DISPLACED BALANCED INCOMPLETE BLOCK DESIGNS

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CHAPTER I

INTRODUCTION

Although in many experimental situations the number of experimental units and their grouping into blocks are totally under the control of the experimenter, frequently he is unable to influence these without sacrifice of experimental material and precision or practicability. Thus incomplete block designs have been developed to cover those experimental situations in which considerations of the availability and natural occurrence of experimental units dictate a block size which is less than the number of treatments.

Experimental situations in which considerations of availability and natural occurrence of experimental units lead to blocks whose size exceeds the number of treatments are also of interest.

These situations may occur when the experimental units exist naturally in groups even before the experimenter conceives of the problem. For example, an experiment in which t treatments of some sort are to be applied to newborn animals might be concerned with a species for which it is quite likely that the number of offspring in a litter exceeds t. Presuming that the principal cost associated with an experimental unit is incurred prior to its birth, the experimenter might not favor the discarding of litter members which would be necessary for the application of a randomized complete block design. Another example is available from certain horticultural experiments in which considerable expense is involved in bringing plants to the blossoming stage. Here, with interest being in the effect on fruit of treatments applied to blossoms, and with each truss as a block, it would seem quite likely that block size would exceed the number of treatments and that the sacrifice of experimental material should be minimized. A number of situations occurring in industrial experimentation are characterized by a continuance of overhead costs whether or not a service is used. These also may often be viewed as situations with fewer treatments than plots in a block. For example, suppose there exists interest in the differential effects of six production methods where each method requires an hour of operating time and each is done by a single operator. Here, it would seem reasonable and desirable to designate operators as blocks and, since machine operators are usually assured by contract of eight-hour workdays, the number of experimental units per block would be eight, two greater than the number of treatments. In this case there would be a uniformity of block size and current wages would discourage the obtaining of blocks of size six through the dismissal of the workers after six hours.

The problem of experimentation with blocks of size greater than the number of treatments has apparently received less attention than that of experimentation with blocks of size less than the number of treatments. It has, however, been considered recently by authors working with differing interests and points of view. S. C. Pearce (9) considered comparative experiments in which one treatment, the control, was logically different from the others. An example is given in which four treatments, A, B, C, and D, were different weedkillers and the control, 0, was no treatment. The purpose of the experiment was to compare these new weedkillers not with each other but with the control; hence it follows that the control be more highly replicated than the other treatments. With four blocks available, each of seven plots, the experiment was designed thus:

Block 1: O O A A B C D

Block 2: O O A B C D D

Block 3: O O A B B C D

Block 4: O O A B C C D.

In a situation with the control introduced only in order to demonstrate the consequences of doing nothing, it would be replicated fewer times than the others. Pearce develops the application of a device, supplemented balance, to this sort of problem. A design has the supplemental balance property if blocks are of constant size, all treatments are replicated r times except the supplementary one (the control) which has r_o replicates and each pair of treatments occur together in the same block λ times unless one of the pair is the supplementing one in which case there are λ_o occurrences. Supplemented balance in designs of several different types is considered.

In a later paper Pearce (10) concerns himself with possible methods for designing experiments in which naturally occurring blocks are of varying sizes. Both the case in which all treatments are to be compared with equal precision and the case in which one treatment is a "control" are considered and computational procedures for their analysis are provided.

W. T. Federer (2) encounters the problem of experimentation with blocks which have more plots than there are treatments by way of concern with a common problem in plant breeding and in biochemical research, namely, the evaluation of new strains of treatments. Since it is felt that making small scale preliminary tests on "new" treatments in combination with standard experiments on other treatments would achieve a greater degree of efficiency in the use of resources, he defines an "augmented design" to this end. Such a design is defined to be any standard design with "new" treatments added to the complete block, the incomplete block, the row, the column, etc. An example is given of a randomized block experiment for "standard" treatments augmented by a set of "new" treatments each of which appears only once in the experiment. The analysis is carried out and standard errors are given for comparisons of two standard treatments, two new treatments in the same block and in different blocks, and a standard treatment with a new treatment. The adjusted treatment sum of squares is partitioned into variability among standard treatments, among new treatments within blocks, and between standard and new treatments within blocks. A second example is given in which a balanced lattice design is augmented. Here again, each new treatment appears only once and variances for treatment comparisons are given as in the first example. An appendix to the paper contains the generalized AOV for all designs with one-way elimination of heterogeneity for both the fixed and mixed models.

P. W. M. John (6) has examined this sort of experimental situation with special attention to the problem in which two treatments are to be compared at different levels of another factor where the block size is three. General directions for analyses in such situations are given.

A later paper by John (7) defines an "extended complete block design" to be one in which the block size exceeds the number of treatments, t, but is less than 2t and where each block contains each treatment at least once with some treatments duplicated in each block according to some balanced pattern. The paper then treats that situation in which each block contains a complete replicate plus a block from a Balanced Incomplete Block design and also the special case in which there is only one extra plot in each block. The designs are specified by v treatments, each replicated r times, b blocks each of size k, and by the incidence matrix N which is v x b with each element a one or a two. Where T is the vector of treatment totals, B the vector of block totals, G the grand total of the observations, j a vector of 1's, and $A = rI_v - \frac{NN'}{k}$, the intrablock estimates are shown to be given by $A\hat{\tau} = T - \frac{NB}{k}$ and the interblock estimates by $A\tilde{\tau} = NB - \frac{rGj}{b}$.

The efficiency factor for the intrablock analysis is given as $E = \frac{v\lambda^*}{rk}$ where $\lambda^* = 2r - b + \lambda$ and λ is the parameter of the incomplete block design. The intrablock and interblock analyses and the combination of estimates are described and a numerical example is given to illustrate the method of analysis.

The concern of the present investigation is with the sort of experimental situations discussed and exemplified above, broadly, those in which the number of treatments is exceeded by block size. Unlike the work of Pearce (9) and Federer (2), the emphasis is not upon screening situations and no particular treatment is singled out for special attention, but rather interest is confined to situations in which it is thought desirable to estimate all possible treatment differences with equal precision.

Also, consideration will be limited to those experimental situations in which blocks are of constant size. This condition would frequently be met in industrial contexts; and although its strict realization in biological experimentation would seem less likely, variability in litter size and truss size, for example, may be small enough so that the condition could be achieved by discarding far fewer experimental units than would be necessary for the application of the more usual designs. The circumstances encountered in experimentation which are to be dealt with are much like those which were of interest to John (7), but no restriction of block size to being less than twice the number of treatments is imposed and the necessity for the occurrence of a complete replicate in a block is not invoked.

Generally then, the concern here is with the provision of easily applied methods of analyses, along with information on their effectiveness, for those experimental situations which are characterized by experimental units of high cost which group naturally into constant-sized blocks so that costs are principally associated with blocks as a whole and so that the block size exceeds the number of treatments.

The designs developed here to meet such situations may be termed Displaced Balanced Incomplete Block designs and are those obtained by the displacement of each one and each zero in existing Balanced Incomplete Block designs by the non-negative integers n_1 and n_2, respectively.

These designs share with Balanced Incomplete Block designs the properties of balance and simplicity of analysis. As would be expected, many of the formulae derived in the next chapters in connection with the development of the analysis are markedly similar to corresponding formulae in the analysis of standard Balanced Incomplete Block designs.

Chapter II contains the definition and description of these Displaced designs, their intrablock analysis, and tables indicating "best" designs for certain experimental situations. Chapter III treats the recovery of interblock information from this sort of design and the combining of intrablock and interblock information. Chapter IV contains the summary and conclusions along with recommendations for further study for this and related areas of experimental design.

A final example will now be given to illustrate those qualities of experimental situations which are in mind and to exhibit an application of a Displaced Balanced Incomplete Block design.

An experimenter investigating the claim that the many different laundry detergents on the market differ only in packaging and advertising found that there were eleven brands in the top sales category. He decided to use these for experimentation; and in order to make his conclusions more meaningful, he decided to have housewives use these detergents to do their laundries in a setting which would allow for control and observation. To this end, he arranged to lease for a day a commercial self-service laundry and advertised free laundering facilities, including detergent, to the community's housewives in exchange for the opportunity of having a trained home economist determine the quality of each wash job. The laundry was equipped with machines of the same design and manufacturer, and it was found that fifty-five machines were located so as to be readily accessible. The experience of the laundry operator indicated that these machines would be in constant use or demand on the day of the experiment. The laundry was in operation from 8 a.m. until 11 p.m., and a complete washing cycle on each machine was of twenty-five minutes duration. With an allowance of slightly more than four minutes for the loading and unloading operations, it was found that thirty-one washings could be expected from each machine during the test day. It was decided to use each machine as a block, an experimental unit being a machine load of laundry. The observed response would be, of course, the quality measure of the wash job. Thus the situation is one with eleven treatments, fifty-five blocks, and thirty-one experimental units per block. Since the lease was for the whole day, the major overhead cost continues whether or not observations are taken.

If a randomized complete block design were to be used, blocks of size twenty-two could be obtained by discarding nine experimental units from each block. The variance of an estimate of the difference in two detergent means would be given by $\frac{\sigma^2}{55}$ where σ^2 is the variance associated with experimental units within blocks.

An alternative to this would be the splitting of each block into three groups of ten experimental units each, with one experimental unit being discarded. The experiment could then be viewed as fifteen sub-experiments each with eleven treatments, eleven blocks, and ten plots per block. Each of these fifteen sub-experiments could be analyzed as a Balanced Incomplete Block design, and the fifteen independent estimates of a treatment difference could be used to form a combined estimate. This estimate would have a variance of $\frac{\sigma^2}{74.25}$. Since the partitioning of blocks would probably be done arbitrarily, it would not necessarily follow that the σ^2 here would be smaller than that appearing in the randomized complete block approach.

The approach of John may most obviously be extended to cover situations, such as this one, in which block size lies between twice the number of treatments and three times the number of treatments, by putting on two complete replicates and adding a Balanced Incomplete Block design, that is, by having each treatment appear at least twice in each block with some treatments appearing three times such that the triplicated treatments form a Balanced Incomplete Block design. The attempt to apply this adaptation of John's extended complete block designs to the present situations leads to a search of available Balanced Incomplete Block plans for one with eleven treatments, fifty-five blocks, and nine experimental units per block. The search for such a design would be fruitless and it would be concluded that this approach would be inapplicable.

It will be seen in the chapters which follow that a Displaced Balanced Incomplete Block design can be applied in this situation without the loss of any experimental units and that if this is done, the variance of an estimate of treatment differences would be $\frac{\sigma^2}{77.35}$.

Also of interest here is the fact that had it been possible to apply the earlier-mentioned adaptation of the extended complete block designs such an application would have been picked up in considering the applicable Displaced Balanced Incomplete Block designs.

CHAPTER II

DISPLACED BALANCED INCOMPLETE BLOCK DESIGNS AND THEIR INTRABLOCK ANALYSIS

In the present chapter the definition and description of Displaced Balanced Incomplete Block designs is preceded by a brief discussion of the general two-way classification of which they are a special case. Significant and well-known properties of this wider class are given and used to develop the special analysis of Displaced Balanced Incomplete Block designs. The problem of choosing among such displaced designs as are applicable in specific experimental situations is considered.

The General Two-Way Classification

The scalar model for the general two-way classification without interaction is given by:

$$y_{ijk} = \mu + \tau_i + \beta_j + e_{ijk}; \quad \begin{array}{l} i = 1, \ldots, t \\ j = 1, \ldots, b \\ k = 1, \ldots, c_{ij} \\ E(e_{ijk}) = 0 \\ E(e_{ijk}e_{i'j'k'}) = \begin{cases} \sigma^2; & i=i', j=j', k=k' \\ 0; & \text{otherwise} \end{cases} \end{array}$$

in which τ_i represents the effect of treatment i and β_j represents the effect of block j. The number of applications of treatment i in block j is given by c_{ij} and the matrix $C = (c_{ij})$ is termed the incidence matrix.

Displaced Balanced Incomplete Block Designs

Definition 2.1: A Displaced Balanced Incomplete Block design, abbreviated DBIBD, is a connected, two-way design with the following properties:

1. Each treatment is applied n_1 or n_2 times in a block ($n_i \geq 0$).

2. Replacement of n_1 by unity and n_2 by zero results in a Balanced Incomplete Block Design (BIBD).

The above definition implies the existence of a set of constants associated with any given DBIBD. These are: the number of treatments, the number of blocks, the number of experimental units per block, the number of applications in the experiment of each treatment, and the number of distinct conjunctions in the same block of each pair of treatments. These constants are denoted t, b, k, r, and λ, respectively.

Some clarification of the definition may be obtained by considering a simple example. Suppose that, in an experimental situation with four treatments to be investigated, there are six blocks available for experimentation, each containing ten plots. The layout exemplifying an applicable DBIBD might appear as below:

Treatments

Blocks	1	2	3	4
1	3	3	2	2
2	3	2	3	2
3	3	2	2	3
4	2	3	3	2
5	2	3	2	3
6	2	2	3	3

$t = 4$

$b = 6$

$k = 10$

$r = 15$

$\lambda = 37$

The values of t, b, k, and r are easily observed. The value for λ may be obtained by selecting any pair of treatments, say Treatment 1 and Treatment 2, and noting the number of distinct conjunctions of these in each block. In Block 1, Treatment 1 occurs three times and each occurrence is accompanied by three distinct occurrences of Treatment 2; thus, there exist nine distinct conjunctions of Treatment 1 and Treatment 2 in Block 1. In Block 2, Treatment 1 again occurs three times, but each occurrence is accompanied by only two occurrences of Treatment 2; thus, there exist six distinct conjunctions of the two treatments in Block 2. Similarly, it may be seen that in Blocks 3, 4, 5, and 6 there exist, respectively, six, six, six, and four distinct conjunctions of Treatment 1 and Treatment 2. Hence, the number of distinct conjunctions in the same block of the first two treatments is thirty-seven. That this value remains the same for any pair of treatments may be verified.

The properties implied by the definition of a DBIBD may be stated in the notation of the general two-way classification as follows:

1. $c_{ij} = n_1 \geq 0$ or $c_{ij} = n_2 \geq 0$
2. $c_{.j} = \sum_i c_{ij} = k, \forall j$
3. $c_{i.} = \sum_j c_{ij} = r, \forall i$
4. $\sum_j c_{ij}c_{i'j} = \lambda, \forall i \neq i'$

It follows directly from the definition of a DBIBD that, in a given BIBD, displacement of the 1's and 0's by n_1's and n_2's, respectively, results, if the design is connected, in a DBIBD. It is this feature for which the DBIBD is named. In this sense, every DBIBD may be thought of as having been generated by a BIBD.

<u>Definition 2.2</u>: That BIBD obtained from a particular DBIBD by replacing n_1 with unity and n_2 with zero is the generating BIBD of the given DBIBD.

The constants associated with a BIBD, number of treatments, number of blocks, number of experimental units per block, number of blocks in which each treatment appears, and number of conjunctions, are denoted here by t^*, b^*, k^*, r^*, and λ^*.

A number of useful identities involving the constants of a DBIBD and those of its generating BIBD are immediate. These are given below:

1. $t = t^*$, $b = b^*$
2. $r^*t = bk^*$, $rt = bk = c_{..}$
3. $\lambda^*(t - 1) = r^*(k^* - 1)$
4. $k = n_1k^* + (t - k^*)n_2 = n_2t + (n_1 - n_2)k^*$
5. $r = n_1r^* + (b - r^*)n_2 = n_2b + (n_1 - n_2)r^*$.

An important relationship involving λ and λ^* is given as Theorem 1 in the next section.

Intrablock Analysis of DBIBD's

The general two-way classification model may be written in matrix notation as: $Y = \mu j + X_1\tau + X_2\beta + e$. In which case, the normal equations are given by $A\hat{\tau} = q$ where $A = X_1'X_1 - X_1'X_2(X_2'X_2)^{-1}X_2'X_1$ and $q = X_1'Y - X_1'X_2(X_2'X_2)^{-1}X_2'Y$.

The intrablock estimates of the treatment effects in a DBIBD are obtained by way of simplifications of A and q of the normal equations for the general two-way classification.

As in the BIBD, we have the following relationships:

$$X_1'X_1 = rI_t,\ X_2'X_2 = kI_b, \text{ and } X_1'X_2 = C.$$

Hence, $A = rI_t - \frac{1}{k} CC'$.

Any off-diagonal element of CC' is apparently λ. That is, if $CC' = (b_{ij})$, then $b_{ij} = \lambda$, $i \neq j$. This, along with the following theorem, provides a useful relationship between λ and λ^*.

<u>Theorem 2.1</u>: $b_{ij} = \lambda^*(n_1 - n_2)^2 + n_2(2r - bn_2)$, $i \neq j$

<u>Proof</u>:
$$
\begin{aligned}
b_{ij} &= \lambda^* n_1^2 + [2(r^* - \lambda^*)]n_1 n_2 + [b - \lambda^* - 2(r^* - \lambda^*)]n_2^2 \\
&= \lambda^* n_1^2 + (b - \lambda^*)n_2^2 + 2(r^* - \lambda^*)n_2(n_1 - n_2) \\
&= \lambda^*(n_1^2 - n_2^2) + bn_2^2 + 2n_2(n_1 - n_2)\left[\frac{r - n_2 b}{n_1 - n_2} - \lambda^*\right] \\
&= \lambda^*(n_1^2 - n_2^2) + bn_2^2 + 2n_2[r - n_2 b - \lambda^*(n_1 - n_2)] \\
&= \lambda^*(n_1^2 - n_2^2) + bn_2^2 - 2bn_2^2 + 2rn_2 - 2n_2\lambda^*(n_1 - n_2) \\
&= \lambda^*(n_1^2 - n_2^2) + 2rn_2 - bn_2^2 - 2n_2\lambda^*(n_1 - n_2) \\
&= \lambda^*(n_1 - n_2)[(n_1 + n_2) - 2n_2] + n_2(2r - bn_2) \\
&= \lambda^*(n_1 - n_2)^2 + n_2(2r - bn_2)
\end{aligned}
$$

The desired relationship between λ and λ^* is then $\lambda = \lambda^*(n_1 - n_2)^2 + n_2(2r - bn_2)$.

Another theorem useful in the simplification of $CC' = (b_{ij})$ is given below.

<u>Theorem 2.2</u>: $b_{ii} = rk - \lambda(t - 1)$

<u>Proof</u>:
$$
\begin{aligned}
b_{ii} &= r^* n_1^2 + (b - r^*)n_2^2 \\
&= r^*(n_1^2 - n_2^2) + bn_2^2 \\
&= (r - bn_2)(n_1 + n_2) + bn_2^2 \\
&= (r - bn_2)n_1 + n_2 r \\
&= (r - bn_2)n_1 + [n_2 bk - n_2 rt] + n_2 r
\end{aligned}
$$

$$= [(r - bn_2)(n_1 - n_2) + (r - bn_2)n_2]$$

$$+ \left[n_2 b\left(\frac{k - n_2 t}{n_1 - n_2}\right)(n_1 - n_2) + n_2^2 bt \right] - n_2 rt + n_2 r$$

$$= (r - bn_2)(n_1 - n_2) + n_2 bk^*(n_1 - n_2) + (r - bn_2)n_2$$

$$+ n_2^2 bt - n_2 rt + n_2 r$$

$$= [(r - bn_2)(n_1 - n_2)k^* - (r - bn_2)(n_1 - n_2)(k^* - 1)]$$

$$+ n_2 bk^*(n_1 - n_2) + (r - bn_2)n_2 - n_2 t(r - bn_2) + n_2 r$$

$$= rk^*(n_1 - n_2) - (r - bn_2)(n_1 - n_2)(k^* - 1)$$

$$- n_2(r - bn_2)(t - 1) + n_2 r$$

$$= rk^*(n_1 - n_2) - r^*(n_1 - n_2)^2(k^* - 1) - n_2(r - bn_2)(t - 1)$$

$$+ n_2 r + (n_2 rt - n_2 rt)$$

$$= rn_2 t + rk^*(n_1 - n_2) - (n_1 - n_2)^2 r^*(k^* - 1)$$

$$- n_2(r - bn_2)(t - 1) - n_2 r(t - 1)$$

$$= r[n_2 t + k^*(n_1 - n_2)] - \lambda^*(n_1 - n_2)^2 \frac{r^*(k^* - 1)}{\lambda^*}$$

$$- n_2(2r - bn_2)(t - 1)$$

$$= r[n_2 t + k^*(n_1 - n_2)] - [\lambda^*(n_1 - n_2)^2 + n_2(2r - bn_2)](t - 1)$$

$$= rk - \lambda(t - 1).$$

Taking account of the above theorems allows the expression $CC' = (rk - \lambda t)I + \lambda J$. Hence, $A = rI - \frac{1}{k}[(rk - \lambda t)I + \lambda J]$

$$= \frac{1}{k}[rk - (rk - \lambda t)]I - \frac{1}{k}\lambda J$$

$$= \frac{\lambda t}{k}(I - \frac{1}{t}J).$$

The rank of A is seen to be $t - 1$ and the imposition of the usual restriction $j_t^1 \hat{\tau} = 0$ results in the normal equations; $\frac{\lambda t}{k} \hat{\tau} = q$. The q vector may be expressed as $T - \frac{1}{k} CB$ where $T = (T_i)$ is the vector of treatment totals and $B = (B_j)$ is the vector of block totals. Hence, $\hat{\tau}_i = \frac{k}{\lambda t}(T_i - \frac{1}{k} \sum_j c_{ij} B_j)$.

As in the BIBD, $E\ q = \frac{\lambda t}{k}(I - \frac{1}{t} J)\tau$ and $Var\ q = \frac{\lambda t}{k}(I - \frac{1}{t} J)\sigma^2$. Hence, $E\hat{\tau} = \tau - \bar{\tau}.j_1^t$ and $Var\ \hat{\tau} = \frac{k}{\lambda t}(I - \frac{1}{t} J)\sigma^2$. Also $E(\hat{\tau}_i - \hat{\tau}_{i'}) = \tau_i - \tau_{i'}$ and $Var\ (\hat{\tau}_i - \hat{\tau}_{i'}) = \frac{k \cdot 2\sigma^2}{\lambda t}$.

Unlike the BIBD, it is possible with the DBIBD to provide a test for the interaction of blocks and treatments. The recognition of the possible existence of such interaction would seem desirable in many experimental situations.

With the DBIBD scalar model as, $y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + e_{ijk}$ and letting η be the number of empty cells, the intrablock analysis of variance table for testing $H_o : (\tau\beta)_{11} = (\tau\beta)_{12} = \cdots = (\tau\beta)_{tb}$ is as follows:

Source	df	S.S.	M.S.
Total	n..	$\sum_{ijk} y_{ijk}^2$	
Blocks (Unadj.)	b	$\frac{1}{k}\sum_j B_j^2$	
Treatments (Adj.)	t-1	$\frac{k}{\lambda t}\sum_i [T_i - \frac{1}{k}\sum_j c_{ij}B_j]^2$	
Blocks x Treatments	$(b-1)(t-1) - \eta$	$\sum_{ij} \frac{y_{ij.}^2}{n_{ij}} - \frac{1}{k}\sum_j B_j^2 - \frac{k}{\lambda t}\sum_i [T_i - \frac{1}{k}\sum_j c_{ij}B_j]^2$	I
Intrablock Error	$n.. - (bt-\eta)$	$\sum_{ijk} y_{ijk}^2 - \sum_{ij} \frac{y_{ij.}^2}{n_{ij}}$	E

The value of the F-statistic for testing the hypothesis of no interaction is given by $\frac{I}{E}$.

"Best" DBIBD for Given Numbers of Treatments and Block Sizes

In a given experimental situation to which a DBIBD is applicable, it is clear that more than one such design may be applied. For example, in a situation with t = 5, k = 7, and b = 10, both of the following two layouts are applicable DBIBD's.

Blocks	Treatments 1	2	3	4	5
1	1	1	1	1	3
2	1	1	1	3	1
3	1	1	3	1	1
4	1	3	1	1	1
5	3	1	1	1	1
6	1	1	1	1	3
7	1	1	1	3	1
8	1	1	3	1	1
9	1	3	1	1	1
10	3	1	1	1	1

Blocks	Treatments 1	2	3	4	5
1	2	2	1	1	1
2	1	1	2	2	1
3	1	2	1	1	2
4	2	1	2	1	1
5	1	1	1	2	2
6	2	1	1	2	1
7	1	2	2	1	1
8	1	1	2	1	2
9	2	1	1	1	2
10	1	2	1	2	1

The remainder of the present chapter is a consideration of the criteria for and identification of "best" designs for particular situations and culminates in a catalog of "best" designs for given t, k, and bk. This catalog contains entries for number of treatments from three to fifteen. For each value of t, k is allowed to range from t + 1 through 3t. The values of bk which appear are all those possible in view of the necessity of the existence of generating BIBD's.

Procedure

The following steps constitute the procedure used in the development of the catalog and were taken for each value of t:

1. A listing of all possible BIBD's with the given t value was made. This was done by searching the indexes of Cochran and Cox (1) and Fisher and Yates (3). For each design in the catalog there is a reference to one or another of these sources or an entry of "Unreduced." All Cochran and Cox references are given as 11.xx where xx is the design number. Fisher and Yates references are identifiable as those with just a design number. Those BIBD obtainable by forming all possible combinations of the t numbers in groups of size k are unreferenced and indicated by "Unreduced."

2. Since any DBIBD generated by a BIBD that has a complement could equally well be generated by that complement simply by interchanging the roles of n_1 and n_2, pairs of designs appearing in the list which were complements were identified and one of each pair eliminated from the list. Also removed from the list were those BIBD's which could be obtained by combining others given or complements of others given. This resulted in a shortest list of BIBD's which would generate all possible DBIBD's.

3. For each value of k, every design in the shortest list was considered separately and all pairs (n_1, n_2) which when applied to that design would produce a DBIBD with the given k value were identified.

4. The "best" of the designs obtained in Step 3, i.e., the "best" DBIBD with a given value of k and generated by a given BIBD, was selected by the criterion discussed below. Thus for each value of t, each value of k, and for each possible underlying BIBD, a "best" DBIBD was found. These were listed and appear in the appendix along with those values of bk which are attained by a single occurrence of the design and corresponding criteria values.

5. Finally for given values of t, k, and bk, all surviving DBIBD's were considered. These included the "best" from each generating BIBD which could produce a DBIBD with the proper constants and also designs resulting from combinations of these. A criterion, discussed below, was applied to identify that design or combination of designs which appears in the catalog.

Criterion for selection of "best" DBIBD for fixed t, k, and generating BIBD

Since the primary interest of an investigator would quite likely be in the estimation of treatment differences, the minimization of the variance of this estimator is used as the basis of the criterion. It has been shown that Var $(\tau_i - \tau_j) = \frac{2\sigma^2 k}{\lambda t}$ for a DBIBD. Since the fixing of k and t allows only λ to vary, the criterion becomes that of choosing the design which maximizes λ. The following discussion provides a refinement of this criterion in terms of the relative sizes of n_1 and n_2.

The expression $\lambda = \lambda^*(n_1 - n_2)^2 + n_2(2r - bn_2)$ may be viewed as a function of n_1 and n_2 and with fixed k we have also $n_1 = \frac{k - (t - k^*)n_2}{k^*}$. Hence, $\lambda = f(n_2) = \lambda^*\left(\frac{k - (t - k^*)n_2}{k^*} - n_2\right)^2 + n_2(2r - bn_2)$

$$= \frac{\lambda^*}{k^{*2}}(k - tn_2)^2 + 2rn_2 - bn_2^2$$

$$= \frac{\lambda^*}{k^{*2}}(k^2 - 2ktn_2 + t^2n_2^2) + 2rn_2 - bn_2^2$$

$$= \left(\frac{\lambda^* t^2}{k^{*2}} - b\right)n_2^2 - \left(\frac{2kt\lambda^*}{k^{*2}} - 2r\right)n_2 + \frac{\lambda^* k^2}{k^{*2}}.$$

It may be noted that the above is the equation of a parabola with axis parallel to the $f(n_2)$ axis.

Taking first and second derivatives achieves the following results:

$$\frac{df(n_2)}{dn_2} = 2\left(\frac{\lambda^* t^2}{k^{*2}} - b\right)n_2 - 2\ \frac{kt\lambda^*}{k^{*2}} - r$$

$$\frac{d^2f(n_2)}{dn_2^2} = 2\left(\frac{\lambda^* t^2}{k^{*2}} - b\right) = \frac{2}{k^{*2}}(\lambda^* t^2 - bk^{*2}) = \frac{2}{k^{*2}}(\lambda^* t^2 - r^* tk^*)$$

$$= \frac{2t}{k^{*2}}(\lambda^* t - r^* k^*) = \frac{2t}{k^{*2}}[(\lambda^* + r^* k^* - r^*) - r^* k^*]$$

$$= \frac{2t}{k^{*2}}(\lambda^* - r^*).$$

Hence, $\frac{df(n_2)}{dn_2} = 0$ implies $n_2 = \frac{rk^{*2} - kt\lambda^*}{bk^{*2} - \lambda^* t^2} = n^*$.

Also, $\frac{d^2 f(n_2)}{dn_2^2} < 0$, since $\lambda^* < r^*$.

It is desirable at this point to give two short lemmas which will then be used to simplify the expression for n^*.

<u>Lemma 1</u>: $\frac{k^*}{t - k^*} = \frac{r^*}{b - r^*}$, since $r^* t = k^* b$.

<u>Lemma 2</u>: $(t - k^*)\lambda^* t - (b - r^*)k^{*2} = (\lambda^* - r^*)(t - k^*)$

<u>Proof</u>: $(t - k^*)\lambda^* t - (b - r^*)k^{*2} = (t - k^*)(\lambda^* + r^* k^* - r^*) - bk^{*2}$
$+ r^* k^{*2}$

$= (\lambda^* t + r^* k^* t - r^* t - k^*\lambda^* - r^* k^{*2} + r^* k^*) - r^* k^* t + r^* k^{*2}$

$= \lambda^* t - r^* t - k^*\lambda^* + r^* k^*$

$= \lambda^*(t - k^*) - r^*(t - k^*) = (\lambda^* - r^*)(t - k^*)$

Now $bk^{*2} - \lambda^* t^2 = r^* tk^* - \lambda^* t^2 = t(r^* k^* - \lambda^* t)$

$= t(r^* k^* - \lambda^* - r^* k^* + r^*) = t(r^* - \lambda^*)$

and $rk^{*2} - \lambda^* tk = [n_1 r^* + (b - r^*)n_2]k^{*2} - [n_1 k^* + (t - k^*)n_2]\lambda^* t$

$= (b - r^*)\left[\frac{r^*}{b - r^*} n_1 + n_2\right]k^{*2} - (t - k^*)\left[\frac{k^*}{t - k^*} n_1 + n_2\right]\lambda^* t$

$= \left(\frac{k^*}{t - k^*} n_1 + n_2\right)[(b - r^*)k^{*2} - (t - k^*)\lambda^* t]$ by Lemma 1

$= \left(\frac{k^*}{t - k^*} n_1 + n_2\right)[(r^* - \lambda^*)(t - k^*)]$ by Lemma 2

$= (r^* - \lambda^*)\left[k^* n_1 + (t - k^*)n_2\right] = (r^* - \lambda^*)k.$

Hence, $n^* = \frac{rk^{*2} - \lambda^* tk}{bk^{*2} - \lambda^* t^2} = \frac{(r^* - \lambda^*)k}{(r^* - \lambda^*)t} = \frac{k}{t}$.

So it is seen that if $f(n_2) = \left(\frac{\lambda^* t^2}{k^{*2}} - b\right) n_2{}^2 - \left(\frac{2kt\lambda^*}{k^{*2}} - 2r\right) n_2 + \frac{\lambda^* k^2}{k^{*2}}$ is considered to be a continuous function of n_2, it is the equation of a parabola with axis parallel to the $f(n_2)$ axis, opening downward and with vertex at $n_2 = n^* = \frac{k}{t}$. Since $n_1 = \frac{k - (t - k^*)n_2}{k^*}$; where $n_2 = \frac{k}{t}$,

$n_1 = \frac{k - (t - k^*)k/t}{k^*} = \frac{kt - (t - k^*)k}{k^* t} = \frac{k}{t} = n^*$. It follows that, for a fixed k and with $\lambda = f(n_1, n_2)$ considered as a continuous function of n_1 and n_2, $\lambda = f(n_1, n_2)$ achieves its maximum value at (n^*, n^*).

The domain of $f(n_1, n_2)$ is composed of points (n_1, n_2) where n_1 and n_2 are integral; $n_1 \geq 0$, $n_2 \geq 0$. It follows from the parabolic nature of $f(n_2)$ that λ achieves its maximum value for a fixed k at a point (n_1, n_2) in the domain such that the absolute distance of (n^*, n^*) from (n_1, n_2) is less than the absolute distance from any other point in the domain. Letting the distance from (n^*, n^*) to (n_1, n_2) be d gives

$$d^2 = (n^* - n_1)^2 + (n^* - n_2)^2 = n_1{}^2 + n_2{}^2 + 2n^{*2} - 2n^*(n_1 + n_2)$$

$$= n_1{}^2 + n_2{}^2 + \frac{2k^2}{t^2} - \frac{2k}{t}(n_1 + n_2) = \frac{1}{t^2}[t^2(n_1{}^2 + n_2{}^2) + 2k^2 - 2kt(n_1 + n_2)]$$

and $[t^2(n_1{}^2 + n_2{}^2) - 2kt(n_1 + n_2) + 2k^2]$

$$= t^2(n_1{}^2 + n_2{}^2) - 2t[n_2 t + (n_1 - n_2)k^*](n_1 + n_2) + 2[n_2 t + (n_1 - n_2)k^*]^2$$

$$= t^2(n_1{}^2 + n_2{}^2) - 2t[n_1 n_2 t + n_2{}^2 t + (n_1{}^2 - n_2{}^2)k^*] + 2[t^2 n_2{}^2 + (n_1{}^2 + n_2{}^2 - 2n_1 n_2)k^{*2} + 2tk^*(n_1 n_2 - n_2{}^2)]$$

$$= n_1{}^2(t^2 - 2tk^* + 2k^{*2}) - 2n_1 n_2(t^2 + 2k^{*2} - 2tk^*) + n_2{}^2(t^2 - 2t^2 + 2tk^* + 2t^2 + 2k^{*2} - 4tk^*)$$

$$= (n_1 - n_2)^2 (t^2 - 2tk^* + 2k^{*2}).$$

So $d^2 = \frac{t^2 - 2tk^* + 2k^{*2}}{t^2}(n_1 - n_2)^2$ and it is seen that d^2 will be minimized (and λ maximized) by choosing n_1 and n_2 so that the absolute value of their difference is minimized.

It is interesting to note that a necessary and sufficient condition for $n_1 - n_2 = 0$ is that k be a multiple of t.

Criterion for selection of "best" DBIBD for fixed t, b, and bk

Again the basis of the criterion is chosen as the minimization of Var $\widehat{(\tau_i - \tau_j)}$.

Three different situations, resulting from the varying nature of the bk value, can occur where there exist several possible generating BIBD's. These are considered below:

Case 1. The value of bk is attainable only by the single occurrence of one DBIBD or another. Here, the best design may be determined by direct comparison of the λ values associated with all such designs. These λ values are those appearing in the Appendix and that design with the greatest λ value is chosen.

Case 2. The value of bk is attainable only by N_1 repetitions of a DBIBD or by N_2 repetitions of a different DBIBD. Where $N_1 = N_2$ a direct comparison of λ's from the repeated designs will identify the better design. However, where $N_1 \neq N_2$ the λ's associated with designs obtained by repetitions evolve from those of the repeated designs in different ways. Hence, the comparison of λ's from the generating designs is ineffective. In such cases, the efficiency of a design, the ratio of Var $(\widehat{\tau_i - \tau_j})$ from a randomized complete block design with the same number of treatments and number of experimental units to Var $(\widehat{\tau_i - \tau_j})$ from the design under consideration, was used as a criterion value. This property of a design is invariant under repetitions. That DBIBD is selected whose generating design has the greatest efficiency.

Case 3. The value of bk is attainable by N_1 repetitions of one DBIBD and N_2 repetitions of a different DBIBD. Where the efficiency of the larger of these two generating designs exceeds that of the smaller, an estimator obtained by combining the estimator from the N_1 repetitions with that of the N_2 repetitions is called for. The variance of this combined estimator involves a λ from each of the generators and the comparison of this design with other appropriate candidates is achieved by use of the efficiency criterion.

The theorems which follow develop the combined estimator, its variance, and the efficiency of the related design. Corollaries are given which support and refine the efficiency criterion.

<u>Lemma 3</u>: If a DBIBD is given by t, k, n_1, n_2, b_1, r_1, and λ_1 and a second DBIBD given by t, k, n_1, n_2, b_2, r_2, and λ_2 is obtained by repeating the first design N times, then $\lambda_2 = N\lambda_1$.

<u>Proof</u>: By definition, $b_2 = Nb_1$, $r_2 = Nr_1$, and $\lambda_2{}^* = N\lambda_1{}^*$. So

$$\lambda_2 = (n_1 - n_2)^2 \lambda_2{}^* + n_2(2r_2 - b_2n_2) = (n_1 - n_2)^2 N\lambda_1{}^* + n_2(2Nr_1 - Nb_1r_2)$$

$$= N[(n_1 - n_2)^2 \lambda_1{}^* + n_2(2r_1 - b_1n_2)] = N\lambda_1.$$

Theorem 2.3: Where a DBIBD is given by t, k, b_1, r_1, n_{11}, n_{12}, λ_1 and has an efficiency of E_1, a second DBIBD is given by t, k, b_2, r_2, n_{21}, n_{22}, λ_2, and has an efficiency of E_2, and a design is obtained by combining N_1 repetitions of the first with N_2 repetitions of the second; the best linear unbiased estimate of $(\tau_i - \tau_j)$ is given by

$$\widehat{(\tau_i - \tau_j)}_C = \frac{N_1\lambda_1\widehat{(\tau_i - \tau_j)}_{N_1} + N_2\lambda_2\widehat{(\tau_i - \tau_j)}_{N_2}}{N_1\lambda_1 + N_2\lambda_2},$$

where $\widehat{(\tau_i - \tau_j)}_{N_1}$ and $\widehat{(\tau_i - \tau_j)}_{N_2}$ are estimates from the repetitions of the first and second designs, respectively.

Proof: By the lemma: N_1 repetitions of the first design produce a DBIBD with associated constants t, k, N_1b_1, N_1r_1, n_{11}, n_{12}, $N_1\lambda_1$, and N_2 repetitions of the second design produce a DBIBD with associated constants t, k, N_2b_2, N_2r_2, n_{21}, n_{22}, $N_2\lambda_2$. So Var $\widehat{(\tau_i - \tau_j)}_{N_1} = \frac{2\sigma^2 k}{N_1\lambda_1 t}$ and Var $\widehat{(\tau_i - \tau_j)}_{N_2} = \frac{2\sigma^2 k}{N_2\lambda_2 t}$.

By Theorem 18.11 Graybill (4):

$$\widehat{(\tau_i - \tau_j)}_C = \left[\frac{2\sigma^2 k\widehat{(\tau_i - \tau_j)}_{N_1}}{N_2\lambda_2 t} + \frac{2\sigma^2 k\widehat{(\tau_i - \tau_j)}_{N_2}}{N_1\lambda_1 t}\right] \div \left[\frac{2\sigma^2 k}{N_2\lambda_2 t} + \frac{2\sigma^2 k}{N_1\lambda_1 t}\right]$$

$$= \frac{\frac{\widehat{(\tau_i - \tau_j)}_{N_1}}{N_2\lambda_2} + \frac{\widehat{(\tau_i - \tau_j)}_{N_2}}{N_1\lambda_1}}{\frac{1}{N_2\lambda_2} + \frac{1}{N_1\lambda_1}} = \frac{N_1\lambda_1\widehat{(\tau_i - \tau_j)}_{N_1} + N_2\lambda_2\widehat{(\tau_i - \tau_j)}_{N_2}}{N_1\lambda_1 + N_2\lambda_2}$$

Theorem 2.4: The efficiency of the design described in Theorem 2.3 is given by $E_C = \frac{(N_1\lambda_1 + N_2\lambda_2)t^2}{(N_1b_1 + N_2b_2)k^2}$.

Proof: $$\text{Var}\,(\widehat{\tau_i - \tau_j})_C = \frac{N_1^2\lambda_1^2}{(N_1\lambda_1 + N_2\lambda_2)^2}\cdot\frac{2\sigma^2 k}{N_1\lambda_1 t} + \frac{N_2^2\lambda_2^2}{(N_1\lambda_1 + N_2\lambda_2)^2}\cdot\frac{2\sigma^2 k}{N_2\lambda_2 t}$$

$$= \frac{2\sigma^2 k}{t}\left[\frac{N_1\lambda_1 + N_2\lambda_2}{(N_1\lambda_1 + N_2\lambda_2)^2}\right] = \frac{2\sigma^2 k}{t(N_1\lambda_1 + N_2\lambda_2)}.$$

Hence $$E_C = \frac{2\sigma^2 t}{(N_1 b_1 + N_2 b_2)k} \div \frac{2\sigma^2 k}{t(N_1\lambda_1 + N_2\lambda_2)} = \frac{(N_1\lambda_1 + N_2\lambda_2)t^2}{(N_1 b_1 + N_2 b_2)k^2}.$$

Corollary 1: The efficiency of a DBIBD is invariant under repetitions.

Proof: Let the design be given by t, k, b, r, λ and have efficiency E_1, then $E_1 = \frac{\lambda t^2}{bk^2}$.

By the theorem, the efficiency of the design obtained by N repetitions of the given DBIBD is $E = \frac{N\lambda t^2}{Nbk^2} = \frac{\lambda t^2}{bk^2} = E_1$.

Corollary 2: If an experimental situation involves t treatments, $N_1 b_1 + N_2 b_2$ blocks where $N_1 > 0$, $0 < b_2 N_2 < b_1$; k experimental units per block; and there exists a DBIBD with constants t, k, b_1, n_{11}, n_{12}, λ_1, E_1; and a DBIBD with constants t, k, b_2, n_{21}, n_{22}, λ_2, E_2; then that DBIBD obtained by combining N_1 repetitions of the first design and N_2 repetitions of the second design is of greater efficiency than the design obtained by $\frac{N_1 b_1 + N_2 b_2}{b_2}$ repetitions of the second design if and only if $\lambda_1 b_2 > \lambda_2 b_1$.

Proof: $\lambda_1 b_2 > \lambda_2 b_1 \Longleftrightarrow N_1\lambda_1 b_2 + N_2\lambda_2 b_2 > N_1\lambda_2 b_1 + N_2\lambda_2 b_2$

$$\Longleftrightarrow (N_1\lambda_1 + N_2\lambda_2)b_2 > (N_1 b_1 + N_2 b_2)\lambda_2 \Longleftrightarrow \frac{(N_1\lambda_1 + N_2\lambda_2)t^2}{(N_1 b_1 + N_2 b_2)k^2} > \frac{\lambda_2 t^2}{b_2 k^2}$$

$$\Longleftrightarrow E_C > E_2.$$

The eleven tables which follow make up the catalog of "best" designs for given t, k, and bk.

TABLE I

BEST DBIBD'S WITH THREE TREATMENTS, bk EXPERIMENTAL UNITS, AND k EXPERIMENTAL UNITS PER BLOCK

		Associated BIB Design						
k	bk	b	k*	Reps.	Source	n_1	n_2	E
4	12N N>0	3	2	N	Unred.	1	2	.9375
5	15N N>0	3	2	N	Unred.	2	1	.9600
6	6N N>0			N	R.C.B.			1.0000
7	21N N>0	3	2	N	Unred.	2	3	.9796
8	24N N>0	3	2	N	Unred.	3	2	.9844
9	9N N>0			N	R.C.B.			1.0000

TABLE II

BEST DBIBD'S WITH FOUR TREATMENTS, bk EXPERIMENTAL UNITS, AND k EXPERIMENTAL UNITS PER BLOCK

		Associated BIB Designs						
k	bk	b	k*	Reps.	Source	n_1	n_2	E
5	20N N>0	4	3	N	Unred.	1	2	.9600
6	$36N_1+24N_2$ $N_1\geq 0$ $0\leq N_2<3$	6 4	2 3	N_1 N_2	11.1 Unred.	2 2	1 0	$\frac{(13N_1+8N_2)4}{(6N_1+4N_2)9}$
7	28N N>0	4	3	N	Unred.	2	1	.9796
8	8N N>0			N	R.C.B.			1.00
9	36N N>0	4	3	N	Unred.	2	3	.9877
10	$60N_1+40N_2$ $N_1\geq 0$ $0\leq N_2<3$	6 4	2 3	N_1 N_2	11.1 Unred.	3 3	2 1	$\frac{(37N_1+24N_2)4}{(6N_1+4N_2)25}$
11	44N N>0	4	3	N	Unred.	3	2	.9917
12	12N N>0			N	R.C.B.			1.00

TABLE III

BEST DBIBD'S WITH FIVE TREATMENTS, bk EXPERIMENTAL UNITS, AND k EXPERIMENTAL UNITS PER BLOCK

k	bk	b	k*	Reps.	Source	n_1	n_2	E
		Associated BIB Designs						
6	30N N>0	5	4	N	Unred.	1	2	.9722
7	$70N_1+35N_2$ $N_1\geq 0$ $0\leq N_2<2$	10 5	2 4	N_1 N_2	11.2 Unred.	2 1	1 3	$\frac{(19N_1+9N_2)25}{(10N_1+5N_2)49}$
8	$80N_1+40N_2$ $N_1\geq 0$ $0\leq N_2<2$	10 5	2 4	N_1 N_2	11.2 Unred.	1 2	2 0	$\frac{(25N_1+12N_2)25}{(10N_1+5N_2)64}$
9	45N N>0	5	4	N	Unred.	2	1	.9877
10	10N			N	R.C.B.			1.00
11	55N N>0	5	4	N	Unred.	2	3	.9917
12	$120N_1+60N_2$ $N_1\geq 0$ $0\leq N_2<2$	10 5	2 4	N_1 N_2	11.2 Unred.	3 2	2 4	$\frac{(57N_1+28N_2)25}{(10N_1+5N_2)144}$
13	$130N_1+65N_2$ $N_1\geq 0$ $0\leq N_2<2$	10 5	2 4	N_1 N_2	11.2 Unred.	2 3	3 1	$\frac{(67N_1+33N_2)25}{(10N_1+5N_2)169}$
14	70N N>0	5	4	N	Unred.	3	2	.9949
15	15N				R.C.B.			1.00

TABLE IV

BEST DBIBD'S WITH SIX TREATMENTS, bk EXPERIMENTAL UNITS, AND k EXPERIMENTAL UNITS PER BLOCK

		Associated BIB Designs						
k	bk	b	k*	Reps.	Source	n_1	n_2	E
7	42N	6	5	N	Unred.	1	2	.9796
	N>0							
	$120N_1+48N_2$							
8	$N_1\geq 0$	15	2	N_1	11.3	2	1	$\frac{(26N_1+10N_2)9}{(15N_1+6N_2)16}$
	$0\leq N_2<5$	6	5	N_2	Unred.	1	3	
	$90N_1+54N_2$							
9	$N_1\geq 0$	10	3	N_1	11.4	1	2	$\frac{(22N_1+12N_2)4}{(10N_1+6N_2)9}$
	$0\leq N_2<5$	6	5	N_2	Unred.	1	4	
	$150N_1+60N_2$							
10	$N_1\geq 0$	15	2	N_1	11.3	1	2	$\frac{(41N_1+16N_2)9}{(15N_1+6N_2)25}$
	$0\leq N_2<5$	6	5	N_2	Unred.	2	0	
11	66N	6	5	N	Unred.	2	1	.9917
	N>0							
12	12N			N	R.C.B.			1.00
	N>0							
13	78N	6	5	N	Unred.	2	3	.9941
	N>0							
	$210N_1+84N_2$							
14	$N_1\geq 0$	15	2	N_1	11.3	3	2	$\frac{(81N_1+32N_2)9}{(15N_1+6N_2)49}$
	$0\leq N_2<5$	6	5	N_2	Unred.	2	4	
	$150N_1+90N_2$							
15	$N_1\geq 0$	10	3	N_1	11.4	2	3	$\frac{(62N_1+36N_2)4}{(10N_1+6N_2)25}$
	$0\leq N_2<5$	6	5	N_2	Unred.	2	5	
	$240N_1+96N_2$							
16	$N_1\geq 0$	15	2	N_1	11.3	2	3	$\frac{(106N_1+42N_2)9}{(15N_1+6N_2)64}$
	$0\leq N_2<5$	6	5	N_2	Unred.	3	1	

TABLE IV (continued)

Associated BIB Designs

k	bk	b	k*	Reps.	Source	n_1	n_2	E
17	102N N>0	6	5	N	Unred.	3	2	.9965
18	18N			N	R.C.B.			1.00

TABLE V

BEST DBIBD'S WITH SEVEN TREATMENTS, bk EXPERIMENTAL UNITS, AND k EXPERIMENTAL UNITS PER BLOCK

		Associated BIB Designs						
k	bk	b	k*	Reps.	Source	n_1	n_2	E
8	56N, N>0	7	6	N	Unred.	1	2	.9844
	$189N_1+63N_2$							
9	$N_1 \geq 0$	21	2	N_1	11.2a	2	1	$\frac{(34N_1+11N_2)7}{(3N_1+N_2)81}$
	$0 \leq N_2 < 3$	7	6	N_2	Unred.	1	3	
10	70N, N>0	7	3	N	11.7	2	1	.9800
11	77N, N>0	7	3	N	11.7	1	2	.9835
	$252N_1+84N_2$							
12	$N_1 \geq 0$	21	2	N_1	11.2a	1	2	$\frac{(61N_1+20N_2)7}{(3N_1+N_2)144}$
	$0 \leq N_2 < 3$	7	6	N_2	Unred.	2	0	
13	91N, N>0	7	6	N	Unred.	2	1	.9941
14	98N, N>0			N	R.C.B.			1.00
15	105N, N>0	7	6	N	Unred.	2	3	.9956
	$336N_1+112N_2$							
16	$N_1 \geq 0$	21	2	N_1	11.2a	3	2	$\frac{(109N_1+36N_2)7}{(3N_1+N_2)256}$
	$0 \leq N_2 < 3$	7	6	N_2	Unred.	2	4	
17	119N, N>0	7	3	N	11.7	3	2	.9931
18	126N	7	3	N	11.7	2	3	.9938
	$399N_1+133N_2$							
19	$N_1 \geq 0$	21	2	N_1	11.2a	2	3	$\frac{(154N_1+51N_2)7}{(3N_1+N_2)361}$
	$0 \leq N_2 < 3$	7	6	N_2	Unred.	3	1	

TABLE V (continued)

Associated BIB Designs

k	bk	b	k*	Reps.	Source	n_1	n_2	E
20	140N	7	6	N	Unred.	3	2	.9975
21	147N N>0			N	R.C.B.			1.00

TABLE VI

BEST DBIBD'S WITH EIGHT TREATMENTS, bk EXPERIMENTAL UNITS, AND k EXPERIMENTAL UNITS PER BLOCK

		Associated BIB Designs						
k	bk	b	k*	Reps.	Source	n_1	n_2	E
9	72N	8	7	N	Unred.	1	2	.9877
	$280N_1+80N_2$							
10	$N_1 \geq 0$	28	2	N_1	11.9	2	1	$\frac{(43N_1+12N_2)16}{(28N_1+8N_2)25}$
	$0 \leq N_2 < 7$	8	7	N_2	Unred.	1	3	
11	88N	8	7	N	Unred.	1	4	.9256
	$168N_1+96N_2$							
12	$N_1 \geq 0$	14	4	N_1	11.10	1	2	$\frac{(31N_1+16N_2)4}{(14N_1+8N_2)9}$
	$0 \leq N_2 < 7$	8	7	N_2	Unred.	1	5	
13	104N	8	7	N	Unred.	1	6	.8521
	$392N_1+112N_2$							
14	$N_1 \geq 0$	28	2	N_1	11.9	1	2	$\frac{(85N_1+24N_2)16}{(28N_1+8N_2)49}$
	$0 \leq N_2 < 7$	8	7	N_2	Unred.	2	0	
15	120N	8	7	N	Unred.	2	1	.9956
16	16N			N	R.C.B.			1.00
17	136N	8	7	N	Unred.	2	3	.9965
	$504N_1+144N_2$							
18	$N_1 \geq 0$	28	2	N_1	11.9	3	2	$\frac{(141N_1+40N_2)16}{(28N_1+8N_2)81}$
	$0 \leq N_2 < 7$	8	7	N_2	Unred.	2	4	
19	152N	8	7	N	Unred.	2	5	.9751
	$280N_1+160N_2$							
20	$N_1 \geq 0$	14	4	N_1	11.10	2	3	$\frac{(172N_1+48N_2)4}{(14N_1+8N_2)25}$
	$0 \leq N_2 < 7$	8	7	N_2	Unred.	2	6	
21	168N	8	7	N	Unred.	3	0	.9796

TABLE VI (continued)

Associated BIB Designs

k	bk	b	k*	Reps.	Source	n_1	n_2	E
22	$616N_1+176N_2$ $N_1\geq 0$ $0\leq N_2<7$	28 8	2 7	N_1 N_2	11.9 Unred.	2 3	3 1	$\frac{(211N_1+60N_2)16}{(28N_1+8N_2)121}$
23	184N	8	7	N	Unred.	3	2	.9981
24	24N			N	R.C.B.			1.00

TABLE VII

BEST DBIBD'S WITH NINE TREATMENTS, bk EXPERIMENTAL UNITS, AND k EXPERIMENTAL UNITS PER BLOCK

		Associated BIB Designs						
k	bk	b	k*	Reps.	Source	n_1	n_2	E
10	90N	9	8	N	Unred.	1	2	.9900
11	$396N_1+99N_2$ $N_1\geq 0$ $0\leq N_2<4$	36 9	2 8	N_1 N_2	11.3a Unred.	2 1	1 3	$\frac{(53N_1+13N_2)81}{(36N_1+9N_2)121}$
12	$144N_1+108N_2$ $N_1\geq 0$ $0\leq N_2<4$	12 9	6 8	N_1 N_2	11.13 Unred.	1 1	2 4	$\frac{(21N_1+15N_2)9}{(12N_1+9N_2)16}$
13	$234N_1+117N_2$ $N_1\geq 0$ $0\leq N_2<2$	18 9	5 8	N_1 N_2	11.12 Unred.	1 1	2 5	$\frac{(37N_1+17N_2)81}{(18N_1+9N_2)169}$
14	$252N_1+126N_2$ $N_1\geq 0$ $0\leq N_2<2$	18 9	5 8	N_1 N_2	11.12 Unred.	2 1	1 6	$\frac{(43N_1+19N_2)81}{(18N_1+9N_2)196}$
15	$270N_1+180N_2+135N_3$ $N_1\geq 0$ $0\leq N_2<3$ $0\leq N_3<2$	18 12 9	5 6 8	N_1 N_2 N_3	11.12 11.13 Unred.	3 2 1	0 1 7	$\frac{(45N_1+33N_2+21N_3)9}{(18N_1+12N_2+9N_3)25}$
16	144N	9	8	N	Unred.	2	0	.9844
17	153N	9	8	N	Unred.	2	1	.9965
18	18N			N	R.C.B.			1.00
19	171N	9	8	N	Unred.	2	3	.9972
20	180N	9	8	N	Unred.	2	4	.9900
21	$252N_1+189N_2$ $N_1\geq 0$ $0\leq N_2<4$	12 9	6 8	N_1 N_2	11.13 Unred.	2 2	3 5	$\frac{(65N_1+48N_2)9}{(12N_1+9N_2)49}$

TABLE VII (continued)

Associated BIB Designs

k	bk	b	k*	Reps.	Source	n_1	n_2	E
22	$396N_1+198N_2$ $N_1 \geq 0$ $0 \leq N_2 < 2$	18 9	5 8	N_1 N_2	11.12 Unred.	2 2	3 6	$\frac{(107N_1+52N_2)81}{(18N_1+9N_2)484}$
23	$414N_1+207N_2$ $N_1 \geq 0$ $0 \leq N_2 < 2$	18 9	5 8	N_1 N_2	11.12 Unred.	3 2	2 7	$\frac{(117N_1+56N_2)81}{(18N_1+9N_2)529}$
24	$288N_1+216N_2$ $N_1 \geq 0$ $0 \leq N_2 < 4$	12 9	6 8	N_1 N_2	11.13 Unred.	3 3	2 0	$\frac{(85N_1+63N_2)9}{(12N_1+9N_2)64}$
25	$900N_1+225N_2$ $N_1 \geq 0$ $0 \leq N_2 < 4$	36 9	2 8	N_1 N_2	11.3a Unred.	2 3	3 1	$\frac{(277N_1+69N_2)81}{(36N_1+9N_2)625}$
26	234N	9	8	N	Unred.	3	2	.9985
27	27N			N	R.C.B.			1.00

TABLE VIII

BEST DBIBD'S WITH TEN TREATMENTS, bk EXPERIMENTAL UNITS, AND k EXPERIMENTAL UNITS PER BLOCK

Associated BIB Designs

k	bk	b	k*	Reps.	Source	n_1	n_2	E
11	110N	10	9	N	Unred.	1	2	.9917
12	$540N_1+120N_2+180N_3$ $N\geq0$ $0\leq N_2<9$ $0\leq N_3<2$	45 10 15	2 9 6	N_1 N_2 N_3	11.14 Unred. 11.18	2 1 2	1 3 0	$\frac{(64N_1+14N_2+20N_3)25}{(45N_1+16N_2+15N_2)36}$
13	$3\ 0N_1+130N_2$ $N_1\geq0$ $0\leq N_2<3$	30 10	3 9	N_1 N_2	11.15 Unred.	2 1	1 4	$\frac{(50N_1+16N_2)100}{(30N_1+10N_2)169}$
14	$210N_1+140N_2$ $N_1\geq0$ $0\leq N_2<3$	15 10	6 9	N_1 N_2	11.18 Unred.	1 1	2 5	$\frac{(29N_1+18N_2)25}{(15N_1+10N_2)49}$
15	$270N_1+150N_2$ $N_1\geq0$ $0\leq N_2<9$	18 10	5 9	N_1 N_2	11.17 Unred.	2 1	1 6	$\frac{(40N_1+20N_2)4}{(18N_1+10N_2)9}$
16	$240N_1+160N_2$ $N_1\geq0$ $0\leq N_2<3$	15 10	6 9	N_1 N_2	11.18 Unred.	2 1	1 7	$\frac{(38N_1+22N_2)25}{(15N_1+10N_2)64}$
17	$510N_1+170N_2$ $N_1\geq0$ $0\leq N_2<3$	30 10	3 9	N_1 N_2	11.15 Unred.	1 1	2 8	$\frac{(86N_1+24N_2)100}{(30N_1+10N_2)289}$
18	$810N_1+180N_2+270N_3$ $N_1\geq0$ $0\leq N_2<9$ $0\leq N_3<2$	45 10 15	2 9 6	N_1 N_2 N_3	11.14 Unred. 11.18	1 2 1	2 0 3	$\frac{(145N_1+32N_2+47N_3)25}{(45N_1+10N_2+15N_3)81}$
19	190N	10	9	N	Unred.	2	1	.9972
20	200N			N	R.C.B.			1.00
21	210N	10	9	N	Unred.	2	3	.9977

TABLE VIII (continued)

Associated BIB Designs

k	bk	b	k*	Reps.	Source	n_1	n_2	E
	$990N_1+220N_2+330N_3$							
22	$N_1 \geq 0$	45	2	N_1	11.14	3	2	$\frac{(217N_1+48N_2+71N_3)25}{(45N_1+10N_2+15N_3)121}$
	$0 \leq N_2 < 9$	10	9	N_2	Unred.	2	4	
	$0 \leq N_3 < 2$	15	6	N_3	11.18	3	1	
	$690N_1+230N_2$							
23	$N_1 \geq 0$	30	3	N_1	11.15	3	2	$\frac{(158N_1+52N_2)100}{(30N_1+10N_2)529}$
	$0 \leq N_2 < 3$	10	9	N_2	Unred.	2	5	
	$360N_1+240N_2$							
24	$N_1 \geq 0$	15	6	N_1	11.18	2	3	$\frac{(86N_1+56N_2)25}{(15N_1+10N_2)144}$
	$0 \leq N_2 < 3$	10	9	N_2	Unred.	2	6	
	$450N_1+250N_2$							
25	$N_1 \geq 0$	18	5	N_1	11.17	3	2	$\frac{(170N_1+60N_2)4}{(18N_1+10N_2)25}$
	$0 \leq N_2 < 9$	10	9	N_2	Unred.	2	7	
	$390N_1+260N_2$							
26	$N_1 \geq 0$	15	6	N_1	11.18	3	2	$\frac{(101N_1+69N_2)25}{(15N_1+10N_2)169}$
	$0 \leq N_2 < 2$	10	9	N_2	Unred.	2	8	
	$810N_1+270N_2$							
27	$N_1 \geq 0$	30	3	N_1	11.15	2	3	$\frac{(218N_1+72N_2)100}{(30N_1+10N_2)729}$
	$0 \leq N_2 < 3$	10	9	N_2	Unred.	3	0	
	$1260N_1+280N_2+420N_3$							
28	$N_1 \geq 0$	45	2	N_1	11.14	2	3	$\frac{(352N_1+78N_2+116N_3)25}{(45N_1+10N_2+15N_3)196}$
	$0 \leq N_2 < 9$	10	9	N_2	Unred.	2	4	
	$0 \leq N_3 < 3$	15	6	N_3	11.18	3	1	
29	290N	10	9	N	Unred.	3	2	.9988
30	300N			N	R.C.B.			1.00

TABLE IX

BEST DBIBD'S WITH ELEVEN TREATMENTS, bk EXPERIMENTAL UNITS, AND k EXPERIMENTAL UNITS PER BLOCK

		Associated BIB Designs						
k	bk	b	k*	Reps.	Source	n_1	n_2	E
12	132N	11	10	N	Unred.	1	2	.9931
	$715N_1+143N_2$							
13	$N_1\geq 0$	55	2	N_1	11.4a	2	1	$\frac{(76N_1+15N_2)121}{(55N_1+11N_2)169}$
	$0\leq N_2<5$	11	10	N_2	Unred.	1	3	
14	154N	11	10	N	Unred.	1	4	.9541
	$825N_1+165N_2$							
15	$N_1\geq 0$	55	2	N_1	11.4a	3	1	$\frac{(99N_1+19N_2)121}{(55N_1+11N_2)225}$
	$0\leq N_2<5$	11	10	N_2	Unred.	1	5	
16	176N	11	5	N	11.19	2	1	.9883
17	187N	11	5	N	11.19	1	2	.9896
	$990N_1+198N_2$							
18	$N_1\geq 0$	55	2	N_1	11.4a	0	2	$\frac{(144N_1+25N_2)121}{(55N_1+11N_2)324}$
	$0\leq N_2<5$	11	10	N_2	Unred.	1	8	
	$1045N_1+209N_2$							
19	$N_1\geq 0$	55	2	N_1	11.4a	5	1	$\frac{(151N_1+27N_2)121}{(55N_1+11N_2)361}$
	$0\leq N_2<5$	11	10	N_2	Unred.	1	9	
	$1100N_1+220N_2$							
20	$N_1\geq 0$	55	2	N_1	11.4a	1	2	$\frac{(181N_1+36N_2)121}{(55N_1+11N_2)400}$
	$0\leq N_1<5$	11	10	N_2	Unred.	2	0	
21	231N	11	10	N	Unred.	2	1	.9977
22	242N			N	R.C.B.			1.00
23	253N	11	10	N	Unred.	2	3	.9981
	$132N_1+264N_2$							
24	$N_1\geq 0$	55	2	N_1	11.4a	3	2	$\frac{(261N_1+52N_2)121}{(55N_1+11N_2)576}$
	$0\leq N_2<5$	11	10	N_2	Unred.	2	4	

TABLE IX (continued)

Associated BIB Designs

k	bk	b	k*	Reps.	Source	n_1	n_2	E
25	275N	11	10	N	Unred.	2	5	.9856
26	$1430N_1+286N_2$ $N_1\geq 0$ $0\leq N_2<5$	55 11	2 10	N_1 N_2	11.4a Unred.	4 2	2 6	$\frac{(304N_1+60N_2)121}{(55N_1+11N_2)676}$
27	297N	11	5	N	11.19	4	1	.9959
28	308N	11	5	N	11.19	2	3	.9962
29	$1595N_1+319N_2$ $N_1\geq 0$ $0\leq N_2<5$	55 11	2 5	N_1 N_2	11.4a 11.19	1 1	3 4	$\frac{(379N_1+74N_2)121}{(55N_1+11N_2)841}$
30	330N	11	10	N	Unred.	3	0	.9900
31	$1705N_1+341N_2$ $N_1\geq 0$ $0\leq N_2<5$	55 11	2 10	N_1 N_2	11.4a Unred.	2 3	3 1	$\frac{(436N_1+87N_2)121}{(55N_1+11N_2)961}$
32	352N	11	10	N	Unred.	3	2	.9990
33	33N			N	R.C.B.			1.00

TABLE X

BEST DBIBD'S WITH THIRTEEN TREATMENTS, bk EXPERIMENTAL UNITS, AND k EXPERIMENTAL UNITS PER BLOCK

k	bk	b	k*	Reps.	Source	n_1	n_2	E
		Associated BIB Designs						
14	364N	26	6	N	41	0	2	.9286
15	390N	26	3	N	11.21	5	0	.7222
16	$416N_1+208N_2$ $N_1\geq 0$ $0\leq N_2<2$	26 13	3 4	N_1 N_2	11.21 11.22	2 4	1 0	$\frac{(39N_1+16N_2)169}{(26N_1+13N_2)256}$
17	221N	13	4	N	11.22	2	1	.9896
18	234N	13	4	N	11.22	0	2	.9630
19	494N	26	6	N	41	2	1	.9903
20	$520N_1+260N_2$ $N_1\geq 0$ $0\leq N_2<2$	26 13	6 4	N_1 N_2	41 11.22	1 5	2 0	$\frac{(61N_1+25N_2)169}{(26N_1+13N_2)400}$
21	273N	13	4	N	41	3	1	.9728
22	286N	13	4	N	41	1	2	.9938
23	598N	26	3	N	11.21	1	2	.9953
24	$624N_1+312N_2$ $N_1\geq 0$ $0\leq N_2<2$	26 13	6 4	N_1 N_2	41 11.22	4 6	0 0	$\frac{(80N_1+36N_2)169}{(26N_1+13N_2)576}$
25	$650N_1+325N_2$ $N_1\geq 0$ $0\leq N_2<2$	26 13	6 4	N_1 N_2	41 11.22	3 4	1 1	$\frac{(94N_1+46N_2)169}{(26N_1+13N_2)625}$
26	26N			N	R.C.B.			1.00

TABLE X (continued)

Associated BIB Designs

k	bk	b	k*	Reps.	Source	n_1	n_2	E
27	$702N_1+351N_2$ $N_1\geq 0$ $0\leq N_2<2$	26 13	6 4	N_1 N_2	41 11.22	1 0	3 3	$\frac{(110N_1+54N_2)169}{(26N_1+13N_2)729}$
28	$728N_1+364N_2$ $N_1\geq 0$ $0\leq N_2<2$	26 13	6 4	N_1 N_2	41 11.22	0 7	4 0	$\frac{(112N_1+49N_2)169}{(26N_1+13N_2)784}$
29	$754N_1+377N_2$ $N_1\geq 0$ $0\leq N_2<2$	26 13	3 4	N_1 N_2	11.21 11.22	3 5	2 1	$\frac{(129N_1+61N_2)169}{(26N_1+13N_2)841}$
30	390N	13	4	N	11.22	3	2	.9967
31	403N	13	4	N	11.22	1	3	.9875
32	$832N_1+416N_2$ $N_1\geq 0$ $0\leq N_2<2$	26 13	6 4	N_1 N_2	41 11.22	3 8	2 0	$\frac{(157N_1+64N_2)169}{(26N_1+13N_2)1024}$
33	$858N_1+429N_2$ $N_1\geq 0$ $0\leq N_2<2$	26 13	6 4	N_1 N_2	41 11.22	2 6	3 1	$\frac{(167N_1+78N_2)169}{(26N_1+13N_2)1089}$
34	442N	13	4	N	11.22	4	2	.9896
35	455N	13	4	N	11.22	2	3	.9976
36	$936N_1+468N_2$ $N_1\geq 0$ $0\leq N_2<2$	26 13	3 4	N_1 N_2	11.21 11.22	2 0	3 4	$\frac{(199N_1+96N_2)169}{(26N_1+13N_2)1296}$
37	$962N_1+481N_2$ $N_1\geq 0$ $0\leq N_2<2$	26 13	6 4	N_1 N_2	41 11.22	5 7	1 1	$\frac{(202N_1+97N_2)169}{(26N_1+13N_2)1369}$
38	$988N_1+494N_2$ $N_1\geq 0$ $0\leq N_2<2$	26 13	6 4	N_1 N_2	41 11.22	4 5	2 2	$\frac{(220N_1+109N_2)169}{(26N_1+13N_2)1444}$
39	39N				R.C.B.			1.00

TABLE XI

BEST DBIBD'S WITH FIFTEEN TREATMENTS, bk EXPERIMENTAL UNITS, AND k EXPERIMENTAL UNITS PER BLOCK

		Associated BIB Designs						
k	bk	b	k*	Reps.	Source	n_1	n_2	E
16	240N $N>0$	15	7	N	11.25	0	2	.9375
18	630N $N>0$	35	3	N	11.24	2	1	.9921
21	$735N_1+315N_2$ $N_1 \geq 0$ $0 \leq N_2 < 7$	35 15	6 7	N_1 N_2	62 11.25	2 3	1 0	$\frac{(68N_1+27N_2)225}{(35N_1+15N_2)441}$
22	330N $N>0$	15	7	N	11.25	2	1	.9917
23	345N $N>0$	15	7	N	11.25	1	2	.9924
24	$840N_1+360N_2$ $N_1 \geq 0$ $0 \leq N_2 < 7$	35 15	6 7	N_1 N_2	62 11.25	1 0	2 3	$\frac{(89N_1+36N_2)225}{(35N_1+15N_2)576}$
27	945N $N>0$	35	3	N	11.24	1	2	.9965
28	420N $N>0$	15	7	N	11.25	4	0	.9184
29	435N $N>0$	15	7	N	11.25	3	1	.9810
30	30N $N>0$			N	R.C.B.			1.00
31	465N $N>0$	15	7	N	11.25	1	3	.9834
32	480N $N>0$	15	7	N	11.25	0	4	.9375

TABLE XI (continued)

Associated BIB Designs

k	bk	b	k*	Reps.	Source	n_1	n_2	E
33	1155N N>0	35	3	N	11.24	3	2	.9976
35	525N N>0	15	7	N	11.25	5	0	.9184
36	$1260N_1+540N_2$ $N_1\geq 0$ $0\leq N_2<7$	35 15	6 7	N_1 N_2	62 11.25	3 4	2 1	$\frac{(201N_1+84N_2)225}{(35N_1+15N_2)1296}$
37	555N N>0	15	7	N	11.25	3	2	.9971
38	570N N>0	15	7	N	11.25	2	3	.9972
39	$1365N_1+585N_2$ $N_1\geq 0$ $0\leq N_2<7$	35 15	6 7	N_1 N_2	62 11.25	2 1	3 4	$\frac{(236N_1+99N_2)225}{(35N_1+15N_2)1521}$
40	600N N>0	15	7	N	11.25	0	5	.9375
42	$1470N_1+630N_2$ $N_1\geq 0$ $0\leq N_2<7$	35 15	3 7	N_1 N_2	11.24 11.25	2 4	3 2	$\frac{(274N_1+108N_2)225}{(35N_1+15N_2)1764}$
43	645N N>0	15	7	N	11.25	5	1	.9618
44	660N N>0	15	7	N	11.25	4	2	.9945
45	45N			N	R.C.B.			1.00

CHAPTER III

THE RECOVERY OF INTERBLOCK INFORMATION

In the preceding chapter, expressions for estimates of treatment effects in DBIBD's were obtained which were dependent on comparisons within blocks. The present chapter, using an assumption of random block effects, is a development of an interblock analysis. This analysis uses only block totals in the estimation of treatment effects. The problem of combining the intrablock and interblock estimates is also considered.

The Interblock Analysis

Suppose the model for a DBIBD to be as follows:

$$y_{ijk} = \mu + \tau_i + \beta_j + e_{ijk}; \qquad \begin{array}{l} i = 1, \ldots, t \\ j = 1, \ldots, b \\ k = n_1, n_2 \\ \beta_j \sim N(0,\sigma_1^2);\ E(\beta_j\beta_{j^*}) = 0,\ j \neq j^* \\ e_{ijk} \sim N(0,\sigma^2); \\ \quad E(e_{ijk}\cdot e_{i^*j^*k^*}) = \begin{cases} \sigma^2; & i=i^*,\ j=j^*,\ k=k^* \\ 0; & \text{otherwise} \end{cases} \\ E(\beta_j e_{ij^*k}) = 0,\ \forall i,j,j^*,k \end{array}$$

This may again be written in matrix form as: $Y = \mu j + X_1\tau + X_2\beta + e$. Then $X_2'Y$, the vector of block totals, may be simplified as follows:

$$X_2'Y = X_2'(\mu j + X_1\tau + X_2\beta + e) = \mu X_2'j + X_2'X_1\tau + X_2'X_2\beta + X_2'e$$
$$= \mu kj + C'\tau + k\beta + X_2'e.$$

It can be seen then that $E(X_2'Y) = \mu k j_1^b + C'\tau$. Also, Var $(X_2'Y)$

$$= E[(k\beta + X_2'e)\ (k\beta + X_2'e)']$$

$$= E(k^2\beta\beta' + X_2'ee'X_2)$$

$$= k^2\sigma_1^2 I + X_2'X_2\sigma^2 I$$

$$= k(k\sigma_1^2 + \sigma^2)I.$$

The model for block totals, $X_2'Y = \mu k j_1^b + C'\tau + (k\beta + X_2'e)$, may be written as $Z = \mu k j_1^b + C'\tau + \delta$. Then the normal equations for this model would be:

$$\begin{pmatrix} k^2 b & k j_b^1 C' \\ k C j_1^b & CC' \end{pmatrix} \begin{pmatrix} \tilde{\mu} \\ \tilde{\tau} \end{pmatrix} = \begin{pmatrix} k j_b^1 Z \\ CZ \end{pmatrix} = \begin{pmatrix} k j_b^1 X_2'Y \\ CX_2'Y \end{pmatrix}.$$

These may be simplified to:

$$\begin{pmatrix} k^2 b & k r j_t^1 \\ k r j_1^t & (rk - \lambda t)I_t + \lambda J_t^t \end{pmatrix} \begin{pmatrix} \tilde{\mu} \\ \tilde{\tau} \end{pmatrix} = \begin{pmatrix} k y_{\cdot\cdot\cdot} \\ CX_2'Y \end{pmatrix}.$$

Summing the last t rows of the above coefficient matrix, we have:

$$j_t^1[k r j_1^t,\ (rk - \lambda t)I_t + \lambda J_t^t]$$

$$= [rkt,\ (rk - \lambda t)j_t^1 + \lambda t j_t^1].$$

Now, since $rt = bk$, we have $rkt = k^2 b$ and also $(rk - \lambda t)j_t^1 + \lambda t j_t^1 = k r j_t^1$. Hence, there exists a dependency and the rank of the system of normal equations is seen to be at most t.

The solution of the system may be accomplished by addition of the restriction $j_t^1 \tilde{\tau} = 0$. Under this restriction, the normal equations become:

$$\begin{pmatrix} k^2 b & \phi \\ krj_1^t & (rk - \lambda t)I \end{pmatrix} \begin{pmatrix} \tilde{\mu} \\ \tilde{\tau} \end{pmatrix} = \begin{pmatrix} ky\ldots \\ CX_2'Y \end{pmatrix}.$$

Hence, $\tilde{\mu} = \frac{y\ldots}{bk} = \frac{1}{bk} j_b^1 X_2'Y$

and $\tilde{\tau} = \frac{1}{rk - \lambda t} (CX_2'Y - krj_1^t \tilde{\mu})$

$$= \frac{1}{rk - \lambda t} (C - krj_1^t \cdot \frac{1}{bk} j_b^1) X_2'Y$$

$$= \frac{1}{rk - \lambda t} (C - \frac{1}{t} J_t^t C) X_2'Y$$

$$= \frac{1}{rk - \lambda t} (I - \frac{1}{t} J_t^t) CX_2'Y.$$

It can be shown from the above considerations that, as in the case of balanced incomplete block designs, the following important relationships hold:

1. $E\tilde{\tau} = \tau - \bar{\tau}.j_1^t$
2. $\text{Var}\,\tilde{\tau} = \frac{k(\sigma^2 + k\sigma_1^2)}{rk - \lambda t} (I - \frac{1}{t} J)$
3. $\hat{\tau}$ and $\tilde{\tau}$ are uncorrelated.

The Combining of Intrablock and Interblock Estimators

Where $X_1 = (X_{11}, X_{12}, \ldots, X_{1t})$ and $C = \begin{pmatrix} C_1 \\ \vdots \\ C_t \end{pmatrix}$, we have

$\hat{\tau}_i = \frac{k}{\lambda t} (X_{1i}'Y - \frac{1}{k} C_i X_2'Y)$ with $\text{Var}\,\hat{\tau}_i = \frac{k}{\lambda t} (1 - 1/t)\sigma^2$ and $\tilde{\tau}_i = \frac{1}{rk - \lambda t} (1 - 1/t) C_i X_2'Y$ with $\text{Var}\,\tilde{\tau}_i = \frac{k(\sigma^2 + k\sigma_1^2)}{rk - \lambda t} (1 - \frac{1}{t})$ and $\hat{\tau}_i$ and $\tilde{\tau}_i$ are uncorrelated.

By Theorem 18.11 of Graybill (4), the best linear unbiased estimators of $(\tau_i - \bar{\tau}.)$, if the variances are known, is given by:

$$\check{\tau}_i = \frac{\hat{\tau}_i[k(\sigma^2 + k\sigma_1^2)/rk - \lambda t] + \tilde{\tau}_i[k\sigma^2/\lambda t]}{k\sigma^2/\lambda t + [k(\sigma^2 + k\sigma_1^2)/rk - \lambda t]}.$$

Since in most cases the variances appearing in the above expression are unknown, it cannot usually be applied directly.

A very similar situation arises in the combining of intrablock and interblock estimators from BIB designs. If $\hat{\tau}_i*$ and $\tilde{\tau}_i*$ are intrablock and interblock estimators, respectively, in a BIB design, then the best linear unbiased estimator of $\tau_i - \bar{\tau}.$ is given by

$$\check{\tau}_i* = \frac{\hat{\tau}_i*[k^*(\sigma^{*2} + k^*\sigma_1^{*2})/r^* - \lambda^*] + \tilde{\tau}_i*[k^*\sigma^{*2}/\lambda^* t]}{[k^*(\sigma^{*2} + k^*\sigma_1^{*2})/r^* - \lambda^*] + k^*\sigma^{*2}/\lambda^* t}$$

where σ^{*2} is the error variance and σ_1^{*2} is the block variance. As in the DBIBD, σ^{*2} and σ_1^{*2} are most often unknown and the usual procedure has been to consider methods of estimating these variances, or expressions involving them, and thus obtain a random weighting of the intrablock and interblock estimators that has desirable properties.

The similarity of the problem of obtaining a useable combined estimator for a DBIBD to that problem for a BIBD suggests that the approach to the current problem be by way of applying the results and procedures of some investigator, concerned with combining unbiased estimators, who has made particular reference to BIBD's.

In a recent paper, Seshadri (11), concerned with the recovery of interblock information in a BIBD, develops random weights for combining intrablock and interblock estimates which give rise to an estimator uniformly better than either of these in nearly all experiments. Although there have been important earlier papers by Yates (13) and Graybill and Deal (5), this more recent paper, apparently, holds greater promise in terms of range of applicability. The following development of a combined estimator for a DBIBD is along lines suggested by that study.

Consider the following simplification of the expression for the combined estimator in the case where the variances are known:

$$\check{\tau}_i = \frac{\hat{\tau}_i[k(\sigma^2 + k\sigma_1^2)/rk - \lambda t] + \tilde{\tau}_i[k\sigma^2/\lambda t]}{[k\sigma^2/\lambda t] + [k(\sigma^2 + k\sigma_1^2)/rk - \lambda t]}$$

$$= \frac{\hat{\tau}_i[\lambda tk(\sigma^2 + k\sigma_1^2)] + \tilde{\tau}_i[k(rk - \lambda t)\sigma^2]}{(rk - \lambda t)k\sigma^2 + \lambda tk(\sigma^2 + k\sigma_1^2)}$$

$$= \hat{\tau}_i\left[1 - \frac{(rk - \lambda t)k\sigma^2}{(rk - \lambda t)k\sigma^2 + \lambda tk(\sigma^2 + k\sigma_1^2)}\right]$$

$$+ \tilde{\tau}_i \frac{(rk - \lambda t)k\sigma^2}{(rk - \lambda t)k\sigma^2 + \lambda tk(\sigma^2 + k\sigma_1^2)}$$

$$= \hat{\tau}_i + (\tilde{\tau}_i - \hat{\tau}_i)\ \frac{(rk - \lambda t)k\sigma^2}{(rk - \lambda t)k\sigma^2 + \lambda tk(\sigma^2 + k\sigma_1^2)}$$

$$= \hat{\tau}_i + (\tilde{\tau}_i - \hat{\tau}_i)\ \frac{(rk - \lambda t)\sigma^2}{k(r\sigma^2 + \lambda k\sigma_1^2)}.$$

It will now be shown that the use of an unbiased estimate of $\frac{(rk - \lambda t)\sigma^2}{k(r\sigma^2 + \lambda k\sigma_1^2)}$ in the above expression will lead to an unbiased estimate of $\tau_i - \bar{\tau}.$ which is uniformly better than $\hat{\tau}_i$ or $\tilde{\tau}_i$.

We have that $\hat{\tau} \sim N[\tau - j_t^1 \bar{\tau}., \frac{k}{\lambda t} (I - 1/t\ J)\sigma^2]$, $\tilde{\tau} \sim N[\tau - j_t^1 \bar{\tau}., \frac{k}{rk - \lambda t} (I - 1/t\ J)\ (\sigma^2 + k\sigma_1^2)]$, and that these are independent. Therefore,

$$(\tilde{\tau} - \hat{\tau}) \sim N\left\{\phi, \left[\frac{k}{\lambda t}\sigma^2 + \frac{k}{rk - \lambda t}(\sigma^2 + k\sigma_1^2)\right](I - 1/t\ J)\right.$$

$$\left. = \frac{k^2(r\sigma^2 + \lambda t\sigma_1^2)}{\lambda t(rk - \lambda t)}(I - 1/t\ J)\right\}$$

Also, $\frac{\lambda t(rk - \lambda t)}{k^2(r\sigma^2 + \lambda t\sigma_1^2)} \cdot \frac{k^2(r\sigma^2 + \lambda t\sigma_1^2)}{\lambda t(rk - \lambda t)} (I - 1/t\ J) = (I - 1/t\ J)$

is independent and of rank t - 1. So by Theorem 4.8 of Graybill (4),

$$\frac{\lambda t(rk - \lambda t)}{k^2(r\sigma^2 + \lambda t\sigma_1^2)} \sum_{i=1}^{t} (\tilde{\tau}_i - \hat{\tau}_i)^2 = (\tilde{\tau} - \hat{\tau})' \left[\frac{\lambda t(rk - \lambda t)}{k^2(r\sigma^2 + \lambda t\sigma_1^2)} I \right] (\tilde{\tau} - \hat{\tau})$$

is distributed as $\chi^2(t - 1)$.

Further, where s^2 is the intrablock error sum of squares and f is the degrees of freedom for intrablock error, s^2/σ^2 is distributed as $\chi^2(f)$ and is independent of $\frac{\lambda t(rk - \lambda t)}{k^2(r\sigma^2 + \lambda t\sigma_1^2)} \sum_{i=1}^{t} (\tilde{\tau}_i - \hat{\tau}_i)^2$.

Now the ratio of these chi-square variables divided by their respective degrees of freedom will be distributed as F(f,t - 1).

So $E\left[\frac{s^2 k^2(r\sigma^2 + \lambda t\sigma_1^2)\ (t - 1)}{\sigma^2 \sum_{i=1}^{t} (\tilde{\tau}_i - \hat{\tau}_i)^2 f\lambda t(rk - \lambda t)} \right] = \frac{t - 1}{t - 3}$

and $E\left[\frac{s^2 k(t - 3)}{f\lambda t \sum_{i=1}^{t} (\tilde{\tau}_i - \hat{\tau}_i)^2} \right] = \frac{(rk - \lambda t)\sigma^2}{k(r\sigma^2 + \lambda t\sigma_1^2)}$. Therefore, we obtain the estimate, $\check{\tau}_i = \hat{\tau}_i + (\tilde{\tau}_i - \hat{\tau}_i) \frac{s^2 k(t - 3)}{f\lambda t \sum_{i=1}^{t} (\tilde{\tau}_i - \hat{\tau}_i)^2}$.

Now $E\,\check{\tau}_i = E\,\hat{\tau}_i + k(t - 3)\sigma^2\, E\left[\frac{\tilde{\tau}_i - \hat{\tau}_i}{\sum_{i=1}^{t} (\tilde{\tau}_i - \hat{\tau}_i)^2} \right]$ and $(\tilde{\tau}_i - \hat{\tau}_i)$ is a normal variate with mean zero and variance $k^2(r\sigma^2 + \lambda t\sigma_1^2)/\lambda t(rk - \lambda t)$.

Since $\frac{\tilde{\tau}_i - \hat{\tau}_i}{\sum_{i=1}^{t} (\tilde{\tau}_i - \hat{\tau}_i)^2}$ is an odd function of $(\tilde{\tau}_i - \hat{\tau}_i)$ and

$-\infty < \frac{\tilde{\tau}_i - \hat{\tau}_i}{\sum_{i=1}^{t} (\tilde{\tau}_i - \hat{\tau}_i)^2} < \infty$, $E\left[\frac{\tilde{\tau}_i - \hat{\tau}_i}{\sum_{i=1}^{t} (\tilde{\tau}_i - \hat{\tau}_i)^2} \right] = 0$ and $\check{\tau}_i$ is unbiased.

Also $\text{Var}\,\check{\tau}_i = E(\check{\tau}_i - E\,\check{\tau}_i)^2$

$$= E[(\hat{\tau}_i - E\,\hat{\tau}_i) + k(t-3)s^2(\tilde{\tau}_i - \hat{\tau}_i)/f\lambda t \sum_{i=1}^{t}(\tilde{\tau}_i - \hat{\tau}_i)^2]^2$$

$$= E(\hat{\tau}_i - E\,\hat{\tau}_i)^2 + [2k(t-3)/f\lambda t]\; E[s^2(\tilde{\tau}_i - \hat{\tau}_i)/\sum_{i=1}^{t}(\tilde{\tau}_i - \hat{\tau}_i)^2]$$

$$+ [k(t-3)/f\lambda t]^2\; E[s^2(\tilde{\tau}_i - \hat{\tau}_i)/\sum_{i=1}^{t}(\tilde{\tau} - \hat{\tau})^2].$$

The first of these terms is immediately $k\sigma^2/\lambda t$; the second and third terms may be evaluated by using the same transformation as that employed by Seshadri (11) and are

$-[2k(t-3)/f\lambda t]\,[f(rk - \lambda t)\,(\sigma^2)^2/k(t-1)\,(r\sigma^2 + \lambda t\sigma_1^{\,2})]$ and $[k(t-3)/f\lambda t]^2\,[f(f+2)\,(\sigma^2)^2/t-1]$ $[\lambda t(rk - \lambda t)/k^2(r\sigma^2 + \lambda t\sigma_1^{\,2})\,(t-3)]$, respectively. So $\text{Var}\,\check{\tau}_i$ is given by

$$\frac{k\sigma^2}{\lambda t} - \frac{2(t-3)\,(rk-\lambda t)\,(\sigma^2)^2}{\lambda t(t-1)\,(r\sigma^2 + \lambda t\sigma_1^{\,2})} + \frac{(t-3)\,(f+2)\,(rk-\lambda t)\,(\sigma^2)^2}{\lambda t(t-1)\,f\,(r\sigma^2 + \lambda t\sigma_1^{\,2})}$$

and $\text{Var}\,\check{\tau}_i - \text{Var}\,\hat{\tau}_i = -\dfrac{(t-3)\,(rk-\lambda t)\,(\sigma^2)^2\,(f-2)}{\lambda t(t-1)\,f\,(r\sigma^2 + \lambda t\sigma_1^{\,2})}$

Consider $rk - \lambda t$. By Theorem 2.2,

$$rk - \lambda t = r^*n_1^{\,2} + (b - r^*)n_2^{\,2} - [\lambda^*(n_1 - n_2)^2 + n_2(2r - bn_2)]$$

$$= r^*n_1^{\,2} + (b - r^*)n_2^{\,2}$$

$$- [\lambda^*n_1^{\,2} + \lambda^*n_2^{\,2} - 2\lambda^*n_1n_2 + bn_2^{\,2} + 2n_1n_2r^* - 2n_2^{\,2}r^*]$$

$$= (r^* - \lambda^*)n_1^{\,2} + [b - r^* - \lambda^* - b + 2r^*]n_2^{\,2} - 2(r^* - \lambda^*)n_1n_2$$

$$= (r^* - \lambda^*)\,(n_1 - n_2)^2 \geq 0.$$

Thus, where $n_1 \neq n_2$, $t > 3$, and $f > 2$, the variance of the combined estimator is seen to be exceeded by the variance of the intrablock estimator.

Therefore, for those designs appearing in the catalog of Chapter II, excluding those in Table I and those which reduce to randomized complete blocks, the combined estimator should be used. There exist no exceptions other than those noted, since $b(k - t) - \eta > 2$ in every case.

CHAPTER IV

SUMMARY AND EXTENSIONS

The concern of this thesis is with those experimental situations characterized by experimental units of high cost which group naturally into constant sized blocks such that costs are principally associated with blocks as a whole and such that block size exceeds the number of treatments. The intent is to provide easily applied methods of analysis for such situations and to present information on their effectiveness.

In Chapter II, Displaced Balanced Incomplete Block Designs are defined to be those connected two-way classification designs which are such that each treatment is applied either n_1 or n_2 times in a block and with the additional property that replacement of n_1 by unity and n_2 by zero would result in a Balanced Incomplete Block Design. A set of constants associated with any such design is identified and identities relating these are found and used to derive expressions for intrablock estimates of treatment effects and their variances.

Recognition is accorded the fact that in an experimental situation to which a DBIBD is applicable more than one such design might fit. Considerations of the criteria for and identification of "best" designs for intrablock analysis in particular situations is therefore included. Those DBIBD's which are "best" for given numbers of treatments, experimental units per block, and experimental units in the experiment are presented in tabular form. These tables are for all possible values of t (number of treatments) between three and fifteen and for block sizes from t + 1 to 3t.

The assumption of random block effects makes possible the development, in Chapter III, of an analysis of DBIBD's based on block totals. A procedure for combining these interblock estimators with those from the intrablock analysis is developed and its applicability discussed.

Several directions for further study and extension have been suggested by the investigations reported here. Concern with experimental situations in which the balance condition could be replaced by one of partial balance might give rise to a useful research, paralleling this one, on what might be called, "Displaced Partially Balanced Incomplete Block Designs." Comparison of the properties of such designs with those proposed by Pearce (9) and Federer (2) would be interesting.

Also of possible interest would be the development of more general methods for generating and analyzing balanced designs in which block size exceeds the number of treatments. Such a development might provide useful means for the analysis of those experiments which fail the condition of becoming BIBD's with the replacement of n_1 and n_2 by unity and zero, respectively.

Another possible source of fruitful study might be the investigation of special properties possessed by DBIBD's which have been generated by special kinds of BIBD's, for example, properties peculiar to displaced designs generated from resolvable BIBD's.

The investigation of special methods for combining intrablock and interblock estimators in designs characterized by extra large block size could also be a worthy area of inquiry.

A potential application of the work done here to experiments using a factorial arrangement of treatments might be profitably explored. That is, in a two-factor experiment with one factor at t levels and the second factor at b levels, the number of experimental units available for experimentation might be such that if all were used there would result unequal replications of the treatment combinations. For example, suppose a factor A to be at four levels and a factor B to be at six levels and further suppose the availability of sixty experimental units. Rather than replicate each treatment combination twice and use only forty-eight experimental units, an experimenter might achieve maximum usage of his resources and retain some kind of balance by repeating some treatment combinations twice and others three times in the manner suggested by the following layout:

		A			
		0	1	2	3
	0	3	3	2	2
	1	3	2	3	2
B	2	3	2	2	3
	3	2	3	3	2
	4	2	3	2	3
	5	2	2	3	3

An intended similarity of the above with the layout exemplifying a DBIBD which appears early in Chapter II should be noted. The estimation of treatment effects and interactions in such a factorial arrangement might also closely resemble the results obtained in this paper and some additional value of the tables of "best" designs might be recognized.

SELECTED BIBLIOGRAPHY

(1) Cochran, W. G., and Cox, G. Experimental Designs. New York: John Wiley and Sons, 1957.

(2) Federer, W. T. "Augmented Designs With One-Way Elimination of Homogeneity," Biometrics, 17(1961), 447-472.

(3) Fisher, R. A., and Yates, F. Statistical Tables for Biological, Agricultural, and Medical Research. New York: Hafner Publishing Company, Inc., 1963.

(4) Graybill, F. A. Introduction to Linear Statistical Models, Vol. I. New York: McGraw-Hill Book Company, Inc., 1961.

(5) Graybill, F. A., and Deal, R. B. "Combining Unbiased Estimators," Biometrics, 15(1959), 543-550.

(6) John, P. W. M. "Testing Two Treatments When There Are Three Experimental Units In Each Block," Applied Statistics, 11(1962), 164-169.

(7) John, P. W. M. "Extended Complete Block Designs," Australian Journal of Statistics, 5(1963), 147-152.

(8) Kempthorne, O. The Design and Analysis of Experiments. New York: John Wiley and Sons, 1952.

(9) Pearce, S. C. "Supplemented Balance," Biometrika, 47(1960), 263-271.

(10) Pearce, S. C. "Experimenting With Blocks of Natural Size," Biometrics, 20(1964), 699-706.

(11) Seshadri, V. "Combining Unbiased Estimators," Biometrics, 19(1963), 163-170.

(12) Seshadri, V. "Constructing Uniformly Better Estimators," Journal of the American Statistical Association, 58(1963), 172-175.

(13) Yates, F. "The Recovery of Inter-Block Information in Balanced Incomplete Block Designs," Annals of Eugenics, 10(1940), 317-325.

APPENDIX

TABLE XII

BEST DBIBD'S WITH THREE TREATMENTS FOR GIVEN NUMBERS OF EXPERIMENTAL UNITS PER BLOCK AND GIVEN GENERATING BIB DESIGNS

BIB Design

Unreduced
($b=3$, $k^*=2$, $r^*=2$)

k	(n_1, n_2)	bk	λ	E
4	(1,2)	12	5	.9375
5	(2,1)	15	8	.9600
6	(2,2)	18	12	1.0000
7	(2,3)	21	16	.9796
8	(3,2)	24	21	.9844
9	(3,3)	27	27	1.0000

TABLE XIII

BEST DBIBD'S WITH FOUR TREATMENTS FOR GIVEN NUMBERS OF EXPERIMENTAL UNITS PER BLOCK AND GIVEN GENERATING BIB DESIGNS

	BIB Designs							
	11.1 ($b=6, k^*=2, r^*=3$)				Unreduced ($b=4, k^*=3, r^*=3$)			
k	(n_1, n_2)	bk	λ	E	(n_1, n_2)	bk	λ	E
5					(1,2)	20	6	.9600
6	(2,1)	36	13	.9630	(2,0)	24	8	.8889
7					(2,1)	28	12	.9796
8	(2,2)	48	24	1.0000	(2,2)	32	16	1.0000
9					(2,3)	36	20	.9877
10	(3,2)	60	37	.9867	(3,1)	40	24	.9600
11					(3,2)	44	30	.9917
12	(3,3)	72	54	1.0000	(3,3)	48	36	1.0000

TABLE XIV

BEST DBIBD'S WITH FIVE TREATMENTS FOR GIVEN NUMBERS OF EXPERIMENTAL UNITS PER BLOCK AND GIVEN GENERATING BIB DESIGNS

	BIB Designs							
	11.2 ($b=10, k^*=2, r^*=4$)				Unreduced ($b=5, k^*=4, r^*=4$)			
k	(n_1, n_2)	bk	λ	E	(n_1, n_2)	bk	λ	E
6	(0,2)	60	12	.8333	(1,2)	30	7	.9722
7	(2,1)	70	19	.9694	(1,3)	35	9	.9184
8	(1,2)	80	25	.9766	(2,0)	40	12	.9375
9	(3,1)	90	30	.9259	(2,1)	45	16	.9877
10	(2,2)	100	40	1.0000	(2,2)	50	20	1.0000
11	(1,3)	110	46	.9504	(2,3)	55	24	.9917
12	(3,2)	120	57	.9896	(2,4)	60	28	.9722
13	(2,3)	130	67	.9911	(3,1)	65	33	.9763
14	(4,2)	140	76	.9694	(3,2)	70	39	.9949
15	(3,3)	150	90	1.0000	(3,3)	75	45	1.0000

TABLE XV

BEST DBIBD'S WITH SIX TREATMENTS FOR GIVEN NUMBERS OF EXPERIMENTAL UNITS PER BLOCK AND GIVEN GENERATING BIB DESIGNS

	BIB Designs											
	11.3 (b=15, k^*=2, r^*=5)				11.4 (b=10, k^*=3, r^*=5)				Unreduced (b=6, k^*=5, r^*=5)			
k	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E
7									(1,2)	42	8	.9796
8	(2,1)	120	26	.9750					(1,3)	48	10	.9375
9					(1,2)	90	22	.9778	(1,4)	54	12	.8889
10	(1,2)	150	41	.9840					(2,0)	60	16	.9600
11									(2,1)	66	20	.9917
12	(2,2)	180	60	1.0000	(2,2)	120	40	1.0000	(2,2)	72	24	1.0000
13									(2,3)	78	28	.9941
14	(3,2)	210	81	.9918					(2,4)	84	32	.9796
15					(2,3)	150	62	.9920	(2,5)	90	36	.9600
16	(2,3)	240	106	.9938					(3,1)	96	42	.9844
17									(3,2)	102	48	.9965
18	(3,3)	270	135	1.0000	(3,3)	180	90	1.0000	(3,3)	108	54	1.0000

TABLE XVI

BEST DBIBD'S WITH SEVEN TREATMENTS FOR GIVEN NUMBERS OF EXPERIMENTAL UNITS PER BLOCK AND GIVEN GENERATING BIB DESIGNS

	BIB Designs											
	11.7 (b=7,k*=3,r*=3)				Unreduced (b=7,k*=6,r*=6)				11.2a (b=21,k*=2,r*=6)			
k	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E
8	(0,2)	56	8	.8750	(1,2)	56	9	.9844	(4,0)	168	16	.5833
9	(3,0)	63	9	.7778	(1,3)	63	11	.9506	(2,1)	189	34	.9794
10	(2,1)	70	14	.9800	(1,4)	70	13	.9100	(0,2)	210	40	.9333
11	(1,2)	77	17	.9835	(1,5)	77	15	.8678	(3,1)	231	49	.9449
12	(0,3)	84	18	.8750	(2,0)	84	20	.9722	(1,2)	252	61	.9884
13	(3,1)	91	23	.9527	(2,1)	91	24	.9941	(4,1)	273	66	.9112
14	(2,2)	98	28	1.0000	(2,2)	98	28	1.0000	(2,2)	294	84	1.0000
15	(1,3)	105	31	.9644	(2,3)	105	32	.9956	(0,3)	315	90	.9333
16	(4,1)	112	34	.9297	(2,4)	112	36	.9844	(3,2)	336	109	.9935
17	(3,2)	119	41	.9931	(2,5)	119	40	.9689	(1,3)	357	121	.9769
18	(2,3)	126	46	.9938	(3,0)	126	45	.9722	(4,2)	378	136	.9794

TABLE XVI (continued)

	BIB Designs											
	11.7 (b=7,k*=3,r*=3)				Unreduced (b=7,k*=6,r*=6)				11.2a (b=21,k*=2,r*=6)			
k	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E
19	(1,4)	133	49	.9501	(3,1)	133	51	.9889	(2,3)	399	154	.9954
20	(4,2)	140	56	.9800	(3,2)	140	57	.9975	(5,2)	420	165	.9625
21	(3,3)	147	63	1.0000	(3,3)	147	63	1.0000	(3,3)	441	189	1.0000

TABLE XVII

BEST DBIBD'S WITH EIGHT TREATMENTS FOR GIVEN NUMBERS OF EXPERIMENTAL UNITS PER BLOCK AND GIVEN GENERATING BIB DESIGNS

	BIB Designs											
	11.9 ($b=28, k^*=2, r^*=7$)				11.10 ($b=14, k^*=4, r^*=7$)				Unreduced ($b=8, k^*=7, r^*=7$)			
k	(n_1, n_2)	bk	λ	E	(n_1, n_2)	bk	λ	E	(n_1, n_2)	bk	λ	E
9									(1,2)	72	10	.9877
10	(2,1)	280	43	.9829					(1,3)	80	12	.9600
11									(1,4)	88	14	.9256
12	(3,1)	336	60	.9524	(1,2)	168	31	.9841	(1,5)	96	16	.8889
13									(1,6)	104	18	.8521
14	(1,2)	392	85	.9913					(2,0)	112	24	.9796
15									(2,1)	120	28	.9956
16	(2,2)	448	112	1.0000	(2,2)	224	56	1.0000	(2,2)	128	32	1.0000
17									(2,3)	136	36	.9965
18	(3,2)	504	141	.9947					(2,4)	144	40	.9877
19									(2,5)	152	44	.9751

TABLE XVII (continued)

	BIB Designs											
	11.9 (b=28,k*=2,r*=7)				11.10 (b=14,k*=4,r*=7)				Unreduced (b=8,k*=7,r*=7)			
k	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E
20	(1,3)	560	172	.9829	(2,3)	280	87	.9943	(2,6)	160	48	.9600
21									(3,0)	168	54	.9796
22	(2,3)	616	211	.9965					(3,1)	176	60	.9917
23									(3,2)	184	66	.9981
24	(3,3)	672	252	1.0000	(3,3)	336	126	1.0000	(3,3)	192	72	1.0000

TABLE XVIII

BEST DBIBD'S WITH NINE TREATMENTS FOR GIVEN NUMBERS OF EXPERIMENTAL UNITS PER BLOCK AND GIVEN GENERATING BIB DESIGNS

	BIB Designs															
	11.3a (b=36,k*=2,r*=8)				11.12 (b=18,k*=5,r*=10)				11.13 (b=12,k*=6,r*=8)				Unreduced (b=9,k*=8,r*=8)			
k	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E
10	(5,0)	360	25	.5625	(2,0)	180	20	.9000					(1,2)	90	11	.9900
11	(2,1)	396	53	.9855									(1,3)	99	13	.9669
12	(6,0)	432	36	.5625	(0,3)	216	27	.8438	(1,2)	144	21	.9844	(1,4)	108	15	.9375
13	(3,1)	468	72	.9586	(1,2)	234	37	.9852					(1,5)	117	17	.9053
14	(0,2)	504	84	.9643	(2,1)	252	43	.9872					(1,6)	126	19	.8724
15	(4,1)	540	93	.9300	(3,0)	270	45	.9000	(2,1)	180	33	.9900	(1,7)	135	21	.8400
16	(1,2)	576	123	.9834	(0,4)	288	48	.8438					(2,0)	144	28	.9844
17	(5,1)	612	116	.9031	(1,3)	306	62	.9654					(2,1)	153	32	.9965
18	(2,2)	648	144	1.0000	(2,2)	324	72	1.0000	(2,2)	216	48	1.0000	(2,2)	162	36	1.0000
19	(6,1)	684	141	.8788	(3,1)	342	78	.9723					(2,3)	171	40	.9972
20	(3,2)	720	135	.7594	(4,0)	360	80	.9000					(2,4)	180	44	.9900
21	(0,3)	756	189	.9643	(1,4)	378	93	.9490	(2,3)	252	65	.9949	(2,5)	189	48	.9796
22	(4,2)	792	212	.9855	(2,3)	396	107	.9948					(2,6)	198	52	.9669
23	(1,3)	828	232	.9868	(3,2)	414	117	.9953					(2,7)	207	56	.9527
24	(5,2)	864	249	.9727	(4,1)	432	123	.9609	(3,2)	288	85	.9961	(3,0)	216	63	.9844
25	(2,3)	900	277	.9972	(1,5)	450	130	.9360					(3,1)	225	69	.9936
26	(6,2)	936	288	.9586	(2,4)	468	148	.9852					(3,2)	234	75	.9985
27	(3,3)	972	324	1.0000	(3,3)	486	162	1.0000	(3,3)	324	108	1.0000	(3,3)	243	81	1.0000

TABLE XIX

BEST DBIBD'S WITH TEN TREATMENTS FOR GIVEN NUMBERS OF EXPERIMENTAL UNITS PER BLOCK AND GIVEN GENERATING BIB DESIGNS

	BIB Designs																			
	11.14 (b=45,k*=2,r*=9)				11.15 (b=30,k*=3,r*=9)				11.17 (b=18,k*= ,r*=9)				11.18 (b=15,k*=6,r*=9)				Unreduced (b=10,k*=9,r*=9)			
k	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E
11																	(1,2)	110	12	.9917
12	(2,1)	540	64	.9877	(4,0)	360	32	.7407					(2,0)	180	20	.9259	(1,3)	120	14	.9722
13					(2,1)	390	50	.9862									(1,4)	130	16	.9467
14	(3,1)	630	85	.9637	(0,2)	420	56	.9524					(1,2)	210	29	.9864	(1,5)	140	18	.9184
15					(5,0)	450	50	.7407	(2,1)	270	40	.9877					(1,6)	150	20	.8889
16	(0,2)	720	112	.9722	(3,1)	480	74	.9635					(2,1)	240	38	.9896	(1,7)	160	22	.8594
17					(1,2)	510	86	.9919									(1,8)	170	24	.8304
18	(1,2)	810	145	.9945	(6,0)	540	72	.7407					(1,3)	270	47	.9671	(2,0)	180	32	.9877
19					(4,1)	570	102	.9418									(2,1)	190	36	.9972
20	(2,2)	900	180	1.0000	(2,2)	600	120	1.0000	(2,2)	360	72	1.0000	(2,2)	300	60	1.0000	(2,2)	200	40	1.0000
21					(0,3)	630	126	.9524									(2,3)	210	44	.9977
22	(3,2)	990	217	.9963	(5,1)	660	134	.9229					(3,1)	330	71	.9780	(2,4)	220	48	.9917
23					(3,2)	690	158	.9956									(2,5)	230	52	.9830
24	(4,2)	1080	256	.9877	(1,3)	720	170	.9838					(2,3)	360	86	.9954	(2,6)	240	56	.9722
25					(6,1)	750	170	.9067	(3,2)	450	112	.9956					(2,7)	250	60	.9600
26	(1,3)	1170	301	.9895	(4,2)	780	200	.9862					(3,2)	390	101	.9961	(2,8)	260	64	.9467
27					(2,3)	810	218	.9968									(3,0)	270	72	.9877
28	(2,3)	1260	352	.9977	(0,4)	840	224	.9524					(2,4)	420	116	.9864	(3,1)	280	78	.9949
29					(5,2)	870	246	.9750									(3,2)	290	84	.9988
30	(3,3)	1350	405	1.0000	(3,3)	910	270	1.0000	(3,3)	540	162	1.0000	(3,3)	450	135	1.0000	(3,3)	300	90	1.0000

TABLE XX

BEST DBIBD'S WITH ELEVEN TREATMENTS FOR GIVEN NUMBERS OF EXPERIMENTAL UNITS PER BLOCK AND GIVEN GENERATING DESIGNS

	BIB Designs											
	11.4a ($b=55, k^*=2, r^*=10$)				11.19 ($b=11, k^*=5, r^*=5$)				Unreduced ($b=11, k^*=10, r^*=10$)			
k	(n_1, n_2)	bk	λ	E	(n_1, n_2)	bk	λ	E	(n_1, n_2)	bk	λ	E
12	(6,0)	660	36	.5500	(0,2)	132	12	.9167	(1,2)	132	13	.9931
13	(2,1)	715	76	.9893					(1,3)	143	15	.9763
14	(7,0)	770	49	.5500					(1,4)	154	17	.9541
15	(3,1)	825	99	.9680	(3,0)	165	18	.8800	(1,5)	165	19	.9289
16	(8,0)	880	64	.5500	(2,1)	176	23	.9883	(1,6)	176	21	.9200
17	(4,1)	935	124	.9439	(1,2)	187	26	.9896	(1,7)	187	23	.8754
18	(0,2)	990	144	.9778	(0,3)	198	27	.9167	(1,8)	198	25	.8488
19	(5,1)	1045	151	.9202					(1,9)	209	27	.8227
20	(1,2)	1100	181	.9955	(4,0)	220	32	.8800	(2,0)	220	36	.9900
21	(6,1)	1155	180	.8980	(3,1)	231	39	.9728	(2,1)	231	40	.9977
22	(2,2)	1210	220	1.0000	(2,2)	242	44	1.0000	(2,2)	242	44	1.0000

TABLE XX (continued)

	BIB Designs											
	11.4a (b=55,k*=2,r*=10)				11.19 (b=11,k*=5,r*=5)				Unreduced (b=11,k*=10,r*=10)			
k	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E
23	(7,1)	1265	211	.5032	(1,3)	253	47	.9773	(2,3)	253	48	.9981
24	(3,2)	1320	261	.9969	(0,4)	264	48	.9167	(2,4)	264	52	.9931
25	(8,1)	1375	244	.8589	(5,0)	275	50	.8800	(2,5)	275	56	.9856
26	(4,2)	1430	304	.9893	(4,1)	286	59	.9600	(2,6)	286	60	.9763
27	(0,3)	1485	324	.9778	(3,2)	297	66	.9959	(2,7)	297	64	.9657
28	(5,2)	1540	349	.9793	(2,3)	308	71	.9962	(2,8)	308	68	.9541
29	(1,3)	1595	379	.9914	(1,4)	319	74	.9679	(2,9)	319	72	.9417
30	(6,2)	1650	396	.9680	(0,5)	330	75	.9167	(3,0)	330	81	.9900
31	(2,3)	1705	436	.9981	(5,1)	341	83	.9500	(3,1)	341	87	.9958
32	(7,2)	1760	445	.9561	(4,2)	352	92	.9883	(3,2)	352	93	.9990
33	(3,3)	1815	495	1.0000	(3,3)	363	99	1.0000	(3,3)	363	99	1.0000

TABLE XXI

BEST DBIBD'S WITH THIRTEEN TREATMENTS FOR GIVEN NUMBERS OF EXPERIMENTAL UNITS PER BLOCK AND GIVEN GENERATING BIB DESIGNS

	BIB Designs											
	11.21 (b=26, k^*=3, r^*=6)				11.22 (b=13, k^*=4, r^*=4)				41 (b=26, k^*=6, r^*=12)			
k	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E
14									(0,2)	364	28	.9286
15	(5,0)	390	25	.7222								
16	(2,1)	416	39	.9902	(4,0)	208	16	.8125				
17					(2,1)	221	22	.9896				
18	(6,0)	468	36	.7222	(0,2)	234	24	.9630	(3,0)	468	45	.9028
19	(3,1)	494	54	.9723					(2,1)	494	55	.9903
20	(0,2)	520	60	.9750	(5,0)	260	25	.8125	(1,2)	520	61	.9912
21	(7,0)	546	49	.7222	(3,1)	273	33	.9728	(0,3)	546	63	.9286
22	(4,1)	572	71	.9535	(1,2)	286	37	.9938				
23	(1,2)	598	81	.9953								
24	(8,0)	624	64	.7222	(6,0)	312	36	.8125	(4,0)	624	80	.9028
25	(5,1)	650	90	.9360	(4,1)	325	46	.9568	(3,1)	650	94	.9776

TABLE XXI (continued)

	BIB Designs											
	11.12 (b=26,k*=3,r*=6)				11.22 (b=13,k*=4,r*=4)				41 (b=26,k*=6,r*=12)			
k	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E
26	(2,2)	676	104	1.0000	(2,2)	338	52	1.0000	(2,2)	676	104	1.0000
27	(9,0)	702	81	.7222	(0,3)	351	54	.9630	(1,3)	702	110	.9808
28	(6,1)	728	111	.9203	(7,0)	364	49	.8125	(0,4)	728	112	.9286
29	(3,2)	754	129	.9970	(5,1)	377	61	.9429				
30	(0,3)	780	135	.9750	(3,2)	390	69	.9967	(5,0)	780	125	.9028
31	(7,1)	806	134	.9063	(1,3)	403	73	.9875	(4,1)	806	143	.9672
32	(4,2)	832	156	.9902	(8,0)	416	64	.8125	(3,2)	832	157	.9966
33	(1,3)	858	166	.9908	(6,1)	429	78	.9311	(2,3)	858	167	.9968
34	(8,1)	884	159	.8940	(4,2)	442	88	.9896	(1,4)	884	173	.9728
35	(5,2)	910	155	.9816	(2,3)	455	94	.9976	(0,5)	910	175	.9286
36	(2,3)	936	199	.9981	(0,4)	468	96	.9630	(6,0)	936	180	.9028
37	(9,1)	962	186	.8831	(7,1)	481	97	.9211	(5,1)	962	202	.9591
38	(6,2)	988	216	.9723	(5,2)	494	119	.9812	(4,2)	988	220	.9903

TABLE XXI (continued)

	BIB Designs											
	11.21 (b=26,k*=3,r*=6)				11.22 (b=13,k*=4,r*=4)				41 (b=26,k*=6,r*=12)			
k	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E
39	(3,3)	1014	234	1.0000	(3,3)	507	117	1.0000	(3,3)	1014	234	1.0000

TABLE XXII

BEST DBIBD'S WITH FIFTEEN TREATMENTS FOR GIVEN NUMBERS OF EXPERIMENTAL UNITS PER BLOCK AND GIVEN GENERATING BIB DESIGNS

	BIB Designs											
	11.24 ($b=35, k^*=3, r^*=7$)				62 ($b=35, k^*=6, r^*=14$)				11.25 ($b=15, k^*=7, r^*=7$)			
k	(n_1, n_2)	bk	λ	E	(n_1, n_2)	bk	λ	E	(n_1, n_2)	bk	λ	E
16									(0,2)	240	16	.9375
17												
18	(2,1)	630	50	.9921	(0,2)	630	48	.9524				
19												
20												
21	(3,1)	735	67	.9767	(2,1)	735	68	.9913	(3,0)	315	27	.9184
22									(2,1)	330	32	.9917
23									(1,2)	345	35	.9924
24	(0,2)	840	88	.9821	(1,2)	840	89	.9933	(0,3)	360	36	.9375
25												
26												

TABLE XXII (continued)

	BIB Designs											
	11.24 (b=35, k*=3, r*=7)				62 (b=35, k*=6, r*=14)				11.25 (b=15, k*=7, r*=7)			
k	(n_1, n_2)	bk	λ	E	(n_1, n_2)	bk	λ	E	(n_1, n_2)	bk	λ	E
27	(1,2)	945	113	.9965	(3,1)	945	111	.9788				
28									(4,0)	420	48	.9184
29									(3,1)	435	55	.9810
30	(2,2)	1050	140	1.0000	(2,2)	1050	140	1.0000	(2,2)	450	60	1.0000
31									(1,3)	465	63	.9834
32									(0,4)	480	64	.9375
33	(3,2)	1155	169	.9976	(1,3)	1155	167	.9858				
34												
35									(5,0)	525	75	.9184
36	(4,2)	1260	200	.9921	(3,2)	1260	201	.9970	(4,1)	540	84	.9722
37									(3,2)	555	91	.9971
38									(2,3)	570	96	.9972
39	(1,3)	1365	235	.9932	(2,3)	1365	236	.9975	(1,4)	585	99	.9763

TABLE XXII (continued)

	BIB Designs											
	11.24 (b=35,k*=3,r*=7)				62 (b=35,k*=6,r*=14)				11.25 (b=15,k*=7,r*=7)			
k	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E	(n_1,n_2)	bk	λ	E
40									(0,5)	600	100	.9375
41												
42	(2,3)	1470	274	.9985	(4,2)	1470	272	.9913	(6,0)	630	108	.9184
43									(5,1)	645	119	.9654
44									(4,2)	660	128	.9917
45	(3,3)	1575	315	1.0000	(3,3)	1575	315	1.0000	(3,3)	675	135	1.0000

VITA

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