus, by 3.4, p belongs to $\mathcal{C}(S_1)$. If card $S_1 > m$, there is a subset S_2 of S_1 such that card $S_2 <$ card S_1 and $p \in \mathcal{C}(S_2)$. By applying (b) and (3.4) repeatedly, we can construct a chain

$$S_0 \supset S_1 \supset S_2 \supset \cdots \supset S_k,$$

where $0 \leq k \leq$ (card S_0)-m, such that S_{i+1} is a proper subset of S_i for $i = 0,1,2,\ldots,k-1$, $p \in \mathcal{C}(S_i)$ for $i = 0,1,\ldots,k$, and card $S_k \leq m$. Thus, $\mathcal{C}$ has Caratheodory number $c \leq m$.

By using 3.5 and 3.6, it is possible to give a new proof that the Helly property follows from the Radon property. The setting is more general than that used for the classical proof of the Helly theorem from the Radon theorem but less general than the setting used by Levi.

3.9. THEOREM. Let $(X, \mathcal{C})$ be a closure space such that $\mathcal{C}$ has the separation property and contains all 1-element subsets of X. If $\mathcal{C}$ has Radon number r, then $\mathcal{C}$ has Helly number $h \leq r-1$.

Proof. Let S be an r-element subset of X. Since $\mathcal{C}$ has Radon number r, then by 3.5 (b) there is a proper subset A of S and a point p in X such that each $\mathcal{C}$-half-space which contains either A or $S \sim A$ also contains p. Since each $\mathcal{C}$-half-space containing r-1 members of S must contain either A or $S \sim A$, then each $\mathcal{C}$-half-space containing r-1 members of S must also contain p. By 3.6 $\mathcal{C}$ has Helly number $h \leq r-1$.

The similarity of the (b) statements in 3.5, 3.6, and 3.8 indicate that the Caratheodory, Radon, and Helly properties are closely related in spaces which satisfy the hypotheses of those theorems and that conditions can be imposed to give additional implications among these properties. Some results of this type are given below and are based on the following observations.

(a) In E^n, for $n = 1,2,3$, there is an integer k such that if S is a subset which contains at least k elements, there is a subset A of S such that card $A \leq 2$ and $(\text{conv } A) \cap (\text{conv } (S \sim A)) \neq \emptyset$.

(b) In a linear space, if card $A \leq 2$, $A \subset S$, and $(\text{conv } A) \cap (\text{conv } (S \sim A)) \neq \emptyset$, then

$$\text{conv } S = \bigcup\{\text{conv } ((S \sim A) \cup \{x\}): x \in A\}.$$

(c) In E^n, for $n = 1,2,3$, if S is a finite subset which has a Radon partition and $p \in \text{conv } S$, then S has a Radon partition (S_1, S_2) such that

$\bigcap\{C\colon C \text{ is a half-space, } S_1 \subset C, \text{ and } p \notin C\} \cap S_2 \neq \emptyset$.

Observation (a) is a direct consequence of the Radon theorem, (b) is proved by an algebraic argument, and (c) follows from (a) and (b). The previously unpublished theorem of B. J. Birch mentioned on page 118 of [4] shows that (a) is not true for $n > 3$.

<u>3.10. THEOREM</u>. Let $(X, \mathcal{C})$ be a domain finite closure space in which $\mathcal{C}$ has the separation property, contains all 1-element subsets, and satisfies the following additional hypothesis (property (c) above): If $p \in \mathcal{C}(S)$, where S is a finite point set which has a Radon partition, there exists a Radon partition (S_1, S_2) of S such that

$\bigcap\{C\colon C \text{ is a } \mathcal{C}\text{-half-space, } S_1 \subset C, \text{ and } p \notin C\} \cap S_2 \neq \emptyset$.

Then, if $\mathcal{C}$ has Radon number r, it has Caratheodory number c and $c \leq r-1$.

Proof. Let R be a subset of X and $p \in \mathcal{C}(R)$. Since $(X, \mathcal{C})$ is domain finite, there is a minimal finite subset S of R such that $p \in \mathcal{C}(S)$. If card $S \geqslant r$, then S has a Radon partition; so by hypothesis S has a Radon partition (S_1, S_2) for which there is a point

$s \in \bigcap\{C\colon C \text{ is a } \mathcal{C}\text{-half-space}, S_1 \subset C, \text{ and } p \notin C\} \cap S_2$.

If C is a $\mathcal{C}$-half-space containing $S \sim \{s\}$, then $S_1 \subset C$, and from the choice of s it follows that $p \in C$. Thus, $p \in \mathcal{C}(S \sim \{s\})$, contradicting the minimality of S. Therefore, card $S < r$, and $\mathcal{C}$ has Caratheodory number $c \leqslant r-1$.

3.11. THEOREM. Let $(X, \mathcal{C})$ be a domain finite closure space which contains all 1-element subsets and satisfies the following two conditions (properties (a) and (b) above):

(a) There exists a positive integer k such that if S contains at least k members of X, there is a subset A of S such that card $A \leqslant 2$ and $\mathcal{C}(A) \cap \mathcal{C}(S \sim A) \neq \emptyset$.

(b) If A and B are subsets of X such that card $A \leqslant 2$ and $\mathcal{C}(A) \cap \mathcal{C}(B) \neq \emptyset$, then

$$\mathcal{C}(A \cup B) = \bigcup\{\mathcal{C}(B \cup \{x\})\colon x \in A\}.$$

Then $\mathcal{C}$ has Caratheodory number $c \leqslant k-1$, Radon number $r \leqslant k$, and Helly number $h \leqslant k-1$.

Proof. Let T be a subset of X. If p belongs to $\mathcal{C}(T)$, then by the domain finiteness there is a smallest finite subset S of T such that $p \in \mathcal{C}(S)$. If card $S \geqslant k$, let A be the subset of S which exists by (a) and let $B = S \sim A$. If card A = 1, then $A \subset \mathcal{C}(B)$ and $p \in \mathcal{C}(S) = \mathcal{C}(B)$, contradicting the choice of S. If card A = 2 and $A = \{x,y\}$, then by (b), p belongs to either $\mathcal{C}(B \cup \{x\})$ or $\mathcal{C}(B \cup \{y\})$. Hence, p belongs to the $\mathcal{C}$-hull of a set whose cardinality is less than that of S. This again contradicts the choice of S; hence, card $S \leq k-1$, which proves the existence of the Caratheodory number c with $c \leq k-1$. Hypothesis (a) clearly implies that $\mathcal{C}$ has Radon number $r \leq k$, and the existence of the Helly number h with $h \leq r-1 \leq k-1$ follows immediately from 2.9.

In the process of formulating hypotheses for 3.10 and 3.11 which would be consistent with the usual convexity in familiar spaces, a characterization of the existence of a Radon partition of a set of points in a linear space was found. The Radon theorem says that a set of at least n+2 points in E^n has a Radon partition; however, a set having fewer than n+2 points may also have a Radon partition. The following result gives a necessary and sufficient condition, which depends on neither the number of points nor the dimension of the space, for a finite set of points in a linear space to have a Radon partition. It will be noted that the condition is closely related to the Caratheodory property.

<u>3.12. THEOREM</u>. A finite set $S = \{x_1, x_2, \ldots, x_k\}$ in a linear space has a Radon partition if and only if there is a point p such that for some j, $1 \leq j \leq k$,

$$p = \sum_{i=1}^{k} a_i x_i = \sum_{\substack{i=1 \\ i \neq j}}^{k} b_i x_i,$$

where $\sum_{i=1}^{k} a_i = 1 = \sum_{\substack{i=1 \\ i \neq j}}^{k} b_i$ and for all i, $a_i > 0$ and $b_i \geqslant 0$.

<u>Proof</u>. Suppose $S = \{x_1, x_2, \ldots, x_k\}$ has a Radon partition. Let $p = \sum_{i=1}^{k} \frac{1}{k} x_i$. Since S has a Radon partition, there are real numbers $b_1, b_2, \ldots, b_k$ not all zero but with sum zero such that $\sum_{i=1}^{k} b_i x_i = 0$. Let $W = \{i: b_i < 0\}$ and let $j \in W$ such that $\frac{1/k}{b_j} \geqslant \frac{1/k}{b_i}$ for all $i \in W$. Then

$$p = \sum_{i=1}^{k} \left[\frac{1}{k} - \left(\frac{1/k}{b_j}\right) b_i\right] x_i,$$

where all the coefficients are non-negative and sum to one. But the coefficient of x_j is zero, so we have

$$p = \sum_{i=1}^{k} \frac{1}{k} x_i = \sum_{\substack{i=1 \\ i \neq j}}^{k} \left[\frac{1}{k} - \left(\frac{1/k}{b_j}\right) b_i\right] x_i$$

with the coefficients satisfying the conditions required by the theorem.

Conversely, suppose there exists a point p as given in the theorem. Then

$$c_j x_j + \sum_{\substack{i=1 \\ i \neq j}}^{k} c_i x_i = 0,$$

where $c_i = a_i - b_i$ for $i \neq j$ and $c_j = a_j$. Let $U = \{i: c_i < 0\}$ and $V = \{i: c_i \geqslant 0\}$. Since $\sum_{i=1}^{k} c_i = 0$ and $a_j > 0$, then both U and V are non-empty and $\sum_{i \varepsilon U} (-c_i) = \sum_{i \varepsilon V} c_i \neq 0$. Let $M = \sum_{i \varepsilon V} c_i$. Then

$$\sum_{i \varepsilon U} \left(\frac{-c_i}{M}\right) x_i = \sum_{i \varepsilon V} \left(\frac{c_i}{M}\right) x_i,$$

from which we conclude that (S_1, S_2), with $S_1 = \{x_i \varepsilon S: i \varepsilon U\}$ and $S_2 = \{x_i \varepsilon S: i \varepsilon V\}$, is a Radon partition of S.

CHAPTER IV

CONVEXITY STRUCTURES IN E^n

In chapters II and III relationships between various properties were investigated in the abstract setting of closure spaces. In this chapter, however, the setting is specialized to n-dimensional euclidean space and the effect of additional conditions on $\mathcal{C}$ is examined. After some preliminary results this leads to a characterization of the collection of all convex sets. One result in this direction was obtained by Dvoretsky [5] and provides the following converse of Helly's theorem:

4.1. THEOREM. Let $X = E^n$ and let $\mathcal{C}$ be a family of subsets of X such that

(a) each member of $\mathcal{C}$ is either an n-dimensional compact subset of X or is affinely equivalent to such a set, and

(b) $\mathcal{C}$ has Helly number n+1.

Then all members of $\mathcal{C}$ are convex.

In the above theorem, $\mathcal{C}$ is not necessarily closed under intersections and may not generate a $\mathcal{C}$-hull for subsets of X. However, the theorem is of interest since it uses the Helly property as one of the hypotheses. In the results which follow, $\mathcal{C}$ does generate a $\mathcal{C}$-hull for subsets of X, and a separation property, rather than the Helly property, is used.

4.2. DEFINITION. If $(E^n, \mathcal{C})$ is a closure space, then $\mathcal{C}$ is closed under isometries if for every C in $\mathcal{C}$ and every isometry α of E^n onto itself, the image of C under α, denoted by $\alpha(C)$, belongs to $\mathcal{C}$.

4.3. DEFINITION. A flat is a translate of a linear subspace in E^n whose dimension is the dimension of the subspace. (The dimension of a flat F is denoted by dim F.) In particular, a 1-dimensional flat is a line. The symbol extv S denotes the minimal flat containing a set S; however, extv $\{x,y\}$ for $x \neq y$ is a line and will be denoted simply by $L(x,y)$. A hyperplane in E^n is an (n-1)-dimensional flat.

Topological terms, with reference to the usual topology, will be used to simplify the description of certain sets. For example, if H is a hyperplane in E^n, then each component of $E^n \sim H$ is an open half-space whereas an open half-space together with its bounding hyperplane is a closed half-space. A half-space is any set which contains an open half-space and is contained in a closed half-space. These terms are used with reference to the classical structure in E^n whereas the term $\mathcal{C}$-half-space will continue to be used in accordance with definition 3.1.

4.4. DEFINITION. If F and G are flats in E^n with F a subset of G and 1+dim F = dim G, then F is a <u>hyperplane of</u> G.

<u>4.5. DEFINITION</u>. If A and B are sets in E^n, then A and B can be <u>weakly separated in</u> F = extv $(A \cup B)$ if there is a hyperplane H of F such that if C and D are the open half-spaces of F determined by H then $A \subset (H \cup C)$ and $B \subset (H \cup D)$.

<u>4.6. LEMMA</u>. Let $(E^n, \mathcal{C})$ be a closure space in which $\mathcal{C}$ is closed under isometries. If $C \in \mathcal{C}$, x and y (with $x \neq y$) belong to C, and there exists a hyperplane H of extv C such that x and y belong to H and C is contained in one of the closed half-spaces of extv C determined by H, then $\mathcal{C}(\{x,y\})$ is a subset of $L(x,y)$.

<u>Proof</u>. Let x, y, C, and H satisfy the hypotheses of the lemma. Since C is contained in one of the closed half-spaces of extv C determined by H, then for every z which does not belong to $L(x,y)$ there is an isometry α of E^n onto itself such that $\alpha(L(x,y)) = L(x,y)$ and $z \notin \alpha(C)$. (The isometry α is explicitly determined in the next paragraph.) Since $\mathcal{C}$ is closed under isometries, and $z \notin L(x,y)$ implies $z \notin \alpha(C)$ for some isometry α, then

$$\mathcal{C}(\{x,y\}) \subset \bigcap\{\alpha(C) : \alpha(L(x,y)) = L(x,y)\} \subset L(x,y).$$

Thus, $\mathcal{C}(\{x,y\})$ is a subset of $L(x,y)$.

In determining the isometry α for the above proof, we can assume that E^n = extv C, since any isometry on extv C can be extended to an isometry on E^n. If $z \notin C$, α may be taken as the identity map; and if $z \in C \sim H$, α may be taken as the reflection in H. If $z \in (C \cap H) \sim L(x,y)$, let u be the point in $L(x,y)$ such that $L(z,u)$ is orthogonal to $L(x,y)$. Choose v to be the point in the open half-space which is determined by the hyperplane H and which does not contain C such that $L(v,u)$ is orthogonal to both $L(x,y)$ and $L(z,u)$ and the euclidean distances $d(z,u)$ and $d(v,u)$ are equal. Then let w be the midpoint of the segment zv, and let G be the hyperplane through w orthogonal to $L(z,w)$. We choose α to be the reflection in G. Since $\alpha(v) = z$ and $v \notin C$, then $z \notin \alpha(C)$; and since $L(x,y) \subset G$, then $\alpha(L(x,y)) = L(x,y)$.

<u>4.7. THEOREM</u>. Let $(E^n, \mathcal{C})$ be a closure space with the following properties.

(a) $\mathcal{C}$ is closed under isometries.

(b) If x is a point and C is a non-empty member of $\mathcal{C}$ such that $x \notin C$, then $\{x\}$ and C can be weakly separated in extv $(\{x\} \cup C)$.

(c) $\mathcal{C}$ contains all 1-element subsets of E^n.

Then all members of $\mathcal{C}$ are convex.

<u>Proof</u>. Let C be a member of $\mathcal{C}$, suppose that C is not convex, let x and y be elements of C such that xy is not contained in C, and let z belong to $xy \sim C$. Since $z \in$ extv C, then $\{z\}$ and C can be weakly separated in extv C by a hyperplane H of extv C; let F and G be the open half-spaces of extv C determined by H such that $F \cup H$ and $G \cup H$ contain C and $\{z\}$ respectively. Since $F \cup H$ is convex and $\{x,y\} \subset C \subset F \cup H$, then $z \in xy \subset F \cup H$. But $\{z\} \subset G \cup H$

and $(F \cup H) \cap (G \cup H) = H$, from which it follows that z is in H. Since xy is a subset of $F \cup H$ and z is in the intersection of xy and H, then xy is a subset of H from the simple linearity properties of lines and hyperplanes in extv C. The hypotheses of lemma 4.6 are satisfied; hence, $\mathcal{C}(\{x,y\})$ is a subset of $L(x,y)$. Since $\{z\}$ and $\mathcal{C}(\{x,y\})$ are members of $\mathcal{C}$ and

$$\text{extv } (\{z\} \cup \mathcal{C}(\{x,y\})) = L(x,y),$$

there exists a hyperplane (point) of $L(x,y)$, say $w \neq z$, which weakly separates $\{z\}$ and $\mathcal{C}(\{x,y\})$. Thus, w simultaneously belongs to xz and yz, which is impossible since z is in the relative interior of xy. Therefore, C is convex.

<u>4.8. REMARK</u>. The preceding theorem gives conditions under which all the members of $\mathcal{C}$ are convex in E^n. However, $\mathcal{C}$ may not contain all the convex sets. In particular, since it is known that the semispaces form an intersection basis for convex sets in E^n and $\mathcal{C}$ is closed under isometries, <u>either $\mathcal{C}$ contains no semispace or $\mathcal{C}$ contains all the convex sets</u>. Examples in which $\mathcal{C}$ contains some, but not all, convex sets are given in (c) and (d) following 2.2.

We state a simple corollary of theorem 4.7 which proves the existence of finite Radon and Helly numbers when $\mathcal{C}$ has the properties as stated in the theorem.

<u>4.9. COROLLARY</u>. Under the hypotheses of theorem 4.7, $\mathcal{C}$

has Radon number $r \leq n+2$ and Helly number $h \leq n+1$.

Proof. Let S be a set of at least n+2 points in E^n. By the Radon theorem, there are disjoint, complementary subsets S_1 and S_2 of S such that conv $S_1 \cap$ conv $S_2 \neq \emptyset$. Since all members of $\mathcal{C}$ are convex, the $\mathcal{C}$-hull of a set S contains the convex hull of S. Thus $\emptyset \neq ((\text{conv } S_1) \cap (\text{conv } S_2)) \subset (\mathcal{C}(S_1) \cap \mathcal{C}(S_2))$, from which it follows that (S_1, S_2) is a Radon partition of S with respect to $\mathcal{C}$. This proves that $\mathcal{C}$ has Radon number $r \leq n+2$. The existence of the Helly number $h \leq n+1$ follows from 2.9.

In the following theorem, additional restrictions are placed on $\mathcal{C}$ in order to insure that $\mathcal{C}$ contains all convex sets. Then, by observing that the collection of all convex sets in E^n satisfies the conditions on $\mathcal{C}$, we conclude that the theorem gives a characterization of the collection of convex sets in E^n. The added hypotheses are those suggested by Ellis [7] in order to assure the existence of $\mathcal{C}$-half-spaces. First, we need a definition.

4.10. DEFINITION. If $(X, \mathcal{C})$ is a closure space, S is a subset of X, and p is a member of X, then the $\mathcal{C}$-join of p and S is the set $\cup\{\mathcal{C}(\{x,p\}) : x \in \mathcal{C}(S)\}$, denoted by $\mathcal{C}$-jn pS.

4.11. THEOREM. Let $(E^n, \mathcal{C})$ be a closure space with the following properties.

(a) $\mathcal{C}$ is closed under isometries.

(b) If x is a point and C is a non-empty member of $\mathcal{C}$ such that if $x \notin C$, then $\{x\}$ and C can be weakly separated in extv $(\{x\} \cup C)$.

(c) $\mathcal{C}$ contains all 1-element subsets of E^n.

(d) $(E^n, \mathcal{C})$ is domain finite.

(e) If F is a finite subset of E^n and p is a point in E^n, then $\mathcal{C}(F \cup \{p\}) \subset \mathcal{C}$-jn pF.

(f) If a belongs to $\mathcal{C}(\{b,p\})$ and c belongs to $\mathcal{C}(\{d,p\})$, then $\mathcal{C}(\{a,d\})$ and $\mathcal{C}(\{b,c\})$ have a non-empty intersection.

Then $\mathcal{C}$ is precisely the family of convex subsets of E^n.

Proof. By 4.7, all members of $\mathcal{C}$ are convex. Conversely, suppose there is a convex set C in E^n such that C is not in $\mathcal{C}$. Then C is a proper subset of $\mathcal{C}(C)$. Since $(E^n, \mathcal{C})$ is domain finite, for each x in $\mathcal{C}(C)$ there exists a finite set of points $\{x_1,x_2,\ldots,x_k\}$ such that x belongs to $\mathcal{C}(\{x_1,x_2,\ldots,x_k\})$. For each x in $\mathcal{C}(C)$ let $n(x)$ be the least integer such that x is in the $\mathcal{C}$-hull of $n(x)$ members of C. Let

$$m = \min \{n(x) : x \in (\mathcal{C}(C) \sim C)\}.$$

Since all 1-element subsets are members of $\mathcal{C}$, it follows that $m \geqslant 2$. Let q be a member of $\mathcal{C}(C) \sim C$ and $S = \{d_1,d_2,\ldots,d_m\}$ be a subset of C such that S contains m elements and q belongs to $\mathcal{C}(S)$. Let $F = S \sim \{d_m\}$. By the choice of m, it follows that $\mathcal{C}(F)$ is a subset of C. If $m > 2$, then $\mathcal{C}(\{a,d_m\})$ is a subset of C for all a in

$\mathcal{C}(F)$, and consequently, $\mathcal{C}$-jn d_mF is a subset of C. Thus, q belongs to $\mathcal{C}(F \cup \{d_m\})$ but does not belong to $\mathcal{C}$-jn d_mF, which contradicts hypothesis (e). We conclude that $m = 2$ and that q belongs to $\mathcal{C}(\{d_1,d_2\}) \sim C$.

Now, we shall show that $\mathcal{C}(\{d_1,d_2\})$ is a subset of $L(d_1,d_2)$. If $n = 1$ the assertion is trivial, for $\mathcal{C}(\{d_1,d_2\}) \subset E^1 = L(d_1,d_2)$. For $n \geqslant 2$ the theorem of Ellis [7] can be applied to two disjoint members of $\mathcal{C}$ in order to show the existence of non-empty complementary $\mathcal{C}$-half-spaces C_1 and C_2. Since all members of $\mathcal{C}$ are convex, C_1 and C_2 are complementary convex sets in E^n and, thus, must contain an open half-space and be contained in a closed half-space. If $n = 2$, one of the sets C_1 or C_2 must be either a closed half-space or a semispace. If C_1 is a semispace, then by remark 4.8, $\mathcal{C}$ contains all convex sets and $\mathcal{C}(\{d_1,d_2\}) =$ conv $\{d_1,d_2\} \subset L(d_1,d_2)$. If C_1 is a closed half-space, we can assume, by hypothesis (a), that d_1 and d_2 lie in the line bounding C_1. Then, by lemma 4.6, since $\{d_1,d_2\} \subset C_1 \varepsilon \mathcal{C}$, $\mathcal{C}(\{d_1,d_2\}) \subset L(d_1,d_2)$. If C_2 is a semispace or a closed half-space, the results are the same. If $n > 2$, the boundary of either C_1 or C_2 must contain a line. Again by hypothesis (a), we can assume that $L(d_1,d_2)$ is a subset of the boundary of either C_1 or C_2, from which it follows by lemma 4.6, and thus in all cases, that $\mathcal{C}(\{d_1,d_2\})$ is a subset of $L(d_1,d_2)$.

Since q belongs to $\mathcal{C}(\{d_1,d_2\})$, q belongs to $L(d_1,d_2)$. Suppose $q \notin d_1d_2$, in which case either d_1 is in qd_2 or d_2 is in qd_1. If d_1 is in qd_2, then by the convexity of members of $\mathcal{C}$, d_1 is also in $\mathcal{C}(\{q,d_2\})$. But $\mathcal{C}(\{q,q\}) \cap \mathcal{C}(\{d_1,d_1\}) = \{q\} \cap \{d_1\} = \emptyset$, which contradicts hypothesis (f). If d_2 is in qd_1, we arrive at a similar contradiction. Thus, $q \in d_1d_2 \subset C$, a contradiction. Therefore, $\mathcal{C}$ must contain all convex sets, and the proof is complete.

As noted earlier, the hypotheses of theorem 4.11 are consistent since they are satisfied when $\mathcal{C}$ is the collection of all convex sets in E^n. The following examples show that hypotheses (a) through (f) in 4.11 are independent, with each example satisfying all the hypotheses except the one having the same label as the example. The properties of each example will be verified only in the more difficult cases.

(a) Let $X = E^2$. Let $\mathcal{C}$ consist of all semispaces with boundaries parallel to one of the coordinate axes together with all intersections of such sets. Hypothesis (a) is not satisfied since $\mathcal{C}$ is not closed under rotations; (b) is satisfied since all members of $\mathcal{C}$ are convex; and (c) follows from the fact that complements of semispaces can be expressed as intersections of semispaces and thus are in $\mathcal{C}$. If x is in the interior of the $\mathcal{C}$-hull of some set S, then x is in the $\mathcal{C}$-hull of at most four members of S; if x is in the boundary of $\mathcal{C}(S)$, then x is in the $\mathcal{C}$-hull of at most two members of S. Thus, $(X, \mathcal{C})$ is domain finite. Hypotheses (e) and (f) are proved by considering several cases determined by the position of points in the plane.

(b) Let $X = E^n$ and let $\mathcal{C}$ be the collection of all subsets of X. Hypothesis (b) is not satisfied, but the remaining hypotheses can be verified using the fact that $\mathcal{C}(S) = S$ for all subsets S in X.

(c) Let $X = E^n$ and let $\mathcal{C} = \{E^n, \emptyset\}$. Hypothesis (b) is vacuously satisfied, (c) is not satisfied, and the remaining hypotheses follow from the fact that if S is a non-empty subset of X, $\mathcal{C}(S) = X$.

(d) Let $X = E^1$ and let $\mathcal{C}$ be the closed convex sets in E^1. The space $(X, \mathcal{C})$ is not domain finite since 0 is in $\mathcal{C}(\{\frac{1}{n}: n = 1,2,3,\ldots\})$ but is not in the $\mathcal{C}$-hull of any finite subset. Hypothesis (b) is satisfied since all members of $\mathcal{C}$ are convex; hypotheses (e) and (f) follow from the fact that the closed convex hull of a finite set is identical with the convex hull. (This example may be extended easily to E^n for $n > 1$.)

(e) Let $X = E^2$ and let $\mathcal{C} = \{X, \emptyset\} \cup \{C: C$ is a singleton or C is a 1-dimensional convex set in $E^2\}$. If F consists of two points and p is a point which is not collinear with those in F, then $\mathcal{C}(F \cup \{p\}) = E^2$ and $\mathcal{C}$-jn $pF =$ conv $(F \cup \{p\})$. Thus, hypothesis (e) is not satisfied. Hypothesis (f) is satisfied since the $\mathcal{C}$-hull of two points coincides with the ordinary convex hull of the points. $\mathcal{C}$ has Caratheodory number 3; so $(X, \mathcal{C})$ is domain finite. (This example may also be extended to E^n for $n > 2$.)

(f) Let $X = E^1$ and let $\mathcal{C} = \{X, \emptyset\} \cup \{C: C$ is a singleton}. Hypotheses (a), (b), and (c) are clearly satisfied, (d) follows from the fact that $\mathcal{C}$ has Caratheodory number 2, and (e) follows from the fact that a $\mathcal{C}$-hull is $\emptyset$, X, or a singleton. However, (f) is not satisfied if we let $a = d = 0$, $p = 1$, and $b = c = 2$.

In closing, perhaps one other comment concerning the hypotheses of theorem 4.11 is in order. Hypothesis (c) determines the $\mathcal{C}$-hulls of singletons, (f) places restrictions on the $\mathcal{C}$-hulls of 2-element subsets, (e) uses $\mathcal{C}$-hulls of 2-element subsets to control $\mathcal{C}$-hulls of finite subsets, and (d) expresses $\mathcal{C}$-hulls of infinite subsets in terms of $\mathcal{C}$-hulls of finite subsets.

APPENDIX

SYMBOLS USED IN THE TEXT

Symbol	Meaning
$\emptyset$	Empty set
E^n	n-dimensional euclidean space
$x \in S$	x is an element of S
$A \subset B$	A is a subset of B
$\bigcap \mathcal{F}$	$\bigcap \{F: F \in \mathcal{F}\}$
$\mathcal{C}(S)$	$\bigcap \{C \in \mathcal{C}: S \subset C\}$
card S	Cardinality of S
[ ]	Greatest integer function
$A \sim B$	Set of elements in A but not in B
$A \cup B$	Union of A and B
$A \cap B$	Intersection of A and B
conv S	Convex hull of a set S
dim S	Dimension of a set S
extv S	Linear extension of a set S
xy	Closed line segment with endpoints x and y
x_i	i-th coordinate of a point x in E^n
L(x,y)	Line containing distinct points x and y
$\mathcal{C}$-jn pS	$\mathcal{C}$-join of a point p and a set S; that is, the set $\bigcup \{\mathcal{C}(\{x,p\}): x \in \mathcal{C}(S)\}$

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