THE EXISTENCE CRITERIA OF ONE-GENERAL CONSTRAINT MECHANISMS

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LIST OF SYMBOLS

a_i	ith kinematic link
b	Kutzbach's parameter
f_k	Degrees of freedom of kth class
F	Degrees of freedom of mechanisms
F_c	Degrees of freedom of chain
h_w	Number of active constraints
H	Number of passive constraints
$[I]$	Unit matrix
m	Number of general constraints
$[M]$	Coefficient matrix of differential displacement
n	Number of links
$[N]$	Null matrix
N_R	Number of rotating freedoms
N_T	Number of translation freedoms
N_H	Number of helical freedoms
p_k	Number of class k pairs
$[P]$	Operator matrix
r	Rank of a matrix
s_i	ith kink link
$[T_i(\theta)]$	Screw matrix of rotation and translation
u_i	Number of constraints
z	Number of loops

α_i	ith twist angles
$\hat{\alpha}$	Dual angle
θ_i	ith angular displacement parameter
τ_x	Translation about x axis
τ_y	Translation about y axis
τ_z	Translation about z axis
σ	Dual operator
ω_x	Rotation about x axis
ω_y	Rotation about y axis
ω_z	Rotation about z axis

CHAPTER I

INTRODUCTION

In the principal areas of research in the science of mechanisms, the vast domain of space mechanisms with or without general constraints is virtually unexplored. The formation and application of the different concepts utilized in the areas of type synthesis and classification of mechanisms only magnify the awareness of the lack of knowledge of the constrained or unconstrained space mechanism domain. An examination of this domain within the limits of the current existence criteria, discloses the work of many distinguished kinematicians and mathematicians.

Most of the literature on the theory of classification of space mechanisms shows a primary concern for the adaptation of suitable mathematical relationships for defining and determining the degrees of mobility of a space mechanism. The most notable efforts include the adaptation of the kinematic notations of the kinematic pairs. The preliminary thoughts concerning the definition of kinematic pairs and their classification were given by Rankine in his book, "Machinery and Millwork", published in 1869. However, a systematic approach was proposed by Reuleaux [1][1] in 1876. Reuleaux introduced the concept of the lower and higher pairs and classified the existing pairs accordingly.

[1]Numbers in small brackets refer to similarly numbered references in the bibliography.

He then demonstrated a synthesis technique for constructing a kinematic chain using the kinematic pairs.

During this period when Grübler, Bricard, Alt, and Kutzbach were concerned about the theoretical approach to the determination of the degree of mobility of a spatial kinematic chain, two Russian kinematicians, Assur and Malytcheff, also were developing new concepts and approaches to this subject. Assur [13] developed the concept of the open chain and utilized this concept for structure classification. It is noted in the Russian and Rumanian literature that A. P. Malytcheff [14] had derived one of Kutzbach's mobility relationships in 1923. Nevertheless, neither Kutzbach nor Malytcheff were able to provide sufficient theoretical justification for the existence of the so-called "paradoxial" mechanisms, that is, the Bennett mechanism [6], the Goldberg mechanism [15], or the Bricard six-link mechanism [5], which defied all the known criteria for mobility. It should be noted, however, that it was Kutzbach's mobility relationship that led Kraus [16], [17], [18], [19] in 1940 and Macmillan [20] in 1956 to propose a number synthesis theory for space mechanisms as well as for plane mechanisms.

To account for the existence of the paradoxial mechanisms, Artobolevski [21] and Dobrovol'ski [22] introduced the concept of the general constraints. That is, some mechanisms must contain certain geometric conditions in addition to the constraints imposed by the kinematic pairs in order to obtain mobility. They therefore modified the Malytcheff mobility criterion by introducing a new parameter signifying the existence of the general constraints in the space mechanisms.

Although a rational procedure for determining the existence of the general constraints was not provided by Artobolevski and Dobrovol'ski, several number synthesis approaches based on this concept of general constraints have been proposed by other kinematicians. Among these are Popov [23], Pisarev [24], [25], Lifshits [26], and Bugaievski, Bogdan and Pelecudi [27]. All of these number-synthesis techniques simply involve the different possible interpretations of the structural relationship of Artobolevski and Dobrovol'ski.

Though Reuleaux had already established some of the fundamental concepts of space mechanisms, most of the early work was focused on the planar mechanisms. In 1883 Grashof proposed the mobility criteria for the planar four-link chain. In the same year Grübler proposed another approach for a synthesis technique suited especially for four or more links. Two mathematicians Chebychev (1869) and Sylvester (1874) proposed an approach similar to that of Grübler. In their approach, the development of the classification theory proceeds from the number of degrees of freedom permitted by the kinematic pairs connecting successive links and leads to the degree of freedom of the chain.

Grübler [2], [3], [4], who proposed a criterion to determine the degree of mobility of the planar chain, in 1917 extended his theory to the spatial kinematic chain with revolute pairs. But, Bricard [5] pointed out the weakness of this mobility criterion by claiming that the criterion did not justify the existence of Bennett's four-link four-revolute mechanisms [6] and Bricard's six-link six-revolute space mechanism [7]. However, Alt [8] in 1928 was able to establish with the help of Grübler's criterion that for a constrained motion the total number of degrees of freedom of the pairs must be seven. Based on this evaluation, Alt then proposed that there are three types of four-link and four types of three-link space mechanisms. Thus, it was indirectly established that the pairs can be substituted for links and vice-versa.

In 1928, Kutzbach in his first paper [9] established an analogy between a hydraulic press and a mechanical kinematic chain to propose a scheme to determine the degrees of mobility of a kinematic chain having pairs other than the revolute pairs. However, this theory had its limitations. In 1933 [10], he established a mathematical relationship for the degree of mobility of a spatial kinematic chain and in 1937 presented his theory for the degree of mobility of a kinematic chain with pairs having passive degrees of freedom [11], [12].

Kolchin [28], however, has introduced a seemingly contradictory concept of passive constraints and proposed that mechanisms can possess both passive as well as general constraints, thus implying that general constraints alone are not sufficient to define mobility.

This introduction of the passive constraint concept was an attempt to account for the existence of the so-called paradoxial mechanisms. However, it is another indication of the apparent weakness of all the foregoing mobility criteria; that is, none have presented a means for identifying the geometric conditions that determine the general constraints.

In order to shed new light on the idea of general constraints, Moroshkin [32] completely ignored the theories of Kutzbach, Artobolevski and Dobrovol'ski, and Kolchin. He proposed an analytical scheme based on the number of closed loops of the kinematic chain and the number of independent transformation equations. Thus, the degrees of freedom of the entire chain becomes a function not only of the number and class of the kinematic pairs but of the rank r of the transformation matrix. Although Moroshkin's technique is cumbersome and has not been fully applied, it suggests another parameter analogous to the general constraints.

Sharikov [33] introduced the classical theory of screws to define the existence of constraints in space mechanisms. He developed the concept of the reciprocal screw to account for the degrees of freedom and the nature of the general motion of the chain. The approach provides a theoretical justification for the existence of the paradoxial mechanisms and the number of reciprocal screws is correlated with the parameters in previous theories that define the number of general constraints.

An analogous approach for justifying the existence of the paradoxial mechanisms was developed by Vionea and Atanasiu [34]. Their technique also involves the theory of classical screws and establishes that the rank Q of the matrix of the coefficients of the unknowns in a system of equations describing the angular velocities of the relative helicoidal movements is analogous to the general constraint parameter.

Summarizing briefly, the major effort in type and number synthesis of the planar and spatial mechanisms is confined to the following:

(1) Classification of the kinematic pairs and pair-mechanisms.

(2) Development of suitable mobility criteria and the general classification of the mechanisms.

(3) Developments of rational procedures to evaluate the number of general and passive constraints.

The progressive development that took place in the past century is neither exhaustive nor sufficient enough to regard it as a significant contribution. Yet the field of classification of mechanisms and number synthesis has created sufficient academic interest to pursue a number of studies of the existence criteria of thousands of mechanisms with or without any general constraints. The present study is an investigation of the existence criteria of the one general constraint mechanism. However, there are a number of objectives that must be met in undertaking such a study:

(1) The development of a suitable mathematical model is necessary to identify the existence of the general or passive constraints and the class of the mechanism. An ideal mathematical model is not only needed to define the existence and the class or the family of the mechanism but it also must define the mobility region, dead centers and the limit positions.

(2) The development of the existence criteria relating the kinematic parameters of the representative mechanism is of vital importance in identifying all the mechanisms in a given family. It is recognized that a closure condition must be known for each family of mechanisms. Any random combination of the kinematic parameters such as the kinematic link, the kink-link or the skew angles is not expected to yield a mechanism. In the present study of the existence criteria of one-general constraint mechanisms, the six-link, six-revolute mechanism appears to be a representative mechanism for obtaining the closure conditions.

(3) The development of a method of substituting various classes of kinematic pairs for the revolute pairs will then be expected to identify the additional mechanisms of the same family. Once the closure conditions relating the kinematic parameters are obtained for a representative mechanism, such as the six-revolute, six-link mechanism, then the other mechanism of the same family can be obtained by substituting kinematic pairs either of the same class or of the different class.

These objectives place an extremely severe requirement on the development of the efficient mathematical model. In the following chapter the works of some of the outstanding German and Russian kinematicians have been explored. The remaining chapters discuss the results of the principal objectives discussed above.

CHAPTER II

CLASSIFICATION OF MECHANISMS

Classification of Pairs and Pair Mechanisms

The kinematic pairs of a mechanism are the pairs of contacting elements of two joining links. A minimum of one point contact is required, and, therefore, each pair of elements, depending on their geometric shape, has a maximum of five degrees of freedom. That is, theoretically they may at most permit rotation about three coordinate axes or may permit translation along three coordinate axes and rotation about two coordinate axes. However, one degree of freedom of translation is destroyed on an axis normal to the surface because of the contact, and, therefore, with five degrees of freedom the pair can permit rotation about three coordinate axes and translation along two coordinate axes. Clearly, with one point contact one constraint is imposed and the degree of freedom of the pair is reduced by one. When an element, otherwise free in space, makes two point contact, it automatically introduces two constraints on its motion and as a consequence two degrees of freedom are destroyed.

A pair may have the maximum of five and minimum of one point contact. Correspondingly, the pair may have the maximum of five and minimum of one degree of freedom.

The classification of pairs may follow from any one of the factors described above. That is, the pairs may be classified according to the number of points of contact it makes, according to its number of degrees of freedom, or according to the number of constraints imposed on it.

The Russian kinematicians prefer to classify the pairs according to the number of constraints imposed on the pair. There are five classes of pairs as the pair can have the maximum of five and minimum of one constraint. Class I pair will impose one constraint, class II pair will impose two constraints, class III pair will impose three constraints, etc. Thus, based on the number of constraints, a pair may be classified into any one of the five classes.

The German kinematicians Kraus [16] and Altman [38] prefer to classify the pairs based on the number of points of contact. There are five classes of pairs as the pair can have the maximum of five and minimum of a one point contact. Thus, class I pairs have a one point contact, class II pairs have a two point contact, class II pairs have three point contact, etc. Thus, using Kraus and Altman's approach, a pair may be classified into any one of the five classes.

The English literature lists the approach shown by Harrisberger [29], who suggested the classification of pairs by their number of degrees of freedom. Here again, there are five classes of pairs as the pair can have the maximum of five and minimum of one degree of freedom. Thus, class I pairs have one degree of freedom, class II pairs have two degrees of freedom, class III pairs have three degrees of freedom, etc.

The classification of pairs as shown by Harrisberger is presented in Table I. The number of freedoms of rotation, translation, and

TABLE I

CLASSIFICATION OF KINEMATIC PAIRS

Class	Degrees of Freedom f	Degrees of Constraint u	Number of Point-Contact	Class Symbol	Type Number R T H	Type Symbol	Name	Type of Content	Contact Classification
I	1	5	5	P_1	1 0 0	R	Revolute	Surface	Lower
					0 1 0	P	Prism	Surface	Lower
					0 0 1	H	Helix	Surface	Lower
II	2	4	4	P_2	2 0 0	T	Torus	Line	Higher
					1 1 0	C	Cylinder	Surface	Lower
					1 0 1	T_H	Torus-helix	Line	Higher
					0 2 0	-	-	-	-
					0 1 1	-	-	-	-
III	3	3	3	P_3	3 0 0	S	Sphere	Surface	Lower
					2 1 0	S_S	Sphere-slotted cylinder	Line	Higher
					2 0 1	S_{SH}	Sphere-slotted helix	Line	Higher
					1 2 0	P_L	Plane	Surface	Lower
					0 2 1	-	-	-	-
					0 1 1	-	-	-	-
IV	4	2	2	P_4	3 1 0	S_G	Sphere-Groove	Line	Higher
					3 0 1	S_{GH}	Sphere-Grooved helix	Line	Higher
					2 2 0	C_P	Cylinder-plane	Line	Higher
					1 2 1	-	-	-	-
					2 1 1	-	-	-	-
V	5	1	1	P_5	3 2 0	S_P	Sphere-plane	Point	Higher
					2 2 1	-	-	-	-
					3 1 1	-	-	-	-

helical motion of each "type" of pair in Table I are described by the three digit number

$$N_R N_T N_H$$

where N_R = number of rotating freedoms

N_T = number of translation freedoms

N_H = number of helical freedoms.

Each type of pair, within a class, is determined by the particular pair of basic geometric shapes which define the manner of practical construction of the pair to achieve the defined function. Therefore, it is convenient to identify pair type by the letter symbols shown in Table I which define the fundamental geometric shape of the known physically realizable paired elements.

Note in Table I, there are eight types of pairs for which physically realizable geometric shapes are unknown. It is possible that the relative motion between two links described by the unknown pair types could be achieved by "pair mechanisms"; that is, a combination of several pair elements which would allow the desired relative motion. For example, a Hooke's joint is a pair mechanism which functions as a class III pair of the 300 type.

Table I is based on an observation that a pair can have a maximum of three freedoms of rotation about mutually perpendicular axes, a maximum of two freedoms of translation along two mutually perpendicular axes in a plane perpendicular to the common normal, and one freedom of helical motion along an axis. Theoretically, one would expect a pair to perform these independent translations and three independent helical type of motion. However, physically realizable shapes of the pairs producing such motion are unknown. As of now, such motions are anticipated only from the pair mechanisms.

Grübler's Theory of Determining the Degrees of Mobility of a Spatial Kinematic Chain [2, 3]

The classification of pairs leads immediately to the classification of kinematic chains and to the determination of their degrees of freedom for movability. Six independent parameters are required to define the position of a link in space: for instance, three parameters define the position of any point in the body, two more give the direction of a line fixed in the body and the sixth defines the rotation of the body about this line. Alternately stated, a link in space has six degrees of freedom. With n free links, $6n$ degrees of freedom are possible. However, if these links are connected in any particular manner, permitting motion at each joint, then the number of degrees of freedom of the chain of these n links is reduced. The reduction in the degrees of freedom of the links is dependent upon the class and number of kinematic pairs that are used to connect the links. For class I pairs, there are five constraints imposed on the freedom of the link; when class II pairs are used, four constraints are imposed on the freedom of the links, etc. Thus, the total remaining freedoms of the kinematic chain would be

$$F_C = 6n - (\text{total number of constraints imposed by all the pairs}). \qquad (2.1)$$

If n number of links is connected by g number of pairs, thereby imposing $u_1, u_2, \ldots\ldots u_{g-1}, u_g$ number of constraints, then Equation (2.1) becomes

$$F_C = 6n - \sum_{k=1}^{g} u_k \tag{2.2}$$

$$F_C = 6n - \sum_{k=1}^{g} (6 - f_k) \tag{2.3}$$

where f_k designates the number of degrees of freedom of the kth pair and can be obtained from

$$f_k = 6 - u_k$$

When one of the links is fixed, six degrees of freedom of the chain are destroyed and the number of degrees of freedom of the kinematic chain is given by

$$F = 6n - \sum_{k=1}^{g} (6 - f_k) - 6$$

or

$$F = 6(n - g - 1) + \sum_{k=1}^{g} f_k \tag{2.4}$$

Equation (2.4) provides a tool to determine the degrees of mobility of a spatial kinematic chain. Grübler's relationship for determining the degrees of mobility of a spatial kinematic chain having all revolute pairs (with one degree of freedom) can be obtained by considering $\Sigma f_k = g$ in Equation (2.4). For a constrained Grübler's spatial chain, i.e., $F = 1$ and $\Sigma f_k = g$, Equation (2.4) becomes

$$5g - 6n + 7 = 0 \qquad (2.5)$$

The values g and n, satisfying Equation (2.5) can be obtained from

$$g = 7 + 6\lambda \qquad (2.6)$$

and

$$n = 7 + 5\lambda \qquad (2.7)$$

where $\lambda = 0, 1, 2, \ldots. k$.

The different values of λ specify the number of supplementary moving polygons. When $\lambda = 0$, we get $g = 7$, and $n = 7$, i.e., the kinematic chain of constrainted motion has a maximum of seven links connected by seven class I pairs.

Alt, who was aware of Grübler's finding, pointed out that the kinematic chain with higher pairs can be constructed. This may be done be removing some of the links and substituting higher pairs for these links in such a manner that the sum of the degrees of freedom of all the pairs is seven. Thus, he showed that there are three different kinds of four-link and four different kinds of three-link kinematic chains, all of which have $\Sigma f_k = 7$.

Harrisberger [29] extended this principle of substituting links for pairs and pairs for links. The process of substitution may proceed in a manner so that either the number of pairs or the number of links increases or decreases; but, the sum of the degrees of freedom of all the pairs of the kinematic chains must remain invariant. The simplest possible chain appears to be the one with seven links connected by the seven class I pairs. As there are three different types of class I pairs, one can obtain 36 different kinds of mechanisms having seven links connected by seven pairs of the class I. From the $7p_1$ chain, we can remove two class I pairs (p_1) and substitute one class II pair (p_2). Thus, we have a six-link chain, five of which are connected by the class I pairs and the sixth link is connected by a class II pair. There again, one can obtain 63 different kinds of mechanisms as there are three different types of class I pairs and three different types of class II pairs. Proceeding in this manner, substituting links for appropriate pairs, we obtain altogether thirteen different types and four hundred and thirty-five different kinds of mechanisms all of which are constrained and have seven as the sum of the degrees of freedom of all the pairs.

Malytcheff's Mobility Criterion [14]

This criterion for determining the degrees of mobility of space mechanisms considers the number of kinematic pairs and the number of links of a closed kinematic chain. The proposed criterion is based on the fact that a rigid link free in space can be subjected to six different types of motion, consisting of three independent translations and three independent rotations about an arbitrary set of three rectangular coordinate axes. Therefore, a link free in space has six degrees of freedom. For n links of a kinematic chain, a total of $6n$ degrees of freedom is possible. In a mechanism, however, one link is always kept fixed and therefore only a total of $6(n-1)$ degrees of freedom is possible. When the links are paired by any of the pairs among the five classes of pairs, as suggested by Harrisberger [29], each pair will destroy one or more of the freedom of relative motion of the links. Therefore, for a mechanism the total number of degrees of freedom can be determined by

$$F = 6(n-1) - 5p_1 - 4p_2 - 3p_3 - 2p_4 - 1p_5 \qquad (2.8)$$

where F = degrees of freedom of the mechanisms with n links

p_k = number of class k pairs where $k = 1, 2, 3, 4, 5$.

Kutzbach's Criterion to Determine the Degrees of Mobility of a Spatial Kinematic Chain [11, 12]

Kutzbach [11] described the mobility equation in a somewhat different manner. He stated that the degrees of freedom of a kinematic chain are dependent upon its type of motion. Thus, he expressed the mobility equation as

$$F = b(n - 1) - \Sigma u_k \qquad (2.9)$$

where b = degrees of motion, ($b = 6$ for space motion and $b = 3$ for a plane motion)

n = number of links of the kinematic chain

Σu_k = the total number of constraints imposed by the pairs.

When the kinematic chain is operating in a plane, b takes the value of three. When, however, the same chain is operating in space, b takes the value of six. He also stated that the number of constraints imposed by the pairs also changes correspondingly. The relationship describing the degrees of motion (b), the degrees of freedom of the pairs (f_k), and the number of constraints (u_k) imposed on the pairs is given by

$$f_k + u_k = b \tag{2.10}$$

Substituting Equation (2.10), Equation (2.9) becomes

$$F = b(n - 1) - \Sigma(b - f_k) \tag{2.11}$$

In his latter publication, Kutzbach [12] introduced the concept of active constraints and redefined the relationship described by Equation (2.10) as

$$u_w + h_w = b \tag{2.12}$$

where h_w = number of active constraints

Substituting Equation (2.12), Equation (2.11) becomes

$$F = b(n - 1) - \Sigma(b - h_w) \tag{2.13}$$

The number of active constraints must be computed for each pair. Kutzbach illustrated the use of h_w by considering an example of the spatial four-link mechanism RSSR. The coupler of this mechanism is connected to the input and the follower-link using the two spherical pairs. Due to this special connection of this mechanism, the coupler is able to rotate freely about its own axis, thereby introducing an idle constraint. Since each spheric pair has three constraints on its motion, the two spherical pairs, together, are expected to have a total of six constraints. However, due to the special connectivity, an idle constraint of one degree is induced on the mechanism. Thus the parameter h_w for the two spheric pairs is expected to take a value of seven.

Artobolevski, Dobrovol'ski's Criterion [21, 22]

These authors introduced the concept of general constraints and modified the mobility criterion of Malytcheff by introducing the relationship

$$F = (6 - m)(n - 1) - \Sigma(6 - m - k)p_k \qquad (2.14)$$

where m represents the number of general constraints.

A space mechanism can have a minimum of zero and a maximum of four general constraints. The existence of one or more general constraint, i.e., $(m > 0)$, imposes a restriction on the general motion of the mechanism and in turn on the geometrical configuration of the mechanism. Thus, the existence of one general constraint provides a mechanism having a specific orientation of the axes of its pairs and having a general motion consisting of either three rotations and two translations or two rotations and three translations along a set of three cartesian coordinates.

Based on this concept of general constraints, Artobolevski and Dobrovol'ski proposed a scheme for classifying the existing mechanisms. A kinematic chain can be classified into any one of the five classes which correspond to the five different values of the general constraints. The "zero family" mechanisms consist of a group of mechanisms which have no general constraints, i.e., $m = 0$; the first family mechanisms consist of a group of mechanisms which have one general constraint, etc. Observe that the mobility equations derived by Kutzbach and Malytcheff correspond to the value of $m = 0$.

The mechanisms which do not belong to the zero family obey different mobility relationships. These mobility relationships are tabulated in Table II. Notice that the mechanisms with higher values of general constraints do not permit chains containing pairs of higher classes. For example, family I does not permit mechanisms with class V pairs, family II does not permit mechanisms with class V and class IV pairs, etc.

The family I mechanisms have one general constraint. That is, the mechanisms have a motion capability which may consist of three rotations and two translations or two rotations and three translations. The family II mechanisms with two general constraints have three rotations and one translation, two rotations and two translations, or one rotation and three translations. The family III mechanisms with three general constraints have three rotations, two rotations, and one translation, one rotation and two translations, or three translations. Finally, the family IV mechanisms with four general constraint have two rotations, or one rotation and one translation.

Kolchin's Approach to Construct an Extended Structural Classification of Mechanisms

Artobolevski and Dobrovol'ski introduced the concept of the general constraints in the mechanisms. Based on this concept, discussed earlier, these kinematicians then proposed the five well-known families of mechanisms. Kolchin, however, has proposed that among the predefined general constraints, there are other types of constraints which remain inactive or unoperational in the movement of the mechanisms.

TABLE II.

CLASSIFICATION OF MECHANISMS BASED ON THE NUMBER OF GENERAL CONSTRAINTS.

Family	Number of General Constraints	Possible Type of Motion	Permitted Classes of Pairs	Mobility Relationship	Example of Mechanism
0	0	3R, 3T	p_1, p_2, p_3, p_4, p_5	$F_0 = 6(n-1)-5p_1-4p_2-3p_3$ $-2p_4-p_5$	RSCR
I	1	3R, 2T; 2R, 3T	p_1, p_2, p_3, p_4	$F_1 = 5(n-1)-4p_1-3p_2-2p_3-p_4$	Bricard's Six-Bar
II	2	3R, 1T; 2R, 2T; 1R, 3T	p_1, p_2, p_3	$F_2 = 4(n-1)-3p_1-2p_2-p_3$	Goldberg's Five-Bar
III	3	3R; 2R, 1T; 1R, 2T; 3T	p_1, p_2	$F_3 = 3(n-1)-2p_1-p_2$	Plane four-bar, Spherical mechanism, Bennett's mechanism
IV	4	2R; 1R, 1T; 2T	p_1	$F_4 = 2(n-1)-p_1$	Differential Screw

He named these inactive or unoperational constraints as the "passive" or "idle" constraints and designated them by a symbol H, where H can be obtained from Equation (2.15)

$$F_m = 6N - \Sigma(6 - k)p_k + mz \tag{2.15}$$

or

$$F_m = F_o + mz \tag{2.16}$$

or

$$F_m = F_o + H \tag{2.17}$$

where $H \gtrless mz$, and z denotes the number of closed loops in a kinematic chain.

Because m can take values 0, 1, 2, 3, or 4, H/z can also take the same values. However, Kolchin has proposed that, depending on the nature of the passive constraints,

(a) H/z can be greater than m

(b) H/z can be equal to m

(c) H/z can be less than m.

Based on these different values of the ratio H/z, Kolchin divided further the five families of mechanisms into series. This division of families into series is based on the relationship given by

$$\frac{6 - H/z}{m}$$

where m = 0, 1, 2, 3, or 4

H/z = 0, 1, 2, 3, or 4

The classification scheme proposed by Kolchin is given in Table III. Observe that each family of mechanisms is subdivided into series.

TABLE III

CLASSIFICATION OF MECHANISMS INTO FAMILIES AND

SERIES AS PROPOSED BY KOLCHIN

General Constraints / Passive Constraints	Description	Zero Family	First Family	Second Family	Third Family	Fourth Family	Description
Series 0	Basic H = mz	6/0	6/1	6/2	6/3	6/4	Unlimited
Series 1	Limited	5/0	5/1	5/2	5/3	5/4	Mechanisms
Series 2	Mechanisms	4/0	4/1	4/2	4/3	4/4	H < mz
Series 3	H > mz	3/0	3/1	3/2	3/3	3/4	
Series 4		2/0	2/1	2/2	2/3	2/4	Basic H = mz
	Type of Motion	3R, 3T.	3R, 2T; 2R, 3T.	3R, 1T; 2R, 2T; 1R, 3T.	3R; 2R, 1T; 1R, 2T; 3T	2R; 1R, 1T; 2T.	
			Space Mechanisms		Space Plane	Plane	

Thus, series one in each family does not have any passive constraints; the series two has one passive constraint, etc. The diagonal elements of this classification table have their number of passive constraints equal to the number of general constraints. These diagonal elements represent what is called the basic mechanisms. The series with $H/z > m$ are considered to represent the special mechanisms. Finally, the series with $H/z < m$ are considered to represent the unlimited mechanisms.

All the zero family mechanisms are characterizied to have the motion with three components of rotation and three components of translation. The groups of mechanisms with one general constraint, i.e., of family one, are characterized to have motion with either three components of rotation and two components of translation or two components of rotation and three components of translation.

Very little is known of the passive constraint. Kolchin, however, attempted to make a distinction between the passive and general constraints by suggesting that the existence of the passive constraints imposes a restriction only on the geometrical configuration of the mechanism and not on the general motion of the mechanism. Clearly, Kolchin's theory of passive constraints runs into an apparent contradiction with the theory of general constraints proposed by Artobolevski and Dobrovol'ski.

Moroshkin's Criterion [32]

This approach is based on the number of closed loops of a system of kinematic chain λ and the number of independent transformation equations. Accordingly, if q is the total number of kinematic pairs and n is the total number of links, then the number of closed loops can be given as

$$z = q - n. \tag{2.18}$$

Furthermore, if p_k be the number of kinematic pairs of class k belonging to the system of chain λ, then the equation of kinematic pairs determine 6p. Euler coordinates $\bar{q}_1 \ldots \bar{q}_{6p}$ of the system λ as a function of the

$$N = \sum_k k p_k \tag{2.19}$$

Lagrangian coordinates $q_1 \ldots q_n$. The latter are related by the transformation equations. For each of the z independent simple closed loops of λ, there are twelve transformation equations. Thus, $q_1 \ldots q_n$ obey $K = 12(q - n)$ equations. However, Moroshkin claims that the number of independent equations cannot be greater than $6z$ and, therefore, the degrees of freedom of the entire chain can be given by

$$F = \Sigma\, k p_k - r \tag{2.20}$$

where r is the rank of the number of independent transformation equations.

Sharikov's Criterion [33]

This was the first method to introduce the classical theory of screws to define the existence of constraints in the space mechanisms.

A classical screw is an axis of translation and rotation. If a rigid body is acted upon by a force and a couple about screw β and as a result of this action, the body displaces and rotates about screw α, then the work done on the body can be expressed as

$$W = A \left\{ (p_\alpha + p_\beta) \cos \theta - d \sin \theta \right\} \tag{2.21}$$

where A = constant

p_α = pitch of the screw α

p_β = pitch of the screw β

θ = angle between the screws α and β

and d = the common normal between the screws α and β.

If, however, the body remains in equilibrium, then according to the principle of virtual velocities, the work done in small displacement against the external forces must be zero, i.e.,

$$(p_\alpha + p_\beta) \cos \theta - d \sin \theta = 0 \tag{2.22}$$

The screws α and β which satisfy the above relationship are called reciprocal screws.

According to the proposed approach of Sharikov, a kinematic chain is translated into a system of classical screws. This system of classical screws is then examined for an absence or presence of one or more number of reciprocal screws. The determination of the reciprocal screws, however, utilizes the methods of descriptive geometry.

The theory of classical screws proposes the five families of mechanisms similar to those proposed by Artobolevski and Dobrovol'ski. According to the theory, the motion of a body can be considered in general as composed of screw motion, that is, the motion consisting of independent rotation and translation. The existence of six components of motion, three rotations and three translations, can be represented by a maximum of six classical screws. An absence or presence of one or more number of classical screws creates correspondingly the existence of one or more number of reciprocal screws. Then, when the number of classical screws is six, the number of reciprocal screws is zero. When, however, the number of the classical screws is five, then there exists one reciprocal screw. Similarly, there exists two reciprocal screws corresponding to four classical screws.

The existence of the number of reciprocal screws establishes the basis of the classification of mechanisms. The zero family mechanisms are characterized to have zero number of reciprocal screws; the family one mechanisms are characterized by the existence of one reciprocal screw, etc.

Sharikov's classification scheme is presented in Table IV. Examination of the different possible combinations of the orientation of the classical screws or pairs shows certain patterns. For example, the zero family mechanisms need no specific orientation of the axes of the pairs. Family I mechanisms are proposed to have axes of the pairs intersecting by three into two points either at a finite or at infinite distance. The family II mechanisms are composed of three sub-families and the axes of the pairs generate two hyperboloids with two common generators.

It should be remarked that this proposed classification scheme is by no means exhaustive since mechanisms are known to exist outside the classification of families and sub-families.

TABLE IV

CLASSIFICATION OF MECHANISMS BASED ON THE CLASSICAL THEORY OF SCREWS

Family	Number of Reciprocal Screws	Examples of Mechanisms	Geometrical Locus of the Axes of Pairs
0	0	7R Spatial Chain	Arbitrary location in space
I	1	6R Bricard Mechanisms	Two bundles of lines, three in each, with centers located at a finite or infinitely extended distance
II	2	5R Goldberg's Mechanism	Two hyperboloids with two common producers
III	3	4R Bennett's Mechanism	Surface of hyperboloid
		4R Spherical Mechanism	Bundle of lines with center located at a finite distance
		4R Plane Mechanism	Bundle of parallel lines
		4P Space Mechanism	Pairs located arbitrarily on an infinitely extended plane
IV	4	Plane Mechanism with sliding pairs	Parallel lines, located on an infinitely extended plane

Vionea and Atanasiu's Criterion [34]

This is also an approach based on the classical screws. Accordingly a set of homogeneous coordinates u_i, $v_,$, w_i, l_i, m_i, n_i of a helicoidal screw movement is defined. If j is the number of screws situated on the curve Γ_1 and ℓ the number of kinematic parameters of a closed chain and if $\omega_1, \omega_2, \ldots \omega_r, \omega_{j+1} \ldots \omega_\ell$ are the angular velocities of the possible relative helicoidal movements, then according to the theory of composition of relative movements, a system of linear and homogeneous in $\omega_1, \omega_2, \ldots \omega_j$ equations can be obtained. These equations are:

$$
\begin{aligned}
\omega_1 u_1 + \ldots\ldots\ldots + \omega_j u_j &= 0 \\
\omega_1 v_1 + \ldots\ldots\ldots + \omega_j v_j &= 0 \\
\omega_1 w_1 + \ldots\ldots\ldots + \omega_j w_j &= 0 \\
\omega_1 l_1 + \ldots\ldots\ldots + \omega_j l_j &= 0 \\
\omega_1 m_1 + \ldots\ldots\ldots + \omega_j m_j &= 0 \\
\omega_1 n_1 + \ldots\ldots\ldots + \omega_j n_j &= 0
\end{aligned}
\qquad (2.23)
$$

If Q is the rank of the matrix of the coefficients of the unknowns, then degrees of freedom of the kinematic chain are given by

$$F = \Sigma \; kp_k - Q \qquad (2.24)$$

The proposed approach of Vionea and Atanasiu suggests a possible classification of mechanisms into five families. When the rank Q of the matrix of the coefficients of the unknown is six, then the mechanism satisfying this matrix belongs to the zero family. Similarly, when

Q takes the value five, then the mechanism under consideration belongs to the family one.

It should be remarked that the proposed approach has been applied to investigate the existence criteria of the family III mechanisms. Furthermore, due to the analytical nature of the mathematical method, a number synthesis of the space mechanism becomes virtually impossible.

Dimentberg's Theory of Passive Constraints [46,47]

This approach is an alternative of finding the existence of general constraints. Accordingly, the method of determining the passive or general constraints is based on a philosophy that under the influence of the passive constraints the space mechanism, such as an RRRRRC, will cease to function in the form in which it is described, but instead it will operate as an RRRRRR mechanism. Thus, the existence of passive constraints has imposed some geometrical requirement on the configuration of RRRRRC mechanisms, and this requirement has, in turn, made the cylindric pair function like a revolute pair. Let θ_6 and S_6 be the angular and linear displacement at the cylindric pair. Then the condition of passive constraints is described by

$$\frac{dS_6}{d\theta_1} = 0 \tag{2.25}$$

where θ_1 is the input angular displacement of the mechanism RRRRRC.

Dimentberg applied the dual number algebra to study the conditions of passive constraints. However, the theory of dual number algebra was developed by A. P. Kotelnikoff in 1895 [48].

To demonstrate the practicality of this tool, let us consider an example of imposing one passive coupling on a mechanism selected from the zero family. Consider, for instance, a mechanism shown in Figure 1, and schematically described as R-C-C-C. At the joint 1 we have a revolute pair. The joints 2, 3, and 4 consist of the cylindric pairs. Let α_o, β_o, γ_o, and δ_o be the skew angles of the axes 2, 3, 4, and 1, and α, β, γ, δ be the common normals between the joints 1-2, 2-3, 3-4, and 4-1. Let $\hat{u}_1$, $\hat{u}_2$, $\hat{u}_3$ and $\hat{u}_4$ be the unit vectors associated with the axes 1, 2, 3, 4, such that

$$\hat{u}_1 \cdot \hat{u}_2 = \text{Cos}\ \hat{\alpha} \tag{2.26}$$

$$\hat{u}_2 \cdot \hat{u}_3 = \text{Cos}\ \hat{\beta} \tag{2.27}$$

$$\hat{u}_3 \cdot \hat{u}_4 = \text{Cos}\ \hat{\gamma} \tag{2.28}$$

and

$$\hat{u}_4 \cdot \hat{u}_1 = \text{Cos}\ \hat{\delta} \tag{2.29}$$

where

$$\hat{\alpha} = \alpha_o + \sigma\ \alpha \tag{2.30}$$

$$\hat{\beta} = \beta_o + \sigma\ \beta \tag{2.31}$$

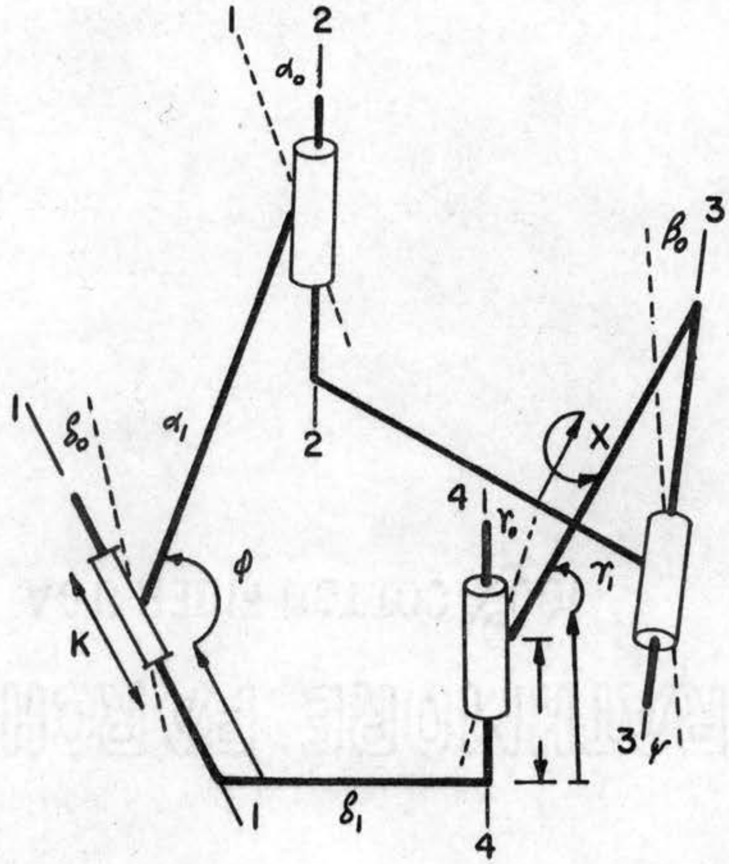
$$\hat{\gamma} = \gamma_o + \sigma\ \gamma \tag{2.32}$$

and

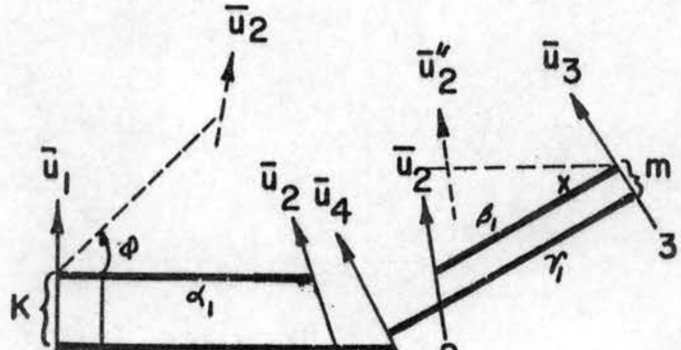
$$\hat{\delta} = \delta_o + \sigma\ \delta \tag{2.33}$$

The joints 2, 3, and 4 are capable of accepting one passive coupling. Let us consider a case where one passive coupling is introduced in the joint 3; that is, after the passive coupling of one translation is introduced, the pair at the joint 3 operates as if it is a revolute pair.

However, relationship between the $\hat{\Phi}$ the input at the joint 1 and $\hat{\chi}$ the output at the joint 3 needs to be derived before introducing the passive coupling at this joint. This relationship can be derived in a following manner.



Rccc ~~RRCC~~ Mechanism



RRCC Mechanism disconnected at the joint 2 and the links α_1 and β_1 are folded as shown.

Figure 1. RRCC Mechanism

Let us disconnect the mechanism at joint 2 and rotate the links 1-2 and 3-2 around the axes 1 and 3 so that they are superimposed respectively on the links 1-4 and 3-4. After this, rotate link 1-2 about axis 1 by an angle $\hat{\varphi} = \varphi_0$ and link 3-2 about an axis 3 by a Dual angle $\hat{\chi} = \chi_0 + \chi_1$ so that the unit vectors u'_2 and u''_2 of the axes 2' and 2" form the same Dual angle with axis 4 after rotation. If this condition is fulfilled, then without varying $\hat{\varphi}$ and $\hat{\chi}$, it is possible to superimpose these axes by giving the motion (helical) in the joint 4. Thus, the two unit vectors $\hat{u}'_2$ and $\hat{u}''_2$ become one and the same unit vector.

Let the vectors of final rotations be $\hat{u}_1 \hat{\Phi}$ and $\hat{u}_3 \hat{Y}$ where

$$\hat{Y} = Y_0 + \sigma Y_1 = \tan 1/2 \hat{\chi} = \tan 1/2 \chi_0 + \sigma \frac{m}{2} (1 + \tan^2 \frac{\chi_o}{2})$$

$$= Y_0 + \frac{\sigma m}{2} (1 + Y^2_0)$$

According to the two rotations of $\hat{u}_2$, about axes 1 and 3, we get

$$\hat{u}'_2 = \frac{1}{1 + \hat{\Phi}^2} \left[(1 - \hat{\Phi}^2) \hat{u}_2 + 2(\hat{u}_1 \cdot \hat{u}_2) \hat{u}_1 \hat{\Phi}^2 + 2(\hat{u}_1 \times \hat{u}_2) \hat{\Phi} \right] \quad (2.34)$$

and

$$\hat{u}''_2 = \frac{1}{1 + \hat{Y}^2} \left[(1 - \hat{Y}^2) \hat{u}_2 + 2(\hat{u}_3 \cdot \hat{u}_2) \hat{u}_3 \hat{Y}^2 + 2(\hat{u}_3 \times \hat{u}_2) \hat{Y} \right] \quad (2.35)$$

Equations (2.34) and (2.35) are however related by one conditions, i.e.,

$$\hat{u}'_2 \cdot \hat{u}_4 = \hat{u}''_2 \cdot \hat{u}_4 \quad (2.36)$$

Equation (2.36) can be solved using the following relationships

$$\hat{u}_2 \cdot \hat{u}_4 = \text{Cos } (\hat{\delta} - \hat{\alpha})$$

or

$$\hat{u}_2 \cdot \hat{u}_4 = \text{Cos } (\hat{\gamma} - \hat{\beta})$$

$$\hat{u}_1 \cdot \hat{u}_4 = \text{Cos}\ \hat{\delta}$$

$$\hat{u}_3 \cdot \hat{u}_4 = \text{Cos}\ \hat{\gamma}$$

$\hat{u}_4 \cdot (\hat{u}_1 \times \hat{u}_2) = 0$ the unit vectors are

$\hat{u}_4 \cdot (\hat{u}_3 \times \hat{u}_2) = 0$ linearly dependent

The resulting expression can be written as

$$\left\{ \text{Cos}\ (\hat{\beta} + \hat{\gamma}) - \text{Cos}(\hat{\delta} - \hat{\alpha}) + \text{Cos}(\hat{\beta} + \hat{\gamma}) - \text{Cos}(\hat{\delta} + \hat{\alpha}) \quad \hat{\Phi}^2 \right\} \hat{Y}^2$$
$$+ \ \text{Cos}(\hat{\beta} - \hat{\gamma}) - \text{Cos}(\hat{\delta} - \hat{\alpha}) + \text{Cos}(\hat{\beta} - \hat{\gamma}) - \text{Cos}(\hat{\delta} + \hat{\alpha}) \quad \hat{\Phi}^2 = 0 \tag{2.37}$$

Equation (2.37) can be briefly expressed as

$$(\hat{a} + \hat{A}\ \hat{\Phi}^2)\ \hat{Y}^2 + (\hat{b} + \hat{B}\ \hat{\Phi}^2) = 0 \tag{2.38}$$

where $\hat{a} = a_0 + \sigma a_1 = \text{Cos}(\beta_0 + \gamma_0) - \text{Cos}(\delta_0 - \alpha_0)$

$$+ \sigma\ [- (\beta_1 + \gamma_1)\ \text{Sin}\ (\beta_0 + \gamma_0) + (\delta_1 - \alpha_1)\ \text{Sin}(\delta_0 - \alpha_0)]$$

$$\hat{A} = A_0 + \sigma A_1 = \text{Cos}(\beta_0 + \gamma_0) - \text{Cos}(\delta_0 + \alpha_0)$$
$$+ \sigma\ [- (\beta_1 + \gamma_1)\ \text{Sin}(\beta_0 + \gamma_0) + (\delta_1 + \alpha_1)\ \text{Sin}(\delta_0 + \alpha_0)]$$

$$\hat{b} = b_0 + \sigma b_1 = \text{Cos}(\beta_0 - \gamma_0) - \text{Cos}(\delta_0 - \alpha_0)$$
$$+ \sigma\ [- (\beta_1 - \gamma_1)\ \text{Sin}(\beta_0 - \gamma_0) + (\delta_1 - \alpha_1)\ \text{Sin}(\delta_0 - \alpha_0)]$$

$$\hat{B} = B_0 + \sigma B_1 = \text{Cos}(\beta_0 - \gamma_0) - \text{Cos}(\delta_0 + \alpha_0)$$
$$+ \sigma\ [- (\beta_1 - \gamma_1)\ \text{Sin}(\beta_0 - \gamma_0) + (\delta_1 + \alpha_1)\ \text{Sin}(\delta_0 + \alpha_0)]$$

Equation (2.38) is the relationship between the input rotation $\hat{\Phi}$ at the joint 1 and the output rotation $\hat{Y}$ at the joint 3. It should be noted, however, that joint 1 consists of a revolute pair and therefore $\hat{\Phi} = \Phi_0$. When the condition of passive coupling is forced at the joint

3, we have $\hat{Y} = Y_0$. Therefore, for the condition of passive coupling, we have

$$(\hat{a} + \hat{A}\ \Phi_0^2)\, Y_0^2 + \hat{b} + \hat{B}\ \Phi_0^2 = 0 \tag{2.39}$$

Separating the real and imaginary part of Equation (2.39), we get

$$(a_0 + A_0\ \Phi_0^2)\, Y_0^2 + (b_0 + B_0\ \Phi_0^2) = 0 \tag{2.40}$$

and

$$(a_1 + A_1\ \Phi_0^2)\, Y_0^2 + (b_1 + B_1\ \Phi_0^2) = 0 \tag{2.41}$$

Equation (2.40) and (2.41) must be solved simultaneously. This condition can be expressed in the form of determinant.

$$\begin{vmatrix} a_0 + A_0\ \Phi_0^2 & b_0 + B_0\ \Phi_0^2 \\ a_1 + A_1\ \Phi_0^2 & b_1 + B_1\ \Phi_0^2 \end{vmatrix} = 0 \tag{2.42}$$

$$(a_0 + A_0\ \Phi_0^2)(b_1 + B_1\ \Phi_0^2) - (b_0 + B_0\ \Phi_0^2)(a_1 + A_1\ \Phi_0^2) = 0 \tag{2.43}$$

Rearranging the above equation, we get

$$\begin{aligned}(A_0B_1 - B_0A_1)\ \Phi_0^4 + (a_0B_1 + A_0b_1 - b_0A_1 - B_0a_1)\ \Phi_0^2 \\ + (a_0b_1 - b_0a_1) = 0\end{aligned} \tag{2.44}$$

This fourth degree polynomial must be equated to zero identically, that is, all the coefficients of this polynomial must be equated to zero. Thus, we have

$$A_0B_1 - B_0A_1 = 0 \tag{2.45}$$

$$a_0B_1 + A_0b_1 - b_0A_1 - B_0a_1 = 0 \tag{2.46}$$

and

$$a_0b_1 - b_0a_1 = 0 \tag{2.47}$$

First, consider Equation (2.46). Substituting the corresponding quantities for a_0, B_1, A_0, b_1, etc., we get

$$a_oB_1 + A_ob_1 - b_oA_1 - B_oa_1 =$$
$$[\mathrm{Cos}(\beta_o + \gamma_o) - \mathrm{Cos}(\delta_o - \alpha_o)][-(\beta_1 - \gamma_1)\,\mathrm{Sin}(\beta_o - \gamma_o)$$
$$+ (\delta_1 + \alpha_1)\,\mathrm{Sin}(\delta_o + \alpha_o)] + [\mathrm{Cos}(\beta_o + \gamma_o) - \mathrm{Cos}(\delta_o + \alpha_o)] \times$$
$$[-(\beta_1 - \gamma_1)\,\mathrm{Sin}(\beta_o - \gamma_o) + (\delta_1 - \alpha_1)\,\mathrm{Sin}(\delta_o - \alpha_o)]$$
$$- [\mathrm{Cos}(\beta_o - \gamma_o) - \mathrm{Cos}(\delta_o - \alpha_o)][-(\beta_1 + \gamma_1)\,\mathrm{Sin}(\beta_o + \gamma_o)$$
$$+ (\delta_1 + \alpha_1)\,\mathrm{Sin}(\delta_o + \alpha_o)] - [\mathrm{Cos}(\beta_o - \gamma_o) - \mathrm{Cos}(\delta_o + \alpha_o)] \times$$
$$[-(\beta_1 + \gamma_1)\,\mathrm{Sin}(\beta_o + \gamma_o) + (\delta_1 - \alpha\delta)\,\mathrm{Sin}(\delta_o - \alpha_o)] \equiv 0 \qquad (2.48)$$

Clearly, this equation satisfies identically. Therefore, let us consider the other conditions given by Equations (2.45) and (2.47). Thus, we get

$$[\mathrm{Cos}(\beta_o + \gamma_o) - \mathrm{Cos}(\delta_o - \alpha_o)][(\beta_1 - \gamma_1)\,\mathrm{Sin}(\beta_o - \gamma_o)$$
$$- (\delta_1 - \alpha_1)\,\mathrm{Sin}(\delta_o - \alpha_o)] - [\mathrm{Cos}(\beta_o - \gamma_o) - \mathrm{Cos}(\delta_o - \alpha_o)] \times$$
$$[(\beta_o + \gamma_o)\,\mathrm{Sin}(\beta_o + \gamma_o) - (\delta_1 - \alpha_1)\,\mathrm{Sin}(\delta_o - \alpha_o)] = 0 \qquad (2.49)$$

and

$$[\mathrm{Cos}(\beta_o + \gamma_o) - \mathrm{Cos}(\delta_o - \alpha_o)][(\beta_1 - \gamma_1)\,\mathrm{Sin}(\beta_o - \gamma_o)$$
$$- (\delta_1 + \alpha_1)\,\mathrm{Sin}(\delta_o + \alpha_o)] - [\mathrm{Cos}(\beta_o - \gamma_o) - \mathrm{Cos}(\delta_o + \alpha_o)] \times$$
$$[(\beta_1 + \gamma_1)\,\mathrm{Sin}(\beta_o + \gamma_o) - (\delta_1 + \alpha_1)\,\mathrm{Sin}(\delta_o + \alpha_o)] = 0 \qquad (2.50)$$

Rearranging Equations (2.49) and (2.50) we get

$$\frac{\mathrm{Sin}\ \beta_o\ \mathrm{Sin}\ \gamma_o}{\mathrm{Sin}\ \alpha_o\ \mathrm{Sin}\ \delta_o} = \frac{\beta_1\ \mathrm{Cos}\ \beta_o\ \mathrm{Sin}\ \gamma_o + \gamma_1\ \mathrm{Cos}\ \gamma_o\ \mathrm{Sin}\ \beta_o}{\delta_1\ \mathrm{Cos}\ \delta_o\ \mathrm{Sin}\ \alpha_o + \alpha_1\ \mathrm{Cos}\ \alpha_o\ \mathrm{Sin}\ \delta_o} \qquad (2.51)$$

and

$$\frac{\mathrm{Sin}\ \beta_o\ \mathrm{Sin}\ \gamma_o}{\mathrm{Sin}\ \alpha_o\ \mathrm{Sin}\ \delta_o} = \frac{\beta_1\ \mathrm{Sin}\ \gamma_o\ \mathrm{Cos}\ \gamma_o + \gamma_1\ \mathrm{Sin}\ \beta_o\ \mathrm{Cos}\ \beta_o}{\delta_1\ \mathrm{Sin}\ \alpha_o\ \mathrm{Cos}\ \alpha_o + \alpha_1\ \mathrm{Sin}\ \delta_o\ \mathrm{Cos}\ \delta_o} \qquad (2.52)$$

TABLE V

APPARENT CORRELATION BETWEEN THE DIFFERENT MOBILITY CRITERIA

	Artobolevski and Dobrovol'ski's Criterion				Sharikov's Criterion		Viones and Atanasiu's Criterion	
Kutzbach's parameter b	General Constraints m	The possible component's of the general motion R = rotation; T = translation	Kolchin's number of passive constraints when $H^{**} = m$	Moroshkin's number of independent transformation equations	Number of reciprocal screws	Geometrical locus of the axes of screws with either zero pitch (revolute pairs) or infinite pitch (prismatic pair)	Parameter Q the rank of the matrix of the screws	Ensemble's of straight lines
6	0	3R, 3T	0	6	0	Arbitrary location in space	6^{***}	
5^{*}	1	3R, 2T; 2R, 3T	1	5	1	Two bundles of lines, with centers located at a finite or infinitely extended distance	5^{***}	
4^{*}	2	3R, 1T; 2R, 2T; 1R, 3T	2	4	2	Two hyperboloids with two common producers	4^{***}	
3	3	3R; 2R, 1T; 1R, 2T; 3T	3	3	3	(a) Surface of hyperboloid e.g., 4R Bennett mechanism (b) Bundles of lines with center located at a finite distance, e.g., 4R spherical mechanism (c) Bundle of parallel lines, e.g., 4R plane mechanism (d) Pairs located arbitrarily on an infinitely extended plane, e.g., 4P space mechanism	3	(a) generators of the same family of a ruled quadric surface (b) generators of the same family of a hyperbolic paraboloid (c) three straight lines at infinity or the ensemble of all the lines at infinity (d) ensemble of all coplanar lines (e) Star of concurrent lines, etc.
2	4	2R; 1R, 1T; 2T	4	2	4	Parallel lines located on an infinite extended plane, e.g., plane mechanism with sliding pair	2	(a) the planar cone of straight lines concurrent in 0 and coplanar (b) Two straight lines parallel and situated in P

*These values were not proposed by Kutzbach.

**H > m or H < m contradicts the proposed concept of general constraints.

***Vionea and Atanasiu did not investigate the ensembles of straight lines for these values.

Equations (2.51) and (2.52) represent the necessary conditions for having one passive pair at the joint 3.

However, there is one objection to this method of finding the conditions of passive or general constraints because it is also able to generate mechanisms which are characterized by more than one general constraint. Apart from this, if one were to study the constraint conditions on mechanisms such as RRRCC or RRRRRC, the mathematics involved requires the examination of the roots of a determinant equations having an order as high as thirty-two.

Similarities in the Criteria of General Constraints

All of the proposed mobility criteria have a correlation with one another. In Table V it can be seen that the Kutzbach parameter b, which defines the total freedom possible, correlates with Moroshkin's parameter r, which is the rank of the independent transformation equations, and with Vionea and Atanasiu's parameter Q, which is the rank of the matrix of the coefficients associated with the classical screws. Table V also shows that Artobolevski and Dobrovol'ski's parameter m, which designates the number of general constraints, is analogous to Sharikov's parameter S, which is the number of reciprocal screws. Furthermore, Kutzbach's parameter b, Moroshkin's parameter r, Vionea and Atanasiu's parameter Q, Artobolevski and Dobrovol'ski's parameter m and Sharikov's parameter S are inter-related. This relationship can be expressed in terms of two parameters A and B where $B = Q = r = b$, and $B = S = m$, so that these parameters satisfy the condition

$$A + B = 6 \tag{2.53}$$

Thus, each of these mobility criteria establish similar relationships between the freedom of the mechanism and the parameter defining the general constraints. The only exception among the studies is the proposal by Kolchin [28]. His contention that there are passive constraints or passive freedom conditions that can exist other than, or in addition to, the conditions defined as general constraints appears to contradict all of the other theories. Since each of the above criterion arrive at similar conclusions from totally different paths, it raises some doubt that Kolchin's parameter H is valid. However, until general constraints are defined, there is no way to refute the possibility of other "special" constraints in addition to "general" constraints.

Nature of One General Constraint

The concept of general constraints suggests that there are certain specific geometrical conditions which must be imposed on a kinematic chain if it is to have one degree of freedom. According to the mobility criterion of Artobolevski and Dobrovol'ski, a six-link six-revolute kinematic chain can have one degree of freedom if it has one general constraint. The exact nature of this one general constraint is not completely known although Artobolevski [21] and Dobrovol'ski [22], Altman [35], [36], [37], [38], [39], Franke [40], Sharikov [33], and Vionea and Atanasiu [34] have each contributed some views about it. Artobolevski and Dobrovol'ski proposed that the one general constraint is defined by a specific orientation of the axes of the pairs. They contend that the condition for mobility of the six-link six-revolute mechanism is determined when one set of three revolute axes intersect at a common finitely located point and the remaining three revolute axes intersect at a second finitely located point. Franke, Vionea and Atanasiu also established the same conditions for the one general constraint as Artobolevski and Dobrovol'ski. However, Altman and Sharikov pointed out that the two intersection points could be located at a finite or at infinite distance.

Ironically, this criterion of intersections of axes fails to account for several six-link mechanisms which are known to function with one degree of freedom. For example, Sarrus's[2] six-link mechanism [41] has its six axes intersecting by pairs at three distinct points. The articulated six-link mechanism of Bricard [7] and Ladopoulou [42] have every combination of two of the axes intersecting in six different points. Thus, the criteria of intersecting axes is neither necessary nor sufficient to describe the nature of one general constraints for a six-link six-revolute mechanism.

Scope of One General Constraint Domain

When there are no general constraints (m=0), the Artobolevski-Dobrovol'ski mobility criterion reduces to the Malytcheff criterion. Harrisberger[3] [29] showed that there are 13 different types and 435 different kinds of single-loop, single degree of freedom space chains which do not have general constraints. In a similar manner it is

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[2]The name "Sarrus" is spelled quite often as "Sarrut".

[3]Reference [29] is in error due to the omission of one type of chain described by the combination $1p_1 + 1p_2 + 1p_4$ and various counting errors.

possible to survey the one general constraint domain to determine the types and kinds of chains that could exist.

The existence of one general constraint is specified in the Artobolevski equation when parameter m equals 1. Consequently, the mobility criterion of Artobolevski for all mechanisms having one general constraint is

$$F = 5(n-1) - 4p_1 - 3p_2 - 2p_3 - 1p_4 \tag{2.54}$$

Observe that the existence of one general constraint eliminates all kinematic pairs having five degrees of freedom. A maximum of six links is possible only when the class I pairs are employed in the synthesis of a kinematic chain. The six links may include a variety of combinations of both the kinematic-link and the kink-link components. Similarly, when class I and class II pairs are used, the number of permissible links is five. That is, the kinematic chain contains four class I pairs and one class I pair. Continuing in this manner, one can obtain two types of four-link chains containing either two class I pairs and two class II pairs or three class I pairs and one class III pair.

According to the classification of kinematic pairs of Harrisberger [29], there are three types of class I pairs, three types of class II pairs, four types of class III pairs, and three types of class IV pairs. Thus, in the one general constraint domain, there are 28 kinds of chains with six links, 45 kinds of chains with five links, 76 kinds of chains with four links, etc. Table VI is a summary of a survey of the types and kinds of single degree of freedom, single-loop chains requiring one general constraint for mobility. Observe that there are eight different types of chains and 212 different kinds. It should also

TABLE VI

TYPES AND KINDS OF SINGLE DEGREE OF FREEDOM KINEMATIC CHAINS HAVING ONE GENERAL CONSTRAINT

No. Links	Type	No.	Kinds
6	$6p_1$	28	6R, 6P, 6H, 5R + [1P, 1H], 4R + [2P, 2H, 1P + 1H], 3R + [3P, 3H, 2P + 1H, 1P + 2H], 2R + [4P, 4H, 3P + 1H, 2P + 2H, 1P + 3H], 1R + [5P, 5H, 4P + 1H, 3P + 2H, 2P + 3H, 1P + 4H], 5P + 1H, 4P + 2H, 3P + 3H, 2P + 4H, P + 5H
5	$4p_1 + 1p_2$	45	4R + [1C, 1T, $1T_H$], 3R + [1P, 1H] + [1C, 1T, $1T_H$], 2R + [2P, 2H, 1P + 1H] + [1C, 1T, $1T_H$], 1T + [3P, 3H, 2P + 1H, 1P + 2H] + [1C, 1T, $1T_H$], [4P, 4H, 3P + 1H, 2P + 2H, 1P + 3H] + [1C, 1T, $1T_H$]
4	$2p_1 + 2p_2$	36	2R + [2C, 2T, $2T_H$, 1C + 1T, 1C + $1T_H$, 1T + $1T_H$], 1R + [1P, 1H] + [2C, 2T, $2T_H$, 1C + 1T, 1C + $1T_H$, 1T + $1T_H$], [2P, 2H, 1P + 1H] + [2C, 2T, $2T_H$, 1C + 1T, 1C + $1T_H$, 1T + $1T_H$]
4	$3p_1 + 1p_3$	40	3R + [1S, $1S_S$, $1S_{SH}$, $1P_L$], 2R + [1P + 1H] + [1S, $1S_S$, $1S_{SH}$, $1P_L$], 1R + [2P, 2H, 1H + 1P] + [1S, $1S_S$, $1S_{SH}$, $1P_L$], [3P, 3H, 2P + 1H, 1P + 2H] + [1S, $1S_S$, $1S_{SH}$, $1P_L$]
3	$2p_1 + 1p_4$	18	2R + [$1S_G$, $1S_{HG}$, $1C_p$], 1R + [1P, 1H] + [$1S_G$, $1S_{HG}$, $1C_p$], [2P, 2H, 1P + 1H] + [$1S_G$, $1S_{HG}$, $1C_p$]
3	$1p_1 + 1p_2 + 1p_3$	36	1R + [1C, 1T, $1T_H$] + [1S, $1S_S$, $1S_{SH}$, $1P_L$], [1P, 1H] + [1C, 1T, $1T_H$] + [1S, $1S_S$, $1S_{SH}$, $1P_L$]
3	$3p_2$	3	3C, 3T, $3T_H$
2	$1p_2 + 1p_4$	6	[1C, 1T, $1T_H$] + [$1S_G$, $1S_{HG}$, $1C_p$]

Total 8 types and 212 kinds

The following abbreviations are used

R = Revolute;	P = Prism;	H = Helix
T = Torus;	C = Cylinder;	T_H= Torus-helix
S = Sphere;	S_S= Sphere Slotted Helix;	P_L= Plane
S_G= Sphere Groove;	S_{GH} = Sphere Grooved Helix;	C_p= Cylinder-plane

be noted that each of the mechanisms from this group could possibly have up to six kinematic inversions, but there is no assurance that each of them would also have a single degree of freedom.

Although the mobility criterion for one general constraint indicates that in addition to the six-link six-revolute mechanism there are more than 200 other mechanisms that have one general constraint, physical models of most of these mechanisms are not known since we know nothing of the geometric conditions which create the general constraints. We have no way of knowing how to assemble these mechanisms so they will have constrained mobility, except by trial and error.

CHAPTER III

THEORY OF IDENTIFYING THE EXISTENCE OF GENERAL CONSTRAINTS

The examination of the number of existing theories makes us aware of the complexity of the problem in identifying and determining the degrees of motion of kinematic chains. These problems become more involved when the chains having more than four physical links are under consideration. The explicit governing conditions that identify the existence of one or two general constraints are, therefore, not readily obtainable with the approaches examined in the previous chapter. For instance, the approaches suggested by Vionea and Atanasiu and Sharikov are primarily of analytical nature; that is, the application of either of these approaches is expected to point out an existence or non-existence of a mechanism. Although the approach suggested by Dimentberg promises an explicit governing condition, the mathematics of determining the one general constraint condition requires the examination of the roots of a determinant equation of order thirty-tow. Such mathematical approaches of examining the roots of the higher order determinant equations may be expected to lead to all types of erroneous results.

The classical theories defining the degrees of mobility predicts thousands of mechanisms having general constraints whose value varies from a minimum of zero to a maximum of four. However, all of the governing conditions that define a spatial kinematic chain as a mechanism are not known. It is generally believed that such governing conditions are relatively simple for the unconstrained mechanisms and that they become more complex for the mechanisms having one or more general constraints. It should be noted, however, that even these simple governing conditions are not known. Thus, some of the fundamental problems, such as the maximum number of permissible sliding or helical pairs in a spatial mechanism, remain to be solved. However, among these fundamental problems the one of considerable importance is that of examining the governing conditions defining one or more general constraints. Under the ideal situation, this examination of the governing conditions of the general constraints is expected to reveal,

(a) the closure condition for a chain, that is, a set of parameters associated with each link in order to form a closed kinematic chain configuration,

(b) the mobility of the chain when one of the links is fixed,

(3) the limit positions and the dead center of the mechanism.

In the sections to follow, a general theory of examining the existence or nonexistence of a general constraint is developed.

Development of the Theory of Identifying the Existence of General Constraints

Under the ideal condition, a true space mechanism is expected to have a general motion consisting of three rotations (ω_x, ω_y, ω_z) and three translations (τ_x, τ_y, τ_z), along a set of three independent axes x, y, and z. The underlying philosophy of the one general constraint then would state that for some specific geometric configuration of a chain the total number of components of its general motion is either three rotations (ω_x, ω_y, ω_z) and two translations, such as (τ_x, τ_y), (τ_x, τ_z) or (τ_y, τ_z), or two rotations, such as (ω_x, ω_y), (ω_y, ω_z), or (ω_x, ω_z) and three translations (τ_x, τ_y, τ_z).

With a starting assumption of the six components of the general motion, one is expected to set up six simultaneous independent equations relating the six parameters of the general motion ω_x, ω_y, ω_z, τ_x, τ_y, τ_z. It is possible to arrive at this set of six equations by considering the physical significance of the general constraints. For instance, according to F. M. Dimentberg, the existence of one general constraint is expected to impose a condition on a cylinder pair of a mechanism described by a combination RRRRRC. Observe that the first revolute pair R is the input pair and the cylinder pair C is the output pair. The imposed condition of one general constraint on the cylinder pair can be described mathematically as

$$\frac{dS_6}{d\theta_1} = 0 \tag{3.1}$$

where S_6 is the translation permitted by the cylinder pair and θ_1 is the rotation at the input pair. Note that this relationship, given by Equation (3.1), is expected to be true for a total possible range of θ_1. Integration of Equation (3.1) with respect to θ_1 results in

$$S_6 = \text{constant} \tag{3.2}$$

The physical interpretation of the Equation (3.2) suggests that the cylinder pair is made passive for its translational movement; that is, the activity of the cylinder pair is confined to a pure rotation. This condition of restraining the cylinder pair to a pure rotation will then describe the mechanism RRRRRC as a RRRRRR mechanism. Thus, the existence of the condition given by Equation (3.1) in a mechanism such as the 6R mechanism describes the existence of one general constraint. Similarly, the existence of two simultaneous conditions similar to that of Equation (3.1) in a mechanism RRRCC induces the existence of two general constraints and the resulting mechanism can be described as a RRRRR mechanism.

The general mathematical tool that lends itself to induce the mathematical conditions given either by Equation (3.1) or Equation (3.2) and also abide by the general philosophy of the general constraints is the three-by-three screw matrix. This three-by-three screw matrix is composed of a product of two three-by-three dual matrices both describing a rotation and translation of a rigid body, one about the x axis and the other describing about the z axis. Thus, the resultant product of these two three-by-three dual matrices is expected to describe a rotation and translation of a free body about some third instantaneous axis called a screw axis. This screw matrix is given by

$$T_i(\hat{\theta}) = \begin{bmatrix} \mathrm{Cos}\,\hat{\theta}_i & -\mathrm{Sin}\,\hat{\theta}_i\,\mathrm{Cos}\,\hat{\alpha}_i & \mathrm{Sin}\,\hat{\theta}_i\,\mathrm{Sin}\,\hat{\alpha}_i \\ \mathrm{Sin}\,\hat{\theta}_i & \mathrm{Cos}\,\hat{\theta}_i\,\mathrm{Cos}\,\hat{\alpha}_i & -\mathrm{Cos}\,\hat{\theta}_i\,\mathrm{Sin}\,\hat{\alpha}_i \\ 0 & \mathrm{Sin}\,\hat{\alpha}_i & \mathrm{Cos}\,\hat{\alpha}_i \end{bmatrix} \qquad (3.3)$$

where $\hat{\theta}_i$ and $\hat{\alpha}_i$ are the "dual angles" where (see Appendix A)

$$\hat{\theta}_i = \theta_i + \sigma\, s_i$$

$$\hat{\alpha}_i = \alpha_i + \sigma\, a_i$$

where a_i, α_i, θ_i, and s_i are the physical parameters associated with a link of a kinematic chain. These parameters a_i, α_i, θ_i, and s_i and their relationships to one another are shown in Figure 2. Observe that the parameter a_i represents the kinematic link of a chain, α_i the twist angles between the axes, θ_i the angle between the kinematic link and s_i the offset distance along the axis between the two common perpendiculars of the two connected links. This distance can be physically interpreted as a kink in the kinematic link.

According to the mobility criteria, when the mechanism has no general constraints, i.e., m = 0, it can be shown that

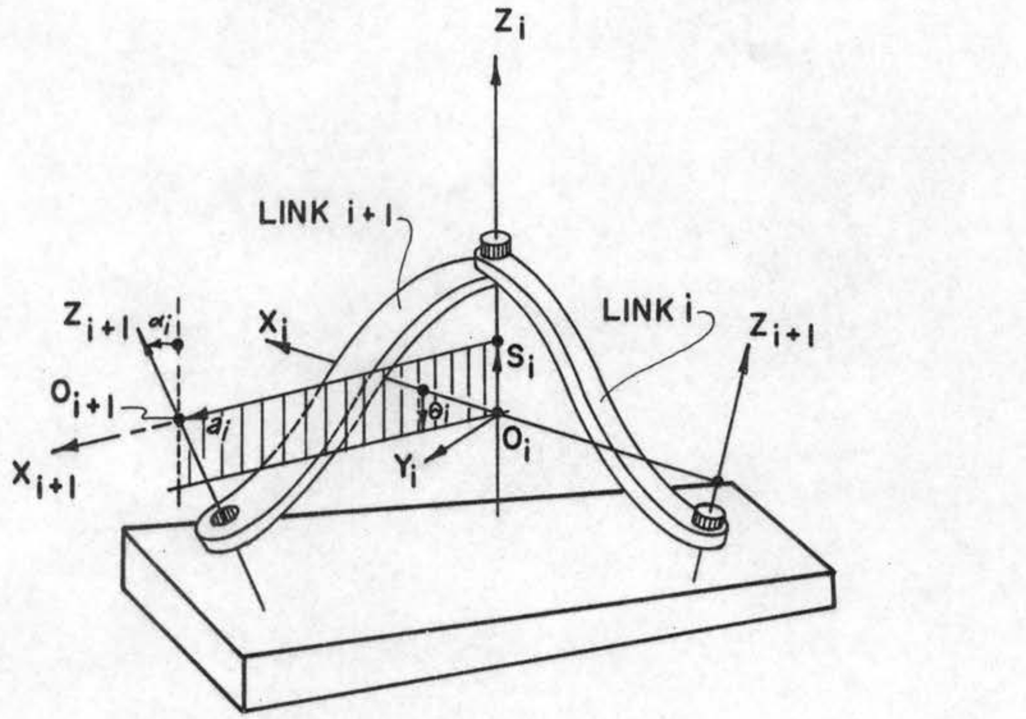
$$\Sigma\, kp_k = 7 \tag{3.4}$$

Thus, when all the pairs are the revolute pairs, i.e., k = 1, then the total required number of links are seven. Thus, corresponding to the seven links of a closed chain, seven screw matrices are required to describe the motion of this mechanism. However, because the chain is a closed loop, the seven screw matrices are related. This relationship is described by

$$[T_1][T_2][T_3][T_4][T_5][T_6][T_7] = [I] \tag{3.5}$$

where the matrix [I] is the unit matrix. Observe that each of the matrices $[T_i]$ involve a_i, α_i, s_i, and θ_i.

In order to check for the mobility of a kinematic chain, displacement analysis of the mechanism of this kinematic chain must be possible. The displacement analysis of a mechanism is performed by determining the displacements of all the follower and coupler links by giving any arbitrary displacement to any one of the links and naming that link as



i-1, i, & i+1 ARE SUCCESSIVE PAIRS IN A KINEMATIC LOOP

Z_i = CHARACTERISTIC MOTION AXIS FOR PAIR i

X_i = COMMON PERPENDICULAR BETWEEN Z_{i+1} AND Z_i

Y_i = AXIS TO FORM RIGHT-HANDED CARTESIAN SYSTEM, $X_i Y_i Z_i$
(POSITIVE SENSE BASED ON CHOSEN ORIENITATIONS OF X_i + Z_i)

a_i = LENGTH OF COMMON PERPENDICULAR FROM Z_i TO Z_{i+1}

α_i = ANGLE FROM POSITIVE Z_i TO POSITIVE Z_{i+1}
(POSITIVE SENSE IS CCW ABOUT POSITIVE X_{i+1})

θ = ANGLE FROM POSITVE X_i TO POSITIVE X_{i+1}
(POSITIVE SENSE IS CCW ABOUT POSITIVE Z_i)

S = DISTANCE ALONG Z_i FROM X_i TO X_{i+1}
(POSITVE SENSE IS THAT OF POSITIVE Z_i)

Figure 2. Kinematic Notations

the input link. In a single-loop mechanism, one of the links connected to the fixed link of a mechanism can be an input link. Then the other link connected to the fixed link becomes a follower and the intermediate links become the coupler links.

Let the input link of the 7R mechanism be displaced through an angle θ_1 such that the coupler and the follower links experience a differential displacement in their original positions described by $\hat{\theta}_2$, $\hat{\theta}_3$, $\hat{\theta}_4$, $\hat{\theta}_5$, $\hat{\theta}_6$ and $\hat{\theta}_7$. In this event the matrices $[T_i]$, $(i \geq 2)$ must accommodate this change. Thus

$$T_1\ T_2(\hat{\theta}_2 + d\hat{\theta}_2)T_3(\hat{\theta}_3 + d\hat{\theta}_3)T_4(\hat{\theta}_4 + d\hat{\theta}_4)T_5(\hat{\theta}_5 + d\hat{\theta}_5)\ \ldots . \cong I \qquad (3.6)$$

Using the Taylor series expansion and neglecting all the higher order terms, the matrix $T_i(\hat{\theta}_i + d\hat{\theta}_i)$ yields the following result

$$T(\hat{\theta}_i + d\hat{\theta}_i) = T(\hat{\theta}_i) + \frac{\partial T(\hat{\theta}_i)}{\partial\ \hat{\theta}_i}\ d\hat{\theta}_i \qquad (3.7)$$

Thus, Equation (3.6) becomes

$$T(\hat{\theta}_i + d\hat{\theta}_i) = \begin{bmatrix} \text{Cos}\ \hat{\theta}_i & -\text{Sin}\ \hat{\theta}_i\ \text{Cos}\ \hat{\alpha}_i & \text{Sin}\ \hat{\theta}_i\ \text{Sin}\ \hat{\alpha}_i \\ \text{Sin}\ \hat{\theta}_i & \text{Cos}\ \hat{\theta}_i\ \text{Cos}\ \hat{\alpha}_i & -\text{Cos}\ \hat{\theta}_i\ \text{Sin}\ \hat{\alpha}_i \\ 0 & \text{Sin}\ \hat{\alpha}_i & \text{Cos}\ \hat{\alpha}_i \end{bmatrix} + \begin{bmatrix} -\text{Sin}\ \hat{\theta}_i & -\text{Cos}\ \hat{\theta}_i\ \text{Cos}\ \hat{\alpha}_i & \text{Cos}\ \hat{\theta}_i\ \text{Sin}\ \hat{\alpha}_i \\ \text{Cos}\ \hat{\theta}_i & -\text{Sin}\ \hat{\theta}_i\ \text{Cos}\ \hat{\alpha}_i & \text{Sin}\ \hat{\theta}_i\ \text{Sin}\ \hat{\alpha}_i \\ 0 & 0 & 0 \end{bmatrix} d\hat{\theta}_i \qquad (3.8)$$

Observe, however, that the second part of the Equation (3.8) is a product of an operator matrix $[P]$ with the original matrix $[T_i]$ where the operator matrix $[P]$ is defined as

$$[P] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.9}$$

Thus, the product [P][T] gives

$$[PT] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{Cos}\ \hat{\theta}_i & -\mathrm{Sin}\ \hat{\theta}_i\ \mathrm{Cos}\ \hat{\alpha}_i & \mathrm{Sin}\ \hat{\theta}_i\ \mathrm{Sin}\ \hat{\alpha}_i \\ \mathrm{Sin}\ \hat{\theta}_i & \mathrm{Cos}\ \hat{\theta}_i\ \mathrm{Cos}\ \hat{\alpha}_i & -\mathrm{Cos}\ \hat{\theta}_i\ \mathrm{Sin}\ \hat{\alpha}_i \\ 0 & \mathrm{Sin}\ \hat{\theta}_i & \mathrm{Cos}\ \hat{\alpha}_i \end{bmatrix}$$

$$= \begin{bmatrix} -\mathrm{Sin}\ \hat{\theta}_i & -\mathrm{Cos}\ \hat{\theta}_i\ \mathrm{Cos}\ \hat{\alpha}_i & \mathrm{Cos}\ \hat{\theta}_i\ \mathrm{Sin}\ \hat{\alpha}_i \\ \mathrm{Cos}\ \hat{\theta}_i & -\mathrm{Sin}\ \hat{\theta}_i\ \mathrm{Cos}\ \hat{\alpha}_i & \mathrm{Sin}\ \hat{\theta}_i\ \mathrm{Sin}\ \hat{\alpha}_i \\ 0 & 0 & 0 \end{bmatrix}$$

Rewriting in terms of the operator matrix, Equation (3.6) becomes

$$\begin{aligned} T(\hat{\theta}_i + d\hat{\theta}_i) &= T_i + PT_i\ d\hat{\theta}_i \\ &= [I + Pd\hat{\theta}_i]\ T_i \end{aligned} \tag{3.10}$$

Substituting for each of the $T(\hat{\theta}_i + d\hat{\theta}_i)$, Equation (3.6) yields,

$$\begin{aligned} T_1(I + Pd\hat{\theta}_2)T_2(I + Pd\hat{\theta}_3)T_3(I + Pd\hat{\theta}_4)T_4(I + Pd\hat{\theta}_5)T_5 \quad \times \\ (I + Pd\hat{\theta}_6)T_6(I + Pd\hat{\theta}_7)T_7 \ \cong\ I \end{aligned} \tag{3.11}$$

Expanding the above equation with the assumption that $d\hat{\theta}_2$, $d\hat{\theta}_3, \ldots,$ $d\hat{\theta}_7$ are small in magnitude, Equation (3.11) simplifies to the following:

$$\begin{aligned} &[T_1T_2T_3T_4T_5T_6T_7] + [T_1PT_2T_3T_4T_5T_6T_7]d\hat{\theta}_2 + [T_1T_2PT_3T_4T_5T_6T_7]d\hat{\theta}_3 + \\ &+ [T_1T_2T_3PT_4T_5T_6T_7]d\hat{\theta}_4 + [T_1T_2T_3T_4PT_5T_6T_7]d\hat{\theta}_5 + \\ &+ [T_1T_2T_3T_4T_5PT_6T_7]d\hat{\theta}_6 + [T_1T_2T_3T_4T_5T_6PT_7]d\hat{\theta}_7 \cong [I] \end{aligned} \tag{3.12}$$

Let

$$[Q_1] = [T_1 T_2 T_3 T_4 T_5 T_6 T_7] \tag{3.13}$$

$$[Q_2] = [T_1 P T_2 T_3 T_4 T_5 T_6 T_7] \tag{3.14}$$

$$[Q_3] = [T_1 T_2 P T_3 T_4 T_5 T_6 T_7] \tag{3.15}$$

$$[Q_4] = [T_1 T_2 T_3 P T_4 T_5 T_6 T_7] \tag{3.16}$$

$$[Q_5] = [T_1 T_2 T_3 T_4 P T_5 T_6 T_7] \tag{3.17}$$

$$[Q_6] = [T_1 T_2 T_3 T_4 T_5 P T_6 T_7] \tag{3.18}$$

and

$$[Q_7] = [T_1 T_2 T_3 T_4 T_5 T_6 P T_7] \tag{3.19}$$

Then Equation(3.12) can be written as

$$[Q_1] + [Q_2]d\hat{\theta}_2 + [Q_3]d\hat{\theta}_3 + [Q_4]d\hat{\theta}_4 + [Q_5]d\hat{\theta}_5 + [Q_6]d\hat{\theta}_6 + [Q_7]d\hat{\theta}_7 \cong I$$

or

$$\sum_{i=2}^{7} [Q_i]d\hat{\theta}_i \cong [I] - [Q_1] \tag{3.20}$$

Equation (3.20) appears to be relatively simple in the form shown here. However, it is apparent simplicity is destroyed if the nature of the screw matrix $[T_i]$ is taken into consideration. Observe that each of the terms in the $[T_i]$ matrix is a dual quantity. Thus, using the dual angle algebra and expanding each of the terms, after substituting

$$\text{Cos } \hat{\theta}_i = \text{Cos } (\theta_i + \sigma s_i) = \text{Cos } \theta_i - \sigma s_i \text{ Sin } \theta_i \tag{3.21}$$

$$\text{Sin } \hat{\theta}_i = \text{Sin } (\theta_i + \sigma s_i) = \text{Sin } \theta_i + \sigma s_i \text{ Cos } \theta_i \tag{3.22}$$

$$\text{Sin } \hat{\alpha}_i = \text{Sin } (\alpha_i + \sigma a_i) = \text{Sin } \alpha_i + \sigma a_i \text{ Cos } \alpha_i \tag{3.23}$$

$$\text{Cos } \hat{\alpha}_i = \text{Cos } (\alpha_i + \sigma a_i) = \text{Cos } \alpha_i - \sigma a_i \text{ Sin } \alpha_i \tag{3.24}$$

the screw matrix $[T_i]$ decomposes into two matrices as follows

$$[T_i] = \begin{bmatrix} \text{Cos } \theta_i & -\text{Cos } \alpha_i \text{ Sin } \theta_i & \text{Sin } \alpha_i \text{ Sin } \theta_i \\ \text{Sin } \theta_i & \text{Cos } \alpha_i \text{ Cos } \theta_i & -\text{Sin } \alpha_i \text{ Cos } \theta_i \\ 0 & \text{Sin } \alpha_i & \text{Cos } \alpha_i \end{bmatrix}$$

$$+ \sigma \begin{bmatrix} -s_i\text{Sin}\theta_i & a_i\text{Sin}\alpha_i\text{Sin}\theta_i - s_i\text{Cos}\theta_i\text{Cos}\alpha_i & a_i\text{Cos}\alpha_i\text{Sin}\theta_i + s_i\text{Cos}\theta_i\text{Sin}\alpha_i \\ s_i\text{Cos}\theta_i & -a_i\text{Sin}\alpha_i\text{Cos}\theta_i - s_i\text{Sin}\theta_i\text{Cos}\alpha_i & -a_i\text{Cos}\alpha_i\text{Cos}\theta_i + s_i\text{Sin}\alpha_i\text{Sin}\theta_i \\ 0 & a_i\text{Cos}\alpha_i & -a_i\text{Sin}\alpha_i \end{bmatrix} \quad (3.25)$$

i.e.,

$$[T_i] = [R_i] + \sigma[D_i] \quad (3.26)$$

where $[R_i]$ represents the real part and $[D_i]$ represents the dual part of the matrix $[T_i]$. Observe that the real matrix $[R_i]$ represents a pure rotation. Furthermore, the real matrix $[R_i]$ is an orthogonal matrix but the dual part matrix $[D_i]$ does not have the same property.

In view of the existing property of the screw matrix $[T_i]$, described by Equation (3.26), the Equations (3.15) to (3.19) need to be simplified. For instance, consider Equation (3.13) which gives

$$[Q_i] = [T_1T_2T_3T_4T_5T_6T_7] \quad (3.13)$$

Substituting $[T_i] = [R_i] + \sigma[D_i]$, Equation (3.13) becomes

$$[Q_1] = [R_1 + \sigma D_1][R_2 + \sigma D_2][R_3 + \sigma D_3][R_4 + \sigma D_4][R_5 + \sigma D_5] \times$$
$$[R_6 + \sigma D_6][R_7 + \sigma D_7] \quad (3.27)$$

Simplifying the above relationship, keeping in mind that $\sigma^2 = 0$, we get

$$[Q_1] = [R_1R_2R_3R_4R_5R_6R_7] + \sigma \begin{bmatrix} [D_1R_2R_3R_4R_5R_6R_7] + \\ [R_1D_2R_3R_4R_5R_6R_7] + \\ [R_1R_2D_3R_4R_5R_6R_7] + \\ [R_1R_2R_3D_4R_5R_6R_7] + \\ [R_1R_2R_3R_4D_5R_6R_7] + \\ [R_1R_2R_3R_4R_5D_6R_7] + \\ [R_1R_2R_3R_4R_5R_6D_7] \end{bmatrix} \quad (3.28)$$

Similarly, each of the matrices Q_i(3x3) can be simplified. Thus,

$$[Q_2] = [T_1PT_2T_3T_4T_5T_6] = [R_1PR_2R_3R_4R_5R_6] + \sigma \begin{bmatrix} [D_1PR_2R_3R_4R_5R_6R_7] + \\ [R_1PD_2R_3R_4R_5R_6R_7] + \\ [R_1PR_2D_3R_4R_5R_6R_7] + \\ [R_1PR_2R_3D_4R_5R_6R_7] + \\ [R_1PR_2R_3R_4D_5R_6R_7] + \\ [R_1PR_2R_3R_4R_5D_6R_7] + \\ [R_1PR_2R_3R_4R_5R_6D_7] \end{bmatrix} \quad (3.29)$$

$$[Q_3] = [T_1T_2PT_3T_4T_5T_6T_7] = [R_1R_2PR_3R_4R_5R_6R_7] + \sigma \begin{bmatrix} [D_1R_2PR_3R_4R_5R_6R_7] + \\ [R_1D_2PR_3R_4R_5R_6R_7] + \\ [R_1R_2PD_3R_4R_5R_6R_7] + \\ [R_1R_2PR_3D_4R_5R_6R_7] + \\ [R_1R_2PR_3R_4D_5R_6R_7] + \\ [R_1R_2PR_3R_4R_5D_6R_7] + \\ [R_1R_2PR_3R_4R_5R_6D_7] \end{bmatrix} \quad (3.30)$$

$$[Q_4] = [T_1T_2T_3PT_4T_5T_6T_7] = [R_1R_2R_3PR_4R_5R_6R_7] + \sigma \begin{bmatrix} [D_1R_2R_3PR_4R_5R_6R_7] + \\ [R_1D_2R_3PR_4R_5R_6R_7] + \\ [R_1R_2D_3PR_4R_5R_6R_7] + \\ [R_1R_2R_3PD_4R_5R_6R_7] + \\ [R_1R_2R_3PR_4D_5R_6R_7] + \\ [R_1R_2R_3PR_4R_5D_6R_7] + \\ [R_1R_2R_3PR_4R_5R_6D_7] \end{bmatrix} \tag{3.31}$$

$$[Q_5] = [T_1T_2T_3T_4PT_5T_6T_7] = [R_1R_2R_3R_4PR_5R_6R_7] + \sigma \begin{bmatrix} [D_1R_2R_3R_4PR_5R_6R_7] + \\ [R_1D_2R_3R_4PR_5R_6R_7] + \\ [R_1R_2D_3R_4PR_5R_6R_7] + \\ [R_1R_2R_3D_4PR_5R_6R_7] + \\ [R_1R_2R_3R_4PD_5R_6R_7] + \\ [R_1R_2R_3R_4PR_5D_6R_7] + \\ [R_1R_2R_3R_4PR_5R_6D_7] \end{bmatrix} \tag{3.32}$$

$$[Q_6] = [T_1T_2T_3T_4T_5PT_6T_7] = [R_1R_2R_3R_4R_5PR_6R_7] + \sigma \begin{bmatrix} [D_1R_2R_3R_4R_5PR_6R_7] + \\ [R_1D_2R_3R_4R_5PR_6R_7] + \\ [R_1R_2D_3R_4R_5PR_6R_7] + \\ [R_1R_2R_3D_4R_5PR_6R_7] + \\ [R_1R_2R_3R_4D_5PR_6R_7] + \\ [R_1R_2R_3R_4R_5PD_6R_7] + \\ [R_1R_2R_3R_4R_5PR_6D_7] \end{bmatrix} \tag{3.33}$$

$$[Q_7] = [T_1 T_2 T_3 T_4 T_5 T_6 P T_7] = [R_1 R_2 R_3 R_4 R_5 R_6 P R_7] + \sigma \begin{bmatrix} [D_1 R_2 R_3 R_4 R_5 R_6 P R_7] + \\ [R_1 D_2 R_3 R_4 R_5 R_6 P R_7] + \\ [R_1 R_2 D_3 R_4 R_5 R_6 P R_7] + \\ [R_1 R_2 R_3 D_4 R_5 R_6 P R_7] + \\ [R_1 R_2 R_3 R_4 D_5 R_6 P R_7] + \\ [R_1 R_2 R_3 R_4 R_5 D_6 P R_7] + \\ [R_1 R_2 R_3 R_4 R_5 R_6 P D_7] \end{bmatrix} \tag{3.34}$$

Observe that $[Q_i]$ matrices have been decomposed into a set of real matrices and dual part matrices. Denoting the real and dual part components of $[Q_i]$ by $[A_i]$ and $[B_i]$ we obtain

$$[Q_i] = [A_i] + \sigma[B_i]$$

Thus, for i = 2,

$$[A_2] = [R_1 P R_2 R_3 R_4 R_5 R_6] \tag{3.35}$$

and

$$[B_2] = \begin{bmatrix} [D_1 P R_2 R_3 R_4 R_5 R_6 R_7] + [R_1 P D_2 R_3 R_4 R_5 R_6 R_7] + \\ [R_1 P R_2 D_3 R_4 R_5 R_6 R_7] + [R_1 P R_2 R_3 D_4 R_5 R_6 R_7] + \\ [R_1 P R_2 R_3 R_4 D_5 R_6 R_7] + [R_1 P R_2 R_3 R_4 R_5 D_6 R_7] + \\ [R_1 P R_2 R_3 R_4 R_5 R_6 D_7] \end{bmatrix} \tag{3.36}$$

Observe that the matrices $[R_i]$, $[D_i]$, and $[P]$ have each three rows and three columns. Therefore, the product matrices $[A_i]$ and $[B_i]$ must also have three rows and three columns.

Using this notation, Equation (3.20) can be rewritten as

$$\sum_{i=2}^{7} [A_i + \sigma B_i] d\hat{\theta}_i \cong [I] - [A_1 + \sigma B_1] \tag{3.37}$$

i.e.,

$$\sum_{i=2}^{7} [A_i] d\hat{\theta}_i + \sigma \sum_{i=2}^{7} [B_i] d\hat{\theta}_i \cong [I] - [A_1] - \sigma[B_1] \tag{3.38}$$

Recall that each of the dual angle $\hat{\theta}_i$ can be written as

$$\hat{\theta}_i = \theta_i + \sigma s_i \tag{3.39}$$

Differentiating both the sides, we get

$$d\hat{\theta}_i = d\theta_i + \sigma d\ s_i \tag{3.40}$$

Observe, however, that if s_i is not a variable, then

$$d\hat{\theta}_i = d\theta_i \tag{3.41}$$

The case in which s_i becomes variable is the one in which a kinematic chain has a cylinder pair. For the seven-link mechanism to move with one degree of freedom, all the kinematic pairs are the revolute pairs, and therefore, all the s_i are of constant values. Thus, Equation (3.39) becomes

$$\sum_{i=2}^{7} [A_i] d\theta_i + \sigma \sum_{i=2}^{7} [B_i] d\theta_i \cong [I] - [A_1] - \sigma[B_1] \tag{3.42}$$

Separating the real and dual part of the Equation (3.42), we get a set of two equations which are

$$\sum_{i=2}^{7} [A_i] d\theta_i \cong [I] - [A_1] \tag{3.43}$$

and

$$\sum [B_i] d\theta_i \cong - [B_1] \tag{3.44}$$

Since each of the $[A_i]$ and $[B_i]$ have three rows and three columns, thereby having nine elements, Equations (3.43) and (3.44) together represent a set of eighteen equations in six unknowns $d\theta_2$, $d\theta_3$, $d\theta_4$, $d\theta_5$, $d\theta_6$, and $d\theta_7$. Corresponding to these six unknowns, therefore, a minimum of six independent equations must exist in order that the kinematic chain of 7R moves with one degree of freedom when one of the links is fixed. The following is the procedure to obtain a set of six independent equations from the set of these eighteen equations.

Recall that $[P]$ is an anti-symmetric matrix. Because of this property, the product matrix

$$[G] = [Z][P][Z]^t \tag{3.45}$$

is also anti-symmetric, where $[Z]^t$ is a transpose of any matrix $[Z]$. Now, consider any one of the matrices $[Q_i]$ given by Equations (3.13) to (3.19), say $[Q_2]$ then

$$[Q_2] = [T_1PT_2T_3T_4T_5T_6] \tag{3.46}$$

Let $[T_1]^{-1}$ be the inverse of $[T_1]$. Then Equation (3.46) can be rewritten as

$$[Q_2] = [T_1P][T_1^{-1}T_1][T_2T_3T_4T_5T_6T_7] \tag{3.47}$$

i.e.,

$$[Q_2] = [T_1PT_1^{-1}][T_1T_2T_3T_4T_5T_6T_7] \tag{3.48}$$

Observe, however, that from Equation (3.5)

$$[T_1T_2T_3T_4T_5T_6T_7] = [I]$$

and therefore,

$$[Q_2] = [T_1PT_1^{-1}] \tag{3.49}$$

If the screw matrix were to describe only pure rotation, then from the definition it is known that the screw matrix is an orthogonal matrix. Therefore,

$$[T_1]^{-1} = [T_1]^t \tag{3.50}$$

Thus, Equation (3.49) can be rewritten as

$$[Q_2] = [T_1 P T_1^{\ t}] \tag{3.51}$$

Comparing the two equations, (3.51) and (3.45), we deduce that the matrix $[Q_2]$ must be an anti-symmetric matrix, i.e.,

$$[Q_2] = \begin{bmatrix} 0 & q_{12} & q_{13} \\ -q_{12} & 0 & q_{23} \\ -q_{13} & -q_{23} & 0 \end{bmatrix} \tag{3.52}$$

Clearly, Equation (3.52) suggests that out of the nine elements only three elements are independent under a complete closure condition. That is, when

$$[T_1 T_2 T_3 T_4 T_5 T_6 T_7] = [I]$$

However, since $[Q_2]$ decomposes into the real and the dual components, there are altogether twelve independent elements available to obtain the set of simultaneous relationships in $d\theta_i$ described by the Equations (3.43) and (3.44). Thus, Equation (3.52) can be written as

$$[Q_2] = \begin{bmatrix} 0 & q_{12} & q_{13} \\ -q_{12} & 0 & q_{23} \\ -q_{13} & -q_{23} & 0 \end{bmatrix}$$

i.e.,

$$[Q_2] = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix} + \sigma \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ -b_{12} & b_{22} & b_{23} \\ -b_{13} & -b_{23} & b_{33} \end{bmatrix} \qquad (3.53)$$

It can be seen that the similar relationships can be derived for the product matrices $[Q_i]$ where i takes the value one through seven. Observe that all the diagonal elements of each of the real part matrices are zero, but the diagonal elements of the matrix $[Q_i]$ may not be zero. This is due to the fact that dual part matrix is not an orthogonal matrix. These elements, however, do become zero under special conditions. These governing special conditions are yet not known.

The problem of obtaining the number of independent equations from the set of twelve equations becomes complicated. However, the principle of transference as proposed by A. P. Kotelnikoff [48] is applied. Accordingly, the number of independent equations obtained from real part and from dual part matrices must be the same. Since there are only off-diagonal elements contributing the three independent equations from the real part matrix, then the application of the "principle of transference" suggests that there are three independent dual part equations obtained from the off-diagonal elements of the dual part of the matrix $[Q_i]$.

Thus, each of $[A_i]$ and $[B_i]$ of Equations (3.43) and (3.44) under the closure condition contributes three elements to form a set of six independent equations. Furthermore, these contributed elements of $[A_i]$ and $[B_i]$ are, in fact, the off-diagonal elements. Therefore, Equations (3.43) and (3.44) may be written as

$$\sum_{i=2} A_{ijk}\, d\theta_i \cong -[A_{1jk}] \tag{3.54}$$

$$\sum_{i=2} B_{ijk}\, d\theta_i \cong -[B_{1jk}] \tag{3.55}$$

where j and k denote respectively the rows and columns of the ith matrix. Equations (3.54) and (3.55) can be futher modified if we consider the conditions under which they are derived. Recall that these equations are the result of the assumption that a closure condition for a kinematic chain is achieved. Under this assumption

$$[Q_1] = [T_1 T_2 T_3 T_4 T_5 T_6 T_7] = [I] \tag{3.56}$$

$$= [A_{1ji}] + \sigma [B_{jk}] \tag{3.57}$$

Since the unit matrix $[I]$ is a real matrix, then equating the real and the dual parts we get

$$[A_{1jk}] = [I] \tag{3.58}$$

and

$$[B_{1jk}] = [N] \tag{3.59}$$

where the matrix $[N]$ is the null matrix. Equation (3.58) indicates that all the off-diagonal elements of the matrix $[A_{1jk}]$ are zero. Furthermore, Equation (3.59) indicates that all the elements of the matrix $[B_{1jk}]$ are zero. Consequently, Equations (3.54) and (3.55) become a set of six simultaneous homogeneous equations. These equations may be written in the matrix form as

$$\begin{bmatrix} a_{212} & a_{312} & a_{412} & a_{512} & a_{612} & a_{712} \\ a_{213} & a_{313} & a_{413} & a_{513} & a_{613} & a_{713} \\ a_{223} & a_{323} & a_{423} & a_{523} & a_{623} & a_{723} \\ b_{212} & b_{312} & b_{412} & b_{512} & b_{612} & b_{712} \\ b_{213} & b_{313} & b_{413} & b_{513} & b_{613} & b_{713} \\ b_{223} & b_{323} & b_{423} & b_{523} & b_{623} & b_{723} \end{bmatrix} \begin{bmatrix} d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \\ d\theta_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (3.60)$$

i.e.,

$$[M][\Delta\theta] = [0] \qquad (3.61)$$

where the matrix $[M]$ is the coefficient of the differentials of the angular displacements of the links 2, 3, 4, 5, 6, and 7, and the column matrix $[\Delta\theta]$ is the differential displacements. When the closure condition is obtained after giving a differential displacement to these links, the angular positions θ_2, θ_3, θ_4, θ_5, θ_6, and θ_7 of these links are described by their corresponding exact values. Consequently, the column matrix $[\Delta\theta]$ must consist of a null vector in order to satisfy Equation (3.60). The coefficient matrix $[M]$, however, remains nonsingular. Since there are six independent rows, the rank of this matrix must be six.

The coefficient matrix $[M]$ plays a significant role in answering some of the basic issues related to the mobility of a kinematic chain. Observe that this matrix has six rows and six columns. These six columns correspond to the six unknown dependent displacements. In general, the number of columns of the coefficients matrix and the number of dependent displacements of a single-loop mechanism are related. This relationship can be expressed as

$$\text{Number of columns} = (\text{Total number of linear and angular displacements}) - 1 \tag{3.62}$$

The above relationship stems out clearly from the fact that in a mechanism a kinematic pair of one degree of freedom is used for the input motion and the motion at the other kinematic pair is simply dependent on the motion of the input pair. Thus, in the 7R chain θ_1 is the angular motion at the input pair and the angular motions θ_2, θ_3, θ_4, θ_5, θ_6, and θ_7 are simply dependent on the input motion.

The application of Equation (3.62) suggests that in the six-link Bricard mechanism, where all the kinematic pairs are the revolute pairs, the number of columns of the coefficient matrix [M] is five. Similarly, the Goldberg five-link and the Bennett four-link mechanism will have, respectively, four and three columns in the coefficient matrix [M].

The rows of the coefficient matrix, however, exhibit altogether different properties. These properties appear to correlate with the basic concept of the general constraints. The number of independent rows that can be obtained for a mechanism is entirely dependent upon the specific configuration of the mechanism.

Observe that the total number of rows are six and that they are not related in any manner with either the total number of links or the total number of kinematic pairs of a mechanism. Note that the first three rows in the matrix [M] are obtained from the real part of the $[Q_i]$ matrix and that the last three rows are obtained from the dual part of the matrix $[Q_i]$. It has been observed, however, that it is the specific geometric configuration of the mechanism that decides on the number of independent real and dual rows of the coefficient matrix [M].

Coefficient Matrix [M] for the Spherical Four-Link Mechanism

A specific configuration does exist wherein all the dual components assume zero values. That is,

$$\hat{\alpha}_i = \alpha_i + \sigma(0) \tag{3.63}$$

and

$$\hat{\theta}_i = \theta_i + \sigma(0) \tag{3.64}$$

Such a configuration can be described on a sphere, for instance, the spherical four-link mechanism. In this case, all the three equations obtained from the dual components of the matrices $[Q_i]$ are zero, thus leaving only the first three real row vectors in the coefficient matrix [M]. Since there are four revolute pairs, the application of Equation (3.62) suggests that there are only three columns in the matrix [M]. Thus, for a spherical four-link mechanism the coefficient matrix [M] is expected to take the following form:

$$[M]_{\text{spherical 4R}} = \begin{bmatrix} a_{212} & a_{312} & a_{412} & 0 & 0 & 0 \\ a_{213} & a_{313} & a_{413} & 0 & 0 & 0 \\ a_{223} & a_{323} & a_{423} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.65}$$

Clearly, the rank of the coefficient matrix [M] for a spherical four-link mechanism is three. Theoretically, the components of the general motion of a spherical mechanism are the three rotations about three non-planar axes. The existence of three real part equations is due to the existence of only the real part in the dual angles $\hat{\alpha}$ and $\hat{\theta}$. As a result of this condition, only pure rotations are accomplished. These pure rotation components are then described by the existence of the three real part row vectors.

Coefficient Matrix [M] for a Plane Four-Link Mechanism

Another classical example that can be considered to study the correlation of the number of real and dual rows of the coefficient matrix with the components is that of the general motion of a plane mechanism which can be described by one rotation and two translations, a consequence of having all the axes of the revolute pairs parallel. Accordingly, three independent equations can be expected from the coefficient matrix [M]. Furthermore, due to the general motion of one rotation and two translations, it can be predicted that out of the three rows of the coefficient matrix [M], one row must consist of the elements from the real part of the matrices $[Q_i]$ and two rows must consist of elements from the dual part of the matrices $[Q_i]$.

It should be remarked, however, that such a set of equations cannot be intuitively established. For this reason, a numerical example is considered. The following are the parameters of any arbitrarily selected four-link plane-mechanism for which the closure conditions are known.

$a_1 = 4, \quad a_2 = 4, \quad a_3 = 4, \quad a_4 = 2$

$\alpha_1 = 0, \quad \alpha_2 = 0, \quad \alpha_3 = 0, \quad \alpha_4 = 0$

$\theta_1 = 30^\circ, \quad \theta_2 = 126.76^\circ, \quad \theta_3 = 86.67^\circ, \quad \theta_4 = 116.56^\circ$

$s_1 = 0 \quad s_2 = 0 \quad s_3 = 0 \quad s_4 = 0$

The coefficient matrix [M] then becomes

$$[M]_{\theta_1 = 30^\circ} = \begin{array}{c} \left[\begin{array}{cccccc} -1.0 & -1.0 & -1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ -3.4641 & 0.2115 & 2.000 & 0 & 0 & 0 \\ -2.000 & -3.5778 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \text{Real part} \\ \\ \\ \text{Dual part} \\ \\ \end{array}$$

The second set of closure conditions can be described by the following angular displacements of the links,

$$\theta_1 = 60^\circ, \quad \theta_2 = 112.30^\circ, \quad \theta_3 = 97.18^\circ, \quad \theta_4 = 90.51^\circ$$

The coefficient matrix [M] then takes the following form

$$[M]_{\theta_1 = 60^\circ} = \left[\begin{array}{cccccc} -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ -2.0 & 1.9639 & 2.0 & 0 & 0 & 0 \\ -3.4641 & -3.9998 & 0.0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \text{Real} \\ \\ \\ \text{Dual} \\ \\ \end{array}$$

The third set of closure conditions can be described by the following angular displacements of the links,

$$\theta_1 = 90^\circ, \quad \theta_2 = 97.42^\circ, \quad \theta_3 = 112.02^\circ, \quad \theta_4 = 60.55^\circ$$

The coefficient matrix [M] then takes the following form

$$
[M]_{\theta_1 = 90^\circ} = \begin{array}{c} \left[\begin{array}{cccccc} -1.0 & -1.0 & -1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.9664 & 2.0 & 0 & 0 & 0 \\ -4.0 & -3.4832 & 0.0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \text{Real} \\ \\ \\ \text{Dual} \\ \\ \end{array}
$$

Observe that in each of the three matrices $[M]_{\theta_1 = 30^\circ}$, $[M]_{\theta_1 = 60^\circ}$, and $[M]_{\theta_1 = 90^\circ}$ there is a striking resemblence in the nature of the real part row vectors. The first row vector of the real part of these matrices is identical and the other two real part row vectors are, in fact, the null vectors. Furthermore, the dual part first row vector is also a null vector in each of these matrices. The last two dual part row vectors, however, exhibit different properties.

The invariant nature of the real part first row vector indicates that the row vectors can be expected to represent the instantaneous screw axes of rotations. In a plane mechanism there exists one axis about which the mechanism executes a rotation and there exists two axes along which the mechanism executes two translations, and the axis of rotation is normal to the plane of translation. The invariant nature of the first row vector of the real part of the matrix [M] directly relates to this concept of the axis of rotation. The first dual part row vector then indicates that the translation does not take place along this axis. Furthermore, the existence of the last two dual part row vectors explains the existence of the two instantaneous axes along which the mechanism executes two translations.

Finally, the last two real part null vectors establish a further support in viewing the coefficient matrix [M] as the matrix of the instantaneous screw axes.

The orientation of the screw axes varies as the input displacement, θ_1, takes different values. However, the screw axes can be rotated into a position where orientation is independent of the different values of the input displacement. This process of rotation of the screw axes then involves finding the Eigen values and the Eigen vectors of a real matrix. For instance, consider the matrix [F] composed of the last two dual part row vectors of the matrix $[M]_{\theta_1=30°}$. Then

$$[F] = \begin{bmatrix} -3.4641 & 0.2115 & 2.0 \\ -2.0 & -3.5778 & 0 \end{bmatrix}$$

Now consider the product matrix $[F][F]^t$ which is

$$[FF^t] = \begin{bmatrix} -3.4641 & 0.2115 & 2.0 \\ -2.0 & -3.5778 & 0 \end{bmatrix} \begin{bmatrix} -3.4641 & -2.0 \\ 0.2115 & -3.5778 \\ 2.0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 16.42298 & 6.17149 \\ 6.17149 & 16.80064 \end{bmatrix}$$

Normalizing the product matrix $[FF^t]$ we get

$$[FF^t] = \begin{bmatrix} 1.0 & \dfrac{6.17149}{\sqrt{16.42298 \times 16.80065}} \\ \dfrac{6.17149}{\sqrt{16.42298 \times 16.80065}} & 1.0 \end{bmatrix}$$

i.e.,

$$[FF^t] = \begin{bmatrix} 1.0 & 0.37153 \\ 0.37153 & 1.0 \end{bmatrix}$$

The process of finding the Eigen values and the Eigen vector then requires solving the linear equations having the form

$$\begin{bmatrix} 1.0 - \lambda & 0.37153 \\ 0.37153 & 1.0 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

where λ is called the Eigen value and the column matrix $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is called the Eigen vector. The Eigen values are found by solving for the roots of the determinant

$$\begin{bmatrix} 1.0 - \lambda & 0.37153 \\ 0.37153 & 1.0 - \lambda \end{bmatrix} = 0$$

i.e., $(1.0 - \lambda)\ (1.0 - \lambda) - (0.37153)^2 = 0$

i.e., $\lambda^2 - 2\lambda + 0.86197 = 0$

Solution of the above equation gives two distinct roots

$$\lambda_1 = 0.62845$$

and

$$\lambda_2 = 1.37155$$

The Eigen vector corresponding to λ_1 and λ_2 are

$$0.37155\ x_1^{(1)} + 0.37153\ x_2^{(1)} = 0$$

$$0.37155\ x_1^{(2)} - 0.37153\ x_2^{(2)} = 0$$

The solution of these equations gives the two Eigen vectors which are

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The principal axes of these vectors are

$$(1/\sqrt{2},\ -\ 1/\sqrt{2}) \quad \text{and} \quad (1/\sqrt{2},\ \ 1/\sqrt{2})\ .$$

Thus, for $\theta_1 = 30^\circ$, the mechanism has three translational axes whose direction cosines are

$$\left.\begin{matrix} (\ 0, & 0, & 0\ ) \\ (\ 0, & 1/\sqrt{2}, & -\ 1/\sqrt{2}) \\ (\ 0, & 1/\sqrt{2}, & 1/\sqrt{2}) \end{matrix}\right\} \quad (\theta_1 = 30^\circ)$$

Similar computation of the Eigen values and the Eigen vectors for $\theta_1 = 60$ and $\theta_1 = 90$ gives the following set of translational axes whose direction cosines are

$$\left.\begin{matrix} (\ 0, & 0, & 0\ ) \\ (\ 0, & 1/\sqrt{2}, & -\ 1/\sqrt{2}) \\ (\ 0, & 1/\sqrt{2}, & 1/\sqrt{2}) \end{matrix}\right\} \quad (\theta_1 = 60^\circ)$$

and

$$\left.\begin{matrix} (\ 0, & 0, & 0\ ) \\ (\ 0, & 1/\sqrt{2}, & -\ 1/\sqrt{2}) \\ (\ 0, & 1/\sqrt{2}, & 1/\sqrt{2}) \end{matrix}\right\} \quad (\theta_1 = 90^\circ)$$

Examination of the three sets of the direction cosines of the Eigen vectors of the last two dual part row vectors points out their invariant characteristic, thus identifying their existence in the coefficient matrix [M] as the instantaneous screw axes.

Similar computations of the real part row vectors provide the three invariant vectors whose direction cosines are

$$\left.\begin{array}{c}(1,\ 0,\ 0)\\(0,\ 0,\ 0)\\(0,\ 0,\ 0)\end{array}\right\}$$

Observe that the above equation states that there is only one real axis about which rotation takes place. Furthermore, this axis is normal to the plane of the axes of translation since it satisfies the orthogonality conditions. This normality condition of the rotation axes to the plane of translational axes satisfies identically the theory of the plane motion.

Coefficient Matrix [M] for the Plane Slider-Crank Mechanism

The coefficient matrix [M] for a plane slider-crank mechanism with the following kinematic parameters

$\alpha_1 = 0^\circ$,	$\alpha_2 = 0^\circ$,	$\alpha_3 = 90^\circ$,	$\alpha_4 = -90^\circ$
$a_1 = 3$,	$a_2 = 4$,	$a_3 = 0$,	$a_4 = 0$
$\theta_1 = 143^\circ$,	$\theta_2 = -196.203^\circ$,	$\theta_3 = 53.203^\circ$,	$\theta_4 = 0^\circ$
$s_1 = 0$,	$s_2 = 0$,	$s_3 = 0$	$s_4 = -1.397621$

takes the following form

$$[M]_{\left\{\begin{matrix} R_3P \\ \text{Plane} \end{matrix}\right\}\theta_1 = 143^\circ} = \left[\begin{array}{cccccc} -1.0 & -1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 2.39597 & 0 & -1.0 & 0 & 0 & 0 \\ -1.805445 & 1.397621 & 0 & 0 & 0 & 0 \end{array}\right]$$

Note that the plane slider-crank mechanism also has three components of general motion. These are one rotation and two translations.

Coefficient Matrix [M] for the 7R Space Mechanism

In the 7R mechanism, the number of unknown displacements to be obtained are six for every input displacement. Correspondingly, the number of columns of the coefficient matrix are six due to the six unknowns. Thus, the rank of the coefficient matrix is six. For this reason, one can expect the matrix [M] to consist of six nonvanishing row vectors, three real part row vectors from the matrices $[A_i]$ and three dual part row vectors from the matrices $[B_i]$. For instance, consider the following parameters of the seven link mechanism:

$a_1 = 0$	$\alpha_1 = -90^\circ$	$s_1 = 0$	$\theta_1 = 270^\circ$
$a_2 = 0$	$\alpha_2 = 90^\circ$	$s_2 = 2.0''$	$\theta_2 = 270^\circ$
$a_3 = 2.0''$	$\alpha_3 = -90^\circ$	$s_3 = 4.0''$	$\theta_3 = 270^\circ$
$a_4 = 0$	$\alpha_4 = 90^\circ$	$s_4 = 0$	$\theta_4 = 90^\circ$
$a_5 = 2.0''$	$\alpha_5 = -90^\circ$	$s_5 = 2.0''$	$\theta_5 = 0^\circ$
$a_6 = 0$	$\alpha_6 = 90^\circ$	$s_6 = 0$	$\theta_6 = 90^\circ$
$a_7 = 2.0''$	$\alpha_7 = -90^\circ$	$s_7 = 2.0''$	$\theta_7 = 0^\circ$

The coefficient matrix [M] under the complete closure condition becomes

$$[M]_{\theta_1 = 270^\circ} = \begin{bmatrix} 0.000 & -0.003 & -1.000 & 0.003 & -1.000 & 0.000 \\ 0.000 & 1.000 & -0.003 & 0.000 & 0.000 & -1.000 \\ -1.000 & 0.000 & 0.003 & 1.000 & 0.003 & 0.000 \\ 0.000 & -2.000 & -0.012 & -4.000 & -0.006 & -2.000 \\ 0.000 & -0.006 & -0.000 & -0.006 & 2.000 & 0.000 \\ 0.000 & 0.000 & -4.000 & 0.012 & -2.000 & 0.000 \end{bmatrix}$$

(7R mechanism)

Observe that the six row vectors of the coefficient matrix [M] of the 7R mechanism are independent. Corresponding to these three real part vectors, which represent the screw axes of rotations, three Eigen vectors can be determined. Similarly, corresponding to the three dual part vectors, which represent the screw axes of translations, three Eigen vectors can be determined.

Coefficient Matrix [M] for the Six-Link 6R Space Mechanism

The existing literature on the classification of mechanisms describes three elementary models of the six-link six-revolute mechanism. These elementary models are shown in Figures 3, 4, and 5. The six-link mechanism shown in Figure 3 is called Franke's "wirbelkette". According to the kinematic notations, all its kinematic links are equal, i.e., a_i = constant; all the kink-links are zero, i.e., $s_i = 0$; and the absolute values of the twist angles are 90°, i.e., $|\alpha_i| = 90^\circ$. Let us assume the following values of its parameters.

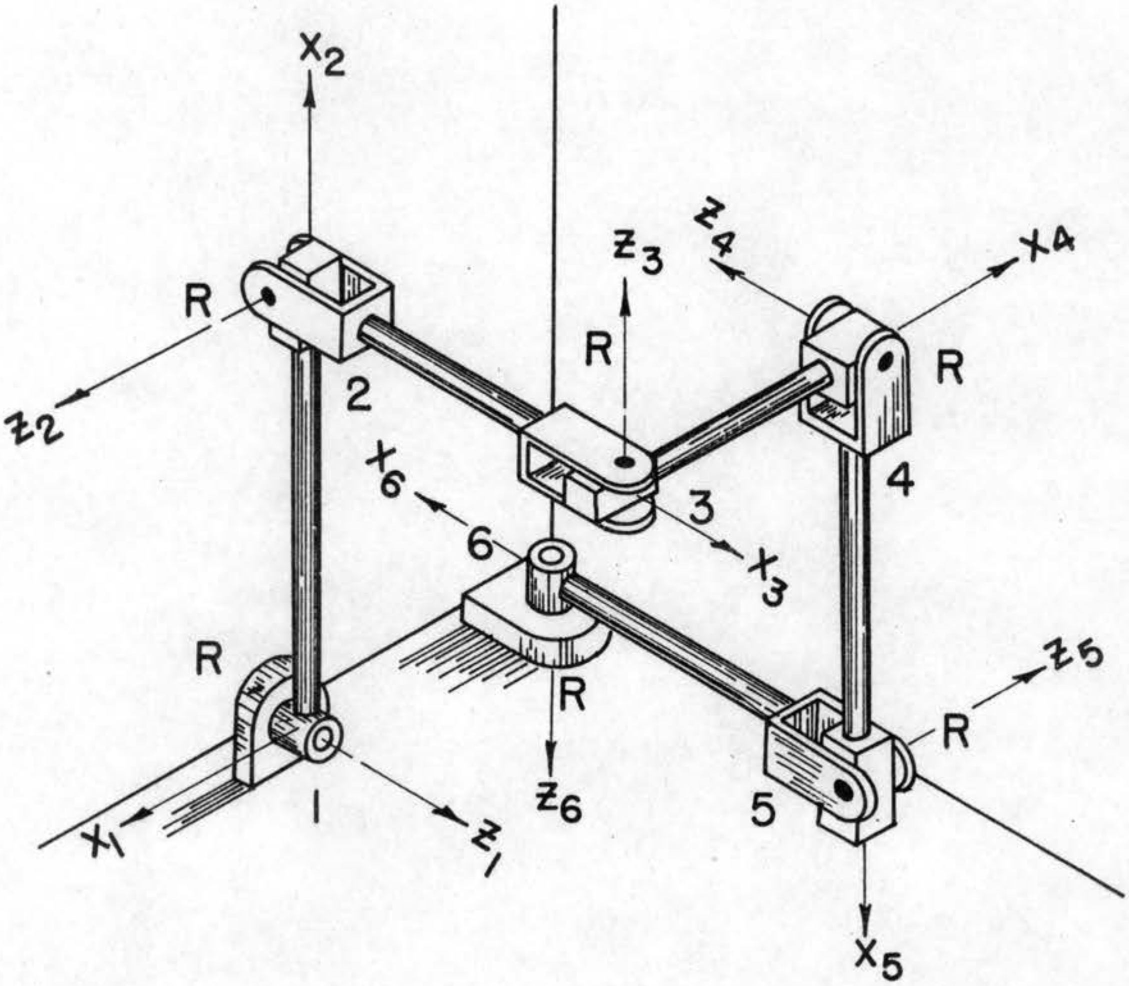


Figure 3. Franke's "Wirbelkette"
Note that all the kinematic-links
are zero.

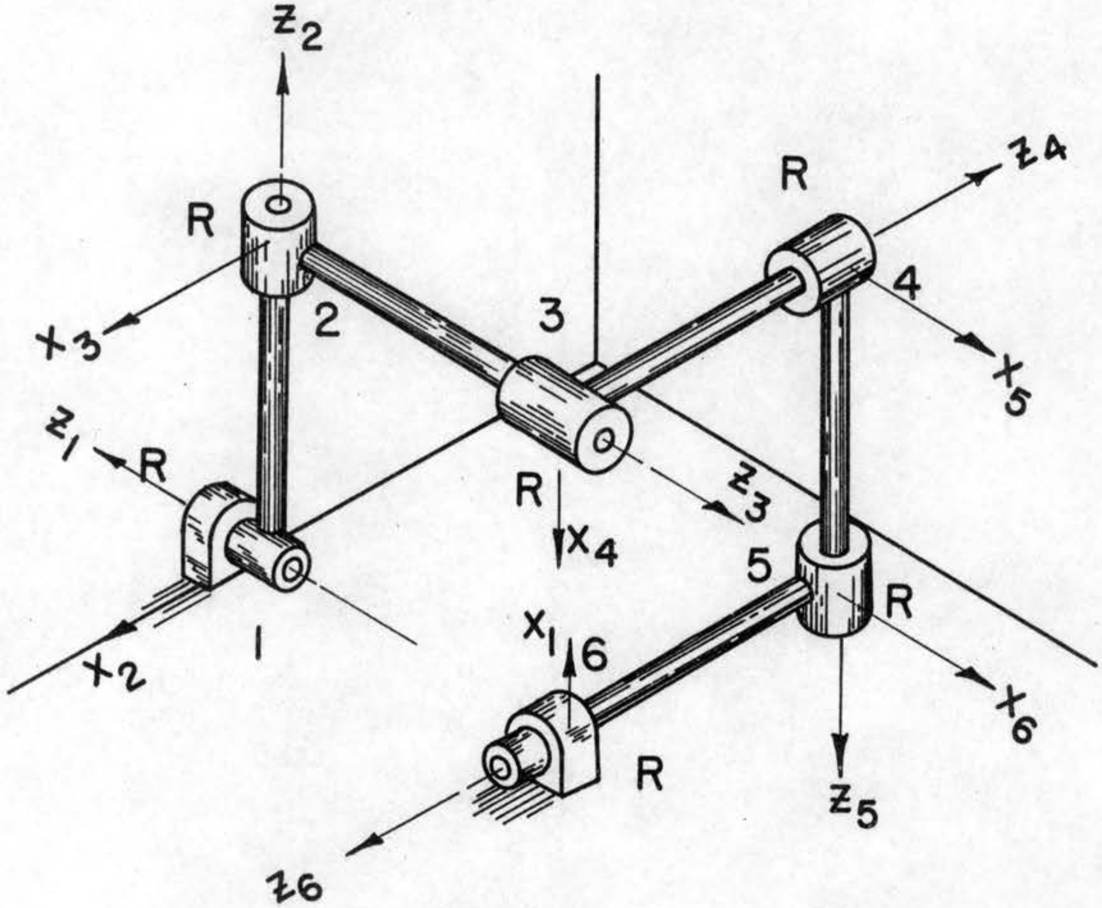


Figure 4. Bricard's Articulated Six-Link Mechanism. Note that all the kinematic-links are zero.

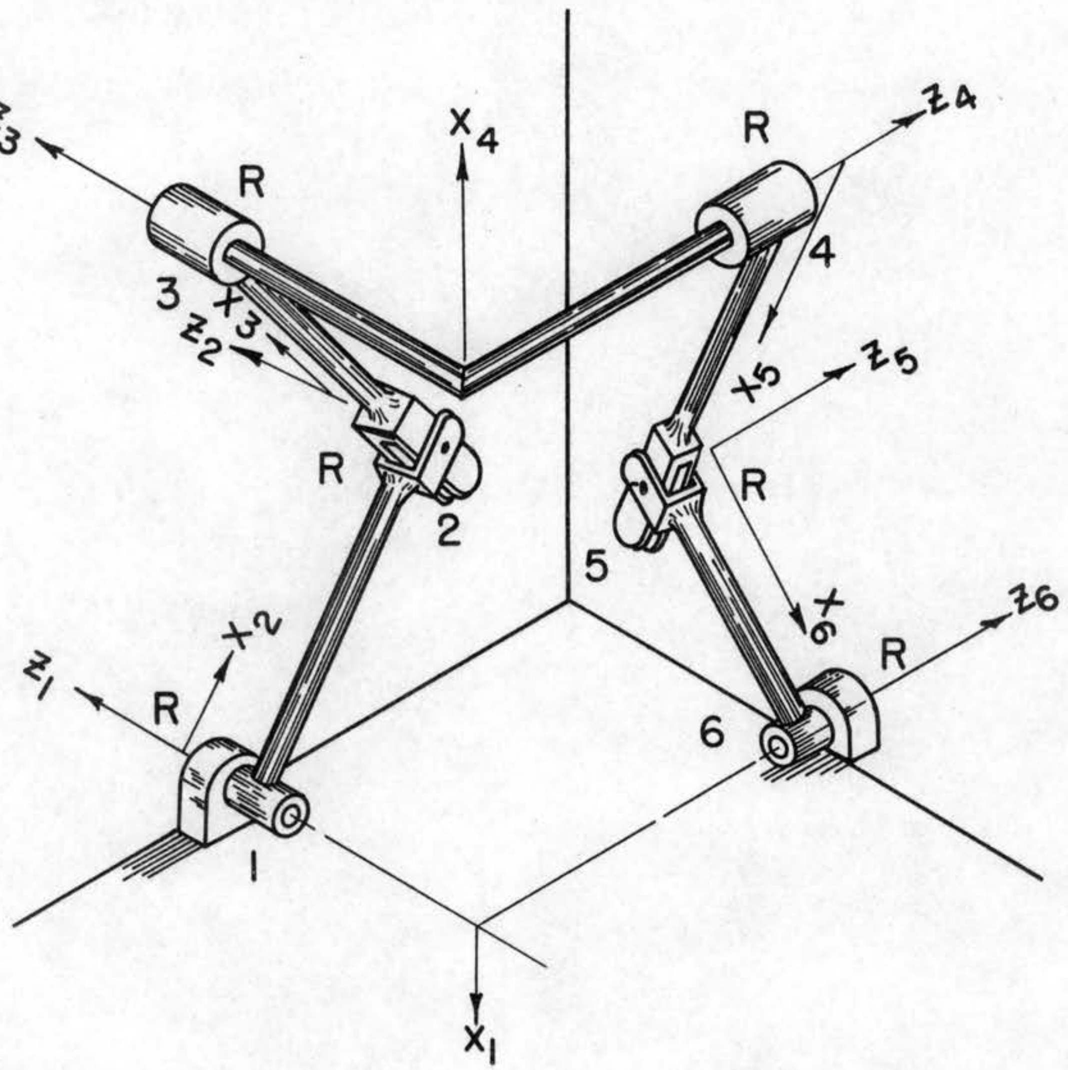


Figure 5. Sarrus' Six-Link Mechanism
Note that two of the kinematic-links and two of the kink-links are of zero length.

$a_1 = 5''$ $\alpha_1 = -90^\circ$ $s_1 = 0$ $\theta_1 = 90^\circ$

$a_2 = 5''$ $\alpha_2 = -90^\circ$ $s_2 = 0$ $\theta_2 = 270^\circ$

$a_3 = 5''$ $\alpha_3 = -90^\circ$ $s_3 = 0$ $\theta_3 = 270^\circ$

$a_4 = 5''$ $\alpha_4 = 90^\circ$ $s_4 = 0$ $\theta_4 = 90^\circ$

$a_5 = 5''$ $\alpha_5 = 90^\circ$ $s_5 = 0$ $\theta_5 = 270^\circ$

$a_6 = 5''$ $\alpha_6 = 90^\circ$ $s_6 = 0$ $\theta_6 = 270^\circ$

The coefficient matrix [M] for the Franke's "wirbelkette" then becomes

$$
\underset{\text{(Franke's 6R)}}{[M]_{\theta_1 = 90}} = \begin{bmatrix} 0.0 & 0.0 & -1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 1.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ -5.0 & 0.0 & 0.0 & 0.0 & 5.0 & 0.0 \\ 0.0 & 0.0 & 5.0 & -5.0 & 0.0 & 0.0 \\ 0.0 & 5.0 & -5.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}
$$

Observe that the last column of the coefficient matrix $[M]_{\theta_1 = 90^\circ}$ is filled with the elements having zero values. Thus, the rank of this matrix is five. However, there does exist three Eigen vectors describing the rotations of the six-link mechanism. The principal axes are

$$
\left.\begin{matrix} (1, & 0, & 0) \\ (0, & 1, & 0) \\ (0, & 0, & 1) \end{matrix}\right\} \text{(the principal axes of rotations)}
$$

Observe that there are three distinct dual part row vectors. Corresponding to these row vectors there exists three Eigen vectors describing the possible translations of the six-link mechanism. The principal axes are

$$\left.\begin{array}{l}(1,\ 0,\ 0)\\(0,\ 1/\sqrt{2},\ -1/\sqrt{2})\\(0,\ 1/\sqrt{2},\ 1/\sqrt{2})\end{array}\right\}\quad \text{(the principal axes of translations)}$$

The possible existence of these three vectors of translation will be discussed later.

The six-link mechanism shown in Figure 4 is called the Bricard's articulated six-link. According to the kinematic notations, all its kinematic links have zero value, i.e., $a_i = 0$; all the kink-links are of equal lengths, i.e., s_i = constant; and all the values of twist angles are -90°. Let us assume the following numerical values for these parameters.

$$\begin{array}{llll}
a_1 = 0 & \alpha_1 = -90^\circ & s_1 = 4'' & \theta_1 = 60^\circ \\
a_2 = 0 & \alpha_2 = -90^\circ & s_2 = 4'' & \theta_2 = 26.89^\circ \\
a_3 = 0 & \alpha_3 = -90^\circ & s_3 = 4'' & \theta_3 = 251.31^\circ \\
a_4 = 0 & \alpha_4 = -90^\circ & s_4 = 4'' & \theta_4 = 60.0^\circ \\
a_5 = 0 & \alpha_5 = -90^\circ & s_5 = 4'' & \theta_5 = 26.89^\circ \\
a_6 = 0 & \alpha_6 = -90^\circ & s_6 = 4'' & \theta_6 = 251.31^\circ
\end{array}$$

The coefficient matrix [M] for this articulated six-link then becomes

$$[M]_{\theta_1 = 60^\circ} = \begin{bmatrix}
0.0000 & 0.8918 & 0.4286 & -0.3204 & 0.000 & 0.0 \\
0.5000 & -0.3918 & 0.8918 & 0.0000 & -1.000 & 0.0 \\
0.8661 & 0.2262 & -0.1449 & -0.9473 & 0.000 & 0.0 \\
0.0000 & -1.8097 & 3.9590 & 3.7892 & 0.000 & 0.0 \\
-3.4641 & -3.9942 & -1.8097 & 0.0000 & 0.000 & 0.0 \\
2.0000 & 0.2164 & 0.5714 & -1.2815 & 0.000 & 0.0
\end{bmatrix}$$

(Articulated six-link)

Note that in both Franke's six-link and Bricard's articulated six-link there are three distinct principal axes of rotation and three principal axes of translation. Recall that such a situation is examined in the case of the 7R mechanism, for which the coefficient matrix [M] has nonvanishing six-row vectors and nonvanishing six-column vectors. The existence of nonvanishing six-column vectors determines the rank of the coefficient matrix [M]. Since the rank of the coefficient matrix of the six-link mechanism is five, only five of the six-row vectors can be utilized for the determination of the principal axes of translation and rotation. Accordingly, one of the row vectors of the coefficient matrix [M] of any six-link mechanism cannot contribute any independent relationship other than what has been established by the other five row vectors. Correspondingly, the principal axis that corresponds to such a row vector does not perform either a rotation or a translation. That is, one principal axis is simply made passive. In general, one can expect either a principal axis of rotation or a principal axis of translation to become passive for the six-link kinematic chain in order that it can exist as a one degree of freedom mechanism. Fortunately, however, due to the nature of axes of rotation, whenever a rotation axis of the six-link is made passive the real part row vector of the coefficient matrix vanishes, thus leaving five nonvanishing row vectors and five nonvanishing column vectors in the coefficient matrix [M] with five unknowns. The mechanism that satisfies such a condition of having one of the real part vanishing row vector is called the Sarrus' six-link mechanism, shown in Figure 5.

The concept of the existence of the number of passive axes of translation or the vanishing axes of rotation correlates with Sharikov's concept of the reciprocal screw. Recall that according to this concept, a six-link mechanism has one reciprocal screw (axis) about which either the six-link mechanism does not have either a rotation or translation. Since there are three principal axes of translations, any one of these three axes can become passive in order that a six-link chain exists as a mechanism. This possibility of passivity of the principal axes then correspondingly establishes a criterion for the existence of the different kinds of six-link mechanisms. Regardless of the further subdivision based on which of the principal axes became passive, the principal divisions of the six-link mechanism are the following:

(a) six-link mechanisms having three principal axes of rotation and two principal axes of translation,

(b) six-link mechanisms having two principal axes of rotation and three principal axes of translation, e.g., Sarrus' six-link mechanism. Note that one of the principal axes of rotation in the Sarrus' mechanism becomes a null axis. The Sarrus' mechanism has the following kinematic parameters:

$a_1 = 3$	$\alpha_1 = 0^\circ$	$s_1 = 2.0''$	$\theta_1 = 170^\circ$
$a_2 = 2$	$\alpha_2 = 0^\circ$	$s_2 = 0.0$	$\theta_2 = 20^\circ$
$a_3 = 0$	$\alpha_3 = -90^\circ$	$s_3 = -2.00''$	$\theta_3 = 350^\circ$
$a_4 = 3$	$\alpha_4 = 0^\circ$	$s_4 = 2.00''$	$\theta_4 = 170^\circ$
$a_5 = 3$	$\alpha_5 = 0^\circ$	$s_5 = 0.0$	$\theta_5 = 20^\circ$
$a_6 = 0$	$\alpha_6 = -90^\circ$	$s_6 = -2.0''$	$\theta_6 = 350^\circ$

The coefficient matrix [M] for the Sarrus' mechanism then takes the following form:

$$[M]_{\theta_1 = 170^\circ} = \begin{bmatrix} -1.000 & -1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & -1.000 & -1.000 & -1.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & -5.909 & -2.594 & 0.000 & 0.000 \\ 2.954 & 5.989 & 0.000 & 0.000 & 0.000 & 0.000 \\ -0.521 & 0.000 & 0.000 & 0.000 & 0.521 & 0.000 \end{bmatrix}$$

(Sarrus' Six-Bar)

Observe that one row vector of rotation is a null vector. The following are the principal axes of rotations.

$$\left.\begin{matrix} (1, & 0, & 0) \\ (0, & 1, & 0) \\ (0, & 0, & 0) \end{matrix}\right\} \text{Principal axes of rotations}$$

Since there are three dual part row vectors, three principal axes of translation must exist correspondingly. Thus, the total components of general motion are five, viz., two rotations and three translations.

Coefficient Matrix [M] for the 4R Bennett Mechanism [6]

This "paradoxical" four-link four-revolute space mechanism was discovered by a mathematician named Bennett in 1903. The orientations of the axes of the revolute pairs are related to the corresponding link lengths. Thus, for the mobility of the Bennett mechanism, the following conditions must be satisfied:

(1) Opposite link lengths are equal, that is,

$$a_1 = a_3 \quad \text{and} \quad a_2 = a_4$$

(2) Opposite twist angles are equal, that is,

$$\alpha_1 = \alpha_3 \quad \text{and} \quad \alpha_2 = \alpha_4$$

and

(3) The adjacent twist angles and link lengths must satisfy the relationship

$$\frac{a_1}{\sin \alpha_1} = \pm \frac{a_2}{\sin \alpha_2} .$$

For the computation of the coefficient matrix, let us assume the following values of these parameters:

$$\begin{array}{llll} a_1 = 8 & \alpha_1 = 90^\circ & s_1 = 0 & \theta_1 = 60^\circ \\ a_2 = 4 & \alpha_2 = 30^\circ & s_2 = 0 & \theta_2 = 216.8698^\circ \\ a_3 = 8 & \alpha_3 = 90^\circ & s_3 = 0 & \theta_3 = -60.0^\circ \\ a_4 = 4 & \alpha_4 = 30^\circ & s_4 = 0 & \theta_4 = -216.8698^\circ \end{array}$$

The coefficient matrix [M] for these set of parametric values takes the following form:

$$[M]_{\theta_1 = 60^\circ} = \begin{bmatrix} 0.0 & -0.40 & -0.866025 & 0.0 & 0.0 & 0 \\ -0.50 & -0.6928 & 0.500 & 0.0 & 0.0 & 0 \\ -0.866 & -0.50 & 0 & 0.0 & 0.0 & 0 \\ 7.999 & 4.7569 & 2.0 & 0.0 & 0.0 & 0 \\ 0.0 & -2.40 & 3.4641 & 0.0 & 0.0 & 0 \\ 0 & 0 & 0 & 0.0 & 0.0 & 0 \end{bmatrix}$$

Since there are three unknown angular displacement parameters, the rank of the coefficient matrix [M] of the Bennett mechanism must be three. Observe, however, that we have five nonvanishing row vectors in the coefficient matrix. Since the mechanism is neither a plane four-link nor a spherical four-link mechanism, the general motion of this Bennett mechanism must be two rotations and one translation. Consequently, the coefficient matrix [M] has one passive rotation and one passive translation vector.

Coefficient Matrix [M] for the 5R Goldberg Space Mechanism

The Goldberg five-link five-revolute space mechanism was discovered by M. Goldberg in 1943. This mechanism was constructed by combining two Bennett mechanisms in series. A typical set of parametric values of the Goldberg mechanism can be as follows:

$$
\begin{array}{llll}
a_1 = 8 & \alpha_1 = 90^\circ & s_1 = 0 & \theta_1 = 30^\circ \\
a_2 = 8 & \alpha_2 = 60^\circ & s_2 = 0 & \theta_2 = 197.589^\circ \\
a_3 = 8 & \alpha_3 = 90^\circ & s_3 = 0 & \theta_3 = 310.204^\circ \\
a_4 = 4 & \alpha_4 = 30^\circ & s_4 = 0 & \theta_4 = 149.996^\circ \\
a_5 = 4 & \alpha_5 = 30^\circ & s_5 = 0 & \theta_5 = 32.209^\circ
\end{array}
$$

The coefficient matrix [M] corresponding to these parametric values then takes the following form:

$$
[M]_{\theta_1 = 30^\circ} = \begin{bmatrix}
0.00 & -0.824 & -0.540 & 0.866 & 0.0 & 0.0 \\
0.866 & -0.566 & 0.796 & 0.500 & 0.0 & 0.0 \\
-0.5 & -0.019 & -0.272 & 0.00 & 0.0 & 0.0 \\
8.00 & 0.197 & 6.374 & 1.999 & 0.0 & 0.0 \\
0.00 & -0.330 & 3.676 & 3.464 & 0.0 & 0.0 \\
0.00 & 1.244 & -1.891 & 0.00 & 0.0 & 0.0
\end{bmatrix}
$$

Since there are only four unknown angular displacement parameters corresponding to every assumed input displacement parameter, the rank of the coefficient matrix is four. Furthermore, due to the three non-vanishing real part row vectors, the mechanism is expected to indicate the existence of two passive screw axes of translations. Thus, the Goldberg mechanism is expected to have three active screw axes of rotations and one active axes of translations.

Estimation of the Displacement Parameters

The displacement parameters for a given angular or linear displacement need to be estimated in order to arrive at the coefficient matrix [M]. Thus, for instance, in the 7R mechanism, for every input angular displacement θ_1, six angular displacements θ_2, θ_3, θ_4, θ_5, θ_6, and θ_7 need to be estimated. In general, parameters such as a_i, α_i, and s_i are normally not known, especially when one is searching for a combination of parameters that will give a closure condition for different input displacements. Therefore, any random combination of these parameters is likely to generate either structures or a configuration which tends to remain open-ended. Under these circumstances it is difficult to arrive at a unique solution of the displacement parameters for every assumed input displacement. Thus, the estimation of the displacement parameters requires that a complete closure condition of the kinematic chain be calculated for every position. To accomplish this, the diagonal elements of both the dual and real part matrices of the product matrix $[Q_i]$ need to be considered simultaneously with the off diagonal elements of the coefficient matrix [M]. Thus, Equations (3.43) and (3.44) are required to retain the diagonal and one side of the off-diagonal elements. Since there are three diagonal elements in the matrices $[A_i]$ and $[B_i]$, the total number of equations obtained from these two sets of matrices are twelve. These equations may be expressed in a matrix form as follows:

$$\begin{bmatrix} a_{211} & a_{311} & a_{411} & a_{511} & a_{611} & a_{711} \\ a_{212} & a_{312} & a_{412} & a_{512} & a_{612} & a_{712} \\ a_{213} & a_{313} & a_{413} & a_{513} & a_{613} & a_{713} \\ a_{222} & a_{322} & a_{422} & a_{522} & a_{622} & a_{722} \\ a_{223} & a_{323} & a_{423} & a_{523} & a_{623} & a_{723} \\ a_{233} & a_{333} & a_{433} & a_{533} & a_{633} & a_{733} \\ b_{211} & b_{311} & b_{411} & b_{511} & b_{611} & b_{711} \\ b_{212} & b_{312} & b_{412} & b_{512} & b_{612} & b_{712} \\ b_{213} & b_{313} & b_{413} & b_{513} & b_{613} & b_{713} \\ b_{222} & b_{322} & b_{422} & b_{522} & b_{622} & b_{722} \\ b_{223} & b_{323} & b_{423} & b_{523} & b_{623} & b_{723} \\ b_{233} & b_{333} & b_{433} & b_{533} & b_{633} & b_{733} \end{bmatrix} \begin{bmatrix} d\theta_2 \\ d\theta_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \\ d\theta_7 \end{bmatrix} = \begin{bmatrix} 1 - a_{111} \\ - a_{112} \\ - a_{113} \\ 1 - a_{122} \\ - a_{123} \\ 1 - a_{133} \\ - b_{111} \\ - b_{112} \\ - b_{113} \\ - b_{122} \\ - b_{123} \\ - b_{133} \end{bmatrix} \tag{3.66}$$

i.e.,

$$[U][\Delta\theta] = [V] \tag{3.67}$$

where the matrix $[U]$ represents the coefficient of the diagonal and off-diagonal elements of the matrices $[A_i]$ and $[B_i]$ ($i > 2$) and the column matrix $[V]$ represents the diagonal and off-diagonal elements of the matrices $[A_1]$ and $[B_1]$. The above set of twelve equations has only six unknowns $d\theta_2$, $d\theta_3$, $d\theta_4$, $d\theta_5$, $d\theta_6$, and $d\theta_7$. Therefore, the rank of the matrix $[U]$ must be six. The estimation of these unknowns then must proceed in a manner similar to that being used by the "least-square technique". Accordingly, multiplying both sides of Equation (3.67) by a transpose of matrix $[U]$, we get

$$[U]^t[U][\Delta\theta] = [U]^t[V] \tag{3.68}$$

Let $[W] = [U]^t[U]$ and let $[W]^{-1}$ be the inverse of $[W]$. Then multiplying both sides of Equation (3.68) by $[W]^{-1}$, we get

$$[W]^{-1}[W][\Delta\theta] = [W]^{-1}[U]^t[V] \tag{3.69}$$

But

$$[W]^{-1}[W] = [I] \tag{3.70}$$

where the matrix $[I]$ is the unit matrix. Therefore, Equation (3.69) becomes

$$[\Delta\theta] = [W]^{-1}[U]^t[V] \tag{3.71}$$

Thus, the unknown column matrix $[\Delta\theta]$ is evaluated using the relationship given by Equation (3.71). If for a given combination of a_i, α_i, and s_i, the input link of a mechanism is rotated from an initial position θ_1 to θ_1', the corresponding values of $\theta_i (i > 2)$ will change under a complete closure condition of the mechanism. However, the final angular positions of the follower links are obtained by assuming their initial values and computing their exact values by an iterative procedure. At each iteration, successive values of $d\theta_i$ are calculated using the relationship given by Equation (3.71). These computed values of $d\theta_i$ are then added to the previous values of $\theta_i (i > 2)$. Thus, if $\theta_i (i \geq 2)$ are initial values and $d\theta_i$ are calculated values, then new assumed values $\theta_i (i \geq 2)$ can be obtained from

$$\theta_i' = \theta_i + d\theta_i \quad \text{(for } i \geq 2\text{)} \tag{3.72}$$

Thus, at each iteration, new values of $\theta_i (i \geq 2)$ are estimated until these values obtain a stability, in which case the process of iteration achieves a convergence, and the differential displacements $d\theta_i$ vanish at the final stage of iteration. However, such convergence is only possible when the assumed combination of a_i, α_i, and s_i satisfies the requirements of closure conditions and the closed kinematic chain is a mechanism when one of the links is fixed. Observe that when a complete convergence occurs and all the θ_i's obtain their exact values satisfying the closure condition then all the diagonal coefficients of the matrices $[A_i]$ are zero. Consequently, the coefficient matrix $[M]$ can be obtained from the coefficient matrix $[U]$. Furthermore, under the complete closure conditions, the column matrix $[V]$ becomes a column matrix of null vector. The number of active screw axes of rotations and translations will then decide the class of the mechanism.

Let us consider a numerical example to illustrate the technique of estimating the dependent angular parameters. For instance, consider the Bricard's articulated six-link mechanism which does not obey any of the existing hypotheses for the one general constraint. The following are the parametric values of this mechanism:

$$
\begin{array}{lll}
a_1 = 0 & \alpha_1 = -90^\circ & s_1 = 4'' \\
a_2 = 0 & \alpha_2 = -90^\circ & s_2 = 4'' \\
a_3 = 0 & \alpha_3 = -90^\circ & s_3 = 4'' \\
a_4 = 0 & \alpha_4 = -90^\circ & s_4 = 4'' \\
a_5 = 0 & \alpha_5 = -90^\circ & s_5 = 4'' \\
a_6 = 0 & \alpha_6 = -90^\circ & s_6 = 4''
\end{array}
$$

Let the input angular displacement $\theta_i = 60^\circ$ and let us assume the following unknown angular displacements, i.e., let

$\theta_2 = 338^\circ$, $\theta_3 = 305^\circ$, $\theta_4 = 99^\circ$, $\theta_5 = 338^\circ$, and $\theta_6 = 291^\circ$.

With these values the coefficient matrix [U] and the matrix [V] can be computed. Thus, the coefficient matrix [U] takes the following form:

$$[U]_{\theta_1 = 60^\circ} = \begin{bmatrix} 0.00038 & -0.79421 & 0.263224 & -0.306753 & 0.03896 & 0.0000 \\ -0.02297 & 0.46257 & -0.17506 & 0.22113 & 0.0000 & 0.0000 \\ 0.49947 & 0.34665 & 0.36309 & -0.79912 & -0.51555 & 0.0000 \\ 0.03979 & -0.78504 & 0.30293 & -0.28433 & 0.00000 & 0.0000 \\ 0.86511 & 0.22323 & -0.86300 & -0.48077 & 0.85686 & 0.0000 \\ -0.04057 & -0.00808 & 0.00688 & 0.04289 & -0.00077 & 0.0000 \\ 0.45666 & -0.85655 & -3.04138 & -3.15446 & -0.51919 & 0.0000 \\ 0.18216 & 1.08421 & 1.91310 & 1.46848 & -0.00000 & 0.0000 \\ -3.45974 & -1.70608 & 0.94536 & 1.33646 & -0.23389 & 0.0000 \\ -0.05205 & -1.32735 & -3.37091 & -2.46200 & 0.00000 & 0.0000 \\ 1.99910 & 3.57144 & -1.21535 & 0.55244 & -0.14156 & 0.0000 \\ -0.32384 & 0.33249 & 0.85001 & -0.10881 & -0.91871 & 0.0000 \end{bmatrix}$$

The column matrix [V] takes the following form:

$$[V]_{\theta_1 = 60^\circ} = \begin{bmatrix} 0.48445 \\ -0.85597 \\ -0.03896 \\ 0.48502 \\ -0.02435 \\ 0.00105 \\ -0.23389 \\ 0.11724 \\ 0.51919 \\ -0.19897 \\ -0.77333 \\ -0.00140 \end{bmatrix}$$

Then the matrix [W] can be obtained as follows:

$$[W] = [U]^t[U] = \begin{bmatrix} 17.31716 & 12.81069 & -7.41445 & -5.33974 & 1.07046 \\ 12.81069 & 21.07862 & 3.27365 & 7.61881 & -0.36822 \\ -7.41445 & 3.27365 & 28.43566 & 21.12174 & -0.16785 \\ -5.3397 & 7.61788 & 21.12174 & 21.36713 & 1.33498 \\ 1.07046 & -0.36823 & -0.16785 & 1.33497 & 2.18985 \end{bmatrix}$$

The product of the two matrices $[U]^t[V]$ is given by the matrix

$$[U]^t[V] = \begin{bmatrix} -3.45688 \\ -4.22613 \\ 3.46706 \\ 1.23375 \\ 0.12886 \end{bmatrix}$$

Finally, the column matrix $[\Delta\theta]$ can be computed from the relationship

$$[\Delta\theta] = [W]^{-1}[U]^t[V] \tag{3.71}$$

Thus,

$$\underset{\text{(in radians)}}{[\Delta\theta]} = \begin{bmatrix} 0.407059 \\ -0.611801 \\ -0.033670 \\ 0.443092 \\ -0.515712 \end{bmatrix}$$

The estimated θ_i then can be computed by adding the computed differential displacements to the assumed values, i.e.,

$$\theta_i' = \theta_i(\text{assumed}) + d\theta_i(\text{computed})$$

Thus

$$\theta_2' = 361.3227$$
$$\theta_3' = 269.94635$$
$$\theta_4' = 97.07085$$

$$\theta_5' = 363.3872$$
$$\theta_6' = 261.45188$$

The coefficient matrix [U] and its transpose are recomputed with the corresponding values of θ_i' and $\theta_1 = 60^\circ$. Then, another set of $d\theta_i$ are computed. At every stage of the iteration, these values of differential displacement become smaller and smaller if the closure condition of the mechanism for this particular value of $\theta_1 = 60^\circ$ exists. The rate at which the convergence occurs depends upon how close the assumed values are. An example of this convergence is shown in Table VII.

Observe that at each successive iteration, the column matrix $[\Delta\theta]$ approaches to a column null matrix. At the same time the unknown displacements θ_i arrive steadily at their true values which corresponds to the input displacement θ_1. At the last iteration when the column matrix $[\Delta\theta]$ becomes a column null matrix, all those row vectors of the coefficient matrix [U], which correspond to the diagonal elements of the matrices $[A_i]$, also become null vectors. Consequently, the coefficient matrix [U] degenerates into the coefficient matrix [M]. For the mechanism under consideration, this coefficient matrix [M] has been examined earlier.

Note that when a complete convergence is established the diagonal elements of the matrix $[B_i]$ may or may not become zero. This existence of the diagonal elements in $[B_i]$ matrices is due to its non-orthogonal property. In some special cases, however, this matrix does become orthogonal, and in turn the diagonal elements reduce to zero.

TABLE VII

ESTIMATION OF THE θ_i ($i > 2$) for $\theta_1 = 60^\circ$ OF THE ARTICULATED BRICARD MECHANISM

Iteration	$[\Delta\theta]$	Estimated θ_i ($i > 2$)
1	0.235266	θ_2 = 374.8025
	-0.286395	θ_3 = 253.5371
	-0.442558	θ_4 = 71.7141
	0.272505	θ_5 = 379.0006
	-0.175539	θ_6 = 251.3942
2	0.19000	θ_2 = 385.6891
	-0.04150	θ_3 = 251.1591
	-0.20063	θ_4 = 60.2184
	0.13593	θ_5 = 386.7894
	-0.04239	θ_6 = 248.9649
3	0.021056	θ_2 = 386.9855
	0.002763	θ_3 = 251.3174
	-0.003898	θ_4 = 59.9951
	0.002132	θ_5 = 386.9115
	0.040381	θ_6 = 251.2786
4	0.000059	θ_2 = 386.8989
	-0.000056	θ_3 = 251.3142
	0.000084	θ_4 = 59.9999
	-0.000220	θ_5 = 386.8989
	0.000622	θ_6 = 251.3142
5	0.00000	θ_2 = 386.8989
	0.00000	θ_3 = 251.3142
	0.00000	θ_4 = 60.0000
	0.00000	θ_5 = 386.8989
	0.00000	θ_6 = 251.3142

Technical Problems Associated With the Iterative Method

The development of the numerical method is based on the expansion of each of the terms of the screw matrix $[T_i]$ according to the Taylor series expansion. Since all the higher order terms are neglected in this expansion, the process of convergence demands the values of the unknown displacement parameters to be assumed too close to their true values. With larger deviations of the assumed values, the number of iterations required for the convergence is large. In general, it has been observed that on an average every ten degree deviation of the assumed value requires one iteration. However, if a closure condition exists for a mechanism, the method does arrive at the solution regardless of the maximum deviation between the assumed and the exact values of the displacement parameters.

It should be noted, however, that the method of estimation of these unknowns is based on the least-square technique. This technique is capable of producing the exact answer when it exists as well as the answer wherein the deviation becomes minimum. In both the instances, the convergence is guaranteed. However, in solving the problems pertaining to the estimation of the unknown displacement parameters of a mechanism, the estimated parameter must satisfy the closure conditions; that is, the row vectors of the matrix [U] corresponding to diagonal elements of the matrices $[A_i]$ must become null vectors.

This type of convergence, where the row vectors of the matrix $[U_i]$ corresponds to diagonal elements of the matrix $[A_i]$ do not become null vectors, are in some cases due to an incorrect sign associated with the parameters of a mechanism.

The condition of a dead-center of a mechanism does represent a closure condition of the mechanism. Therefore, whenever a dead-center is found for the mechanism, the method of estimating unknown parameters should converge. However, the coefficient matrix [M] of the mechanism becomes singular. Thus, the singularity of matrix does not permit the system to converge and the unknown parameter will never obtain a stable solution.

The limit position of a mechanism is recognized as if the mechanism does not form a close chain. Thus, the closure conditions are never satisfied. In this event this iterative procedure produces a divergent system. The unique solution of the unknown displacement parameters is therefore not possible.

Finally, if for some combination of the paramters, the kinematic chain becomes a structure, then the coefficient matrix [M] becomes singular. However, since the procedure of estimating the displacement parameter is based on an initial assumed value, the coefficient matrix [U] does not have singularity. As the number of iteration increases, the non-singular matrix [U] becomes unstable and the system of independent equations representing the coefficient matrix [V] becomes divergent. The nature of the divergent matrix can be detected at the earlier stages of the iterative procedure. If either the determinant of the matrix [W] is extremely large or the determinant of the matrix $[W]^{-1}$ is extremely small, then the system in most cases becomes divergent at the later stage. It is also advisable to examine at every iteration the difference matrix [L] given by

$$[L] = [W] - [W^{-1}]^{-1}$$

If the difference matrix $[L]$ has elements which represent "finite" quantities, then the original matrix $[W]$ is in general a singular matrix. For further complex problems in detecting the singularity of the approximate matrix reference [49] must be consulted.

CHAPTER IV

THE SIX-LINK MECHANISM

The development of the theory of determining the existence or nonexistence of one or more general constraints makes it possible to examine the characteristic performance of the nature of general constraints. The present investigation is, however, confined to the examination of the nature of one general constraint.

According to the theory developed in the last chapter, the existence of one general constraint degenerates the six-by-six coefficient matrix [M] into a five-by-five non-singular matrix. The existence of the numerical real part row vectors corresponds to the number of rotation components of the general motion. If, however, all the real part row vectors are nonvanishing, then there does exist one passive dual part row vector. If, however, one real part row vector is a null vector, then all the threel dual part row vectors must be active because the rank of the coefficient matrix [M] cannot otherwise be five.

The procedure of arriving at the coefficient matrix [M] is, however, numerical. This numerical technique operates with the coefficient matrix [U] and in turn with the product matrix [W]. If the rank of the product matrix is six, the rank of the coefficient matrix [M] is six. If the rank of the product matrix [W] is five, then the rank of the coefficient matrix [M] is five, in which case the mechanism giving such a coefficient matrix [M] has one general constraint. Note, however, that since the numerical method is iterative and the product matrix [W] is computed initially with the approximate information of the dependent displacement parameters, the product matrix [W] will diverge under the condition of its singularity and therefore the determinant of the product matrix either becomes extremely large or extremely small. Both of these properties are attributed to the singularity of the product matrix [W]. Thus, what is expected to happen to the product matrix [W] according to the theory is translated in terms of divergence and convergence of the product matrix [W].

The method of determining the existence of the six-link mechanism, therefore, becomes of analytical nature. A set of twenty-four parametric values of a six-link chain are assumed. The product matrix [W] is computed with the specified value of the input displacement θ_1 and the approximate values of the dependent angular displacements θ_2, θ_3, θ_4, θ_5 and θ_6. The exact values of the dependent displacements are computed using the iterative procedure and with the assumption that the rank of the product matrix [W] is five. The successive iterations of the product matrix [W] are expected to lead to any one of the following three results:

1. exact convergence
2. pseudo convergence
3. divergence

The exact convergence of the system can be identified by the fact that the column matrix [V] degenerates into column null vector. Consequently, the dependent displacement parameters achieve their exact values corresponding to the complete closure condition of the chain specified by the input displacement parameter.

Since the convergence of the product matrix [W] is arrived with the assumption that the rank of the matrix [W] is five, and since the computed dependent displacement parameters do satisfy the complete closure condition, the assumed six-link chain yields a six-link mechanism.

The pseudo convergence and the divergence of the product matrix [W] are somewhat related. The pseudo convergence is quite often encountered either because the closure conditions are examined in the region past beyond the limit position but relatively close to it or because of the inexact information of one of the parameters, for instance, a kinematic-link of the six-link chain.

In either of these cases, there is an element of doubt concerning the existence of the six-link mechanism and therefore a second closure condition must be examined.

The divergence of the product matrix [W] indicates that closure conditions are being examined in the region of a limit position or that the six-link chain is a structure. Thus, the divergence of the product matrix requires the examination of a second set of closure conditions.

Whenever an exact convergence of the product matrix [W] is established for an artibtrarily selected kinematic parameter of a six-link chain, it can then be deduced that such a chain is expected to yield a six-link mechanism. However, for a complete assurance and as a part of a good practice, a six-link chain is tested for a second independent complete closure condition once the first closure conditions are established.

The first closure conditions are, however, difficult to achieve. The following approach is adopted in the present investigation of a six-link chain. At the first attempt, six closure conditions corresponding to the six input angular positions, $\theta_1 = 0°, 60°, 120°, 180°, 240°, 300°$, are examined. If a complete closure condition is achieved at any one of the positions, then the chain is tested for a second independent closure condition. If, however, a complete closure condition does not exist in the previous investigation, then a second set of the six input angular positions, $\theta_1 = 30°, 90°, 150°, 210°, 270°, 330°$, is examined for the closure conditions. If successful results were not obtained with the second set of the input angular positions, then a third set of twelve input angular positions, $\theta_1 = 15°, 45°, 75°, 105°, 135°, 165°, 195°, 225°, 285°, 315°, 345°$, are tested for the complete closure conditions. If after trying these three sets a complete closure condition is not obtained, then the six-link chain is pronounced as a structure.

Parameters of the Six-Link Mechanism

According to the kinematic notation of Denavit and Hartenberg [43], the following are the twenty-four parameters associated with the six-link mechanism.

(1) The kinematic links: There are six parametric values of the kinematic links. These are denoted by a_1, a_2, a_3, a_4, a_5, and a_6. The numerical values of these parameters are conventionally kept positive.

(2) The twist angles: There are six parametric values of twist angles. These angles measure the degree of skewness in the orientation

of two successive kinematic pairs. The twist angles can take either a positive or a negative value.

(3) The angular displacements: There are six parametric values of the angular displacements. These are θ_1, θ_2, θ_3, θ_4, θ_5, and θ_6. In a mechanism when one of the links adjacent to the fixed link is given an angular displacement θ_1, then the values of the other angular displacements θ_2, θ_3, θ_4, θ_5, and θ_6 are dependent on the input displacement. Thus, any arbitrary value of θ_1 can be assumed and corresponding values of θ_2, θ_3, θ_4, θ_5, and θ_6 must be determined.

(4) The kink-links: There are six parametric values of the kink-link components. These links are the off-set distance between the two kinematic links, and are denoted by s_1, s_2, s_3, s_4, s_5, and s_6. The values of these parameters can be either positive or negative.

From the twenty-four parameters described above, there are only eighteen parameters that govern the closure condition and mobility of the six-link mechanism. Once it is established that the 6R chain is a mechanism, then the dependent displacement parameters can be evaluated for the different values of the input displacements.

Parametric Study of the Six-Link

It has been examined that there are eighteen parameters of the six-link mechanism, twelve of which can assume either positive or negative signs in order to build a closed kinematic chain. Thus, when the associated signs are taken into consideration, the total number of parametric values that need consideration is thirty. If a thorough study of these parameters is planned without giving any other considerations, then the present investigation of examining the governing conditions would nearly involve, with a first degree of approximation, a combination of thirty factorial parametric values. On the other hand, if higher percentages of these thirty factorial parametric values of the six-link do yield the six-link mechanism, then any random set of these eighteen parameters should also yield a six-link mechanism. However, in view of the fact that there are only three elementary models of the six-link mechanism that are known to exist and that more than a hundred kinematicians have wondered about their existence, such a plan of studying the thirty factorial combinations not only proves to be impractical but also proves to be unintelligent. Thus, the problem of studying these parameters of the six-link mechanism is more complex and it needs a more careful thinking, planning, observing every available information on hand, analyzing every existing combination that defines the existence of the six-link mechanism, and interpreting every available information in a manner that a new set of combinations of these parameters would yield a new six-link mechanism.

The problem of determining the governing conditions of the existence of the six-link mechanism is somewhat analogous to the problem of determining a location of a particular city in the map of the world, especially when the latitude and the longitude was difficult to obtain. Perhaps, one intelligent way to get around to this problem is to inquire into its possible existence in the south or the north of the hemisphere. After dividing the world into two halves, perhaps one may divide the proper half into another half by inquiring whether this particular city exists in the east or the west. Thus, proceeding in this manner and examining every answer to every question asked, it is possible to locate the particular city on the map of the world, provided, of course, there does exist a source which is capable of giving the correct answer to every question.

The analogy of locating a city on the map of the world and determining the governing conditions then suggests that only those combinations should be examined which contributes new information. The existence of the three different six-link mechanisms provides a good start for such an investigation. These three mechanisms are:

(1) Franke's "wirbelkette". This mechanism has twist angles as follows:

$$\alpha_1 = -90^\circ \quad \alpha_2 = -90^\circ \quad \alpha_3 = -90^\circ$$
$$\alpha_4 = 90^\circ \quad \alpha_5 = 90^\circ \quad \alpha_6 = 90^\circ$$

All the kinematic links are equal, that is,

$$a_1 = a_2 = a_3 = a_4 = a_5 = a_6$$

and all the kink-links are zero, that is,

$$s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = 0$$

The mechanism is shown in Figure 3.

(2) Sarrus' six-link mechanism. In this mechanism, four of the twist angles are zero; two of the twist angles are of -90° value. Two kinematic links and two kink-links are zero. The mechanism is shown in Figure 5.

(3) Bricard's articulated six-link mechanism. In this mechanism, all the kinematic links have zero values; all the kink-links are positive and equal and all the twist angles are of -90° value.

The existence of these three six-link mechanisms provides a good start for exploring the other possible combinations of the parametric values. In the following section, these mechanisms are investigated with a wide variety of combinations and permutations of the parametric values.

Variation in Franke's "Wirbelkette"

Variation in the Twist Angles

There are primarily six types of variations that can be studied with the twist angles and with their appropriate signs. The first type of variation is concerned with the different possible values of twist angles. For instance, in the Franke's "wirbelkette" the twist angles 1 to 6 have the following pattern:

$$-90^\circ,\ -90^\circ,\ -90^\circ,\ 90^\circ,\ 90^\circ,\ 90^\circ\ .$$

The first three twist angles have a negative sign and the last three have a positive sign associated with their values. The absolute values of the twist angles are, however, equal. Following the same pattern, the other possible values of the twist angles can be investigated. Thus, for instance, the twist angles 1 to 6 may have values such as

$$-80^\circ,\ -80^\circ,\ -80^\circ,\ 80^\circ,\ 80^\circ,\ 80^\circ$$

$$-70^\circ,\ -70^\circ,\ -70^\circ,\ 70^\circ,\ 70^\circ,\ 70^\circ$$

etc.

The method developed in the last chapter can now be utilized to examine the possible existence of a six-link mechanism having a set of six twist angles similar to those described above and the other parameters are the same as those of Franke's mechanism. That is, all the kinematic links are equal and all the kink-links are zero.

The results of this investigation are presented in Table VIII. The results of the first nine sets of combinations indicate that Franke's six-link mechanism exists with the twist angles given by a set

$$-\alpha,\ -\alpha,\ -\alpha,\ \alpha,\ \alpha,\ \alpha \tag{4.1}$$

Observe that in Table VIII we have not attempted to examine any set in which the twist angles have zero value. Therefore, it must be noted that in the above set $\alpha_i \neq 0$. The limit values of α_i will be examined at a later stage. Observe that in Table VIII, each set is examined for a minimum of two input angular displacements.

The second type of variation in the Franke's six-link mechanism is described by sets 10-18. Observe that the twist angles 1 and 4, 2 and 5, and 3 and 6 have the same absolute values but opposite signs. The sign permutation is followed in the same manner as that of the original Franke's six-link mechanism. Furthermore, note that in each of these seven sets the twist angles are given different values. The examination of this second variation in the twist angles indicates that Franke's mechanism exists with the twist angles given by a set

$$-\alpha,\ -\beta,\ -\gamma,\ \alpha,\ \beta,\ \gamma \tag{4.2}$$

Here again, the lower limits of α, β, and γ are not examined. Note that a minimum of two closure conditions are reported for each set of combinations.

The third type of variation that is considered in Table VIII is the cyclic permutation of the last three twist angles. Accordingly,

TABLE VIII

VARIATION OF THE TWIST ANGLES IN THE FRANKE'S "WIRBELKETTE"

Sets		1	2	3	4	5	6
	a_i	5.	5.	5.	5.	5.	5.
	α_i	-90.	-90.	-90.	90.	90.	90.
1	s_i	0.	0.	0.	0.	0.	0.
	θ_i	90.	270.	270.	90.	270.	270.
		120.	0.	240.	0.	240.	0.
		5.	5.	5.	5.	5.	5.
		-80.	-80.	-80.	80.	80.	80.
2		0.	0.	0.	0.	0.	0.
		60.	250.13	277.24	98.42	277.24	250.13
		30.	242.7	330.92	112.39	300.92	242.27
		5.	5.	5.	5.	5.	5.
		-70.	-70.	-70.	70.	70.	70.
3		0.	0.	0.	0.	0.	0.
		60.	249.	263.05	88.66	263.05	249.
		30.	242.05	280.01	107.47	280.01	242.06
		5.	5.	5.	5.	5.	5.
		-60.	-60.	-60.	60.	60.	60.
4		0.	0.	0.	0.	0.	0.
		60.	247.38	254.18	80.42	254.18	247.38
		30.	241.73	265.69	103.	265.69	241.73
		5.	5.	5.	5.	5.	5.
		-50.	-50.	-50.	50.	50.	50.
5		0.	0.	0.	0.	0.	0.
		60.	245.54	248.52	73.73	245.53	245.54
		30.	241.73	265.69	103.	265.69	241.73
		5.	5.	5.	5.	5.	5.
		-40.	-40.	-40.	40.	40.	40.
6		0.	0.	0.	0.	0.	0.
		30.	240.94	249.35	95.86	249.35	240.94
		60.	243.74	244.87	68.54	244.87	243.74
		5.	5.	5.	5.	5.	5.
		-30.	-30.	-30.	30.	30.	30.
7		0.	0.	0.	0.	0.	0.
		30.	240.56	244.92	93.31	244.92	240.56
		60.	242.18	242.52	64.69	242.52	242.18

TABLE VIII (continued)

Sets		1	2	3	4	5	6
	a_1	5.	5.	5.	5.	5.	5.
	α_1	-20.	-20.	-20.	20.	20.	20.
8	s_1	0.	0.	0.	0.	0.	0.
	θ_1	30.	240.26	242.08	91.47	242.08	240.26
		60.	240.99	241.05	62.04	241.05	240.99
		5.	5.	5.	5.	5.	5.
		-10.	-10.	-10.	10.	10.	10.
9		0.	0.	0.	0.	0.	0.
		30.	240.06	240.51	90.37	240.51	240.07
		60.	240.25	240.25	60.51	240.25	240.25
		5.	5.	5.	5.	5.	5.
		-120.	-120.	-120.	120.	120.	120.
10		0.	0.	0.	0.	0.	0.
		30.	118.26	265.69	256.99	265.69	118.26
		60.	112.61	254.18	279.58	254.18	112.62
		5.	5.	5.	5.	5.	5.
		-160.	-160.	-160.	160.	160.	160.
11		0.	0.	0.	0.	0.	0.
		60.	119.	241.05	297.95	241.05	119.
		90.	117.96	240.28	328.47	240.28	117.96
		5.	5.	5.	5.	5.	5.
		-90.	-80.	-70.	90.	80.	70.
12		0.	0.	0.	0.	0.	0.
		30.	235.38	316.23	110.25	283.03	249.26
		60.	241.36	289.95	95.18	261.54	260.85
		5.	5.	5.	5.	5.	5.
		-80.	-70.	-60.	80.	70.	60.
13		0.	0.	0.	0.	0.	0.
		30.	232.42	295.55	104.67	260.67	253.47
		60.	237.77	276.86	84.15	246.36	263.68
		5.	5.	5.	5.	5.	5.
		-70.	-60.	-50.	70.	60.	50.
14		0.	0.	0.	0.	0.	0.
		30.	229.28	282.80	98.74	244.76	257.83
		60.	233.30	270.58	74.0	236.02	265.99

TABLE VIII (continued)

Sets		1	2	3	4	5	6
	a_i	5.	5.	5.	5.	5.	5.
	α_i	-60.	-50.	-40.	60.	50.	40.
15	s_i	0.	0.	0.	0.	0.	0.
	θ_i	60.	227.76	269.43	64.36	228.07	268.81
		30.					
		5.	5.	5.	5.	5.	5.
		-50.	-40.	-30.	50.	40.	30.
16		0.	0.	0.	0.	0.	0.
		30.	230.61	270.83	67.76	228.76	269.83
		60.	220.1	272.67	56.84	220.45	272.97
		5.	5.	5.	5.	5.	5.
		-40.	-30.	-20.	40.	30.	20.
17		0.	0.	0.	0.	0.	0.
		30.	211.10	298.33	41.06	208.22	298.24
		60.	210.01	278.95	50.71	211.49	280.65
		5.	5.	5.	5.	5.	5.
		-30.	-20.	-10.	30.	20.	10.
18		0.	0.	0.	0.	0.	0.
		30.	190.85	321.68	24.41	192.02	323.47
		60.	192.99	294.33	34.34	196.3	305.26
		5.	5.	5.	5.	5.	5.
		-90.	-80.	-70.	90.	80.	70.
19		0.	0.	0.	0.	0.	0.
		60.	241.36	289.95	95.18	261.54	260.85
		30.	235.38	316.23	110.25	283.03	249.26
		5.	5.	5.	5.	5.	5.
		-90.	-80.	-70.	70.	90.	80.
20		0.	0.	0.	0.	0.	0.
		60.	246.58	278.93	110.25	262.83	247.48
		30.	238.44	302.75	121.24	284.48	239.24
		5.	5.	5.	5.	5.	5.
		-90.	-80.	-70.	80.	70.	90.
21		0.	0.	0.	0.	0.	0.
		30.	243.81	286.69	109.53	290.98	248.64
		60.	255.08	266.62	91.65	270.16	262.20

TABLE VIII (continued)

Sets		1	2	3	4	5	6
22	a_i	5.	5.	5.	5.	5.	5.
	α_i	-90.	90.	-80.	80.	-70.	70.
	s_i	0.	0.	0.	0.	0.	0.
	θ_i	30.	236.02	13.65	235.29	48.46	233.11
		60.	97.	74.22	98.77	47.48	101.42
23		5.	5.	5.	5.	5.	5.
		-90.	-80.	-70.	70.	80.	90.
		0.	0.	0.	0.	0.	0.
		30.	242.74	288.49	116.87	288.50	242.74
		60.	253.92	267.56	102.47	267.57	253.92
24		5.	5.	5.	5.	5.	5.
		-90.	-80.	-70.	90.	70.	80.
		0.	0.	0.	0.	0.	0.
		30.	239.5	302.47	106.0	284.49	253.85
		60.	248.06	279.11	87.86	262.58	268.48
25		5.	5.	5.	5.	5.	5.
		-90.	-80.	-70.	80.	90.	70.
		0.	0.	0.	0.	0.	0.
		30.	235.15	316.31	118.05	283.15	241.38
		60.	240.97	289.98	106.57	261.87	249.44

the values of the first three angles and their signs are kept unchanged while the last three twist angles are permuted cyclically in the above combination. Thus, the following combination will result:

$$-\alpha,\ -\beta,\ -\gamma,\ \alpha,\ \beta,\ \gamma \tag{4.3}$$

$$-\alpha,\ -\beta,\ -\gamma,\ \gamma,\ \alpha,\ \beta \tag{4.4}$$

$$-\alpha,\ -\beta,\ -\gamma,\ \beta,\ \gamma,\ \alpha \tag{4.5}$$

These are the only three independent permutations that can be obtained. A set of representative values of the sets described by Equations (4.3), (4.4), and (4.5) are tabulated as the sets 19, 20, and 21 in Table VIII. The other possible values of α, β, and γ are not considered because of the findings described by the first two primary types of variations. Note again that the complete closure conditions exist for these types of variation.

The fourth type of variation that is considered in Table VIII is the case in which the two adjacent twist angles are equal in the magnitude but opposite in sign. Such a combination can be described as

$$-\alpha,\ \alpha,\ -\beta,\ \beta,\ -\gamma,\ \gamma \tag{4.6}$$

Set 22 in Table VIII describes such a permutation of the representative values of the twist angles. Observe that complete closure conditions are obtained for this combination. Thus, the combination given by Equation (4.6) describes six-link mechanisms heretofore unknown.

The combination given by Equation (4.5) suggests to investigate a combination such as

$$-\alpha,\ -\beta,\ -\gamma,\ \gamma,\ \beta,\ \alpha \tag{4.7}$$

and permute again cyclically the last three twist angles. Such a permutation yields

$$-\alpha,\ -\beta,\ -\gamma,\ \alpha,\ \gamma,\ \beta \tag{4.8}$$

$$-\alpha,\ -\beta,\ -\gamma,\ \beta,\ \alpha,\ \gamma \tag{4.9}$$

Sets 23, 24, and 25 in Table VIII describe the representative values of these combinations of the twist angles. Observe there are closure conditions in these sets. Thus, the permutation of the type described by Equations (4.1) to (4.9) are the different variations of the Franke's six-link mechanism. Note that in these twenty-three sets of combinations, all the kinematic links of the six-link mechanism are equal and that all the kink-links components are zero.

The successful findings of the above results should not mislead the reader. Even with extreme care and precautions, it may still be possible to arrive at a wrong conclusion. For instance, the cyclic permutation of the combination given either by Equation (4.2) or by (4.7) does not lead to the conclusion that the cyclic permutation of the combination given by Equation (4.6) is possible. Some of the possible permutations of this equation can be described as

$$-\alpha,\ -\beta,\ \alpha,\ \beta,\ -\gamma,\ \gamma$$

$$-\alpha,\ -\beta,\ \alpha,\ -\gamma,\ \beta,\ \gamma$$

$$-\alpha,\ -\beta,\ \beta,\ \alpha,\ -\gamma,\ \gamma$$

$$-\alpha,\ -\beta,\ \beta,\ -\gamma,\ \alpha,\ \gamma$$

Note that the closure conditions are not possible for these permutations, thus indicating that the six-link mechanism does not exist for these cases.

The existence and nonexistence of the six-link mechanism is shown schematically in Figure 6. Observe that there is a definite order of the permutation of the signs of the twist angles. Note that either three positive or negative signs associated with the twist angles appear successively or alternately.

The importance of the signs associated with Franke's six-link mechanism must be recognized. According to the kinematic notations, there does exist a choice of selecting the direction of the z axes, and therefore, the twist angles may be represented according to the individual's choice. However, it has been observed that the Franke's six-link mechanism does not exist as a six-link mechanism when all the twist angles have positive values, that is, when the combinations such as $\alpha, \alpha, \alpha, \alpha, \alpha, \alpha$ or $\alpha, \beta, \gamma, \alpha, \beta, \gamma$ exist.

Finally, with the present sign convention of the twist angles, and with their apparent relationship such as

$$|\alpha + \beta + \gamma| = |\alpha + \beta + \gamma|$$

it may appear that a six-link mechanism exists for a combination

$$-\alpha_1, \ -\alpha_2, \ -\alpha_3, \ \alpha_4, \ \alpha_5, \ \alpha_6$$

where

$$|\alpha_1 + \alpha_2 + \alpha_3| = |\alpha_4 + \alpha_5 + \alpha_6|$$

However, the present investigation suggests that a six-link chain yields a structure rather than a mechanism.

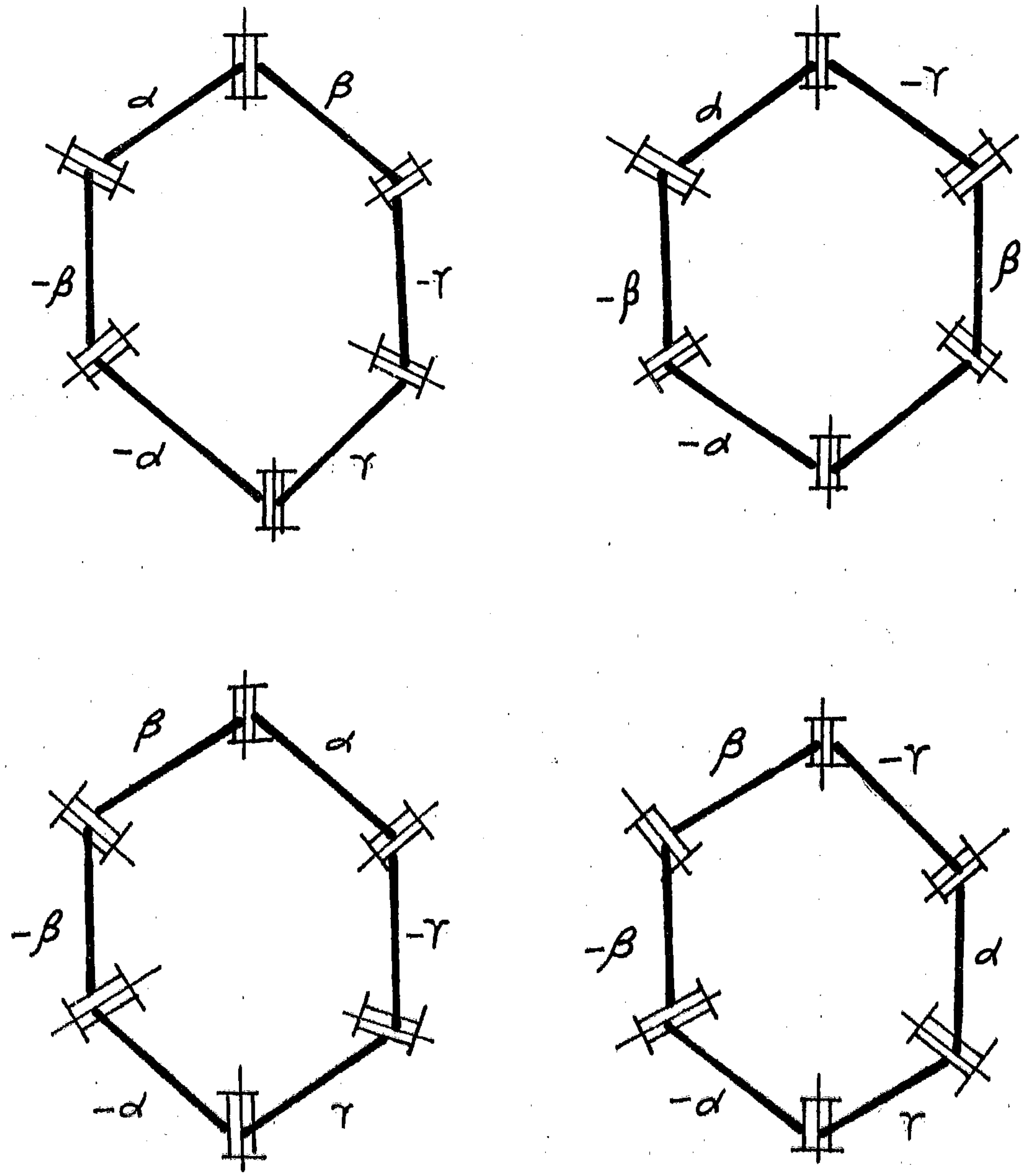


Figure 6(b). Structures (F = 0)

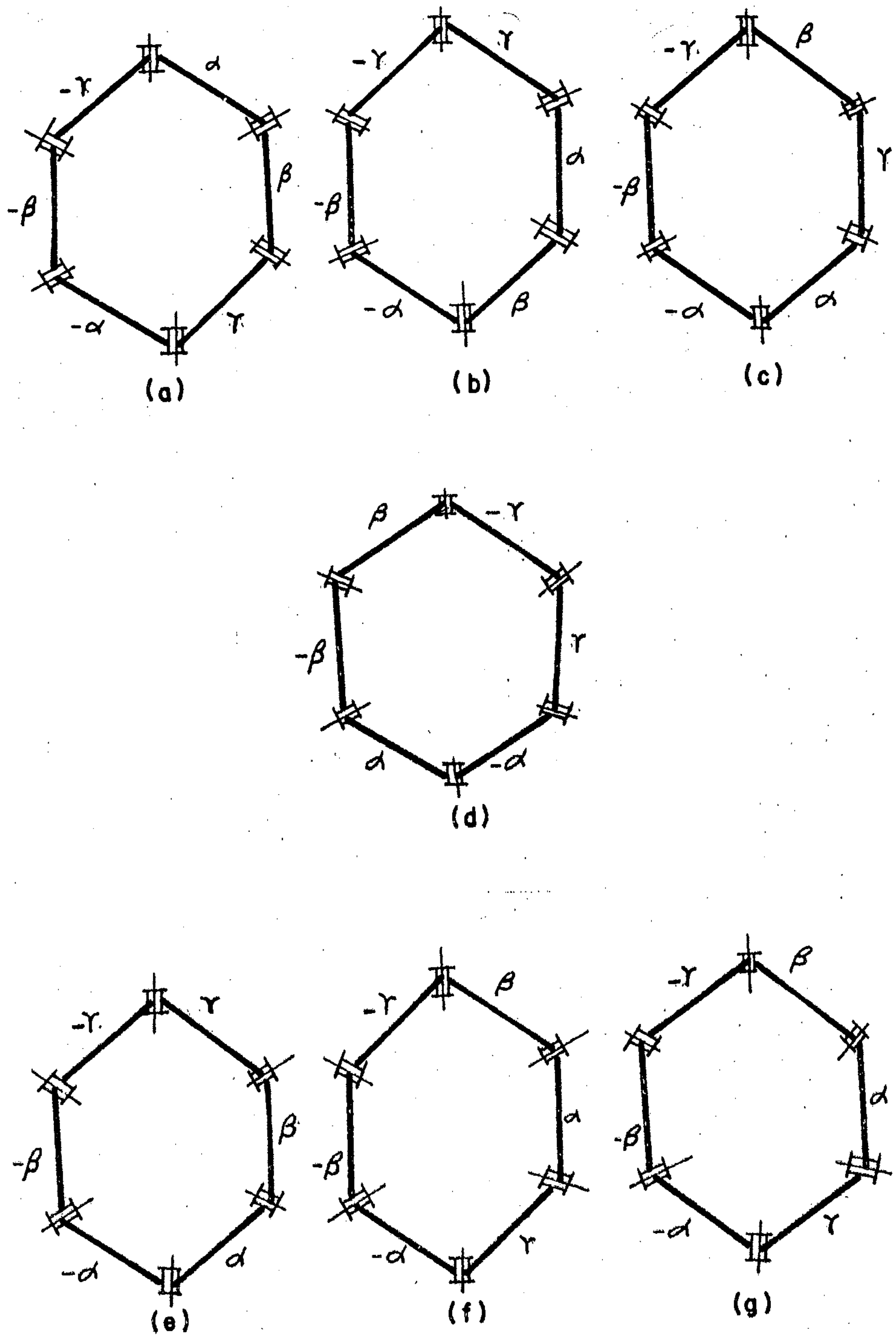


Figure 6(a). Mechanisms (F = 1)

Variation in the Kinematic-Link Lengths

The study of the variation of the kinematic-link in the Franke's six-link mechanism provides a wide variety of mechanisms. In the previous sections on the study of the variation of twist angles, the parametric values of the kinematic-links were kept invariant. All the kink-links were assumed to be of zero values.

This section is devoted to the study of the relationship between the kinematic link and the twist angles of the six-link mechanisms which are similar to construction to Franke's "wirbelkette".

Recall that all the kinematic-link lengths of the Franke's mechanism are equal and have nonzero values. If one of the kinematic-link lengths is assumed to have a zero value, then the mechanism does not assemble into a closed chain. If, however, the opposite link lengths are assumed to have zero values, then a closed configuration of the mechanism can be accomplished. The results of this investigation are presented in Table IX. Observe that the sets 1, 2, 3, 4, 5, and 6 indicate two distinctly different closure conditions of these mechanisms. The results of this investigation can be summarized by the following combinations.

$$\begin{matrix} -\alpha, & -\alpha, & -\alpha, & \alpha, & \alpha, & \alpha \\ 0, & a, & a, & 0, & a, & a \end{matrix} \tag{4.10}$$

$$\begin{matrix} -\alpha, & -\alpha, & -\alpha, & \alpha, & \alpha, & \alpha \\ a, & 0, & a, & a, & 0, & a \end{matrix} \tag{4.11}$$

$$\begin{matrix} -\alpha, & -\alpha, & -\alpha, & \alpha, & \alpha, & \alpha \\ a, & a, & 0, & a, & a, & 0 \end{matrix} \tag{4.12}$$

TABLE IX

VARIATION OF THE TWIST ANGLES, AND KINEMATIC LINKS IN THE FRANKE'S SIX-LINK MECHANISM

Sets		1	2	3	4	5	6
	a_i	0.	5.	5.	0.	5.	5.
	α_i	-90.	-90.	-90.	90.	90.	90.
1	s_i	0.	0.	0.	0.	0.	0.
	θ_i	30.	210.	306.87	149.99	329.99	306.87
		60.	240.	278.21	119.99	299.99	278.21
		5.	0.	5.	5.	0.	5.
		-90.	-90.	-90.	90.	90.	90.
2		0.	0.	0.	0.	0.	0.
		30.	15.54	195.54	329.99	195.54	15.54
		60.	324.73	215.26	60.	215.26	324.73
		5.	5.	0.	5.	5.	0.
		-90.	-90.	-90.	90.	90.	90.
3		0.	0.	0.	0.	0.	0.
		30.	306.86	329.99	149.99	306.86	209.99
		60.	234.92	263.0	122.07	301.47	262.74
		0.	5.	5.	0.	5.	5.
		-80.	-80.	-80.	80.	80.	80.
4		0.	0.	0.	0.	0.	0.
		30.	262.89	301.88	122.32	263.19	230.31
		60.	249.5	258.60	97.34	277.07	258.55
		5.	5.	0.	5.	5.	0.
		-80.	-80.	-80.	80.	80.	80.
5		0.	0.	0.	0.	0.	0.
		30.	195.64	339.0	29.99	340.02	195.64
		60.	258.57	277.21	97.45	277.21	258.95
		5.	0.	5.	5.	0.	5.
		-80.	-80.	-80.	80.	80.	80.
6		0.	0.	0.	0.	0.	0.
		30.	195.64	339.03	30.0	339.0	195.64
		60.	216.66	311.88	59.99	311.88	216.66
		1.	5.	6.	3.	2.	7.
		-90.	-90.	-90.	90.	90.	90.
7		0.	0.	0.	0.	0.	0.
		30.	208.77	321.06	130.21	341.63	286.26
		60.	217.57	293.05	109.77	325.93	272.49

TABLE IX (continued)

Sets		1	2	3	4	5	6
	a_i	2.	3.	7.	4.	4.	4.
	α_i	-90.	-90.	-90.	90.	90.	90.
8	s_i	0.	0.	0.	0.	0.	0.
	θ_i	30.	246.34	309.13	139.97	314.59	266.30
		60.	360.0	206.44	180.0	146.44	180.0
		1.	4.	6.	5.	3.	3.
		-90.	-90.	-90.	90.	90.	90.
9		0.	0.	0.	0.	0.	0.
		30.	243.26	289.07	149.06	299.71	283.64
		60.					
		1.	5.	6.	3.	2.	7.
		-80.	-80.	-80.	80.	80.	80.
10		0.	0.	0.	0.	0.	0.
		30.	215.61	288.08	120.85	305.24	275.80
		60.	224.09	270.59	99.27	297.72	265.98
		2.	3.	7.	4.	4.	4.
		-80.	-80.	-80.	80.	80.	80.
11		0.	0.	0.	0.	0.	0.
		30.	273.35	262.39	133.99	256.22	251.29
		60.	341.93	203.52	194.28	143.88	172.53
		1.	4.	6.	5.	3.	3.
		-80.	-80.	-80.	80.	80.	80.
12		0.	0.	0.	0.	0.	0.
		30.	262.98	236.85	134.19	241.36	279.11
		60.					
		6.	7.	5.	10.	3.	1.
		-90.	-90.	-90.	90.	90.	90.
13		0.	0.	0.	0.	0.	0.
		60.	180.	165.57	179.99	225.57	359.99
		90.					
		6.	7.	5.	10.	3.	1.
		-80.	-80.	-80.	80.	80.	80.
14		0.	0.	0.	0.	0.	0.
		30.	234.01	253.86	136.59	268.50	288.45
		60.					

TABLE IX (continued)

Sets		1	2	3	4	5	6
	a_i	4.	5.	6.	4.	5.	6.
	α_i	-90.	-80.	-70.	90.	80.	70.
15	s_i	0.	0.	0.	0.	0.	0.
	θ_i	30.	232.02	309.80	112.19	281.21	257.27
		90.	254.0	269.06	68.06	235.76	279.14
		4.	5.	6.	5.	6.	4.
		-90.	-80.	-70.	80.	70.	90.
16		0.	0.	0.	0.	0.	0.
		30.	262.88	268.92	122.47	262.89	254.28
		90.	311.52	242.11	31.49	222.14	325.07
		4.	5.	6.	6.	4.	5.
		-90.	-80.	-70.	70.	90.	80.
17		0.	0.	0.	0.	0.	0.
		30.	238.12	295.72	131.40	272.86	250.59
		60.	251.76	267.32	117.83	244.65	265.33
		4.	4.	5.	5.	6.	6.
		-90.	90.	-80.	80.	-70.	70.
18		0.	0.	0.	0.	0.	0.
		30.0	99.87	46.72	112.94	21.75	124.43
		90.0	256.28	70.09	243.48	111.11	234.39
		4.	5.	6.	6.	5.	4.
		-90.	-80.	-70.	70.	80.	90.
19		0.	0.	0.	0.	0.	0.
		30.	252.32	267.14	132.89	267.14	252.32
		60.	270.0	244.09	109.21	244.09	270.14
		4.	5.	6.	5.	4.	6.
		-90.	-80.	-70.	80.	90.	70.
20		0.	0.	0.	0.	0.	0.
		30.	229.22	310.04	118.96	282.59	250.87
		90.	249.38	265.55	82.88	251.39	267.4
		4.	5.	6.	4.	6.	5.
		-90.	-80.	-70.	90.	70.	80.
21		0.	0.	0.	0.	0.	0.
		30.	248.30	292.11	113.40	272.46	258.07
		90.0	277.29	260.43	50.34	232.56	299.73

TABLE IX (continued)

Sets		1	2	3	4	5	6
	a_i	4.	5.	5.	6.	6.	4.
	α_i	90.	-80.	80.	-70.	70.	90.
22	s_i	0.	0.	0.	0.	0.	0.
	θ_i	30.	93.31	37.56	121.98	46.31	111.24
		90.	273.46	103.53	232.14	115.86	253.15

If three of the kinematic-link lengths are assumed to have zero values, then the mechanism becomes a structure. If, however, four of the kinematic-link lengths are assumed to have zero values, then the mechanism becomes a two-link chain and therefore it behaves as a kinematic pair. These results are summarized schematically in Figure 7.

The fact that the opposite link lengths can become zero and that with a minimum of four kinematic links the mechanism does operate with one degree of freedom leads to an investigation of the sum of the first and last three link lengths. This investigation can be described by the combination

$$\begin{gathered} -\alpha,\ -\alpha,\ -\alpha,\ \alpha,\ \alpha,\ \alpha \\ a_1 + a_2 + a_3 = a_4 + a_5 + a_6 \end{gathered} \tag{4.13}$$

where a_1, a_2, a_3, a_4, a_5, and a_6 are kinematic-link lengths. The sets 7, 8, 9, 10, 11, and 12 of Table IX describe the variations given by Equation (4.13). Note that this type of variation does promise a six-link mechanism.

The combination described by Equation (4.13) suggests an investigation of the possibilities described by Equation (4.14) which is

$$\begin{gathered} -\alpha,\ -\alpha,\ -\alpha,\ \alpha,\ \alpha,\ \alpha \\ a_1^2 + a_2^2 + a_3^2 = a_4^2 + a_5^2 + a_6^2 \end{gathered} \tag{4.14}$$

where a_1, a_2, a_3, a_4, a_5, and a_6 are the kinematic-link lengths. Sets 13 and 14 represent the parametric values of the combination given by the above equation. Note that this type of combination does provide a mechanism. The results of the above investigation provides an obvious general form of the combination, such as

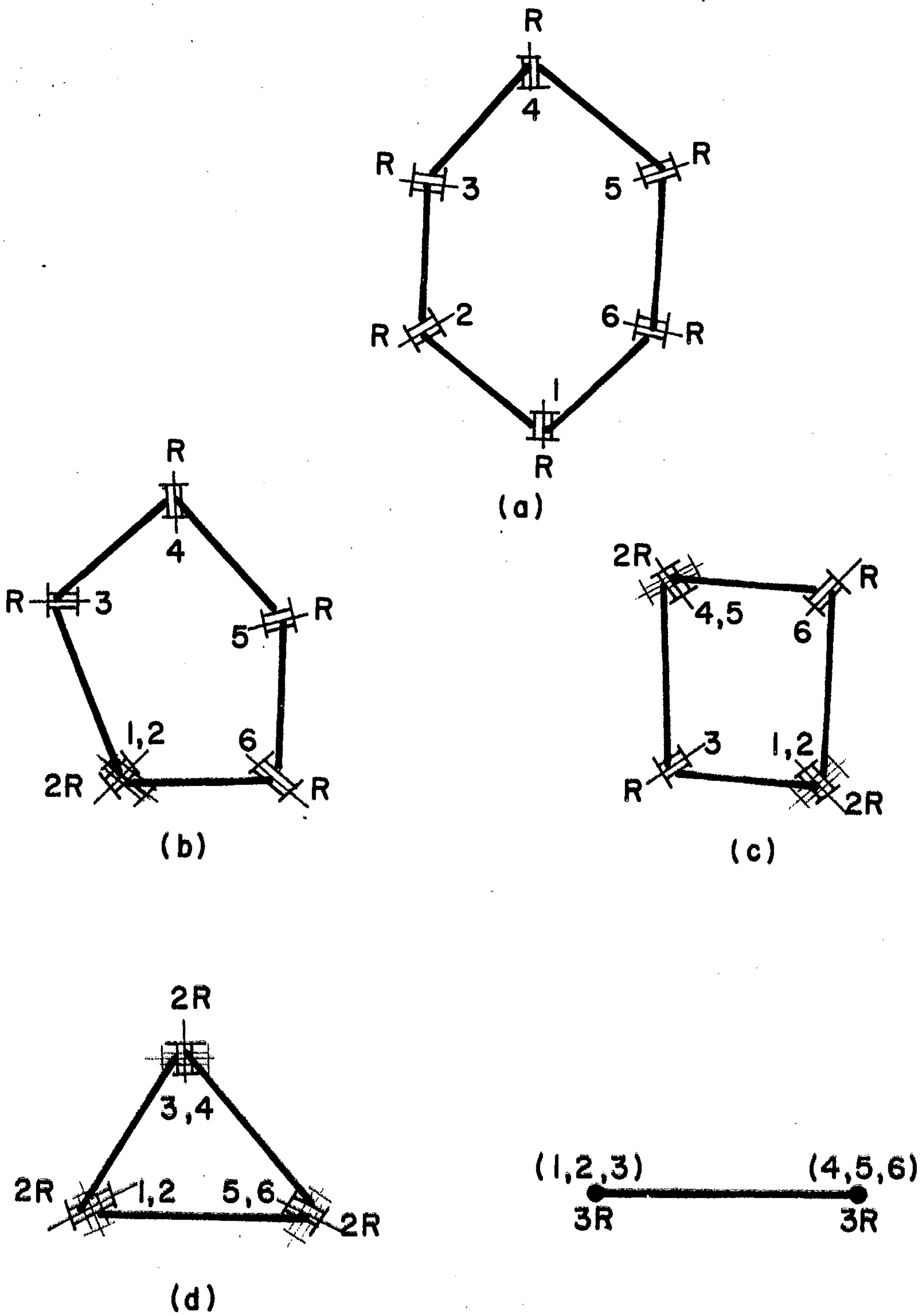


Figure 7. Degenerate Forms of Franke's
Six-Link Chains.

$$\begin{array}{c} -\alpha,\ -\alpha,\ -\alpha,\ \alpha,\ \alpha,\ \alpha \\ a_1{}^k + a_2{}^k + a_3{}^k = a_4{}^k + a_5{}^k + a_6{}^k \end{array} \tag{4.15}$$

where k can take the values other than zero.

The different variations studied by the combinations described by the Equations (4.10), (4.11), (4.12), (4.13), (4.14), and (4.15) do not consider the variations of the possible different values of the twist angles. The results of the previous section can be utilized. Consider, for instance, the set of combinations of the twist angles described by Equation (4.2) which is

$$-\alpha,\ -\beta,\ -\gamma,\ \alpha,\ \beta,\ \gamma \tag{4.2}$$

Some of the possible sets of kinematic links which can be combined with the above variations are

$$a,\ a,\ a,\ a,\ a,\ a$$

and

$$a_1,\ a_2,\ a_3,\ a_1\ a_2\ a_3$$

Consider, for instance, the following simultaneous variations of the kinematic-link and the twist angles

$$\begin{array}{c} -\alpha,\ -\beta,\ -\gamma,\ \alpha,\ \beta,\ \gamma \\ a_1,\ a_2,\ a_3,\ a_1,\ a_2,\ a_3 \end{array} \tag{4.16}$$

Equation (4.16) indicates that for the six-link mechanism under consideration the first and the fourth, the second and the fifth, and the third and the sixth two of three kinematic parameters, the kinematic link and the twist angles are the same. The third parameter, the kink-link, is assumed to be zero for each of the links.

Set 15 in Table IX is the result of an investigation of this type of combination. Note that this combination does yield a mechanism.

However, in view of the results of the previous section, the permutations of the twist angles provide two more sets of combinations. These are

$$\begin{array}{c} -\alpha,\ -\beta,\ -\gamma,\ \beta,\ \gamma,\ \alpha \\ a_1,\ a_2,\ a_3,\ a_2,\ a_3,\ a_1 \end{array} \tag{4.17}$$

and

$$\begin{array}{c} -\alpha,\ -\beta,\ -\gamma,\ \gamma,\ \alpha,\ \beta \\ a_1,\ a_2,\ a_3,\ a_3,\ a_1,\ a_2 \end{array} \tag{4.18}$$

The parametric values of Equation (4.17) and (4.18) are described by the sets 16 and 17 in Table IX. Observe that these types of combinations do yield a six-link mechanism.

The existence of the six-link mechanism described by the combinations given by Equations (4.16), (4.17), and (4.18) leads us to consider the similar combinations such as

$$\begin{array}{c} -\alpha,\ \alpha,\ -\beta,\ \beta,\ \gamma,\ -\gamma \\ a_1,\ a_1,\ a_2,\ a_2,\ a_3,\ a_3 \end{array} \tag{4.19}$$

$$\begin{array}{c} -\alpha,\ -\beta,\ -\gamma,\ \gamma,\ \beta,\ \alpha \\ a_1,\ a_2,\ a_3,\ a_3,\ a_2,\ a_1 \end{array} \tag{4.20}$$

$$\begin{array}{c} -\alpha,\ -\beta,\ -\gamma,\ \beta,\ \alpha,\ \gamma \\ a_1,\ a_2,\ a_3,\ a_2,\ a_1,\ a_3 \end{array} \tag{4.21}$$

$$\begin{array}{c} -\alpha,\ -\beta,\ -\gamma,\ \alpha,\ \gamma,\ \beta \\ a_1,\ a_2,\ a_3,\ a_1,\ a_3,\ a_2 \end{array} \tag{4.22}$$

The parametric values of Equations (4.19), (4.20), (4.21), and (4.22) are described by the sets 18, 19, 20, and 21. Observe that in each of these cases, the six-link chain does exist as a mechanism.

It should be remarked that the order in which the signs appear with the twist angle is extremely important. The negative signs may appear either with the first three or the last three twist angles for the cases described by Equations (4.16), (4.17), (4.18), (4.20), (4.21), and (4.22). For the case described by Equation (4.19), the negative signs appear with the first, third, and fifth or with the second, fourth, and sixth twist angles. For instance, the combination described by Equation (4.19) can be described equally well by the following combination

$$\begin{matrix} \alpha, & -\beta, & \beta, & -\gamma, & \gamma, & -\alpha \\ a_1, & a_2, & a_2, & a_3, & a_3, & a_1 \end{matrix} \tag{4.23}$$

A numerical case of this type of combination is illustrated by set 22. Observe, again, that whenever a cyclic symmetry is observed, a six-link space chain appears to yield a six-link mechanism.

It should be remarked that a six-link chain having the following combination

$$\begin{matrix} -\alpha, & -\beta, & -\gamma, & \alpha, & \beta, & \gamma \\ a_1, & a_2, & a_3, & a_4, & a_5, & a_6 \end{matrix}$$

where $a_1 + a_2 + a_3 = a_4 + a_5 + a_6$ does not yield a six-link mechanism. The same type of results were obtained in the other similar combinations and the permutations of the combination.

Variation in the Kink-Links of the Franke's Six-Link Mechanism

The present section is devoted to a study of the existence and nonexistence of the kink-link components in the Franke's six-link mechanism. In the case of Franke's original mechanism, all the kink-link components have zero values, (see Figure 3). From the geometry of the figure, however, it appears that at least one closure condition can be achieved if all the kink-links are made equal in length and measured along the z axes. Thus, the six kink-link components are

$$s,\ s,\ s,\ -s,\ -s,\ -s$$

Since the first closure condition is obtained by visualizing geometrically, it becomes necessary to examine a closure condition at the second input angular displacement. The combination of a six-link chain under consideration can be described by the following combination of the twist angles, kinematic-links and kink-links.

$$\begin{array}{c} -\alpha,\ -\alpha,\ -\alpha,\ \alpha,\ \alpha,\ \alpha \\ a,\ a,\ a,\ a,\ a,\ a, \\ s,\ s,\ s,\ -s,\ -s,\ -s \end{array} \tag{4.23}$$

The parametric combinations described by Equation (4.23) can be rewritten to have the following form

$$\begin{array}{c} -\alpha,\ -\alpha,\ -\alpha,\ \alpha,\ \alpha,\ \alpha \\ a_1,\ a_2,\ a_3,\ a_1,\ a_2,\ a_3 \\ s_1,\ s_2,\ s_3,\ -s_1,\ -s_2,\ -s_3 \end{array} \tag{4.24}$$

This type of combination indicates that the six-link chain under consideration has its kink-links equal in magnitude but opposite in signs.

The numerical values of this type of combination are tabulated in Table X. Observe that sets 1, 2, 3, 4, 5, 6, and 7 consider different values of the twist angles.

The different closure conditions obtained for these sets of values indicate that the combinations described by Equation (4.24) yield a six-link mechanism.

It has been observed that the Franke's mechanism can exist without any kink-links. In fact, it has been shown that this mechanism can exist even when two of the opposite kinematic-links have zero magnitude. Therefore, it can be predicted that a six-link chain is expected to exist as a mechanism with the following combinations in which two of the opposite links are of zero length.

$$\begin{gathered} -\alpha,\ -\alpha,\ -\alpha,\ \alpha,\ \alpha,\ \alpha \\ 0,\ a,\ a,\ 0,\ a,\ a \\ s_1,\ s_2,\ s_3,\ -s_1, -s_2,\ -s_3 \end{gathered} \tag{4.25}$$

The comparison of the two types of combinations given by Equations (4.10) and (4.24) indicates that the above combination is expected to yield a six-link mechanism. The above combination can be further modified to the following

$$\begin{gathered} -\alpha,\ -\alpha,\ -\alpha,\ \alpha,\ \alpha,\ \alpha \\ 0,\ a,\ a,\ 0,\ a,\ a \\ s_1,\ s_2,\ 0,\ s_1,\ s_2,\ 0 \end{gathered} \tag{4.26}$$

$$\begin{gathered} -\alpha,\ -\alpha,\ -\alpha,\ \alpha,\ \alpha,\ \alpha \\ 0,\ a,\ a,\ 0,\ a,\ a \\ 0,\ 0,\ s_3,\ 0,\ 0,\ -s_3 \end{gathered} \tag{4.27}$$

TABLE X

VARIATION OF THE TWIST ANGLES, KINEMATIC LINKS AND KINK-LINKS OF THE FRANKE'S SIX-LINK MECHANISM

Sets		1	2	3	4	5	6
	a_i	4.	5.	6.	4.	5.	6.
	α_i	-90.	-90.	-90.	90.	90.	90.
1	s_i	1.	2.	3.	-1.	-2.	-3.
	θ_i	60.0	232.22	302.85	102.27	311.66	256.48
		90.	270.	270.	90.	270.	270.
		4.	5.	6.	4.	5.	6.
		-80.	-80.	-80.	80.	80.	80.
2		1.	2.	3.	-1.	-2.	-3.
		60.	235.15	279.97	93.64	292.82	250.92
		90.	279.52	254.57	69.66	244.56	272.14
		4.	5.	6.	4.	5.	6.
		-70.	-70.	-70.	70.	70.	70.
3		1.	2.	3.	-1.	-2.	-3.
		60.	240.88	263.54	85.26	274.92	247.58
		90.	273.10	249.25	57.69	235.17	270.34
		4.	5.	6.	4.	5.	6.
		-50.	-50.	-50.	50.	50.	50.
4		1.	2.	3.	-1.	-2.	-3.
		60.	245.95	247.73	70.72	251.94	244.82
		90.	248.25	245.85	54.88	232.98	254.61
		4.	5.	6.	4.	5.	6.
		-40.	-40.	-40.	40.	40.	40.
5		1.	2.	3.	-1.	-2.	-3.
		60.	244.84	244.46	66.03	246.49	243.54
		90.	239.25	242.16	57.46	234.20	246.35
		4.	5.	6.	4.	5.	6.
		-30.	-30.	-30.	30.	30.	30.
6		1.	2.	3.	-1.	-2.	-3.
		60.	243.02	242.47	62.98	243.22	242.22
		90.	235.11	236.53	58.91	235.43	240.07
		4.	5.	6.	4.	5.	6.
7		-20.	-20.	-20.	20.	20.	20.
		1.	2.	3.	-1.	-2.	-3.
		60.	241.38	241.15	61.19	241.31	241.07
		90.	234.77	231.54	58.21	236.16	236.21

TABLE X (continued)

Sets		1	2	3	4	5	6
	a_i	0.	5.	5.	0.	5.	5.
	α_i	-90.	-90.	-90.	90.	90.	90.
8	s_i	1.	2.	0.	-1.	-2.	0.
	θ_i	30.	176.89	332.72	150.0	310.0	332.72
		60.	219.20	284.49	119.99	320.79	284.49
		0.	5.	5.	0.	5.	5.
		-90.	-90.	-90.	90.	90.	90.
9		0.	0.	1.	0.	0.	-1.
		30.	208.63	307.23	143.58	323.56	307.29
		60.	241.29	280.92	116.38	296.35	280.91
		0.	5.	5.	0.	5.	5.
		-90.	-90.	-90.	90.	90.	90.
10		1.	0.	0.	-1.	0.	0.
		30.	199.10	314.68	149.99	340.89	314.68
		60.	233.10	280.21	119.99	306.89	280.21
		0.	5.	5.	0.	5.	5.
		-90.	-90.	-90.	90.	90.	90.
11		0.	1.	0.	0.	-1.	0.
		30.	200.02	314.0	149.99	339.97	314.0
		60.	234.26	279.87	119.99	305.73	279.87
		0.	0.	5.	0.	0.	5.
		-90.	-90.	-90.	90.	90.	90.
12		1.	0.	0.	-1.	0.	0.
		30.	247.0	220.66	150.0	193.0	220.66
		60.	338.19	249.43	120.0	201.80	249.43
		0.	0.	5.	0.	0.	5.
		-90.	-90.	-90.	90.	90.	90.
13		0.	0.	1.	0.	0.	-1.
		30.	164.86	317.68	30.	15.13	222.32
		60.					
		0.	0.	0.	0.	0.	0.
		-90.	-90.	-90.	90.	90.	90.
14		3.	3.	3.	-3.	-3.	-3.
		90.	270.	270.	90.	270.	270.
		120.	307.39	267.10	52.60	240.0	293.63

TABLE X (continued)

Sets		1	2	3	4	5	6
	a_i	0.	0.	0.	0.	0.	0.
	α_i	-90.	-80.	-70.	90.	80.	70.
15	s_i	1.	2.	3.	-1.	-2.	-3.
	θ_i	60.	228.85	321.79	79.0	274.99	235.47
		90.					
		0.	0.	0.	0.	0.	0.
		-90.	-80.	-70.	80.	70.	90.
16		1.	2.	3.	-2.	-3.	-1.
		30.	214.58	226.19	357.58	110.97	144.46
		60.	208.89	306.92	44.53	318.71	221.87
		0.	0.	0.	0.	0.	0.
		-90.	-80.	-70.	70.	90.	80.
17		1.	2.	3.	-3.	-1.	-2.
		30.	225.02	331.25	88.56	303.38	219.09
		60.	220.42	283.68	97.07	289.34	254.20
		0.	0.	0.	0.	0.	0.
		-90.	90.	-80.	80.	-70.	70.
18		-3.	3.	-2.	2.	-1.	1.
		30.	169.97	124.28	35.44	261.91	191.81
		60.					
		0.	0.	0.	0.	0.	0.
		-90.	-80.	-70.	90.	70.	80.
19		3.	2.	1.	-3.	-1.	-2.
		30.	244.39	297.76	109.89	297.15	255.34
		60.	235.88	275.80	96.47	269.96	269.98
		0.	0.	0.	0.	0.	0.
		-90.	-80.	-70.	80.	90.	70.
20		3.	2.	1.	-2.	-3.	-1.
		30.	131.0	58.66	0.194	266.72	236.90
		60.	229.08	313.54	91.51	274.75	233.29
		0.	0.	0.	0.	0.	0.
		-90.	-90.	-90.	90.	90.	90.
21		3.	4.	0.	-3.	-4.	0.
		60.	228.19	160.09	300.0	131.81	199.9
		90.	270.	270.	90.	270.	270.

TABLE X (continued)

Sets		1	2	3	4	5	6
	a_i	0.	0.	0.	0.	0.	0.
	α_i	-90.	-90.	-90.	90.	90.	90.
22	s_i	3.	0.	4.	-3.	0.	-4.
	θ_i	60.	281.68	311.82	300.0	78.31	48.18
		90.	270.	270.	90.	270.	270.
		0.	0.	0.	0.	0.	0.
		-90.	-90.	-90.	90.	90.	90.
23		0.	3.	4.	0.	-3.	-4.
		90.	270.	270.	90.	270.	270.

$$\begin{matrix} -\alpha, & -\alpha, & -\alpha, & \alpha, & \alpha, & \alpha \\ 0, & a, & a, & 0, & a, & a \\ 0, & s_2, & 0, & 0, & -s_2, & 0 \end{matrix} \tag{4.28}$$

and

$$\begin{matrix} -\alpha, & -\alpha, & -\alpha, & \alpha, & \alpha, & \alpha \\ 0, & a, & a, & 0, & a, & a \\ s_1, & 0, & 0, & -s_1, & 0, & 0 \end{matrix} \tag{4.29}$$

These combinations, (4.26), (4.27), (4.28), and (4.29), are described by considering the appropriate numerical values associated with sets 8, 9, 10, and 11 of Table X. Observe that in each of these combinations, a kinematic chain of six-links yields a six-link mechanism.

The importance of the existence of the kink-links is realized when four of the six kinematic links of a six-link chain have zero link length. For instance, consider the following combinations of the kinematic-links and kink-links.

$$\begin{matrix} -\alpha, & -\alpha, & -\alpha, & \alpha, & \alpha, & \alpha \\ 0, & 0, & a, & 0, & 0, & a \\ s_1, & 0, & 0, & -s_1, & 0, & 0 \end{matrix} \tag{4.30}$$

$$\begin{matrix} -\alpha, & -\alpha, & -\alpha, & \alpha, & \alpha, & \alpha \\ 0, & 0, & a, & 0, & 0, & a \\ 0, & s_2, & 0, & 0, & -s_2, & 0 \end{matrix} \tag{4.31}$$

$$\begin{matrix} -\alpha, & -\alpha, & -\alpha, & \alpha, & \alpha, & \alpha \\ 0, & 0, & a, & 0, & 0, & a \\ 0, & 0, & s_3, & 0, & 0, & -s_3 \end{matrix} \tag{4.32}$$

Sets 12 and 13 are examples of the combinations described by the Equations (4.30), (4.31), and (4.32). Observe that the existence of the two opposite kink-links with a minimum of two opposite equal kinematic-links yields a mechanism. It should be noted here that these mechanisms have four physical links. The vanishing of the four kinematic-links and four kink-links places two revolute pairs at the two opposite vertices of the six-link mechanism. Consequently, such a combination of two revolute pairs can be replaced by substituting the kinematic pairs having two degrees of freedom. For instance, the two intersecting revolute pairs can be substituted by a kinematic pair having rotations about two independent axes, viz., a slotted sphere.

The different variations of the kinematic-link and the kink-links and their importance in constructing a six-link mechanism lead to the problem of examining the existence of a six-link chain having all the kinematic-links of zero length and all the kink-links are of finite length. Consider, for instance, the following combinations

$$\begin{array}{c} -\alpha,\ -\alpha,\ -\alpha,\ \alpha,\ \alpha,\ \alpha \\ 0,\ 0,\ 0,\ 0,\ 0,\ 0 \\ s,\ s,\ s,\ -s,\ -s,\ -s \end{array} \tag{4.33}$$

Observe that in the above combination, all the twist angles and the kink-links are equal. The numerical values of this combination are given in set 14, Table X. Note that the four closure conditions are obtained for this type of the six-link chain. Thus, a six kink-link mechanism having all the kinematic-links of zero length exists.

If we examine all the previous kink-links combinations, we observe that all six twist angles are equal in magnitude. The study of the variation of twist angles and the kinematic-link, therefore, suggests the examination of the following combinations:

$$\begin{matrix} -\alpha, & -\beta, & -\gamma, & \alpha, & \beta, & \gamma \\ 0, & 0, & 0, & 0, & 0, & 0 \\ s_1, & s_2, & s_3, & -s_1, & -s_2, & -s_3 \end{matrix} \tag{4.34}$$

$$\begin{matrix} -\alpha, & -\beta, & -\gamma, & \beta, & \gamma, & \alpha \\ 0, & 0, & 0, & 0, & 0, & 0 \\ s_1, & s_2, & s_3, & -s_2, & -s_3, & -s_1 \end{matrix} \tag{4.35}$$

$$\begin{matrix} -\alpha, & -\beta, & -\gamma, & \gamma, & \alpha, & \beta \\ 0, & 0, & 0, & 0, & 0, & 0 \\ s_1, & s_2, & s_3, & -s_3, & -s_1, & -s_2 \end{matrix} \tag{4.36}$$

$$\begin{matrix} -\alpha, & -\beta, & -\gamma, & \gamma, & \beta, & \alpha \\ 0, & 0, & 0, & 0, & 0, & 0 \\ s_1, & s_2, & s_3, & -s_3, & -s_2, & -s_1 \end{matrix} \tag{4.37}$$

$$\begin{matrix} -\alpha, & -\beta, & -\gamma, & \beta, & \alpha, & \gamma \\ 0, & 0, & 0, & 0, & 0, & 0 \\ s_1, & s_2, & s_3, & -s_2, & -s_1, & -s_3 \end{matrix} \tag{4.38}$$

$$\begin{matrix} -\alpha, & -\beta, & -\gamma, & \alpha, & \gamma, & \beta \\ 0, & 0, & 0, & 0, & 0, & 0 \\ s_1, & s_2, & s_3, & -s_1, & -s_3, & -s_2 \end{matrix} \tag{4.39}$$

and

$$-\alpha,\ \alpha,\ -\beta,\ \beta,\ -\gamma,\ \gamma$$
$$0,\ 0,\ 0,\ 0,\ 0,\ 0 \qquad (4.40)$$
$$s_1,\ -s_1,\ s_2,\ -s_2,\ s_3,\ -s_3$$

Equations (4.34), (4.35), (4.36), (4.37), (4.38), (4.39), and (4.40) represent the seven characteristic permutations of the twist angles. Observe, however, that the kink-links, their magnitude and signs, are also permuted correspondingly. The examination of the sets 15 through 20 in Table X proves that the above combinations do yield a six kink-link mechanism.

The limiting conditions under which a kink-link chain can be assembled to form a six or less number of kink-link mechanism can be investigated by considering the following combinations:

$$-\alpha,\ -\alpha,\ -\alpha,\ \alpha,\ \alpha,\ \alpha$$
$$0,\ 0,\ 0,\ 0,\ 0,\ 0 \qquad (4.41)$$
$$s_1,\ s_2,\ 0,\ -s_1,\ -s_2,\ 0$$

$$-\alpha,\ -\alpha,\ -\alpha,\ \alpha,\ \alpha,\ \alpha$$
$$0,\ 0,\ 0,\ 0,\ 0,\ 0 \qquad (4.42)$$
$$s_1,\ 0,\ s_3,\ -s_1,\ 0,\ -s_3$$

$$-\alpha,\ -\alpha,\ -\alpha,\ \alpha,\ \alpha,\ \alpha$$
$$0,\ 0,\ 0,\ 0,\ 0,\ 0 \qquad (4.43)$$
$$0,\ s_2,\ s_3,\ 0,\ -s_2,\ -s_3$$

Sets 21, 22, and 23 of Table X show the results of this investigation. Observe that the successful results obtained for these combinations indicate that two of the opposite kinks can be assumed to have a zero kink-link. Consequently, the six kink-link mechanism reduces to a four kink-link mechanism, having two revolute pairs at the opposite vertex. Note that these kinematic pairs are connected by a kink-link and kinematic-links, both having a zero length. Therefore, such a combination of the two intersecting revolute pairs can be substituted by a kinematic pair having two independent rotations, for instance, the slotted sphere.

It should be noted that a minimum of four kink-links must exist in a mechanism having all the kinematic-links of zero length.

The striking similarities in the behavior of the kink-links and the kinematic-links in building the six-link mechanism immediately lead to the problem of examining the existence of the six-link mechanism having the following combination

$$-\alpha,\ -\alpha,\ -\alpha,\ \alpha,\ \alpha,\ \alpha$$

$$0,\ 0,\ 0,\ 0,\ 0,\ 0$$

$$s_1,\ s_2,\ s_3,\ -s_4,\ -s_5,\ -s_6$$

where the six kink-links are related as follows

$$|s_1 + s_2 + s_3| = |s_4 + s_5 + s_6|$$

Note that the kink-links chain, having the above combination, yields a structure rather than a mechanism. Thus, the kink-links and the kinematic-links are playing their independent role at this stage of the combination. Though these two types of parameters, the kink-links and kinematic-links, help build a kinematic chain, they do not seem to be related to each other when mobility of the six-link chain is the major issue. For instance, consider the following apparent relationship between the kink-links and the kinematic-links of the Franke's six-link.

Figure 8 shows the Franke's six-link having the various combinations of the kink-links and the kinematic-links. Suppose along axis z_1 (Figure 8a) we introduce a kink-link of length s_1 and make the corresponding change in the kinematic-link a_2 so that one complete closure condition is known. Thus, the kinematic-link a_2 will be altered in its length to a_2' (Figure 8b) given by the following relationship.

$$a_2' = a_2 + s_1$$

If a similar change is made along the z_2 axis and in the kinematic-link a_3 (Figure 8c) so that

$$a_3' = a_3 + s_2$$

Similar changes between the kink-links and kinematic-links will yield the relationship

$$a_1' = a_i \pm s_{i-1}$$

If such changes are made in the kinematic-links to accommodate the existence of the kink-link and if such a kinematic chain is examined for a closure condition, then the product matrix [W] becomes divergent. Thus, the apparent simple relationship known to be giving a closed chain does not yield the closure condition. Therefore, such a closed chain must be a structure.

The above investigation of this simple relationship leads to a conclusion that both the kinematic-links and kink-links play their independent role when the mobility of a close chain is the principal issue.

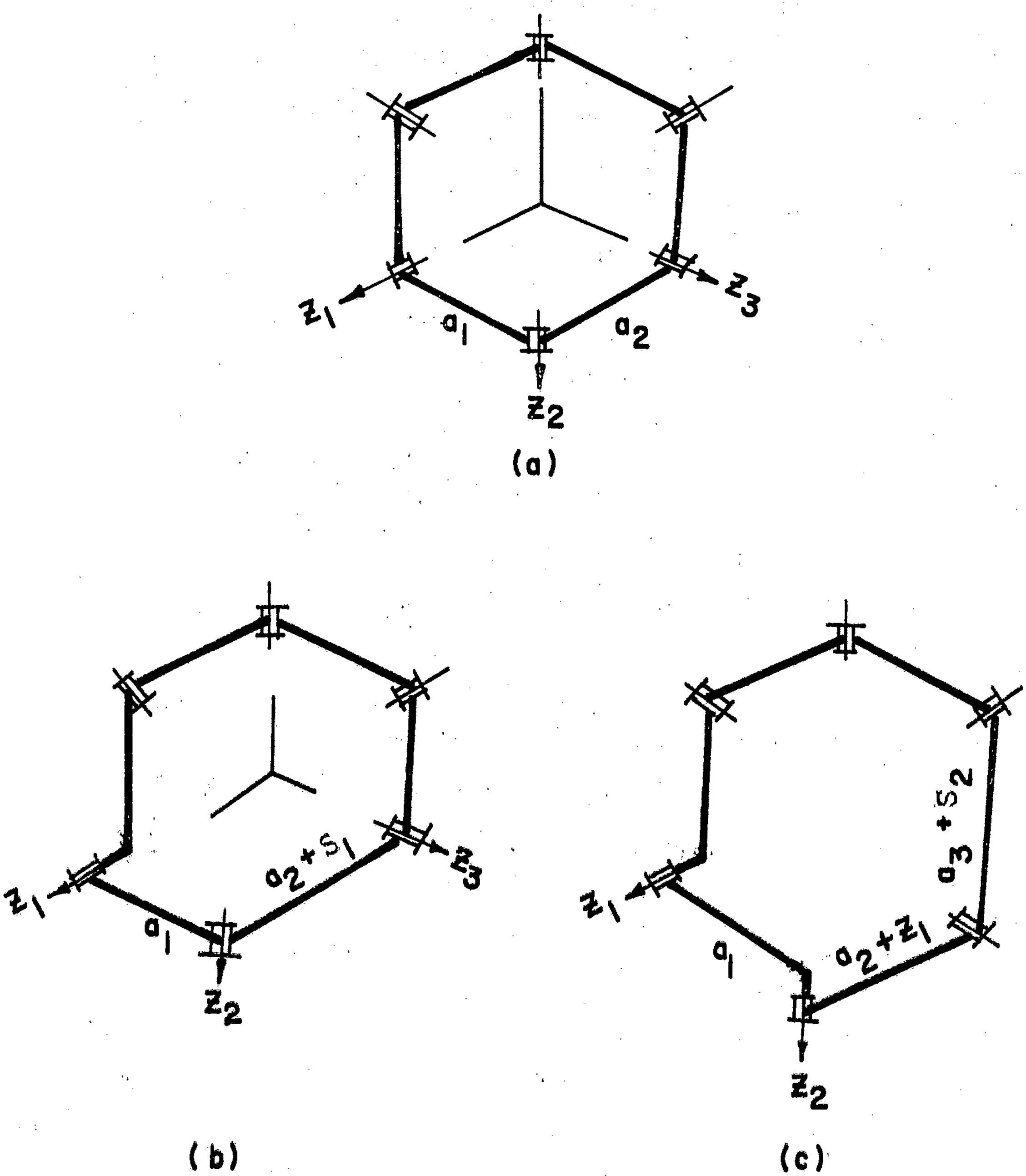


Figure 8. Apparent Relationships Between the Kink-Links and Kinematic Links.

It appears that they are both rather related to the twist angles of a chain.

Variation in the Bricard's Articulated Six-Link Mechanism

The Bricard articulated six-link mechanism is defined by the following kinematic parameters.

$$\begin{array}{c} -90^\circ,\ -90^\circ,\ -90^\circ,\ -90^\circ,\ -90^\circ,\ -90^\circ \\ 0,\ 0,\ 0,\ 0,\ 0,\ 0 \\ s,\ s,\ s,\ s,\ s,\ s \end{array} \tag{4.44}$$

Observe that all the twist angles are equal, all the kinematic-links are of zero length, and all the kink-links are of equal length.

It should be noted that Bricard's six-link mechanism is similar in construction to the Franke's kink-link mechanism. In fact, all the results obtained for the Franke's mechanism are similar to those obtained for this Bricard mechanism. The difference, however, exists in the signs of the twist angles and in the signs of the kink-links.

The general notations to describe the Bricard's kink-link six-link mechanism can be expressed as

$$\begin{array}{c} -\alpha,\ -\alpha,\ -\alpha,\ -\alpha,\ -\alpha,\ -\alpha \\ 0,\ 0,\ 0,\ 0,\ 0,\ 0 \\ s,\ s,\ s,\ s,\ s,\ s \end{array} \tag{4.45}$$

Sets 1, 2, and 3 of Table XI show numerical examples satisfying the conditions described by Equation (4.45). These conditions may be generalized as was done in the Franke's mechanism by the following

TABLE XI

VARIATION OF THE BRICARD'S ARTICULATED SIX-LINK MECHANISM

Sets		1	2	3	4	5	6
	a_i	0.	0.	0.	0.	0.	0.
	α_i	-90.	-90.	-90.	-90.	-90.	-90.
1	s_i	4.	4.	4.	4.	4.	4.
	θ_i	90.	0.	270.	90.	0.	270.
		60.	26.89	251.31	60.	26.89	251.31
		0.	0.	0.	0.	0.	0.
		-80.	-80.	-80.	-80.	-80.	-80.
2		4.	4.	4.	4.	4.	4.
		90.	338.70	300.44	90.0	338.70	300.44
		60.	20.88	270.17	60.0	20.83	270.17
		0.	0.	0.	0.	0.	0.
		-70.	-70.	-70.	-70.	-70.	-70.
3		4.	4.	4.	4.	4.	4.
		60.	357.72	302.71	60.0	357.72	302.71
		120.	57.24	304.07	118.52	57.36	299.96
		0.	0.	0.	0.	0.	0.
		-80.	-70.	-60.	-80.	-70.	-60.
4		4.	5.	6.	4.	5.	6.
		60.	349.59	295.34	60.	349.59	295.34
		120.	52.57	294.61	119.58	52.57	294.61
		5.	5.	5.	5.	5.	5.
		-90.	-90.	-90.	-90.	-90.	-90.
5		4.	4.	4.	4.	4.	4.
		60.	281.27	130.44	60.0	281.26	130.44
		90.	102.68	269.99	89.99	102.68	269.99
		5.	0.	0.	5.	0.	0.
		0.	-90.	-90.	0.	-90.	-90.
6		3.	0.	3.	3.	0.	3.
		160.	200.	245.11	160.0	200.	245.11
		180.	180.	241.92	180.0	180.0	241.93
		5.	0.	0.	5.	0.	0.
		0.	-90.	-90.	0.	-90.	-90.
7		3.	0.	4.	3.	0.	4.
		160.	200.	245.11	160.	200.	245.11
		180.	180.	241.92	180.	180.	241.92

$$-\alpha,\ -\beta,\ -\gamma,\ -\alpha,\ -\beta,\ -\gamma$$
$$0,\ 0,\ 0,\ 0,\ 0,\ 0 \qquad (4.46)$$
$$s_1,\ s_2,\ s_3,\ s_1,\ s_2,\ s_3$$

Set 4 of Table XI shows that the condition given by Equation (4.46) does yield the Bricard kink-link mechanism. The permutation of twist angles along with the kink-links in the above equation is possible. Such permutation will yield the similar conditions described by Equations (4.34) to (4.39). The Bricard articulated mechanism does exist under these conditions.

In the variational study of Franke's six-link mechanism, a general model was obtained by introducing the kink-links. Thus, the existence of the kink-links in Franke's six-link mechanism then yields a six-link mechanism with all eighteen parameters. Similarly, a general model of Bricard's articulated six-link mechanism can be obtained if the mechanism exists with the following conditions

$$-\alpha,\ -\alpha,\ -\alpha,\ -\alpha,\ -\alpha,\ -\alpha$$
$$a,\ a,\ a,\ a,\ a,\ a \qquad (4.47)$$
$$s,\ s,\ s,\ s,\ s,\ s$$

The numerical illustration shown in the set 5 suggests that the general model described by Equation (4.47) is possible from the Bricard mechanism.

The general model described by Equation (4.47) does exist in some of the limiting cases, when two of the opposite kinematic links are of non-zero but of equal values in their length and two of the opposite kink-links are zero. Such a six-link mechanism can be described by the following combination

$$-\alpha,\ -\alpha,\ -\alpha,\ -\alpha,\ -\alpha,\ -\alpha$$

$$a,\ 0,\ 0,\ a,\ 0,\ 0 \qquad (4.48)$$

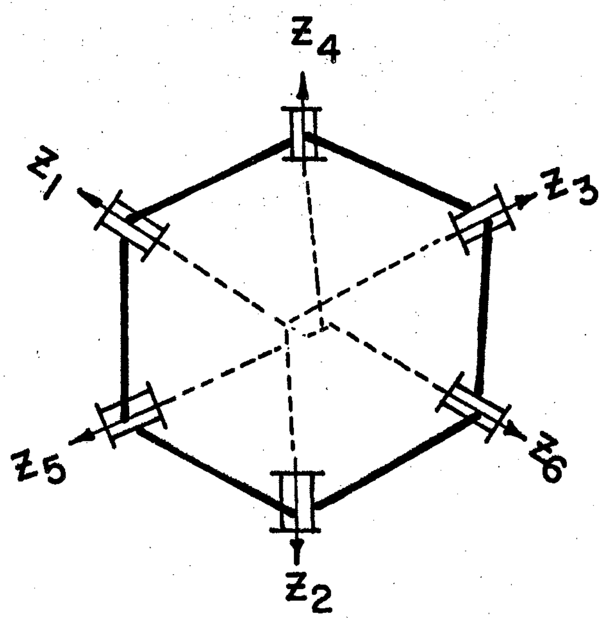
$$s,\ 0,\ s,\ s,\ 0,\ s$$

Sets 6 and 7 show that the above combination does yield a six-link mechanism which is generated from Bricard's articulated mechanism.

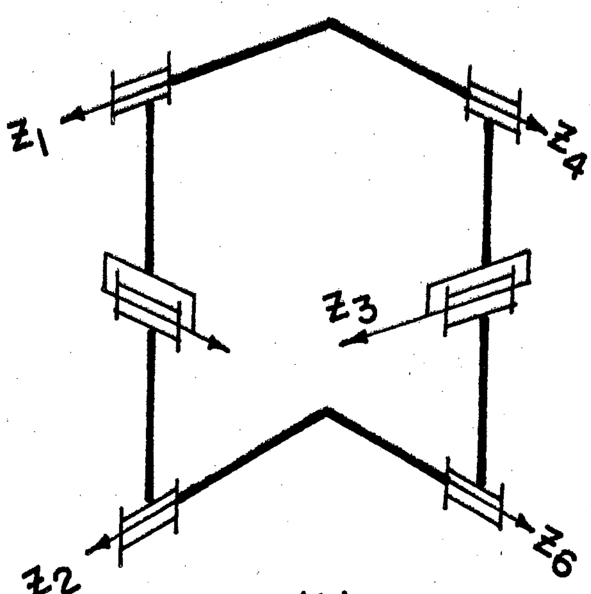
A limited investigation was made of the Bricard mechanism primarily because of the observation that it is similar in construction to the Franke's mechanism and was found to be giving similar conditions for the existence of the mechanism. The only difference between the two mechanisms is in the signs of the twist angles. Observe that all the twist angles are either of positive or negative values in the case of Bricard's mechanism. However, in the Franke's kink-link mechanism either the first three or the alternate three twist angles are negative values. The other three twist angles are always positive.

Relationship Between the Franke's Six-Link and Bricard's Kink-Link Mechanism

The similar behavior of the Franke's kink-link mechanism and the Bricard's articulated mechanism indicates a possible relationship between these two mechanisms. Such a relationship becomes more obvious when the geometry of the Franke's six-link mechanism is considered. When all the kink-links are zero, then two pairs of three alternate axes intersect in two finitely located points as shown in Figure 9a. When the same mechanism is reconstructed so that the two finitely located points of intersection now lie at infinity, then the kinematic notations become



(a)



(b)

Figure 9. Relationship Between the Franke's and Bricard's Six-Link Mechanism.

$$\begin{array}{c} -\alpha,\ -\alpha,\ -\alpha,\ -\alpha,\ -\alpha,\ -\alpha \\ a,\ a,\ 0,\ a,\ a,\ 0 \\ -s,\ 0,\ -s,\ s,\ 0,\ s \end{array} \qquad (4.49)$$

The second case of Franke's six-link mechanism having the two points intersecting at infinity is shown in Figure 9b where $\alpha = 90^\circ$.

Set 1 of Table XII shows that the combination described by Equation (4.49) yields a six-link mechanism. The examination of the kinematic notations of the mechanism shown in Figure 9b indicates clearly that this mechanism is one of the degenerate cases of the combination described by the general model of the Bricard six-link mechanism given by Equation (4.47).

The mechanism considered in set 1 is especially suitable for studying the limiting values of twist angles. A six-link mechanism exists when two of the opposite twist angles are zero. For such a mechanism, the existence of kink-link becomes essential. The kinematic notations of such mechanisms can be described by the following combination.

$$\begin{array}{c} -\alpha,\ -\alpha,\ 0,\ -\alpha,\ -\alpha,\ 0 \\ a,\ a,\ 0,\ a,\ a,\ 0 \\ -s,\ 0,\ s,\ -s,\ 0,\ s \end{array} \qquad (4.50)$$

It has been noted earlier that a six-link mechanism exists with a minimum of four kink-links or four kinematic-links. Thus, the condition described by Equation (4.50) can be rewritten to take into account the absolute minimum requirements for a six-link mechanism. Such a combination of the kinematic parameters can be described by the following.

TABLE XII

RELATIONSHIP BETWEEN FRANKE'S AND BRICARD'S SIX-LINK MECHANISM

Sets		1	2	3	4	5	6
	a_i	3.	3.	0.	3.	3.	0.
	α_i	-90.	-90.	-90.	-90.	-90.	-90.
1	s_i	-4.	0.	4.	-4.	0.	4.
	θ_i	140.	65.46	40.	220.	294.54	320.
		160.	37.76	20.	200.	322.74	340.
		0.	0.	0.	0.	0.	0.
		90.	90.	0.	90.	90.	0.
2		-3.	0.	3.	-3.	0.	3.
		30.	280.	339.0	21.0	280.	330.
		60.	0.0	99.74	254.43	0.0	294.17
		3.	3.	0.	3.	3.	0.
		0.	0.	-90.	0.	0.	-90.
3		2.	0.	-2.	2.	0.	-2.
		170.	20.	350.	170.	20.	350.
		140.	80.	320.	140.	80.	320.
		3.	3.	0.	3.	3.	0.
		0.	0.	-80.	0.	0.	-80.
4		2.	0.	-2.	2.	0.	-2.
		170.0	20.0	350.	170.	20.	350.0
		140.0	80.0	320.0	140.0	80.0	320.0
		3.	3.	0.	3.	3.	0.
		0.	0.	-70.	0.	0.	-70.
5		2.	0.	-2.	2.	0.	-2.
		170.0	20.0	350.0	170.0	20.0	350.0
		140.	80.0	320.0	140.0	80.0	320.0
		3.	3.	0.	3.	3.	0.
		0.	0.	-40.	0.	0.	-40.
6		2.	0.	-2.	2.	0.	-2.
		170.0	20.0	350.0	170.0	20.0	350.0
		140.0	80.0	320.0	140.0	80.0	320.0
		3.	3.	0.	3.	3.	0.
		0.	0.	-30.	0.	0.	-30.
7		2.	0.	-2.	2.	0.	-2.
		170.0	20.	350.0	170.0	20.0	350.0
		140.0	80.0	320.0	140.0	80.0	320.0

TABLE XII (continued)

Sets		1	2	3	4	5	6
	a_i	3.	3.	0.	3.	3.	0.
	α_i	0.	0.	-20.	0.	0.	-20.
8	s_i	2.	0.	-2.	2.	0.	-2.
	θ_i	170.0	20.	350.0	170.0	20.0	350.0
		140.0	80.0	320.0	140.0	80.0	320.0
		3.	3.	3.	3.	3.	3.
		0.	0.	-80.	0.	0.	-80.
9		2.	0.	-2.	2.	0.	-2.
		170.0	20.	350.0	170.0	20.0	350.0
		140.0	80.0	320.0	220.	280.0	40.0
		3.	3.	3.	3.	3.	3.
		0.	0.	-80.	0.	0.	-80.
10		2.	2.	-4.	2.	2.	-4.
		170.0	20.0	350.0	170.0	20.0	350.0
		140.0	80.0	320.0	220.0	280.0	40.0

$$-\alpha,\ -\alpha,\ 0,\ -\alpha,\ -\alpha,\ 0$$
$$0,\ 0,\ 0,\ 0,\ 0,\ 0 \tag{4.51}$$
$$-s,\ 0,\ s,\ -s,\ 0,\ s$$

Since it does not really matter in the above combination if the twist angles are taken to be of positive values, set 2 in Table XII shows the illustrative example of such a degenerate case.

In the above example, only two of the opposite twist angles assume zero value. The second limiting case can be considered in which four of the twist angles assume zero values. However, in such a case a minimum of two kinematic-links must exist in order to obtain a mechanism.

The existing literature on the six-link mechanism cites a case of such a six-link mechanism in which four of the twist angles assume zero values. The mechanism can be described by the following combinations.

$$0,\ 0,\ -\alpha,\ 0,\ 0,\ -\alpha$$
$$a,\ a,\ 0,\ a,\ a,\ 0 \tag{4.52}$$
$$s,\ 0,\ -s,\ s,\ 0,\ -s$$

The more general combinations are:

$$0,\ 0,\ -\alpha,\ 0,\ 0,\ -\alpha$$
$$a_1,\ a_2,\ a_3,\ a_1,\ a_2,\ a_3 \tag{4.53}$$
$$s_1,\ s_2,\ -(s_1+s_2),\ s_4,\ s_5,\ (-s_4+s_5)$$

Sets 3-10 are the mechanisms described by the combinations given by Equations (4.52) and (4.53).

The Existence Criteria of the Six-Link Mechanism

In the previous section the different conditions under which a six-link mechanism exists were examined. The literature on the six-link mechanism has emphasized that the existence of this mechanism is either due to a symmetry about a plane or line or due to the "ad-hoc" criterion of the intersection of a pair of three axes into two points, located at a finite distance or at infinity. While such criteria are able to justify the existence of some of the mechanisms examined in the previous chapter they fail to account for the existence of the others.

The mathematics of the general constraints suggests that a six-link mechanism exists because of its specific geometry which in turn is responsible for producing a general motion consisting of either three rotations and two translations or two rotations and three translations. Existence criteria such as these do not help to build six-link mechanisms though they do provide a necessary and sufficient mathematical reason for their existence.

Note that such a mathematical criteria is translated from the specific geometry of the mechanism. The Bennett mechanism, which is noted to have three general constraints and the geometry that helps to build the mechanism is given by

$$a_1 = a_3, \qquad a_2 = a_4$$

$$\alpha_1 = \alpha_3, \qquad \alpha_2 = \alpha_4$$

and

$$\frac{a_1}{\sin \alpha_1} = \pm \frac{a_2}{\sin \alpha_2}$$

where a_1, a_2, a_3, a_4 and α_1, α_2, α_3, α_4 are the kinematic-links and the twist angles. Goldberg [15] was able to provide a similar geometrical relationship to build the five-link mechanism. Therefore, it is not too unrealistic to expect a set of mathematical relationships that will help build a six-link mechanism.

The findings of the previous section may be briefly summarized as follows:

(1) When all the twist angles are equal, Franke's six-link mechanism exists provided

$$a_1 + a_2 + a_3 = a_4 + a_5 + a_6 \tag{4.53}$$

$$a_1^2 + a_2^2 + a_3^2 = a_4^2 + a_5^2 + a_6^2 \tag{4.54}$$

A similar relationship does not exist between the kink-links and the twist angles.

(2) When the twist angles are different, then the kinematic-links and the kink-links must observe the following relationships:

$$\hat{a}_i = \hat{a}_j \qquad i = 1, 2, 3 \tag{4.55}$$

$$|\hat{d}_i| = |\hat{d}_j| \qquad j = 4, 5, 6 \tag{4.56}$$

where

$$\hat{a}_i = \alpha_i + \sigma a_i \tag{4.57}$$

$$\hat{d}_i = \alpha_i + \sigma d_i \tag{4.58}$$

(3) The following are the seven basic permutations of the twist angles

$$-\alpha,\ -\beta,\ -\gamma,\ \alpha,\ \beta,\ \gamma \tag{4.59}$$

$$-\alpha,\ -\beta,\ -\gamma,\ \beta,\ \gamma,\ \alpha \tag{4.60}$$

$$-\alpha,\ -\beta,\ -\gamma,\ \gamma,\ \alpha,\ \beta \tag{4.61}$$

$$-\alpha,\ \alpha,\ -\beta,\ \beta,\ -\gamma,\ \gamma \tag{4.62}$$

$$-\alpha,\ -\beta,\ -\gamma,\ \gamma,\ \beta,\ \alpha \tag{4.63}$$

$$-\alpha,\ -\beta,\ -\gamma,\ \beta,\ \alpha,\ \gamma \tag{4.64}$$

$$-\alpha,\ -\beta,\ -\gamma,\ \alpha,\ \gamma,\ \beta \tag{4.65}$$

(4) The 6R mechanism exists either with a minimum of four opposite kinematic-links or with a minimum of four opposite kink-links. The mechanism also exists with a minimum of two opposite kinematic-links and two opposite kink-links.

(5) The kinematic-link lengths are always positive.

(6) The kink-links may be either positive or negative. A definite relationship between the twist angles and the signs of the kink-link does not seem to exist. There is, however, a rule-of-thumb which follows: The signs of the kink-links may be taken as opposite to the signs of the twist angles.

(7) The limiting case of the six-link mechanisms have the opposite twist angles of zero values. A minimum of four opposite twist angles may assume a zero value. When the twist angles assume zero values, then the six-link mechanism degenerates. When the twist angles are assumed to be of zero value, then corresponding adjustment is required to assume finite kink-links.

The seven points described above appear to be the governing conditions and are extremely useful in building an empirical relationship between the twist angles, kinematic-links and kink-links of a six-link mechanism. It should be remarked that, in general, there is still no rational way of obtaining such a relationship. The present investigation on the six-link mechanism has relied heavily on all the possible available information regarding mathematical relationships between the kinematic parameters of the six-link mechanism. Perhaps the most important contribution that has been made in this area was by F. M. Dimentberg [46, 47] and Michael Goldberg [15].

Goldberg contends that the six-link mechanism must be related to the Bennett mechanism and Dimentberg derived a relationship for a four-link mechanism having one constraint. However, such a relation appears to take a form described below.

$$\frac{f_1(a,\alpha) + f_2(a,\alpha)}{f_3(a,\alpha) + f_4(a,\alpha)} = \frac{f_1(\alpha)\, f_2(\alpha)}{f_3(\alpha)\, f_4(\alpha)} \tag{4.66}$$

If the information contributed by Goldberg and Dimentberg were placed together, then it is possible to generalize nearly a hundred functions, all of which may claim to be governing the conditions of the existence of the six-link mechanism. Simply by the process of trial and error and by the process of elimination, it is possible to arrive at satisfactory results.

The empirical conditions that appear to govern the existence of a six-link mechanism is given by the following:

$$|\alpha_1 + \alpha_2 + \alpha_3| = |\alpha_4 + \alpha_5 + \alpha_6| \tag{4.67}$$

$$\frac{a_1 \text{ Cosec } \alpha_1 + a_2 \text{ Cosec } \alpha_2 + a_3 \text{ Cosec } \alpha_3}{a_4 \text{ Cosec } \alpha_4 + a_5 \text{ Cosec } \alpha_5 + a_6 \text{ Cosec } \alpha_6} = \pm \frac{\text{Cosec } \alpha_1 \text{ Cosec } \alpha_2 \text{ Cosec } \alpha_3}{\text{Cosec } \alpha_4 \text{ Cosec } \alpha_5 \text{ Cosec } \alpha_6} \tag{4.68}$$

Equation (4.68) may be written to take a more general form such as

$$\frac{a_1^k \text{ Cosec } \alpha_1 + a_2^k \text{ Cosec } \alpha_2 + a_3^k \text{ Cosec } \alpha_3}{a_4^k \text{ Cosec } \alpha_4 + a_5^k \text{ Cosec } \alpha_5 + a_6^k \text{ Cosec } \alpha_6} = \pm \frac{\text{Cosec } \alpha_1 \text{ Cosec } \alpha_2 \text{ Cosec } \alpha_3}{\text{Cosec } \alpha_4 \text{ Cosec } \alpha_5 \text{ Cosec } \alpha_6} \tag{4.69}$$

where $k \geq 0$. The present investigation has examined the case where either $k = 1$, or $k = 2$.

The relationship between the kink-links and the twist angles is given by

$$|\hat{d}_i| = |\hat{d}_j| \qquad \text{for} \quad i = 1, 2, 3 \quad j = 4, 5, 6 \tag{4.70}$$

where

$$\hat{d}_i = \pm \alpha_i \mp \sigma d_i \tag{4.71}$$

and

$$\hat{d}_j = \pm \alpha_j \mp \sigma d_j \tag{4.72}$$

It should be noted that an empirical relationship similar to Equation (4.68) does exist for the kink-link six-link mechanism. However, the empirical relationship needs to be modified because it is noted earlier that the summation law in Equation (4.53) and (4.54) does not exist for the kink-links. Such a modified relationship is given by

$$d_i \text{ Cosec } \alpha_i = \pm \left[\frac{\text{Cosec } \alpha_1 \text{ Cosec } \alpha_2 \text{ Cosec } \alpha_3}{\text{Cosec } \alpha_4 \text{ Cosec } \alpha_5 \text{ Cosec } \alpha_6} \right] d_j \text{ Cosec } \alpha_j \tag{4.72}$$

The following points must be observed before constructing the six-link mechanism. In order to construct a Franke's six-link mechanism, Equations (4.67) and (4.68) must be satisfied simultaneously. Furthermore, three of the twist angles must have negative values, and only cyclic and symmetric permutations are possible. Similar rules hold also for the kink-link six-link mechanism.

The use of these empirical relationships is illustrated by considering the following sets of computed values:

twist angles:	-80°, -80°, -80°, 80°, 80°, 80°	
kinematic-link:	$4''$, $4''$, $4''$, $4''$, $4''$, $4''$	(1)
kink-link:	0°, 0°, 0°, 0°, 0°, 0°	
	-80°, -82°, -78°, 80°, 80°, 80°	
	$4''$, $4.0270''$, $3.97792''$, $0''$, $0''$, $0''$	(2)
	0°, 0°, 0°, 0°, 0°, 0°	
	-80°, -84°, -76°, 80°, 80°, 80°	
	$4''$, $4.05978''$, $3.96094''$, $4''$, $4''$, $4''$	(3)
	0°, 0°, 0°, 0°, 0°, 0°	
	-80°, -86°, -74°, 80°, 80°, 80°	
	$4''$, $4.09798''$, $3.94888''$, $4''$, $4''$, $4''$	(4)
	0°, 0°, 0°, 0°, 0°, 0°	
	-80°, -88°, -76°, 80°, 80°, 80°	
	$4''$, $4.14188''$, $3.94162''$, $4''$, $4''$, $4''$	(5)
	0°, 0°, 0°, 0°, 0°, 0°	
	-80°, -90°, -70°, 80°, 80°, 80°	
	$4''$, $4.19202''$, $3.93920''$, $4''$, $4''$, $4''$	(6)
	0°, 0°, 0°, 0°, 0°, 0°	

Figure 10 shows the displacement analysis of the six-link mechanism described by combination 6. The kinematic-links were obtained from the degenerate form of Equation (4.68):

$$a_2 \text{ Cosec } \alpha_2 = \left[- \frac{\text{Cosec } \alpha_1 \text{ Cosec } \alpha_2 \text{ Cosec } \alpha_3}{\text{Cosec } \alpha_4 \text{ Cosec } \alpha_5 \text{ Cosec } \alpha_6} \right] a_5 \text{ Cosec } \alpha_5$$

$$a_3 \text{ Cosec } \alpha_3 = \left[- \frac{\text{Cosec } \alpha_1 \text{ Cosec } \alpha_2 \text{ Cosec } \alpha_3}{\text{Cosec } \alpha_4 \text{ Cosec } \alpha_5 \text{ Cosec } \alpha_6} \right] a_6 \text{ Cosec } \alpha_6$$

Other permutations described earlier are also expected to yield six-link mechanisms. For instance, consider the following combination of twist angles.

$$\begin{array}{l} -80^\circ,\ -85^\circ,\ -75^\circ,\ 80^\circ,\ 80^\circ,\ 80 \\ a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6 \\ 0,\ 0,\ 0,\ 0,\ 0,\ 0 \end{array} \qquad (7)$$

We need to find the magnitude of the kinematic-link which gives a six-link mechanism. Let

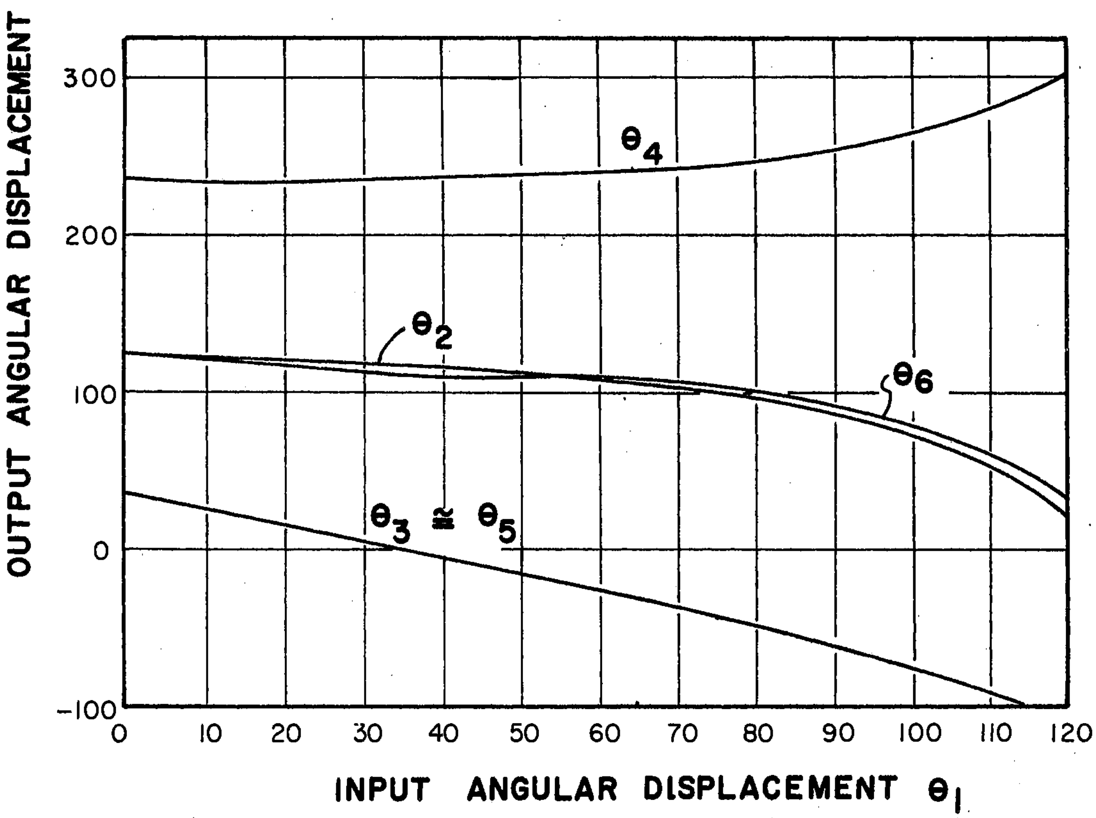
$$a_1 = a_4 = a_5 = a_6 = 4.0''$$

and let $\dfrac{a_2}{\text{Sin } \alpha_2} = \dfrac{a_3}{\text{Sin } \alpha_3}$ be an additional condition. Then Equation (4.68) gives,

$$a_2 = 4.094165$$

$$a_3 = 3.969802$$

The displacement analysis of this mechanism can be carried out similarly as shown in Figure 11.



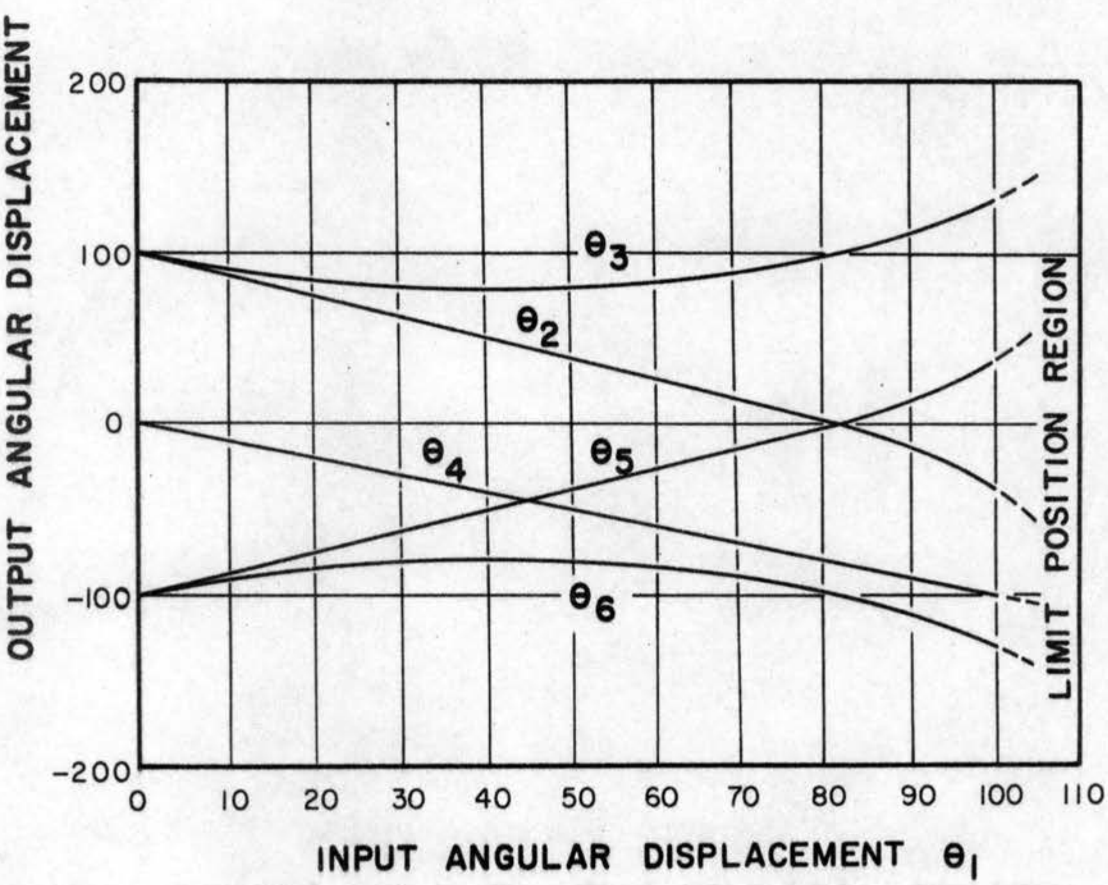
The kinematic parameters of the mechanism are:

α: -80°, -90°, -70°, 80°, 80°, 80°

a: 4.0, 4.19202, 3.9392, 4.0, 4.0, 4.0

s: 0, 0, 0, 0, 0, 0

Figure 10. Displacement Analysis of the Synthesized 6R Mechanism



The kinematic parameters of the mechanism are:

α: -80°, -80°, -80°, 80°, 80°, 80°

a: 0, 0, 0, 0, 0, 0

s: 4", 4", 4", 4", 4", 4"

Figure 11. Displacement Analysis of the Synthesized Kink-Link 6R Mechanism

The kink-link six-link mechanism can be similarly constructed using the relationship given by Equation (4.72). However, the mobility region decreases considerably and therefore it is advisable to use Equations (4.70) in order to build an useful mechanism. Figure 11 shows the displacement analysis of a kink-link mechanism.

It should be noted that whenever the kinematic-links are computed by the empirical equation then the remainder of the computation of the displacement analysis should be carried out using double-precision calculations or else the displacement parameters may not be accurate in the third and fourth decimal places. The column matrix [V] then, on an average, takes the form that resembles nearly ideal conditions. For instance, the column matrix [V] for case 6 takes the following form when θ_1 = 100 at the final stage of iteration:

$$\underset{\text{Case 6}}{[V]_{\theta_1 = 100^\circ}} = \begin{bmatrix} 0.000000 \\ -0.000005 \\ 0.000193 \\ 0.000000 \\ 0.000162 \\ 0.000000 \\ 0.000000 \\ -0.000054 \\ 0.000026 \\ -0.000000 \\ 0.000026 \\ -0.000000 \end{bmatrix}$$

Under the complete ideal condition it must become a column null vector. The difference is due to lack of precision in the computation.

CHAPTER V

THE SCOPE OF ONE GENERAL CONSTRAINT

In the previous chapter the nature of one general constraint was examined. This study of one general constraint was centered around the very basic issues that define the mobility of a six-link chain. This study disclosed the relationships between the kinematic parameters of the six-link six-revolute mechanism. The six-link mechanism, however, represents just one of the many other undiscovered mechanisms having one general constraint. According to the mobility criteria one general constraint, there is a possibility of the existence of nearly two hundred mechanisms having a wide variety of number of kinematic-links, kink-links, twist angles, and kinematic pairs having one or more number of degrees of freedom. Table VI shows the different types and kinds of chains which are likely to generate mechanisms if proper conditions of their existence are known.

One possible interpretation of the problem of determining the other types of mechanisms, such as RRRRRP, RRRRRH, RRRRC, etc., is to plan a study similar to the one conducted in the last chapter for the six-link mechanism. Fortunately, however, there does exist an alternate approach by which the existence of the other types of mechanisms can be formulated. This alternate approach involves relating the revolute pairs with the other kinematic pairs, such as the prism pair, the helical pair, the cylinder pair, et. al.

Substitution of the Prism Pair

In Table I there are three class I pairs each having one degree of freedom. These are the revolute pairs, the prism pair, and the helical pair. Each of these pairs are described symbolically as

$$\hat{\theta} = \theta_o + \sigma s_o \tag{5.1}$$

where $\hat{\theta}$ represents the dual rotation. Observe that the dual rotation $\hat{\theta}$ has two parameters θ_o and s_o. The revolute pair is described by the above dual notation when the parameter s_o is assumed to be a constant.

The prism pair is also described using this dual notation when the parameter θ_o is assumed to be a constant. In each of these cases, the axis of the rotation and the axis of translation are the same.

Differentiating both the sides of Equation (5.1) with respect to time t, we get

$$\dot{\hat{\theta}} = \dot{\theta}_o + \sigma \dot{s}_o \tag{5.2}$$

Let $\qquad \dot{\theta} = \omega \quad$ then $\quad \dot{s}_o = \bar{\omega} \times \bar{\gamma}$

then

$$\hat{\omega} = \omega_o + \sigma(\bar{\omega}_o \times \bar{\gamma}) \tag{5.3}$$

Observe that Equation (5.3) provides a physical interpretation to Equation (5.1). The real part of this equation represents a rotation and the dual part represents the translation. Furthermore, the dual part of Equation (5.3) indicates that the axis of rotation must be normal to the plane of translation.

This physical interpretation of Equation (5.1) suggests a possible orientation of the axis of the prism pair to be substituted for a given revolute pair of a kinematic chain.

Thus, according to the interpretation of Equation (5.3), the axis of the substituting prism pair must be normal to the axis of the revolute pair. For instance, consider the plane four-link mechanism in which there are four revolute pairs. One of these pairs can be substituted by a prism pair whose axis of translation must be normal to the axis of the revolute pair. Such a substitution of a revolute pair by a prism pair yields a plane slider-crank mechanism.

In the case of the 7R spatial mechanism, theoretically, there is a possibility of replacing all the seven revolute pairs by seven prism pairs. However, such a kinematic chain of seven prism pairs cannot be expected to have a general motion consisting of three rotations and three translations. Therefore, it becomes necessary to determine the maximum possible number of prism pairs permissible in a kinematic chain having a general motion consisting of three rotations and three translations. For this purpose, consider Equation (3.37) in which

$$\sum_{i=2}^{7} [A_i + \sigma B_i]\, d\theta_i \cong [I] - [A_1 + \sigma B_1] \tag{3.37}$$

Consider, for instance, that the seventh revolute pair is to be substituted by a prism pair. Then the above equation may be rewritten as

$$\sum_{i=2}^{6} [A_i + \sigma B_i]\, d\theta_i + [A_7 + \sigma B_7]\, d\hat{\theta}_7 = [I] - [A_1 + B_1] \tag{5.4}$$

where

$$\hat{\theta}_7 = \theta_7 + \sigma\, s_7 \tag{5.5}$$

Since s_7 is the only variable in a prism pair, then

$$d\hat{\theta}_7 = \sigma \; d \; S_7 \tag{5.6}$$

Substituting Equation (5.7), Equation (5.4) becomes

$$\sum_{i=2}^{6} [A_i + \sigma B_i] \, d\theta_i + [A_7 + \sigma B_7][\sigma \; d \; s_7] = [I] - [A_1 + \sigma B_1] \tag{5.7}$$

Noting that $\sigma^2 = 0$, the above equation simplifies to the following:

$$\sum_{i=2}^{6} [A_i] \, d\theta_i + \sigma \sum_{i=2}^{6} [B_i] \, d\theta_i + [A_7] \; d \; s_7 \quad = [I] - [A_1 + \sigma B_1] \tag{5.8}$$

Consequently, the coefficient matrix [M] takes the following form:

$$[M]_{R_6P} = \begin{bmatrix} a_{212} & a_{312} & a_{412} & a_{512} & a_{612} & 0 \\ a_{213} & a_{313} & a_{413} & a_{513} & a_{613} & 0 \\ a_{223} & a_{323} & a_{423} & a_{523} & a_{623} & 0 \\ \hline b_{212} & b_{312} & b_{412} & b_{512} & b_{612} & a_{712} \\ b_{213} & b_{313} & b_{413} & b_{513} & b_{613} & a_{713} \\ b_{223} & b_{323} & b_{423} & b_{523} & b_{623} & a_{723} \end{bmatrix} \tag{5.9}$$

Observe that the last column consists of three elements having zero values. These three elements are in turn the last elements of the three real-part row-vectors. Furthermore, the last elements of the three dual-part row-vectors are the same as those of the last elements of the three real-part row-vectors representing the 7R mechanism. Thus, in case of the mechanism RRRRRRP, the first three elements of the last column of the coefficient matrix [M] representing the 7R mechanism are displaced by three rows.

If a mechanism represented by a combination RRRRRPP were to be described by the coefficient matrix [M], then it takes the following form

$$[M]_{R_5P_2} = \begin{bmatrix} a_{212} & a_{312} & a_{412} & a_{512} & 0 & 0 \\ a_{213} & a_{313} & a_{413} & a_{513} & 0 & 0 \\ a_{223} & a_{323} & a_{423} & a_{523} & 0 & 0 \\ b_{212} & b_{312} & b_{412} & b_{512} & a_{612} & a_{712} \\ b_{213} & b_{313} & b_{413} & b_{513} & a_{613} & a_{713} \\ b_{223} & b_{323} & b_{423} & b_{523} & a_{623} & a_{723} \end{bmatrix} \tag{5.10}$$

Observe again that the existence of a prism pair reduces the length of the real-part row-vectors. In the case of 6R + 1P mechanisms, the real-part row-vector consists of five elements; and in the present case where the mechanism has two prism pairs, the real-part row-vectors each have four non-zero elements.

In a space mechanism with zero general constraints, the general motion consists of three rotations and three translations. It has been shown in Chapter III that the three real-part row-vectors of the coefficient matrix [M] represent the three rotations and the three dual-part row-vectors of the coefficient matrix [M] represent the three translations. Thus, the coefficient matrix [M] divides itself into two sub-matrices, each having three rows and six columns. Since there are three independent rotations and translations, the rank of each of these sub-matrices must be three. Equations (5.9) and (5.10) show that the existence of the prism pair in a mechanism reduces the size of the real-part sub-matrix of [M]. With one prism pair, this real part sub-matrix has three rows and five columns; with two prism pairs, the sub-matrix has three rows and four columns. Since the rank of this sub-matrix is three for a zero family mechanism, the sub-matrix must have a minimum of three rows and three columns. That is, the coefficient matrix [M] may take the following form in the limit conditions.

$$[M]_{limit} = \begin{bmatrix} a_{212} & a_{312} & a_{412} & 0 & 0 & 0 \\ a_{213} & a_{313} & a_{413} & 0 & 0 & 0 \\ a_{223} & a_{323} & a_{423} & 0 & 0 & 0 \\ b_{212} & b_{312} & b_{412} & a_{512} & a_{612} & a_{712} \\ b_{213} & b_{313} & b_{413} & a_{513} & a_{613} & a_{713} \\ b_{223} & b_{323} & b_{423} & a_{523} & a_{623} & a_{723} \end{bmatrix} \quad (5.11)$$

The limiting case described by Equation (5.11) corresponds to a mechanism having four turning pairs and three sliding pairs, i.e., the mechanism having the combination RRRRPPP.

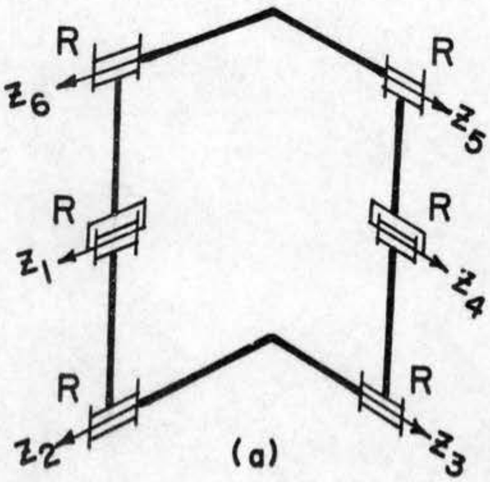
Since in a mechanism the input displacement is independent of the dependent displacements and the coefficient matrix [M] is independent of the input displacement, a prism pair can be employed to give the displacement to the other dependent links. Therefore, a maximum of a four prism pair can be employed in a 7R mechanism to substitute four turning pairs, provided one of the prism pair is employed for the input displacement. Such mechanisms may be described by combinations PPRRPPP, PRPRPRP, etc.

Note, however, that if a prism pair is not employed as the input pair and if the turning pair is the input pair, then the maximum number of prism pairs that can be employed to substitute the turning pairs in the 7R mechanism must be three. The coefficient matrix [M] will become singular for the case RRRPPPP.

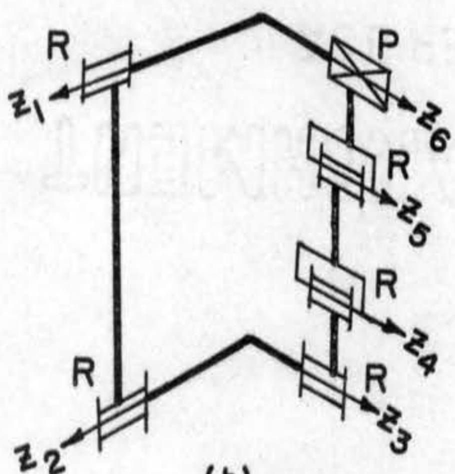
It must be noted that in the above derivation of Equation (5.11) the orientation of the axes of the three turning pairs must be such that the real part sub-matrix of three rows and three columns must be nonsingular. Furthermore, the existence of the prism pairs must not produce two or more identical columns in the dual part sub-matrix of the coefficient matrix [M]. If such a case exists, then the coefficient matrix [M] will become singular.

Fortunately, however, the problems associated with the orientation of the axes of the kinematic pairs in a mechanism with zero general constraints are not as complex as they are for the mechanisms having one or more general constraints. Consequently, the problem of substituting the prism pairs for the revolute pairs needs a careful consideration. For instance, consider the Sarrus' six-link six-revolute mechanism shown in Figure 12a. Note that in the Sarrus' six-link mechanism the axes of the turning pairs 6, 1, and 2 are parallel and that the axes of the turning pairs 3, 4, and 5 are parallel. If it is desired to substitute the turning pair at the joint 6 by a prism pair, then the prism pair at this joint must be in a plane normal to the axis of the turning pair at the joint 6. This resulting mechanism RRRRRP is shown in Figure 12b. The displacement analysis of this mechanism is shown in Figure 13. Observe that the mechanism is a rocker-rocker type. That is, the input crank does not make a total rotation of 360°.

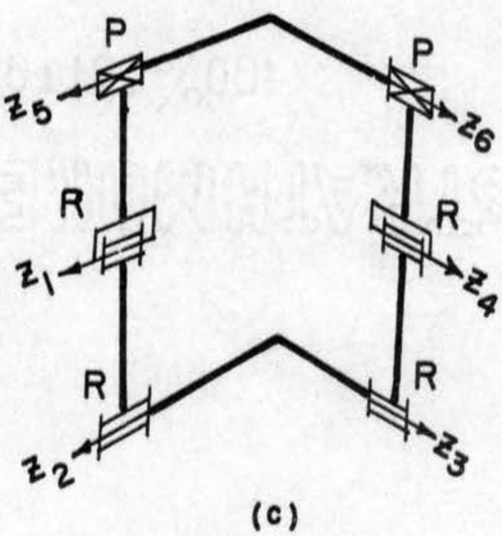
The Sarrus' mechanism is also capable of having a second prism pair. In Figure 12b the prism pair at the joint 6 is substituted so that its axis lies parallel to the axes of the turning pairs 4 and 5. Similarly, the revolute pair at the joint 5 of the Sarrus' mechanism



Sarrus 6R Mechanism

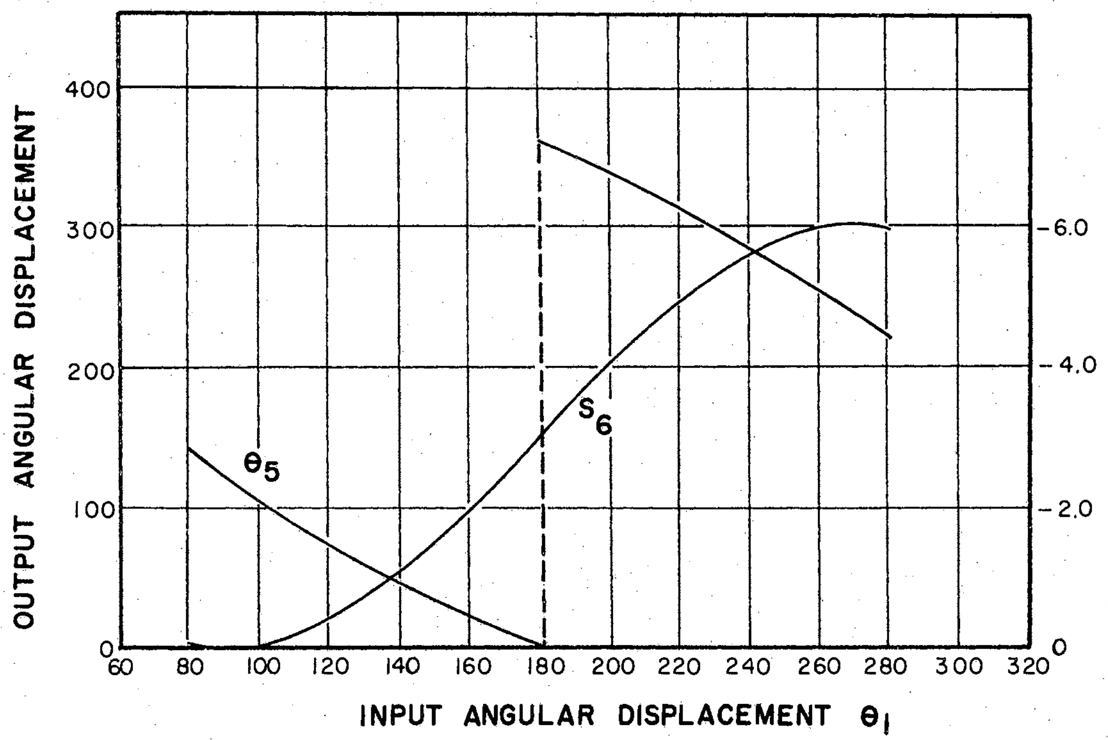


RRRRRP Mechanism



PRRRRP Mechanism

Figure 12. Substitution of a Prism Pair in the Sarrus' Mechanism.



The kinematic parameters of the mechanism are:

α: 0, -90°, 0, 0, 0, -90°

a: 3", 0", 1", 1", 1", 0"

s: 2", -2", 3", 0", 0", s_6

Figure 13. Displacement Analysis of RRRRRP Mechanism

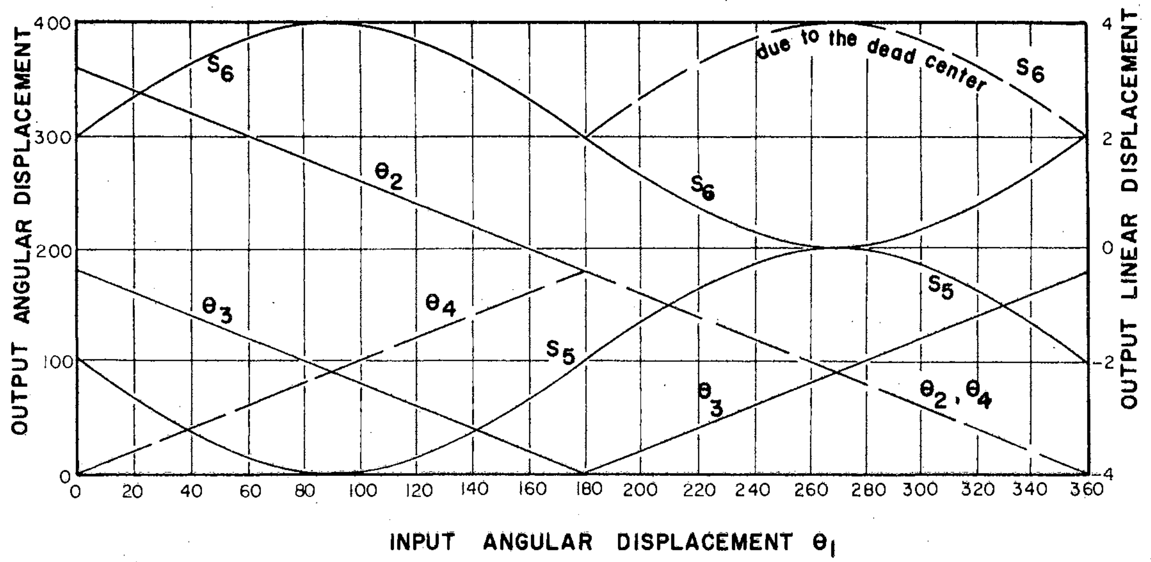
can be substituted by a prism pair in such a manner that its axis of translation lies parallel to the axes of the turning pairs at the joints 1 and 2. The resulting Sarrus' mechanism with two prism pairs is shown in Figure 12c. The displacement analysis of this mechanism is shown in Figure 14. The mechanism is a space mechanism having two slider pairs. Figure 14 shows that the mechanism has dead-centers at $\theta_1 = 0$ and $\theta_1 = 180^\circ$.

It has been noted earlier that the Sarrus' mechanism has a general motion of two rotations and three translations. Therefore, one is led to believe that a maximum of three prism pairs can be substituted for three turning pairs. This assumption would have been true if the mechanism under consideration were to belong to a family having no constraints. However, the solution of this problem becomes relatively simple if we examine the coefficient matrix [M] of the Sarrus' mechanism.

The coefficient matrix [M] for the Sarrus' mechanism having six turning pairs takes the following form for $\theta_1 = 170^\circ$.

$$[M]_{\substack{\theta_1 = 170^\circ \\ 6R}} = \begin{bmatrix} -1.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0 & 0 & -1.0 & -1.0 & -1.0 & 0.0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5.90884 & -2.9544 & 0 & 0.0 \\ 2.9544 & 5.9088 & 0.0 & 0.0 & 0 & 0.0 \\ -0.5209 & 0.0 & 0.0 & 0.5209 & 0 & 0 \end{bmatrix}$$

When one of the revolute pairs is substituted for a prism pair as in the case of Figure 12b, the coefficient matrix [M] for this mechanism RRRRRP takes the following form for $\theta_1 = 170^\circ$.



The kinematic parameters of the mechanisms are:

α: 0, -90°, 0, 0, -90°, 0

a: 2", 0", 2", 2", 0", 2"

s: 0", 2", 2", 0", s_5, s_6

Figure 14. Displacement Analysis of RRRRPP Mechanism Note that the dashed curve s_6 is due to the dead center at $\theta_1 = 180°$.

$$[M]_{\substack{\theta_1 = 170^\circ \\ R_5P}} = \left[\begin{array}{cccccc} -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0 & -1.0 & -1.0 & -1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -2.9544 & -1.9772 & -1.0 & 0 & 0 \\ 2.9544 & 0.0 & 0.0 & 0.0 & -1.0 & 0 \\ -0.52094 & 0.0 & -0.2123 & 0.0 & 0.0 & 0 \end{array}\right]$$

Observe that the sixth column of both the matrices is a null vector. This is due to the fact that the mechanism has a total of six kinematic pairs. However, the fifth column of both the matrices are different. This is because the second case pertains to the mechanism having a prism pair. Note that the first three elements of the fifth column of coefficient matrix [M] representing the 6R Sarrus mechanism appear to be displaced downward by three rows in the coefficient matrix [M] representing the RRRRRP mechanism. Observe that the substitution of the prism pair for revolute pairs does not alter the general motion of the mechanism. The coefficient matrix of the mechanism RRRRRP has the same number of nonvanishing real and dual row-vectors as those for RRRRRR mechanism. That is, the mechanism has two rotations and three translations.

For the mechanism RRRRPP shown in Figure 12c, the coefficient matrix takes the following form.

$$[M]_{\substack{\theta_1 = 10^\circ \\ R_4P_2}} = \left[\begin{array}{cccccc} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -1.9696 & 0 & 0 & -1 & 0 \\ -1.96961 & 0 & 0 & 1 & 0 & 0 \\ -0.347296 & -2.0 & -2.3473 & 0 & 0 & 0 \end{array}\right]$$

Here again the effect of introducing the prism pair results in a displacement of the first three elements of the columns 4 and 5 by three rows. Note that the existence of the two prism pairs does not alter the general motion of the mechanism; that is, the mechanism has a general motion of two rotations and three translations.

If a third prism pair is to be substituted for a turning pair, it can be introduced either at the joint 3 or 4 of Figure 12c. It can be seen from the above coefficient matrix that such an introduction of a prism pair is expected to retain the two rotation components of the general motion. However, since the introduction of the prism pair displaces the elements of the corresponding column by three rows, the resulting coefficient matrix $[M]_{R_3P_3}$ will have two identical columns and, therefore, will become singular. Thus, a maximum of only two prism pairs can be introduced in the Sarrus' mechanism.

Substitution of the Helical Pair

The problem of substituting the helical pair for a revolute pair brings us back to consider Equation (5.1) which is

$$\hat{\theta} = \theta_o + q\, s_o \tag{5.1}$$

The helical pair is capable of executing a rotation and a translation about the same axis. However, the rotation and the translation are related. This relationship is given by

$$\frac{d\theta_o}{ds_o} = \text{constant} \neq 0 \tag{5.12}$$

Observe that the helical pair has the ~~essential~~ feature of the turning pair as well as those of the prism pair. That is, whenever a helical pair is employed to substitute for a turning pair, the axis of the helical pair must lie in a plane where the mechanism executes a rotation and a translation simultaneously. For instance, consider the plane four-link mechanism examined in Chapter III. It has been noted there that the plane four-link mechanism has one rotation and two translations. Furthermore, the axis of rotation is normal to the plane of translation. Since the mechanism does not have the axis of rotation lying in the plane of translation, the turning pairs cannot be substituted by a helical pair.

In case of a zero family mechanism having three rotations and three translations for its general motions, the requirements for the substitution of helical pairs are met more readily. However, since the substituted helical pair allows translatory motion in addition to the rotary motion of the revolute pair, it must satisfy the requirements specified for the prism pairs. Furthermore, since only one of the variables can be kept independent and if translation is kept independent, then the coefficient matrix [M] takes the same form as that shown in Equation (5.9) for the R_6H mechanism. Thus, the coefficient matrix will become

$$[M]_{R_6H} = \begin{bmatrix} a_{212} & a_{312} & a_{412} & a_{512} & a_{612} & 0 \\ a_{213} & a_{313} & a_{413} & a_{513} & a_{613} & 0 \\ a_{223} & a_{323} & a_{423} & a_{523} & a_{623} & 0 \\ b_{212} & b_{312} & b_{412} & b_{512} & b_{612} & a_{712} \\ b_{213} & b_{313} & b_{413} & b_{513} & b_{613} & a_{713} \\ b_{223} & b_{323} & b_{423} & b_{523} & b_{623} & a_{723} \end{bmatrix} \qquad (5.13)$$

In view of this development regarding the nature of the helical pair, it can be concluded that the maximum number of permissible helical pairs in a zero family mechanism is three.

Because of the specific orientation of the axes of the turning pairs in the six-link mechanism, the problems involved are as complex as those involved in substituting the prism pair. The Sarrus' mechanism again presents a good example to illustrate the procedure of substituting a helical pair in the six-link mechanism. In Figure 15a the Sarrus' six-link mechanism with the six turning pairs is shown. The helical pair is substituted at the joint 6 of the 6R mechanism. This substitution of the helical pair requires that the axis of helical pair be parallel to the axes of the turning pair at the joints 3, 4, or 5.

The displacement analysis of the mechanism R_5H is shown in Figure 16. Note that the relationship between θ_6, the output rotation of the helical pairs, and θ_1, the input rotation, must be similar to that between s_6, the output translation of the helical pair, and θ_1, the input rotation. This apparent similarity stems from the fact that the rotation and translation produced by the helical pair must satisfy the relationship

$$\frac{d\theta_6}{ds_6} = A = \text{constant}$$

i.e.,

$$\theta_6 = A \; s_6$$

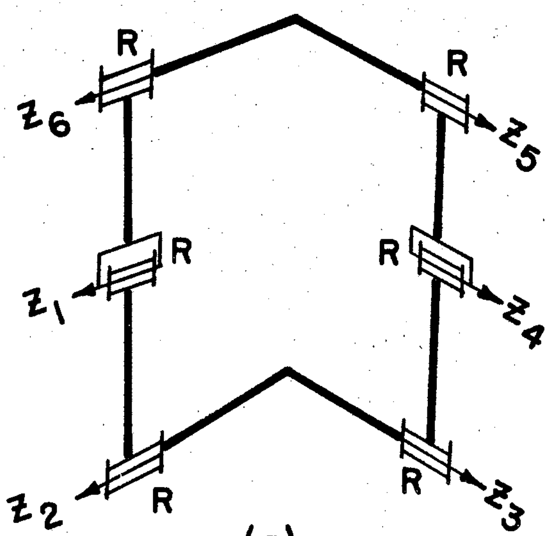
or

$$\theta_6 = A[B \; f(\theta_1)]$$

or

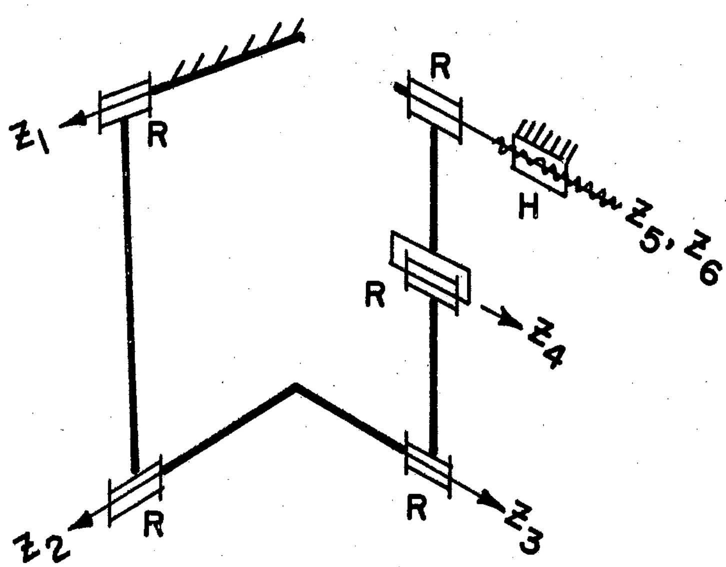
$$\theta_6 = K \; f(\theta_1)$$

where $K = AB$.



(a)

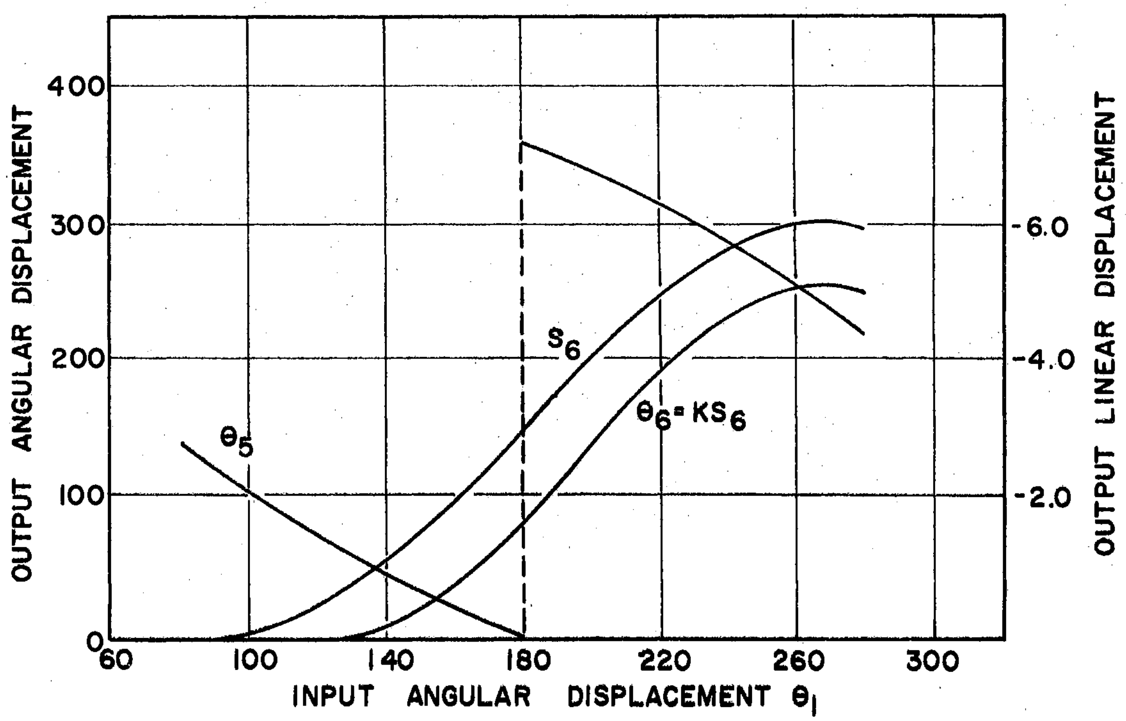
Sarrus' Six-link Mechanism



(b)

RRRRRH Mechanism

Figure 15. Substitution of a Helical Pair in the Sarrus' Mechanism.



The kinematic parameters of the mechanism are:

α: 0, -90°, 0, 0, 0, -90°

a: 3", 0", 1", 1", 1", 0"

s: -2", -2", 3", 0", 0", s_6*

*Note that for the computation of the output at the helical pair, one of two parameters θ_6 and s_6 can be computed. In the above displacement analysis, s_6 is computed for every θ_1, and θ_6 may be computed using

$$\theta_6 = K\, s_6$$

where K is related to the pitch of the helical pair. Since θ_6 differs from s_6 by a constant, its relationship with θ_1 is simlar to that of s_6.

Figure 16. Displacement Analysis of the RRRRRH Mechanism

Substitution of the Torus Pair

The function of the torus pair in a kinematic chain is to provide two rotations in a skew plane. In the torus pair, however, this is achieved by placing two revolute pairs at an angle of 90° and separating the two pairs by a common normal which in turn is the kinematic link.

Franke's six-link mechanism is best suited to illustrate the use of the torus pair since all the skew angles of this mechanism are 90° and all the six turning pairs are separated by six kinematic links.

The limiting case of the torus pair is the case where the kinematic link between the two revolute pairs goes to zero. In this case the torus pair degenerates into the slotted-sphere pair. The existence of a slotted-sphere type of pair permits two rotations about the two independent intersecting axes. In Figure 17 is shown the mechanism which is degenerated from the Franke kink-link six-link mechanism. Observe that two of the opposite links are zero. Furthermore, two of the opposite kink-links are made zero. The kinematic pair of slotted sphere can be introduced at the joints 2, 3, and 5, 6. The displacement analysis of this mechanism is shown in Figure 18.

Substitution of the Cylinder Pair

The function of the cylinder pair is to produce two degrees of motion consisting of a rotation and a translation along the same axis. The rotation of the cylinder pair is independent of its translation. This function of the cylinder pair can be described by Equation (5.1) which is

$$\hat{\theta} = \theta_o + \sigma s_o \tag{5.1}$$

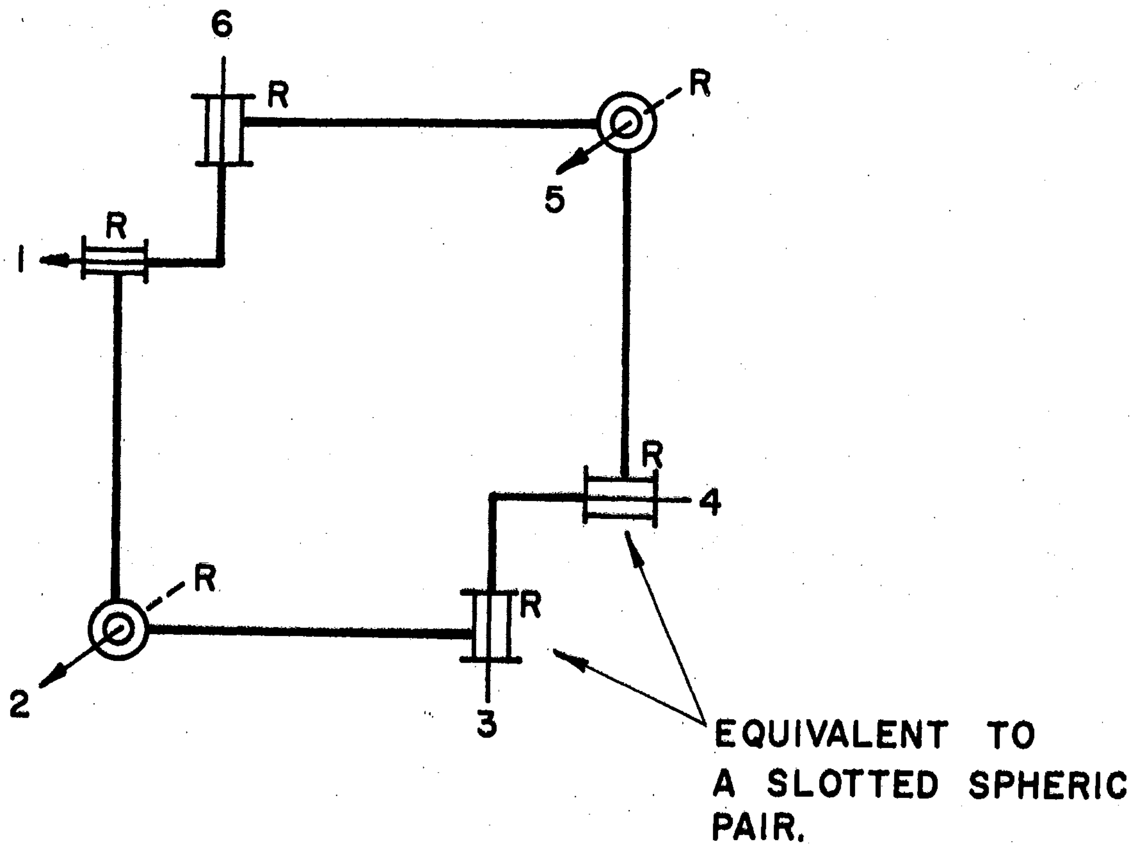
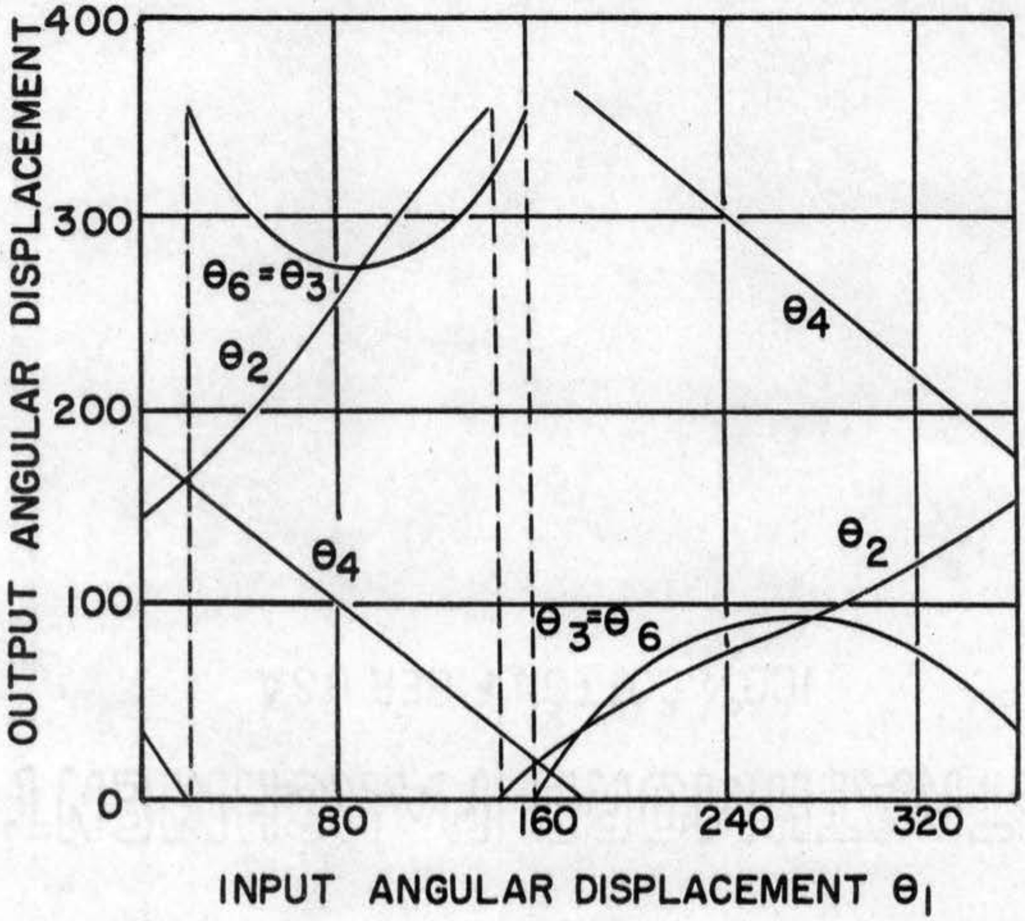


Figure 17. Degenerate Franke's Six-Link Mechanism that is Equivalent to RS_LRRR Mechanism.



The kinematic parameters of the mechanism are:

α: -90°, -90°, -90°, 90°, 90°, 90°

a: 0, 5", 5", 0, 5", 5"

s: 1", 2", 0, -1", -2", 0

Figure 18. Displacement Analysis of RS_LRRR Mechanism

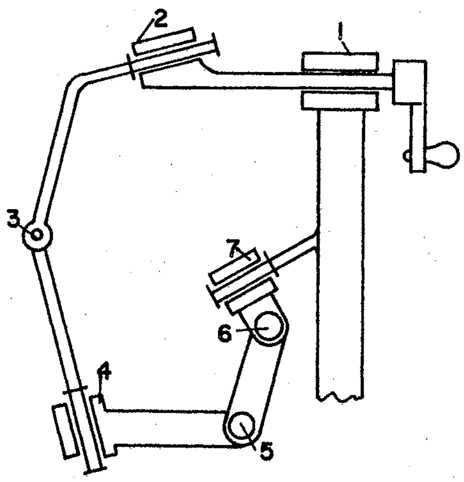
The substitution of a cylinder pair in a kinematic chain will require two turning pairs. The first turning pair may be retained in its original position to produce the rotation of the cylinder pair. Then the second turning pair must substitute for the translatory motion of the cylinder pair. That is, a prism pair must be substituted for the second revolute pair in such a manner that the axis of translation also becomes the axis of rotation of the first turning pair.

It has been shown earlier that the prism pair can be substituted for a revolute pair in a kinematic chain provided the axis of the prism pair is normal to the axis of the revolute pair. Since the cylinder pair requires the axis of the rotation and the axis of the translation to be the same, then either of the axes of the two revolute pairs which are to be replaced by a cylinder pair must intersect at right angles or must be along the two 90° skew lines.

Thus, the requirements of replacing two revolute pairs by the cylinder pair are the same as those for the torus pair even though the kinematic behavior of these pairs are different. The torus pair is required to execute two rotations and the cylinder pair is required to execute a rotation and a translation. Therefore, the coefficient matrix $[M]$ for a mechanism having a cylinder pair is different from that of a mechanism having a torus pair.

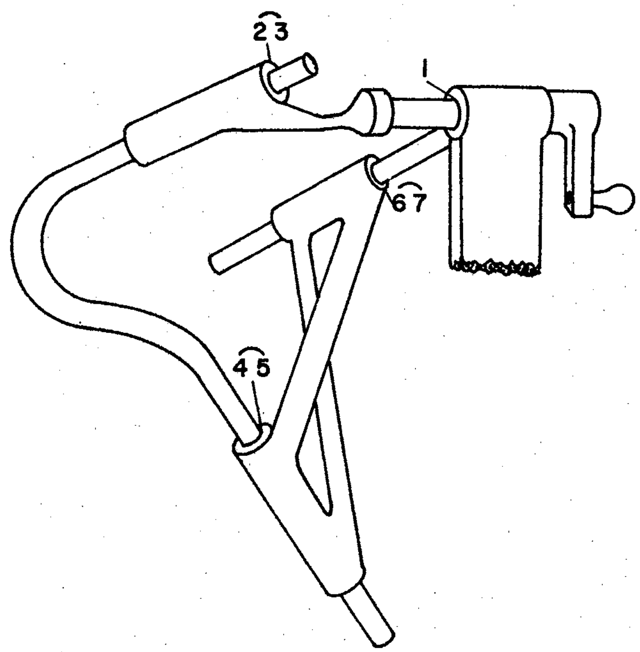
This concept of substituting a cylinder pair for two turning pairs whose axes are skew by 90° was somewhat vaguely mentioned by Franke, who suggested the two equivalent mechanisms shown in Figure 19.

The characteristic behavior of the coefficient matrix $[M]$ can be studied by considering Equation (3.37) which is



(a)

7R Mechanism



(b)

RCCC Mechanism

Figure 19. Franke's Equivalent Mechanisms.

$$\sum_{i=2}^{7} [A_i + \sigma B_i]\, d\hat{\theta}_i \cong [I] - [A_1 + \sigma B_1] \tag{3.37}$$

The above equation pertains to a mechanism having seven turning pairs. If two of the turning pairs are replaced by a cylinder pair, then the total number of kinematic pairs are six instead of seven. Therefore, the above equation can be rewritten as

$$\sum_{i=2}^{6} [A_i + \sigma B_i]\, d\hat{\theta}_i \cong [I] - [A_1 + \sigma B_1] \tag{5.14}$$

If the above equation describes a mechanism RRRRRC where the output is a rotation and translation, then

$$d\hat{\theta}_2 = d\theta_2 + \sigma(0) \tag{5.15}$$

$$d\hat{\theta}_3 = d\theta_3 + \sigma(0) \tag{5.16}$$

$$d\hat{\theta}_4 = d\theta_4 + \sigma(0) \tag{5.17}$$

$$d\hat{\theta}_5 = d\theta_5 + \sigma(0) \tag{5.18}$$

and

$$d\hat{\theta}_6 = d\theta_6 + \sigma\, d\, s_6 \tag{5.19}$$

Equation (5.19) is different from the others because it describes the differential displacement of the cylinder pair of the mechanism RRRRRC. Using the above relationships and noting that $\sigma^2 = 0$, Equation (5.14) is simplified to the following:

$$\sum_{i=2}^{6} [A_i]d\theta_i + \sigma[A_6]ds_6 + \sigma \sum_{i=2}^{6} [B_i]d\theta_i \cong [I] - [A_1 + \sigma B_1] \tag{5.20}$$

The coefficient matrix [M] for the RRRRRC mechanism then takes the following form:

$$[M]_{R_5C} = \begin{bmatrix} a_{212} & a_{312} & a_{412} & a_{512} & a_{612} & 0 \\ a_{213} & a_{313} & a_{413} & a_{513} & a_{613} & 0 \\ a_{223} & a_{323} & a_{423} & a_{523} & a_{623} & 0 \\ b_{212} & b_{312} & b_{412} & b_{512} & b_{612} & a_{612} \\ b_{213} & b_{313} & b_{413} & b_{513} & b_{613} & a_{613} \\ b_{223} & b_{323} & b_{423} & b_{523} & b_{623} & a_{623} \end{bmatrix} \qquad (5.21)$$

Equation (5.21) describes the RRRRRC mechanism and appears to be similar to Equation (5.9) which describes RRRRRRP mechanism. The difference in these two equations is due to the fact that in a cylinder pair the rotation and the translation are along the same axis. Thus, the last three elements of the sixth column are the same as the first three elements of the fifth column.

If, however, a space mechanism has two cylinder pairs, for example the RRRCC mechanism, then the coefficient matrix [M] takes the following form:

$$[M]_{R_3C_2} = \begin{bmatrix} a_{212} & a_{312} & a_{412} & a_{512} & 0 & 0 \\ a_{213} & a_{313} & a_{413} & a_{513} & 0 & 0 \\ a_{223} & a_{323} & a_{423} & a_{523} & 0 & 0 \\ b_{212} & b_{312} & b_{412} & b_{512} & a_{412} & a_{512} \\ b_{213} & b_{313} & b_{412} & b_{513} & a_{413} & a_{513} \\ b_{223} & b_{323} & b_{423} & b_{523} & a_{423} & a_{523} \end{bmatrix} \qquad (5.22)$$

Here again the two equations, (5.22) and (5.10), appear to be similar in form. The difference is that the cylinder pair has a translation along the axis of rotation.

Finally, the coefficient matrix [M] for a space mechanism without general constraints, the RCCC, takes the following form:

$$[M]_{RCCC} = \left[\begin{array}{cccccc} a_{212} & a_{312} & a_{412} & 0 & 0 & 0 \\ a_{213} & a_{313} & a_{413} & 0 & 0 & 0 \\ a_{223} & a_{323} & a_{423} & 0 & 0 & 0 \\ \hline b_{212} & b_{312} & b_{412} & a_{212} & a_{312} & a_{412} \\ b_{213} & b_{313} & b_{413} & a_{213} & a_{313} & a_{413} \\ b_{223} & b_{323} & b_{423} & a_{223} & a_{323} & a_{423} \end{array}\right] \qquad (5.23)$$

The above coefficient matrix is the limiting conditions for the maximum number of cylinder pairs that can exist in a space mechanism with no general constraints. Further modification of this matrix yields a singularity condition.

The displacement analysis of the RCCC mechanism has been performed in many different ways using the different analytical techniques. However, Uicker, Denavit and Hartenberg [50] were among the first ones to carry out numerical analysis of a particular RCCC mechanism shown in Figure 20. These results were confirmed by A. T. Yang [51], who applied the dual quaternions for obtaining the explicit displacement relationships.

The method developed in the present work is applied to this particular RCCC mechanism. The results of the displacement analysis are tabulated in Table XIII. Note that these results confirm the investigation carried out both by Uicker and Yang.

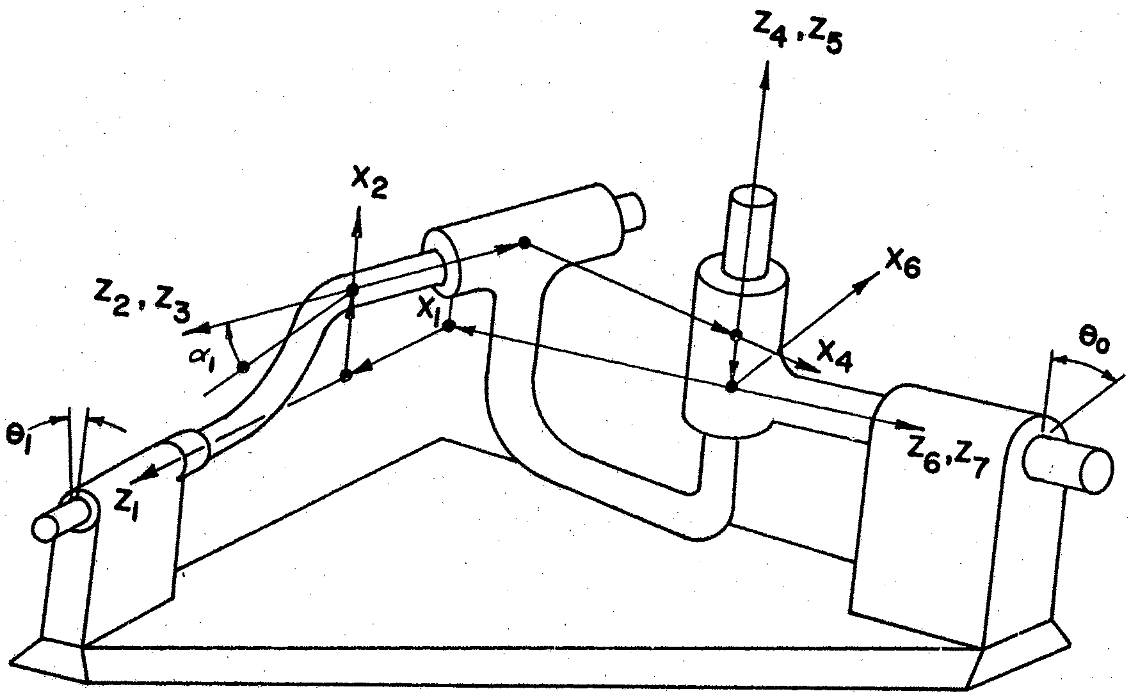


Figure 20. Yang's and Uicker's RCCC Mechanism.

TABLE XIII

DISPLACEMENT ANALYSIS OF YANG'S AND

UICKER'S RCCC MECHANISM

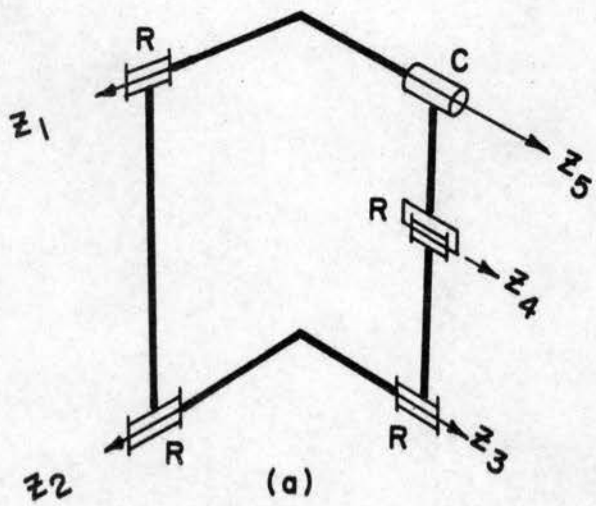
θ_1	θ_4	s_4
0	144.209377	-0.115081
20	131.899738	-0.920543
40	116.674592	-1.770566
60	101.194976	-2.248310
80	87.219700	-2.259417
100	75.723766	-1.888758
120	67.559073	-1.262205
140	64.213796	-0.529173
160	68.596581	0.011077
180	83.700148	-0.173163
200	105.329823	-0.842910
220	124.052093	-1.085719
240	136.989077	-0.937881
260	145.467159	-0.663168
280	150.868462	-0.367654
300	153.853981	-0.084375
320	154.370251	0.150238
340	151.599628	0.220370
360	144.209385	-0.115081

The substitution of a cylinder pair for the two revolute pairs in a mechanism having one general constraint presents the same problem as the one for substituting a prism pair. The coefficient matrix [M] for each six-revolute mechanism must be examined before and after the substitution of a cylinder pair. The existence of a cylinder pair must not change the characteristic components of the general motion.

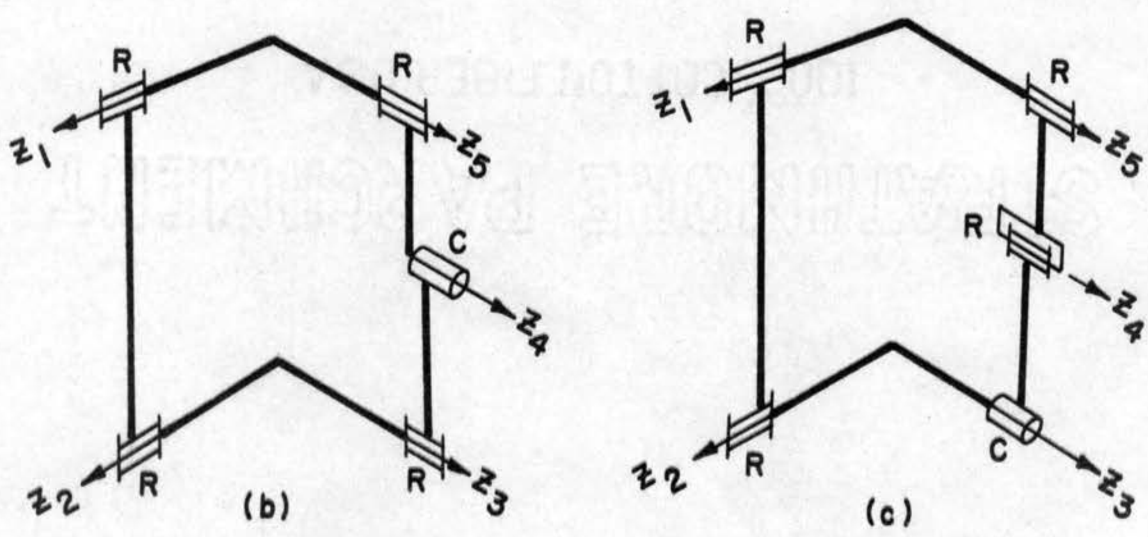
For example, consider the mechanism shown in Figure 12b. Here one prism pair is substituted for the turning pair at the joint 6 of the six-link mechanism of Figure 12a. The axis of the substituted prism pair is parallel to the axis of the revolute pairs at the joints 3, 4, and 5. Since the axes of rotation and translation are parallel, any of the turning pairs can be combined with the prism pair so that the resultant pair is a cylinder pair. Thus, from Figure 12b there is a possibility of obtaining three different mechanisms having one cylinder pair and four revolute pairs. These three mechanisms are shown in Figures 21a, 21b, and 21c and can be schematically described as RRRRC, RRRCR, and RRCRR mechanisms. Figure 22 shows the displacement analysis of the RRRRC mechanism. The coefficient matrix [M] for this mechanism takes the following form for

$$\theta_1 = 100^\circ$$

$$[M]_{\substack{R_4C \\ \theta_1 = 100^\circ}} = \left[\begin{array}{cccccc} -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0 & -1.0 & -1.0 & -1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ \hline 0.0 & -0.69459 & -3.4729 & 0.0 & 0.0 & 0.0 \\ 0.69459 & 0.0 & 0.0 & 0.0 & -1.0 & 0.0 \\ -3.93923 & 0.0 & 1.96961 & 0.0 & 0.0 & 0.0 \end{array}\right]$$



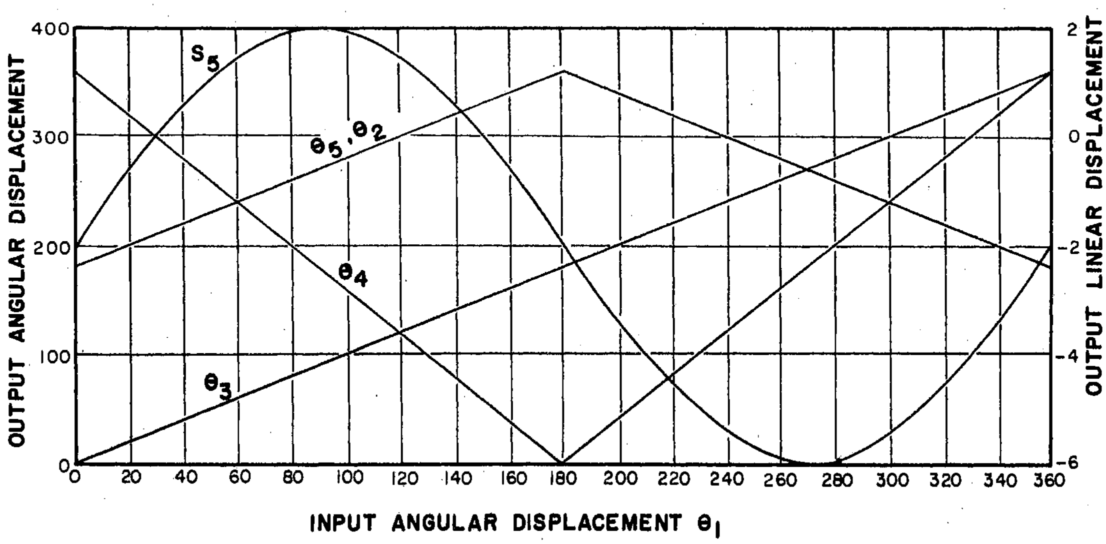
RRRRC Mechanism



RRRCR Mechanism

RRCRR Mechanism

Figure 21. Possible Types of One General Constraint Mechanisms with a Cylinder Pair.



The kinematic parameters of the mechanism are:

α: 0, -90°, 0, 0, -90°

a: 4", 0, 2", 2", 0

s: 3", -3", 2", 0, s_5

Figure 22. Displacement Analysis of the RRRRC Mechanism

Observe that in the above matrix there are five distinct independent equations. The analyses of this matrix shows that the RRRRC mechanism has two rotations and three translations as its general motion. This mechanism was recently reported by Harrisberger and Soni [52].

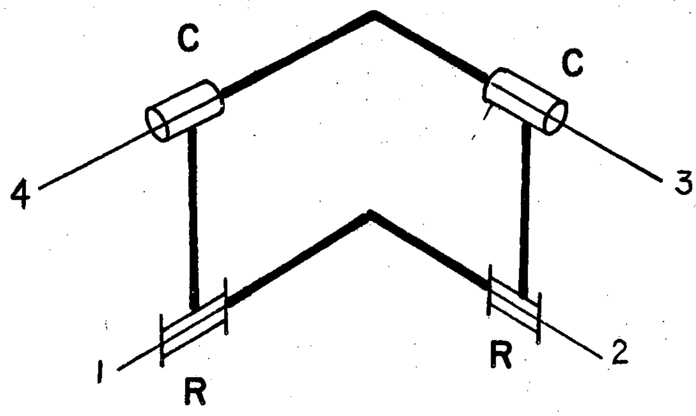
In the RRRRC mechanism, two of the revolute pairs at the joint 2 and 3 can be replaced to give the RCRC mechanism as shown in Figure 23a. In the case of RRRCR mechanism, Figure 21b, the two revolute pairs at the joint 2 and 3 can be replaced by a cylinder pair to yield the RCCR mechanism as shown in Figure 23b.

Figure 24 shows the displacement analysis of the mechanism RRCC. The coefficient matrix [M] for this mechanism takes the following form for

$$\theta_1 = 270^\circ$$

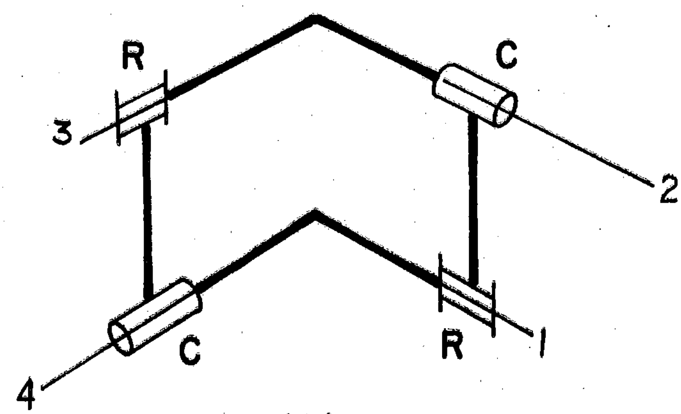
$$[M]_{\substack{R_2 C_2 \\ \theta_1 = 270^\circ}} = \left[\begin{array}{cccccc} 0 & 0 & -1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1.0 & -1.0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1.0 & 0 \\ -3.0 & 0 & 3.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.0 & 0 & 0 \end{array}\right]$$

The examination of the coefficient matrix shows that the substitution of two cylinder pairs does not change the components of the general motion. That is, the RRCC mechanism has two rotations and three translations for its general motion.



(a)

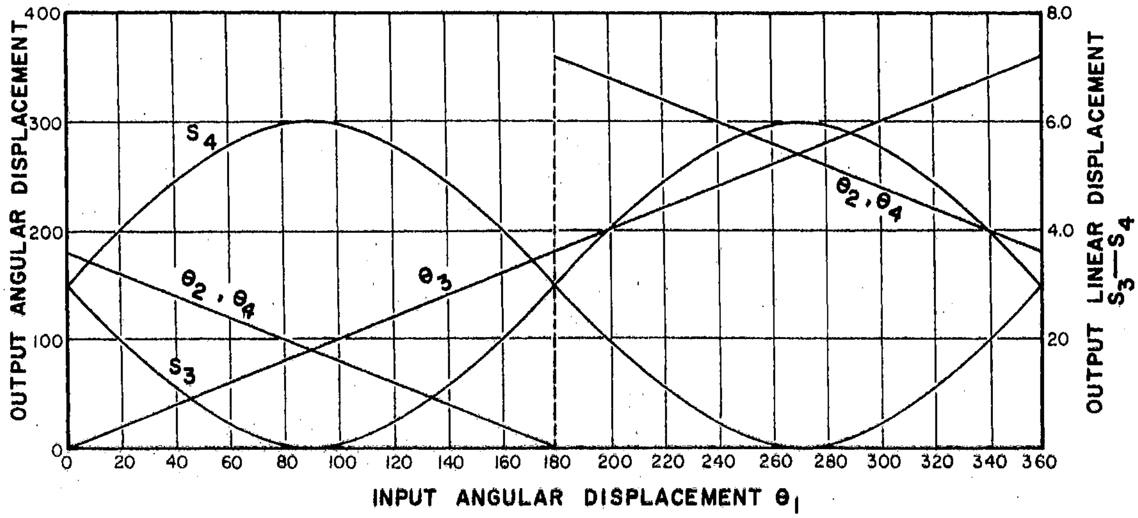
RRCC Mechanism



(b)

RCRC Mechanism

Figure 23. Possible Types of Mechanisms With Two Cylinder Pairs



The kinematic parameters of the mechanism are:

α: -90°, 0, -90°, 0

a: 0, 3", 0, 3"

s: -3, 3, s_3, s_4

Figure 24. Displacement Analysis of the RRCC Mechanism

Substitution of the Spheric Pair

The spheric pair belongs to the class three pairs and is capable of having three degrees of freedom defined by three independent rotations. Thus, the spheric pair can be represented by the dual vector as follows.

$$\hat{\theta}_{ijk} = (\theta_i, \theta_j, \theta_k) + \sigma(0) \tag{5.24}$$

where i, j, and k are the unit vectors associated with the three independent axes of rotation.

From the definition of the spheric pair, it is clear that a spheric pair can be substituted for three revolute pairs provided the three axes of these revolute pairs are not coplanar. Note that the existence of the spheric pair in a mechanism does not change the form of the coefficient matrix [M]. This is due to the fact that the existence of the spheric pair is a special case in which the three nonplanar axes of the revolute pair are intersecting in a finitely located point. Note that the criterion of intersection of these three axes forces the removal of two of the adjacent kinematic links. Thus, the coefficient matrix [M] for the 7R mechanism and the coefficient matrix [M] for the RRRRS mechanism basically differ by these two physical constant representing the two removed kinematic-links.

The substitution of the spheric pair in the zero-family mechanism does not present any problem. However, the family one mechanism must be examined carefully before a spheric pair is used to substitute the three revolute pairs. For instance, the Sarrus' mechanism is not capable of accepting a spheric pair because of not having the three revolute pair axes intersecting in a finitely located point. On the other hand, Franke's "wirbelkette" is a representative example to illustrate the substitution of the spheric pair for the three intersecting revolute pairs.

In Figure 25a the degenerate form of a general Franke's six-link mechanism is shown. The general six-link mechanism has six nonzero kinematic-links and six nonzero kink-links. The degenerate form shown in Figure 25a is obtained by removing four of the kinematic links and two of the kink-links. Observe that this mechanism has two joints 3 and 6 at which three axes of the revolute pairs intersect in two finitely located points. Thus, either the revolute pairs at the joint 1, 6, and 5 or at the joints 2, 3, and 5 can be replaced by a spheric pair. The displacement analysis of the mechanism shown in Figure 25b, obtained from Figure 25a, is shown in Figure 26.

Other Class Three Kinematic Pairs

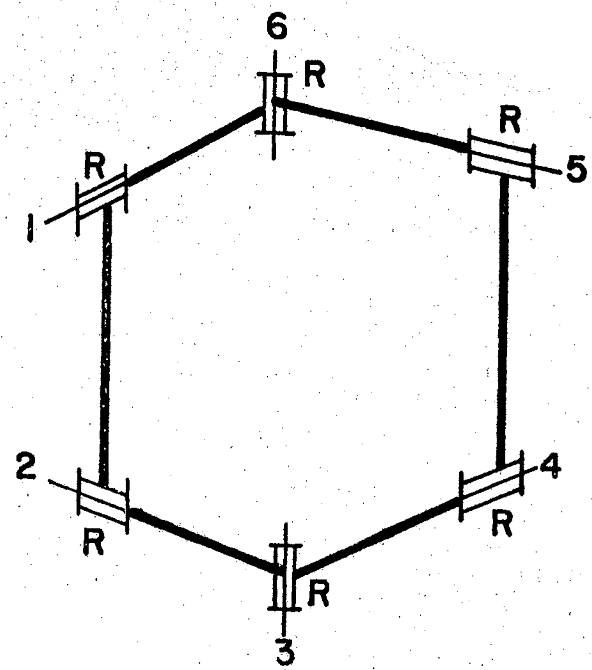
Besides the spheric pair, there are three other kinematic pairs in the class three pairs. These are the slotted sphere-cylinder pair, the slotted sphere-helix pair, and the plane pair.

The slotted sphere-cylinder kinematic pair has three degrees of freedom described by two rotations and one translation. Thus, this kinematic pair can be represented mathematically as

$$\hat{\theta}_{ij} = (\theta_i, \theta_j) + \sigma(s_i) \tag{5.25}$$

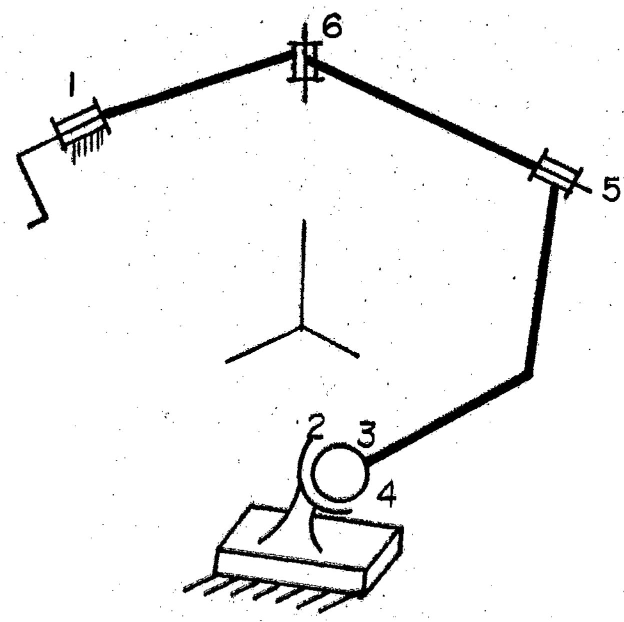
or

$$\hat{\theta}_{ij} = (\theta_i, \theta_j) + \sigma(s_j) \tag{5.26}$$



(a)

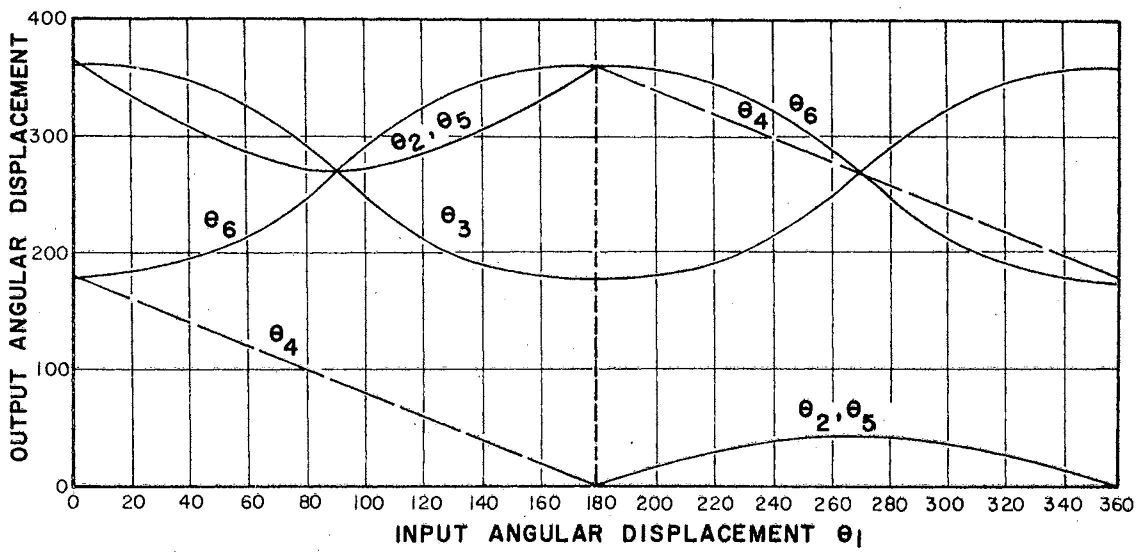
Degeneral Form of General Model of Six-Link Mechanism Capable of Accepting a Spherical Pair Either at the Joint 6 or at the Joint 3.



(b)

Equivalent RRRS Mechanism

Figure 25. RRRS Mechanism



The kinematic parameters of the mechanism are:

α: -90°, -90°, -90°, 90°, 90°, 90°

a: 4", 0, 0, 4", 0, 0

s: 3", 0, 3", -3", 0, -3"

Figure 26. Displacement Analysis of the RRRS Mechanism Shown in Figure 25a.

From the definition of the slotted-cylinder pair, it is clear that the $RRRRS_C$ mechanism, where S_C represents the slotted-cylinder pair, is mathematically equivalent to either a RRRRRC mechanism or RRRRRRP mechanism where the axis of the prism pair is parallel to the axis of the preceding revolute pair.

The sphere-helix pair has three degrees of freedom described by two rotational and one helical movements. This pair can be described mathematically as

$$\hat{\theta}_{ijk} = (\theta_i, \theta_j, \theta_k) + \sigma(s_k) \qquad (5.27)$$

where i, j, and k are the three unit ortogonal vectors and

$$\frac{d\theta_k}{ds_k} = \text{constant.} \qquad (5.28)$$

Here again the definition of the sphere-helix pair indicates that the RRRRRRH mechanism is mathematically equivalent to the $RRRRS_H$ mechanism where S_H represents the sphere-helix pair.

The plane-kinematic pair has three degrees of freedom described by one rotation and two translations. This pair can be described mathematically as

$$\hat{\theta}_{ijk} = (\theta_i) + \sigma(s_i, s_j) \qquad (5.29)$$

The mathematical definition of the plane kinematic pair indicates that the RRRRRPP mechanism, where the axes of the two prism pairs are intersecting, is mathematically equivalent to the $RRRRP_L$ mechanism where P_L represents the plane kinematic pair.

It should be remarked that the problems involved in substituting the slotted sphere-cylinder pair, the sphere-helix pair, and the plane pair are similar to those involved in substituting either a prism pair or the cylinder pair. Since each of these cases are considered in great length in the previous section, it does not seem necessary to reconsider them again.

The present discussion has considered the substitution of the lower class kinematic pairs only primarily because these pairs are capable of transmitting higher forces. The higher pairs, especially of class four and five, demand extremely severe requirements in order to be substituted for the revolute pairs. Furthermore, these kinematic pairs are more complex in structure and geometry than the basic elementary pairs such as the revolute pair, the prism pair, and the cylinder pair.

CHAPTER VI

SUMMARY AND CONCLUSIONS

The present investigation is a step in an attempt to open the mysteries of general constraints and passive freedom. However, before such a step can be taken it is necessary to examine the state of the art. Several leading kinematicians have made observations on the nature of the general constraints and accordingly have proposed schemes to identify the existence of the general or passive constraints. Since these observations were limited to the schemes proposed by these kinematicians, they only provided a partial solution to the existing dilemma of identifying the existence of general constraints.

While each of these criteria may prove to be necessary, none were found to be sufficient. Consequently, those who observed the state of this art reexamined their own proposed criteria and came up with the new ones. For instance, Kutzbach proposed in 1932 a mathematical relationship which was reviewed in 1936. The mobility criteria of Malytcheff was reviewed by Artobolevski and Dobrovol'ski. Kolchin, however, was able to make some of his own observations, and as a consequence, the mobility criteria of Artobolevski and Dobrovol'ski was modified by introducing an extra parameter called the passive constraints.

While these kinematicians have modified the mobility criteria from one form to another and have introduced new parameters, none have presented a rational procedure to determine their existence. Therefore, a reader is always left to a choice of selecting the form of the mobility criteria. However, until a rational procedure is discovered, the number synthesis or the type synthesis of the space mechanism virtually remains unexplored.

The need for establishing the rational procedure of identifying and determining the number of general constraints or passive constraints was recognized years ago. Recently, Sharikov, one of the former students of Artobolevski, attempted to introduce a method based on the classical theory of screws. The method, however, utilizes descriptive geometry and, therefore, has its limitation. A rigorous mathematical approach to determine the existence of the general constraint is suggested by the two Rumanians, Vionea and Atanansiu. Unfortunately, their investigation does not proceed beyond the family of mechanisms having less than three general constraints.

The survey of the existing literature points out the striking correlation between the existing mobility criteria as shown in Table V. All the existing approaches, except for the Kolchin's approach, classifies the mechanisms into the five families of mechanisms. The zero family mechanisms have no specific constraints regarding the orientation of the axes of the kinematic pairs. The family one mechanisms have one general constraint; that is, the orientation of the axes of the kinematic pairs must observe a specific law or laws. Such laws are neither sufficient nor necessary for the existence of a mechanism having one general constraint. For instance, Sharikov's classical theory of screws has hypothesized that the six-link six-revolute mechanism exists if a pair of three axes intersects in two distinct points located either at a finite or at infinite distance. The articulated six-link mechanism of Bricard then becomes an exception to such hypothesis.

Clearly, one is led to conclude that either there was something misleading in the method of investigating the nature of one general constraint or the classical theory of screws does not provide a proper mathematical model.

The study of the nature of one general constraint may also have been conducted by the method proposed by F. M. Dimentberg. However, the proposed method leads to an examination of the root of a polynomial of order thirty-two. Clearly, such an investigation might lead to all types of erroneous results.

A need for a rational procedure to study the number of general constraints in a mechanism was recognized. Chapter III of the present investigation is completely devoted to the development of the theory of identifying the existence of general constraints.

The method of investigating the existence of general constraints concentrates on examining the rank of a coefficient matrix [M]. This matrix [M] is obtained by giving a differential displacement to the screw matrices describing the closure condition of a space mechanism. The differential displacement provides a set of twelve simultaneous non-homogeneous equations. When a complete closure condition for a mechanism is established, the matrix representing the twelve simultaneous linear equations degenerates to yield the coefficient matrix [M].

If for a given mechanism the rank of this coefficient matrix [M] is six, then the mechanism under consideration is free from any general constraints. If, however, the rank of the coefficient matrix [M] is five, then the mechanism under consideration has one general constraint. If the rank of the coefficient matrix [M] is four, three, or two, then correspondingly the mechanism under consideration has either two, three, or four general constraints.

The most remarkable characteristic of the coefficient matrix [M] is that it consists of two types of equations which in turn describe the instantaneous axes either of rotations or translations. For instance, in case of a plane four-link four-revolute mechanism, the rank of the coefficient matrix [M] is three. Furthermore, this matrix consists of three equations, two of which describe the instantaneous axes of translations and the other describes the instantaneous axis of rotation. The principal axes of rotation and translation of this mechanism are determined by computing the Eigen-vectors.

The examination of the classical "paradoxical" mechanisms such as the Bennett mechanism and Goldberg five-link mechanism revealed the other properties of the [M] matrix. The rank of the coefficient matrix [M] in the case of the Bennett mechanism is three. Accordingly, this matrix must consist of three nonvanishing equations. Instead, the coefficient matrix [M] has five nonvanishing equations. Since the rank of the matrix is three, only three of the five equations are necessary to describe the Bennett mechanism. That is, two of the five equations may conveniently be ignored. Since the existence of these two added equations does not contribute any new information to the coefficient matrix [M] and their withdrawal does not produce any singularity in the coefficient matrix [M], these two additional equations are regarded as passive. If the principal axes of rotations and translations are computed, then correspondingly there will be two passive axes about which one rotation and one translation component of the general motion will be found to have zero values.

The existence and nonexistence of one or more number of passive equations in the coefficient matrix opens the door to a great many number of basic questions related to the nature of general constraints. Due to the analytical nature of the present method, it is not possible to state the factors that control their existence. Since theoretically it is possible to expect a maximum of six compatible equations in r unknowns where r also represents the rank of the coefficient matrix [M], then a maximum of 6-r and a minimum of zero number of passive axes must correspondingly exist for a particular family of mechanisms. Furthermore, since the family of the mechanism does not seem to depend on the number of compatible equations, the information provided by the existence of the passive axes must provide a new dimension to these basic issues of the nature and characteristic of the general constraint mechanisms. The present investigation was, however, confined to the study of the six-link mechanism, and therefore, these questions are purposely left aside for future studies.

The six-link six-revolute mechanism is noted to have one general constraint because the rank of the coefficient matrix is five. It has been observed that the six-link mechanism can be classified into two groups of mechanisms. This classification is based on the components of the general motion. The coefficient matrix has six equations in five unknowns when the mechanism is describing three rotations and two translations. However, the coefficient matrix has five equations in five unknowns when the mechanism is describing two rotations and three translations. Since the row vector describing rotation vanishes to zero, it is concluded that for the case in which six equations exist with five unknowns one of the row vectors describing translation must be passive.

A mechanism may be subclassified depending upon the number and type of passive axes it produces. For this purpose, however, an efficient method of detecting the passivity must be formulated. The present investigation was confined to the study of the one general constraint mechanism, and therefore, no effort was made to develop an elegant and efficient method for detecting which of the axes are passive. Instead, the problem is considered to be of secondary importance for the present investigation.

The method of arriving at the coefficient matrix [M] is iterative. A set of kinematic parameters, viz., the kinematic link, the kink-links, the type of pairs and twist angles between the two successive axes, is expected to be known for a kinematic chain. Then for any assumed input displacement, a complete closure condition is determined. If a complete closure condition exists for any arbitrarily selected input position, then the kinematic chain is a mechanism. If the assumed parameters were to yield a structure, then the iterative process does not converge, even for the specific position where the chain forms a close configuration. Whenever the iterative process does yield a complete convergence for an arbitrarily selected input parameter, then before announcing this particular chain as a mechanism the chain is invariably tested for a second closure condition.

There were, however, some technical problems associated with the iterative process, especially when the product matrix [W] was singular. The singularity conditions exist in three situations. These situations exist when the chain is either examined in the region beyond the limit position, or the chain has a dead-center position. The singularity condition also exists when the chain is a structure. These cases were handled very carefully by examining the complete region of mobility of the chain. That is, a minimum of twelve independent closure conditions were examined for the convergence of the product matrix [W].

The method developed for identifying and determining the existence of the general constraints also provided the answers for the mobility region of the mechanisms. The limit-position and dead-centers of any mechanism can be determined by the computer within a fraction of a minute once the chain is determined to be a mechanism. Thus, the advantage of the developed method was recognized from the very early stage of its development.

This method was used to examine the governing conditions under which a six-link six-revolute chain exists as a mechanism. The six-link six-revolute chain was selected because it represented the family of mechanisms having one general constraint. Furthermore, if the governing conditions of this mechanism are once discovered, then the other mechanisms obtained by substituting the other types of pairs can also be discovered simply by relating the revolute pairs with the other kinematic pairs.

The most difficult part of the present investigation is involved in making a proper decision. There are two ways in which a study can be conducted to investigate the governing conditions of the six-link mechanism. The six-link six-revolute mechanism is capable of having eighteen parameters, twelve of which may have either positive or negative values. Therefore, in order to arrive at an explicit governing condition, the behavior of a total of thirty parameters must be studied.

If a total variational study of these parameters is planned, then nearly thirty factorial six-link chains must be examined for the closure conditions. The computation required for the six independent closure conditions of a chain takes on an average of six and a half minutes on the IBM 7040. Therefore, if such a procedure would have been adopted to examine the governing conditions, then the present investigation would not have come to an end in the present century.

In view of the above statements perhaps the procedure adopted in the present investigation for examining the governing conditions of the existence of the six-link mechanism can be more appreciated. The procedure is based on an observation that three elementary types of the six-link mechanisms that could exist with a minimum of kinematic parameters are known. These are the Franke's six-link, the Bricard's articulated six-link and Sarrus' six-link mechanism. Franke's six-link mechanism has all kink-links of zero length, Bricard's articulated six-link mechanism has all kinematic links of zero values, and the Sarrus' mechanism is a combination of both the kink-links and the kinematic links.

The adopted procedure for determining the governing conditions then is centered around these three elementary models. A variational study was planned to vary the eighteen parameters in such a manner to obtain the general and the degenerate cases of the six-link mechanism. The present investigation examined nearly three hundred and fifty different six-link chains. It should be noted that only one-fourth of these chains generated a six-link mechanism.

The present investigation indicates that the existence of the six-link mechanism is due to a mathematical equality rather than physical symmetry. This mathematical equality takes into account the permutations of the kinematic parameters.

One of the most interesting points that is observed in the investigation of the six-link mechanism is the relationship between the physical symmetry of the mechanism and its mobility region. The majority of the six-link mechanisms appear to be either of rocker-rocker type or crank-rocker type. The mobility region, however, may be enlarged if the mechanism has a higher order of symmetry.

The successful results obtained for the governing conditions of the existence of a six-link mechanism led to an investigation relating the turning pairs to the other kinematic pairs. This investigation, however, was confined to relating only the lower pairs; that is, the kinematic pairs such as the prism pair, the helical pair, the cylinder pair, the torus pair, and the spherical pair.

According to the mobility criterion of one general constraint, only the class five kinematic pairs are not permissible. However, the mobility criterion does not take into account the governing conditions of one general constraint, and therefore, it can be predicted that all the kinematic pairs from class one to class four need to be examined.

The present investigation is confined to the useful lower pairs only. The other lower kinematic pairs having a combination of a helical pair and sphere, a cylinder and sphere are not considered primarily because they demand extremely severe requirements in order to replace the revolute pairs.

The problem becomes more complex when a mechanism of one general constraint is under consideration. The six-link mechanism which has been found to exist with a wide variety of combinations of kink-links and kinematic links, however, appears to be more suitable for adopting kinematic pair mechanisms rather than the kinematic pair. For instance, the six-link mechanism can more readily accept the Hookes-joint type of pair mechanism than the spherical pair, even though the function of both of these pairs is to produce three rotations.

The method of replacing the turning pairs by the other lower kinematic pairs having one, two, and three degrees of freedom is presented in Chapter V. The existing dilemma concerning the maximum number of prism pairs and helical pairs is resolved for the zero family space mechanism. The coefficient matrix [M] shows that a zero family space mechanism with a turning pair for an input displacement is capable of having a maximum of three prism pairs. A maximum of four prism pairs can be permitted provided one of the prism pairs is employed for the input displacement. Similarly, a helical pair can be substituted for a revolute pair.

The method of substituting other classes and types of pairs for a revolute pair is suitable for the zero family mechanism only. That is, the method is independent of the theory of the general constraints.

Therefore, whenever one turning pair is replaced by the other, the resulting chain is expected to yield a mechanism. However, there is no complete assurance that the resulting mechanism will still belong to the same family as it did before the substitution. Therefore, at each stage of substitution, the coefficient matrix must be examined for a possible degeneration of a mechanism to a lower group.

The present investigation then can be briefly summarized as follows:

(1) A mathematical procedure was developed to identify the number of general constraints in a mechanism. The method also provides a complete displacement analysis of a mechanism, and identifies the existence of dead-centers and limit-positions.

(2) A procedure for the analysis of the six-link mechanism and an existence criteria was developed.

(3) A method was shown for substituting various types and kinds of kinematic pairs for a revolute pair of a kinematic chain. This development leads to the other types and kinds of mechanisms belonging to the family of six-link mechanisms.

The outcome of the present investigation leads to the key that opens the mysteries of the world of mechanisms with or without general constraints. According to the mobility criterion, there are five families of mechanisms. The present investigation has simply considered the mysteries of the mechanisms with one general constraint. Similar studies are now possible to unlock the mysteries of mechanisms either free from general constraint (m = 0) or having two, three or four general constraints (m = 2, 3, 4).

Harrisberger [29] had predicted, based on the available information of the mobility criteria, the existence of nearly five hundred space mechanisms free from any general constraints. Since the mobility criteria are not capable of providing an insight to the closure conditions of these mechanisms, a scientific study similar to the present investigation must be planned to discover the existence criteria of the zero family space mechanism. The present investigation indicates that any random orientation of the kinematic pairs in the 7R mechanism does not necessarily yield a space mechanism. Instead, it forecasts a definite relationship between the twist angles, the kinematic link and the kink-links.

Recently, an effort was made by Dobrjanskyj and Freudenstein [53] to extend the work of Harrisberger [29]. According to these authors, pair inversion of Harrisberger's five hundred mechanisms produces nearly four thousand mechanisms. However, Dobrjanskyj and Freudenstein completely ignored the basic issues of the existence criteria. In view of the established fact concerning the maximum number of prism pairs and helical pairs, nearly half the mechanisms claimed by Dobrjanskyj and Freudenstein have no basis for existance as zero family space mechanisms, and for the other half, closure conditions are unspecified.

The present investigation has presented a method of obtaining other types and kinds of mechanisms described in Table VI. This table must be revised with the proper modification of the pair inversions and their corresponding existence criteria must be developed. It is expected that such a study will produce many useful mechanisms having one general constraint.

It has been observed that in the one general constraint domain, the mechanisms having a higher order of symmetry appear to produce a constant velocity output. The Cardan mechanism, for instance, has been used over the century for obtaining a constant velocity output in a skew plane. The present investigation has identified a large number of mechanisms which are symmetric and have a constant velocity output.

One unexplored area in the domain of one general constraint mechanisms is an investigation of the coupler curves of the four-link mechanisms and their coupler cognates since a proper existence criteria is not known. The present investigation now makes it possible to explore this area. It appears that the next fruitful areas of research are the following:

(1) Pair inversion study and the existence criteria of the different types and kinds of mechanisms.

(2) Complete investigation of the symmetric mechanisms having six, five, and four links and producing constant velocity output.

(3) There are two types of cognates. These are Robert's cognates and Soni's cognates [54]. The Robert's cognates are the mechanisms which generate the same coupler curve as does the source mechanism. The Soni cognates are the mechanisms which generate the same output motion of the follower as does the source mechanism. The importance of this type of research hardly needs to be emphasized, especially when all the practical two-loop configurations can exist either with a coupler-drive or with the follower-drive.

(4) The mechanisms with two, three, and four general constraints are virtually unknown, primarily because all the necessary and sufficient existence criteria are not known. Once the existence criteria are discovered by using the technique developed in Chapter III, the studies proposed in points 1, 2, and 3 above may be organized to determine their practical applicability.

(5) The three general constraint mechanisms appear to have a wide variety of practical applicability. For instance, the four-link plane mechanism and its related multi-loop mechanisms are used extensively in industry. The spherical four-link mechanism having three rotations for its general motion are being found to have a wide variety of practical application. The Bennett mechanism, which also belongs to this group, can be used to produce a constant velocity output in a skew plane. Yet the application of this mechanism is virtually unexplored.

The present theory of identifying the existence of general constraints predicts the existence of the four-link mechanisms such as **PPPP**, **RPPP**, and **HPPP**. The exact existence criteria of these mechanisms are not known. However, it appears that these mechanisms are capable of producing a translatory motion in a skew plane. That is, they are space models of a plane slider-crank.

In view of the five areas of future research proposed, the outcome of the present investigation appears to be "a drop in a bucket". Yet, it should be clear that it is the "drop" that promises the kinematicians a journey into the mysterious world of space mechanisms just waiting to be discovered.

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APPENDIX A

ALGEBRA OF DUAL NUMBERS AND DUAL VECTORS

The dual number is defined as

$$\hat{x} = x_0 + \sigma x_1$$

where

x_0 = real part

x_1 = imaginary part

and $\sigma^2 = 0$

Properties of Dual Numbers:

(1) $\hat{x} = 0$ when $x_0 = 0$ and $x_1 = 0$

(2) $\hat{x} = \hat{y}$ when $x_0 = y_0$ and $x_1 = y_1$

Addition and Subtraction:

(3) $\hat{x} + \hat{y} = (x_0 + \sigma x_1) + (y_0 + \sigma y_1) = (x_0 + y_0) + \sigma(x_1 + y_1)$

(4) $\hat{x} - \hat{y} = (x_0 + \sigma x_1) - (y_0 + \sigma y_1) = (x_0 - y_0) + \sigma(x_1 - y_1)$

Multiplication and Division

(5) $\hat{x}\hat{y} = (x_0 + \sigma y_1)(y_0 + \sigma y_1) = x_0 y_0 + \sigma(x_1 y_0 + x_0 y_1)$

$$= x_0 y_0 \left\{1 + \sigma \left(\frac{x_1}{x_0} + \frac{y_1}{y_0}\right)\right\}$$

(6) $$\frac{\hat{x}_o}{\hat{y}} = \frac{x_0 + \sigma x_1}{y_0 + \sigma y_1} = \frac{x_0 \left(1 + \sigma \frac{x_1}{y_0}\right)}{y_0 \left(1 + \sigma \frac{y_1}{y_0}\right)} = \frac{x_0}{y_0} \; \frac{\left(1 + \sigma \frac{x_1}{y_1}\right)\left(1 - \sigma \frac{y_1}{y_1}\right)}{\left[1 - \sigma^2 \left(\frac{y_1}{y_0}\right)^2\right]}$$

$$= \frac{x_0}{y_0} \left\{1 + \sigma\left(\frac{x_1}{x_0} - \frac{y_1}{y_0}\right)\right\}$$

(7) $\hat{x}^n = (x_o + \sigma x)^n = x_o{}^n \, (1 + \sigma \frac{x_1}{x_o})^n = x_o^n \left\{ 1 + {}^n x_1 \;\; \sigma \frac{x_1}{x_o} \right\}$

$= x_o{}^n \left\{ 1 + \sigma^n \frac{x_1}{x_o} \right\} = x_o{}^n + \sigma n_1 \, x_o{}^{n - 1}$

(8) The expression of any function of Dual numbers $x_o + \sigma x_1$ is obtained using Taylor series expansion as

$$f(\hat{x}) = f(x_o + \sigma x_1) = f(x_o) + \sigma x_1 \frac{\delta f(a_o)}{\delta a_o}$$

Trigonometric and Exponential relationship:

If we assign a dual angle $\hat{x} = x_o + \sigma x_1$, formed by two straight lines of space, where x_o is the normal angle between the unit vector axes of the straight lines and x_1 is the shortest distance between the straight lines, then the trigonometrical function of the dual angles can be expressed as

(9) $\mathrm{Sin}(x_o \pm \sigma x_1) = \mathrm{Sin}\ x_o\ \mathrm{Cos}(\sigma x_1) \pm \mathrm{Cos}\ x_o\ \mathrm{Sin}\ (\sigma x_1)$

$= \mathrm{Sin}\ x_o \pm \sigma x_1\ \mathrm{Cos}\ x_o$

(10) $\mathrm{Cos}\ (x_o \pm \sigma x_1) = \mathrm{Cos}\ x_o \mp \sigma x_1\ \mathrm{Sin}\ x_o$

(11) $\tan\ (x\sigma \pm \sigma x_1) = \tan\ x_o \mp \sigma \frac{x_1}{\mathrm{Cos}^2\ x_o}$

$= \tan\ x_o \pm \sigma x_1\ (1 + \tan^2\ x_o)$

(12) $\mathrm{Cos}\ (x_o \pm \sigma x_1) = \mathrm{Cos}\ x_o \mp \sigma \frac{x_1}{\mathrm{Sin}^2\ x_o}$

$= \mathrm{Cos}\ x_o \mp \sigma x_1\ (1 + \mathrm{Cos}^2\ x_o)$

(13) $e^{x_o} + \sigma x_1 = e^{x_o}\ e^{\sigma x_1} = e^{x_o}\ (1 + \sigma x_1)$

(14) $\ell n(x_o + \sigma x_1) = \ell n \left\{ x_o (1 + \sigma \frac{x_1}{x_o}) \right\} = \ell n\ n_o + \sigma \frac{x_1}{x_o}$

It should be noted that all identities of ordinary algebra and trigonometry and also all formulas of differential and integral calculus are maintained in the algebra of dual numbers.

Let us consider a polynomial having dual numbers as the coefficient. If the right hand side of this polynomial is equated to zero, then

$$\hat{a}x^n + \hat{b}n^{n-1} + \hat{c}x^{n-2} + \ldots \hat{s}x + \hat{t} = 0$$

where $\hat{a} = a_o + \sigma a_1$, $\hat{b} = b_o + \sigma b_1$, $\hat{c} = c_o + \sigma c_1$, ..., $\hat{t} = t_o + \sigma t_1$.

However the property of the dual number requires that

(15) $a_o x_o{}^n + b_o x_o{}^{n-1} + \ldots s_o x_o + t_o = 0$, and

(16) $n\, a_o x_o{}^{n-1} + (n - 1)\, b_o x_o{}^{n-2} + (n - 2)\, c_o x_o{}^{n-3} + \ldots s_o]\, x_1$
$+ [a_1 x_o{}^n + b_1 x_o{}^{n-1} + \ldots s_1 x_o + t_1] = 0$

Let us consider a special case of the complex quadratic

$$\hat{a}\hat{x}^2 + \hat{b}\hat{x} + \hat{c} = 0 \qquad \text{where } \hat{x} = x_o + \sigma x_1$$

Then according to identities (15) and (16) we have

(17) $a_o x_o{}^2 + b_o x_o + c_o = 0$

(18) $(2a_o x_o + b_o)\, x_1 + (a_1 x_o{}^2 + b_1 x_o + c_1) = 0$

from where, we get

(19) $x_o = \dfrac{-b_o \pm \sqrt{b_o{}^2 - 4a_o c_o}}{2a_o}$

(20) $x_1 = \dfrac{-1}{2a_o x_o + b_o}\,(a_1 x_o{}^2 + b_1 x_o + c_1)$

In order that the equation has real root, it is necessary that $x_1 = 0$ and at the same time x_o must satisfy the two equations:

(21) $a_o x_o{}^2 + b_o x_o + c_o = 0$

(22) $a_1 x_o{}^2 + b_1 x_o + c_1 = 0$

(23) That is, $(a_o c_1 - a_1 c_o)^2 - (a_o b_1 - a_1 b_o)(b_o c_1 - b_1 c_o) = 0$

This identity (23) is a necessary and sufficient condition for the presence of real roots of the equation with complex coefficients. The identity (23) can be rewritten in the form of a determinant as

$$\begin{vmatrix} a_0 & b_0 & c_0 & 0 \\ 0 & a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 & 0 \\ 0 & a_1 & b_1 & c_1 \end{vmatrix} = (a_0c_1 - a_1c_0)^2 - (a_0b_1 - a_1b_0)(b_0c_1 - b_1c_0) = 0$$

Dual Vector:

The dual vector is defined as

(24) $\hat{A} = \bar{a}_0 + \sigma a_1$

where a_0 is the real part and a_1 is the imaginary part of the dual vector. Here again

$$\sigma^2 = 0$$

The operation on complex vectors is formally not distinguished from the operation on ordinary vectors.

The dual vector can be considered as a screw which has two components. The real part of the dual vector can be considered as the angular velocity of a link about an axis and the imaginary part as the translatory velocity along the same axis. Thus

(25) $\hat{A} = \hat{S} = \bar{W} + \sigma\bar{T}$

where $\hat{S}$ = screw

$\bar{W}$ = angular velocity.

APPENDIX B

COMPUTER PROGRAM

The computer program listed on the following pages is based on the method developed in the Chapter III. The program output consists of the following:

(1) Initial input screw matrices

(2) Coefficient matrix [M] at every stage of the iteration

(3) Inverse of the coefficient matrix

(4) Determinant of the coefficient matrix

(5) Estimated displacement parameters.

The program input consists of the following:

(1) Defining the type of mechanism

(2) Providing the exact values of the invariant kinematic parameters

(3) Initial estimate of variant kinematic parameters

```
  0 $IBFTC DKNAME   DECK
  1          DIMENSION KSB(10),KSA(10),KYA(10),KYB(10)
  2          DIMENSION ST(20)
  3          DIMENSION TRB(6,6)
  4          DIMENSION ARTX(38),ARTB(38),CTP(10)
  5          DIMENSION AX(20),AY(20),X(20),Y(20),A(10,3,3),B(10,3,3)
  6          DIMENSION P(3,3),D(10,3,3),E(10,3,3),BB(10,3,3)
  7          DIMENSION AA(10,3,3),AM(12,6),AP(12,1),AT(6,12),TX(6,6),TY(6,1)
 10          DIMENSION TB(6,6),DEY(6,1)
 11          DIMENSION ATX(38),ATB(38)
 12 6040     FORMAT(1H0,4HY(I),5X,6F12.6)
 13 6030     FORMAT(1H0,4HX(I),5X,6F12.6)
 14 6020     FORMAT(1H0,5HAY(I),5X,6F12.6)
 15 6010     FORMAT(1H0,5HAX(I),5X,6F12.6)
 16 6000     FORMAT(1H1,6I4,F12.6)
 17 6050     FORMAT(1H0,7I4)
 20 2010     FORMAT(9I4,2F12.6)
 21 3010     FORMAT (12I4)
 22 2020     FORMAT(7F10.6)
 23 2030     FORMAT (7F10.6)
 24 2130     FORMAT(1H1,12HINPUT MATRIX)
 25 2140     FORMAT(1H ,7F12.6)
 26 2190     FORMAT(1H0,5HDET.=F20.6,5X,3HIE=I2)
 27 2200     FORMAT(1H0,43HRH MATRIX, DEVIATIONS, AND ESTIMATED THETAS)
 30 2210     FORMAT(1H ,3F12.6)
 31 2220     FORMAT(1H0,10HITERATION=I2)
 32 5000     READ(5,2010)JMAX,IJA,KK,KM,ITT,JB,ICS,JJQS,IGE,DEL,DELX
 44          READ(5,3010) LMN,(KSB(J),J=1,LMN),  (KYB(J),J=1,LMN)
 56          READ(5,3010)LPJ,(KSA(J),J=1,LPJ),(KYA(J),J=1,LPJ)
 70          WRITE(6,6050) LMN,(KSB(J),J=1,LMN),(KYB(J),J=1,  LMN)
101          WRITE( 6,6050) LPJ,(KSA(J),J=1,LPJ),(KYA(J),J=1,LPJ)
112          READ(5,2020)(AX(I),I=1,JMAX )
117          READ(5,2030)(AY(I),I=1,JMAX)
124          READ(5,2030)(X(I),I=1,JMAX)
131          READ(5,2030)(Y(I),I=1,JMAX)
136          WRITE(6,6000)JMAX,IJA,KK,KM,ITT,JB,DEL
137          WRITE(6,6010)(AX(I),I=1,JMAX)
144          WRITE(6,6020)(AY(I),I=1,JMAX)
151          WRITE(6,6030)(X(I),I=1,JMAX)
156          WRITE(6,6040)(Y(I),I=1,JMAX)
163          KT=0
164          DO 5  I=1,JMAX
165          X(I)=X(I)*3.141592654/180.
166          AX(I)=AX(I)*3.141592654/180.
167 5        ST(I)=AX(I)
171          DELX=DELX*3.141592654/180.0
172          DO 8050 JKLT=1, JJQS
173          IF (JKLT.EQ.1) GO TO  777
176          X(1)=X(1)+DELX
177 777      CONTINUE
200          IT=0
201 2000     DO 10 I=1,JMAX
202          AX(I)=ST(I)
203          A(I,1,1)=COS(X(I))
204          A(I,1,2)=-(SIN(X(I)))*(COS(AX(I)))
```

```
      A(I,1,3)=SIN(X(I)) *(SIN(AX(I)))
      A(I,2,1)=SIN(X(I))
      A(I,2,2)=COS(X(I))*COS(AX(I))
      A(I,2,3)=-(COS(X(I)))*SIN(AX(I))
      A(I,3,1)=0.
      A(I,3,2)=SIN(AX(I))
      A(I,3,3)=COS(AX(I))
      B(I,1,1)=-Y(I)*SIN(X(I))
      B(I,1,2)=-Y(I)*COS(AX(I))*COS(X(I))+SIN(X(I))*SIN(AX(I))*AY(I)
      B(I,1,3)=Y(I)*COS(X(I))*SIN(AX(I))+AY(I)*SIN(X(I))*COS(AX(I))
      B(I,2,1)=Y(I)*COS(X(I))
      B(I,2,2)=-Y(I)*SIN(X(I))*COS(AX(I))-AY(I)*COS(X(I))*SIN(AX(I))
      B(I,2,3)=Y(I)*SIN(X(I))*SIN(AX(I))-AY(I)*COS(X(I))* CCS(AX(I))
      B(I,3,1)=0.
      B(I,3,2)=AY(I)*COS(AX(I))
      B(I,3,3)=-AY(I)*SIN(AX(I))
   10 CONTINUE
      P(1,1)=0.0
      P(1,2)=-1.0
      P(1,3)=0.
      P(2,1)=1.0
      P(2,2)=0.0
      P(2,3)=0.
      P(3,1)=0.
      P(3,2)=0.
      P(3,3)=0.
      DC 40 I=1,JMAX
      DO 20 J=1,JMAX
      DO 20 K=1,3
      DO 20 L=1,3
      IF(I.EQ.J) GO TO 30
      D(J,K,L)=A(J,K,L)
      GO TO 20
   30 D(J,K,L)=B(I,K,L)
   20 CONTINUE
      DO 50 K=1,3
      DO 50 J=1,3
   50 E(I,K,J)=0.
      MAX=JMAX-1
      DO 60 J=1,MAX
      DO 70 K=1,3
      DC 70 M=1,3
      DO 70 L=1,3
      JM=J+1
   70 E(I,K,M)=E(I,K,M)+D(J,K,L)*D(JM,L,M)
      IF(J.EQ.(JMAX-1)) GO TO 60
      DO 80 K=1,3
      DO 80 M=1,3
   80 D(JM,K,M)=E(I,K,M)
      DC 81 K=1,3
      DO 81 M=1,3
   81 E(I,K,M)=0.
   60 CONTINUE
   40 CONTINUE
      DO 90 J=1,3
```

```
      DO 90 K=1,3
   90 BB(1,J,K)=0.0
      DO 100 K=1,3
      DO 100 M=1,3
      DO 100 I=1,JMAX
  100 BB (1,K,M)=BB(1,K,M)+E(I,K,M)
      DO 120 J=1,JMAX
      DO 120 K=1,3
      DO 120 L=1,3
      D(J,K,L)=A(J,K,L)
  120 CONTINUE
      DO 130 K=1,3
      DO 130 J=1,3
  130 AA(1,K,J)=0.
      MAX=JMAX-1
      DO 140 J=1,MAX
      DO 150 K=1,3
      DO 150 M=1,3
      DO 150 L=1,3
      JM=J+1
  150 AA(1,K,M)=AA(1,K,M)+D(J,K,L)*D(JM,L,M)
      IF(J.EQ.(JMAX-1)) GO TO 140
      DO 160 K=1,3
      DO 160 M=1,3
  160 D(JM,K,M)=AA(1,K,M)
      DO 161 K=1,3
      DO 161 M=1,3
  161 AA(1,K,M)=0.
  140 CONTINUE
      DO 170 N=2,JMAX
      DO 180 I=1,JMAX
      MAX=JMAX+1
      DO 190 J=1,MAX
      DO 190 K=1,3
      DO 190 L=1,3
      IF(J-N)210,220,230
  210 IF(I.EQ.J)GO TO 215
      D(J,K,L)=A(J,K,L)
      GO TO 190
  215 D(J,K,L)=B(J,K,L)
      GC TO 190
  220 D(J,K,L)=P(K,L)
      GO TO 190
  230 IF(J.EQ.(I+1)) GO TO 235
      D(J,K,L)=A(J-1,K,L)
      GO TO 190
  235 D(J,K,L)=B(J-1,K,L)
  190 CONTINUE
      DO 240 K=1,3
      DO 240 J=1,3
  240 E(I,J,K)=0.
      DO 250 J=1,JMAX
      DO 260 K=1,3
      DO 260 M=1,3
      DO 260 L=1,3
```

```
443            JM=J+1
444 260        E(I,K,M)=E(I,K,M)+D(J,K,L)*D(JM,L,M)
450            IF(J.EQ.(JMAX))GO TO 250
453            DO 270  K=1,3
454            DO 270 M=1,3
455 270        D(JM,K,M)=E(I,K,M)
460            DO 271 K=1,3
461            DO 271 M=1,3
462 271        E(I,K,M)=0.
465 250        CONTINUE
467 180        CONTINUE
471            DO 280 J=1,3
472            DO 280 K=1,3
473 280        BB(N,J,K)=0.
476            DO 290 K=1,3
477            DO 290 M=1,3
500            DO 290 I=1,JMAX
501 290        BB(N,K,M)=BB(N,K,M)+E(I,K,M)
505 170        CONTINUE
507            DO 300 N=2 ,JMAX
510            MAX=JMAX+1
511            DO 310 J=1,MAX
512            DO 310 K=1,3
513            DO 310 L=1,3
514            IF(J-N) 320 ,330,340
515 320        D(J,K,L)=A(J,K,L)
516            GO TO 310
517 330        D(J,K,L)=P(K,L)
520            GC TO 310
521 340        JL=J-1
522            D(J,K,L)=A(JL,K,L)
523 310        CONTINUE
527            DC 350 K=1,3
530            DO 350 J=1,3
531 350        AA(N,J,K)=0.0
534            DO 360 J=1,JMAX
535            DO 370 K=1,3
536            DO 370 M=1,3
537            DO 370 L=1,3
540            JM=J+1
541 370        AA(N,K,M)=AA(N,K,M)+D(J,K,L)*D(JM,L,M)
545            IF(J.EQ.JMAX) GO TO 360
550            DO 390 K=1,3
551            DO 390 M=1,3
552 390        D(JM,K,M)=AA(N,K,M)
555            DO 391 K=1,3
556            DO 391 M=1,3
557 391        AA(N,K,M)=0.
562 360        CONTINUE
564 300        CONTINUE
566            JMAN=IGE-IJA
567            DO 495 I=1,12
570            DO 495 J=1,6
571 495        AM(I,J)=0.
574            J=1
```

```
      DO 500 K=1,3
      DO 520 N=K,3
      DO 515 I=1,JMAN
      IF(KK.EQ.0)GO TO 505
      IF(K.EQ.1)GO TO 530
      GO TO 505
530   IF(N.EQ.2)GO TO 540
      GO TO 505
540   N=N+1
505   IF(I.EQ.1)GO TO 510
      AM(J,I-1)=AA(I,K,N)
      GO TO 515
510   IF(K.EQ.N) GO TO 516
      GO TO 517
516   AP(J,I)=       1.0 - AA(I,K,N)
      GO TO 515
517   AP(J,I)=-AA(I,K,N)
515   CONTINUE
520   J=J+1
500   CONTINUE
      DO 560 K=1,3
      DO 570 N=K,3
      DO 585 I=1,7
      IF(KM.EQ.0)GO TO 600
      IF(K.EQ.1)GO TO 610
      GO TO 600
610   IF(N.EQ.2)GO TO 620
      GO TO 600
620   N=N+1
600   IF(I.EQ.1) GO TO 605
      IF (I.GT.JMAN) GO TO 590
      GO TO 580
605   AP(J,I)=-BB(I,K,N)
      GO TO 585
590   II=I-1
      IF(ICS.EQ.1) GO TO 588
      IJ=I
      GO TO 589
588   IJ=I-IJA
589   CONTINUE
      AM(J,II)=AA(IJ,K,N)
      GO TO 585
580   AM(J,I-1)=BB(I,K,N)
585   CONTINUE
570   J=J+1
560   CONTINUE
      JA=J-1
      WRITE(6,2130)
      DO 561 I=1,JA
561   WRITE(6,2140)(AM(I,J),J=1,6),AP( I,1)
      DO 630 I=1,JA
      DO 630 J=1,6
630   AT(J,I)=AM(I,J)
      DO 640 I=1,6
      DO 640 J=1,6
```

```
640   TX(I,J)=0.
      DO 650 I=1,JB
      DO 650 M=1,JB
      DO 650 J=1,JA
650   TX(I,M)=TX(I,M)+AT(I,J)*AM(J,M)
      ATX(1)=JB
      ATX(2)=JB
      K=2
      DO 648 I=1,JB
      DO 648 J=1,JB
      K=K+1
648   ATX(K)=TX(I,J)
      CALL INVERX(ATX,ATB,DET,IE)
      KPI=JB*JB+2
      I=1
      J=1
      ARTX(1)=JB
      ARTX(2)=JB
      DO 672 K=3,KPI
      TB(I,J)=ATB(K)
      ARTX(K)=ATB(K)
      IF(J.EQ.JB) GO TO 673
      J=J+1
      GO TO 672
673   J=1
      I=I+1
672   CONTINUE
      WRITE(6,2190)DET,IE
      CALL INVERX(ARTX,ARTB,DER,IR)
      I=1
      J=1
      DO 875   K=3,KPI
      TRB(I,J)=ARTB(K)
      IF(J.EQ.JB) GO TO 873
      J=J+1
      GO TO 875
873   J=1
      I=I+1
875   CONTINUE
      WRITE(6,2190) DER,IR
      DO 655 I=1,JB
      J=1
655   TY(I,J)=0.
      DO 660 I=1,JB
      J=1
      DO 660 M=1,JA
660   TY(I,J)=TY(I,J)+AT(I,M)*AP(M,J)
      DO 670 I=1,JB
      J=1
670   DEY(I,J)=0.
      DO 680 I=1,JB
      J=1
      DO 680 M=1,JB
680   DEY(I,J)=DEY(I,J)+TB(I,M)*TY(M,J)
      DO 690 I=1,LMN
```

```
      IQP=KSB(I)
      IQR=KYB(I)
  690 X(IQP)=X(IQP)+DEY(IQR,1)
      DO 691    I=1,LPJ
      IQP=KSA(I)
      IQR=KYA(I)
  691 Y(IQP)=Y(IQP)+DEY(IQR,1)
      WRITE(6,2200)
      DO 661 I=1,JB
      J=I+1
      CTP(J)=X(J)*180./3.141592654
  661 WRITE(6,2210)TY(I,1),DEY(I,1),CTP(J)
      XFR=X(1)*180./3.141592654
      WRITE(6,6030)XFR,(X(I),I=2,JMAX)
      WRITE(6,6040)(Y(I),I=1,JMAX)
      IT=IT+1
      WRITE(6,2220) IT
      J=0
      DO 710 I=1,JB
      IF(ABS(DEY(I,1)).LE.DEL) GO TO 708
      GO TO 710
  708 J=J+1
  710 CONTINUE
      IF(J.EQ.JB) GO TO 8050
      IF(IT.GT.ITT) GO TO      8050
      GC TO 2000
 8050 CONTINUE
      GO TO 5000
      END
```

```
$IBFTC INVERX
      SUBROUTINE INVERX(A,B,DET,IE)
      DIMENSION A(1),B(1)
      DET = 1.0
      N = A(1)
      L10 = N**2 + 2
      DO 1 I = 1,L10
    1 B(I) = 0.
      B(1) = N
      B(2) = N
      L9 = N + 1
      DO 2 I = 3,L10,L9
    2 B(I) = 1.0
      JK = N - 1
      J = 3
      N1 = 3
      N2 = N + 2
      JO = N - 1
      J2 = N + 3
      J4 = 3
      DO 300 L1 = 1,JK
      NR = (J + N - 2)/(N + 1)
      NR1 = NR
      NRI = N - NR
      JN1 = J + N
      IF (NRI .LT. 1) GO TO 900
      IF (NRI .GT. 1) GO TO 804
  800 AMAX = ABS (A(J))
      AMXA = ABS (A(JN1))
      IF (AMAX .GE. AMXA) GO TO 900
  801 N5 = J - NR + 1
      N6 = N5 + N - 1
      IAD = N
  802 DO 803 IT = N5,N6
      IT6 = IT + IAD
      ATEM = A(IT)
      A(IT) = A(IT6)
      A(IT6) = ATEM
      ATEM = B(IT)
      B(IT) = B(IT6)
  803 B(IT6) = ATEM
      GO TO 900
  804 J11 = J + N + 1
      J10 = J + N
      AMAX = ABS (A(J))
      DO 807 IT = 1,NRI
      AMXA = ABS (A(J10))
      IF (AMAX .GE. AMXA)GO TO 806
  805 AMAX = AMXA
      NR1 = (J11 + N - 2)/(N + 1)
  806 J10 = J10 + N
  807 J11 = J11 + N + 1
      N5 = J - NR + 1
      N6 = N5 + N - 1
      ITEM = NR1 - NR
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      IAD = ITEM*N
      IF (IAD  .GT. 0 )   GO TO 802
  900 CONTINUE
      DENOM = A(J)
      IF (DENOM  .EQ. 0.0) GO TO 51
   50 IF (IAD .GT. 0 ) GO TO 701
  700 DET = DET*DENOM
      GO TO 702
  701 DET = DET*(-DENOM)
  702 DO 100 J1 = N1,N2
      A(J1) = A(J1)/DENOM
  100 B(J1) = B(J1)/DENOM
      J3 = J4
      N3 = N2 + 1
      N4 = N2 + N
      DO 200 L = 1,J0
      AMULT = A(J2)
      DO 101 J1 = N3,N4
      A(J1) = A(J1) - AMULT*A(J3)
      B(J1) = B(J1) - AMULT*B(J3)
  101 J3 = J3 + 1
      J2 = J2 + N
      J3 = J4
      N3 = N3 + N
  200 N4 = N4 + N
      N1 = N1 + N
      N2 = N2 + N
      J0 = J0 - 1
      J = J + N + 1
      J2 = J + N
  300 J4 = J4 + N
      DENOM = A(J)
      IF (DENOM .EQ. 0.0) GO TO 51
   60 A(J) = A(J)/DENOM
      DET = DET*DENOM
      LT = J - N + 1
      DO 400 J1 = LT,J
  400 B(J1) = B(J1)/DENOM
      J0 = JK
      J2 = J - N
      J4 = J - N + 1
      N2 = J2 - N
      DO 600 L1 = 1,JK
      J3 = J4
      N3 = N2 + 1
      N4 = N2 + N
      DO 500 L = 1,J0
      AMULT = A(J2)
      DO 401 J1 = N3,N4
      A(J1) = A(J1) - AMULT*A(J3)
      B(J1) = B(J1) - AMULT*B(J3)
  401 J3 = J3 + 1
      J3 = J4
      J2 = J2 - N
      N3 = N3 - N
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  500 N4 = N4 - N
      N2 = N2 - N
      J0 = J0 - 1
      J = J - N - 1
      J2 = J - N
  600 J4 = J4 - N
      IE = 1
  703 RETURN
   51 IE = 0
      GO TO 703
      END
```

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