A METHOD OF CONSTRAINED RANDOMIZATION

FOR 2^n FACTORIALS

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CHAPTER I

INTRODUCTION

Sequential Experimentation

The concept of sequential experimentation is subject to the interpretation of the individual. To some persons it might suggest experimentation in which treatments were applied to each experimental unit in a sequential manner. Detection of effects due to this sort of sequential ordering has been investigated by Prairie and Zimmer [9]. Other persons might visualize experimentation in which results of applied treatments became available for observation in a sequential manner. This sort of situation would include experiments in which the treatments were applied to the experimental units in a sequential manner, with results of one treatment being observed before application of the following treatment. This is the sort of sequential experimentation in which the material presented in this thesis could be useful.

A great amount of this type of experimentation is related to improvement of manufacturing processes. Since most manufacturing processes are controlled by a number of factors, the experiment is usually designed to determine which combination of the possible levels of the controlling factors will give optimum output of the product.

A natural method of experimentation is a factorial arrangement of treatments where each factor is set at each of two or more levels with all possible combinations of the other factor levels. A combination of factor levels is called a treatment combination if all factors under consideration are specified. In an exploratory type of experiment it is ordinarily sufficient to use only two levels of each of the factors. This is known as a 2^n factorial, where n is the number of factors. On the basis of the information gained from a 2^n factorial, one can obtain a related set of optimum operating conditions (optimum as indicated by this factorial). Additional experimentation can be done on selected factors at the discretion of the experimenter with other levels of these factors to find an improved optimum set of operating conditions.

A definition of the terminology to be used in describing sequential experiments is now given.

<u>Definition 1-1</u>. A <u>treatment run</u> will be the performance of an operation under the conditions specified by a particular treatment combination. An <u>experiment</u> will consist of a sequence of runs. Usually the number of runs in the sequence will be specified before the experiment begins.

Suppose the various treatment combinations (also denoted by tc's) are run sequentially on the same piece or pieces of equipment. That is, the process is a continuing one and the various control settings or conditions, i. e. tc's, are imposed on the process without shutdown, or with only a pause in the operation of the equipment. This sequential ordering of the tc's is a basis for several potential difficulties. Each of these difficulties has very serious implications in both the performance of the physical operation as well as the interpretation of the analysis of the experiment.

One of the problems created by the sequential ordering of the treatment runs is the phenomena of incompatibility of adjacent tc's.

Definition 1-2. Two treatment combinations will be called adjacent if one follows immediately after the other in the sequential order of running the experiment.

Often the process under investigation or the nature of the equipment being used prevents one from having certain sets of operating conditions (tc's) adjacent in the sequential order of running the experiment. It may also be difficult to "line out" the apparatus if many factors are changed from one run to the next. Thus in the face of an incompatibility condition, the usual requirement of random assignment of tc's to the experimental units (process runs in the industrial context) is restricted. This thesis deals with several aspects of this incompatibility condition on adjacent treatment combinations.

In particular, this work is concerned with 2^n factorials when the incompatibility condition restricts the number of factor levels which may be changed from run to run.

Another inherent problem of sequential processes is a "learning" or "wear" phenomena. Learning is any systematic change in the process not attributable to the applied treatments directly controlled by the experimenter. This aspect of sequential experimentation has been investigated in a number of papers, Daniel and Wilcoxon [2], Hill [5], and Cox [1].

In industrial experimentation results are usually available in a relatively short time. The period required for the experimentation is measured in hours or days rather than weeks or months. This property makes it practicable to look at the results of sets of runs of tc's or individual tc's before continuing with other runs or to run a fractional replicate of the complete factorial. One then obtains a statistical analysis of the results before continuing with additional fractions to complete the factorial. Among the many useful papers regarding this sort of analysis one finds material by Floyd [4], Hunter [6], and Davis and Hay [3].

Discussion of the Problem

Consider an experimental situation as described previously where a 2^n factorial design of some type is to be run sequentially with the compatibility condition that no more than Δ factor levels may be changed between adjacent tc's. In order that a statistical analysis of the experimental results have good properties, the usual approach of the statistician is to require full randomization. Under full randomization the properties of the statistical analysis depend only upon the assumptions made concerning the mathematical model used to describe the experimental process, see Kempthorne [7] or Ostle [10]. Obviously this will not be possible in the situation stated above unless $\Delta = n$. If $\Delta < n$, then only something less than the usual type of full randomization may be done by the experimenter. It is the purpose of this writer to investigate this problem for arbitrary values of n and Δ. A method of constrained randomization will be given for complete factorial and fractional replications in randomized block designs and for split-plot designs. As used in this thesis, constrained randomization consists of a method of randomization for any given sequence which will not destroy the compatibility properties of adjacent tc's. The experimenter obtains a sequence of the tc's which satisfies the compatibility criterion, and then performs a series of operations at random on the sequence. The operations are restricted to those which preserve the compatibility of adjacent tc's. The set of sequences obtained in this manner using constrained randomization is a subset of the set of possible sequences obtained using complete randomization. Other discussions of types of constrained randomization may be found in Kempthorne [8] and in Daniel and Wilcoxon [2].

A statistical analysis based on the constrained randomization technique used will be developed for the designs discussed using both the infinite model and a randomization model. Attention is given to estimation of main effects and interactions, estimation of the variances of these estimates, and estimation of experimental error.

In order to clarify the following discussion, several definitions will be made.

Definition 1-3. Two treatment combinations have order of adjacency Δ if the number of factor levels which are different in the two adjacent tc's is equal to Δ.

Consider a 2^n factorial type experiment with treatment combination i denoted by $(x_{i1}x_{i2}\ldots x_{in})$, where the x_{ij}'s are either 0's or 1's and similarly for adjacent treatment combinations i'. The order of adjacency, Δ, is given by the sum,

$$\Delta = \sum_j |x_{ij} - x_{i'j}| .$$

Definition 1-4. A sequence of treatment combinations is called a Δ order sequence if the order of adjacency for every pair of adjacent tc's in the sequence is equal to Δ.

Notice that restricting the value of Δ to be $< n$ in a 2^n factorial experiment induces a compatibility condition on the sequence, as mentioned earlier.

Definition 1-5. An operational sequence will be any sequence of the tc's which satisfies the particular compatibility requirement imposed on the design by the experimenter and/or the experimental process.

An operational sequence is, therefore, one which may actually be run by the experimenter in the process under investigation.

The constrained randomization experimental designs developed in this thesis are applicable in a sequential process. The process under investigation is a factorial (2 level) experiment with a compatibility requirement on adjacent tc's. The compatibility condition requires that the sequence of tc's must be a Δ order sequence.

Three types of factorials are discussed. Methods of constrained randomization for full 2^n factorials in blocks and for 2^{n-p} fractional replicates are presented in Chapter III. Chapter IV contains constrained randomization for split-plot designs of a factorial. The constrained randomization methods may be easily used by the experimenter.

The methods of analysis developed for these designs under constrained randomization will be related to different assumptions regarding the population of inference.

For both models unbiased estimates of main effects and interactions are found. An estimate of experimental error is obtained and used to estimate variances of main effect and interaction estimates.

Example

Consider a 2^2 factorial experiment in a randomized block design with three complete blocks to be run. Let the compatibility conditions require that each sequence be of order $\Delta = 1$.

A 2^2 factorial experiment consists of the tc's (00), (01), (10), (11). There are 24 different possible sequences of these four tc's. Some of these will have order $\Delta = 1$, and some will not. By listing all 24 possible sequences it is found that the eight operational sequences which follow have order $\Delta = 1$.

00	00	01	01	10	10	11	11
01	10	00	11	00	11	01	10
11	11	10	10	01	01	00	00
10	01	11	00	11	00	10	01

The other 16 non-operational sequences have at least one adjacency with order $\Delta = 2$. The sequences

00	00	01	01	10	10	11	11
01	10	00	11	00	11	01	10
10	01	11	00	11	00	10	01
11	11	10	10	01	01	00	00

all have orders of adjacency $\Delta = 1$ for tc 1 and tc 2, $\Delta = 2$ for tc 2 and tc 3, and $\Delta = 1$ for tc 3 and tc 4. The sequences

00	00	01	01	10	10	11	11
11	11	10	10	01	01	00	00
01	10	00	11	00	11	01	10
10	01	11	00	11	00	10	01

all have orders of adjacency $\Delta = 2$ for tc 1 and tc 2, $\Delta = 1$ for tc 2 and tc 3, and $\Delta = 2$ for tc 3 and tc 4. Thus the experimenter might pick at random with replacement three of the operational sequences in the first group. One sequence would be selected for each block or replication which is to be run.

In the 2^2 factorial experiment it was not a difficult task to list all possible sequences and then separate the operational sequences, which satisfied the compatibility condition. In a larger experiment this method is not practical; in a 2^3 experiment there are $8!$ possible sequences. More refined methods of finding sequences which satisfy particular compatibility conditions are presented in Chapter II.

CHAPTER II

FINDING OPERATIONAL SEQUENCES

This chapter deals with the problem of finding an operational sequence of order Δ for a 2^n factorial. An example is provided which demonstrates the difficulty of obtaining an operational sequence and a geometric interpretation of the problem is given.

Preliminary Considerations

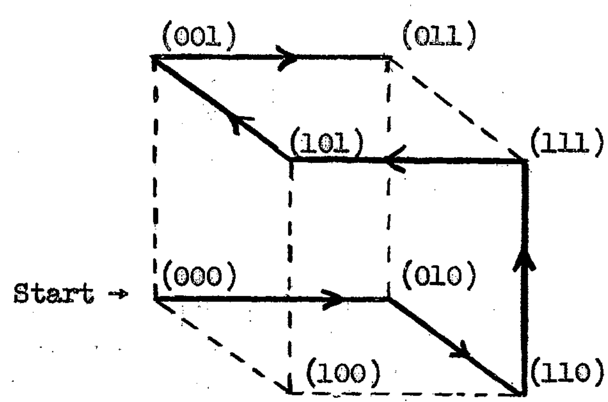
Any method of finding an operational sequence for a 2^n factorial must use all 2^n treatment combinations. The order of adjacency of every pair of adjacent tc's must be equal to Δ. (The degree of randomness used in obtaining the sequence will relate to the validity and generality of the analysis of the results of the experiment.)

One may think of the 2^n tc's as points in a finite geometric space. Finding an operational sequence is equivalent to finding a path which connects the vertices of the space. The path must be unbroken, connect all of the vertices, and intersect each vertex only once. To satisfy the compatibility condition, the distance, measured along the lattice lines, between adjacent intersected points must be equal to Δ.

The major pitfall awaiting the individual constructing an operational sequence of tc's according to some random scheme is now related. After arranging some portion of the operational sequence and arriving at, say, the jth ordered tc in the sequence, no other tc can be found among the remaining unused tc's which satisfies the compatibility condition. That is, for each of the unused tc's, the order of adjacency with tc j is not equal to Δ.

For example, consider a 2^3 factorial experiment in a full replicate design with $\Delta = 1$. Suppose the following operational sequence has been arranged: 000,010,110,111,101,001,011. The only remaining tc is 100. However, the order of adjacency of 011 and 100 is $\Delta = 3$, not $\Delta = 1$.

Geometrically this problem can be thought of in the following manner. After some portion of the path connecting the vertex points in the finite geometry is completed, all of the remaining points in the space are at a distance not equal to Δ from the end point of the path completed. Considering the above example of a 2^3 factorial experiment, the following graph illustrates the problem in a geometric sense. Solid lines with arrows denote the connect path. Again the lattice line distance from 011 to 100 is 3 units rather than 1 unit.



Thus it becomes clear that a method of finding a random operational sequence is needed which will avoid the type of impassé which was just illustrated. This will be accomplished by using a method of constrained randomization on a set of transformation generators.

Definition 2-1. A transformation generator (tg) is an operational sequence with the first tc being (00· · ·0), the low level of each factor, which will generate a set of operational sequences under constrained randomization.

The notation $TG(2^n, \Delta)$ will be used to identify a set of transformation generators for particular values of n and Δ. Using completely random selection from the set of generated operational sequences, the analysis of the data obtained from the selected sequences will be shown to possess good statistical properties.

Definition 2-2. A pair of operational sequences are isomorphic if one may be obtained from the other by the methods of constrained randomization. Otherwise, they are not isomorphic.

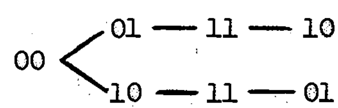
The methods of constrained randomization presented in the subsequent chapters require that one have a set of transformation generators.

Such a set of transformation generators may consist of only one sequence of tc's or it may consist of a large number of sequences of tc's. From an intuitive point of view, one would probably desire a large or even a maximal set of tg's. However, this is not required in order to perform an analysis of the results.

There are two requirements which must be met in order for a sequence to belong to the set of transformation generators. Each sequence must be an operational sequence and no pair of tg's should be isomorphic under the methods of constrained randomization. Thus it is required that one be able to find operational sequences by some method, which need not be random. That is, one need only be able to arrange all the tc's into sequences which satisfy the compatibility condition. Then all such sequences which have $(00\cdots 0)$ as their first tc and which are not isomorphic to one another make up the set of transformation generators. The remainder of this chapter deals with finding transformation generators.

Branch Diagrams

One could find a set of sequences which includes the maximal set of transformation generators by constructing a branch diagram for any particular 2^n experiment. This would be done in a manner similar to that shown for a 2^2 with $\Delta = 1$ below.



From this one gets the two operational sequences 00, 01, 11, 10 and 00, 10, 11, 01. Now the maximal set of tg's for this situation, $TG(2^2, \Delta = 1)$, can consist of either of these but not both, since they are isomorphic under the method of constrained randomization presented in Chapter III. This method of finding all tg's, in fact finding an even larger set of sequences of tc's, has two defects. It is unnecessary to have a set of sequences larger than the maximal set of tg's for constrained randomization. Also, the number of possible branches on the diagram becomes unmanageably large for even a 2^3 experiment with the compatibility requirement that the sequence be of order $\Delta \leq 2$.

Adjacency Tables

The intuition and natural caution of the experimenter would possibly cause him to desire a maximal set of transformation generators for constrained randomization. However, it will be shown that one does not need to have a maximal or even a large set of tg's. Since one or only a few tg's will be sufficient, one might list all of the treatment combinations and the proceed to arrange them into a Δ order sequence. This process would be facilitated by the use of a tabular listing of tc's with order of adjacency Δ. An example of such a table is given below. An x in the intersection of any row and column indicates that the tc's listed in that row and column have order of adjacency equal to Δ.

For a 2^3 experiment with $\Delta = 1$, the following table indicates tc's with the proper order of adjacency.

$2^3, \ \Delta = 1$

000							
x	001						
x		010					
	x	x	011				
x				100			
	x			x	101		
		x		x		110	
			x		x	x	111

Note that to find a transformation generator one would always start with $(00\cdot\cdot\cdot0)$ as the first tc in the sequence. Then using the table above, or a similar one for other values of n or Δ, the sequence could be completed by consulting the table at each step to find adjacent tc's.

Of course in arranging a sequence satisfying the compatibility condition, one might find himself confronted with the same situation as in the earlier examples. A number of tc's may still remain to be used, none of which has order of adjacency equal to Δ with respect to the last tc in the partially arranged sequence. In this case, one only wants an operational sequence and nothing is sacred concerning the order in which the tc's are encountered. Thus it will be satisfactory to rearrange the sequence already partially completed until all of the tc's have been used and the final sequence is of order Δ. For instance, one might have the sequence 000, 010, 110, 111, 101, 001, 011 for a 2^3 with $\Delta = 1$, leaving the tc 100 left over. Suppose that 100 is inserted after 110, in the fourth process run (plot). Then one has the partial sequence 000, 010, 110, 100. Next must follow 101, then either 111 or 001. In either case the sequence may be finished using all eight tc's and having order $\Delta = 1$. Other possibilities obviously exist and would lead to other operational sequences, which could be used as transformation generators.

It was this method which was used to prepare Tables VI through XI in the appendix. The first five tables or non-maximal portions of them could have been prepared in this manner. However, for these five tables the IBM 7040 computer at Oklahoma State University was employed to find all possible operational sequences for a given 2^n experiment and compatibility condition Δ.

Computer Use

Essentially the computer program was used to find all possible branches of the branch diagram. Then transformation generators, that is, operational sequences which are not isomorphic under constrained randomization, were found and are listed in the appendix. These are necessarily maximal sets of tg's since all possible operational sequences were searched by the computer to find the set of tg's.

The computer program for a typical situation may be found in Appendix A.

The computer was not used for finding the sets of tg's after Table V in the Appendix B because the computer time required to find a maximal set of tg's was prohibitive. Also, since a maximal set is not required for constrained randomization, it was deemed sufficient to provide a set of representative tg's.

The methods of constrained randomization using sets of tg's are presented in the next chapters.

CHAPTER III

CONSTRAINED RANDOMIZATION FOR A 2^{n-p} FACTORIAL IN A RANDOMIZED NON-CONSECUTIVE BLOCK DESIGN

This chapter contains methods of constrained rancomization for 2^{n-p} factorials in randomized complete block designs, where each block is independent of the others. The methods are applicable for $n \geq 2$ and $0 \leq p < n$. A discussion of sets of unique transformation generators used in constrained randomization is included with references to the appropriate tables of these sets of generators in Appendix B. The analysis of each of these designs is given, both for infinite model and for randomization model assumptions. The model which a particular experimenter may use will be determined by the process under investigation and the population to which he wishes to draw inference.

Transformation Generators and Constrained Randomization

As previously defined, an arrangement of the treatment combinations into a sequence which satisfies the compatibility condition is called an operational sequence. A transformation generator is a given operational sequence of treatment combinations which is used in the constrained randomization technique to generate additional operational sequences. It is necessary to have a set of transformation generators, each of which is unique under the method of constrained randomization which follows. The statistical analysis presented in subsequent sections of this thesis will be shown to be valid for any set of transformation generators. Thus a set consisting of only one transformation generator will be sufficient for constrained randomization.

Sets of unique transformation generators for the various constrained experimental designs discussed in this thesis are given in the appendix.

Constrained randomization for a 2^{n-p} factorial in non-consecutive replicates is performed according to the following outline.

(1) For each replication of the basic design a single transformation generator is chosen at random from the appropriate set of unique transformation generators in the appendix. The proper set of transformation generators is identifiable by the value of n and the compatibility condition.

(2) For each replication, randomly assign the n factors under investigation to the n pseudo factor names $x_1, x_2, \cdots, x_n$ in the 2^{n-p} treatment combinations. Note that the assignment is only done once in each replication. Thus each tc in a given replication has the same assignment of real factor names.

(3) Randomly choose one of the base 2 numbers which represents a tc used in the factorial experiment. Then combine this number with each tc using vector addition modulo 2. This step effectively does a random assignment of the high and the low levels of each factor to the pseudo level names 0 and 1 in the 2^{n-p} tc's, where the high levels are then renamed 1 and the low levels renamed 0.

Example 3-1. As an example of this technique consider the following randomization obtained for one replication of a 2^3 factorial experiment when the compatibility condition requires that the order of adjacency be equal to 1, that is, a full 2^3 with $\Delta = 1$.

From Table II in Appendix B, one finds a maximal set of transformation generators, namely $TG(2^3, \Delta = 1)$:

(1) 000, 001, 011, 010, 110, 100, 101, 111

(2) 000, 001, 011, 010, 110, 111, 101, 100

(3) 000, 001, 011, 111, 101, 100, 110, 010

One of these generators is selected by a random process. Suppose it is

(2) 000, 001, 011, 010, 110, 111, 101, 100.

Following step (2) in the constrained randomization process, one randomly assigns the real factor names A, B, C to the pseudo factor names x_1, x_2, x_3 in the 2^3 sequence. Suppose that one obtained the following: $A = x_2$; $B = x_1$; and $C = x_3$. The operational sequence would then be arranged into the following form: 000, 001, 101, 100, 110, 111, 011, 010. Note that this is still an operational sequence, i.e., that $\Delta = 1$ for every adjacency.

To complete the constrained randomization procedure, one now chooses at random one of the base 2 numbers 000, 001, 010, 011, 100, 101, 110, 111. Suppose that it is 101. Then 101 is added componentwise modulo 2 to each of the tc's in the operational sequence obtained in step (2). The result is 101, 100, 000, 001, 011, 010, 110, 111. This is the operational sequence which would be run in the experimental situation under investigation by the experimenter. Note that the compatibility requirement, $\Delta = 1$, still holds for this sequence. A theorem formalizing this observation follows.

Theorem 3-1. The sequence of treatment combinations resulting from constrained randomization is an operational sequence. If the set of unique transformation generators is maximal, then the operational sequence obtained is equivalent to randomly choosing an operational sequence from the totality of all such sequences of tc's.

Proof of Theorem 3-1. The set of transformation generators is a set of operational sequences by definition. Thus after step (1) of the constrained randomization procedure, one has an operational sequence.

Step (2) is a renaming of the pseudo factors and therefore does not disturb the property of being an operational sequence. This is a consequence of the fact that every tc in the sequence receives identically the same assignment of real factor names. Thus for every pair of adjacent tc's i and i', factors x_{ij} and $x_{i'j}$ which corresponded position-wise before assignment of factor names in step (2) still correspond position-wise after assignment of real factor names $x_{ij'}$ and $x_{i'j'}$. Hence the sum

$$\begin{aligned}\Delta &= \sum_j |x_{ij} - x_{i'j}| \\ &= \sum_{j'} |x_{ij'} - x_{i'j'}|,\end{aligned}$$

where j' is the new name under step (2) of j. This relationship will hold for every pair of adjacent tc's in the sequences.

The randomization which is to be performed in step (3) of the constrained randomization procedure also preserves the compatibility condition and thus the property of being an operational sequence. This is seen by considering two adjacent tc's $(x_1 x_2 \cdots x_n)$ and $(y_1 y_2 \cdots y_n)$ which satisfy the compatibility requirement, having order of adjacency

$< \Delta$. That is,

$$\Sigma |x_i - y_i| < \Delta .$$

Using the method of randomization in step (3), one obtains the tc $(x_1+a_1, x_2+a_2, \cdots, x_n+a_n)$ and the adjacent tc $(y_1+a_1, y_2+a_2, \cdots, y_n+ a_n)$ where $(a_1a_2\cdots a_n)$ is the randomly chosen base 2 number. From these two transformed adjacent tc's the following relation is found.

$$\Sigma |(x_i+a_i) - (y_i+a_i)| = \Sigma |x_i - y_i| < \Delta.$$

Hence the new sequence of transformed tc's also satisfies the compatibility condition. This completes the first portion of the proof.

The second statement in the theorem is simply a clarification of the notion of a maximal set of transformation generators. A set of such generators will not be called maximal unless it generates all possible operational sequences under the method of constrained randomization.

A result which is basic in the development of the theory of the randomization model is presented in the following theorem.

<u>Theorem 3-2.</u> Over all possible constrained randomizations of a given operational sequence each treatment combination appears an equal number of times in each position in the sequence. Since a transformation generator is an operational sequence, the same result holds for a tg.

<u>Proof of Theorem 3-2.</u> Consider any operational sequence which results from the constrained randomization procedure carried out in steps (1)

and (2). Such an operational sequence of tc's has a total of 2^{n-p} possible randomizations under the procedure in step (3). A different randomization occurs for each of the base 2 numbers in the factorial experiment. Let $y_1 y_2 \cdots y_n$ denote any one of these numbers.

Consider any tc denoted by $(x_1 x_2 \cdots x_n)$ in any position in the operational sequence obtained at step (2). The n-tuple $(x_1 x_2 \cdots x_n)$ is itself a particular base 2 number. For any choice of $(y_1 y_2 \cdots y_n)$ the vector sum

$$(x_1 x_2 \cdots x_n) + (y_1 y_2 \cdots y_n) = (x_1+y_1,\ x_2+y_2,\ \cdots,\ x_n+y_n)$$
$$= (y_1' y_2' \cdots y_n')\text{modulo } 2,$$

where $(y_1' y_2' \cdots y_n')$ is again a base 2 number.

Note that if $(y_1 y_2 \cdots y_n) \neq (z_1 z_2 \cdots z_n)$ then $\sum_i |y_i - z_i| \neq 0$. For any n-tuple $(x_1 x_2 \cdots x_n)$, one then has

$$(x_1 x_2 \cdots x_n) + (y_1 y_2 \cdots y_n) \neq (x_1 x_2 \cdots x_n) + (z_1 z_2 \cdots z_n)$$

or

$$(x_1+y_1,\ x_2+y_2,\ \cdots,\ x_n+y_n) \neq (x_1+z_1,\ x_2+z_2,\ \cdots,\ x_n+z_n)$$

where the addition is component-wise modulo 2. Since if one has equality, then

$$\sum_i |(x_i+y_i) - (x_i+z_i)| = 0$$

or

$$\sum_i |x_i+y_i-x_i-z_i| = \sum_i |y_i-z_i| = 0 ,$$

which is a contradiction.

Since in step (3) of the constrained randomization, one has 2^{n-p} distinct numbers $(y_1y_2\cdots y_n)$, there will be 2^{n-p} distinct sums.

$$(x_1x_2\cdots x_n) + (y_1y_2\cdots y_n) = y_1'y_2'\cdots y_n') \ .$$

Now the only possible values of $(y_1'y_2'\cdots y_n')$ are in the set of 2^{n-p} numbers in base 2, so it follows that each treatment combination (base 2 number) will appear in a particular position in the set of operational sequences obtained in step (3) exactly once.

Since this result was obtained for any operational sequence found after step (2), one must now consider the set of all possible randomizations obtained after step (1). For a given transformation generator from step (1) there are $n!$ possible randomizations in step (2). Thus over the set of $n!2^{n-p}$ possible randomizations of a given transformation generator, every tc will appear in a particular position in the set of these operational sequences $n!$ times. This completes the proof of Theorem 3-2.

The statistical analysis of this experimental design will now be investigated using the randomization model technique.

The Randomization Model

The population of inference under the randomization model is the experimental units (process runs) actually used, or the larger population from which experimental units were chosen at random. Extending this inference to any other population is a matter for the judgment of the experimenter. Conceptually every tc can be applied to every process run in each replication. Let y_{ijk} denote the population response (conceptual yield) to treatment combination k, on experimental unit j

in replication i. Here there are 2^{n-p} treatment combinations, $k = 1, 2, \cdots, 2^{n-p}$, 2^{n-p} experimental units (process runs) in each replication, $j = 1, 2, \cdots, 2^{n-p}$, and, suppose, r replications, $i = 1, 2, \cdots, r$, of the design.

Consider the identity

$$y_{ijk} = \bar{y}_{...} + (\bar{y}_{i..} - \bar{y}_{...}) + (\bar{y}_{ijk} - \bar{y}_{ij.}) + (\bar{y}_{ij.} - \bar{y}_{i..}) .$$

Assuming additivity of treatment effects, let $y_{ijk} - \bar{y}_{ij.} = t_k$, $\bar{y}_{...} = \mu$, and $\bar{y}_{i..} - \bar{y}_{...} = b_i$. Note that $\sum_i b_i = \sum_k t_k = 0$. Then one may write

$$y_{ijk} = \mu + b_i + t_k + (\bar{y}_{ij.} - \bar{y}_{i..}) .$$

Now, in fact, in the real world one only observes the yield of one tc on any given process run. Thus one response is observed for each tc in every replication. To relate the conceptual population of responses to the observed responses, consider the random variable δ_{ij}^k defined as follows.

$$\delta_{ij}^k = \begin{cases} 1 & \text{if tc } k \text{ is on process run } j \text{ of rep } i \\ 0 & \text{otherwise.} \end{cases}$$

Now for a given i and j there are 2^{n-p} δ's. Only one of these is equal to 1. If tc k is on process run j then $\delta_{ij}^k = 1$ and $\delta_{ij}^{k'} = 0$ for all other k'. Note that $\sum_j \delta_{ij}^k = \sum_k \delta_{ij}^k = 1$. Thus the observed response y_{ik} to tc k on rep i is given by

$$y_{ik} = \sum_j \delta_{ij}^k y_{ijk} = \mu + b_i + t_k + \sum_j \delta_{ij}^k (\bar{y}_{ij.} - \bar{y}_{i..})$$

$$= \mu + b_i + t_k + e_{ik} ,$$

where $e_{ik} = \sum_j \delta_{ij}^k (\bar{y}_{ij.} - \bar{y}_{i..})$. Note $\sum_k e_{ik} = 0$. The properties of this model for this design will be investigated in the following lemmas and theorems.

Lemma 3-1. Under constrained randomization in a 2^{n-p} factorial in a randomized complete block design, the random variable δ_{ij}^k introduced in the randomization model has the following distributional properties:

(1) $E[\delta_{ij}^k] = E[(\delta_{ij}^k)^2] = 1/2^{n-p}$;

(2) $E[\delta_{ij}^k \delta_{ij'}^k] = 0, \quad j \neq j'$;

(3) $E[\delta_{ij}^k \delta_{i'j'}^{k'}] = 1/2^{2n-2p}, \quad i \neq i'$;

(4) $E[\delta_{ij}^k \delta_{ij}^{k'}] = 0, \quad k \neq k'$;

(5) $0 \leq E[\delta_{ij}^k E_{ij'}^{k'}] \leq 1/2^{n-p}, \quad k \neq k', \quad j \neq j'$.

Proof of Lemma 3-1. Each part will be considered in turn. Note that all probabilities are a result of the method of constrained randomization which was postulated previously. That is, constrained randomization was applied independently in each replication. A given treatment appears only once in any given replication and a given process run receives only one treatment, all subject to the primary constraint requiring that no more than Δ factors be allowed to change from tc to adjacent tc.

Proof of (1). Using the usual definition of expectation,

$$E[\delta_{ij}^i]^2 = 1 \cdot \text{Prob}(\delta_{ij}^{k^2} = 1) + 0 \cdot \text{Prob}(\delta_{ij}^{k^2} = 0)$$

$$= 1 \cdot \text{Prob}(\delta_{ij}^k = 1)$$

$$= E[\delta^k_{ij}]$$

$$= \text{Prob(tc } k \text{ is on process run } j) .$$

Now by Theorem 3-1, every tc appears on process run j with equal frequency. Hence the probability tc k is on process run j is 1/(the number of tc's). Thus,

$$E[\delta^k_{ij}] = 1/2^{n-p} ,$$

Proof of (2). For $j \neq j'$,

$$E[\delta^k_{ij}\delta^k_{ij'}] = \text{Prob}(\delta^k_{ij'} = 1, \ \delta^k_{ij} = 1)$$

$$= \text{Prob}(\delta^k_{ij'} = 1 \mid \delta^k_{ij} = 1)\cdot\text{Prob}(\delta^k_{ij} = 1)$$

$$= 0\cdot(1/2^{n-p})$$

$$= 0 ,$$

since tc k cannot be on process run j' if it is on process run j.

Proof of (3). For $i \neq i'$,

$$E[\delta^k_{ij}\delta^k_{i'j'}] = \text{Prob}(\delta^k_{i'j'} = 1 \mid \delta^k_{ij} = 1)\cdot\text{Prob}(\delta^k_{ij} = 1)$$

$$= \text{Prob}(\delta^k_{i'j'} = 1)\cdot\text{Prob}(\delta^k_{ij} = 1) ,$$

since randomization is independent in replications i and i'. But

$$P(\delta^k_{i'j'} = 1) = \text{Prob}(\delta^k_{ij} = 1) = (1/2^{n-p}) ,$$

so

$$E[\delta^k_{ij}\delta^k_{i'j'}] = (1/2^{n-p})(1/2^{n-p})$$

$$= 1/2^{2n-2p} .$$

Proof of (4). For $k \neq k'$,

$$E[\delta_{ij}^{k}\delta_{ij}^{k'}] = \text{Prob}(\delta_{ij}^{k'} = 1 \mid \delta_{ij}^{k} = 1)\cdot\text{Prob}(\delta_{ij}^{k} = 1)$$

$$= 0\cdot(1/2^{n-p})$$

$$= 0$$

since tc k and tc k' are not both on process run j.

Proof of (5). For $k \neq k'$, $j \neq j'$,

$$E[\delta_{ij}^{k}\delta_{ij'}^{k'}] = \text{Prob}(\delta_{ij'}^{k'} = 1 \mid \delta_{ij}^{k} = 1)\cdot\text{Prob}(\delta_{ij}^{k} = 1) .$$

Now

$$0 \leq \text{Prob}(\delta_{ij'}^{k'} = 1 \mid \delta_{ij}^{k} = 1) \leq 1,$$

and

$$\text{Prob}(\delta_{ij}^{k} = 1) = 1/2^{n-p} ,$$

so

$$0 \leq E[\delta_{ij}^{k}\delta_{ij'}^{k'}] \leq 1/2^{n-p} .$$

The value 0 is actually assumed by the conditional probability for j' adjacent to j and for k' not compatible with k. The value 1 is also assumed by the conditional probability for certain 2^{n-p} designs for particular j, j', k, and k'. For instance, in a 2^2 with $\Delta = 1$, if $j = 1$, $k = 00$ then $j' = 3$ and $k' = 11$, the expression $P(\delta_{i3}^{11} = 1 \mid \delta_{i1}^{00} = 1)$ is equal to 1.

As a further illustration of the impossibility of finding a simple expression for $\text{Prob}(\delta_{ij'}^{k'} = 1 \mid \delta_{ij}^{k} = 1)$, consider the following table of conditional probabilities constructed for a 2^3 factorial with $\Delta = 1$. The table was constructed from a list of all operational sequences for this factorial and the table entries are $\mathrm{Prob}(\delta_{ij'}^{k'} = 1 \mid \delta_{i1}^{000} = 1)$.

k'

j'	001	010	011	100	101	110	111
2	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{6}$	0	0	0
3	0	0	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	0
4	$\frac{2}{9}$	$\frac{2}{9}$	0	$\frac{2}{9}$	0	0	$\frac{1}{6}$
5	0	0	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	0
6	$\frac{2}{9}$	$\frac{2}{9}$	0	$\frac{2}{9}$	0	0	$\frac{1}{6}$
7	0	0	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{6}$	0
8	$\frac{2}{9}$	$\frac{2}{9}$	0	$\frac{2}{9}$	0	0	$\frac{1}{6}$

This completes the proof of Lemma 3-1.

The following lemma relates the distributional properties of δ_{ij}^{k} to the randomization model, and is basic to the remainder of the material developed for the design.

Lemma 3-2. Let $s^2 = \sum_j (\bar{y}_{ij.} - \bar{y}_{i..})^2$. Then

(1) $E[e_{ik}] = 0$,

(2) $E[e_{ik}^2] = s^2/2^{n-p}$,

(3) $E[e_{ik}e_{i'k}] = E[e_{ik}, e_{i'k'}] = 0, \quad i \neq i'$,

(4) $-s^2/2^{n-p} \leq E[e_{ik}, e_{ik'}] \leq 0; \quad k \neq k'$.

Proof of Lemma 3-2.

Proof of (1). Consider

$$E[e_{ik}] = E[\sum_j \delta^k_{ij}(\bar{y}_{ij.} - \bar{y}_{i..})] .$$

Using Lemma 3-1, (1),

$$E[e_{ik}] = (1/2^{n-p}) \sum_j(\bar{y}_{ij.} - \bar{y}_{i..})$$
$$= 0 .$$

Proof of (2). Consider

$$E[e^2_{ik}] = E[\sum_j \delta^k_{ij}(\bar{y}_{ij.} - \bar{y}_{i..})^2 +$$
$$\sum_j \sum_{j'}{}_{j \neq j'} \delta_{ij}\delta_{ij'}(\bar{y}_{ij.} - \bar{y}_{i..})(\bar{y}_{ij'.} - \bar{y}_{i..})]$$

Using Lemma 3-1, (1) and (2),

$$E[e^2_{ik}] = (1/2^{n-p}) \sum_j(\bar{y}_{ij.} - \bar{y}_{i..})^2$$
$$= s^2/2^{n-p} .$$

Proof of (3). For $i \neq i'$,

$$E[e_{ik}, e_{i'k'}] = E[\sum_j \sum_{j'} \delta^k_{ij}\delta^{k'}_{i'j'}(\bar{y}_{ij.} - \bar{y}_{i..})(\bar{y}_{i'j'.} - \bar{y}_{i'..})].$$

Using Lemma 3-1, (3) and the fact that $\sum_{j'} (\bar{y}_{i'j'.} - \bar{y}_{i'..}) = 0$,

$$E[e_{ik}, e_{i'k'}] = 0 .$$

Proof of (4). For $k \neq k'$, consider

$$E[e_{ik}, e_{ik'}] = E[\sum_j \delta^k_{ij}\delta^{k'}_{ij}(\bar{y}_{ij.} - \bar{y}_{i..})^2 +$$
$$\sum_j \sum_{j'}{}_{j \neq j'} \delta^k_{ij}\delta^{k'}_{ij'}(\bar{y}_{ij.} - \bar{y}_{i..})(\bar{y}_{ij'.} - \bar{y}_{i..})] .$$

From Lemma 3-1, (4) and (5), and using the fact that

$$\sum_{\substack{j' \\ j \neq j'}} (\bar{y}_{ij'.} - \bar{y}_{i..}) = -(\bar{y}_{ij.} - \bar{y}_{i..}) ,$$

it is seen that

$$- s^2/2^{n-p} \leq E [e_{ik}, e_{ik'}) \leq 0 .$$

This completes the proof of the lemma.

Theorem 3-3. If $i \neq i'$, $k \neq k'$, then

(1) $E [y_{ik}] = \mu + b_i + t_k$,

(2) $Var (y_{ik}) = s^2/2^{n-p}$,

(3) $Cov (y_{ik}, y_{i'k}) = Cov (y_{ik}, y_{i'k'}) = 0$,

(4) $- s^2/2^{n-p} \leq Cov (y_{ik}, y_{ik'}) \leq 0$.

Proof of Theorem 3-3.

Proof of (1). Consider

$$E [y_{ik}] = E [\mu + b_i + t_k + e_{ik}]$$
$$= \mu + b_i + t_k .$$

Proof of (2). Consider

$$Var (y_{ik}) = E [e_{ik}^2]$$
$$= s^2/2^{n-p} .$$

Proof of (3). Since

$$Cov (y_{ik}, y_{ik'}) = E [e_{ik}, e_{ik'}],$$

$$- s^2/2^{n-p} \le \text{Cov}\ (y_{ik}, y_{ik'}) \le 0 \ .$$

Proof of (3). Consider

$$\text{Cov}\ (y_{ik}, y_{i'k}) = E\,[e_{ik}, e_{i'k}] = 0$$

and

$$\text{Cov}\ (y_{ik}, y_{i'k'}) = E\,[e_{ik}, e_{i'k'}] = 0 \ .$$

This completes the proof of the theorem.

Corollary 3-1. The following properties follow directly,

(1) $E\,[\bar{y}_{.k}] = \mu + t_k$,

(2) $E\,[\bar{y}_{i.}] = \mu + b_i$,

(3) $\text{Var}\ (\bar{y}_{.i}) = s^2/r2^{n-p}$,

(4) $- s^2/r2^{n-p} \le \text{Cov}\ (\bar{y}_{.k}, \bar{y}_{.k'}) \le 0 \ , \quad k \ne k'$.

Proof of Corollary 3-1.

Proof of (1). Consider

$$\begin{aligned} E\,[\bar{y}_{.k}] &= E\,[(1/r) \textstyle\sum_i y_{ik}] \\ &= (1/r) \textstyle\sum_i E\,[y_{ik}] \\ &= (1/r) \textstyle\sum_i (\mu + b_i + t_k) \\ &= \mu + t_k \ . \end{aligned}$$

Proof of (2). Consider

$$E\,[\bar{y}_{i.}] = (1/2^{n-p}) \textstyle\sum_i (\mu + b_i + t_k)$$

$$= \mu + b_i .$$

Proof of (3). Consider

$$\begin{aligned} \mathrm{Var}\,(\bar{y}_{.k}) &= (1/r^2)[\textstyle\sum_i \mathrm{Var}\,(y_{ik}) + \sum_i \sum_{i'} \mathrm{Cov}\,(y_{ik}, y_{i'k})] \\ &= (1/r^2)[\textstyle\sum_i s^2/2^{n-p} + \sum_i \sum_{i'} 0] \\ &= s^2/r2^{n-p} . \end{aligned}$$

Proof of (4). For $i \neq i'$, $k \neq k'$,

$$\begin{aligned} \mathrm{Cov}\,(\bar{y}_{.k}, \bar{y}_{.k'}) &= (1/r^2)[\textstyle\sum_i \mathrm{Cov}\,(y_{ik}, y_{ik'}) + \\ &\qquad \textstyle\sum_i \sum_{i'} \mathrm{Cov}\,(y_{ik}, y_{i'k'})] \\ &= (1/r^2)[\textstyle\sum_i \mathrm{Cov}\,(y_{ik}, y_{ik'}) + \sum_i \sum_{i'} 0] . \end{aligned}$$

Thus one finds that

$$- s^2/r2^{n-p} \leq \mathrm{Cov}\,(\bar{y}_{.k}, \bar{y}_{.k'}) \leq 0 .$$

This completes the proof of the corollary.

One of the essential properties of any experimental design is that of giving unbiased estimates of the treatment effects. A theorem is now stated regarding this property.

Theorem 3-4. An unbiased estimate of the effect of any treatment combination k is given by $\bar{y}_{.k} - \bar{y}_{..}$.

Proof of Theorem 3-4. Consider

$$\begin{aligned} E\,[\bar{y}_{.k} - \bar{y}_{..}] &= E\,[\bar{y}_{.k}] - E\,[\bar{y}_{..}] \\ &= \mu + t_k - \mu \\ &= t_k . \end{aligned}$$

This completes the proof of Theorem 3-4.

Since the basic design is a 2^{n-p} factorial, one wishes to estimate factorial effects. A factorial effect estimate is given by a linear contrast of the t_k. Using the notation $X_1X_2\cdots X_n$ to denote such a factorial effect, then

$$X_1X_2\cdots X_n = \sum_k \pi_k t_k,$$

where $\pi_k = \pm (1/2^\beta)$ and $\sum_k \pi_k = 1$. In the expression for π_k the + or - sign is determined by the tc, k. The value of β is determined by the type of factorial effect that is being found and the particular tc, k. If $X_1X_2\cdots X_n$ is a main effect or interaction, then all $\beta = 2^{n-p-1}$. Thus an unbiased estimate of $X_1X_2\cdots X_n$ is given by

$$\sum_k \pi_k(\bar{y}_{.k} - \bar{y}_{..}) = \sum_k \pi_k \bar{y}_{.k} .$$

The experimenter who uses a statistical design to estimate treatment effects ordinarily also requires estimates of the variance of these estimates of treatment effects. In order to obtain estimates of these variances, consider the following theorems.

<u>Theorem 3-5</u>. The variance of an estimated main effect or interaction $X_1X_2\cdots X_n$ is bounded by

$$0 \leq \text{Var}\,(X_1X_2\cdots X_n) \leq s^2(2^{n-p-1} + 1)(r2^{2n-2p-2})^{-1}.$$

The following lemma is essential to the proof of the theorem.

<u>Lemma 3-3</u>. Let $\alpha_i = a$ for $i = 1, \cdots, 2^{n-1}$ and let $\alpha_i = -a$ for $i = 2^{n-1} + 1, \cdots, 2^n$, where a is any positive constant. In the expression

$$(\Sigma \alpha_i)^2 = \sum_i \alpha_i^2 + \sum_i \sum_{\substack{i' \\ i \neq i'}} \alpha_i \alpha_{i'}$$

there are $2^n(2^n - 1)$ products $\alpha_i \alpha_{i'}$ where $i \neq i'$. Of these $2^n(2^{n-1} - 1)$ are positive and 2^{2n-1} are negative.

Proof of Lemma 3-3. Consider products $\alpha_i \alpha_{i'}$ where $i \neq i'$. There are 2^n ways to choose α_i and $2^n - 1$ ways to choose $\alpha_{i'}$, thus there are $2^n(2^n-1)$ such products.

In order for the product to be positive both α_i and $\alpha_{i'}$ are positive; this can happen in $2^{n-1}(2^{n-1} - 1)$ ways. Just as many products exist where α_i and $\alpha_{i'}$ are both negative so the number of positive products, $\alpha_i \alpha_{i'}$, is $2(2^{n-1})(2^{n-1} - 1) = 2^n(2^{n-1} - 1)$.

The product $\alpha_i \alpha_{i'}$ is negative if and only if one α is negative and one positive. This happens in $2(2^{n-1})(2^{n-1})$ ways. Thus there are a total of 2^{2n-1} negative products. Note that $2^n(2^{n-1} - 1) + 2^{2n-1} = 2^n(2^n - 1)$ total products with $i \neq i'$ as found previously. This completes the proof of Lemma 3-3.

The proof of Theorem 3-5 follows immediately.

Proof of Theorem 3-5. It is desired to get bounds on the variance of a main effect or interaction,

$$\text{Var}\ (X_1 X_2 \cdots X_n) = \sum_k \pi_k^2\ \text{Var}\ (\bar{y}_{.k}) + \sum_k \sum_{\substack{k' \\ k \neq k'}} \pi_k \pi_{k'}\ \text{Cov}\ (\bar{y}_{.k}, \bar{y}_{.k'}).$$

By Corollary 3-1,

$$\text{Var}\ (\bar{y}_{.k}) = s^2/r2^{n-p}$$

and

$$- s^2/r2^{n-p} \leq \text{Cov}\ (\bar{y}_{.k}, \bar{y}_{.k'}) \leq 0 .$$

In any contrast one-half the π_i are negative and one-half are positive. Thus 2^{n-1-p} of the π_i are negative and 2^{n-1-p} of the π_i are positive. Hence by Lemma 3-3, $2^{n-p}(2^{n-1-p} - 1)$ products $\pi_k \pi_{k'}$ where $k \neq k'$ are positive, and $2^{2n-2p-1}$ products are negative. Using these known facts, an upper bound on $\text{Var}(X_1 X_2 \cdots X_n)$ will be obtained by using $\text{Cov}(\bar{y}_{.k}, \bar{y}_{.k'}) = -\sigma^2/r2^{n-p}$ when $\pi_k \pi_{k'}$ is negative and using $\text{Cov}(\bar{y}_{.k}, \bar{y}_{.k'}) = 0$ when $\pi_k \pi_{k'}$ is positive. Note also that $|\pi_k| = 1/2^{n-p-1}$ for a main effect or interaction so the actual upper bound on the variance is

$$\text{Var}(X_1 X_2 \cdots X_n) = \sum_k \pi_k^2 (s^2/r2^{n-p}) + \sum_k \sum_{k'} \pi_k \pi_{k'} \text{Cov}(y_{.k}, y_{.k'})$$
$$\leq [s^2/(r)(2^{n-p})(2^{2n-2p-2})][2^{n-p} + 2^{2n-2p-1} + 0].$$

From this one gets

$$\text{Var}(X_1 X_2 \cdots X_n) \leq s^2(2^{n-p-1} + 1)/(r2^{2n-2p-2}) .$$

A lower bound for $\text{Var}(X_1 X_2 \cdots X_n)$ is given by

$$\text{Var}(X_1 X_2 \cdots X_n) \geq [s^2/(r)(2^{n-p})(2^{2n-2p-2})][2^{n-p} - 0 - (2^{n-p})(2^{n-p} - 1)] ,$$

or

$$\text{Var}(X_1 X_2 \cdots X_n) \geq 0 \geq s^2(2 - 2^{n-p-1})/(r2^{2n-2p-2}) .$$

This completes the proof of Theorem 3-5.

Consider the difference in the two bounds found on the variance in Theorem 3-6, Upper bound - Lower bound = d. Then

$$d = s^2(2^{n-p-1} + 1)/r2^{2n-2p-2} .$$

Clearly as $n - p$ gets large, the difference d approaches 0. For $n - p \geq 4$ the difference is not large, $d < (1/7)(s^2/r)$. Thus the bounds are sufficiently close together to be useful in finding the variance of a main effect or interaction. Note that by increasing r, the number of replications, this difference, the upper bound on the variance, is made smaller.

Now that an expression which bounds the variance of any main effect or interaction has been found, it is desired to find an estimator of this variance. That is, one now needs an estimate of s^2 which appears in the expression for Var $(X_1 X_2 \cdots X_n)$.

Consider the usual analysis of variance tableau for a blocked experimental design based on the observed responses.

Source	df	Sum of Squares	E [Sum of Squares]
Total	$r2^{n-p} - 1$	$\sum_i \sum_k (y_{ik} - y_{..})^2$	$2^{n-p} \sum b_i + r\sum_k t_k^2 + rs^2$
Blocks	$r - 1$	$\sum_i \sum_k (\bar{y}_{i.} - \bar{y}_{..})^2$	$2^{n-p} \sum_i b_i^2$
Treatments	$2^{n-p} - 1$	$\sum_i \sum_k (\bar{y}_{.k} - \bar{y}_{..})^2$	$r\sum_k t_k^2 + s^2$
Error	$(r-1)(2^{n-p} - 1)$	$\sum_i \sum_k (\bar{y}_{ik} - \bar{y}_{i.} - \bar{y}_{.k} + \bar{y}_{..})^2$	$s^2(r - 1)$

The expectations of the sums of squares in the AOV are found using the distributional properties given in Lemma 3-2, Theorem 3-3, and Corollary 3-1.

For example, consider the expectation of the total sum of squares,

$$E\left[\sum_i \sum_k (y_{ik} - \bar{y}_{..})^2\right] = E\left[\sum_i \sum_k (b_i + t_k + e_{ik})^2\right]$$

$$= E[2^{n-p} \sum_i b_i^2 + r\sum_k t_k^2 + 0 + 2 \sum_i \sum_k b_i e_{ik} +$$

$$2 \sum_i \sum_k t_k e_{ik} + \sum_i \sum_k e_{ik}^2]$$

$$= 2^{n-p} \sum_i b_i^2 + r \sum_k t_k^2 + rs^2 .$$

From this analysis of variance it is seen that to get an estimate of the variance of $X_1 X_2 \cdots X_n$ one uses the expression

$$1/(r - 1) \sum_i \sum_k (y_{ik} - \bar{y}_{i.} - \bar{y}_{.k} + \bar{y}_{..})^2$$

as an estimate of s^2. Also, if in fact all treatment effects are zero, i.e., all $t_k = 0$ then the design gives an unbiased test of treatment effects in the analysis of variance.

Infinite Model

If an experimenter can meet the assumptions necessary to use the analysis based on the infinite model, then the results in the following pages may be used. For infinite model analysis one assumes the model

$$y_{ik} = \mu + b_i + t_k + e_{ik},$$

where $i = 1, \cdots, r$, $k = 1, \cdots, 2^{n-p}$, y_{ik} is the observed response to treatment k in block i, μ is the overall mean, b_i is the effect of block i, t_k is the effect of treatment k, e_{ik} is the failure of observed response y_{ik} to be explained by μ, b_i and t_k. The errors e_{ik} are assumed to be distributed normally and independently with mean 0 and variance σ^2.

Theorem 3-6. An unbiased estimate of any main effect or interaction $(X_1 X_2 \cdots X_n)$ where $X_1 X_2 \cdots X_n = \sum_k \pi_k t_k$ is given by $\sum_k \pi_k \bar{y}_{.k}$.

Proof of Theorem 3-6. Consider

$$E[\sum_k \pi_k \bar{y}_{.k}] = \sum_k (\pi_k E[\bar{y}_{.k}])$$

$$= \sum_k (\pi_k E[\mu + t_k + \bar{b}_. + \frac{1}{r}\sum_i e_{ik}])$$

$$= \sum_k [\pi_k (\mu + t_k + \bar{b}_.)]$$

$$= (\mu + \bar{b}_.)\sum_k \pi_k + \sum_k \pi_k t_k$$

$$= \sum_k \pi_k t_k .$$

This completes the proof of Theorem 3-6.

Theorem 3-7. The variance of an estimated main effect or interaction $X_1X_2\cdots X_n$ is given exactly by

$$\text{Var}(X_1X_2\cdots X_n) = \frac{\sigma^2}{r}\sum_k \pi_k^2 = (2^n + 1)\ \sigma^2/r(2^{n-1}).$$

Proof of Theorem 3-7. Consider

$$\text{Var}(X_1X_2\cdots X_n) = \text{Var}(\sum_k \pi_k \bar{y}_{.k})$$

$$= \sum_k \pi_k^2 \text{Var}(\bar{y}_{.k}) + \sum_k \sum_{\substack{k' \\ k \neq k'}} \pi_k \pi_{k'} \text{Cov}(\bar{y}_{.k}, \bar{y}_{.k'}).$$

Now

$$\text{Var}(\bar{y}_{.k}) = E[\bar{y}_{.k} - E(\bar{y}_{.k})]^2$$

$$= E[\frac{1}{r}\sum_i e_{ik}]^2$$

$$= (1/r^2)\sum_i E[e_{ik}^2] + (1/r^2)\sum_i \sum_{\substack{i' \\ i \neq i'}} E[e_{ik}e_{i'k}]$$

$$= (1/r^2)\sum_i \sigma^2 + 0$$

$$= \sigma^2/r .$$

Also,

$$\text{Cov}(\bar{y}_{.k}, \bar{y}_{.k'}) = E[\bar{y}_{.k} - E(\bar{y}_{.k})][\bar{y}_{.k'} - E(\bar{y}_{.k'})] ,$$

So

$$\begin{aligned}\text{Cov}(y_{.k}, y_{.k'}) &= E[(1/r^2) \sum_i e_{ik} \sum_{i'} e_{i'k'}] \\ &= (1/r^2) \sum_i \sum_{i'} E[e_{ik}e_{i'k'}] \\ &= 0 .\end{aligned}$$

Thus using these two expressions one gets

$$\begin{aligned}\text{Var}(X_1X_2\cdots X_n) &= \sum_k \pi_k^2(1/r)\sigma^2 + \sum_k \sum_{k'} \pi_k\pi_{k'}(0) \\ &= (\sigma^2/r) \sum_k \pi_k^2 \\ &= (\sigma^2/r)(1/2^{n-1-p})^2(2^{n-p})2^{n-p} + 1)/2 \\ &= (2^{n-p} + 1)\sigma^2/r(2^{n-1-p}) .\end{aligned}$$

This completes the proof of Theorem 3-7.

To find an estimator of this variance of a main effect or interaction, one needs an estimate of σ^2. Consider the analysis of variance tableau for a replicated experimental design.

Source	df	Sums of Squares	E [Sums of Squares]
Total	$r2^{n-p} - 1$	$\sum_i\sum_k (y_{ik} - \bar{y}_{..})^2$	
Replications	$r - 1$	$\sum_i\sum_k (\bar{y}_{i.} - \bar{y}_{..})^2$	$(r-1)\sigma^2 + 2^{n-p}\Sigma b_i^2$
Treatments	$2^{n-p} - 1$	$\sum_i\sum_k (\bar{y}_{.k} - \bar{y}_{..})^2$	$(2^{n-p} - 1)\sigma^2 + r\Sigma t_k^2$
Error	$(r-1)(2^{n-p} - 1)$	$\sum_i\sum_k (y_{ik} - \bar{y}_{i.} - \bar{y}_{.k} + \bar{y}_{..})^2$	$\sigma^2(r-1)(2^{n-p} - 1)$

This may be found in Ostle [10].

Thus one may use $(1/[r-1][2^{n-p}-1] \sum_i \sum_k (y_{ik} - \bar{y}_{i.} - \bar{y}_{.k} + \bar{y}_{..})^2$ as an estimate of σ^2 in the expression for $\text{Var}(X_1X_2 \cdots X_n)$ in order to estimate the variance of an estimate of a main effect or interaction.

The material presented in this chapter deals with constrained randomization in a non-consecutive replication design. A method of constrained randomization for consecutive replication of 2^{n-p} factorials will be given in Chapter IV, and it will be shown that the statistical analysis based on a randomization model is identical with that just presented.

CHAPTER IV

CONSTRAINED RANDOMIZATION FOR OTHER 2^n FACTORIAL EXPERIMENTS

This chapter contains methods of constrained randomization for several types of factorial designs which are different than that given in Chapter III. Constrained randomization for 2^{n-p} factorials in randomized replication designs is discussed. In consecutive replication designs it is assumed that the sequences within all reps have the same order, Δ, and that the order of the adjacency between all reps is also Δ. That is, the same compatibility condition is in effect between all reps and within all reps.

Unblocked 2^{n-p} factorials with r replications of each tc are discussed briefly.

Split-plot designs of several types for 2^{n-p} factorials are discussed and methods of constrained randomization are presented for each. The randomization model is developed giving unbiased estimates of factorial effects. Methods for estimating the variances of main effect and interaction estimates are presented, and an analysis of variance tableau is given with estimates of variances indicated.

The 2^{n-p} Factorial in a Randomized Consecutive Replication Design

If the replications of the factorial experiment are to be run in consecutive order immediately after one another, then the constrained randomization procedure is as given below.

(1) For each replication of the experiment a single tg is chosen at random from a set of tg's, $TG(2^{n-p}, \Delta)$.

(2) Randomly assign the n real factors being investigated to the pseudo factor names $x_1, x_2, \cdots, x_n$. This is done independently in each replication.

(3) For the first replication do steps (1) and (2), then go on to the second replication. For all replicates after the first, find an "eligible" set of tc's, those tc's which are Δ adjacent to the last tc in the previous replicate. If the compatibility condition is $\leq \Delta$, then the last tc in the previous replicate is included in the eligible set.

(4) Select a tc at random from the set of eligible tc's. This tc (base 2 number) is then combined with each tc in the particular replication using vector addition modulo 2.

(5) When all r replications have been randomized in the manner of steps (1)-(4) above, then choose a tc at random from the entire set of tc's in the experiment. This tc is then combined with every tc in the entire extended sequence of all replications using vector addition modulo 2.

In order to simplify the arithmetic needed in actual practice, steps (3)-(5) above may be replaced by the following.

For the first replication do steps (1) and (2) and then pick a tc from the set of all those used in the factorial. Combine this tc with each tc in the first rep using vector addition modulo 2. After this proceed as in steps (3) and (4) until all r replications have been formed and then stop, omitting step (5).

Example 4-1. Suppose one wished to run a second replication of the experiment in Example 3-1 without a shutdown in the process. The operational sequence which was found in Example 3-1 using constrained randomization for the first rep was

101, 100, 000, 001, 011, 010, 110, 111.

To get the second operational sequence one applies constrained randomization for consecutive replication. In step (1) suppose that one picks sequence (3) from the set $TG(2^3, \Delta = 1)$,

000, 001, 011, 111, 101, 100, 110, 010.

Suppose that in step (2) the real factors are assigned to the pseudo factors as follows: $A = x_1$, $B = x_2$, $C = x_3$. The resulting sequence is still

000, 001, 011, 111, 101, 100, 110, 010.

Step (3). Since the first sequence ended with tc 111 the set of eligible base 2 numbers for use in step (4) are those whose order of adjacency with 111 gives $\Delta = 1$. This eligible set consists of the tc's 011, 101, 110. Suppose that in step (4) 101 is selected at random. The operational sequence for the second replication becomes

101, 100, 110, 010, 000, 001, 011, 111.

Note that when the second replicate immediately follows the first, the experimenter is essentially running a sequence of 16 (that is, $2^{n-p} + 2^{n-p}$) tc's with order $\Delta = 1$. A theorem important to the development of the randomization model will now be stated.

Theorem 4-1. Over all possible constrained randomizations of a given operational sequence of consecutive replications each tc appears an equal number of times in each position in each of the replications of the extended sequence.

Proof of Theorem 4-1. The proof of this theorem is an immediate extension of Theorem 3-2. Each of the consecutive replications in the extended sequence is an operational sequence. Consequently, Theorem 3-2 holds for each of the consecutive replications. This completes the proof of Theorem 4-1.

Using this theorem, the randomization model will be developed for consecutive replications of a 2^{n-p} factorial.

Since the consecutive replication design is a blocked design, the same model will be used as was used in Chapter III. The population response, y_{ijk}, and the random variable, δ^k_{ij}, are defined as in Chapter III. Then

$$y_{ik} = \sum_j \delta^k_{ij} y_{ijk}$$

$$= \mu + b_i + t_k + e_{ik},$$

where $e_{ik} = \sum_j \delta^k_{ij}(\bar{y}_{ij.} - \bar{y}_{i..})$.

Much of the material regarding the randomization model for consecutive replication will be the same as for the design in Chapter III with non-consecutive reps. The only difference in the two designs is that in non-consecutive replication the constrained randomization is done independently in the various replicates and in consecutive replication it is not done independently. Thus the distributional properties of the random variable δ_{ij}^{k} will be somewhat different. The material presented on the randomization model for consecutive replications will, however, be essentially identical to that for non-consecutive replications, and all of the results obtained in Chapter III will be valid for the consecutive replication design. Unbiased estimates of main effects and interactions may be found in the usual manner (see Theorem 3-3). In addition the expression found for the variance of such an estimate in Theorem 3-5 will be valid. The AOV and the expectations of the sums of squares will not be affected by the dependence of the randomization procedure in consecutive replications. Consequently the analysis and the interpretation of the analysis will be identical for consecutive replications under constrained randomization to the analysis and interpretation found for non-consecutive replications.

However, since constrained randomization for consecutive replications is not done independently in the various replicates. The distributional properties of δ_{ij}^{k} similar to those presented in Lemma 3-1 will now be given.

Lemma 4-1. For consecutive replications

(1) $E\,[\delta_{ij}]^2 = 1/2^{n-p}$,

(2) $E\,[\delta_{ij}^{k}\delta_{ij'}^{k}] = 0, \quad j \neq j'$,

(3) $0 \leq E\,[\delta_{ij}^{k}\delta_{i'j'}^{k'}] \leq 1/2^{n-p}, \quad i \neq i'$,

(4) $E\,[\delta_{ij}^{k}\delta_{ij}^{k'}] = 0, \quad k \neq k'$,

(5) $0 \le E[\delta^k_{ij}\delta^{k'}_{ij'}] \le 1/2^{n-p}$, $j \ne j'$, $k \ne k'$.

Proof of Lemma 4-1. The proofs of (1), (2), (4) and (5) are identical to those given in Lemma 3-1.

Proof of (3). For $i \ne i'$,

$$E[\delta^k_{ij}\delta^k_{i'j'}] = \text{Prob}(\delta^{k'}_{i'j'} = 1 \mid \delta^k_{ij} = 1)\cdot\text{Prob}(\delta^k_{ij} = 1) .$$

Now

$$0 \le \text{Prob}(\delta^k_{i'j'} = 1 \mid \delta^k_{ij} = 1) < 1,$$

so

$$0 \le E[\delta^k_{ij}\delta^k_{i'j'}] \le 1/2^{n-p} .$$

This completes the proof of Lemma 4-1.

A lemma containing results basic to the randomization model for consecutive replication follows.

Lemma 4-2. Where $s^2 = \sum_j (\bar{y}_{ij.} - \bar{y}_{i..})^2$,

(1) $E[e_{ik}] = 0$,

(2) $E[e^2_{ik}] = s^2/2^{n-p}$,

(3) $E[e_{ik}e_{i'k'}] = 0$, $i \ne i'$,

(4) $-s^2/2^{n-p} \le E(e_{ik}e_{ik'}] \le 0$, $k \ne k'$.

Proof of Lemma 4-2. In this lemma (1), (2) and (4) follow immediately, being identical with results (1), (2) and (4) in Lemma 3-2.

Proof of (3). For $i \ne i'$,

$$E[e_{ik}e_{i'k'}] = E[\sum_j \sum_{j'} \delta_{ij}^k \delta_{i'j'}^k (\bar{y}_{ij.} - \bar{y}_{i..})(\bar{y}_{i'j'.} - \bar{y}_{i'..})].$$

Using Lemma 4-1, (3), and the fact that $\sum_j (\bar{y}_{ij.} - \bar{y}_{i..}) = 0$,

$$E[e_{ik}e_{i'k'}] = 0 .$$

This completes the proof of Lemma 4-2.

This lemma contains results identical to those in Lemma 3-2. Also the same observation model was derived for consecutive replication as for non-consecutive replication. Consequently since the development of the randomization model was based entirely on Lemma 3-2, the same development will be valid in the model for consecutive replication based on Lemma 4-2. Thus the statistical analyses for consecutive and for non-consecutive replication are done in the same manner and the results have the same statistical properties.

This completes the presentation and discussion of 2^{n-p} factorials in consecutive replication designs.

Completely Random 2^{n-p} Factorials

Usually when the treatment combinations are to be run sequentially one would block them into replications if possible. Then the blocking would provide protection against any "learning" effect or gradual change in the process being investigated which was not recognized and taken into account. Because of this the unblocked design is mentioned only briefly.

If there is no reason to block the experiment but rather one only desires that each tc be replicated, say r times, then one must form an operational sequence containing $r2^{n-p}$ tc's. In this situation the sets of transformation generators provided in the appendix would not be utilized. One would need to find a set of tg's in which each tc was encountered r times. These encounters could be isolated or any combination of them could be sequentially adjacent if the compatibility condition reads $\leq \Delta$.

With a set of transformation generators in hand one may simply follow the same method of constrained randomization as originally presented in Chapter III.

Several possible tg's for a 2^3 with two replications of each tc and with $\Delta \leq 1$ are listed.

(1) 000,000,001,001,011,011,010,010,110,110,100,100,101,101,111,111 .

Note that this is equivalent to an experiment with repeated sampling.

(2) 000,001,011,010,110,100,101,111,111,101,100,110,010,011,001,000 .

(3) 000,001,101,100,101,100,110,010,011,111,110,111,011,001,000,010 .

Obviously many more possibilities exist and may be found by the methods presented in Chapter II.

Since the use of a completely random design in a sequential experiment is rather unlikely, the details of the randomization model are not presented. If it is deemed unnecessary to block the design, then perhaps the assumption of an infinite model will be reasonable as well.

Split-Plot Designs

Split-plot designs of many types can be visualized by considering various compatibility conditions on the tc's. There might be one compatibility condition on the main-plot treatment combinations, another on the subplot treatment combinations, and still a third condition relating to the adjacency of main-plots. The order of adjacency of main-plots is determined by the main-plot treatments as well as the sub-plot treatment combinations which are made adjacent by the junction of the main-plots. Various classes of designs will be discussed for split-plot designs for 2^n factorials with three possible compatibility conditions. The order of adjacency of main-plots will be denoted by Δ. The order of the sequence of main-plot treatments will be denoted by Δ_m, and the order of the sequence of sub-plot treatment combinations will be denoted by Δ_s, necessarily $\Delta \geq \Delta_m$.

If $\Delta_m \leq \Delta < \Delta_m + \Delta_s$ then the class of designs will be called class (1) split-plot designs. If $\Delta_m + \Delta_s \leq \Delta$ then the class of designs will be called class (2) split-plot designs. In class (1) split-plot designs, the randomization of the sub-plot tc's is not independent of the sub-plot tc's in the adjacent main-plots. This dependency causes one to use consecutive randomization procedures for sub-plot randomization. In class (2) split-plot designs the randomization of sub-plot tc's is done independently within each main-plot and one may use non-consecutive randomization for sub-plots.

A special case of split-plot designs, called class (0), will be discussed first as a particular type of ordinary non-consecutive replication discussed earlier in Chapter III. If the main-plot treatment consists of a single factor or of more than one factor applied in a split- ... -split-plot manner, then the design is a special case of the previously presented material on non-consecutive replication. These designs are a subset of the previously presented material. The set of possible arrays of tc's is a subset of the possible arrays of tc's obtained for ordinary blocked designs. Thus by restricting the tg's to those which list pseudo factors in a split-plot manner the earlier discussion may be utilized. A short discussion including an example of this situation is given relating these split-plots to the presentation in Chapter III.

Sets of tg's for a number of 2^n factorial experiments in a split-plot design with order Δ may be found in Appendix B. If a set of tg's is not found for the particular value of n and Δ desired then such a set of tg's may be found using the methods in Chapter II.

Constrained randomization of a 2^n factorial experiment in a split-plot design of class (0) may be done as follows:

(1) A single tg is chosen at random from the set of tg's identified for split-plot designs for each replication of the design.

(2) For each replication one assigns the main-plot factor to x_1 in the tc's $(x_1 x_2 \cdots x_n)$. Then if there is a second split-plot factor it is assigned to x_2, etc., until the split-plot factors have been assigned to the first factor names. Then one randomly assigns the remaining sub-plot factors to the remaining pseudo factors.

(3) For each replication randomly choose one of the base 2 numbers which represents a tc used in the factorial experiment. Combine this number with each tc using vector addition modulo 2.

An example is given showing this technique.

Example 4-1. Consider a 2^3 with factor A as a main-plot treatment with $\Delta \leq 2$. Following step (1) in the constrained randomization procedure, a tg is chosen from the set of tg's numbers 1-54 in Table III in Appendix B. Suppose it is tg number (52):

000,011,001,010,111,101,110,100. Step (2) requires that the main-plot factor A be assigned to x_1. Then randomly assign B to x_3 and C to x_2. The operational sequence is now 000,011,010,001,111,110,101, 100.

In step (3) suppose that tc 110 is selected. When 110 is added component-wise modulo 2 to each tc in the sequence one obtains the operational sequence actually used in the experiment. It is 110,101, 100,111,001,000,011,010. Note that factor A is still in a split plot and that the sequence is of order $\Delta \leq 2$. This result is formalized in the following theorems.

<u>Theorem 4-2</u>. The sequence of tc's resulting from constrained randomization for a split-plot design is an operational sequence.

<u>Proof of Theorem 4-2</u>. This theorem is a corollary of Theorem 3-1, since constrained randomization for a split-plot design is a particular case of constrained randomization for a 2^{n-p} factorial in blocks.

<u>Theorem 4-3</u>. Over all possible constrained randomizations of a given tg each tc appears an equal number of times in each position in the sequence.

<u>Proof of Theorem 4-3</u>. This theorem is a corollary of Theorem 3-2.

Theorems 4-2 and 4-3 give results identical to Theorems 3-1 and 3-2. Thus the analysis of results from these split-plot designs can be based entirely upon the randomization model presented in Chapter III.

In class (1) and class (2) split-plot designs there are three compatibility conditions. The order of the sequence of main-plot tc's, Δ_m, the order of adjacency of main-plots, Δ, and the order of the sequences of sub-plot tc's within each main-plot, Δ_s, are these compatibility conditions. In either case, $\Delta \geq \Delta_m$. For either class (1) or class (2) designs the randomization of main-plot treatment combinations may be done either before or after the randomization of sub-plot treatments and the two randomization procedures are done independently.

The constrained randomization procedure for main-plot tc's is done according to the method given for randomized blocks in Chapter III.

In class (2) split-plot designs where $\Delta_m + \Delta_s \leq \Delta$ the relation of Δ_s and Δ_m to Δ implies that the constrained randomization for sub-plot tc's is done independently within each main-plot. Thus, for each main-plot the constrained randomization of sub-plot tc's is done according to the method given for non-consecutive replication of 2^n factorials given in Chapter III.

In class (1) split-plot designs where $\Delta_m \leq \Delta < \Delta_m + \Delta_s$, in order that randomization of sub-plot treatment combinations be independent of the randomization of main-plot treatments, the sub-plot randomization must be done according to the method or randomization for consecutive blocks given earlier in this chapter. The set of "eligible" tc's in step (3) will be restricted to those which have order of adjacency $\Delta - \Delta_m$ with respect to the last sub-plot tc in the previous main-plot. Using this restriction on the set of eligible tc's for the consecutive replicate randomization of sub-plot treatments, it is seen that either sub-plot treatments or main-plot treatments may be assigned to their respective experimental units first, and in this manner the independence of the two procedures is insured.

Due to the similarity of non-consecutive and consecutive replication designs the development of the randomization model for classes (1) and (2) split-plot designs will be done simultaneously.

For either class (1) or class (2) split-plot designs let y_{ighjk} denote the population response in a split-plot design. The subscripts refer to the following: replications, $i = 1, \cdots, r$; main-plots, $g = 1, \cdots, 2^m$; sub-plots, $h = 1, \cdots, 2^{n-m}$; main-plot treatment combinations, $j = 1, \cdots, 2^m$; and sub-plot treatment combinations, $k = 1, \cdots, 2^{n-m}$. Under the assumption of additivity of treatment effects one may write $y_{ighjk} = t_{jk} + x_{igh}$. Consider the identity

$$y_{ighjk} = (\bar{t}_{..} + \bar{x}_{...}) + (\bar{x}_{i..} - \bar{x}_{...}) + (\bar{t}_{j.} - \bar{t}_{..}) + (\bar{x}_{ig.} - \bar{x}_{i..})$$

$$+ (\bar{t}_{.k} - \bar{t}_{..}) + (t_{jk} - \bar{t}_{j.} - \bar{t}_{.k} + \bar{t}_{..}) + (x_{igh} - \bar{x}_{ig.}) .$$

Let $\mu = \bar{t}_{..} + \bar{x}_{...}$, $b_i = \bar{x}_{i..} - \bar{x}_{...}$, $t_j = \bar{t}_{j.} - \bar{t}_{..}$, $s_k = \bar{t}_{.k} - \bar{t}_{..}$ and $(ts)_{jk} = t_{jk} - \bar{t}_{j.} - \bar{t}_{.k} + \bar{t}_{..}$. Then

$$y_{ighjk} = \mu + b_i + t_j + (\bar{x}_{ig.} - \bar{x}_{i..}) + s_k + (ts)_{jk} + (x_{igh} - \bar{x}_{ig.}) .$$

In the real world one only observes one yield from a particular experimental unit. Let y_{ijk} denote the observed yield of treatment jk on replication i. To relate y_{ijk} to y_{ighjk} consider the random variables defined as follows:

$$\delta_{ig}^{j} = \begin{cases} 1 & \text{if tc } j \text{ is on main-plot } g \text{ of rep } i, \\ 0 & \text{otherwise ;} \end{cases}$$

$$\gamma_{igh}^{k} = \begin{cases} 1 & \text{if tc } k \text{ is on sub-plot } h \text{ in main-plot } g \text{ of rep } i , \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\sum_h \gamma_{igh}^k = \sum_g \delta_{ig}^j = \sum_k \gamma_{igh}^k = \sum_j \delta_{ig}^j = 1$. Since the constrained randomization employed in the main-plots is independent of that employed in the sub-plots, the random variables δ and γ are independent.

Then

$$\begin{aligned} y_{igh} &= \sum_g \sum_h \delta_{ig}^j \gamma_{igh}^k y_{ighjk} \\ &= \mu + b_i + t_j + \sum_g \delta_{ig}^j (\bar{x}_{ig.} - \bar{x}_{i..}) + s_k + (ts)_{jk} \\ &\quad + \sum_g \sum_h \delta_{ig}^j \gamma_{igh}^k (x_{igh} - \bar{x}_{ig.}) \ . \end{aligned}$$

Thus

$$y_{ijk} = \mu + b_i + t_j + e_{ij} + s_k + (ts)_{jk} + \eta_{ijk}$$

where $e_{ij} = \sum_g \delta_{ig}^j (\bar{x}_{ig.} - \bar{x}_{i..})$ and $\eta_{ijk} = \sum_g \sum_h \delta_{ig}^j \gamma_{igh}^k (x_{igh} - \bar{x}_{ig.})$.

The following means are expressed in terms of the above model:

$$\bar{y}_{ij.} = \mu + b_i + t_k + e_{ij}$$

$$\bar{y}_{.j.} = \mu + t_j + (1/r) \sum_i e_{ij}$$

$$\bar{y}_{...} \equiv \mu$$

$$\bar{y}_{i.k} = \mu + b_i + s_k + (1/2^m) \sum_j \eta_{ijk}$$

$$\bar{y}_{..k} = \mu + s_k + (1/r2^m) \sum_i \sum_j \eta_{ijk}$$

$$\bar{y}_{i..} = \mu + b_i$$

The following lemma giving the distributional properties of the random variables δ_{ig}^j and γ_{igh}^k will be used in developing the randomization model.

Lemma 4-3. The following expectations may be obtained:

(1) $E[\delta^j_{ig}] = E[\delta^j_{ig}]^2 = 1/2^m$;

(2) $E[\delta^j_{ig}\delta^j_{ig'}] = 0, \quad g \neq g'$;

(3) $0 \leq E[\delta^j_{ig}\delta^j_{i'g'}) \leq 1/2^m, \quad i \neq i'$;

(4) $E[\delta^j_{ig}\delta^{j'}_{ig}] = 0, \quad j \neq j'$;

(5) $0 \leq E[\delta^j_{ig}\delta^{j'}_{ig'}] \leq 1/2^m, \quad j \neq j', \quad g \neq g'$;

(6) $E[\gamma^k_{igh}] = E[\gamma^k_{igh}]^2 = 1/2^{n-m}$;

(7) $E[\gamma^k_{igh}\gamma^k_{igh'}] = 0, \quad h \neq h'$;

(8) $0 \leq E[\gamma^k_{igh}\gamma^{k'}_{i'g'h'}] \leq 1/2^{n-m}, \quad ig \neq i'g'$;

(9) $E[\gamma^k_{igh}\gamma^{k'}_{igh}] = 0, \quad k \neq k'$;

(10) $0 \leq E[\gamma^k_{igh}\gamma^{k'}_{igh'}] \leq 1/2^{n-m}, \quad k \neq k', \quad h \neq h'$;

(11) $E[\delta^j_{ig}\gamma^k_{igh}]^2 = E[\delta^j_{ig}\gamma^k_{igh}] = 1/2^n$.

Proof of Lemma 4-3. The constrained randomization is done independently in main-plots and sub-plots. Thus the results stated in (1)-(10) follow immediately from Lemma 3-1 or Lemma 4-1, depending on whether the randomization was done for consecutive or non-consecutive replications.

Proof of (11). To complete the proof of the lemma we have

$$E[\delta^j_{ig}\gamma^k_{igh}]^2 = E[\delta^j_{ig}]^2 \, E[\gamma^k_{igh}]^2 = 1/2^n .$$

The following lemma relates the distributional properties of the random variables δ_{ig}^{j} and γ_{igh}^{k} to the quantities e_{ij} and η_{ijk} in the observation model.

Lemma 4-4. In this lemma let $S^2 = \sum_g (\bar{x}_{ig.} - \bar{x}_{i..})^2$ and $S_s^2 = \sum_g \sum_h (x_{igh} - \bar{x}_{ig.})^2$. Then

(1) $E[e_{ij}] = 0$,

(2) $E[e_{ij}^2]^2 = S^2/2^m$,

(3) $E[e_{ij}e_{i'j'}] = 0$, $i \neq i'$,

(4) $- S^2/2^m \leq E[e_{ij}e_{ij'}] \leq 0$, $j \neq j'$,

(5) $E[\eta_{ijk}] = 0$,

(6) $E[\eta_{ijk}^2] = S_s^2/2^n$,

(7) $E[\eta_{ijk}\eta_{i'j'k'}] = 0$, either $j \neq j'$, or $i \neq i'$,

(8) $- S_s^2/2^n \leq E[\eta_{ijk}\eta_{ijk'}] \leq 0$, $k \neq k'$.

Proof of Lemma 4-4. Statements (1)-(4) follow immediately from Lemma 4-3 in the same manner that the results in Lemma 3-2 were obtained from Lemma 3-1.

Proof of (5). Consider

$$E[\eta_{ijk}^2] = \sum_g \sum_h (x_{igh} - \bar{x}_{ig.}) E[\delta_{ig}^{j}\gamma_{igh}^{k}]$$
$$= 0 .$$

Proof of (6). Consider

$$E[\eta_{ijk}^2] = \sum_g\sum_h (x_{igh} - \bar{x}_{ig.})^2 E[\delta_{ig}^j]^2 E[\gamma_{igh}^k]^2 \cdot$$

$$+ \sum_g\sum_{g'}\sum_h\sum_{h'}{}_{g\neq g'} (x_{igh} - \bar{x}_{ig.})(x_{ig'h'} - \bar{x}_{ig'.}) E[\delta_{ig}^j \delta_{ig'}^j] \cdot$$

$$E[\gamma_{igh}^k \gamma_{ig'h'}^k] + \sum_g\sum_h\sum_{h'}{}_{h\neq h'} (x_{igh} - \bar{x}_{ig.})(x_{igh'} - \bar{x}_{ig.}) \cdot$$

$$E[\delta_{ig}^j]^2 \, E[\gamma_{igh}^k \gamma_{igh'}^k] .$$

Using Lemma 4-3, (1), (2), (6) and (7),

$$E[\eta_{ijk}^2] = (1/2^n) \sum_g\sum_h (x_{igh} - \bar{x}_{ig.})^2 + 0 + 0 + 0$$

$$= S_s^2/2^n .$$

Proof of (7). For $j \neq j'$,

$$E[\eta_{ijk}\eta_{i'j'k'}] = \sum_g\sum_{g'}\sum_h\sum_{h'} (x_{igh} - \bar{x}_{ig.})(x_{i'g'h'} - \bar{x}_{i'g'.}) \cdot$$

$$E[\delta_{ig}^j \delta_{i'g'}^{j'}] \, E[\gamma_{igh}\gamma_{i'g'h'}] .$$

Using Lemma 4-3, (5) and (8) and the fact that $\sum_h (x_{igh} - \bar{x}_{ig.}) = 0$, this expectation is shown to be 0.

Proof of (7). For $i \neq i'$, $E[\eta_{ijk}\eta_{i'j'k'}] = 0$ in the same manner as when $j \neq j'$ using Lemma 4-3, (3), and (8).

Proof of (8). For $k \neq k'$,

$$E[\eta_{ijk}\eta_{ijk'}] = \sum_g\sum_h (x_{igh} - \bar{x}_{ig.})^2 E[\delta_{ig}^j]^2 E[\gamma_{igh}^k]^2 +$$

$$\sum_g\sum_h\sum_{h'}{}_{h\neq h'} (x_{igh} - \bar{x}_{ig.})(x_{igh'} - \bar{x}_{ig.}) E[\delta_{ig}]^2 E[\gamma_{igh}^k \gamma_{igh'}^{k'}] +$$

$$\sum_g\sum_{g'}\sum_h\sum_{h'}{}_{g\neq g'} (x_{igh} - \bar{x}_{ig.})(x_{ig'h} - \bar{x}_{ig'.}) E[\delta_{ig}^j \delta_{ig'}^j] E[\gamma_{igh}^k \gamma_{ig'h}^{k'}].$$

Using Lemma 4-3, (1), (2), and (9),

$$E[\eta_{ijk}\eta_{ijk'}] = 0 + (1/2^m) \sum_g \sum_h \sum_{\substack{h' \\ h \neq h'}} (x_{igh} - \bar{x}_{ig.})(x_{igh'} - \bar{x}_{ig.}) \cdot$$

$$E[\gamma_{igh}^{k} \gamma_{igh'}^{k'}] + 0 + 0 \ .$$

Now using (10) and the fact that $\sum_{\substack{h' \\ h \neq h'}} (x_{igh'} - \bar{x}_{ig.}) = - (x_{igh} - \bar{x}_{ig.})$, bounds are found for $E[\eta_{ijk}\eta_{ijk'}]$,

$$- S^2/2^n \leq E[\eta_{ijk}\eta_{ijk'}] \leq 0 \ .$$

Now the analysis of main-plot treatments is based entirely on the means $\bar{y}_{ij.} = \mu + b_i + t_k + e_{ik}$. Thus, in view of Lemma 4-4, (1)-(4) and this "main-plot model" one sees that the analysis of main-plot effects, $X_1 X_2 \cdots X_m$, is entirely the same as that presented in Chapter III. Thus an unbiased estimate of any main-plot effect, $X_1 X_2 \cdots X_m$, is given by $\sum_j \pi_j \bar{y}_{.j.}$. When $X_1 X_2 \cdots X_m$ is a main effect or interaction $\pi_j = \pm 1/2^{m-1}$ and

$$0 \leq \text{Var}\ (X_1 X_2 \cdots X_m) \leq S^2(2^{m-1} + 1)/(r2^{2m-2}) \ ,$$

where $S^2 = \sum_g (\bar{x}_{ig.} - \bar{x}_{i..})^2$. An estimate of S^2 may be found in the split-plot analysis of variance tableau,

$$\hat{S}^2 = \sum_i \sum_j \sum_k (\bar{y}_{ij.} - \bar{y}_{.j.} - \bar{y}_{i..} + \bar{y}_{...})^2/(r - 1)(2^{n-m}) \ .$$

The analysis of subplot treatments is based on the means $\bar{y}_{i.k} = \mu + b_i + s_k + (1/2^m) \sum_j \eta_{ijk}$. The analysis of this "sub-plot model" is based on Lemma 4-4, (5)-(8). Thus, one sees that this analysis of sub-plot effects, $X_{m+1} X_{m+2} \cdots X_n$, is also entirely the same as that presented in Chapter III. Consequently, an unbiased estimate of any sub-plot effect $X_{m+1} X_{m+2} \cdots X_n$ is given by $\sum_k \pi_k \bar{y}_{..k}$. For a sub-plot main effect or interaction $\pi_k = \pm 1/2^{n-m-1}$ and one obtains

$$0 \le \mathrm{Var}\,(X_{m+1}X_{m+2} \cdots X_n) \le S_s^2(2^{n-m-1} + 1)/r2^{2n-2} ,$$

where $S_s^2 = \sum_g \sum_h (x_{igh} - \bar{x}_{ig.})^2$. An estimate of S_s^2 is found from the split-plot AOV,

$$\hat{S}_s^2 = \sum_i \sum_j \sum_k (y_{ijk} - \bar{y}_{ij.} - \bar{y}_{.jk} + \bar{y}_{.j.})^2/(r - 1) .$$

Factorial interaction effects $X_1X_2 \cdots X_n$, are given by contrasts of $(ts)_{jk}$. That is, $X_1X_2 \cdots X_n = \sum_j \sum_k \pi_{jk}(ts)_{jk}$. The following theorem indicates a method of finding an unbiased estimate of such an effect.

<u>Theorem 4-4.</u> An unbiased estimate of $(ts)_{jk}$ is given by

$$\bar{y}_{.jk} - \bar{y}_{.j.} - \bar{y}_{..k} + \bar{y}_{...} .$$

<u>Proof of Theorem 4-4.</u> Consider

$$E\,[\bar{y}_{.jk} - \bar{y}_{.j.} - \bar{y}_{..k} + \bar{y}_{...}] = \mu + t_j + s_k + (ts)_{jk} - (\mu + t_j) - (\mu + s_k) + \mu$$

$$= (ts)_{jk} .$$

This completes the proof.

An upper bound on the variance of such an interaction estimate may be found from the following expressions,

$$\mathrm{Var}\,(X_1X_2 \cdots X_n) = \mathrm{Var} \sum_j \sum_k \pi_{jk}(\bar{y}_{.jk} - \bar{y}_{.j.} - \bar{y}_{..k} + \bar{y}_{...})$$

$$= \mathrm{Var} \sum_j \sum_k \pi_{jk}[(ts)_{jk} + \frac{1}{r}\sum_i \eta_{ijk} - \frac{1}{r2^m}\sum_i \sum_j \eta_{ijk}]$$

$$= \frac{1}{r^2} E[\sum_i \sum_j \sum_k \pi_{jk}(\frac{1}{r}\eta_{ijk} - \frac{1}{2^m}\sum_{j'} \eta_{ij'k})]^2$$

$$= \frac{1}{r^2} E\left[\sum_i\sum_j\sum_k\sum_{i'}\sum_{j'}\sum_{k'} \pi_{jk}\pi_{j'k'}\left(\eta_{ijk} - \frac{1}{2^m}\sum_{j''}\eta_{ij''k}\right)\cdot\left(\eta_{i'j'k'} - \frac{1}{2^m}\sum_{j'''}\eta_{i'j'''k'}\right)\right]$$

$$= \frac{1}{r^2 2^m}\Big[\sum_i\sum_j\sum_k \pi_{jk}^2 (2^m - 1) E[\eta_{ijk}^2] + \sum_i\sum_j\sum_k\sum_{k'}_{k\neq k'} \pi_{jk}\pi_{jk'}(2^m - 1) E[\eta_{ijk}\eta_{ijk'}] - \sum_i\sum_j\sum_{j'}\sum_k{}_{j\neq j'} \pi_{jk}\pi_{j'k} E[\eta_{ijk}^2] - \sum_i\sum_j\sum_{j'}\sum_k\sum_{k'}{}_{j\neq j',\, k\neq k'} \pi_{jk}\pi_{j'k'} E[\eta_{ijk}\eta_{ijk'}]\Big].$$

Now $E[\eta_{ijk}^2] = \frac{S_s^2}{2^n}$ and $-\frac{S_s^2}{2^n} \le E[\eta_{ijk}\eta_{ijk'}] \le 0$, for $k \neq k'$, so an upper bound on the variance is given by

$$\mathrm{Var}\,(X_1X_2 \cdots X_n) \le \frac{1}{r2^m}\frac{S_s^2}{2^n}\Big[(2^m - 1)\sum_j\sum_k \pi_{jk}^2 - \sum_j\sum_{j'}\sum_k{}_{j\neq j'} \pi_{jk}\pi_{j'k} + (2^m - 1)\sum_j\sum_k\sum_{k'}{}_{k\neq k'} A\,\pi_{jk}\pi_{jk'} - \sum_j\sum_{j'}\sum_k\sum_{k'}{}_{j\neq j',\, k\neq k'} B\,\pi_{jk}\pi_{j'k'}\Big],$$

where

$$A = \begin{cases} 0 & \text{if } \pi_{jk}\pi_{jk'} \ge 0 \\ -1 & \text{if } \pi_{jk}\pi_{jk'} < 0, \end{cases}$$

and

$$B = \begin{cases} -1 & \text{if } \pi_{jk}\pi_{j'k'} > 0 \\ 0 & \text{if } \pi_{jk}\pi_{j'k'} \le 0. \end{cases}$$

An analysis of variance tableau for a split-plot design is given below. This is useful for testing hypotheses and for finding estimates

AOV

Source	df	Sum of Squares	E [Sum of Squares]
Total	$r2^n - 1$	$\sum_i\sum_j\sum_k (y_{ijk} - \bar{y}_{...})^2$	$2^n \sum_i b_i^2 + r2^{n-m} \sum_j t_j^2 + r2^{n-m}S^2 + r2^m \sum_k s_k^2 + r \sum_j\sum_k (ts)_{jk}^2 + r S_s^2$
Blocks	$5 - 1$	$\sum_i\sum_j\sum_k (\bar{y}_{i..} - \bar{y}_{...})^2$	$2^n \sum_i b_i^2$
Main-plot Treatment	$2^m - 1$	$\sum_i\sum_j\sum_k (\bar{y}_{.j.} - \bar{y}_{...})^2$	$r2^{n-m} \sum_j t_j^2 + 2^{n-m}S^2$
Error (a)	$(r - 1)(2^m - 1)$	$\sum_i\sum_j\sum_k (\bar{y}_{ij.} - \bar{y}_{.j.} - \bar{y}_{i..} + \bar{y}_{...})^2$	$(r - 1)\, 2^{n-m}\, S^2$
Sub-Plot Treatments	$2^{n-m} - 1$	$\sum_i\sum_j\sum_k (\bar{y}_{..k} - \bar{y}_{...})^2$	$r2^m \sum_k s_k^2 + \frac{1}{2^m} S_s^2$
Sub-plot Treatments x Main-plot Treatments	$(2^m - 1)(2^{n-m} - 1)$	$\sum_i\sum_j\sum_k (\bar{y}_{.jk} - \bar{y}_{.j.} - \bar{y}_{..k} + \bar{y}_{...})^2$	$r \sum_j\sum_k (ts)_{jk}^2 + \frac{2^m - 1}{2^m} S_s^2$
Error (b)	$2^m(r-1)(2^{n-m} - 1)$	$\sum_i\sum_j\sum_k (y_{ijk} - \bar{y}_{ij.} - \bar{y}_{.jk} + \bar{y}_{.j.})^2$	$(r - 1)\, S_s^2$

of S^2 and S_s^2 which are needed in order to estimate the variances of estimates of factorial effects found previously.

The expectations of the sums of squares may be found using Lemma 4-4. Consider for example the expectation of the sub-plot treatment sum of squares.

$$E\left[\sum_i\sum_j\sum_k (\bar{y}_{..k}-\bar{y}_{...})^2\right] = E\left[\sum_i\sum_j\sum_k \left(s^2 - \frac{2s\sum_{i'}\sum_{j'}\eta_{i'j'k}}{r2^m} + \frac{1}{r^2 2^{2m}}\sum_{i'}\sum_{j'}\sum_{i''}\sum_{j''}\eta_{i'j'k}\eta_{i''j''k}\right)\right]$$

$$= \sum_i\sum_j\sum_k \left\{s^2 - 0 + \frac{1}{r^2 2^{2m}}\left(\sum_{i'}\sum_{j'} E[\eta^2_{i'j'k}] + \sum_{i'}\sum_{\substack{j'}}\sum_{\substack{j'' \\ j'\neq j''}} E[\eta_{i'j'k}\eta_{i'j''k}] + \sum_{\substack{i'}}\sum_{\substack{i'' \\ i'\neq i''}}\sum_{j'}\sum_{j''} E[\eta_{i'j'k}\eta_{i''j''k}]\right)\right\}.$$

Using Lemma 4-4, (6) and (7),

$$E\,[\text{Sub-plot treatment sum of squares}] =$$

$$= \sum_i\sum_j\sum_k \left(s^2 + \frac{1}{r^2 2^{2m}}\sum_{i'}\sum_{j'}\frac{S_s^2}{2^n} + 0 + 0\right)$$

$$= r2^m s^2 + \frac{1}{2^m} S_s^2 \,.$$

This concludes the development of constrained randomization for split-plot designs. A summary of the material developed for constrained randomization procedures is given in the following chapter with one method for extending the results to factorials with factors at more than two levels.

CHAPTER V

SUMMARY AND EXTENSIONS

In this thesis methods of constrained randomization are given for 2^{n-p} factorials in several basic experimental designs. The randomization procedure is restricted by a compatibility condition on adjacent treatment combinations which requires that the number of factor levels which may be changed from tc to tc be equal to Δ, where $\Delta < n-p$. Constrained randomization methods are given for blocked 2^{n-p} factorials in the situation where there is no compatibility condition between adjacent blocks, called non-consecutive replication, and for consecutive replication of 2^{n-p} factorials where tc's which are made adjacent by running the blocks in consecutive order must also satisfy the compatibility condition. The statistical analysis and the interpretation of the results of these designs, based on a randomization model are shown to be identical.

Split-plot designs could have three compatibility conditions, one on adjacent main-plot treatments, a second on adjacent sub-plot treatments within main-plots, and a third condition regarding the adjacency of sub-plot treatments between adjacent main-plots. Methods of constrained randomization are given for 2^{n-p} factorials in three classes of split-plot designs. The three classes of split-plot designs discussed include a class of designs with only one compatibility condition regarding all factors in a tc, while the second class of split-plot designs presented has separate compatibility conditions on main-plot tc's and on sub-plot tc's within main-plots with no requirement on adjacent sub-plot tc's between main-plots. The third class of split-plot designs also has a compatibility condition on adjacent sub-plot tc's between main-plots.

It is interesting that the methods of constrained randomization given for the split-plot designs and also a large portion of the development of the related split-plot randomization model analysis follow as rather straightforward extensions of the previous material concerning 2^{n-p} factorials in blocks.

Examples have been given illustrating the methods of constrained randomization for the various 2^{n-p} factorial designs discussed. These examples discuss and indicate the use of the tables of transformation generators of 2^{n-p} factorial tc's listed in Appendix B, which are used in the constrained randomization procedure.

The material presented in the previous chapters may be immediately extended to factorials of the form $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where p_i and n_i are non-distinct natural numbers, provided that a change of levels in any given factor is counted as one change in determining the order of adjacency of tc's. If, however, the order of adjacency of tc's is determined by the number of levels each factor in the tc changes, then the methods of constrained randomization presented in this thesis are not applicable since they would not preserve this sort of order relation on the operational sequence of tc's. Thus if the determination of the order of adjacency of tc's is done in a manner which discriminates number of levels changed by any given factor or utilizes any type of "degree of difficulty" function for any given factor other than simply denoting a change in levels being made, then in order to arrive at a random operational sequence of tc's some method other than those presented in this thesis would need to be found.

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APPENDIX A

COMPUTER PROGRAM GIVING A MAXIMAL SET OF TRANSFORMATION GENERATORS FOR A 2^3 FACTORIAL WITH $\Delta \leq 2$

```
      DIMENSION IA(16),IND(300,6),
       NM(6,9),NN(1800,9)
40    FORMAT (16I5)
      MP = 6
      M = 8
      MM = M + 1
      READ (5,40)(IA(I),I = 1, M)
      K = 1
      NN(1,1) = 1
      NN(1,2) = IA(1)
      DO 102 I1 = 2,M
      NN(K,3) = IA(I1)
      II = 3
      CALL CHECK (NN,II,K,MD,LD)
      IF(LD.EQ.6) GO TO 102
      IF(MD.EQ.1) GO TO 3
      GO TO 102
3     DO 103 I2 = 2,M
      NN(K,4) = IA(I2)
      II = 4
      CALL CHECK (NN,II,K,MD,LD)
      IF(LD.EQ.6) GO TO 103
      IF(MD.EQ1) GO TO 4
      GO TO 103
      .
      .
      .
8     DO 108 I7 = 2,M
      NN(K,9) = IA(I7)
      II = 9
      CALL CHECK (NN,II,K,MD,LD)
      IF(LD.EQ.6) GO TO 108
      IF(MD.EQ.1) GO TO 75
      GO TO 108
75    KK = K
      K = K + 1
      NN(K,1) = K
      DO 50 J = 2,MM
50    NN(K,J) = NN(KK,J)
108   CONTINUE
107   CONTINUE
106   CONTINUE
105   CONTINUE
104   CONTINUE
103   CONTINUE
102   CONTINUE
      IF(MD.EQ.1) GO TO 199
      K = K - 1
199   DO 200 JJ = 1,K
200   WRITE (6,202)(NN(JJ,J),
       J = 1,MM)
202   FORMAT (1X,17I6)
      LL = 1
      L = 1
299   DO 310 J = 1,MM
310   NM(1,J) = NN(LL,J)
      DO 300 I = 2,MM
      NMA = NM(1,I)/100
      NMB = (NM(1,I) - NMA*100)/10
      NMC = NM(1,I) - NMA*100 -
       NMB 10
      NM(2,I) = NMA*100 + NMC*10 +
       NMB
      NM(3,I) = NMB*100 + NMA*10 +
       NMC
      NM(4,I) = NMB*100 + NMC*10 +
       NMA
      NM(5,I) = NMC*100 + NMA*10 +
       NMB
      NM(6,I) = NMC*100 + NMB*10 +
       NMA
300   CONTINUE
      DO 320 I = 1,MP
      DO 330 IK = 1,K
      IF(NN(IK,1).EQ.0) GO TO 330
      DO 340 J = 2,MM
      IF(NM(I,J).EQ.NN(IK,J)) GO
       TO 321
      GO TO 330
321   IF(J.EQ.9) GO TO 355
340   CONTINUE
355   IND(L,I) = NN(IK,1)
      IF(I.GE.2) GO TO 319
      WRITE (6,501)(NN(IK,JJ),
       JJ = 1,MM)
501   FORMAT (1X, 17I6)
319   NN(IK,1) = 0
      GO TO 320
330   CONTINUE
320   CONTINUE
      WRITE (6,500)(IND(L,J),
       J = 1,MP)
500   FORMAT (1X, 26I5)
      DO 400 I = 1,K
      IF(NN(I,1).EQ.0) GO TO 400
      GO TO 420
400   CONTINUE
```

```
      GO TO 499
420   LL = I
      L = L + 1
      GO TO 299
499   CONTINUE
      STOP
      END
```

CHECK DECK

```
      SUBROUTINE CHECK (NN,II,K,MD,LD)
      DIMENSION NN(1800,9)
      MD = 2
      LD = 2
      J = II - 1
30    IF(NN(K,II).EQ.NN(K,J)) GO TO 31
      IF(J.EQ.2) GO TO 32
      J = J - 1
      GO TO 30
32    J = II - 1
      NA = NN(K,II)/100
      NB = (NN(K,II) - NA*100)10
      NC = NN(K,II) - NA*100 - NB*10
      MA = NN(K,J)/100
      MB = (NN(K,J) - MA*100)/10
      MC = NN(K,J) - MA*100 - MB*10
      MDX = IABS(NA - MA) + IABS(NB - MB)
        + IABS(NC - MC)
      IF(MCX.EQ.1) GO TO 33
      IF(MDX.EQ.2) GO TO 33
      RETURN
33    MD = 1
      RETURN
31    LD = 6
      RETURN
      END
```

DATA INPUT FOR IA(I)

```
000,001,010,011,100,101,110,111
```

APPENDIX B

TRANSFORMATION GENERATORS

In each table the ordered identification numbers 1, 2, ..., N are equivalent pairwise to a corresponding treatment combination in the particular set of tc's under consideration. The order of the set of tc's is the usual increasing order of base 2 numbers. Thus, for example, in a 2^2 factorial 1 = 00, 2 = 01, 3 = 10, 4 = 11.

In each table given, whenever blank spaces are encountered it is to be assumed that the number last listed previously in the same column is the proper entry.

TABLE I

$TB(2^2, \Delta = 1)$; $TG(2^{4-2}, \Delta = 2)$, $I = AB = CD$
(1) 1, 2, 4, 3

TABLE II

$TG(2^3, \Delta = 1)$; $TG(2^{6-3}, \Delta = 2)$, $I = AB = CD = EF$

For class (0) split-plot designs use only tg's (1) and (2).

tg Number	Sequence of tc's
(1)	1, 2, 4, 3, 7, 5, 6, 8
(2)	1, 2, 4, 3, 7, 8, 6, 5
(3)	1, 2, 4, 8, 6, 5, 7, 3

TABLE III

TG(2^3, Δ 2); TG(2^{4-1}, Δ = 2),

I = ABCD; TG(2^{6-3}, Δ 4), I = AB = CD = EF

For class (0) split-plot designs use only tg's (1)-(54),
for split-split-plot designs use tg's (1)-(18).

(1)	1 2 3 4	6 5 8 7	(43)	1 2 4 3	6 4 7 8	(85)	1 2 3 8	4 6 5 7
(2)		6 7 8 5	(44)		5 7 8 6	(86)		4 6 7 5
(3)		7 8 5 6	(45)		5 8 6 7	(87)		4 7 5 6
(4)		7 8 6 5	(46)		5 8 7 6	(88)		4 7 6 5
(5)		8 7 5 6	(47)		7 5 6 8	(89)		5 6 4 7
(6)	1 2 3 4	8 7 6 5	(48)		7 5 8 6	(90)		5 6 7 4
(7)	1 2 4 3	5 6 7 8	(49)		7 6 5 8	(91)		5 7 4 6
(8)		5 6 8 7	(50)		7 6 8 5	(92)		5 7 6 4
(9)		7 8 5 6	(51)		8 5 6 7	(93)		6 4 7 5
(10)		7 8 6 5	(52)		8 5 7 6	(94)		6 5 7 4
(11)		8 7 5 6	(53)		8 6 5 7	(95)		7 4 6 5
(12)	1 2 4 3	8 7 6 5	(54)	1 2 4 3	8 6 7 5	(96)	1 2 3 8	7 5 6 4
(13)	1 4 2 3	5 6 7 8	(55)	1 2 3 5	6 4 7 8	(97)	1 2 4 6	5 3 7 8
(14)		5 6 8 7	(56)		6 4 8 7	(98)		5 3 8 7
(15)		7 8 5 6	(57)		6 7 4 8	(99)		5 7 3 8
(16)		7 8 6 5	(58)		6 7 8 4	(100)		5 7 8 3
(17)		8 7 5 6	(59)		6 8 4 7	(101)		5 8 3 7
(18)	1 4 2 3	8 7 6 5	(60)		6 8 7 4	(102)		5 8 7 3
(19)	1 2 3 4	6 7 5 8	(61)		7 4 6 8	(103)		7 3 5 8
(20)		6 7 8 5	(62)		7 4 8 6	(104)		7 3 8 5
(21)		6 8 5 7	(63)		7 6 4 8	(105)		7 5 3 8
(22)		6 8 7 5	(64)		7 6 8 4	(106)		7 5 8 3
(23)		7 5 6 8	(65)		7 8 4 6	(107)		7 8 3 5
(24)		7 5 8 6	(66)		7 8 6 4	(108)	1 2 4 6	7 8 5 3
(25)		7 6 5 8	(67)		8 4 6 7	(109)	1 2 4 6	8 3 5 7
(26)		7 6 8 5	(68)		8 4 7 6	(110)		8 3 7 5
(27)		8 5 6 7	(69)		8 6 4 7	(111)		8 5 3 7
(28)		8 5 7 6	(70)		8 6 7 4	(112)		8 5 7 3
(29)		8 6 5 7	(71)		8 7 4 6	(113)		8 7 3 5
(30)	1 2 3 4	8 6 7 5	(72)		8 7 6 4	(114)	1 2 4 6	8 7 5 3
(31)	1 2 4 3	5 7 6 8	(73)	1 2 3 7	4 6 8 5	(115)	1 2 4 7	3 5 6 8
(32)		5 7 8 6	(74)		4 8 5 6	(116)		3 5 8 6
(33)		5 8 6 7	(75)		4 8 6 5	(117)		3 8 5 6
(34)		5 8 7 6	(76)		5 6 4 8	(118)		3 8 6 5
(35)		7 5 6 8	(77)		5 6 8 4	(119)		5 3 8 6
(36)		7 5 8 6	(78)		5 8 4 6	(120)		5 6 8 3
(37)		7 6 5 8	(79)		5 8 6 4	(121)		6 5 3 8
(38)		7 6 8 5	(80)		6 4 8 5	(122)		6 5 8 3
(39)		8 5 6 7	(81)		6 5 8 4	(123)		6 8 3 5
(40)		8 5 7 6	(82)		8 4 6 5	(124)		6 8 5 3
(41)		8 6 5 7	(83)		8 5 6 4	(125)		8 3 5 6
(42)		8 6 7 5	(84)		4 6 5 8	(126)	1 2 4 7	8 6 5 3

TABLE III (Continued)

(127)	1 2 4 8	3 5 6 7	(177)	1 4 2 6	5 7 3 8	(227)	1 4 6 5	3 2 8 7
(128)		3 5 7 6	(178)		5 7 8 3	(228)		3 7 8 2
(129)		3 7 5 6	(179)		5 8 3 7	(229)		7 3 2 8
(130)		3 7 6 5	(180)		5 8 7 3	(230)		7 3 8 2
(131)		5 3 7 6	(181)		7 3 5 8	(231)		7 8 2 3
(132)		5 6 7 3	(182)		7 3 8 5	(232)		7 8 3 2
(133)		6 5 3 7	(183)		7 5 3 8	(233)		8 2 3 7
(134)		6 5 7 3	(184)		7 5 8 3	(234)	1 4 6 5	8 7 3 2
(135)		6 7 3 5	(185)		7 8 3 5	(235)	1 4 6 7	3 2 5 8
(136)		6 7 5 3	(186)		7 8 5 3	(236)	1 4 6 7	3 2 8 5
(137)		7 3 5 6	(187)		8 3 5 7	(237		3 5 2 8
(138)	1 2 4 8	7 6 5 3	(188)		8 3 7 5	(238)		3 5 8 2
(139)	1 2 8 3	4 6 5 7	(189)		8 5 3 7	(239)		3 8 2 5
(140)		4 6 7 5	(190)		8 5 7 3	(240)		3 8 5 2
(141)		4 7 5 6	(191)		8 7 3 5	(241)		5 2 3 8
(142)		4 7 6 5	(192)	1 4 2 6	8 7 5 3	(242)		5 2 8 3
(143)		5 6 4 7	(193)	1 4 2 8	3 5 6 7	(243)		5 3 2 8
(144)		5 6 7 4	(194)	1 4 2 8	3 5 7 6	(244)		5 3 8 2
(145)		5 7 4 6	(195)		3 7 5 6	(245)		5 8 2 3
(146)		5 7 6 4	(196)		3 7 6 5	(246)		5 8 3 2
(147)		7 4 6 5	(197)		5 3 7 6	(247)		8 2 3 5
(148)	1 2 8 3	7 5 6 4	(198)		5 6 7 3	(248)		8 2 5 3
(149)	1 2 8 4	3 5 6 7	(199)		6 5 3 7	(249)		8 3 2 5
(150)		3 5 7 6	(200)		6 5 7 3	(250)		8 3 5 2
(151)		3 7 5 6	(201)		6 7 3 5	(251)		8 5 2 3
(152)		3 7 6 5	(202)		6 7 5 3	(252)	1 4 6 7	8 5 3 2
(153)		6 5 3 7	(203)		7 3 5 6	(253)	1 4 6 8	2 3 5 7
(154)		6 5 7 3	(204)	1 4 2 8	7 6 5 3	(254)		2 3 7 5
(155)		6 7 3 5	(205)	1 4 6 2	3 5 7 8	(255)		2 5 3 7
(156)		6 7 5 3	(206)		3 5 8 7	(256)	1 4 6 8	2 5 7 3
(157)		7 3 5 6	(207)		3 7 5 8	(257)		3 2 5 7
(158)	1 2 8 4	7 6 5 3	(208)		3 7 8 5	(258)		3 7 5 2
(159)	1 2 8 7	3 4 6 5	(209)		3 8 5 7	(259)		5 2 3 7
(160)		3 5 6 4	(210)		3 8 7 5	(260)		5 7 3 2
(161)		4 3 5 6	(211)		5 3 7 8	(261)		7 3 2 5
(162)	1 2 8 7	4 6 5 3	(212)		5 3 8 7	(262)		7 3 5 2
(163)	1 4 2 5	3 7 6 8	(213)		5 7 3 8	(263)		7 5 2 3
(164)		3 7 8 6	(214)		5 7 8 3	(264)	1 4 6 8	7 5 3 2
(165)		3 8 6 7	(215)		5 8 3 7	(265)	1 4 8 2	3 5 6 7
(166)		3 8 7 6	(216)		5 8 7 3	(266)		3 5 7 6
(167)		6 7 3 8	(217)		8 3 5 7	(267)		3 7 5 6
(168)		6 7 8 3	(218)		8 3 7 5	(268)		3 7 6 5
(169)		6 8 3 7	(219)		8 5 3 7	(269)		5 3 7 6
(170)		6 8 7 3	(220)		8 5 7 3	(270)		5 6 7 3
(171)		7 3 8 6	(221)		8 7 3 5	(271)		6 5 3 7
(172)		7 6 8 3	(222)	1 4 6 2	8 7 5 3	(272)		6 5 7 3
(173)		8 3 7 6	(223)	1 4 6 5	2 3 7 8	(273)		6 7 3 5
(174)	1 4 2 5	8 6 7 3	(224)		2 3 8 7	(274)	1 4 8 2	6 7 5 3
(175)	1 4 2 6	5 3 7 8	(225)	1 4 6 5	2 8 3 7	(275)	1 4 8 5	2 3 7 6
(176)		5 3 8 7	(226)		2 8 7 3	(276)		2 6 7 3

TABLE III (Continued)

(277)	1 4 8 5 6 2 3 7	(281)	1 4 8 6 2 5 3 7	(285)	1 4 8 6 7 3 2 5
(278)	6 7 3 2	(282)	2 5 7 3	(286)	7 3 5 2
(279)	1 4 8 6 2 3 5 7	(283)	5 2 3 7	(287)	7 5 2 3
(280)	2 3 7 5	(284)	5 7 3 2	(288)	7 5 3 2

TABLE IV

$TG(2^3, \ \Delta \leq 2)$

(1)	1 4 5 2 7 6 3 8	(7)	1 4 6 7 2 3 5 8	(13)	1 8 2 3 5 4 6 7
(2)	1 4 5 2 8 3 6 7	(8)	1 4 6 7 2 3 8 5	(14)	1 8 2 3 5 4 7 6
(3)	1 4 5 8 2 3 6 7	(9)	1 4 6 7 2 5 3 8	(15)	1 8 2 3 6 7 4 5
(4)	1 4 5 8 2 7 6 3	(10)	1 4 6 7 2 5 8 3	(16)	1 8 2 7 4 5 3 6
(5)	1 4 6 3 5 8 2 7	(11)	1 4 6 7 2 8 3 5	(17)	1 8 2 7 4 6 3 5
(6)	1 4 6 3 8 5 2 7	(12)	1 4 6 7 2 8 5 3		

TABLE V

$TG(2^4, \Delta = 1)$

For Class (0) split-plot designs use only tg's (1)-(54), for split-split-plot designs use tg's (1)-(12).

(1)	1	2	4	3	7	5	6	8	16	14	13	15	11	9	10	12
(2)													11	12	10	9
(3)												14	10	9	11	12
(4)													10	12	11	9
(5)					7	8	6	5	13	14	16	15	11	9	10	12
(6)													11	12	10	9
(7)									13	15	16	14	10	9	11	12
(8)													10	12	11	9
(9)	1	2	4	8	6	5	7	3	11	9	10	12	16	14	13	15
(10)													16	15	13	15
(11)									11	12	10	9	13	14	16	15
(12)													13	15	16	14
(13)	1	2	4	3	7	5	6	8	16	12	10	9	11	15	13	14
(14)												14	13	9	11	15
(15)														15	11	9
(16)											11	9	10	14	13	15
(17)												15	13	9	10	14
(18)														14	10	9
(19)												9	13	15	11	12
(20)												12	11	9	13	15
(21)														15	13	9
(22)											13	9	10	12	11	15
(23)									16	15	11	9	13	14	10	12
(24)												12	10	9	13	14
(25)														14	13	9
(26)											13	9	11	12	10	14
(27)					7	8	6	5	13	9	10	12	11	15	16	14
(28)													16	14	11	15
(29)														15	11	14
(30)	1	2	4	3	7	8	6	5	13	9	11	12	10	14	16	15
(31)												15	16	12	10	14
(32)														14	10	12
(33)										14	10	9	11	12	16	15
(34)														15	16	12
(35)												12	16	15	11	9
(36)											16	12	10	9	11	15
(37)										15	11	9	10	12	16	14
(38)														14	16	12
(39)												12	16	14	10	9
(40)											16	12	11	9	10	14
(41)	1	2	4	8	6	5	7	3	11	9	10	14	13	15	16	12
(42)											13	14	10	12	16	15
(43)												15	16	12	10	14
(44)														14	10	12
(45)										12	10	14	16	15	13	9
(46)											16	14	10	9	13	15
(47)												15	13	9	10	14
(48)														14	10	9
(49)										15	13	9	10	12	16	14
(50)														14	16	12
(51)												14	16	12	10	9
(52)											16	12	10	9	13	14
(53)														14	13	9
(54)												14	13	9	10	12
(55)	1	2	4	3	7	5	6	14	10	9	13	15	11	12	16	8
(56)										12	11	9	13	15	16	8
(57)									13	9	10	12	11	15	16	8
(58)										15	11	9	10	12	16	8

TABLE V (Continued)

(59)	1	2	4	3	7	5	13	9	10	12	11	15	16	8	6	14
(60)														14	6	8
(61)										14	6	8	16	12	11	15
(62)														15	11	12
(63)									11	12	10	14	6	8	16	15
(64)										15	16	8	6	14	10	12
(65)												12	10	14	6	8
(66)								14	6	8	16	12	10	9	11	15
(67)												15	11	9	10	12
(68)														12	10	9
(69)								15	11	9	10	12	16	8	6	14
(70)														14	6	8
(71)												14	6	8	16	12
(72)										12	16	8	6	14	10	9
(73)									16	8	6	14	10	9	11	12
(74)														12	11	9
(75)										12	11	9	10	14	6	8
(76)					7	8	6	14	10	9	11	12	16	15	13	5
(77)										12	16	15	11	9	13	5
(78)											10	9	11	15	13	5
(79)										15	11	12	10	9	13	5
(80)							16	12	10	9	11	15	13	5	6	14
(81)														14	6	5
(82)										14	6	5	13	9	11	15
(83)														15	11	9
(84)									11	9	10	14	6	5	13	15
(85)										15	13	5	6	14	10	9
(86)												9	10	14	6	5
(87)								14	6	5	13	9	10	12	11	15
(88)												15	11	9	10	12
(89)														12	10	9
(90)								15	11	9	13	5	6	14	10	12
(91)	1	2	4	3	7	8	16	15	11	12	10	9	13	5	6	14
(92)														14	6	5
(93)												14	6	5	13	9
(94)									13	5	6	14	10	9	11	12
(95)														12	11	9
(96)										9	11	12	10	14	6	5
(97)					7	15	11	9	10	12	16	8	6	5	13	14
(98)														14	13	5
(99)												14	13	5	6	8
(100)									10	14	13	5	6	8	16	12
(101)									13	5	6	8	16	12	10	14
(102)														14	10	12
(103)												14	10	12	16	8
(104)										14	10	12	16	8	6	5
(105)								12	10	9	13	5	6	8	16	14
(106)														14	16	8
(107)												14	16	8	6	5
(108)										14	16	8	6	5	13	9
(109)									16	8	6	5	13	9	10	14
(110)														14	10	9
(111)												14	10	9	13	5
(112)										14	10	9	13	5	6	8
(113)							13	5	6	8	16	12	11	9	10	14
(114)												14	10	9	11	12
(115)														12	11	9
(116)										14	10	9	11	12	16	8
(117)								9	11	12	10	14	16	8	6	5
(118)								14	10	9	11	12	16	8	6	5
(119)							16	8	6	5	13	9	11	12	10	14
(120)												14	10	9	11	12
(121)														12	11	9
(122)										14	10	12	11	9	13	5

TABLE V (Continued)

(123)	1	2	4	3	7	15	16	12	11	9	10	14	13	5	6	8	
(124)								14	10	12	11	9	13	5	6	8	
(125)	1	2	4	8	6	5	7	15	13	9	10	14	16	12	11	3	
(126)										14	16	12	10	9	11	3	
(127)										16	12	10	14	13	9	11	3
(128)											14	13	9	10	12	11	3
(129)								13	9	10	12	11	3	7	15	16	14
(130)											14	16	12	11	3	7	15
(131)															15	7	3
(132)													15	7	3	11	12
(133)										11	3	7	15	16	12	10	14
(134)															14	10	12
(135)											12	10	14	16	15	7	3
(136)									14	10	9	11	3	7	15	16	12
(137)													12	16	15	7	3
(138)											12	16	15	7	3	11	9
(139)												10	9	11	3	7	15
(140)															15	7	3
(141)											15	7	3	11	9	10	12
(142)															12	10	9
(143)									15	7	3	11	9	10	12	16	14
(144)															14	16	12
(145)													12	16	14	10	9
(146)						6	14	10	9	11	3	7	5	13	15	16	12
(147)											12	16	15	13	5	7	3
(148)										13	5	7	3	11	12	16	15
(149)															15	16	12
(150)													15	16	12	11	3
(151)											15	16	12	11	3	7	5
(152)									12	16	15	7	3	11	9	13	5
(153)													5	13	9	11	3
(154)												11	3	7	5	13	9

(155)	1	2	4	8	6	14	10	12	16	15	11	9	13	5	7	3
(156)											13	5	7	3	11	9
(157)												9	11	3	7	5
(158)							13	5	7	3	11	9	10	12	16	15
(159)												15	16	12	10	9
(160)										15	16	12	10	9	11	3
(161)								9	10	12	16	15	11	3	7	5
(162)								15	16	12	10	9	11	3	7	5
(163)							16	12	10	9	11	3	7	5	13	15
(164)														15	13	5
(165)												15	13	5	7	3
(166)											13	5	7	3	11	15
(167)														15	11	3
(168)												15	11	3	7	5
(169)								15	7	3	11	12	10	9	13	5
(170)										5	13	9	10	12	11	3
(171)									11	3	7	5	13	9	10	12
(172)										12	10	9	13	5	7	3
(173)									13	5	7	3	11	9	10	12
(174)														12	10	9
(175)										9	10	12	11	3	7	5
(176)					7	3	11	9	10	12	16	14	6	5	13	15
(177)												15	13	5	6	14
(178)														14	6	5
(179)										14	6	5	13	15	16	12
(180)									13	5	6	14	10	12	16	15
(181)										15	16	12	10	14	6	5
(182)								12	10	9	13	5	6	14	16	15
(183)												15	16	14	6	5
(184)									16	15	13	5	6	14	10	9
(185)												9	10	14	6	5
(186)								15	13	5	6	14	16	12	10	9

TABLE V (Continued)

(187)	1	2	4	8	7	3	11	15	13	9	10	12	16	14	6	5
(188)									16	12	10	9	13	5	6	14
(189)														14	6	5
(190)												14	6	5	13	9
(191)										14	6	5	13	9	10	12
(192)						5	7	14	10	9	13	15	16	12	11	3
(193)										12	16	15	13	9	11	3
(194)									13	9	10	12	16	15	11	3
(195)										15	16	12	10	9	11	3
(196)									16	12	10	9	13	15	11	3
(197)										15	13	9	10	12	11	3
(198)						15	13	5	6	14	16	12	10	9	11	3
(199)							16	12	10	14	6	5	13	9	11	3
(200)								14	6	5	13	9	10	12	11	3
(201)					16	12	10	9	11	3	7	5	6	14	13	15
(202)												15	13	5	6	14
(203)														14	6	5
(204)										15	13	14	6	5	7	3
(205)									13	14	6	5	7	3	11	15
(206)														15	11	3
(207)										15	11	3	7	5	6	14
(208)								14	6	5	7	3	11	9	13	15
(209)														15	13	9
(210)												15	13	9	11	3
(211)											13	9	11	3	7	15
(212)														15	7	3

(213)	1	2	4	8	16	12	10	14	6	5	13	15	7	3	11	9
(214)							11	3	7	5	6	14	10	9	13	15
(215)										15	13	5	6	14	10	9
(216)												9	10	14	6	5
(217)								9	10	14	6	5	13	15	7	3
(218)								15	13	9	10	14	6	5	7	3
(219)						14	6	5	7	3	11	12	10	9	13	15
(220)												15	13	9	10	12
(221)										15	13	9	10	12	11	3
(222)									13	9	10	12	11	3	7	15
(223)														15	7	3
(224)										15	7	3	11	9	10	12
(225)														12	10	9
(226)					16	15	7	3	11	9	13	5	6	14	10	12
(227)										12	10	9	13	5	6	14
(228)														14	6	5
(229)												14	6	5	13	9
(230)								5	6	14	13	9	10	12	11	3
(231)							11	3	7	5	6	14	13	9	10	12
(232)								12	10	9	13	14	6	5	7	3
(233)							13	9	10	12	11	3	7	5	6	14
(234)										14	6	5	7	3	11	12
(235)									11	3	7	5	6	14	10	12
(236)										12	10	14	6	5	7	3
(237)								14	6	5	7	3	11	9	10	12
(238)														12	10	9

TABLE VI

$TG(2^4, \Delta \leq 2)$; $TG(2^{5-1}, \Delta = 2)$, $I = ABCDE$; $TG(2^{7-3}, \Delta \leq 4)$, $I = ABC = DE = FG$

For class (0) split-plot designs use tg's (1)-(10), for split-split-plot designs use tg's (1)-(5).

(1)	1	4	2	3	7	5	6	8	12	10	9	11	15	13	14	16
(2)	1	2	4	3	8	5	7	6	10	9	11	12	14	16	13	14
(3)	1	4	3	2	5	6	8	7	11	9	10	12	16	13	15	14
(4)	1	4	2	3	5	8	6	7	11	10	12	9	13	14	16	15
(5)	1	3	2	4	7	6	8	5	9	12	11	10	13	15	16	14
(6)	1	5	2	4	7	3	8	6	13	11	9	15	12	10	14	16
(7)	1	4	7	3	8	6	2	5	14	9	12	15	13	10	16	11
(8)	1	6	4	2	8	7	3	5	9	14	10	16	12	15	13	11
(9)	1	2	8	5	3	6	4	7	11	16	13	10	14	9	12	15
(10)	1	7	3	8	6	4	2	5	15	12	14	10	11	16	13	9
(11)	1	6	10	12	4	2	14	16	11	3	8	7	13	15	9	5
(12)	1	2	12	15	9	3	8	14	10	6	5	13	16	11	4	7
(13)	1	5	6	14	2	12	4	16	8	15	3	7	13	11	9	10
(14)	1	10	16	14	5	8	12	9	2	6	13	11	7	3	4	15
(15)	1	13	14	6	8	2	10	16	11	4	7	15	5	9	3	12
(16)	1	13	6	10	2	14	9	5	7	3	11	16	12	4	8	15
(17)	1	9	3	12	11	7	16	13	5	8	2	10	4	6	14	15
(18)	1	3	7	16	10	6	4	11	9	14	5	8	2	12	15	13
(19)	1	4	11	15	5	7	13	10	2	12	9	3	8	14	6	16
(20)	1	7	4	10	13	5	15	9	12	3	11	16	8	14	6	2

TABLE VII

$TG(2^4, \Delta \geq 3)$

(1)	1	14	11	2	7	9	4	5	16	3	6	15	10	8	13	12
(2)	1	12	7	2	11	14	4	9	8	13	3	6	15	10	5	16
(3)	1	8	11	2	7	14	3	10	15	6	12	13	4	5	16	9
(4)	1	16	9	8	11	2	13	4	14	3	6	15	10	7	12	5

TABLE VIII

$TG(2^5, \Delta = 1)$

Class (0)

Design	tg
Split-plot	(1)-(2)
Split-split-plot	(1)-(4)
Split-split-split-plot	(1)-(6)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	1	1	1	1	1	1	1
2	2	2	2	2	2	17	2
4	4	4	4	4	4	25	10
3	3	3	3	3	8	9	26
7	7	7	7	7	16	13	18
5	8	5	8	8	12	29	22
6	6	6	6	16	11	31	6
8	5	8	5	12	3	23	14
16	13	16	13	10	7	7	30
14	15	12	14	14	5	15	29
13	16	11	10	6	6	11	31
15	14	15	9	5	14	27	32
11	10	13	11	13	10	19	16
9	9	9	15	15	9	3	8
10	11	10	16	11	13	4	4
12	12	14	12	9	15	2	3
28	28	30	28	25	31	18	19
27	27	29	27	26	32	20	27
25	25	31	25	28	30	24	11
26	26	32	29	32	26	8	15
30	30	28	31	30	18	16	13
32	29	27	32	22	17	14	9
31	31	25	30	18	25	6	25
29	32	26	26	29	29	5	17
21	24	18	18	31	21	21	21
23	22	22	20	27	23	22	5
24	21	21	24	19	19	30	7
22	23	23	22	17	27	32	23
18	19	24	21	18	28	28	24
20	17	20	17	20	20	26	20
19	18	19	19	24	24	10	28
17	20	17	23	23	22	12	12

TABLE IX

$TG(2^5, \Delta \leq 2)$;
$TG(2^{6-1}, \Delta = 2)$,
$I = ABCDEF$

Class (0)

Design	tg
Split-plot	(1)-(6)
Split-split-plot	(1)-(4)
Split-split-split-plot	(1)-(2)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	1	1	1	1	1	1	1
4	3	5	7	2	13	2	2
2	2	2	3	12	6	8	22
3	4	4	8	15	10	16	29
7	7	7	6	9	2	24	13
5	6	3	4	3	14	7	16
6	8	8	2	8	9	31	24
8	5	6	5	14	5	28	19
12	9	13	15	10	7	18	28
10	12	11	12	6	3	10	10
9	11	9	14	5	11	6	4
11	10	15	10	13	16	4	20
15	13	12	11	16	12	20	32
13	15	10	16	11	4	27	15
14	16	14	13	4	8	9	14
16	14	16	9	7	15	14	6
32	22	24	17	31	27	22	21
29	24	23	22	19	25	32	18
30	21	17	20	20	29	12	25
31	23	20	18	28	22	3	30
28	20	22	24	26	20	19	31
27	17	19	23	32	32	25	27
25	19	21	19	24	26	11	11
26	18	18	21	18	17	15	12
22	30	30	25	21	19	29	9
24	31	25	30	30	24	30	26
23	32	28	26	25	31	21	17
21	29	31	32	17	30	13	5
17	26	27	28	27	28	5	23
20	28	32	31	23	18	23	8
18	27	29	29	29	21	17	7
19	25	26	27	22	23	26	3

TABLE X

$TG(2^5, \Delta \leq 3)$

Class (0)

Design	tg
Split-plot	(1)-(4)
Split-split-plot	(1)-(2)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	1	1	1	1	1	1	1
3	2	2	2	15	25	20	10
5	7	16	12	24	17	27	30
7	8	4	4	21	19	15	17
8	4	14	15	13	4	14	13
4	3	11	11	9	3	12	15
6	6	3	5	5	31	4	3
2	5	5	8	18	28	8	8
16	13	13	3	25	10	10	22
9	14	15	13	27	16	18	28
12	16	7	16	32	7	22	32
10	9	9	7	8	5	21	24
13	11	12	14	16	13	9	12
11	12	6	9	3	21	29	6
14	10	8	6	22	26	31	5
15	15	10	10	30	14	24	16
19	32	17	25	17	2	5	31
21	25	24	18	11	8	13	20
24	28	30	32	2	24	30	18
17	26	31	24	14	18	25	14
18	31	26	28	26	12	28	2
23	30	20	29	29	15	3	25
20	29	19	26	6	32	2	29
22	27	23	20	3	30	17	7
28	21	28	22	19	22	7	4
30	17	32	17	28	6	16	19
25	19	29	30	12	9	32	26
32	24	21	21	4	11	26	27
31	23	18	19	23	27	6	11
29	18	27	31	31	29	23	23
27	20	25	23	20	23	19	21
26	22	22	27	10	20	11	9

TABLE XI

For class (0) split-plot designs use tg (1)

TG(2^6, $\Delta = 1$) (1)	(2)	(3)	TG(2^6, $\Delta \leq 2$) (1)	(2)	(3)	TG(2^6, $\Delta \leq 3$) (1)	(2)	(3)
1	1	1	1	1	1	1	1	1
17	33	5	2	41	9	20	23	8
25	49	6	8	34	26	27	32	22
9	17	14	16	2	42	15	44	52
13 35	25 61	13 4	24 45	42 19	57 23	14 41	20 17	57 6
29 36	9 45	29 20	7 39	58 31	17 21	12 43	30 59	28 3
31 52	41 37	61 28	31 40	26 55	2 7	4 60	63 35	44 49
23 56	57 53	53 12	28 64	50 52	50 47	8 59	49 41	15 60
7 64	58 21	37 16	18 58	18 49	58 46	10 39	55 38	35 56
15 60	42 5	45 48	10 41	20 59	18 64	18 36	31 2	11 31
11 59	10 13	47 46	6 37	12 25	24 30	22 50	14 19	14 26
27 63	26 15	15 62	4 38	60 28	40 29	21 56	22 50	62 12
19 55	18 47	31 58	20 46	51 11	39 31	9 63	5 60	55 24
3 39	50 63	63 57	27 61	53 9	51 27	29 61	25 36	43 63
4 47	34 55	64 49	9 54	23 3	60 59	31 53	43 11	9 58
2 45	2 51	32 50	14 56	39 17	20 19	24 45	48 3	17 46
18 37	6 19	24 18	22 63	15 57	12 35	5 34	13 6	4 13
20 33	14 20	8 2	32 59	32 33	8 33	13 54	8 56	40 2
24 41	46 28	40 3	12 52	48 45	16 49	30 58	21 40	50 36
8 42	38 12	56 35	3 44	64 5	32 52	25 64	24 33	19 48
16 34	54 44	55 33	19 60	63 38	48 4	28 48	39 10	25 45
14 38	22 43	39 41	25 48	27 14	56 10	3 42	34 16	41 29
6 40	30 59	7 42	11 47	43 29	55 25	2 52	58 29	39 59
5 48	62 60	23 34	15 35	44 62	37 61	17 55	46 53	54 20
21 46	64 52	21 38	29 33	16 56	41 62	7 40	37 51	30 7
22 62	32 36	17 54	30 50	10 8	11 14	16 38	26 7	32 16
30 61	16 35	25 22	21 53	6 36	43 6	32 46	4 15	21 38
32 57	48 39	27 30	13 49	30 4	3 22	26 62	42 64	61 53
28 58	40 7	19 26	5 55	46 24	5 54	6 49	61 62	42 23
26 50	56 8	51 10	23 51	13 40	13 34	23 51	57 45	34 64
10 54	24 4	52 9	17 36	61 47	45 36	19 35	18 9	18 27
12 53	23 3	60 11	26 34	54 7	15 38	11 37	27 28	51 10
44 49	31 11	44 43	57 42	22 35	63 44	57 47	52 12	47 33
43 51	29 27	36 59	62 43	21 37	53 28	33 44	54 47	37 3

VITA

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