INTRODUCTION TO CONVEX PROGRAMMING

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PREFACE

This dissertation deals with a certain class of nonlinear programming problems known as convex programming problems. The purpose of this paper is to provide an introduction to the theory of convex programming. Mathematical programming in general has risen to a place of importance in the last two decades, beginning with linear programming in the late 1940's and early 1950's and continuing today with nonlinear programming. Much interest and activity has been devoted to nonlinear programming in recent years but because of the very diverse nature of the topic, almost all energy has been directed toward the creation of algorithms for the solution of particular types of problems. This is particularly true also in convex programming.

Not readily available in the literature is a source which gives the theoretical foundation for convex programming in a complete and concise form and specifically formulates the discussion in terms of convex programming. For example, Rockafellar [20] develops the theory of vectors of recession for a convex set and a convex function but does not single out or emphasize that theory which is applicable to convex programming; rather he formulates his results in terms of when a convex function attains its minimum on a convex set. In the section he devotes to convex programming, almost the entire text is in terms of Kuhn-Tucker vectors which contributes nothing to the fundamental theory of convex programming. Karlin [12] talks about concave programming where the objective functions are concave and the goal is to maximize a concave function over the set of feasible solutions. Kuhn and Tucker [15] also formulate their discussion in terms of concave programming and limit themselves to convex programs over E_n^+, the non-negative orthant of E_n. In [16], Kunzi and Krelle characterize an optimal solution for a convex program in terms of the Lagrangian function associated with the convex program but only for a convex program over E_n^+. No mention is made of the more general result that appears in Chapter III of this result. Therefore, it is the express purpose of this paper to fill this gap in the fundamental theory of convex programming.

The desired audience for this paper is any person, involved in mathematical programming or not, who has an adequate background in the undergraduate mathematics courses of linear algebra, elementary topology and analysis (particularly, advanced calculus), and calculus of several variables. The reader should be familar with such concepts as n-dimensional Euclidean space E_n, the inner-product property of this space, basis, subspace, and such topological properties of sets as open, closed, bounded, and compact. The only notation that might be unfamilar is the symbol $A \setminus B$ which means the set of all points in A which are not in B. Also, if x and y are vectors in E_n, then $x \cdot y$ represents the inner-product of x and y.

Chapter I is concerned with the basic definitions of a convex set and a convex function and with the development of the concept of a vector of recession for a convex set and a convex function. In Chapter II, the ideas formulated in Chapter I are used to discuss the feasibility and solvability of convex programs. Also, the relationship of feasible and optimal solutions is investigated. The characterization of optimal solutions for convex programs is the goal of Chapter III. The results in this chapter characterize an optimal solution for a convex program in terms of the Lagrangian function associated with the convex program. This characterization is first given for an arbitrary convex program, and then for differentiable convex programs, and finally for convex programs over the nonnegative orthant E_n^+ of E_n.

The last chapter, Chapter IV, presents a discussion of three algorithms that have been developed for the solution of particular convex programs. The Method of Feasible Directions has had relatively good success in solving certain convex programs through the use of computers according to Dorn [2]. The Cutting Plane Method is more theoretical than applicable but is one of the more well-known convex programming algorithms. The recently developed Sequential Unconstrained Minimization Technique has been very successful in solving convex programs by means of computer.

Finally, all functions discussed in this paper are considered continuous on their domains of definition. If no domain of definition is specified, then the function is considered continuous on E_n.

The author would like to take this opportunity to express his gratitude to Professor E. K. McLachlan, whose guidance and direction were instrumental in the completion of his graduate program, and to Professors Forrest D. Whitfield, Robert T. Alciatore, and W. Ware Marsden for their assistance while serving on his committee.

TABLE OF CONTENTS

CHAPTER I

SOME PROPERTIES OF CONVEX SETS AND CONVEX FUNCTIONS

Introduction

Historically, scientists and economists have been interested in optimization problems which seek to maximize or minimize a numerical function of a number of variables. Optimization problems occurred primarily in the physical sciences and geometry, and the classical optimization techniques, such as differential calculus of variations, were developed to deal with them. However, a new class of optimization problems has received considerable attention in the last two decades, which involve the optimum allocation of limited resources. These are mathematical programming problems which seek to determine values for a specified set of variables which optimize (maximize or minimize) a numerical function of the variables, called the objective function, subject to various constraint relations, called the constraint functions, which are also numerical functions of the variables. A solution to such a mathematical programming problem is a program of action, or a strategy, which is optimal with respect to the imposed limitations given in terms of the constraint functions.

When the objective function and each of the constraint functions are linear functions, the problem is a linear programming problem. This type of problem received a great surge of activity and interest in the late 1940's and early 1950's. However, not all mathematical programming problems that arise are linear. If one or more of the constraint functions, or the objective function, is a nonlinear function, then the problem is called a nonlinear programming problem. Much effort has been devoted recently to the solution of nonlinear programs, but because of the great difficulty of optimizing nonlinear functions in general, emphasis has turned toward the solutions of particular types of nonlinear programming problems. One type of nonlinear programming problem which has received much attention in the last decade is the convex programming problem which is discussed in this paper.

A set C in n-dimensional Euclidean Space E_n is a convex set if and only if for any real number $\lambda \in [0,1]$ and any two points x and y in C, $\lambda x + (1 - \lambda)y$ is in C. Hence a convex set has the property that it always contains the line segment joining any two of its points. A function f defined on a nonempty convex set C is a convex function if and only if for every real number $\lambda \in [0,1]$ and every two points x and y in C,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

The function f is strictly convex if strict inequality holds in the above expression when $\lambda \in [0,1]$ and x is unequal to y. Examples of a convex and a nonconvex set are in Figure 1(a) and 1(b), respectively; examples of a convex, strictly convex, and nonconvex function are found in Figure 1(c), 1(d), and 1(e), respectively.

A convex program, denoted by (P), is comprised of three specific parts: a convex set C in E_n which is its domain; a convex function F defined on C called the objective function for (P); and a finite set of convex functions $\{f_i: i = 1, . . . ,m\}$, each defined on C, called

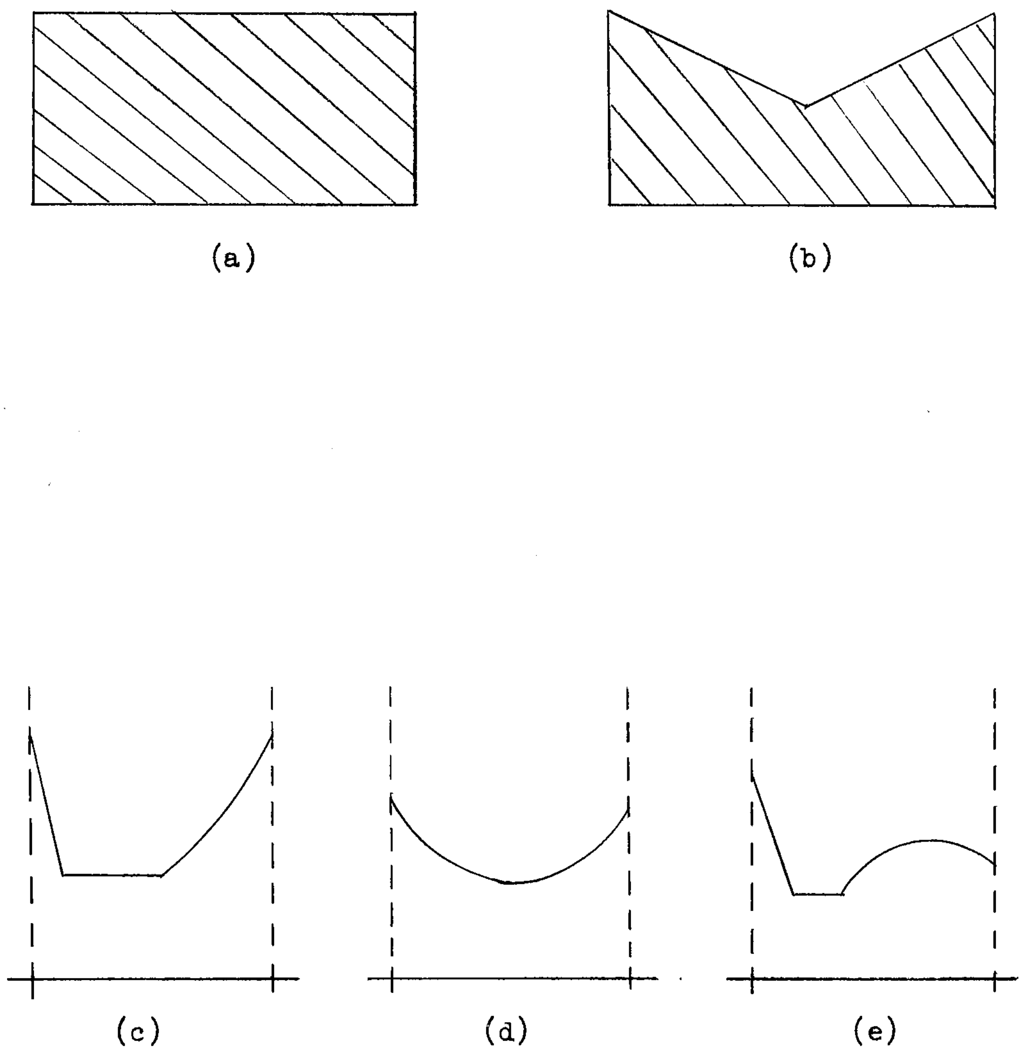


Figure 1.

the constraint functions for (P). Given any convex program, the convex programming problem is to minimize the objective function F over those points in the domain C which satisfy the constraint relations given by $f_i(x) \leq 0$ for $i = 1, \ldots, m$. A more concise statement of the convex programming problem is the following:

$$\min\{F(x)\colon x \in S\} \text{ where } S = \{x \in C\colon f_i(x) \leq 0,\ i = 1, \ldots, m\}.$$

Any point in the set S is called a feasible solution for (P) and S is called the set of feasible solutions. The convex program (P) is feasible if S is a nonempty set, and is solvable if the objective function F attains its minimum in S. That is, (P) is solvable if there exists a point $z \in S$ such that $F(z) \leq F(x)$ for all $x \in S$. A feasible solution z at which F attains its minimum in S is called an optimal solution for (P) and F(z) is called the optimal value for (P).

As an illustration, consider the convex program (P) with domain E_2, objective function $F(x,y) = (x - 2)^2 + (y - 1)^2 + 3$, and constraint functions $f_1(x,y) = x^2 - y$, $f_2(x,y) = x^2 + y - 2$, and $f_3(x,y) = -y$. The set of feasible solutions S for (P) and the graph of the objective function F over S are shown in Figure 2. Notice that the convex program (P) seeks to find that point (x,y) in the set of feasible solutions S such that the square of the distance from (2,1) to (x,y) is equal to or less than the square of the distance from (2,1) to any other point in S. Such a point (x,y) then minimizes the objective function F over S. It turns out that the optimal solution for (P) is the point (1,1) in S. This can be seen intuitively in Figure 3 which shows that (1,1) is the point in S which is "nearest" to (2,1). The next two examples illustrate convex programs based on "real-world"

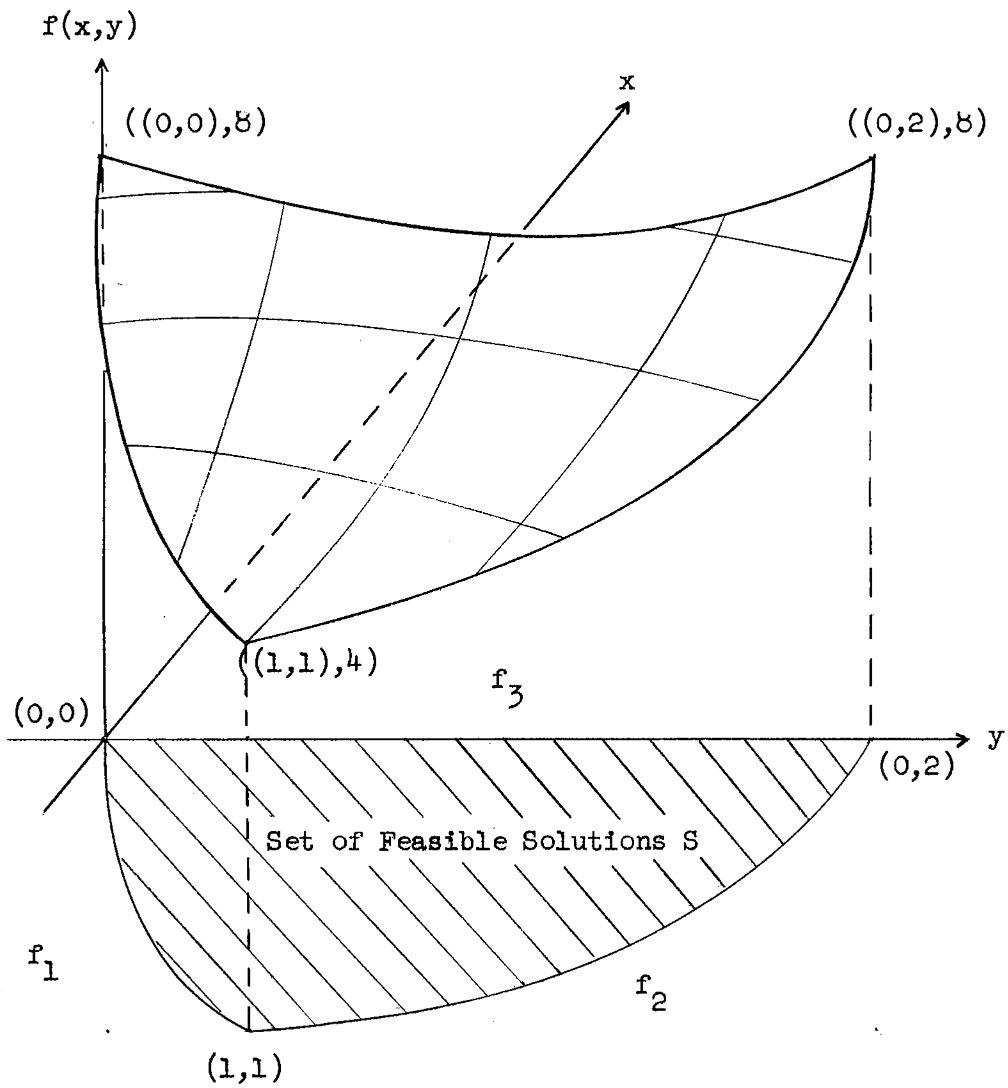


Figure 2.

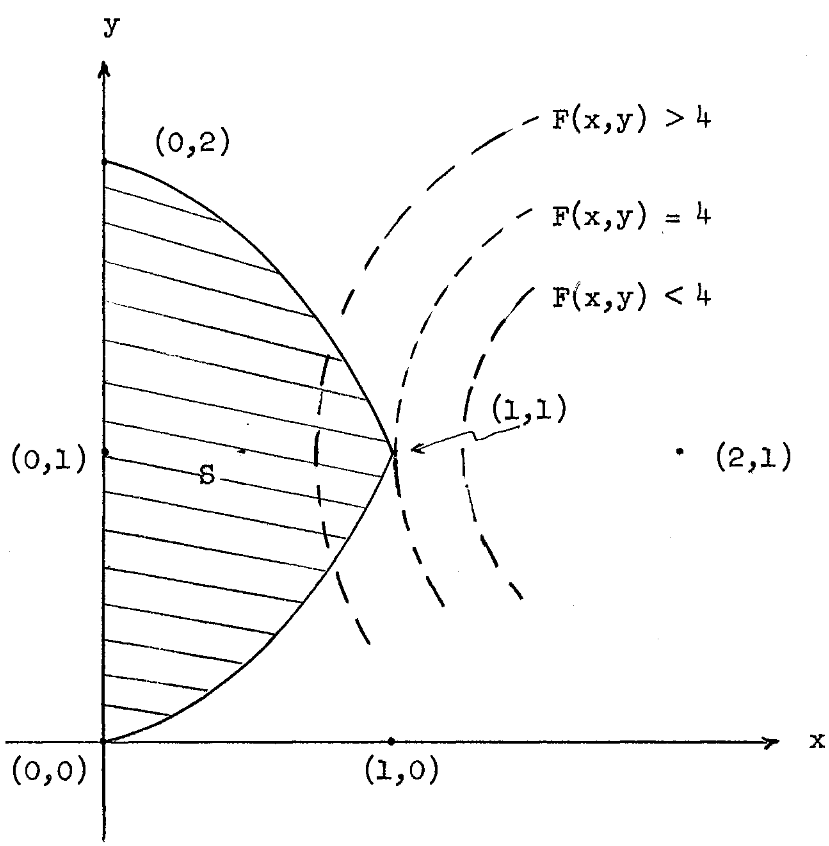


Figure 3.

data.

<u>Example 1.1.</u> (Chemical Equilibrium). Consider a mixture of m chemical elements. Assume that it has been predetermined that the m different types of atoms can combine chemically to produce n compounds. Define

x_j = the number of moles of compound j present in the mixture at equilibrium,

x_0 = the total number of moles in the mixture where $x_0 = \sum_{j=1}^{n} x_j$,

a_{ij} = the number of atoms of element i in a molecule of compound j,

b_i = the number of atomic weights of element i in the mixture.

The mass balance relationships that must hold for the m elements are

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad i = 1, \ldots, m, \qquad (1.1.1)$$

and

$$x_j \geq 0, \quad j = 1, \ldots, n. \qquad (1.1.2)$$

Determination of the composition of the mixture at equilibrium is equivalent to determination of the values of x_j, $j = 1, \ldots, n$, that satisfy (1.1.1) and (1.1.2) and also minimize the total free energy of the mixture. The total free energy of the mixture is given by

$$\sum_{j=1}^{n} x_j[c_j + \ln(x_j/x_0)] \qquad (1.1.3)$$

where $c_j = A_j + \ln P$, A_j is the Gibbs free energy function for the jth compound, and P is the total pressure in atmospheres.

Thus the convex programming problem is to find $(x_1, \ldots, x_n)$ which minimizes the convex objective function (1.1.3) subject to the constraint relations (1.1.1) and (1.1.2).

Example 1.2. (Weapons Assignment). Consider the problem of assigning p types of weapons to q different targets such that weapons cost is minimized and at least a specified expected damage value is inflicted on various targets. Define

x_{ij} = the number of weapons of type i assigned to target j, $i = 1, \ldots, p$ and $j = 1, \ldots, q$,

α_{ij} = the probability that target j will be undamaged by an attack using one unit of weapon i,

d_j = the minimum expected damage to target j,

c_i = the cost per unit of weapon of type i.

The expected damage to target j by the over-all assignment of weapons of all types is

$$1 - \prod_{i=1}^{p} (\alpha_{ij})^{x_{ij}}.$$

Hence constraint relations for expected damage to the various targets are

$$1 - \prod_{i=1}^{p} (\alpha_{ij})^{x_{ij}} \geq d_j, \quad j = 1, \ldots, q, \qquad (1.2.1)$$

$$x_{ij} \geq 0, \quad i = 1, \ldots, p \text{ and } j = 1, \ldots, q. \qquad (1.2.2)$$

The total cost of the assignment of x_{ij} weapons to the various targets is

$$\sum_{i=1}^{p} c_i \sum_{j=1}^{q} x_{ij}. \qquad (1.2.3)$$

Thus the convex programming problem is to find x_{ij}'s which minimize the convex objective function (1.2.3) subject to the constraint relations (1.2.1) and (1.2.2).

A convex program is not necessarily feasible, but even if it is, feasibility does not imply solvability. Certainly if the set of feasible solutions for a convex program (P) is a nonempty closed and bounded set, then the continuity of the objective function implies that (P) is solvable. This implication follows from the fact that a continuous function defined on a compact set attains its minimum there. However, if the set of feasible solutions is nonempty and closed but not bounded, then the continuity of the objective function is not enough to insure that (P) is solvable. If the set of feasible solutions is not bounded, then (P) can be solvable or not solvable as shown by the next two examples.

<u>Example 1.3.</u> Let $C = R$, $F(x) = e^{-x}$, and $f_1(x) = e^{-x} - 1$. Then the set $S = \{x \in R: x \geq 0\}$ is a nonempty closed convex set which is unbounded; that is, there exists no real number M such that $\|x\| \leq M$ for all $x \in S$. Now $\inf\{F(x): x \in S\} = 0$, but for every $x \in S$ it is true that $F(x) > 0$. Consequently, the convex program defined by C, F, and f_1 as given above has set of feasible solutions S but is not solvable.

<u>Example 1.4.</u> Let $C = R$, $F(x) = (x - 5)^2 + 1$, and $f_1(x) = e^{-x} - 1$. Then S is the same as in Example 1.1 and $\inf\{F(x): x \in S\} = 1$. The convex program defined by C, F, and f_1 in this example also has set of feasible solutions S but it is solvable since $F(5) = 1$ and $5 \in S$.

By comparing these two examples, it becomes apparent that both the feasibility and solvability of an arbitrary convex program depend entirely on the convex set and the convex functions which define it. Although the convex program in Example 1.3 is not solvable, by merely replacing the objective function a new convex program is created, as shown in Example 1.4, which is solvable even though the set of feasible solutions is an unbounded set. Therefore any discussion of convex programs and their related convex programming problems is to a very large extent a discussion of those properties of convex sets and convex functions which directly influence the feasibility and solvability of convex programs. The remainder of this chapter is devoted to developing these properties, first of convex sets and then of convex functions. The main goal is to characterize the relationship between unbounded closed convex sets and convex functions defined on them which insures that a convex function defined on an unbounded closed convex set attains its minimum there.

Some Properties of Convex Sets

The objective of this section is to characterize unboundedness in convex sets. The first lemma establishes a simple but fundamental property of an arbitrary collection of convex sets.

Lemma 1.5. Let $\{C_i: i \in I\}$ be a collection of convex sets where I is an arbitrary index set. Then $\cap\{C_i: i \in I\}$ is a convex set.

Proof: If the intersection of the sets is empty, then the conclusion follows trivially. Hence assume the intersection is nonempty and let x and y be in the intersection. Let $\lambda \in [0,1]$ be arbitrary and consider $z = \lambda x + (1 - \lambda)y$. Since x and y are in C_i for each $i \in I$ and each C_i

is convex, it follows that $z \in C_i$ for each $i \in I$. Therefore z is in the intersection of all the convex sets which implies that $\cap\{C_i: i \in I\}$ is a convex set.

Recalling the unbounded convex set in Example 1.3, it seems that a characterization of unboundedness for S is that for any x in S and any $u > 0$ and any $\lambda \geq 0$, the point $x + \lambda u$ is also in S. The convexity of S then implies that the line segment joining x to $x + \lambda u$ is contained in S. Notice that this property can be expressed by saying that for any x in S, there exists a nonnegative vector u such that the closed ray $\{x + \lambda u: \lambda \geq 0\}$ is contained in S. This concept is made more precise in the next definition.

Definition 1.6. Let C be a convex set. Then a vector u is called a vector of recession for C if and only if there exists an $x_0 \in C$ such that $\{x_0 + \lambda u: \lambda \geq 0\}$ is contained in C. The set of vectors of recession for C is denoted by A(C).

It should be pointed out that for any nonempty set C, A(C) is never empty since $u = 0$ is trivially a vector of recession for every set. Throughout the remainder of this paper, any vector of recession mentioned in the discussion is considered a nontrivial vector unless specifically stated otherwise. A useful characterization of vectors of recession for a closed convex set is given by the next theorem.

Theorem 1.7. Let C be a nonempty closed convex set. Then $u \in A(C)$ if and only if there exists a sequence of the form $\lambda_1 x_1, \lambda_2 x_2, \ldots,$ where $x_i \in C$ and $\lambda_i \downarrow 0$ and $\lambda_i x_i$ converges to u.

Proof: Let $u \in A(C)$. Then there exists an $x \in C$ such that the set

$\{x + \lambda u: \lambda \geq 0\}$ is contained in C. In particular, let $\lambda_k = 1/k$ and $x_k = x + ku$, for $k = 1, 2, \ldots,$. Then $\lambda_k \downarrow 0$ as k becomes infinite and $x_k \in C$ for each k. In addition, as k becomes infinite, $\lambda_k x_k$ converges to u which is the desired conclusion.

Now assume there exists a sequence $\{\lambda_k x_k\}$ in C such that $\lambda_k \downarrow 0$, x_k is in C for each k, and $\lambda_k x_k$ converges to a vector u. Let $r > 0$ be an arbitrary but fixed real number. Then for all $\lambda_k \leq 1/r$, it follows that $0 \leq r\lambda_k \leq 1$. Let p be an arbitrary point in C. For each k, x_k is in C so the point $(1 - r\lambda_k)p + r\lambda_k x_k \in C$ by the convexity of C. Since C is a closed set, the limit of $(1 - r\lambda_k)p + r\lambda_k x_k$ as k becomes infinite which is $p + ru$ is in C. Since $r > 0$ is arbitrary, it follows that $\{p + ru: r \geq 0\}$ is contained in C. By Definition 1.6, $u \in A(C)$.

An interesting result concerning vectors of recession is an immediate consequence of the second part of the proof of Theorem 1.7. Notice that p was an arbitrary point in C, yet $\{p + ru: r \geq 0\}$ was contained in C. Does this mean that if C is a nonempty closed convex set, then $u \in A(C)$ implies that $\{x + ru: r \geq 0\} \subset C$ for every x in C? The answer to this question is, yes, as shown by the next corollary.

Corollary 1.8. Let C be a nonempty closed convex set. If $u \in A(C)$, then $\{x + \lambda u: \lambda \geq 0\}$ is contained in C for every x in C.

Proof: Let $u \in A(C)$. Then for some $x_0 \in C$, $\{x_0 + \lambda u: \lambda \geq 0\} \subset C$. Hence $x_0 + u$, $x_0 + 2u, \ldots,$ is a sequence of points in C. Let $\lambda_k = 1/k$ and $x_k = x_0 + ku$, for $k = 1, 2, \ldots,$. Then $\lambda_k x_k$ converges to u as k becomes infinite and it follows from the proof of Theorem 1.7 that for any $x \in C$, $\{x + \lambda u: \lambda \geq 0\}$ is contained in C.

Some examples of convex sets and their associated sets of vectors of recession might help to clarify this concept of unboundedness. The sets discussed are illustrated in Figure 4.

<u>Example 1.9</u>. Let $C_1 = \{(x,y): x > 0,\ y \geq 1/x\}$. Then C_1 is a nonempty closed convex set and $A(C_1) = \{(x,y): x \geq 0,\ y \geq 0\}$.

<u>Example 1.10</u>. Let $C_2 = \{(x,y): y \geq x^2\}$. Then C_2 is also a nonempty closed convex set and $A(C_2) = \{(x,y): x = 0,\ y \geq 0\}$.

<u>Example 1.11</u>. Let $C_3 = \{(x,y): x^2 + y^2 \leq 1\}$. In this case, C_3 is a nonempty closed <u>bounded</u> convex set and hence $A(C_3) = \{0\}$.

<u>Example 1.12</u>. Let $C_4 = \{(x,y): x > 0,\ y > 0\} \cup \{(0,0)\}$. Then C_4 is a nonempty convex set and $A(C_4) = \{(x,y): x \geq 0,\ y \geq 0\} = A(C_1)$.

These examples show that there is a distinct relationship between nonempty convex sets and their sets of vectors of recession. Notice that $C_1 \neq C_4$ yet $A(C_1) = A(C_4)$. It turns out that if C is a nonempty convex set, then A(C) is also a nonempty convex set (it is in fact a convex cone) and if C is closed, then A(C) is also closed. The next theorem makes these assertions precise and appropriate proofs can be found in Rockafellar,[20], Section 8 .

<u>Theorem 1.13</u>. Let C be a nonempty closed convex set. Then A(C) is a closed set with the following properties:

(a). if u_1 and u_2 are in A(C), then $u_1 + u_2$ is in A(C).

(b). if u is in A(C) and $r \geq 0$, then ru is in A(C).

In particular, A(C) is a convex set.

Now consider the collection of convex sets $\{C_i: i \in I\}$ where I is

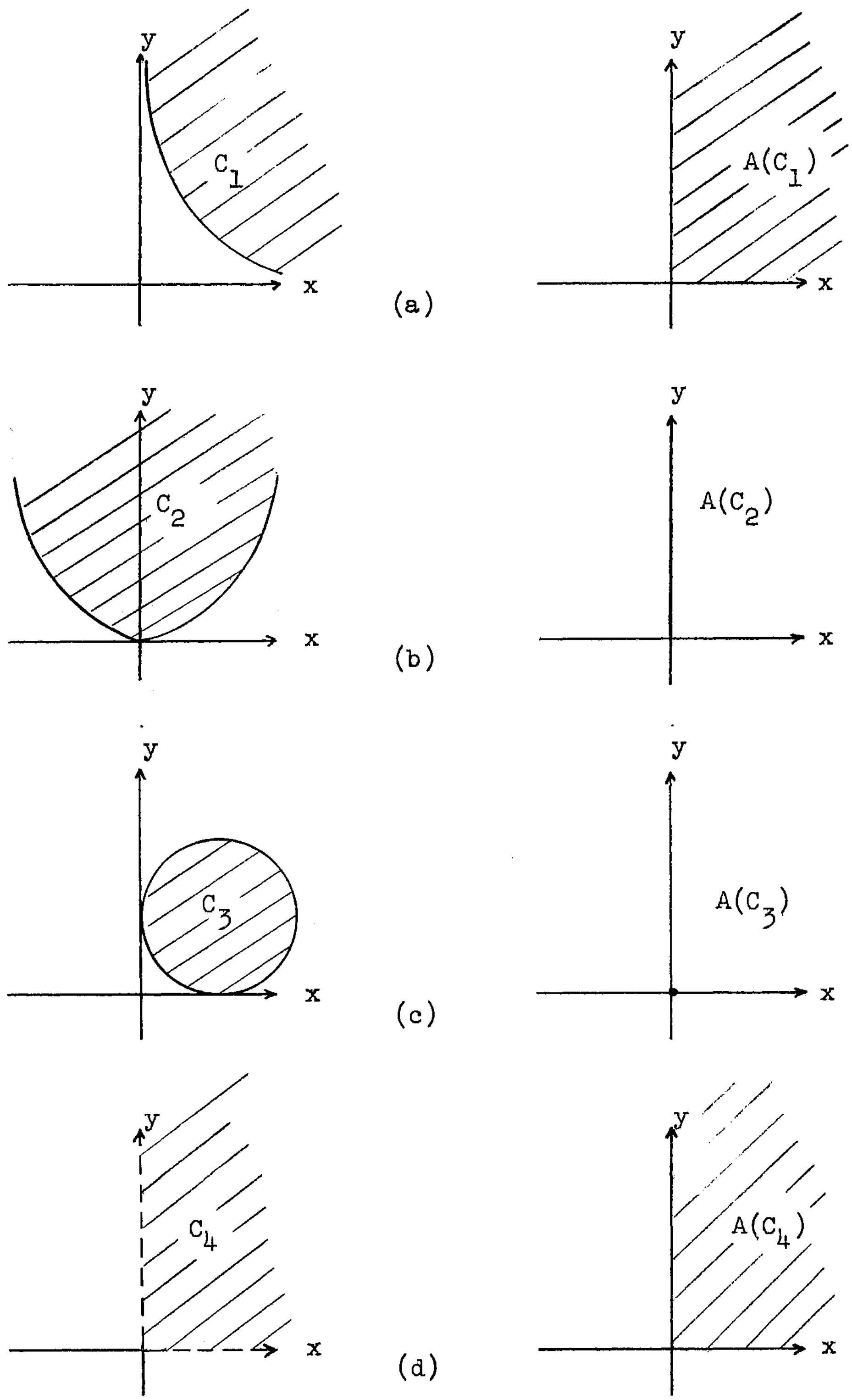


Figure 4.

an arbitrary index set. Lemma 1.5 says that $\cap\{C_i: i \in I\}$ is a convex set. The question then arises as to the relationship between the vectors of recession for the sets C_i and the vectors of recession for the convex set which is their intersection. The following theorem provides an answer for this question.

<u>Theorem 1.14</u>. Let $\{C_i: i \in I\}$ be an arbitrary collection of closed convex sets such that $\cap\{C_i: i \in I\}$ is a nonempty set. Then

$$A(\cap\{C_i: i \in I\}) = \cap\{A(C_i): i \in I\}.$$

Proof: Let $z \in \cap\{C_i: i \in I\}$. If $u \in A(\cap\{C_i: i \in I\})$, then the set $\{z + \lambda u: \lambda \geq 0\} \subset \cap\{C_i: i \in I\}$. Consequently, $u \in A(C_i)$ for each $i \in I$ and it follows that $u \in \cap\{A(C_i): i \in I\}$.

Now let $u \in \cap\{A(C_i): i \in I\}$. If $z \in \cap\{C_i: i \in I\}$, then the set $\{z + \lambda u: \lambda \geq 0\}$ is contained in C_i for each $i \in I$ and it follows that $u \in A(C_i)$ for each $i \in I$. Thus $u \in A(\cap\{C_i: i \in I\})$ and the desired conclusion is immediate.

This discussion of vectors of recession is motivated by a desire to characterize unboundedness in convex sets in a simple and easy to work with manner. Theorem 1.15 below, and Corollary 1.16 which follows it, provide this desired characterization.

<u>Theorem 1.15</u>. Let C be a nonempty closed convex set. Then C is bounded if and only if $A(C) = \{0\}$.

Proof: Suppose that C is bounded. Then there exists a real number M such that $\|x\| \leq M$ for every $x \in C$. Let $x_0 \in C$ and consider the set $-x_0 + C$. It follows directly that $-x_0 + C$ is a nonempty closed convex set. In addition, $-x_0 + C$ is bounded, for if $z \in -x_0 + C$, then there exists an $x \in C$ such that $z = -x_0 + x$. Consequently, evaluating the norm of z, it follows that $\|z\| = \|-x_0 + x\| \leq \|-x_0\| + \|x\| \leq \|x_0\| + M$. The fact that $\|x_0\| + M$ is a fixed real number and that z is arbitrary in $-x_0 + C$ implies that $-x_0 + C$ is bounded. Assume that u is a vector of recession for C. Then for $x_0 \in C$, $\{x_0 + \lambda u: \lambda \geq 0\}$ is contained in C so $-x_0 + \{x_0 + \lambda u: \lambda \geq 0\} = \{\lambda u: \lambda \geq 0\} \subset -x_0 + C$. Since $\|u\| > 0$, there exists a $\lambda_0 > 0$ such that $\|\lambda_0 u\| = \lambda_0\|u\| > \|x_0\| + M$. But $\lambda_0 u$ is in $-x_0 + C$ and thus the boundedness of $-x_0 + C$ is contradicted. Hence $A(C) = \{0\}$.

Now suppose that $A(C) = \{0\}$ and assume that C is unbounded. Then there exists a sequence of vectors $x_1, x_2, x_3, \ldots$, such that $\|x_k\|$ approaches infinity as k becomes infinite. Let $\lambda_k = 1/\|x_k\|$. Then $\lambda_k \downarrow 0$ and $\{\lambda_k x_k\} \subset \{x \in E_n: \|x\| = 1\} = B$. Since B is a compact set, there exists some y in B such that $\lambda_k x_k$ converges to y. By Theorem 1.7, it follows that $y \in A(C)$. But $y \in B$ implies that y is a nontrivial vector which contradicts the assumption that $A(C) = \{0\}$. Hence C is bounded.

The contrapositive of Theorem 1.15 is a more intuitive statement of the conditions which imply that a convex set is unbounded. It is given here as a corollary to the above theorem.

Corollary 1.16. Let C be a nonempty closed convex set. Then C is unbounded if and only if $A(C) \neq \{0\}$.

Another topic with respect to convex sets which has an important influence on the structure of convex programs is conditions which imply that an arbitrary collection of convex sets has a nonempty intersection. The results which follow are used in Chapter II to establish conditions which insure that a feasible convex program is also solvable.

The basic result concerning the intersection of an arbitrary collection of convex sets is known as Helley's Theorem and it is presented here without proof. The interested reader can find several proofs in Valentine [23], Part IV.

Theorem 1.17. (Helley's Theorem). Let $\{C_i: i \in I\}$ be a collection of nonempty closed bounded convex sets in E_n where I is an arbitrary index set. If every subcollection consisting of n + 1 sets has a nonempty intersection, then the entire collection has a nonempty intersection.

If only a finite number of convex sets are involved, then the requirement that the sets be closed and bounded is not necessary to achieve the same conclusion. This result is stated separately as a corollary. A proof can be found in Valentine [23], p. 70, or Rockafellar [20], p. 196.

Corollary 1.18. (Finite Helley's Theorem). Let $\{C_i: i \in I\}$ be a finite collection of convex sets in E_n. If every subcollection of n + 1 sets has a nonempty intersection, then the entire collection has a nonempty intersection.

Unfortunately, if a collection of convex sets is not finite and if, in addition, all of the sets in the collection are not bounded, then Helley's Theorem provides no information as to whether the collection has a nonempty intersection. The next example illustrates this problem.

Example 1.19. In order to construct a collection of nonempty closed convex sets that illustrates the above remarks, consider the following situation. For each real number r, let $C_r = [r,\infty)$. Each C_r is a closed convex set in E_1 and every two sets in the collection $\{C_r: r \in R\}$ have a nonempty intersection. However, $\cap\{C_r: r \in R\}$ is empty.

Notice that the sets in Example 1.19 have vectors of recession in common. For any $u > 0$ and any real number $r \in R$, $\{x + \lambda u: \lambda \geq 0\}$ is contained in C_r for every $x \in C_r$. It is this very property that keeps the collection $\{C_r: r \in R\}$ from having a nonempty intersection. It turns out that if boundedness is replaced in the hypothesis for Helley's Theorem by the requirement that the sets in the collection have no vectors of recession in common, then the conclusion of Helley's Theorem remains valid. This vectors of recession version of Helley's Theorem, with a somewhat difficult proof, can be found in Rockafellar [20], p. 191. Because of the importance of this vectors of recession version to Chapter II of this paper, a direct proof is developed here. The next lemma is essential for this proof.

Lemma 1.20. Let $\{C_i: i \in I\}$ be a collection of nonempty closed convex sets in E_n where I is an arbitrary index set. If every subcollection of $n + 1$ sets has a nonempty intersection and some finite subcollection has a bounded intersection, then the entire collection has a nonempty intersection.

Proof: Let $\{C_1, \ldots, C_k\}$ be the finite subcollection which has a bounded intersection. Since it is contained in the original collection, this finite collection of convex sets has the property that every subcollection of $n + 1$ sets from it has a nonempty intersection so by

Corollary 1.18, it follows that $K = \cap\{C_i: i = 1, \ldots, k\}$ is nonempty. For each $i \in I$, let $D_i = C_i \cap C_1 \cap \ldots \cap C_k = C_i \cap K$. Now the finite collection of sets $\{C_i, C_1, \ldots, C_k\}$ has the property that every subcollection of n + 1 sets from it has a nonempty intersection so by Corollary 1.18, it follows that $D_i \neq \emptyset$ for each $i \in I$. Now consider the following arbitrary collection of n + 1 sets from $\{D_i: i \in I\}$, say $\{D_{i_1}, \ldots, D_{i_{n+1}}\}$. Then

$$D_{i_1} \cap \ldots \cap D_{i_{n+1}} = (C_{i_1} \cap K) \cap \ldots \cap (C_{i_{n+1}} \cap K)$$

$$= (C_{i_1} \cap \ldots \cap C_{i_{n+1}}) \cap K$$

$$= C_{i_1} \cap \ldots \cap C_{i_{n+1}} \cap C_1 \cap \ldots \cap C_k.$$

Now $\{C_{i_1}, \ldots, C_{i_{n+1}}, C_1, \ldots, C_k\}$ is a finite collection of sets with the property that every subcollection of n + 1 sets from it has a nonempty intersection. Hence by Corollary 1.18,

$$C_{i_1} \cap \ldots \cap C_{i_{n+1}} \cap C_1 \cap \ldots \cap C_k = D_{i_1} \cap \ldots \cap D_{i_{n+1}} \neq \emptyset.$$

Thus $\{D_i: i \in I\}$ is a collection of nonempty closed bounded convex sets with the property that every subcollection of n + 1 sets has a nonempty intersection. By Theorem 1.17, it follows that $\cap\{D_i: i \in I\}$ has a nonempty intersection. But

$$\cap\{D_i: i \in I\} = \cap\{C_i \cap K: i \in I\} = \cap\{C_i: i \in I\} \cap K,$$

and the desired conclusion follows.

Certainly the sets in the collection which satisfies the hypothesis of Lemma 1.20 have no vector of recession in common. For if they did, then such a vector of recession would also be a vector of recession for the set $C_1 \cap \ldots \cap C_k$ which would contradict the hypothesis that this set is bounded.

Assuming that the sets in the collection in Lemma 1.20 have no vectors of recession in common, it follows that $\cap\{A(C_i): i \in I\} = \{0\}$. If $B = \{x \in E_n: \|x\| = 1\}$, then B is a nonempty closed bounded set in E_n. For each $i \in I$, let $B_i = B \cap A(C_i)$. Since C_i is closed, $A(C_i)$ is closed for each $i \in I$ by Theorem 1.13; hence B_i is a closed bounded set for each $i \in I$. Thus $\cap\{B_i: i \in I\} = B \cap [\cap\{A(C_i): i \in I\}]$ is an empty set. Since $\{B_i: i \in I\}$ is a collection of closed sets in the compact set B with the property that $\cap\{B_i: i \in I\}$ is empty, there exists a finite subcollection $\{B_1, B_2, \ldots, B_k\}$ such that $B_1 \cap B_2 \cap \ldots \cap B_k = \emptyset$. A proof of this assertion can be found in Gemignani [8], p. 148 . Hence $B \cap A(C_1) \cap \ldots \cap A(C_k)$ is empty and it follows that $A(C_1) \cap \ldots \cap A(C_k) = \{0\}$. Since $C_1 \cap \ldots \cap C_k$ is nonempty by the hypothesis of Lemma 1.20, it follows from Theorems 1.14 and 1.15 that the set $C_1 \cap \ldots \cap C_k$ is a bounded set.

The remarks of the last two paragraphs have proven the following lemma.

Lemma 1.21. Let $\{C_i: i \in I\}$ be a collection of nonempty closed convex sets in E_n where I is an arbitrary index set. Assume that every subcollection of $n + 1$ sets has a nonempty intersection. Then the C_i have no vectors of recession in common if and only if some finite subcollection has a bounded intersection.

The above lemma can now be combined with Lemma 1.20 to give the following important theorem.

Theorem 1.22. Let $\{C_i: i \in I\}$ be a collection of nonempty closed convex sets in E_n where I is an arbitrary index set. Assume that every subcollection of n + 1 sets has a nonempty intersection and that the sets in $\{C_i: i \in I\}$ have no vectors of recession in common. Then the entire collection has a nonempty intersection.

It turns out that the hypothesis of Theorem 1.22 can be relaxed even more and the conclusion will remain valid. If all the sets in a collection of closed convex sets have particular vectors of recession in common, then it is true that they will have a nonempty intersection. The next example illustrates this situation.

Example 1.23. For n = 1, 2, 3, . . . , consider the sets of the form $C_n = \{(x,y): 0 \leq y \leq 1/n\}$. These sets form a collection of nonempty closed convex sets in E_2 and any two of them have a nonempty intersection. Since $A(C_n) = \{(x,y): y = 0\}$ for each positive integer n, these sets have vectors of recession in common, yet the entire collection has a nonempty intersection.

The vectors of recession common to all the sets in Example 1.23 have the property that both $u = (x,0)$ and $-u = (-x,0)$, $x \neq 0$, are vectors of recession for the entire collection. Vectors of recession with this particular property are defined below.

Definition 1.24. Let C be a nonempty convex set. Then u is a vector of linear recession for C if and only if there exists an $x_0 \in C$ such that the line $\{x_0 + \lambda u: \lambda \in R\}$ is contained in C.

From previous results and the above definition, it follows that if u is a vector of linear recession for a nonempty closed convex set C, then the line $\{x + \lambda u: \lambda \in R\}$ is contained in C for every x in C. Also, if C is a nonempty closed convex set, then u is a vector of linear recession for C if and only if $u \in A(C)$ and $-u \in A(C)$. In fact, closed convex sets in E_n which possess vectors of linear recession always contain translates of some k-dimensional subspace of E_n where $1 \le k \le n$. This result is made precise by the next theorem.

Theorem 1.25. Let C be a nonempty closed convex set in E_n. If M is the subspace of E_n spanned by the vectors of linear recession for C, then $x + M \subseteq C$ for every $x \in C$.

Proof: Let M be the subspace described in the hypothesis. It is desirable to get a basis for M where the vectors of the basis are vectors of linear recession for C. This can be done in the following manner. Let u_1 be a vector of linear recession for C. Then $\{u_1\}$ is a linearly independent subset of M which is either a basis for M or not. If it is, then $\{u_1\}$ is the desired basis. If not, then there exists another vector u_2 which is a vector of linear recession for C such that $\{u_1, u_2\}$ is a linearly independent subset of M which is either a basis for M or not. In general, if $\{u_1, u_2, \ldots, u_j\}$ is a set of vectors of linear recession for C which is also a linearly independent subset of M that is not a basis for M, then there exists a vector u_{j+1} which is a vector of linear recession for C such that $\{u_1, \ldots, u_j, u_{j+1}\}$ is a linearly independent subset of M. In this manner, a basis $\{u_1, u_2, \ldots, u_k\}$, where $1 \leq k \leq n$ since E_n is a finite dimensional space, can be found for M such that each u_i is a vector of linear recession for C. Let $z = \lambda_1 u_1 + \cdots + \lambda_k u_k$ be an arbitrary point in M. If $\lambda_i \geq 0$, then $\lambda_i u_i$ is in A(C) by Theorem 1.13. If $\lambda_i < 0$, then rewrite $\lambda_i u_i$ as $(-\lambda_i)(-u_i)$. Since $-\lambda_i > 0$ and $-u_i$ is in A(C), it follows that $\lambda_i u_i = (-\lambda_i)(-u_i) \in A(C)$ by Theorem 1.13. Consequently, it follows by Theorem 1.13 that $z \in A(C)$. Therefore, $x + z = x + (1)z \in C$ for all $x \in C$. Since z is arbitrary in M, it follows that $x + M \subset C$ for all $x \in C$.

Using Theorem 1.25, the hypothesis of Theorem 1.22 can be modified to require that the vectors of recession common to the sets $\{C_i: i \in I\}$ be vectors of linear recession and the conclusion of Theorem 1.22 will still be valid. This result is formulated in the next theorem.

Theorem 1.26. Let $\{C_i: i \in I\}$ be a collection of nonempty closed convex sets in E_n where I is an arbitrary index set. Assume that every subcollection of n + 1 sets has a nonempty intersection and that the only vectors of recession common to all the sets $\{C_i: i \in I\}$ are vectors of linear recession. Then the entire collection has a nonempty intersection.

Proof: Let M be the subspace of E_n spanned by the vectors of linear recession common to all the sets $\{C_i: i \in I\}$ and let M^* be the orthogonal complement to M in E_n. Then M^* is a closed subspace and the direct sum of M and M^* is E_n. For each $i \in I$, let $D_i = C_i \cap M^*$. Then the following are true:

(a). $D_i \neq \emptyset$ for all $i \in I$: For any $z \in C_i$, there exists a $p \in M$

and a $q \in M^*$ such that $z = p + q$. Now $z + M \subset C_i$ by Theorem 1.25 so $p + q + M = q + M \subset C_i$. Since $0 \in M$, it follows that $q \in C_i$. Hence $D_i = C_i \cap M^* \neq \emptyset$ for all $i \in I$.

(b). D_i is a closed convex set: This follows from the fact that C_i and M^* both have these properties.

(c). The D_i have no vectors of recession in common: Assume u is a vector of recession common to all the D_i. Then $u \in A(C_i)$ for all $i \in I$ and $u \in M^*$ and hence u is a vector of recession common to all the C_i. However, $u \in M^*$ implies that $u \notin M$ and thus u is not a vector of linear recession common to all the C_i, which is a contradiction.

(d). Every $n + 1$ subcollection of the D_i have a point in common: Let $\{D_1, \ldots, D_{n+1}\}$ be an arbitrary subcollection of $n + 1$ sets. Then $D_1 \cap \ldots \cap D_{n+1} = (C_1 \cap \ldots \cap C_{n+1}) \cap M^*$. By hypothesis, $C_1 \cap \ldots \cap C_{n+1} \neq \emptyset$ so if $z \in C_1 \cap \ldots \cap C_{n+1}$, then $z + M$ is contained in $C_1 \cap \ldots \cap C_{n+1}$. By part (a) above, it follows that $q \in C_1 \cap \ldots \cap C_{n+1} \cap M^*$ where $z = p + q$. Therefore, $D_1 \cap \ldots \cap D_{n+1}$ is nonempty.

Hence the sets in the collection $\{D_i: i \in I\}$ satisfy the hypothesis of Theorem 1.22 and it follows that $\cap\{D_i: i \in I\} \neq \emptyset$. Consequently, $\cap\{C_i: i \in I\} \neq \emptyset$.

This concludes the essential properties of convex sets needed in a discussion of convex programs. Unbounded convex sets are characterized in terms of vectors of recession and Theorems 1.17, 1.18, 1.20, 1.22, and 1.26 give important information concerning the intersection of an arbitrary or finite collection of certain convex sets. Further development of vectors of recession can be found in Rockafellar [20].

As illustrated by Examples 1.3 and 1.4, a convex function defined on an unbounded convex set does not necessarily attain its minimum there. Therefore the solvability of a convex program must depend in some way on certain properties of the objective function. The next section on convex functions shows that the concept of vector of recession for a convex function is very important, and the results developed for convex functions are used with those for convex sets to prove the important theorems in Chapter II on solvability.

Some Properties of Convex Functions

The objective of this section is to relate the concept of vector of recession to convex functions. This concept is fundamental in discussing the solvability of a convex program.

Associated with any convex function defined on a convex set C in E_n are two distinct convex sets, one in E_n and one in E_{n+1}. The first type of convex set, called a level set, is defined below.

Definition 1.27. Let f be a convex function defined on a convex set C in E_n. If α is any real number, then the level set of f with respect to α, denoted by $\text{lev}_\alpha f$, is defined as follows:

$$\text{lev}_\alpha f = \{x \in C:\ f(x) \le \alpha\}.$$

Notice that the nonempty level sets of f form a collection of sets $\{\text{lev}_\alpha f:\ \alpha \in R\}$ such that $\text{lev}_\alpha f \subset \text{lev}_\beta f$ if and only if $\alpha \le \beta$. Any collection of sets having this property is said to be nested. This property is important in later theorems on the minimization of a convex function f over a convex set C. Two significant properties of level sets are given in the next lemma.

Lemma 1.28. If f is a convex function defined on a nonempty closed convex set C in E_n, then every level set of f is a closed convex set.

Proof: If $\text{lev}_\alpha f$ is empty, then it is trivially a closed convex set. Hence assume that $\text{lev}_\alpha f$ is nonempty and let x and y be in $\text{lev}_\alpha f$ and $\lambda \in [0,1]$. Then

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \leq \lambda\alpha + (1 - \lambda)\alpha = \alpha,$$

so $\text{lev}_\alpha f$ is convex. If $\{x_i\}$ is a sequence in $\text{lev}_\alpha f$ that converges to x in C, then the continuity of f implies that $f(x_i)$ converges to f(x). Since $f(x_i) \leq \alpha$ for all i, it must be true that $f(x) \leq \alpha$. Hence x is in $\text{lev}_\alpha f$ and thus $\text{lev}_\alpha f$ is closed.

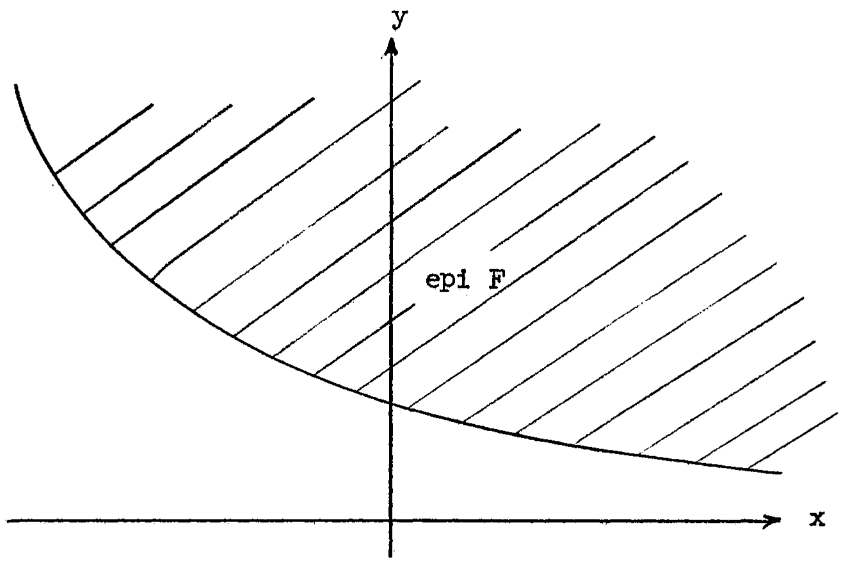
The other convex set associated with a convex function is the set of all points in E_{n+1} which lie on or above the graph of f over C. This set is called the epigraph of f and is denoted by epi f. The next definition makes this concept precise.

Definition 1.29. Let f be a convex function defined on the convex set C in E_n. Then

$$\text{epi } f = \{(x,r) \in E_{n+1}: x \in C,\ r \geq f(x)\}.$$

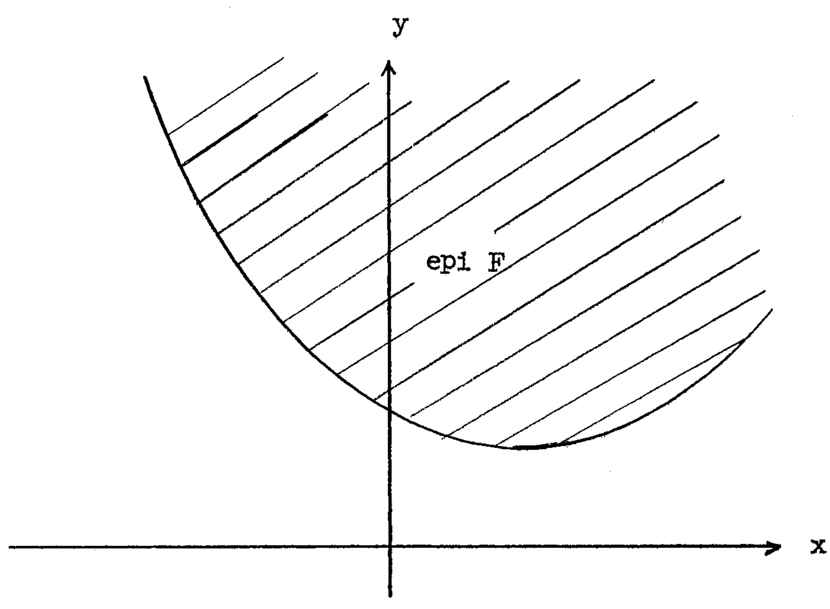
Analogous to Lemma 1.28, epi f is a closed convex set if C is closed in E_n. Examples of epigraphs are given in Figure 5.

From Definition 1.29 and the illustrations in Figure 5, it is apparent that the set of vectors of recession for any epigraph in E_{n+1} contains nontrivial vectors. For example, any vector in E_{n+1} of the form $(0,r)$, $r > 0$, is a vector of recession for every epigraph. However, only particular nontrivial vectors in A(epi f) are of interest,



$F(x) = e^{-x}$

(a)



$F(x) = (x - 5)^2 + 1$

(b)

Figure 5.

and these are vectors of the form $(u,0)$ where $u \in E_n$ and $u > 0$. If such a vector u is in A(epi f), then it follows by Corollary 1.8 that $\{(x,r) + \lambda(u,0): \lambda \geq 0\} = \{(x + \lambda u, r): \lambda \geq 0\}$ is contained in epi f for some $(x,r) \in$ epi f. By Definition 1.28, it then must be true that $f(x + \lambda u) \leq r$ for all $\lambda \geq 0$. This discussion motivates the following definition.

Definition 1.30. Let f be a convex function defined on the convex set C in E_n. Then $u \in E_n$ is a vector of recession for f if and only if $(u,0)$ is in A(epi f). The vectors of recession for f are denoted by A(f).

It is also true that every vector of recession for f is a vector of recession for C, for if $(u,0) \in A(f)$, then $\{(x_0 + \lambda u, r): \lambda \geq 0\}$ contained in epi f for some $(x_0,r) \in$ epi f implies that the set $\{x_0 + \lambda u: \lambda \geq 0\}$ is contained in C and hence $u \in A(C)$. It should be noted that the converse of the above statement is not true, (cf. Figure 5(b).). This observation is important in Chapter II.

The question now arises as to what is the relationship between A(f) and $A(\text{lev}_\alpha f)$ where $\text{lev}_\alpha f$ is a nonempty level set of f. Since every nonempty level set of f is a convex subset of the domain C for f, it is true that $A(\text{lev}_\alpha f) \subset A(C)$ for every nonempty level set $\text{lev}_\alpha f$.

The next theorem shows that a much stronger relationship exists between the vectors of recession for f and those of its nonempty level sets than between A(C) and A(f).

Theorem 1.31. Let f be a convex function defined on the closed convex set C in E_n. Let $L = \{\text{lev}_\alpha f: \alpha \in A\}$ be the collection of all nonempty level sets of f. If α and β are arbitrarily chosen in A, then

$$A(\text{lev}_\alpha f) = A(f) = A(\text{lev}_\beta f).$$

Proof: Let $u \in A(\text{lev}_\alpha f)$. Then $f(x + \lambda u) \le \alpha$ for all $\lambda \ge 0$ and for all $x \in \text{lev}_\alpha f$. For any $x \in \text{lev}_\alpha f$, $(x,\alpha) \in$ epi f so $\{(x + \lambda u, \alpha): \lambda \ge 0\}$ is contained in epi f. Consequently, $u \in A(\text{epi } f)$ and by Definition 1.30, $u \in A(f)$.

Now pick $u \in A(f)$. Then for some $x \in \text{lev}_\beta f$, $\{(x + \lambda u, \beta): \lambda \ge 0\}$ is contained in epi f so $f(x + \lambda u) \le \beta$ for all $\lambda \ge 0$ and for all x in $\text{lev}_\beta f$. Hence $A(f) \subset A(\text{lev}_\beta f)$.

Thus $A(\text{lev}_\alpha f) \subset A(f) \subset A(\text{lev}_\beta f)$. By reversing the roles of α and β in the above proofs, it follows that $A(\text{lev}_\beta f) \subset A(f) \subset A(\text{lev}_\alpha f)$ and the desired conclusion is immediate.

Theorem 1.31 says that if C is a closed convex set, then all the nonempty level sets of f have exactly the same vectors of recession and these vectors of recession are exactly those of f. Hence, if $A(f) = \{0\}$, then all the nonempty level sets of f are bounded sets. Since C is closed, all the nonempty level sets of f are then compact convex sets. In addition, if one of the nonempty level sets of f is a bounded set, then f has no vectors of recession and every nonempty level set of f is bounded.

Suppose now that a vector of recession for some nonempty level set of f is in fact a vector of linear recession for that level set. As discussed previously, this particular vector is also a vector of recession for f. However, there exist special vectors of recession for f which are always vectors of linear recession for the nonempty level sets of f and conversely.

<u>Definition 1.32</u>. Let f be a convex function defined on a convex set C in E_n. Then u is a vector of constant recession for f if and only if there exists an $x_0 \in C$ such that $\{(x_0 + \lambda u, f(x_0)): \lambda \geq 0\} \subset \text{epi } f$ and $\{(x_0 + \lambda(-u), f(x_0)): \lambda \geq 0\} \subset \text{epi } f$.

The following theorem gives two important results concerning vectors of constant recession for a convex function.

<u>Theorem 1.33</u>. Let f be a convex function defined on a closed convex set C in E_n. Then

(a). u is a vector of constant recession for f if and only if $u \in A(f)$ and $-u \in A(f)$.

(b). u is a vector of constant recession for f if and only if there exists an $x_0 \in C$ such that f is constant on the line $\{x_0 + ru: r \in R\}$.

Proof: The assertion in (a) is a direct and immediate consequence of Definition 1.30 and Corollary 1.8.

Now to prove (b). Assume that u is a vector of constant recession for f. Then there exists a point $x_0 \in C$ such that $\{(x_0 + ru, f(x_0)): r \geq 0\} \subset \text{epi } f$ and $\{(x_0 + r(-u), f(x_0)): r \geq 0\}$ is contained in epi f. Suppose there exists an $r \neq 0$ such that $f(x_0 + ru) < f(x_0)$. Without loss of generality, take $r > 0$. Now $-u \in A(f)$ implies that $f(x_0 + r(-u)) \leq f(x_0)$. Also, it follows that $x_0 = (1/2)(x_0 + ru) + (1/2)(x_0 + r(-u))$, so by the convexity of f,

$$\begin{aligned} f(x_0) &= f((1/2)(x_0 + ru) + (1/2)(x_0 + r(-u))) \\ &\leq (1/2)f(x_0 + ru) + (1/2)f(x_0 + r(-u)) \\ &< f(x_0). \end{aligned}$$

This contradiction then implies that $f(x_0 + ru) \geq f(x_0)$ for all $r \in R$. But u a vector of constant recession for f implies that $f(x_0 + ru)$ is equal to or less than $f(x_0)$ for all $r \in R$. Hence it follows that $f(x_0 + ru) = f(x_0)$ for all $r \in R$ and thus f is constant on the line $\{x_0 + ru: r \in R\}$.

If there exists an $x_0 \in C$ such that f is constant on the line $\{x_0 + ru: r \in R\}$ for some nonempty vector $u \in E_n$, then it follows that u must be a vector of constant recession for f by definition.

If the domain of the convex function is a closed convex set in E_n, say C, then a distinct relationship exists between vectors of constant recession for f and vectors of linear recession for C. For if u is a vector of constant recession for f, then $\{(x + ru, f(x)): r \in R\}$ is contained in epi f for every $x \in C$, so it must be true that the set $\{x + ru: r \in R\} \subseteq C$ and hence u is a vector of linear recession for C. This result is stated precisely in the next lemma.

<u>Lemma 1.34</u>. Let f be a convex function defined on a closed convex set C in E_n. If u is a vector of constant recession for f, then u is a vector of linear recession for C.

The converse of Lemma 1.34 is not true as is seen by considering the convex function given by $f(x) = x^2$ defined on the real line R. Every nonzero vector in R is a vector on linear recession for R but f has no vectors of recession.

The next example illustrates a convex function defined on a closed convex set such that the function has a vector of constant recession.

Example 1.35. Let $C = \{(x,y)\colon -1 \le x \le 1\}$. Then C is a nonempty closed convex set in E_2 and any nonzero vector of the form $(0,r)$ is a vector of linear recession for C. Let $f(x,y) = x$ be a convex function defined on C. Then f is constant on every line of the form $L_k = \{(x,y)\colon x = k\}$ where k is a real number in the closed interval $[-1,1]$. Clearly the vector $(0,r)$, $r \neq 0$, is a vector of constant recession for f. Figure 6 illustrates this example.

Notice in Example 1.35 that even though f has a vector of recession, it still attains its minimum on C. In Chapter II, it is shown that a convex function defined on a closed convex set attains its minimum there if the only vectors of recession for the function are vectors of constant recession.

By Theorem 1.31, every vector of constant recession for f is a vector of linear recession for every nonempty level set of f, and conversely. This property is stated in the next theorem.

Theorem 1.36. Let f be a convex function defined on the closed convex set C in E_n. Then u is a vector of constant recession for f if and only if u is a vector of linear recession for some nonempty level set of f.

Proof: If u is a vector of constant recession for f, then $f(x + ru) = f(x)$ for all $x \in C$ and all $r \in R$. Let $x_0 \in C$ be arbitrary and let $f(x_0) = \alpha$. Then $\{x_0 + ru\colon r \in R\} \subset \mathrm{lev}_\alpha f$ and hence u is a vector of linear recession for $\mathrm{lev}_\alpha f$ which is a nonempty set.

If u is a vector of linear recession for some nonempty level set of f, say $\mathrm{lev}_\alpha f$, then $\{x + ru\colon r \in R\} \subseteq \mathrm{lev}_\alpha f$ for every x in $\mathrm{lev}_\alpha f$.

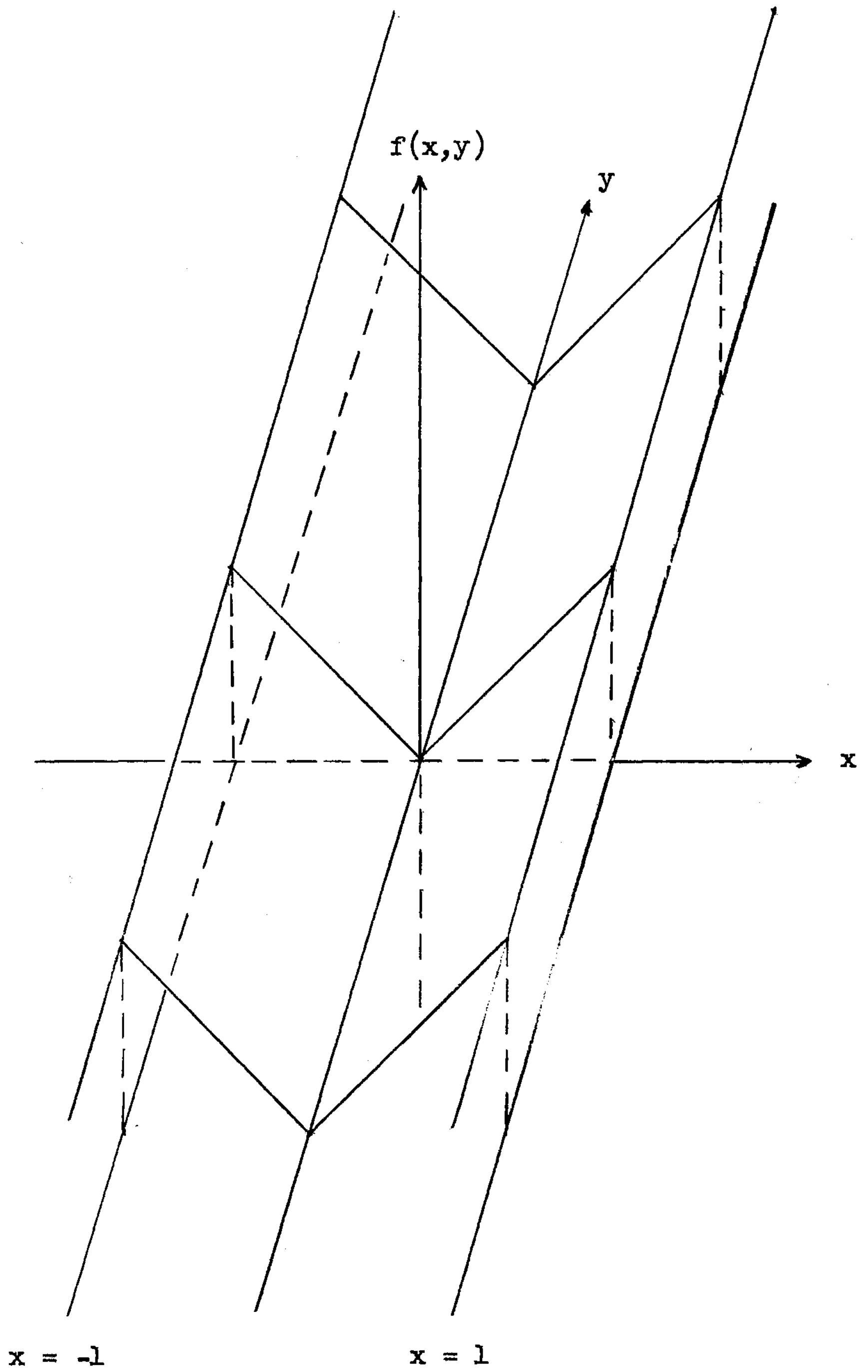


Figure 6.

In particular, for an arbitrary x_0 in $\text{lev}_\alpha f$, it follows that $\{(x_0 + ru, \alpha): r \geq 0\} \subset \text{epi } f$ and $\{(x_0 + r(-u), \alpha): r \geq 0\} \subset \text{epi } f$. Hence $u \in A(f)$ and $-u \in A(f)$ and by Theorem 1.33(a), u is a vector of constant recession for f.

Additional information on convex sets and convex functions can be found in Fenchel [3], Rockafellar [20], and Valentine [23]. Further properties of level sets can be found in the first two of these references and extensive information can be found in the second one concerning vectors of recession for convex sets and functions.

In Chapter II, the information developed in this chapter will be used to discuss the feasibility and solvability of convex programs.

CHAPTER II

FEASIBILITY AND SOLVABILITY OF A CONVEX PROGRAM

In this chapter, the results of Chapter I are used to formulate theorems concerning the feasibility and solvability of a convex program (P). The question of feasibility is discussed first with the main theorem based on the finite version of Helley's Theorem (Corollary 1.18). Next, the major theorems in Chapter I concerning the intersection of certain collections of convex sets are used to establish theorems on the solvability of a convex program (P).

Feasibility of a Convex Program

If (P) is a convex program with domain C, objective function F, and constraint functions $\{f_1, . . . , f_m\}$, then it follows by Lemma 1.28 that $\{x \in C: f_i(x) \leq 0\}$ is a convex set for each $i = 1, . . . , m$. Since the set of feasible solutions for (P) can be expressed as

$$S = \{x \in C: f_1(x) \leq 0\} \cap . . . \cap \{x \in C: f_m(x) \leq 0\},$$

it follows by Lemma 1.5 that S is a convex set. If the domain C of (P) is a closed set, then each of the sets $\text{lev}_0 f_i = \{x \in C: f_i(x) \leq 0\}$ is closed for $i = 1, . . . , m$. Hence, when the domain C of (P) is closed, the set of feasible solutions S for (P) is a closed convex set. In addition, if the set S is nonempty and the constraint functions have no vectors of recession in common, then Theorem 1.14 implies that

$$\begin{aligned} A(S) &= A(\mathrm{lev}_0 f_1 \cap \ldots \cap \mathrm{lev}_0 f_m) \\ &= A(\mathrm{lev}_0 f_1) \cap \ldots \cap A(\mathrm{lev}_0 f_m) \\ &= \{0\}. \end{aligned}$$

Consequently, the set of feasible solutions under these assumptions is a bounded set by Theorem 1.15.

Considering the above remarks on the set of feasible solutions for a convex program (P), it is apparent that asking when the set S is nonempty is equivalent to asking when a finite collection of convex sets in E_n has a nonempty intersection. Certainly, if $\mathrm{lev}_0 f_i$ is empty for one of the constraint functions for (P), then S is trivially empty, and (P) is not feasible. Therefore, the only interesting situation occurs when $\mathrm{lev}_0 f_i$ is nonempty for each of the constraint functions for (P). The most appropriate result to use in this case is the finite version of Helley's Theorem which is Corollary 1.18. Since the domain C of (P) is in E_n, Corollary 1.18 says that (P) is feasible if every subcollection of n + 1 sets from the collection $\{\mathrm{lev}_0 f_i : i = 1, \ldots, m\}$ has a nonempty intersection. This discussion is stated precisely in the next theorem.

Theorem 2.1. Let (P) be a convex program with domain C in E_n. Let $\{f_1, \ldots, f_m\}$ be the set of constraint functions for (P). If every subcollection of n + 1 sets from the collection $\{\mathrm{lev}_0 f_i : i = 1, \ldots, m\}$ has a nonempty intersection, then (P) is feasible.

Proof: The hypothesis of the theorem implies by means of Corollary 1.18 that $\mathrm{lev}_0 f_1 \cap \ldots \cap \mathrm{lev}_0 f_m = S \neq \emptyset$. Hence (P) is feasible.

Solvability of a Convex Program

A convex program (P) is solvable if the objective function F attains its minimum over the set of feasible solutions S for (P). Since previous discussion showed that the set of feasible solutions for a convex program is a convex set, the question of determining when a convex program is solvable is the same as determining when a convex function attains its minimum over a convex set.

If the set of feasible solutions S for (P) is a closed bounded set, then (P) is solvable since all functions considered in this paper are continuous on their domains of definition. In particular, F is continuous on its domain C so it is continuous on the closed bounded set S which is a subset of C. Some basic results concerning the solvability of a convex program are given by the next theorem.

Theorem 2.2. Let (P) be a feasible convex program whose domain C is a closed set. Let $\{f_1, \ldots, f_m\}$ be the set of constraint functions for (P). Assume that one of the following is true.

(a). C is a bounded set, or

(b). The set of constraint functions have no vectors of recession in common, or

(c). $\mathrm{lev}_0 f_i$ is a bounded set for some $i = 1, \ldots, m$.

Then (P) is a solvable convex program.

Proof: The set C is closed which implies that the set of feasible solutions S is closed. If (a) is true, then S is also a bounded set and hence compact. Thus the objective function attains its infimum over S because of its continuity.

Assume that (b) is true. Since S is nonempty by hypothesis, it follows that $A(S) = \{0\}$ as discussed above. Hence S is bounded by Theorem 1.15 and the proof of (a) implies that (P) is solvable.

Assume that (c) is true. Then the fact that S is contained in $\text{lev}_0 f_i$, for each i, implies that S is bounded. Again, the proof of (a) implies that (P) is solvable.

The main property in Theorem 2.2 which implies the solvability of (P) is the closed and bounded property of the set of feasible solutions S. However, not all sets of feasible solutions to a convex program (P) are bounded as was seen in Example 1.3, so it is desirable to construct hypotheses which imply that a convex program (P) is solvable but which does not rely on the compactness of the set of feasible solutions. The following theorem is one of the two main results presented in this chapter regarding the solvability of a convex program (P). Its proof depends entirely on results developed in Chapter I.

<u>Theorem 2.3</u>. Let (P) be a feasible convex program whose domain C is a closed set. Assume that the objective function F of (P) and the set of feasible solutions S for (P) have no vectors of recession in common. Then (P) is a solvable convex program.

Proof: Since (P) is feasible and C is closed, the set of feasible solutions S for (P) is a nonempty closed convex set in E_n. Consider the collection of all nonempty level sets of F over S, denoted by $\{\text{lev}_\alpha F\colon \alpha \in A\}$, where $\text{lev}_\alpha F = \{x \in S\colon F(x) \le \alpha\}$. Since S is a nonempty closed convex set, each of these nonempty level sets is a closed convex set by Lemma 1.28. In fact, each of them is bounded. To prove this assertion, assume that one of the nonempty level sets of F over S is unbounded (hence they all are by Theorem 1.31). By Corollary 1.16, there exists a nontrivial vector $u \in E_n$ such that u is a vector of recession for all the nonempty level sets of F over S, and consequently is a vector of recession for S since every nonempty level set of F over S is contained in S. But u is also a vector of recession for F by Theorem 1.31, and hence u is a vector of recession common to the set of feasible solutions S and the objective function F, which contradicts the hypothesis. Consequently, $\{\text{lev}_\alpha F: \alpha \in A\}$ is a collection of nonempty closed bounded convex sets in E_n with the property that every subcollection of n + 1 sets has a nonempty intersection (since the level sets are nested). It then follows from Helley's Theorem, Theorem 1.17, that $\cap\{\text{lev}_\alpha F: \alpha \in A\}$ is a nonempty set.

Now let $x_0 \in \cap\{\text{lev}_\alpha F: \alpha \in A\}$. Certainly $x_0 \in S$, so it follows that $\beta \leq F(x_0)$ where $\beta = \inf\{F(x): x \in S\}$. Assume that $\beta < F(x_0)$. Then there exists an $\epsilon > 0$ such that $\beta + \epsilon < F(x_0)$. Since β is the infimum of F over S, it must be true that $\text{lev}_{\beta+\epsilon}F$ is a nonempty level set of F over S, and hence $\text{lev}_{\beta+\epsilon}F$ is in $\{\text{lev}_\alpha F: \alpha \in A\}$. Thus x_0 is in $\text{lev}_{\beta+\epsilon}F$, and it follows that $F(x_0) \leq \beta + \epsilon$, which is a contradiction. Therefore, $F(x_0) \leq \beta$ and it follows that $F(x_0) = \beta = \inf\{F(x): x \in S\}$. Hence (P) is solvable.

If F or S or both have no vectors of recession, then the hypothesis of Theorem 2.3 is trivially satisfied and (P) would be solvable. Notice that if any of the three assumptions in the hypothesis of Theorem 2.2 are true, then the set of feasible solutions S for (P) has no vectors of recession and hence is solvable by Theorem 2.3. However, Theorem 2.3 does not require that S be bounded in order that (P) be solvable; only that for every $x \in S$, $F(x + \lambda u)$ be an increasing function of λ for arbitrarily large λ whenever u is a vector of recession for S. If the objective function F has a vector of recession u, then (P) is solvable only if $u \notin A(S)$.

As mentioned previously, the vectors of recession for the set of feasible solutions S for a convex program (P) are exactly those vectors of recession common to all of the constraint functions, provided (P) is feasible. This property allows the formulation of the following corollary to Theorem 2.3.

Corollary 2.4. Let (P) be a feasible convex program whose domain C is closed. Assume that the objective function F of (P) and the constraint functions $\{f_1, \ldots, f_m\}$ of (P) have no vectors of recession in common. Then (P) is a solvable convex program.

Proof: By hypothesis, S is nonempty; hence it follows that $A(S) = A(\mathrm{lev}_0 f_1) \cap \ldots \cap A(\mathrm{lev}_0 f_m)$ by Theorem 1.14, since it is true that $S = \mathrm{lev}_0 f_1 \cap \ldots \cap \mathrm{lev}_0 f_m$. Therefore, any vector of recession for S is a vector of recession common to all of the constraint functions. Assume that the objective function F and S have a vector of recession in common. It now follows that the objective function F and all of the constraint functions have a vector of recession in common which contradicts the hypothesis of the theorem. Hence the objective function F and the set of feasible solutions S have no vectors of recession in common, and it follows from Theorem 2.3 that (P) is a solvable convex program.

Consider again Examples 1.3 and 1.4 of Chapter I. The convex program illustrated in Example 1.3 is not solvable. The reason is that the objective function and the set of feasible solutions have common vectors of recession; that is, $A(S) \cap A(F) \neq \{0\}$ (cf. Figure 5(a) for epi F). However, in Example 1.4, the convex program is solvable. This is due to the fact that the objective function has no vectors of recession and hence Theorem 2.3 is applicable.

Theorem 1.22 shows that under certain assumptions, a collection of nonempty closed convex sets in E_n which have no vectors of recession in common has a nonempty intersection. Theorem 1.26 shows that under the same assumptions, the requirement that the sets in the collection have no vectors of recession in common can be replaced by the alternate requirement that the sets have only vectors of linear recession in common and the collection will still have a nonempty intersection. Observing the method of proof in Theorem 2.3 and recalling Theorem 1.26, it is apparent that the nonempty level sets of the objective function F over the set of feasible solutions S for (P) can be allowed to have vectors of linear recession in common and the conclusion of Theorem 2.3 would still be valid. The following theorem is a result of this discussion.

Theorem 2.5. Let (P) be a feasible convex program whose domain C is a closed set. Assume that the objective function F of (P) has only vectors of constant recession with respect to the set of feasible solutions S for (P). Then (P) is a solvable convex program.

Proof: Consider the collection of all nonempty level sets of F over the set of feasible solutions S for (P), denoted by $\{\text{lev}_\alpha F: \alpha \in A\}$ where each $\text{lev}_\alpha F$ is defined as in the proof of Theorem 2.3. By the argument given in the proof of Theorem 2.3, every set in this collection is a nonempty closed convex set. Now every vector of recession for F is a vector of constant recession so by Theorems 1.34 and 1.31, every vector of recession for F is a vector of linear recession for $\text{lev}_\alpha F$ for every $\alpha \in A$. Since the nonempty level sets of F over S are nested, it follows that every subcollection of n + 1 sets from $\{\text{lev}_\alpha F: \alpha \in A\}$ has a nonempty intersection. Assume that the nonempty closed convex level sets of F over S have a vector of recession in common which is not a vector of linear recession. Then it follows by Theorem 1.31 and the contrapositive of Theorem 1.34 that F has a vector of recession which is not a vector of constant recession. This contradicts the hypothesis of the theorem. Hence $\{\text{lev}_\alpha F: \alpha \in A\}$ is a collection of nonempty closed convex sets in E_n with the properties that the only vectors of recession common to all the sets are vectors of linear recession and every subcollection of n + 1 sets has a nonempty intersection. By Theorem 1.26, it follows that $\cap\{\text{lev}_\alpha F: \alpha \in A\}$ is a nonempty set.

Now let $\beta = \inf\{F(x): x \in S\}$ and let $x_0 \in \cap\{\text{lev}_\alpha F: \alpha \in A\}$. Again x_0 is in S and by the argument given in the proof of Theorem 2.3, it follows that $F(x_0) = \beta$. Consequently, (P) is a solvable convex program.

Notice that the hypothesis of Theorem 2.5 does not require that every vector of recession for the set of feasible solutions S be a vector of constant recession for F; it only requires that every vector of recession for F be a vector of constant recession for F, and hence a vector of linear recession for S. From this discussion, it is possible to write the following corollary to Theorem 2.5.

Corollary 2.6. Let (P) be a feasible convex program whose domain C is a closed set. Assume that the objective function F of (P) and the constraint functions $\{f_1, \ldots, f_m\}$ have only vectors of constant recession in common. Then (P) is a solvable convex program.

Proof: The vectors of recession of the set of feasible solutions S for (P) are exactly those vectors of recession common to all of the constraint functions for (P). Assume now that the objective function F has a vector of recession with respect to S which is not a vector of constant recession for F. Let u be such a vector of recession for F. Then u is a vector of recession for every nonempty level set of F over S and hence is a vector of recession for S. Thus by previous discussion, u is a vector of recession common to all of the constraint functions and it follows that the objective function and the constraint functions have a vector of recession in common which is not a vector of constant recession. This contradicts the hypothesis of the theorem. Therefore, the objective function has only vectors of constant recession with respect to the set of feasible solutions S for (P), and it follows from Theorem 2.5 that (P) is a solvable convex program.

If the only vectors of recession for the nonempty level sets of the objective function F over the set of feasible solutions S are vectors of linear recession, then it follows from Theorem 1.36 that the only vectors of recession for F with respect to S are vectors of constant recession. Theorem 2.5 then implies that (P) is a solvable convex program. This result is stated formally as the next corollary.

Corollary 2.7. Let (P) be a feasible convex program whose domain C is a closed set. Let F be the objective function of (P) and S the set of feasible solutions for (P). Assume that for some real number α,

$\{x \in S\colon F(x) \le \alpha\}$ is a nonempty set whose only vectors of recession are vectors of linear recession. Then (P) is a solvable convex program.

In light of Theorem 2.5, the question arises as to whether the vectors of recession common to a convex function and its convex domain must be vectors of constant recession for the function in order that the function attain its infimum over its convex domain, where the domain is a closed set. A reasonable conjecture is the following:

<u>Conjecture:</u> Let f be a convex function defined on a closed convex set C and let u be a vector of recession common to f and C. Assume that $f(x + ru)$ is a constant function of $r \ge 0$, for all $x \in C$. Then f attains its infimum over C.

This conjecture is false as shown by the following example.

<u>Example 2.8.</u> Let $C = \{(x,y)\colon x > 0,\ y \ge 1/x\}$, and $f(x,y) = x$, where C is the domain of f. Then the only vectors of recession common to f and C are vectors of the form $u = (0,y)$ where $y > 0$. If $z \in C$, then $f(z + ru)$ is a constant function of $r \ge 0$, yet f does not attain its infimum over C.

Suppose now that (P) is a solvable convex program and x_0 is an optimal solution for (P). Then $\{x \in S\colon F(x) \le F(x_0)\}$ is a nonempty level set of F which is convex, and is closed if the domain C of (P) is closed. For an arbitrary convex program (P), let <u>inf F</u> denote the optimal value of (P), and <u>Min F</u> denote the set of all optimal solutions for (P). Then for (P), inf $F = F(x_0)$ and Min $F = \{x \in S\colon F(x) \le F(x_0)\}$. In the arbitrary case, it may be that Min F is empty. However, if (P) is solvable, then all of the nonempty level sets of F over S contain

Min F. Example 2.9 and Figure 7 illustrate this situation.

Example 2.9. Let $C = E_2$, $F(x,y) = x^2 + y^2$, $f_1(x,y) = -x$, and $f_2(x,y) = -y$. Then $S = \{(x,y): x \geq 0, y \geq 0\}$, $\inf F = 0$, and $\text{Min } F = \{(0,0)\}$. The collection of nonempty level sets of F over S can be denoted by $\{\text{lev}_\alpha F: \alpha \geq 0\}$ and Figure 7 illustrates how this collection of closed convex sets contain Min F.

Relationship of Solutions

It turns out that in the situation where (P) is a solvable convex program whose domain is closed, it is possible to use the nested property of the collection of nonempty level sets of the objective function F containing Min F to obtain a feasible solution as close, in the norm of E_n, to the convex set Min F as desired. The following theorem states this property precisely.

Theorem 2.10. Let (P) be a feasible convex program whose domain C is a closed convex set. Assume that the objective function F of (P) and the set of feasible solutions S for (P) have no vectors of recession in common. Then for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every feasible solution x satisfying $F(x) \leq \inf F + \delta$, there exists an optimal solution z such that $\|z - x\| < \epsilon$.

Proof: Theorem 2.3 implies that (P) is solvable and it follows from the proof of Theorem 2.3 that every nonempty level set of F over S is a compact convex set. Let $\epsilon > 0$ be arbitrary but fixed and let $B = \{x \in E_n: \|x\| < 1\}$. Consider the open set $M + \epsilon B$, where $M = \text{Min } F$, and note that M is contained in $M + \epsilon B$. If $S \subset M + \epsilon B$, then the conclusion is trivially true. Therefore, assume that S is not contained

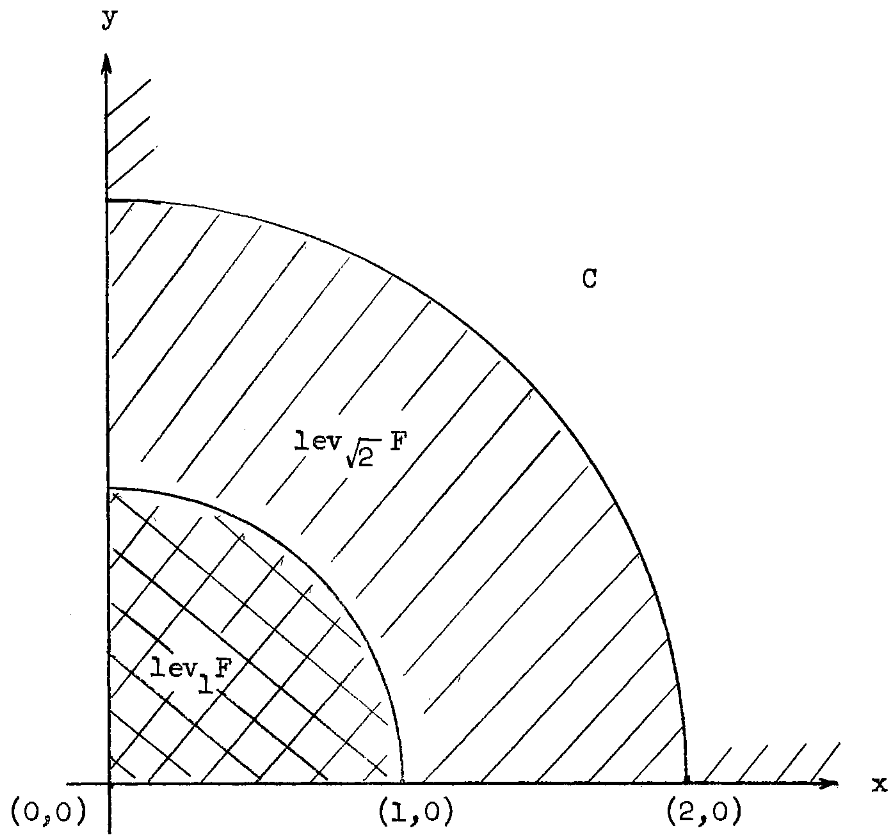


Figure 7.

in $M + \epsilon B$ and consider the nonempty closed set $S \setminus (M + \epsilon B)$. For each $\delta > 0$, let $T_\delta = (S \setminus (M + \epsilon B)) \cap \text{lev}_\alpha F$ where $\alpha = \inf F + \delta$. Since $\text{lev}_\alpha F$ is a compact set for each $\delta > 0$, each T_δ is a compact set. In addition, if $\delta_1 < \delta_2$, then $\alpha_1 = \inf F + \delta_1 < \inf F + \delta_2 = \alpha_2$, so $\text{lev}_{\alpha_1} F \subset \text{lev}_{\alpha_2} F$ and it follows that $T_{\delta_1} \subset T_{\delta_2}$. Assume that T_δ is non-empty for each $\delta > 0$. Then the sets $\{T_\delta : \delta > 0\}$ form a collection of nested nonempty compact sets and it must be true that $\cap\{T_\delta : \delta > 0\}$ is a nonempty set since the collection has the finite intersection property. Let $x_0 \in \cap\{T_\delta : \delta > 0\}$. Then $x_0 \in \text{lev}_\alpha F$ for every $\delta > 0$, so it follows that $x_0 \in \text{Min } F$. But x_0 must also be in $S \setminus (M + \epsilon B)$ and this contradicts the fact that $(\text{Min } F) \cap (S \setminus (M + \epsilon B))$ is an empty set. Consequently, there exists a $\delta_0 > 0$ such that, for $\alpha_0 = \inf F + \delta_0$, $T_{\delta_0} = (S \setminus (M + \epsilon B)) \cap \text{lev}_{\alpha_0} F = \emptyset$; that is, T_{δ_0} is an empty set. Since $\text{lev}_{\alpha_0} F \subset S$ but $\text{lev}_{\alpha_0} F \not\subset (S \setminus (M + \epsilon B))$, and it is true that $S = (S \setminus (M + \epsilon B)) \cup (S \cap (M + \epsilon B))$, it follows that $\text{lev}_{\alpha_0} F \subset (M + \epsilon B)$. Hence, for this arbitrary $\epsilon > 0$, δ_0 is a δ which gives the conclusion of the theorem.

The goal now is to rewrite Theorem 2.10 replacing the requirement that the objective function and the set of feasible solutions have no vectors of recession in common by the alternate requirement that the only vectors of recession for the set of feasible solutions are vectors of constant recession for the objective function. The next lemma is essential to the proof of this alternate form of Theorem 2.10.

Lemma 2.11. Let f be a convex function defined on a closed convex set C in E_n with the property that the only vectors of recession common to f and C are vectors of constant recession for f. If M is the subspace in E_n spanned by the vectors of constant recession for f, then the following statements are true.

(a). If M* is the orthogonal complement of M in E_n, then for any x and y in E_n, $(x + M) \cap (y + M^*)$ is a single point.

(b). If x_0 is an arbitrary point in C, then for every $z \in x_0 + M$, it is true that $f(z) = f(x_0)$.

Proof: To prove (a), note that there exists unique $m_1, m_2 \in M$ and unique $m_1^*, m_2^* \in M^*$ such that for any x, y in E_n, $x = m_1 + m_1^*$, and $y = m_2 + m_2^*$. Hence

$$x + M = m_1 + m_1^* + M = m_1^* + M$$

and

$$y + M^* = m_2 + m_2^* + M^* = m_2 + M^*.$$

Since 0 is in M and in M*, it follows that $m_1^* \in m_1^* + M$, and $m_2 \in m_2 + M^*$. Consequently,

$$(x + M) \cap (y + M^*) = (m_1^* + M) \cap (m_2 + M^*),$$

and hence $m_2 + m_1^* \in (x + M) \cap (y + M^*)$. If $z \in (x + M) \cap (y + M^*)$, then $z = m_1^* + m$, $m \in M$, and $z = m_2 + m^*$, $m^* \in M^*$. Since z can be expressed uniquely as the sum of a point in M and a point in M*, it follows that $m_1^* = m^*$, and $m_2 = m$. Thus $z = m_2 + m_1^*$ and the conclusion of (a) is immediate.

To prove (b), note first that M has a basis $\{u_1, \ldots, u_k\}$, $1 \leq k \leq n$, where each u_i is a vector of constant recession for f. The existence of this basis is guaranteed by the argument given in the proof of Theorem 1.25. By Theorem 1.34, each u_i is a vector of linear recession for C, so if x_0 is in C, it follows by Theorem 1.25 that $x_0 + M$ is contained in C. Since each u_i is a vector of constant recession for f, it is true that $f(x_0 + ru_i) = f(x_0)$ for all $r \in R$ and for each u_i, $1 \leq i \leq k$. If $z \in x_0 + M$, then there exists an $m \in M$ such that $z = x_0 + m$. Since $m \in M$, let $m = \alpha_1 u_1 + \cdots + \alpha_k u_k$ where each u_i is in the basis for M given above and $\alpha_i \in R$. Then

$$\begin{aligned} z &= x_0 + \alpha_1 u_1 + \cdots + \alpha_k u_k \\ &= (1/k)(x_0 + r_1 u_1) + \cdots + (1/k)(x_0 + r_k u_k), \end{aligned}$$

where $r_i = k(\alpha_i)$ for $1 \leq i \leq k$. By the convexity of f, it follows that

$$f(z) \leq (1/k)f(x_0 + r_1 u_1) + \cdots + (1/k)f(x_0 + r_k u_k) = f(x_0)$$

(cf. Rockafellar [20], p. 25, Theorem 4.3). Assume that $f(z) < f(x_0)$. Then consider $z_1 = x_0 - \alpha_1 u_1 - \cdots - \alpha_k u_k$. It follows that z_1 is in $x_0 + M$ and that $(1/2)(z + z_1) = x_0$. Again by the convexity of f,

$$f(x_0) = f((1/2)z + (1/2)z_1) \leq (1/2)f(z) + (1/2)f(z_1) < f(x_0).$$

This contradiction then implies that $f(z) = f(x_0)$. Since z was an arbitrary point in $x_0 + M$, it follows that (b) is true.

The alternate theorem to Theorem 2.10 discussed above can now be stated in the following manner.

<u>Theorem 2.12</u>. Let (P) be a solvable convex program whose domain C is a closed set. Assume that the only vectors of recession for the set of feasible solutions S for (P) are vectors of constant recession for the objective function F of (P). Then for every $\epsilon > 0$, there exists a

$\delta > 0$ such that for every feasible solution x satisfying $F(x) \leq \inf F + \delta$, there exists an optimal solution z such that $\|z - x\| < \epsilon$.

Proof: Let $M + \epsilon B$ be the same set defined in the proof of Theorem 2.10. Then $\text{Min } F \subset M + \epsilon B$ as seen earlier. If S is contained in $M + \epsilon B$, then the conclusion follows trivially. Consequently, assume that the set $S \setminus (M + \epsilon B)$ is nonempty. In addition, let N be the subspace spanned by the vectors of recession common to F and S, which are vectors of constant recession for F and vectors of linear recession for S. Let N^* be the orthogonal complement of N in E_n. If $z \in S \setminus (M + \epsilon B)$, then consider the set $T = (S \setminus (M + \epsilon B)) \cap (z + N^*)$. Since $z \in z + N^*$, T is nonempty. Also, T is bounded, for if it were unbounded, then S and N^* would have a vector of recession in common which would contradict the hypothesis of the theorem. Finally, T is a closed set being the intersection of two closed sets. Since T is a nonempty compact set contained in S and F is a continuous function defined on $T \subset S$, it is true that there exists a point $u \in T$ such that $F(u) = \inf\{F(x): x \in T\}$. Now suppose there exists $t \in S \setminus (M + \epsilon B)$ such that $F(t) < F(u)$. By Lemma 2.11(a), there exists a unique point $w \in (z + N^*) \cap (t + N)$. Since $t + N \subset S$ by Theorem 1.25, it follows that $w \in S$. Suppose now that $(t + N) \cap (M + \epsilon B) \neq \emptyset$, and let y be a point in this intersection. Then there exists an $x_0 \in \text{Min } F$ such that $\|y - x_0\| < \epsilon$. Let $v \in y + N$. Then there exists an $m_0 \in N$ such that $v = y + m_0$. Now $x_0 + m_0 \in \text{Min } F$ since $x_0 + N \subset \text{Min } F$ (cf. Theorems 1.25 and 1.34). It then follows that $v \in (M + \epsilon B)$ since

$$\|v - (x_0 + m_0)\| = \|(y + m_0) - (x_0 + m_0)\| = \|y - x_0\| < \epsilon.$$

Since $y + N = t + N$, it follows that $t + N$ is contained in $M + \epsilon B$ and hence $t \in M + \epsilon B$, which is a contradiction. Consequently, $(t + N) \cap (M + \epsilon B)$ is empty and it follows that $w \notin M + \epsilon B$. Thus $w \in T$ and therefore $F(u) \leq F(w)$. But $w \in t + N$, and N is the subspace spanned by the vectors of constant recession for F, so by Lemma 2.11(b), $F(w) = F(t) < F(u)$. This contradiction then implies that $F(u) \leq F(x)$ for every $x \in S \setminus (M + \epsilon B)$.

If $F(u) = \inf F$, then u must be in Min F which is a contradiction. Hence $F(u) > \inf F$. Let $\delta_0 = (1/2)[F(u) - \inf F]$. For $\alpha = \inf F + \delta_0$, $\text{lev}_\alpha F$ is contained in $(M + \epsilon B)$ since $\text{lev}_\alpha F \subseteq S$, $\text{lev}_\alpha F \cap [S \setminus (M + \epsilon B)] = \emptyset$, and $S = [S \setminus (M + \epsilon B)] \cup [S \cap (M + \epsilon B)]$. Therefore, δ_0 is a δ which gives the conclusion of the theorem.

Theorems 2.10 and 2.12 imply that under certain hypotheses, it is possible to construct a sequence of points in the set of feasible solutions S which converges to an optimal solution. The next corollary gives information concerning this sequence of feasible points.

Corollary 2.13. Let (P) be a solvable convex program whose domain C is a closed set. Let F be the objective function of (P) and S the set of feasible solutions for (P). If $\{x_i\}$ is a sequence in S which converges to an optimal solution for (P), then $\{F(x_i)\}$ converges to $\inf F$.

Proof: The conclusion of the corollary follows directly from the continuity of the objective function.

In review, Theorem 2.1 provides the main result in this chapter on the feasibility of a convex program (P) while Theorems 2.3 and 2.5 are the main theorems concerning solvability. In addition, Theorems

2.10 and 2.12 establish the theoretical basis for the practical solution of convex programs.

Additional information on the feasibility and solvability of convex programs can be found in Bracken [1], Mangasarian [18], and Rockafellar [20].

CHAPTER III

CHARACTERIZATION OF AN OPTIMAL SOLUTION

The objective of this chapter is to present necessary and sufficient conditions for an arbitrary feasible solution for a convex program (P) to be an optimal solution. The main result concerns the situation where the objective function and the constraint functions for an arbitrary convex program (P) are assumed continuous on the domain C of (P) but not necessarily differentiable there. A second result establishes necessary and sufficient conditions for an arbitrary feasible solution for (P) to be an optimal solution under the assumption of differentiability for the objective function and the constraint functions for (P) over the domain C of (P).

Separation Theorem

Before discussing the main theorems of this chapter, a few concepts necessary to their proofs need to be discussed. The first of these is the idea of a hyperplane in E_n. A <u>hyperplane</u> in E_n is a set which results from the translation of a maximal proper subspace of E_n. In E_2, maximal proper subspaces are lines through the origin so it follows that hyperplanes in E_2 are lines; in E_3, maximal proper subspaces are planes through the origin so hyperplanes in E_3 are planes. The characterization of a hyperplane given in the definition is obviously impractical to work with so a more useful one is desirable. It turns out that a much more convenient characterization of hyperplanes in E_n can be given in terms of the inner-product property of this space. The next theorem establishes this characterization and a proof for it can be found in Rockafellar [20], p. 5, Theorem 1.3.

Theorem 3.1. Let H be a hyperplane in E_n. Then there exists a nonzero vector u in E_n and a real number $\beta \in R$ such that $H = \{x \in E_n: x \cdot u = \beta\}$.

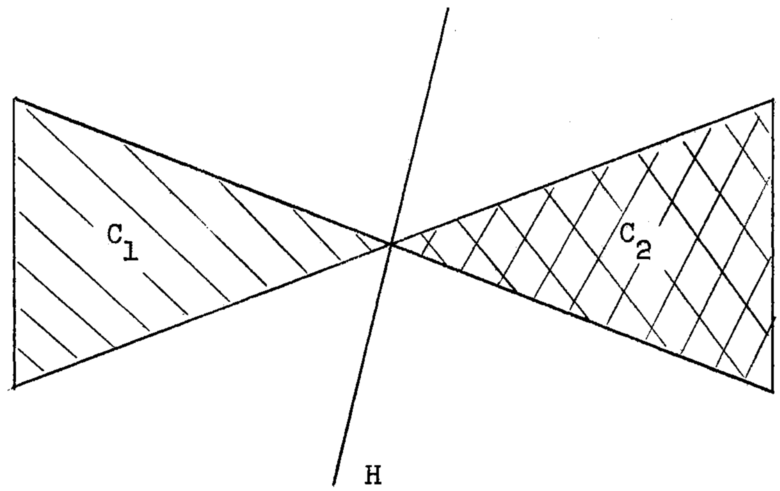
The concept of hyperplane and its characterization in terms of the inner-product of E_n can now be used to state a very important result in the theorey of convex sets needed in the proof of the main theorem in this chapter. This theorem, known as the Separation Theorem for convex sets, is stated as follows:

Theorem 3.2. (Separation Theorem). Let C_1 and C_2 be two convex sets in E_n such that the interior of C_1, int C_1, is nonempty and $(\text{int } C_1) \cap C_2 = \emptyset$. Then there exists a hyperplane $H = \{x \in E_n: x.u = \alpha\}$ in E_n such that $z \cdot u \leq \alpha$ for all $z \in C_1$ and $z \cdot u \geq \alpha$ for all $z \in C_2$.

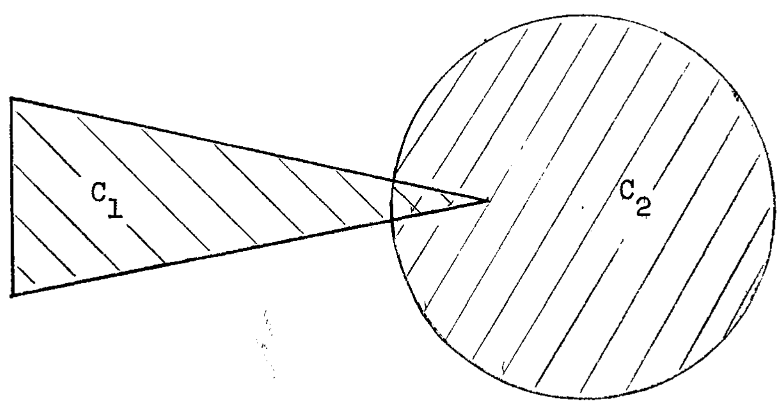
A proof for this theorem can be found in Luenberger [17], p. 133, Theorem 3, or Rockafellar [20], p. 97, Theorem 11.3, or Valentine [23], p. 24, Theorem 2.7. Figure 8(a) is an example of two convex sets in E_2 which can be separated by a hyperplane H while Figure 8(b) is an example of two convex sets in E_2 which cannot be separated by a hyperplane. Notice that in Figure 8(b) the hypothesis of the Separation Theorem is not satisfied since $(\text{int } C_1) \cap C_2$ is a nonempty set.

Lagrangian Function

A second concept that needs to be discussed is that of the



(a)



(b)

Figure 8.

Lagrangian function associated with a convex program (P). If (P) is a convex program with objective function F, constraint functions $\{f_1, \ldots, f_m\}$, and domain C, then the Lagrangian function L for (P) is defined as

$$L(x;u) = F(x) + u_1 f_1(x) + \ldots + u_m f_m(x),$$

where $u = (u_1, \ldots, u_m)$ is in the nonnegative orthant of E_m so that $u \geq 0$. If

$$E_m^+ = \{u = (u_1, \ldots, u_m) \in E_m: u_i \geq 0, i = 1, \ldots, m\},$$

then it follows that the domain of L is the set $C \times E_m^+$. One important property of the function L is the following:

<u>Lemma 3.3</u>. Let u_0 in E_m^+ be arbitrary but fixed. Then $L(x;u_0)$ is a convex function of x on the convex set C in E_n.

Proof: Let x and y be in C and let α be a real number such that $0 < \alpha < 1$. Then for $u_0 = (u_{01}, \ldots, u_{0m})$, and $\beta = 1 - \alpha$,

$$L(\alpha x + \beta y;u_0) = F(\alpha x + \beta y) + \sum_{j=1}^{m} u_{0j} f_j(\alpha x + \beta y)$$

$$\leq \alpha F(x) + \beta F(y) + \sum_{j=1}^{m} u_{0j}[\alpha f_j(x) + \beta f_j(y)]$$

$$= \alpha L(x;u_0) + \beta L(y;u_0).$$

A point $(x^*;u^*)$ in $C \times E_m^+$ is a <u>saddle-point</u> for the Lagrangian function L if and only if $L(x^*;u) \leq L(x^*;u^*) \leq L(x;u^*)$ for all $x \in C$

and for all $u \in E_m^+$. Therefore, if $(x^*;u^*)$ is a saddle-point for L, then for a fixed x^*, $L(x^*;u)$ attains a maximum with respect to u at u^*, and for a fixed u^*, $L(x;u^*)$ attains a minimum with respect to x at x^*. Consequently, $(x^*;u^*)$ is a saddle-point for L if and only if

$$\sup\{L(x^*;u): u \in E_m^+\} = L(x^*;u^*) = \inf\{L(x;u^*): x \in C\}.$$

Main Theorem

Using the concepts discussed previously, the main theorem on the characterization of an optimal solution for a convex program (P) can be written as follows:

Theorem 3.4. Let (P) be a convex program with domain C in E_n, objective function F, and constraint functions $\{f_1, \ldots, f_m\}$. Assume that the set $\{x \in C: f_i(x) < 0, i = 1, \ldots, m\}$ is nonempty. Then x^* in C is an optimal solution for (P) if and only if there exists a u^* in E_m^+ such that $L(x^*;u) \leq L(x^*;u^*) \leq L(x;u^*)$ for all $x \in C$ and for all $u \in E_m^+$.

Proof: Assume there exists an $x^* \in C$ and a $u^* \in E_m^+$ such that $L(x^*;u) \leq L(x^*;u^*) \leq L(x;u^*)$ for all $x \in C$ and for all $u \in E_m^+$. Then for all $u \in E_m^+$,

$$F(x^*) + \sum_{j=1}^{m} u_j f_j(x^*) \leq F(x^*) + \sum_{j=1}^{m} u_j^* f_j(x^*). \qquad (3.4.1)$$

From (3.4.1), it follows that for all $u \in E_m^+$,

$$\sum_{j=1}^{m}(u_j - u_j^*)f_j(x^*) \leq 0. \tag{3.4.2}$$

For each j, consider the vector $(u_1^*, \ldots, u_j^* + 1, \ldots, u_m^*)$ in E_m^+. Substitution of each of these vectors into (3.4.2) for u gives the result that $f_j(x^*) \leq 0$ for $j = 1, \ldots, m$. Consequently, x^* is a feasible solution for (P).

Now $u_j^* \geq 0$ for each j, and $f_j(x^*) \leq 0$ for each j implies that

$$\sum_{j=1}^{m} u_j^* f_j(x^*) \leq 0. \tag{3.4.3}$$

Since the vector $(0, \ldots, 0) \in E_m^+$, it follows from (3.4.2) that

$$\sum_{j=1}^{m} u_j^* f_j(x^*) \geq 0. \tag{3.4.4}$$

Therefore, from (3.4.3) and (3.4.4),

$$\sum_{j=1}^{m} u_j^* f_j(x^*) = 0. \tag{3.4.5}$$

Suppose that for some j, $1 \leq j \leq m$, it is true that $u_j^* f_j(x^*) < 0$. It follows that

$$\sum_{j=1}^{m} u_j^* f_j(x^*) < 0, \tag{3.4.6}$$

and this contradicts (3.4.5). Thus for each $j = 1, \ldots, m$,

$$u_j^* f_j(x^*) = 0. \tag{3.4.7}$$

Using (3.4.5), the inequality $L(x^*;u^*) \leq L(x;u^*)$ can be rewritten as

$$F(x^*) \leq F(x) + \sum_{j=1}^{m} u_j^* f_j(x), \tag{3.4.8}$$

for all $x \in C$. Consider now only those points in the set of feasible solutions S for (P). For each $x \in S$, $f_j(x) \leq 0$, for $j = 1, \ldots, m$. Combining this with the fact that $u_j^* \geq 0$, for each j, it follows that $u_j^* f_j(x) \leq 0$ for each j and for all $x \in S$. Hence,

$$\sum_{j=1}^{m} u_j^* f_j(x) \leq 0, \tag{3.4.9}$$

for all $x \in S$. Therefore, it follows from (3.4.8) that $F(x^*) \leq F(x)$ for all $x \in S$. Thus x^* is an optimal solution for (P).

To complete the proof of the theorem, assume now that x^* in S is an optimal solution for (P). It must be shown that there exists a u^* in E_m^+ such that $(x^*;u^*)$ is a saddle-point for L. Define

$$K_1 = \{(t_0, \ldots, t_m) \in E_{m+1}: t_0 \geq F(x), t_i \geq f_i(x), \text{ for some } x \in C\},$$

$$K_2 = \{(t_0, \ldots, t_m) \in E_{m+1}: t_0 < F(x^*), t_i < 0, i = 1, \ldots, m\}.$$

Since $(F(x^*), f_1(x^*), \ldots, f_m(x^*)) \in K_1$, it follows that $K_1 \neq \emptyset$. Note also that K_2 is nonempty since it is unbounded. Suppose that $K_1 \cap K_2 \neq \emptyset$. Then there exists a vector z in C such that (a). $F(z) \leq t_0 < F(x^*)$

(b). $f_i(z) \leq t_i < 0$, for $i = 1, \ldots, m$.

From (b), it follows that $z \in S$. Thus by (a), x^* is not optimal and this contradicts the hypothesis. Therefore, $K_1 \cap K_2 = \emptyset$. Since K_2 is an open set, the interior of K_2 is exactly K_2 so it is true that $(\text{int } K_2) \cap K_1 = \emptyset$. In addition, K_1 and K_2 are convex sets in E_{m+1} so these two sets satisfy the hypothesis of Theorem 3.2. Hence, there exists a hyperplane $H = \{x \in E_{m+1}: \alpha \cdot x = \beta\}$, where $\alpha \in E_{m+1}$ and $\alpha \neq 0$, such that $\alpha \cdot y_1 \geq \beta \geq \alpha \cdot y_2$ for all $y_1 \in K_1$ and all $y_2 \in K_2$. If $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m)$, then $\alpha_i \neq 0$ for at least one i. Assume that $\alpha_i < 0$. Then for arbitrary $(t_0, \ldots, t_m) \in K_1$ and $(s_0, \ldots, s_m) \in K_2$,

$$\alpha_i(t_i - s_i) + \sum_{\substack{j=1 \\ j \neq i}}^{m} \alpha_j(t_j - s_j) \geq 0. \tag{3.4.10}$$

Since K_2 is unbounded, for any $(t_0, \ldots, t_m) \in K_1$, it is possible to choose $s_i < 0$ so large in absolute value that

$$\alpha_i(t_i - s_i) + \sum_{\substack{j=1 \\ j \neq i}}^{m} \alpha_j(t_j - s_j) < 0. \tag{3.4.11}$$

But (3.4.11) contradicts (3.4.10) so it follows that $\alpha_i \geq 0$ for each i and for at least one $i = 0, 1, \ldots, m$, $\alpha_i > 0$ (since $\alpha \neq 0$).

Let $(F(x), f_1(x), \ldots, f_m(x))$ be in K_1 for some $x \in C$ and consider the vector $(F(x^*), 0, \ldots, 0)$ in $cl(K_2)$, the closure of K_2.

Then there exists a sequence $\{w_n\}$ in K_2 such that $\{w_n\}$ converges to $(F(x^*), 0, \ldots, 0)$. Since $\alpha \cdot w_n \leq \beta$ for each n, it follows by the continuity of the inner-product that for all $x \in C$,

$$\alpha_0 F(x) + \alpha_1 f_1(x) + \ldots + \alpha_m f_m(x) \geq \beta \geq \alpha_0 F(x^*). \tag{3.4.12}$$

If $\alpha_0 = 0$, then for all $x \in C$, (3.4.12) becomes

$$\alpha_1 f_1(x) + \alpha_2 f_2(x) + \ldots + \alpha_m f_m(x) \geq 0. \tag{3.4.13}$$

However, by the hypothesis of the theorem, there exists a $z \in C$ such that $f_j(z) < 0$ for all j. Since $\alpha_j > 0$ for at least one j, it follows that

$$\alpha_1 f_1(z) + \alpha_2 f_2(z) + \ldots + \alpha_m f_m(z) < 0. \tag{3.4.14}$$

Obviously, (3.4.14) contradicts (3.4.13) so it must be true that $\alpha_0 > 0$. Consequently, each term in (3.4.12) can be divided by α_0 resulting in

$$F(x) + \frac{\alpha_1}{\alpha_0} f_1(x) + \ldots + \frac{\alpha_m}{\alpha_0} f_m(x) \geq F(x^*), \tag{3.4.15}$$

for all $x \in C$. Letting $u_j^* = \alpha_j/\alpha_0$ for $j = 1, \ldots, m$, (3.4.15) can be rewritten as

$$L(x;u^*) = F(x) + u_1^* f_1(x) + \ldots + u_m^* f_m(x) \geq F(x^*), \tag{3.4.16}$$

for all $x \in C$. If the point selected in C is x^*, then (3.4.16) implies that

$$\sum_{j=1}^{m} u_j^* f_j(x^*) \geq 0. \tag{3.4.17}$$

But $x^* \in S$ implies that $f_j(x^*) \leq 0$ for all j. Since $u_j^* \geq 0$ for all j, it follows that

$$\sum_{j=1}^{m} u_j^* f_j(x^*) \leq 0. \tag{3.4.18}$$

Thus (3.4.17) and (3.4.18) imply that

$$\sum_{j=1}^{m} u_j^* f_j(x^*) = 0. \tag{3.4.19}$$

It follows directly from (3.4.19) that

$$L(x^*;u^*) = F(x^*) + \sum_{j=1}^{m} u_j^* f_j(x^*) = F(x^*). \tag{3.4.20}$$

Certainly for all $u = (u_1, \ldots, u_m)$ in E_m^+, it is true that

$$\sum_{j=1}^{m} u_j f_j(x^*) \leq 0, \tag{3.4.21}$$

so it follows that for all $u \in E_m^+$,

$$F(x^*) \geq F(x^*) + \sum_{j=1}^{m} u_j f_j(x^*) = L(x^*;u). \tag{3.4.22}$$

Combining the results given in (3.4.16), (3.4.20), and (3.4.22), it follows that

$$L(x^*;u) \leq L(x^*;u^*) \leq L(x;u^*),$$

for all $x \in C$ and for all $u \in E_m^+$, which is the desired conclusion.

A careful examination of the proof of Theorem 3.4 shows that it is possible to characterize the vector u^* in E_m^+ in a manner which allows Theorem 3.4 to be written in an equivalent form. Using (3.4.5) and (3.4.7) from the proof of Theorem 3.4, the following corollary gives this equivalent form.

Corollary 3.5. Let (P) be a convex program with domain C in E_n, objective function F, and constraint functions $\{f_1, \ldots, f_m\}$. Assume that the set $\{x \in C: f_i(x) < 0, i = 1, \ldots, m\}$ is nonempty. Then x^* in C is an optimal solution for (P) if and only if there exists a u^* in E_m^+ with the properties that

(a). $F(x^*) = \inf\{L(x;u^*): x \in C\}$,

(b). $u_j^* f_j(x^*) = 0$, for $j = 1, \ldots, m$,

(c). $f_j(x^*) \leq 0$, for $j = 1, \ldots, m$.

Proof: Assume that x^* is an optimal solution for (P). Then by Theorem 3.4, there exists a $u^* \in E_m^+$ such that $L(x^*;u) \leq L(x^*;u^*) \leq L(x;u^*)$ for all $x \in C$ and for all $u \in E_m^+$. Condition (b) follows from (3.4.7), and condition (c) is a direct consequence of the optimality of x^*. Condition (b) now implies that

$$F(x^*) \leq L(x;u^*)$$

for all $x \in C$. Since x^* is in C, it follows that condition (a) is true.

To complete the proof, assume now that there exists an x^* in C and a u^* in E_m^+ such that (a), (b), and (c) are true. From (a) and (b), it follows that

$$L(x^*;u^*) = F(x^*) \leq L(x;u^*) \qquad (3.5.1)$$

for all $x \in C$. Let $u = (u_1, \ldots, u_m)$ be in E_m^+. Then $u_j \geq 0$, for all j, so it follows that $u_j f_j(x^*) \leq 0$ for all j by condition (c). Certainly, it is true that

$$\sum_{j=1}^{m} u_j f_j(x^*) \leq 0$$

for all $u \in E_m^+$. Therefore,

$$L(x^*;u) = F(x^*) + \sum_{j=1}^{m} u_j f_j(x^*) \leq F(x^*) = L(x^*;u^*). \qquad (3.5.2)$$

Combining (3.5.1) and (3.5.2), it follows that $(x^*;u^*)$ is a saddle-point for L. Hence, x^* is an optimal solution for (P) by Theorem 3.4.

Note that Corollary 3.5 says that if x^* is an optimal solution for (P), then there exists a vector u^* in E_m^+ such that the optimal value for (P) is equal to the infimum of $L(x;u^*)$ over the domain C of (P). Suppose that such a vector $u^* \in E_m^+$ is known for a convex program (P). Then instead of first determining the set of feasible solutions S for (P) and then minimizing the objective function F over S, it hopefully would be possible to determine the minimum set for $L(x;u^*)$

over C and then discard those vectors which did not satisfy certain constraints with the resulting set being exactly Min F. The next definition and theorem establish this procedure precisely.

Definition 3.6. Let (P) be a convex program with domain C in E_n. Then a vector u* in E_m^+ is a solution vector for (P) if and only if

$$\inf\{L(x;u^*): x \in C\} = \inf F.$$

Using Definition 3.6, the above discussion can be formulated into the following theorem.

Theorem 3.7. Let (P) be a solvable convex program with domain C in E_n. If u* in E_m^+ is a solution vector for (P), then

$$\text{Min } F = S \cap M \cap T,$$

where S is the set of feasible solutions for (P),

$M = \{x \in C: u_j^* f_j(x) = 0, \quad j = 1, \ldots, m\}$, and

$T = \{z \in C: L(z;u^*) \leq L(x;u^*), \text{ for all } x \in C\}$.

Proof: Let $t \in$ Min F. Then $F(t) = \inf\{L(x;u^*): x \in C\}$ since u* is a solution vector for (P). Therefore, since $t \in C$,

$$F(t) \leq F(t) + \sum_{j=1}^{m} u_j^* f_j(t),$$

and it follows that

$$\sum_{j=1}^{m} u_j^* f_j(t) \geq 0. \qquad (3.7.1)$$

But t being an optimal solution implies that $t \in S$ so it must be true that

$$\sum_{j=1}^{m} u_j^* f_j(t) \leq 0. \qquad (3.7.2)$$

Now (3.7.1) and (3.7.2) imply that $u_j^* f_j(t) = 0$ for $j = 1, \ldots, m$ so it follows that $t \in M$. Then (3.7.1) and (3.7.2) also imply that $L(t;u^*) = F(t) = \inf F \leq L(x;u^*)$ for all $x \in C$, so $t \in T$. Therefore, Min F is contained in $S \cap M \cap T$.

To show the inclusion in the other direction, let $t \in S \cap M \cap T$. Then $t \in C$ and

$$L(t;u^*) = \inf\{L(x;u^*): x \in C\} = \inf F \qquad (3.7.3)$$

since u* is a solution vector. Since t is in M,

$$\sum_{j=1}^{m} u_j^* f_j(t) = 0.$$

Therefore, (3.7.3) can be rewritten as

$$L(t;u^*) = F(t) = \inf F.$$

Since t is a feasible solution for (P), it must be true that t is also an optimal solution for (P) and hence $t \in$ Min F. The proof of the theorem is now complete.

As an illustation of the concepts presented in Theorem 3.4, consider the following example.

Example 3.8. Let $C = E_1$, $F(x) = e^{-x}$, and $f(x) = e^{-x} - 1$. From Example 1.3, it follows that $S = \{x \in E_1: x \geq 0\}$ and that the convex program so defined is not solvable. This assertion can be verified by Theorem 3.4. Suppose that x^* in S is optimal for this convex program (P). Then by Theorem 3.4, there exists a u^* in E_1^+ such that $L(x^*;u) \leq L(x^*;u^*) \leq L(x;u^*)$ for all x in E_1. From the proof of Theorem 3.4, it follows that $u^*f(x^*) = u^*(e^{-x^*} - 1) = 0$. If $x^* = 0$, then it is not optimal. If $x^* > 0$, then u^* must be 0, and it follows that for any $x > x^*$, $L(x;u^*) = F(x) < F(x^*) = L(x^*;u^*)$ which contradicts the optimality of x^*. Hence (P) cannot be solvable, which is clearly true.

Differentiable Convex Programs

Up to this point, there has been no explicit requirement that the objective function and the constraint functions for (P) be differentiable with respect to an appropriate open subset of E_n. Of course, even if they are, Theorem 3.4, Corollary 3.5, and Theorem 3.7 are still valid. However, in the case where these functions are differentiable, these above results, particularly Corollary 3.5, can be modified to make use of this differentiability property in stating the hypotheses. The goal now is to assume differentiability for the aforementioned functions and present the modified form of Corollary 3.5.

Before turning to this modified theorem, it is necessary to discuss an important property of differentiable convex functions. This property is given by the next theorem, a proof for which can be found in Fleming [7], p. 53, Proposition 9a, or Mangasarian [18], p. 84, Theorem 3.

<u>Theorem 3.9</u>. Let f be differentiable on an open convex set C in E_n. Then f is convex on C if and only if

$$f(x) - f(y) \geq (x - y)\cdot[\text{grad } f(y)]$$

for every x and y in C.

Here, grad f(y) represents the gradient vector of f at y and it is an n-dimensional vector whose components are the partial derivatives f_i of f evaluated at y. Appropriate definitions and results concerning the differential calculus for real valued functions of several variables can be found in Fleming [7], Chapter 2.

The terms defined here are used with respect to the Lagrangian function L for a convex program (P). The partial derivative of L with respect to the i-th variable x_i of the vector $x \in C$ is denoted by L_i; the partial derivative of L with respect to the j-th variable u_j of the vector $u \in E_m^+$ is denoted by L_j. Also, in order to conform to the fact that a differentiable function is defined on an open set, all convex programs (P) whose objective function and constraint functions are differentiable are assumed to have an open convex set for their domain. These preparatory remarks now allow the following theorem to be stated.

<u>Theorem 3.10</u>. Let (P) be a convex program with domain C an open set in E_n, objective function F, and constraint functions $\{f_1, \ldots, f_m\}$. Assume that $\{x \in C: f_j(x) < 0, j = 1, \ldots, m\}$ is nonempty and that F and each f_j are differentiable on C. Then x* in C is an optimal solution for (P) if and only if there exists a u* in E_m^+ such that

(a). $L_j(x^*;u^*) \leq 0, \quad j = 1, \ldots, m,$

(b). $u_j^* L_j(x^*;u^*) = 0, \quad j = 1, \ldots, m,$

(c). $L_i(x^*;u^*) = 0, \quad i = 1, \ldots, n.$

Proof: Assume that x^* is an optimal solution for (P). Then it follows from Theorem 3.4 that for all x in C and for all u in E_m^+,

$$L(x^*;u) \leq L(x^*;u^*) \leq L(x;u^*). \qquad (3.10.1)$$

Since x^* is optimal, $L_j(x^*;u^*) = f_j(x^*) \leq 0$ for all j, so (a) is satisfied.

By (3.4.7), $u_j^* f_j(x^*) = u_j^* L_j(x^*;u^*) = 0$, for $j = 1, \ldots, m$, so (b) is satisfied.

Assume now that for some $i = 1, \ldots, n$, $L_i(x^*;u^*) < 0$. Then by the definition of partial derivative (cf. Fleming [7], p. 37),

$$L_i(x^*;u^*) = \lim_{t \to 0} \frac{L((x_1^*, \ldots, x_i^*+t, \ldots, x_n^*);u^*) - L(x^*;u^*)}{t} < 0.$$

Since C is an open set, there exists a $t_0 > 0$ such that the vector $(x_1^*, \ldots, x_i^*+t_0, \ldots, x_n^*)$ is in C and

$$\frac{L((x_1^*, \ldots, x_i^*+t_0, \ldots, x_n^*);u^*) - L(x^*;u^*)}{t_0} < 0.$$

Therefore, the numerator is negative and hence it must be true that $L((x_1^*, \ldots, x_i^*+t_0, \ldots, x_n^*);u^*) < L(x^*;u^*)$, a contradiction of (3.10.1). Consequently, $L_i(x^*;u^*) \geq 0$ for $i = 1, \ldots, n$.

Assume now that for some $i = 1, \ldots, n$, it is true that

$L_i(x^*;u^*) > 0$. Then there exists a $t_0 > 0$ such that the point $(x_1^*, \ldots, x_i^*-t_0, \ldots, x_n^*)$ is in C and

$$\frac{L((x_1^*, \ldots, x_i^*-t_0, \ldots, x_n^*);u^*) - L(x^*;u^*)}{-t_0} > 0.$$

Again the numerator must be negative and it follows that $L((x_1^*, \ldots, x_i^*-t_0, \ldots, x_n^*);u^*) < L(x^*;u^*)$, which again contradicts (3.10.1). Therefore, $L_i(x^*;u^*) \leq 0$ for $i = 1, \ldots, n$. It follows from this inequality and the one above that $L_i(x^*;u^*) = 0$ for $i = 1, \ldots, n$ so (c) is satisfied.

For the proof in the other direction, assume that there exists a vector x^* in C and a vector u^* in E_m^+ such that (a), (b), and (c) are true. Since u^* is fixed, it follows from Lemma 3.3 that $L(x;u^*)$ is a convex function of x on the convex set C. Since x^* is in C, Theorem 3.9 implies that for all $x \in C$,

$$L(x;u^*) - L(x^*;u^*) \geq (x - x^*)\cdot[\mathrm{grad}_x L(x^*;u^*)], \qquad (3.10.2)$$

where $\mathrm{grad}_x L(x^*;u^*)$ represents the gradient with respect to the vector x of the function $L(x;u^*)$ evaluated at x^*. Since the i-th component of $\mathrm{grad}_x L(x^*;u^*)$ is $L_i(x^*;u^*)$, it follows from (c) that $\mathrm{grad}_x L(x^*;u^*)$ is equal to zero. Consequently, (3.10.2) implies that

$$L(x^*;u^*) \leq L(x;u^*) \qquad (3.10.3)$$

for all $x \in C$.

From (a), $L_j(x^*;u^*) = f_j(x^*) \leq 0$, for $j = 1, \ldots, m$ so x^* is a feasible solution for (P).

From (c), it follows that

$$\sum_{j=1}^{m} u_j^* L_j(x^*;u^*) = \sum_{j=1}^{m} u_j^* f_j(x^*) = 0. \qquad (3.10.4)$$

Observing that for any $u = (u_1, \ldots, u_m) \in E_m^+$, $u_j f_j(x^*) \leq 0$ for $j = 1, \ldots, m$ so it follows that

$$\sum_{j=1}^{m} u_j f_j(x^*) \leq 0 = \sum_{j=1}^{m} u_j^* f_j(x^*). \qquad (3.10.5)$$

By (3.10.4), $L(x^*;u^*) = F(x^*)$. Thus it follows from (3.10.5) that

$$L(x^*;u) = F(x^*) + \sum_{j=1}^{m} u_j f_j(x^*) \leq F(x^*) = L(x^*;u^*) \qquad (3.10.6)$$

for all $u \in E_m^+$. Combining (3.10.3) and (3.10.6), it is apparent that

$$L(x^*;u) \leq L(x+;u^*) \leq L(x;u^*) \qquad (3.10.7)$$

for all $x \in C$ and for all $u \in E_m^+$. Thus for $x^* \in C$, there exists a u^* in E_m^+ such that (3.10.7) is true; it then follows that x^* is an optimal solution for (P) by Theorem 3.4.

Recall from the calculus of several variables that the gradient of a differentiable function evaluated at a point x^* represents a vector which gives the direction from x^* in which the function is increasing the maximum amount. In convex programming, for a point in the domain C of (P), it is of interest to know the direction from this point in which the objective function is decreasing. In the differentiable case, for any point x^* in the domain C of (P), it follows that -grad $F(x^*)$ gives the direction from x^* in which the objective function F is decreasing the maximum amount. With this in mind, notice that Theorem 3.10 implies that x^* in C is optimal for (P) if and only if there exists a u^* in E_m^+ such that x^* is feasible, $u_j^* f_j(x^*) = 0$ for all j, and

$$-\text{grad } F(x^*) = u_1^*[\text{grad } f_1(x^*)] + \ldots + u_m^*[\text{grad } f_m(x^*)].$$

The next two examples illustrate these concepts given by Theorem 3.10.

<u>Example 3.11</u>. Let $C = E_2$, $F(x,y) = x^2 + 1$, $f_1(x,y) = x - 1$, and $f_2(x,y) = -x - 1$. Then it follows that the set of feasible solutions is $S = \{(x,y) \in E_2 : -1 \leq x \leq 1\}$ and the convex program so defined is solvable. In fact, Min $F = \{(x,y) \in E_2 : x = 0\}$. For any optimal solution $x^* = (0,y)$ for (P), $f_1(x^*) = f_2(x^*) = -1$. Since $u_i^* f_i(x^*)$ must be zero for $i = 1, 2$, it follows that $u_1^* = u_2^* = 0$. Noting that grad $F(x,y) = (2x,0)$, it is immediate that grad $F(x^*) = (0,0)$. Hence,

$$-\text{grad } F(x^*) = (0,0) = u_1^*[\text{grad } f_1(x^*)] + u_2^*[\text{grad } f_2(x^*)]$$

since $u^* = (0,0)$. Therefore, if x^* is optimal for (P), then there exists a u^* in E_2^+, namely $u^* = (0,0)$, such that $u_i^* f_i(x^*) = 0$ for $i = 1, 2$, and -grad $F(x^*) = u_1^*[\text{grad } f_1(x^*)] + u_2^*[\text{grad } f_2(x^*)]$. Note also that for all $u = (u_1, u_2) \in E_2^+$, $u_1 f_1(x^*) + u_2 f_2(x^*) \leq 0$. So it follows that for any x^* in Min F, there exists a $u^* = (0,0)$ in E_2^+ such that $L(x^*;u) \leq L(x^*;u^*) \leq L(x;u^*)$ for all $x \in E_2$ and for all $u \in E_2^+$.

<u>Example 3.12</u>. Let $C = E_2$, $F(x,y) = x^2 - y + 2$, $f_1(x,y) = x - y$,

$f_2(x,y) = y - 1$, and $f_3(x,y) = -x$. Then the convex program so defined has $S = \{(x,y) \in E_2\colon 0 \leq x \leq 1,\ x \leq y \leq 1\}$ as its set of feasible solutions and the optimal solution is $x^* = (0,1)$. Also, grad $F(x,y)$ is given by $(2x,-1)$, grad $f_1(x,y) = (1,-1)$, grad $f_2(x,y) = (0,1)$, and grad $f_3(x,y) = (-1,0)$. Since $x^* = (0,1)$, $f_1(x^*) = -1$, so $u_1^* = 0$ since it must be true that $u_i^* f_i(x^*) = 0$ for $i = 1,2,3$. Since $f_2(x^*) = f_3(x^*) = 0$, it is possible for u_2^* and u_3^* to be nonzero. Now $-\text{grad } F(x^*) = (0,1)$. If $u^* = (0, 1, 0)$, then $u_i^* f_i(x^*) = 0$ for $i = 1,2,3$, and

$$-\text{grad } F(x^*) = \sum_{i=1}^{3} u_i^*[\text{grad } f_i(x^*)]. \qquad (3.12.1)$$

This verifies that $x^* = (0,1)$ is an optimal solution for (P) by Theorem 3.10. Figure 9 illustrates this example by graphically depicting (3.12.1). It also shows (0,0) and (1,1) in S cannot be optimal since $-\text{grad } F(x,y)$ cannot be written as a positive linear combination of the gradient vectors for the constraint functions at these points.

Convex Programs Over E_n^+

In a lot of practical problems involving convex programming, it is desirable to minimize the objective function F over only those feasible solutions which lie in the nonnegative orthant E_n^+ of the space E_n. If (P) is a convex program with constraint functions $\{f_1, \ldots, f_m\}$ and it is desired to minimize F over those feasible solutions in E_n^+, let $S^+ = S \cap E_n^+$, and notice that if $f_{m+i}(x) = -x_i$ for

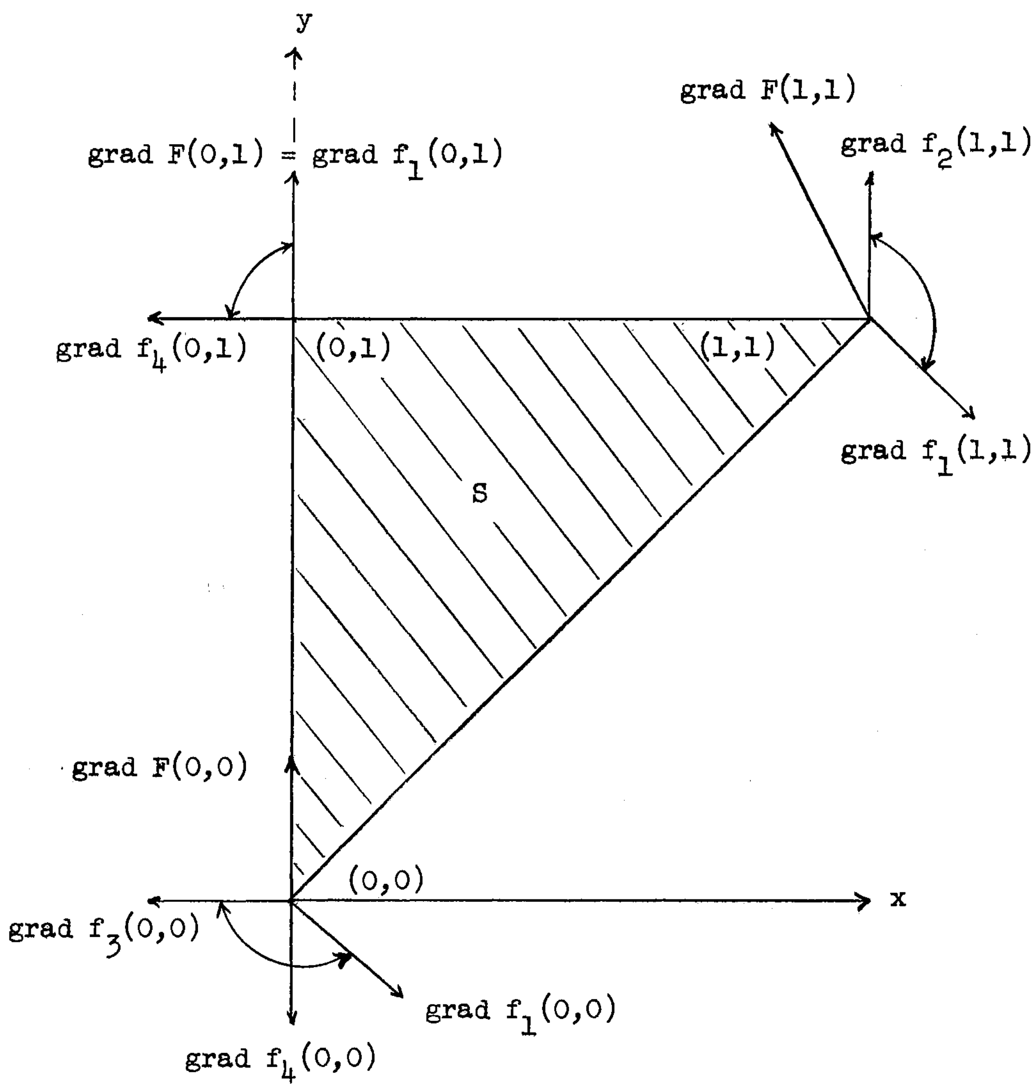

y
grad F(0,1) = grad f1(0,1)
grad F(1,1)
grad f2(1,1)
grad f4(0,1)
(0,1)
(1,1)
grad f1(1,1)
S
grad F(0,0)
(0,0)
x
grad f3(0,0)
grad f1(0,0)
grad f4(0,0)

Figure 9.

$i = 1, \ldots, n$, then

$$E_n^+ = \cap\{\mathrm{lev}_0 f_{m+i}: i = 1, \ldots, n\}.$$

Consequently,

$$S^+ = \{\mathrm{lev}_0 f_j: j = 1, \ldots, m\} \cap \{\mathrm{lev}_0 f_{m+i}: i = 1, \ldots, n\}.$$

Thus, if (P) is considered to have constraint functions

$$\{f_1, \ldots, f_m, f_{m+1}, \ldots, f_{m+n}\},$$

where $f_{m+i}(x) = -x_i$ for $i = 1, \ldots, n$, then it follows that the set of feasible solutions for (P) under this augmented set of constraint functions is exactly S^+; that is, exactly those feasible solutions for (P) under the original set of constraint functions which also lie in E_n^+. Hence, if it is desired to minimize the objective function F of a convex program (P) over the restricted set of feasible solutions S^+ rather than S, it is merely necessary to augment the set of constraint functions for (P) with the n convex functions $\{f_{m+i}: i = 1, \ldots, n\}$ where $f_{m+i}(x) = -x_i$. With this augmented set of constraint functions, the set of feasible solutions for (P) is exactly S^+. For an augmented convex program (P) (meaning that the constraint functions for (P) have been augmented in the manner discussed in the above remarks), the following corollary gives necessary and sufficient conditions for a point x^* in the domain C of (P) to be an optimal solution for (P)

Corollary 3.13. Let (P) be an augmented convex program with domain C an open set in E_n, objective function F, and augmented constraint functions $\{f_1, \ldots, f_m, f_{m+1}, \ldots, f_{m+n}\}$ where $f_{m+i}(x) = -x_i$ for $i = 1, \ldots, n$. Assume that F and each f_j, $j = 1, \ldots, m$, are differentiable over C and that $\{x \in C: f_j(x) < 0, j = 1, \ldots, m+n\}$ is nonempty. Then x* in C is an optimal solution for (P) if and only if there exists a u* in E^+_{m+n} such that

(a). $L_j(x^*;u^*) \leq 0, \quad j = 1, \ldots, m+n,$

(b). $u^*_j L_j(x^*;u^*) = 0, \quad j = 1, \ldots, m+n,$

(c). $L_i(x^*;u^*) = 0, \quad i = 1, \ldots, n.$

Proof: The proof follows directly from Theorem 3.10 when the original set of constraint functions $\{f_1, \ldots, f_m\}$ are replaced by the augmented set of constraint functions $\{f_1, \ldots, f_m, f_{m+1}, \ldots, f_{m+n}\}$. Notice that $L_j(x^*;u^*) = f_j(x^*) \leq 0$, for $j = 1, \ldots, m+1$, forces x* to be in S^+.

Another method is available for finding an optimal solution for (P) over S^+ when the domain of (P) contains E^+_n. Assume that the domain C of (P) contains E^+_n and it is desired to find an optimal solution x* of (P) over S^+. Since E^+_n is a convex set in C, and $S^+ = S \cap E^+_n$, it follows that

$$S^+ = \{x \in C: f_j(x) \leq 0, j = 1, \ldots, m\} \cap E^+_n$$

can be replaced by the equivalent representation

$$S^+ = \{x \in E^+_n: f_j(x) \leq 0, j = 1, \ldots, m\}.$$

From this latter representation of S^+, it is apparent that if the domain of (P) is restricted to E_n^+, then the same set of feasible solutions S^+ results, and an optimal solution x* for (P) over S^+ when the domain of (P) is C is also an optimal solution for (P) over S^+ when the domain of (P) is E_n^+. Since E_n^+ is a convex set, Theorem 3.4 allows the next theorem to be formulated.

Theorem 3.14. Let (P) be a convex program with domain E_n^+ in E_n, objective function F, and constraint functions $\{f_1, \ldots, f_m\}$. Assume that $\{x \in E_n^+: f_j(x) < 0, j = 1, \ldots, m\}$ is nonempty. Then x* in E_n^+ is an optimal solution for (P) if and only if there exists a u* in E_m^+ such that $L(x^*;u) \leq L(x^*;u^*) \leq L(x;u^*)$ for all $x \in E_n^+$ and for all $u \in E_m^+$.

Now let the objective function F and each constraint function f_j, $j = 1, \ldots, m$, be differentiable for a convex program (P) whose domain C is an open set in E_n which contains E_n^+, and suppose that the problem is to find an optimal solution x* for (P) over S^+. Although E_n^+ is not an open set in E_n (it is in fact a closed set), it is still possible to formulate necessary and sufficient conditions for an element x* in E_n^+ to be an optimal solution for (P) over S^+. The next theorem expresses these conditions.

Theorem 3.15. Let (P) be a convex program whose domain C is an open set in E_n which contains E_n^+. Assume that the objective function F and each constraint function f_j, $j = 1, \ldots, m$, are differentiable over C and the $\{x \in E_n^+: f_j(x) < 0, j = 1, \ldots, m\}$ is nonempty. Then x* in E_n^+ is an optimal solution for (P) over S^+ if and only if there exists a u^* in E_m^+ such that

$$\text{(a).}\quad L_i(x^*;u^*) \geq 0, \quad i = 1, \ldots, n,$$

$$\text{(b).}\quad x_i^* \, L_i(x^*;u^*) = 0, \quad i = 1, \ldots, n,$$

$$\text{(c).}\quad L_j(x^*;u^*) \leq 0, \quad j = 1, \ldots, m,$$

$$\text{(d).}\quad u_j^* \, L_j(x^*;u^*) = 0, \quad j = 1, \ldots, m.$$

Proof: Assume that x^* in E_n^+ is an optimal solution for (P) over S^+. Then Theorem 3.14 implies that there exists a u^* in E_m^+ such that

$$L(x^*;u) \leq L(x^*;u^*) \leq L(x;u^*) \tag{3.15.1}$$

for all $x \in E_n^+$ and for all $u \in E_m^+$. Then (3.4.7) in the proof of Theorem 3.4 implies that (d) is true and the optimality of x^* implies that x^* is feasible and hence $L_j(x^*;u^*) = f_j(x^*) \leq 0$, for $j = 1, \ldots, m$, so (c) is true.

Suppose that (a) or (b) is not true. Then either

(1). $x_i^* = 0$ and $L_i(x^*;u^*) < 0$ for some $i = 1, \ldots, n$, or

(2). $x_i^* > 0$ and $L_i(x^*;u^*) \neq 0$ for some $i = 1, \ldots, n$.

is valid. Assume that (1) is true. Then just as in the proof of Theorem 3.10, there exists a $t_0 > 0$ such that

$$\frac{L((x_1^*, \ldots, x_i^*+t_0, \ldots, x_n^*);u^*) - L(x^*;u^*)}{t_0} < 0,$$

which implies that $L((x_1^*, \ldots, x_i^*+t_0, \ldots, x_n^*);u^*) < L(x^*;u^*)$.

Since $x^* \geq 0$, $(x_1^*, \ldots, x_i^*+t_0, \ldots, x_n^*) \in E_n^+$ and this contradicts (3.15.1). Assume now that (2) is true. If $L_i(x^*;u^*) < 0$, then the previous argument again leads to a contradiction. If $L_i(x^*;u^*) > 0$, let $t_0 > 0$ be chosen such that $0 < t_0 < x_i^*$, and

$$\frac{L((x_1^*, \ldots, x_i^*-t_0, \ldots, x_n^*);u^*) - L(x^*;u^*)}{-t_0} < 0.$$

Since $x_i^*-t_0 > 0$, $(x_1^*, \ldots, x_i^*-t_0, \ldots, x_n^*) \in E_n^+$ and it follows that $L((x_1^*, \ldots, x_i^*-t_0, \ldots, x_n^*);u^*) < L(x^*;u^*)$ which is a contradiction. Since (1) and (2) both lead to contradictions, it follows that (a) and (b) are true.

To complete the proof, now let x^* be in E_n^+ and assume that there exists a vector u^* in E_m^+ such that (a) through (d) are true. By (c), x^* is a feasible solution for (P) so the fact that $x^* \in E_n^+$ implies that $x^* \in S^+$. For a fixed vector u^* in E_m^+, $L(x;u^*)$ is a convex function of x over E_n^+ by Lemma 3.3, so it follows from Theorem 3.8 that for all $x \in E_n^+$,

$$L(x;u^*) \geq L(x^*;u^*) + (x - x^*)\cdot[\text{grad}_x L(x^*;u^*)], \qquad (3.15.2)$$

or equivalently,

$$L(x;u^*) \geq L(x^*;u^*) + x\cdot[\text{grad}_x L(x^*;u^*)] - x^*\cdot[\text{grad}_x L(x^*;u^*)].$$

From (b), $x^*\cdot[\text{grad}_x L(x^*;u^*)] = 0$. Since $x \in E_n^+$, $x \geq 0$ and so from (a) it follows that $x\cdot[\text{grad}_x L(x^*;u^*)] \geq 0$. Consequently, it follows that for all $x \in E_n^+$,

$$L(x^*;u^*) \leq L(x;u^*). \qquad (3.15.3)$$

For any $u = (u_1, \ldots, u_m)$ in E_m^+, $u_j \geq 0$ for each j. Combining this with (c), it follows that $u_j L_j(x^*;u^*) = u_j f_j(x^*) \leq 0$ for each j. Hence for all $u \in E_m^+$,

$$\sum_{j=1}^{m} u_j f_j(x^*) \leq 0. \tag{3.15.4}$$

Now by (d),

$$L(x^*;u^*) = F(x^*) + \sum_{j=1}^{m} u_j^* f_j(x^*) = F(x^*) + \sum_{j=1}^{m} u_j^* L_j(x^*;u^*)$$

$$= F(x^*). \tag{3.15.5}$$

Combining (3.15.4) and (3.15.5), it follows that

$$L(x^*;u) = F(x^*) + \sum_{j=1}^{m} u_j f_j(x^*) \leq F(x^*) = L(x^*;u^*) \tag{3.15.6}$$

for all $u \in E_m^+$. It then follows immediately from (3.15.3) and (3.15.6) that for all x in E_n^+ and for all $u \in E_m^+$,

$$L(x^*;u) \leq L(x^*;u^*) \leq L(x;u^*).$$

Theorem 3.14 then implies that x^* in E_n^+ is an optimal solution for (P) over S^+.

This chapter discussed the characterization of an optimal solution for a convex program (P) in terms of the Lagrangian function formed from the objective and constraint functions for (P). It turned out that under certain conditions, the problem of minimizing the objective function of (P) over the set of feasible solutions S could be replaced by the equivalent problem of minimizing the Lagrangian function for (P) with respect to the variable x over the domain C of (P). Theorems 3.4 and 3.10 and Corollary 3.5 are the most general results, while Corollary 3.13 and Theorems 3.14 and 3.15 cover special cases. Theorem 3.15 is an equivalent form of the famous Kuhn-Tucker Theorem (cf. Kuhn and Tucker [15]) which is considered the cornerstone for the theorey of convex programming.

Additional information concerning this topic can be found in Hadley [10], Chapter 2, Karlin [12], Chapter 7, Kuhn and Tucker [15], Kunzi and Krelle [16], Chapter 3, Mangasarian [18], Chapters 5,7,10,11, or Rockafellar [20], Section 28.

Although the goal of the first three chapters was to present an introduction to the theory of convex programming, it is reasonable to expect that such theory is not always convenient or practical to use in solving real life problems. Hence the objective of Chapter IV is to briefly discuss some algorithms developed for solving convex programs.

CHAPTER IV

THREE ALGORITHMS FOR CONVEX PROGRAMS

The objective of this chapter is to discuss three algorithms which have been developed to solve particular types of convex programs. By means of an iterative process, each of these algorithms generate a sequence of feasible solutions which converge to an optimal solution. Since no single algorithm exists which will solve every type of convex program, each algorithm imposes certain requirements upon the convex program (P) in order that it will work. The three algorithms considered here all require that the convex program (P) be solvable and that the objective function be continuous on the domain of (P).

The first algorithm considered is an application of the Method of Feasible Directions developed by Zoutendijk [24]. Although this method can be used to solve convex programs whose constraint functions are nonlinear, the form of the algorithm given here is applicable to a convex program whose constraint functions are linear.

The Cutting Plane Method for solving convex programs is the second algorithm discussed here. This method can be used to solve convex programs whose objective function is linear and whose set of feasible solutions is a particular compact convex set. The iterative process of the Cutting Plane Method generates a sequence (generally infinite) of linear programs, the solutions for which form a sequence of feasible solutions for (P) which converge to an optimal solution for (P). Additional information on the Cutting Plane Method can be found in Kelley [13].

The last method considered is the Sequential Unconstrained Minimization Technique developed by Fiacco and McCormick [5]. According to Bracken [1], this method has had great success in solving many types of nonlinear programs including certain convex programs. This method generates a sequence of feasible solutions for (P) which converge to an optimal solution.

The Method of Feasible Directions

Suppose that (P) is a convex program with domain E_n, objective function F which is differentiable over E_n, and set of feasible solutions given by

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i, \quad i = 1, \ldots, m. \tag{4.1}$$

$$x_j \geq 0, \quad j = 1, \ldots, n. \tag{4.2}$$

Notice that for each i, $1 \leq i \leq m$, f_i given by the expression $f_i(x) = -a_{i1}x_1 - \ldots -a_{in}x_n + b_i$ is convex, and for each j, $1 \leq j \leq n$, $f_{m+j}(x) = -x_j$ defines a convex function f_{m+j}. Then the set of feasible solutions S for (P) is given by

$$S = \{x \in E_n: f_i(x) \leq 0, i = 1, \ldots, m+n\}.$$

Thus (4.1) and (4.2) conform to the standard form for the set of feasible solutions for (P).

Now let $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ be a feasible solution which satisfies (4.1) and (4.2). The Method of Feasible Directions seeks to find a vector (direction) $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$, where $-1 \le s_j^* \le 1$ for each j, and a real number $r > 0$ such that the point $x^{**} = x^* + rs^*$ is a feasible solution for (P) and $F(x^{**}) < F(x^*)$.

Assume first that x^* satisfies the following conditions:

$$\sum_{j=1}^{n} a_{ij}x_j^* = b_i, \quad i = 1, \ldots, k, \tag{4.3}$$

$$\sum_{j=1}^{n} a_{ij}x_j^* > b_i, \quad i = k+1, \ldots, m, \tag{4.4}$$

and

$$x_j^* = 0, \quad j = 1, \ldots, p, \tag{4.5}$$

$$x_j^* > 0, \quad j = p+1, \ldots, n. \tag{4.6}$$

It then follows that if r is sufficiently small, and if

$$\sum_{j=1}^{n} a_{ij}s_j^* \ge 0, \quad i = 1, \ldots, k, \tag{4.7}$$

$$s_j^* \ge 0, \quad j = 1, \ldots, p, \tag{4.8}$$

then $x^{**} = x^* + rs^*$ satisfies (4.1) and (4.2) and hence is a feasible solution.

For those s^* which satisfy (4.7) and (4.8), it is desired to determine that particular vector along which the objective function F attains its maximum decrease from the value $F(x^*)$. Recall that this maximum decrease occurs along the vector -grad $F(x^*)$ if grad $F(x^*) \neq 0$. A verification of this remark can be found in Fleming [7], p. 86. For x^* fixed, consider the following linear function:

$$[\text{grad } F(x^*)]\cdot[s^*]. \tag{4.9}$$

In order to obtain the largest decrease in the objective function F from the value $F(x^*)$, it is necessary to minimize the linear function given in (4.9) over the set of vectors S^* which satisfy (4.7) and (4.8). Notice that this is a linear programming problem (cf. Hadley [11], Chapter 3, for additional information concerning linear programming). The vector s^* which minimizes (4.9) over S^* is called a feasible direction and the point $x^{**} = x^* + rs^*$ gives a smaller value of $F(x)$ for the r that is chosen sufficiently small.

If the solution s^* to the above linear program is the zero vector, then x^* is the solution to the problem of minimizing $F(x)$ over the set of vectors satisfying (4.1) and (4.2).

If $s^* \neq 0$, then the parameter r is chosen in the following manner.

(a). Let

$$r_1 = \text{Min} \begin{cases} (b_i - \sum_{j=1}^{n} a_{ij}x_j^*)/\sum_{j=1}^{n} a_{ij}s_j^*, \text{ if } \sum_{j=1}^{n} a_{ij}s_j^* < 0, \ k+1 \le i \le m. \\ \\ -x_j^*/s_j^*, \text{ if } s_j^* < 0, \ p+1 \le j \le n. \end{cases}$$

It follows directly that if $r \le r_1$, then $x^* + rs^*$ satisfies (4.1) and (4.2).

(b). Note that $F(x^* + rs^*)$ is a function of the single variable r, and that this function is a decreasing function of r for sufficiently small r. To find the minimum value of $F(x^* + rs^*)$ with respect to r, the following expression is solved for r:

$$\frac{d}{dr} F(x^* + rs^*) = [\text{grad } F(x^* + rs^*)]\cdot[s^*] = 0. \qquad (4.10)$$

Let r_2 be the solution to (4.10), if it exists. For $0 \leq r \leq r_2$, $F(x^* + rs^*)$ as a function of r is a decreasing function.

(c). The optimum choice of the parameter r is then

$$r^* = \text{Min } [\ r_1,\ r_2\ ]. \qquad (4.11)$$

The new feasible solution is then

$$x^{**} = x^* + r^*s^*. \qquad (4.12)$$

This process is then iterated until either (1), s^* is exactly zero, or (2), the decrease in the objective function is negligibly small. In his book (cf. [24]), Zoutendijk has shown that this process converges to an optimal solution. Hence, this algorithm can be outlined as follows:

Step 1. Select an arbitrary feasible solution x^*.

Step 2. For this x^*, determine a feasible direction s^*.

Step 3. If $s^* = 0$, then x^* is optimal.

Step 4. If $s^* \neq 0$, determine a feasible parameter r^*.

Step 5. Consider the new feasible solution $x^{**} = x^* + r^*s^*$.

Step 6. Repeat Step 1 through Step 5.

The following simple example is worked by this method in order to illustrate the procedure.

<u>Example 4.1.</u> Let (P) be a convex program such that $C = E_2$,

$F(x,y) = x^2 - y + 2$, $f_1(x,y) = x - y$, $f_2(x,y) = y - 1$, $f_3(x,y) = -x$, and $f_4(x,y) = -y$. Then $S = \{(x,y): 0 \le x \le 1, x \le y \le 1\}$ and from Example 3.12, the point $(0,1)$ is optimal. Note that grad $F(x,y) = (2x,-1)$. Let $x_0 = (1/2,1/2)$ be the initial feasible point arbitrarily chosen. Notice that the constraints for this problem are:

(1). $x - y \le 0$ implies that $-x + y \ge 0$,

(2). $y - 1 \le 0$ implies that $-y \ge -1$,

(3). $-x \le 0$, $-y \le 0$ implies that $x \ge 0$, $y \ge 0$.

The solution proceeds as follows:

Step 1. Since $x_0 = (1/2,1/2)$ satisfies constraint (1) with strict equality, it follows from (4.7) and (4.8) that $s = (s_1, s_2)$ must satisfy the constraints

(a). $-s_1 + s_2 \ge 0$,

(b). $-1 \le s_1 \le 1$,

(c). $-1 \le s_2 \le 1$.

Step 2. Minimize $[\text{grad } F((1/2,1/2))]\cdot[(s_1, s_2)] = s_1 - s_2$ subject to the constraints (a) through (c) above. The solution to this linear programming problem is $s_0 = (-1, 1) = (s_1, s_2)$. Hence s_0 is a feasible direction for x_0.

Step 3. Now $0(x) - 1(y) \ge -1$ is constraint (2) for the problem and $0(s_1) - 1(s_2) = -1 < 0$. Hence $r_1 = [-1 - (-1/2)]/-1 = 1/2$.

Step 4. Setting $[\text{grad } F(x_0 + rs_0)]\cdot[s_0] = 2r - 2 = 0$, it follows that $r_2 = 1$. Therefore, $r^* = \text{Min } (r_1, r_2) = 1/2$ and the new feasible solution is $x_1 = (1/2, 1/2) + (1/2)(-1, 1) = (0, 1)$. Since $s_0 \ne 0$, the procedure must be repeated for the feasible solution x_1.

Step 5. Since x_1 satisfies constraint (2) with strict equality, and the first coordinate of x_1 is zero, the set of possible feasible directions for x_1 is given by the constraints:

(a). $-s_2 \geq 0$, (c). $-1 \leq s_1 \leq 1$,

(b). $s_1 \geq 0$, (d). $-1 \leq s_2 \leq 1$.

Step 6. Minimize $[\text{grad } F((0,1))]\cdot[(s_1, s_2)] = -s_2$ subject to (a) through (d) above. The solution of this linear program is $(s_1, s_2) = (0, 0)$. Thus it follows that x_1 is an optimal solution for (P) since there exists no nontrivial feasible direction for x_1.

The Cutting Plane Method

The Cutting Plane Method can be used for solving convex programs where the objective function is linear and the single constraint function for the program is convex and not linear.

To simplify the statement and proof of this method, consider the following characterization of a nonvertical hyperplane in E_{n+1}. Let G be a convex function defined by G(x) on a nonempty compact convex set C in E_n such that there exists a nonvertical hyperplane of support

$$H_t = \{(x_1, \ldots, x_{n+1}) \in E_{n+1}: a_1^t x_1 + \ldots + a_n^t x_n + a_{n+1}^t x_{n+1} = b^t\}$$

to epi G at the point $(t, G(t))$ at every point $t \in C$. Since H_t is nonvertical, it must be true that $a_{n+1}^t \neq 0$. Therefore, it follows that for each $(x_1, \ldots, x_n) \in C$,

$$x_{n+1} = (1/a^t_{n+1})[-a^t_1 x_1 - \cdots - a^t_n x_n + b^t]$$

such that $(x_1, \ldots, x_n, x_{n+1}) \in H_t$. If $-a^t_i/a^t_{n+1} = \alpha^t_i$ for $1 \le i \le n$, and $b^t/a^t_{n+1} = \beta^t$, then define

$$x_{n+1} = H(x;t) = \alpha^t_1 x_1 + \cdots + \alpha^t_n x_n + \beta^t \qquad (4.13)$$

for each $(x_1, \ldots, x_n) \in C$. Then it follows that the point $(x_1, \ldots, x_n, H(x;t))$ is in H_t. Notice that H defined by $H(x;t)$ is a function whose domain is E_n and whose graph in E_{n+1} is the hyperplane H_t. In addition, $\text{grad}_x H(x;t) = (\alpha^t_1, \ldots, \alpha^t_n)$. Also, note that the point $(t,G(t))$ is in the hyperplane H_t. Thus

$$G(t) = \alpha^t_1 t_1 + \cdots + \alpha^t_n t_n + \beta^t \qquad (4.14)$$

so that

$$\beta^t = G(t) - (\alpha^t_1 t_1 + \cdots + \alpha^t_n t_n). \qquad (4.15)$$

From (4.13) and (4.15), it follows that an alternate expression for H_t is

$$H_t = \{(x_1, \ldots, x_{n+1}) \in E_{n+1}: \alpha^t_1(x_1 - t_1) + \cdots - (x_{n+1} - G(t)) = 0\}$$

where $t = (t_1, \ldots, t_n)$. Consequently, it follows that

$$H(x;t) = x_{n+1} = \alpha^t_1(x_1 - t_1) + \cdots + \alpha^t_n(x_n - t_n) + G(t).$$

The Cutting Plane Method can now be summarized by the following theorem. Notice that the convex program discussed has the n-dimensional convex compact polyhedral set C in E_n as its domain, the linear function $F(x) = c \cdot x$ as its objective function, and the continuous convex function G as its only constraint function.

Theorem. (Cutting Plane Method). Let G be a continuous convex function defined on the n-dimensional convex compact polyhedral set C in E_n such that at every point t in C, there exists a nonvertical hyperplane of support $H_t = \{x \in E_{n+1}: x \cdot a^t = b^t\}$ to epi G at (t,G(t)). Assume that there exists a finite constant K such that for each hyperplane H_t, $\|(a_1^t, \ldots, a_n^t, 0)\| \le K$ for all t in C. Let $F(x) = c \cdot x$ be a linear objective function such that $\|c\| > 0$ is finite and let $S = \{x \in C: G(x) \le 0\}$ with S nonempty. If $t_k \in C_k$ is such that

$$F(t_k) = \inf \{F(x): x \in C_k\}, \quad k = 0, 1, \ldots,$$

where $C_0 = C$ and

$$C_k = C_{k-1} \cap \{x: H(x; t_{k-1}) \le 0\},$$

then the sequence $\{t_k\}$ converges to a point $z \in S$ such that

$$F(z) = \inf \{F(x): x \in S\}. \tag{4.16}$$

Before turning to the proof proper, consider Figure 10 which depicts the above described procedure for k = 0, 1, 2, 3. First, t_0 is determined which minimizes F over C_0. Then the hyperplane H_{t_0} determines the set C_1. Then t_1 minimizes F over C_1 and the hyperplane H_{t_1} determines C_2. Similarly, t_2, H_{t_2}, and C_3 are determined. The procedure then continues in this manner.

Proof: Since C is an n-dimensional convex compact polyhedral set, C can be written as the intersection of a finite number of closed half

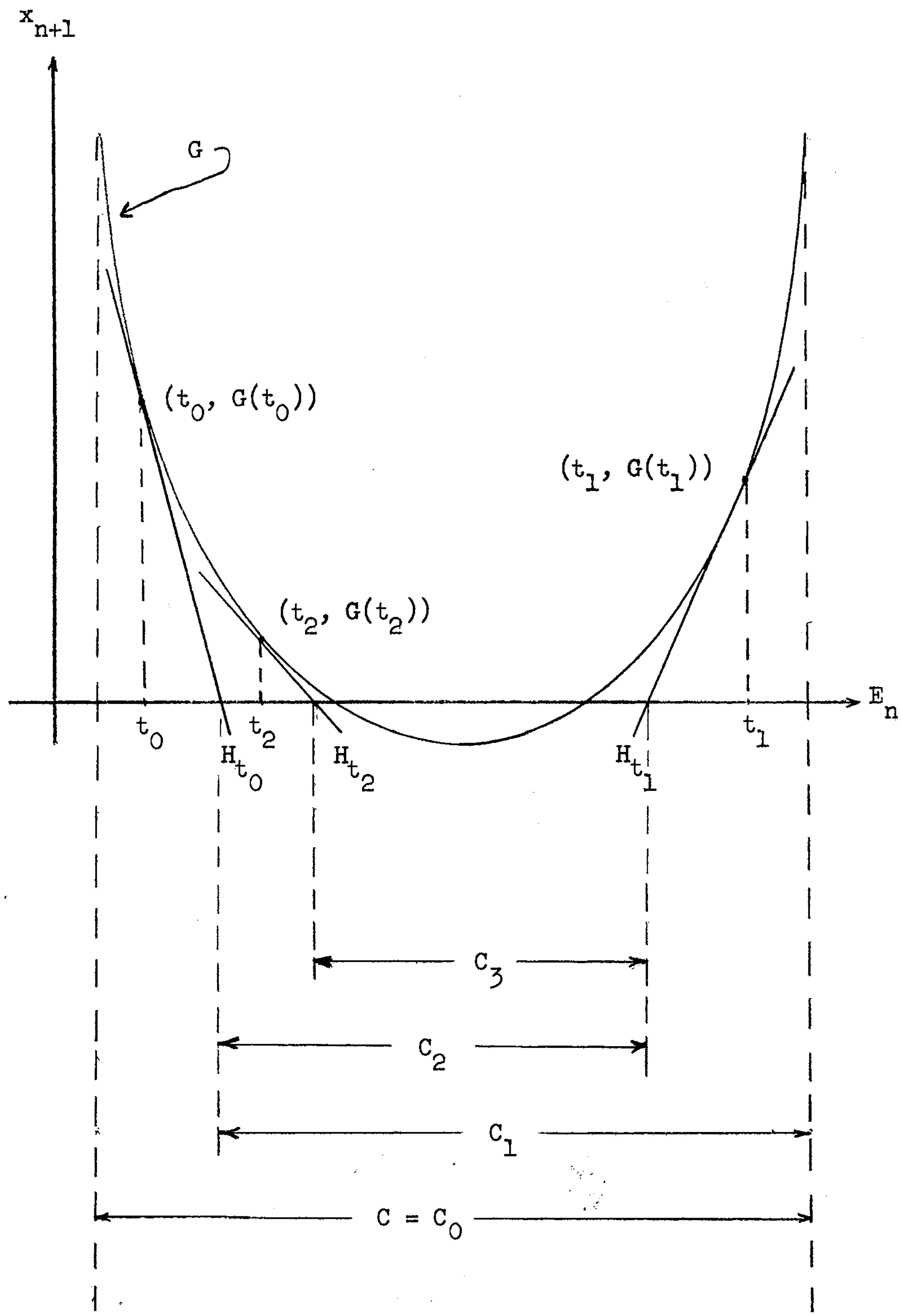


Figure 10.

spaces and hence is closed. Since G is continuous, S is compact so there exists a point $z \in S$ such that (4.16) is true.

Now let t be an arbitrary point in $C \setminus S$. A hyperplane of support to epi G at $(t,G(t))$ can be written as

$$H_t = \{(x;z) \in E_{n+1}: x \in C,\ z = H(x;t) = G(t) + \alpha^t \cdot (x - t),\ \alpha^t = 0\}.$$

(If G is differentiable, then H_t is just the tangent plane to epi G). Since epi G is a convex set, it follows that $H(x;t) \le G(x)$ for all x in C. If $x \in S$, then $G(x) \le 0$ and thus $H(x;t) \le 0$ for all x in S. Since t is not in S, $H(t;t) > 0$. Consider the trace of the hyperplane H_t in E_n; that is, the set $\{x \in E_n: H(x;t) = 0\}$. The above discussion then implies that t and S are separated by the trace of H_t in E_n.

Now let $C = C_0$ and let t_0 be the point in C_0 which minimizes $F(x)$ over C_0. Since C_0 is compact and F is continuous, such a point exists. (assume that t_0 is in $C_0 \setminus S$ for otherwise t_0 is a solution to the problem). Let

$$C_1 = C_0 \cap \{x: H(x;t_0) \le 0\}.$$

Then S is contained in C_1 so C_1 is not empty. Also, C_1 is compact so there exists a t_1 in C_1 which minimizes $F(x)$ over C_1. Note that C_1 contained in C_0 implies that $F(t_1) \ge F(t_0)$, and that $H(t_0;t_0) > 0$ implies that $t_0 \notin C_1$. In general, for $k > 1$, let

$$C_k = C_{k-1} \cap \{x: H(x;t_{k-1}) \le 0\},$$

and let t_k be the point which minimizes $F(x)$ over C_k so that $F(t_k) = \inf\{F(x): x \in C_k\}$. Since $S \subset C_k \subset C_{k-1}$, for all k, it follows that $F(z) \geq F(t_k) \geq F(t_{k-1})$ where z satisfies (4.16). Consider the sequence $\{t_k\}$ generated by the above method. Due to the fact that $\{t_k\}$ is contained in C and C is compact, there exists a point $p \in C$ and a subsequence $\{t_{k_i}\}$ of $\{t_k\}$ such that $\{t_{k_i}\}$ converges to p. A proof of this assertion can be found in Taylor [22], p. 72, Theorem 2.4-H. Now $\{F(t_{k_i})\}$, being a monotone nondecreasing sequence which is bounded above, converges to some value β. Then the continuity of F implies that $F(p) = \beta$ where $\beta \leq F(z)$. If $p \in S$, then $F(p) \geq F(z)$ and it follows that $F(p) = F(z)$. Hence p would be an optimal solution for the problem of minimizing $F(x)$ over S. It remains to be shown that p must be in S.

To show that p is in S, note that if t_k minimizes $F(x)$ over C_k, then $t_k \in C_j$, $j = 0, 1, \ldots, k-1$, so it must be true that

$$H(t_k;t_j) = G(t_j) + \alpha^t \cdot (t_k - t_j), \quad 0 \leq j \leq k-1. \tag{4.17}$$

If $t_k \in S$ for some k, then t_k is a solution. Thus assume $G(t_k) > 0$ for all k. Furthermore, $G(t_{k_i})$ converges to $G(p) \geq 0$. If $p \in S$, then $G(p) = 0$. Hence assume that $G(p) > 0$. Then for $r = (1/2)G(p) > 0$, it is true from the continuity of G that for sufficiently large t_i,

$$0 < r < G(t_{k_i}) \leq \alpha^t \cdot (t_{k_j} - t_{k_i}) \leq K\|t_{k_j} - t_{k_i}\|. \tag{4.18}$$

However, since $\{t_{k_i}\}$ is a convergent sequence, it is a Cauchy sequence and this means that $\|t_{k_j} - t_{k_i}\|$ can be made arbitrarily small for k_j and k_i sufficiently large and this contradicts (4.18). Hence $G(p) = 0$ and thus p is in S, and by the above remarks, p is optimal and satisfies (4.16).

Notice that for any k, C_k is a convex compact polyhedral set and the minimization of F(x) over C_k is a linear program. Hence each t_k is a solution to a linear program associated with C_k and F and it follows that the Cutting Plane Method generates a sequence of linear programs whose solutions form a sequence which contains a subsequence that converges to the optimal solution of the convex program whose objective function is F and whose set of feasible solutions is S.

Example 4.2. To illustrate the computational procedure for the Cutting Plane Method, consider the following problem: Minimize $F(x,y) = x - y$ over the set $S = \{(x,y) \in E_2: G(x,y) = x^2 + 4y^2 - 1 \leq 0\}$. To begin, notice that S is contained in the 2-dimensional convex compact polyhedral set $C = \{(x,y) \in E_2: -1 \leq x \leq 1, -1 \leq y \leq 1\}$. The procedure is then as follows:

Step 1. Solve the linear program:

$$\text{Minimize } F(x,y) \text{ over } C.$$

Then $t_0 = (-1,1)$. Since G is differentiable, H_{t_0} is just the tangent plane to epi G at $(t_0,G(t_0))$ and $H(x;t_0) = -2x + 8y - 6$. Therefore, $C_1 = C \cap \{(x,y): -2x + 8y -6 \leq 0\}$.

Step 2. Solve the linear program:

$$\text{Minimize } F(x,y) \text{ subject to}$$

$$\text{(a). } -1 \leq x \leq 1, \quad \text{(b). } -1 \leq y \leq 1, \quad \text{(c). } -2x + 8y - 6 \leq 0.$$

Then $t_1 = (-1,1/2)$ and $H(x:t_1) = -2x + 4y -3$.

Step 3. Solve the linear program:

Minimize $F(x,y)$ subject to

(a). $-1 \leq x \leq 1$, (c). $-2x + 8y - 6 \leq 0$,

(b). $-1 \leq y \leq 1$, (d). $-2x + 4y - 3 \leq 0$.

Then $t_2 = (-1,1/4)$ and this procedure is continued until t_k is in S or is sufficiently close to S. A criterion for termination of the process would be to require that $G(t_k)$ is smaller than some pre-assigned tolerance. Further details on the Cutting Plane Method can be found in Kelley [13].

Sequential Unconstrained Minimization Technique

The Sequential Unconstrained Minimization Technique makes use of a penalty function formed from the objective and constraint functions of a convex program (P) to create a sequence of unconstrained minimization problems the solutions of which converge to an optimal solution for (P). This method, verified by Fiacco and McCormick [6], can be applied to a convex program when certain hypothesis are satisfied by the objective function and the constraint functions for (P). In what follows, the constraint functions for (P) are referred to as concave functions. Recall that a function f is concave if and only if -f is convex. The Sequential Unconstrained Minimization Technique can be formulated in the following theorem.

<u>Theorem</u>. (Sequential Unconstrained Minimization Technique). Let (P) be a convex program with domain E_n, objective function F, and constraint functions $\{-f_1, \ldots, -f_m\}$, $m > 1$. Assume that the following conditions are valid:

(1). The set $S^* = \{x \in E_n: f_i(x) > 0, i = 1, . . . ,m\} \neq 0$.

(2). The functions $F, f_1, . . . , f_m$ are twice continuously differentiable.

(3). For every integer p, $\{x \in S: F(x) \leq p\}$ is a bounded set where $S = \{x \in E_n: f_i(x) \geq 0, i = 1, . . . ,m\}$.

(4). The function defined by

$$P(x;r) = F(x) + r\sum_{j=1}^{m} 1/f_i(x)$$

is strictly convex in S^* for each $r > 0$.

Then, given $\{r_k\}$ a strictly monotone decreasing sequence of reals such that $\{r_k\}$ converges to zero as k becomes infinite, the following conclusions are true:

(a). The function $P(x;r_k)$ is minimized over S^* at a unique x_k in S^* where $\text{grad}_x P(x_k;r_k) = 0$, and

(b). $\lim_{k\to\infty} P(x_k;r_k) = \inf \{F(x): x \in S\} = v_0 = \lim_{k\to\infty} F(x_k)$.

Proof: Let x^* be a point in S^*. For any k, let $P(x^*;r_k) = M$. Now the set S is the set of feasible solutions for (P) and (3) of the hypothesis implies that the objective function F and S have no vectors of recession in common; hence (P) is solvable and there exists a real number v_0 such v_0 is the optimal value for (P). Next, for the given k, form the sets $R_0 = \{x \in S: F(x) \leq M\}$ and for $i = 1, . . . ,m$, $R_i = \{x \in E_n: r_k/f_i(x) \leq M - v_0\}$. Let $R = \cap\{R_i: i = 1, . . . ,m\}$. The first task is to show that R is nonempty. Since $x^* \in S^*$, it follows that $f_i(x^*) > 0$ for all i and hence $1/f_i(x^*) > 0$ for all i. Since for each k, $r_k > 0$, $r_k[1/f_1(x^*) + \ldots + 1/f_m(x^*)] > 0$. Thus

$$F(x^*) < F(x^*) + r_k[1/f_1(x^*) + \ldots + 1/f_m(x^*)] = P(x^*;r_k) = M,$$

so $x^* \in R_0$. Now let j be an arbitrary index, $1 \leq j \leq m$. Notice from the above equality that

$$r_k/f_j(x^*) = M - [F(x^*) + \sum_{i \neq j} r_k/f_i(x^*)] \tag{4.19}$$

and, since $v_0 \leq F(x^*)$, that

$$v_0 < [F(x^*) + \sum_{i \neq j} r_k/f_i(x^*)].$$

Hence

$$-v_0 > -[F(x^*) + \sum_{i \neq j} r_k/f_i(x^*)]. \tag{4.20}$$

Adding M to both sides of (4.20), and comparing the resulting inequality to (4.19), it follows immediately that

$$r_k/f_j(x^*) < M - v_0. \tag{4.21}$$

Since j was arbitrary, (4.21) holds for $i = 1, \ldots, m$. Hence x^* is in R_i for each i, and thus $x^* \in R$. Therefore, R is nonempty.

Notice that if $x \in R$, then $x \in R_0$ and hence is in S. But x is in R_i for each $i = 1, \ldots, m$, and by the construction of R_i, it must be true that $f_i(x) > 0$ for each i. Otherwise, $r_k/f_i(x)$ is undefined which implies that x is not in R. Thus R is contained in S*.

Also, R_0 and each R_i are closed sets by the continuity of F and f_i, respectively. In addition, R_0 is bounded by (3) of the hypothesis. Consequently, R is a nonempty compact set in E_n.

Since $P(x;r_k)$ is defined over R for each r_k, it is also continuous there by virtue of (1) and (2). The fact that R is compact implies that there exists an $x_k \in R$ such that $P(x_k;r_k) = \inf\{P(x;r_k): x \in R\}$, and the hypothesis that $P(x;r_k)$ is strictly convex over S* implies that $P(x;r_k)$ is strictly convex over R. Hence x_k is unique; that is, the minimum set of $P(x;r_k)$ over R is a singleton set $\{x_k\}$ (a proof of the assertion that a strictly convex function has only one minimum on a convex set where it is bounded below can be found in Kowalik [14], p. 141, Theorem 7). In fact, for each r_k, $P(x;r_k)$ attains its minimum over S* at x_k, and since S* is open, it must be true that $\text{grad}_x P(x;r_k) = 0$. To see that $P(x;r_k)$ also attains its minimum over S* at x_k, note first that $R \subset S^*$ implies that

$$\inf\{P(x;r_k): x \in S^*\} \le \inf\{P(x;r_k): x \in R\}. \qquad (4.22)$$

Now assume that there exists a $z \in S^* \setminus R$ such that

$$P(z;r_k) < \inf\{P(x;r_k): x \in R\} \le M.$$

Then it follows that $P(z;r_k) < P(x^*;r_k) = M$, and since

$$v_0 < P(z;r_k) = F(z) + \sum_{j=1}^{m} r_k/f_j(z) < M, \qquad (4.23)$$

it follows that $F(z) < M$. Consequently, $z \in R_0$. For any j, $1 \le j \le m$, it follows from (4.23) that

$$r_k/f_j(z) = M - [F(z) + \sum_{i \neq j} r_k/f_i(z)] \qquad (4.24)$$

and

$$-v_0 > -[F(z) + \sum_{i \neq j} r_k/f_i(z)]. \qquad (4.25)$$

Combining (4.24) and (4.25), it follows directly that for each j,

$$r_k/f_j(z) < M - v_0.$$

Therefore, $z \in R_i$ for each i and hence $z \in R$, which is a contradiction. Thus

$$\inf\{P(x;r_k): x \in R\} \leq \inf\{P(x;r_k): x \in S^*\}. \qquad (4.26)$$

Consequently, the above assertion then follows from (4.22) and (4.26), which proves (a).

Before proving (b), note that if $k > k^*$, then $r_k < r_{k^*}$ and

$$\begin{aligned} P(x_{k^*};r_k) &= F(x_{k^*}) + r_k[1/f_1(x_{k^*}) + \ . \ . \ . + 1/f_m(x_{k^*})] \\ &< F(x_{k^*}) + r_{k^*}[1/f_1(x_{k^*}) + \ . \ . \ . + 1/f_m(x_{k^*})] \\ &= P(x_{k^*};r_{k^*}). \end{aligned} \qquad (4.27)$$

Therefore, let $\epsilon > 0$ be arbitrary and let $z \in S^*$ such that $F(z) < v_0 + \epsilon/2$. Select k^* such that $r_{k^*} < [\min_i f_i(z)]\epsilon/2m$. Then for $k > k^*$,

$$v_0 \leq \inf \{P(x;r_k): x \in S^*\} = P(x_k;r_k),$$

and since $x_{k^*} \in S^*$, $P(x_k;r_k) \leq P(x_{k^*};r_k)$. By (4.27), it follows that $P(x_{k^*};r_k) < P(x_{k^*};r_{k^*})$, and since $z \in S^*$,

$$P(x_{k^*};r_{k^*}) \leq P(z;r_{k^*}) = F(z) + r_{k^*}[1/f_1(z) + \ldots + 1/f_m(z)]$$

$$< v_0 + \epsilon/2 + ([\min_i f_i(z)]\epsilon/2m)[1/f_1(z) + \ldots + 1/f_m(z)]$$

$$< v_0 + \epsilon.$$

Therefore, for $k > k^*$, $v_0 \leq P(x_k;r_k) < v_0 + \epsilon$. Thus $\lim_{k\to\infty} P(x_k;r_k) = v_0$.

All that needs to be shown now for (b) to be true is that $\lim_{k\to\infty} F(x_k) = v_0$. Note that for $k > k^*$, it is true that

$$v_0 \leq P(x_k;r_k) = F(x_k) + r_k[1/f_1(x_k) + \ldots + 1/f_m(x_k)] < v_0 + \epsilon$$

Subtracting v_0 from all entries in the above expression, it follows that

$$0 \leq [F(x_k) - v_0] + r_k[1/f_1(x_k) + \ldots + 1/f_m(x_k)] < \epsilon.$$

Since $r_k[1/f_1(x_k) + \ldots + 1/f_m(x_k)] > 0$ for all $k > k^*$, it must be true that

$$0 \leq [F(x_k) - v_0] < \epsilon$$

for all $k > k^*$. Hence $\lim_{k\to\infty} F(x_k) = v_0$ which is the desired conclusion.

Therefore, (b) is true.

In light of this theorem, the steps describing the computational algorithm for the Sequential Unconstrained Minimization Technique can be listed as follows:

Step 1. Select a point x* in the interior of the set of feasible solutions.

Step 2. Select the initial value of r.

Step 3. Determine the minimum of $P(x;r_k)$ for the current value of r_k over the interior of the set of feasible solutions.

Step 4. Terminate computations if some final convergence criterion is satisfied. Fiacco and McCormick [4] suggest that one possible convergence criterion is to terminate computations when $r_k[1/f_1(x_k) + \ldots + 1/f_m(x_k)] < \epsilon$ for some predetermined small number $\epsilon > 0$. If such convergence criterion is not satisfied, then go to Step 5.

Step 5. Select $r_{k+1} = r_k/c$ where $c > 1$.

Step 6. Continue procedure from Step 3.

To illustrate the application of this method, consider the following example.

<u>Example 4.3</u>. Let (P) be a convex program with domain E_2. Let $F(x,y) = (1/3)(x + 1)^3 + (1/3)(y + 1)^3$, $f_1(x,y) = x - 1$, and $f_2(x,y) = y - 1$. Then $S = \{(x,y): x \geq 1,\ y \geq 1\}$ and for any $r_k > 0$,

$$P(x;r_k) = (x + 1)^3/3 + (y + 1)^3/3 + r_k/(x - 1) + r_k/(y - 1).$$

Setting $\text{grad}_x P(x;r_k) = 0$ and solving for x_k in terms of r_k, it follows that $x_k = ((r_k^{1/2} + 1)^{1/2},\ (r_k^{1/2} + 1)^{1/2})$. As k becomes infinite, r_k converges to zero and x_k converges to an optimal solution for (P).

Therefore, it follows that

$$\lim_{r_k \to 0} x_k = (1,1)$$

where (1,1) is an optimal solution for (P), and

$$\lim_{r_k \to 0} F(x_k) = \lim_{r_k \to 0} [(r_k^{1/2} + 1)^{1/2} + 1]^3/3 + \lim_{r_k \to 0} [(r_k^{1/2} + 1)^{1/2} + 1]^3/3$$

$$= 16/3$$

where 16/3 is the optimal value of (P).

In general, algorithms for the computational solution of a convex program (P) generate a sequence of feasible solutions for (P) which converge to an optimal solution. Except in particular cases, such as the program in Example 4.1 solved by the Method of Feasible Directions, the iterative procedure is infinite so that some convergence criteria must be used to terminate the computations after a finite number of iterations. According to Fiacco [6] and Bracken [1], the Sequential Unconstrained Minimization Technique has easily been adapted to computer use for the solution of convex programs with nonlinear objective function and linear or nonlinear constraint functions. Much additional information on this method can be found in the recent book by Fiacco and McCormick [6]. Kelley [13], p. 708, states that the Cutting Plane Method for solving convex programs is not always feasible even using computers, since no dependable convergence criterion exists for a large class of convex programs.

More information on algorithms can be found, in addition to those references above, in Kowalik [14] and Kunzi [16].

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