A STUDY OF RECURRING DECIMALS
AND RELATED TOPICS

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CHAPTER I

INTRODUCTION AND STATEMENT OF THE PROBLEM

Introduction

The study of the real numbers has held the interest of man for thousands of years. From the time of Pythagoras it has been known that not all points on a line can be represented by the ratio of two integers. This fact was the impetus for extending the rational number system. The new numbers were named the irrationals and their presence allowed one to say that each point on a line can be denoted by a unique number. This number is called a real number and it will be either rational or irrational.

A study of the real numbers can be done by considering their decimal representation. This gives rise to many interesting questions about the digits used to form their representations. For example, $1/4 = .25$, but can it be expressed as an infinite decimal other than $.25000\cdots$? Also, $.333\cdots$ can be expressed as $1/3$, but can the real numbers $.5363636\cdots$ and $.12345\cdots$ be expressed in the form of a ratio between two integers? If so, how?

The study of representing the integers in bases other than base ten is one of the distinctive characteristics of "modern" elementary mathematics programs found in schools today. If the question is asked, "Why is this studied?," the answer is usually given that it helps the student understand the concept of place value. The place value concept is used to study the rationals as well as integers. Therefore, it is logical to ask what affect does the base of the numeration system have on the rational's "basimal"[1] representation. For example, 1/3 = .333... in base ten, but 1/3 = .25 in base twelve. Notice that in one base the basimal was an infinite repeating one, but in the other it terminated. Why?

Another example of the type of questions asked concerning recurring decimals is why does the decimal representation of 1/7 have a period of six digits and 1/11 has only a two digit period instead of the maximum possibility of ten?

Statement of the Problem

A teacher is reluctant to introduce concepts in which he has had little or no preparation. In his experience as an undergraduate, the secondary teacher finds that the vast majority of his course work is at such a level that it is not applicable to the teaching level he will encounter. If the prospective teacher is fortunate enough to have had a number theory course, he will have some material at his disposal that can be adapted for use by his future students. I. A. Barnett [6] of the University of Cincinatti stated his opinion quite strongly in the American Mathematical Monthly by saying that a course in "... the theory of numbers should be required not only of all mathematics majors, but also of all prospective teachers of elementary-school arithmetic as well as teachers of high-school algebra and geometry." He started his

[1]The author will use the word "basimal" when referring to a numeral from a general base system and will use the word "decimal" when referring to a numeral from the base ten system of numeration.

article by quoting from Hardy's essay "A Mathematician's Apology":

> "The elementary theory of numbers should be one of the very best subjects for early mathematical instruction. It demands very little previous knowledge; its subject matter is tangible and familiar; the processes of reasoning which it employs are simple, general and few; and it is unique among the mathematical sciences in its appeal to natural human curiosity. A month's intelligent instruction in the theory of numbers ought to be twice as instructive, twice as useful, and at least ten times as entertaining as the same amount of Calculus for Engineers."

The purpose of this paper is not to write a text book, but to take one facet of the real numbers, that of recurring decimals, and develop material that could be used in a seminar at the college senior level. The intention is to bring together in one volume certain material that has been written on the subject and topics related to it so the student could have the experience of "using" his mathematical knowledge. The level or difficulty should increase as the reader progresses through the paper. It is expected that many junior and senior high school students, as well as their teachers, will be able to comprehend much of the material of this paper.

Procedure

A survey and analysis of the published results concerning recurring decimals and related topics was made. The Mathematical Review, bibliographies of textbooks and bibliographies of published papers served as primary tools for locating source papers. The material was analyzed and is presented in an expository manner. The material is also organized in an increasing sequence of difficulty. Chapter II provides an introduction to recurring decimals and is intended for the junior high school reader. Chapter III points out many of the properties that recurring decimals possess. Although a few topics of elementary number theory are used, the explanation should be clear to the better senior high student. Chapters IV, V, and part of VI should be understandable to the conscientious college undergraduate. The remainder of Chapter VI is for the student possessing the mathematical maturity of a beginning graduate student.

Scope and Limitations

The published material concerning recurring decimals and related topics is quite extensive, but very uncorrelated. The writer could find no record of the subject being correlated for the various audiences mentioned above. The paper, therefore, will be limited by the level of the intended readers of this paper.

It was the intent of the writer to write a paper which was self-contained with respect to the mathematical background of the different level of readers. For example, some of the elementary results depend on the properties of congruences; therefore, a listing of these properties will be given along with a reference as to where the proofs may be found.

While the development of the material in the paper did not follow the historical development of the subject, the writer has made an effort to show how the subject has evolved.

Expected Outcomes

It is expected that as a result of reading this paper an individual will become aware of how a topic in mathematics grows as mathematicians continue to find the reasons behind the phenomena within the topic. It is also expected that junior and senior high school teachers will find material that can be used as enrichment in their courses and that students studying elementary number theory will be able to understand how the basic theorems of the course can be used to prove theorems about recurring decimals. Finally, it is hoped that this material will stimulate the reader's interest in mathematics.

CHAPTER II

ELEMENTARY INTRODUCTION TO RECURRING DECIMALS

The Division Algorithm and Rational Numbers

Since the system of rational numbers is used throughout the paper, it seems wise to review some definitions and basic properties of this system. Also, since the division algorithm is alluded to later on in the paper, it will be discussed at this point.

Theorem 2.A.[1] (The Division Algorithm) For any two positive integers a and b, there exist unique non-negative (positive or zero) integers q and r with $0 \leq r < b$ such that $a = bq + r$.

The following examples should give the reader a feeling for the division algorithm.

(1) $a = 19$, $b = 5$ implies $q = 3$, $r = 4$ since $19 = 5 \cdot 3 + 4$.

(2) $a = 57$, $b = 12$ implies $q = 4$, $r = 9$ since $57 = 12 \cdot 4 + 9$.

(3) $a = 13$, $b = 17$ implies $q = 0$, $r = 13$ since $13 = 17 \cdot 0 + 13$.

(4) $a = 36$, $b = 9$ implies $q = 4$, $r = 0$ since $36 = 9 \cdot 4 + 0$.

---

[1]The theorems in this paper will be numbered by chapter where the chapter number is followed by either a number or a letter. Those with a number are considered a main part of the paper and will be proved. Those with a letter are background material from number theory and their proofs can be found in most standard textbooks on the subject.

The reader will note that the third example can be generalized such that if $a < b$ then $q = 0$ and $r = a$. Also, it should be noted that the division algorithm as stated above is not as general as found in most textbooks. The condition that a and b be positive integers can be weakened such that they could be any integer, and as a result q could be any integer and $0 \leq r < |b|$. (i.e., $a = -17$, $b = 4$ implies $q = -5$, $r = 3$ since $-17 = 4 \cdot (-5) + 3$). For the purpose of this paper it will not be necessary to consider the more general conditions.

It is the division algorithm that makes the process of division of one positive integer by another a unique process. All of the following examples are mathematically correct, but only one of them can be accepted if division is to be unique.

56	57	54
7)400	7)400	7)400
35	35	35
50	50	50
42	49	28
8	1	22
$400 = 7 \cdot 56 + 8$	$400 = 7 \cdot 57 + 1$	$400 = 7 \cdot 54 + 22$

It is the second example which satisfies the division algorithm. Therefore, it is the one taken as the unique answer to 400 divided by 7.

The study of rational numbers is very broad, and several excellent books have been written for the neophyte mathematics students, such as Niven [24] and Rademacher [26]. Only those definitions, theorems, and properties which relate directly to the topic of this paper will be discussed.

<u>Definition 2.1</u>. A positive rational number is an ordered pair (i.e., (a, b)) of positive integers. This ordered pair is usually expressed as a/b where a is called the numerator and b is called the denominator.

The expression a/b is usually referred to as the rational fraction form or the fractional form of the rational number.

Definition 2.2. The rational numbers a/b and c/d are said to be equialent if and only if ad = bc.

Definition 2.3. A rational number is said to be in lowest terms if and only if the numerator and denominator have no common divisor other than 1. When a general rational number is referred to in the form a/b, it will be understood to be in lowest terms.

One of the most common interpretations of rational numbers is that in which the ordered pair represents a quotient, i.e., $a/b = a \div b$. This interpretation follows from the definition of division since $a = b\cdot(a/b)$.

Terminating and Non-terminating Decimals

The reader is probably familiar with decimal notation for representing rational numbers, but for completeness of the topic it should be reviewed in light of the first section of this chapter.

Definition 2.4. The terminating decimal $.d_1d_2\ldots d_k$ is equal to the rational number whose numerator is the positive integer $d_1d_2\ldots d_k$, where the d_i's are the digits of the integer, and the denominator is 100...0 (k zeros) or 10^k.

It follows from this definition that since 1/8 = 125/1000 then 1/8 = .125. Now this terminating decimal representation could also be found by interpreting 1/8 to mean $1 \div 8$ and use the standard algorithm for decimal division.

For example,

```
     .125
  8)1.000
    8
    --
    20
    16
    --
     40
     40
     ==
```

A second example is $\frac{37}{400} = \frac{37 \cdot 25}{400 \cdot 25} = \frac{925}{10000} = .0925$ and

```
        .0925
  400)37.0000
      36 00
      ------
       1 000
         800
        ----
         2000
         2000
         ====
```

Now consider the fraction 4/7. Does there exist a positive integer K such that $7 \cdot K = 100...0$? Since $100...0 = 10^n = (2 \cdot 5)^n = 2^n 5^n$, there does not exist a K such that $7 \cdot K = 2^n 5^n$. This implies that 4/7 cannot be expressed as a terminating decimal. How is this fact reflected when the algorithm for division is used?

```
      .571428 57...
  7)4.00000000...
    3 5
    ---
     50
     49
     --
      10
       7
      --
       30
       28
       --
        20
        14
        --
         60
         56
         --
          40
          35
          --
           50
           49
           --
            1
```

The reader has noted that the division process of the example appears not to terminate, (i.e., a remainder of zero has not been obtained), but continues indefinitely. Recalling the division algorithm, it is seen that the set of possible remainders when dividing by 7 is {0, 1, 2, 3, 4, 5, 6}. In the example above, the remainders appeared in the following order: 4, 5, 1, 3, 2, 6, 4, 5, 1, ... The fact that the remainders repeat appears to have a bearing on whether or not the decimal representation of the rational number is terminating. This will be the case, and more will be said later in this chapter about this example.

The discussion has been leading to the following theorem:

Theorem 2.1. The rational number a/b has a terminating decimal expansion if and only if the integer b has no prime factors other than 2 or 5.

Proof: First it will be assumed that a/b has a terminating decimal expansion and then show that b has no prime factors other than 2 or 5. From Definition 2.4, it follows that

$$\frac{a}{b} = .d_1d_2 \ldots d_n = \frac{d_1d_2 \ldots d_n}{10^n} = \frac{d_1d_2 \ldots d_n}{2^n \cdot 5^n}.$$

Now if this fraction is not in lowest terms, the reducing of it to lowest terms will not alter the fact that b will not have any prime factors other than 2 or 5. This completes the proof in the only if direction.

For the if part assume b has no prime factors other than 2 or 5. That is, b is equal to a positive integer of the form $2^m \cdot 5^n$. Before continuing with the proof, consider the example $\frac{a}{b} = \frac{7453}{12500} = \frac{7453}{2^2 \cdot 5^5}$.

To convert this to a decimal, simply change it to a fraction which has a denominator that is a power of 10. This can be achieved by multiplying both numerator and denominator by 2^3:

$$\frac{7453}{2^2\cdot 5^5} = \frac{7453\cdot 2^3}{2^5\cdot 5^5} = \frac{59624}{10^5} = .59624.$$

This argument can be generalized from this special case in the following way. Suppose that b is of the form $2^m\cdot 5^n$, where m and n are positive integers or zero. Now, from the law of trichotomy for non-negative integers, one of the three cases, $n = m$, $n < m$, and $n > m$, must hold. When $n = m$ then $2^m\cdot 5^n = 2^m\cdot 5^m = 10^m$, therefore, $a/b = a/10^m$, and the terminating decimal is found by inserting the decimal point in the correct place. The second case where $n < m$ is handled by multiplying both numerator and denominator of the fraction by 5^{m-n}:

$$\frac{a}{b} = \frac{a}{2^m\cdot 5^n} = \frac{a\cdot 5^{m-n}}{2^m\cdot 5^n\cdot 5^{m-n}} = \frac{a\cdot 5^{m-n}}{2^m\cdot 5^m} = \frac{a\cdot 5^{m-n}}{10^m}.$$

Since $m - n$ is positive, 5^{m-n} is an integer, and so $a\cdot 5^{m-n}$ is also an integer, say c. Hence the fraction can be written $a/b = c/10^m$ and the terminating decimal can now be written. The case where $n > m$ is similarly handled.

The following examples are given to illustrate the technique of the proof.

$$\frac{7}{200} = \frac{7}{2^3\cdot 5^2} = \frac{7\cdot 5}{2^3\cdot 5^3} = \frac{35}{10^3} = .035$$

$$\frac{11}{625} = \frac{11}{5^4} = \frac{11\cdot 2^4}{5^4\cdot 2^4} = \frac{176}{10^4} = .0176$$

Recurring[1] Decimals

Rational fractions can now be separated into two types, i.e., those with terminating decimals and those with infinite decimals. It will now be established that each such infinite decimal has a repeating pattern. For example: $3/11 = .272727\cdots$, and $1979/3300 = .59969696\cdots$. For convenience, the notation of placing a dot over the first and last digit of the set of digits which are repeated will be used to denote a recurring decimal: $3/11 = .\dot{2}\dot{7}$, $1979/3300 = .59\dot{9}\dot{6}$, $1/3 = .\dot{3}$, $41/333 = .\dot{1}2\dot{3}$, etc. A second standard notation is the placing of a bar over the set of digits which are repeating: $3/11 = .\overline{27}$, $1979/3300 = .59\overline{96}$, $1/3 = .\overline{3}$, $41/333 = .\overline{123}$, etc. The writer has elected to use the first notation since it is more convenient when the repeating part is quite large. One more word on terminology is needed at this point. The repeating part is referred to as the "period" or "repetend" of the recurring decimal. The writer has elected to use the word "period" for the repeating part.

Recalling the example of where the decimal expansion of $4/7$ was found, it was noted, the set of possible remainders was $\{0, 1, 2, 3, 4, 5, 6\}$. In the actual division process, the remainders occurred in the order 5, 1, 3, 2, 6, 4 and then started to repeat. Therefore, the quotient also started to repeat. The possible remainder 0 could not occur in the process of finding a recurring decimal, since this would terminate the process.

Since $5/6 = .8\dot{3}$, it is noted that first, the period is only one

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[1] The author has elected to use the term recurring instead of periodic or repeating.

digit in size; and second, the period did not start immediately in the place to the right of the decimal point. This example illustrates the fact that not all possible remainders are used before a repeat takes place. Also, since this example has a non-repeating part, it would not be a "pure" recurring decimal. (A "pure" recurring decimal is one whose period starts in the place immediately to the right of the decimal point).

Considering the general case, a/b, the set of possible remainders due to the division algorithm is $\{0, 1, 2, 3, \ldots, b - 2, b - 1\}$, and so a recurrence of the division process is certain, unless the remainder zero occurs and the process terminates. When the division process recurs, a cycle is started and the result is a recurring decimal.

The above argument is half of the proof to the following:

<u>Theorem 2.2</u>. Any rational fraction a/b is expressible as a terminating decimal or an infinite recurring decimal; conversely, any decimal which is either terminating or infinite recurring can be expressed in the form a/b.

The converse deals with two types of decimals, terminating and infinite recurring. The terminating decimals were taken care of by Theorem 2.1. Before the recurring decimals are considered, it would be wise to look at the method in one particular case and then generalize the method to fit any case.

Consider the infinite recurring decimal $x = .73\dot{4}2\dot{6}$. Now the object will be to multiply both members of the equation first by one number and then by another; these numbers will be chosen so that the difference of the two products will be an integer. In this example the numbers 10^5

and 10^2 will serve the purpose because

$$10^5 \cdot x = 100{,}000 \cdot x = 73426.\dot{4}2\dot{6}$$

$$\text{and } 10^2 \cdot x = \quad 100 \cdot x = \quad 73.\dot{4}2\dot{6}$$

so that the difference is $99900 \cdot x = 73353$. Therefore,

$$x = \frac{73353}{99900} = \frac{24451}{33300}$$

which exhibits the fact that x is a rational number.

In the generalization of this method it will be shown that the numbers 10^5 and 10^2 were not "pulled out of the hat" but were chosen systematically. Now any recurring decimal between 0 and 1 can be written in the form

(1) $x = .a_1 a_2 \ldots a_s \dot{b}_1 b_2 \ldots \dot{b}_t$,

where a_1, a_2, a_s represent the s consecutive digits in the non-repeating part and b_1, $b_2 \ldots$, b_t represent the t digits in the period. (In the above example $s = 2$, $t = 3$; $a_1 = 7$, $a_2 = 3$, $b_1 = 4$, $b_2 = 2$, and $b_2 = 6$). Now x is multiplied first by 10^{s+t}, then by 10^s, and then the difference is found; i.e.,

$$10^{s+t} \cdot x = a_1 a_2 \ldots a_s b_1 b_2 \ldots b_t + .\dot{b}_1 b_2 \ldots \dot{b}_t,$$

$$10^s \cdot x = a_1 a_2 \ldots a_s + .\dot{b}_1 b_2 \ldots \dot{b}_t$$

$$(10^{s+t} - 10^s)x = a_1 a_2 \ldots a_s b_1 b_2 \ldots b_t - a_1 a_2 \ldots a_s.$$

Therefore,

(2) $$x = \frac{a_1 a_2 \ldots a_s b_1 b_2 \ldots b_t - a_1 a_2 \ldots a_s}{10^{s+t} - 10^s}$$

which shows x to be rational, since both the numerator and denominator are integers.

An alternate method, which is similar to the process used in Theorem 2.2, is to multiply the number by 10^k, where k is the number of digits in the period, and then subtract the number from this product. For example, consider $.35\dot{6}\dot{2}$;

$$\begin{array}{r}10^2 \cdot x = 35.62\dot{6}\dot{2} \\ x = .35\dot{6}\dot{2} \\ \hline 99x = 35.27\end{array}$$

$$x = \frac{35.27}{99} = \frac{3527}{9900}$$

Note, the difference did not result in an integer. Therefore, it was necessary to multiply both numerator and denominator by 100, so that they both will be integers.

<u>Theorem 2.3</u>. Any rational number of the form a/b, where a and b are in lowest terms and a < b, is equal to a pure recurring decimal if and only if b and 10 have no common factors.

Proof: Assume a/b is equal to a pure recurring decimal, i.e.,

$$\frac{a}{b} = .\dot{d}_1 d_2 \ldots \dot{d}_t.$$

Then using the algebraic process from Theorem 2.2, it is seen that,

$$\frac{a}{b} = \frac{d_1 d_2 \ldots d_t}{99\ldots 9}.$$

Since a/b is in lowest terms, there exists an integer h such that bh = 99...9 (t digits). Now 99...9 is not divisible by 2 or 5, therefore b is not divisible by 2 or 5. This implies b and 10 have no common factor. This completes the proof in one direction. The proof going in the other direction will be done by contradiction. Assume a/b is <u>not</u> equal to a pure recurring decimal, i.e.,

$$\frac{a}{b} = .a_1 a_2 \ldots a_s \dot{b}_1 b_2 \ldots \dot{b}_t$$

Therefore,

$$\frac{10^s a}{b} = a_1 a_2 \ldots a_s + .\dot{b}_1 b_2 \ldots \dot{b}_t,$$

(3) $$= a_1 a_2 \ldots a_s + \frac{b_1 b_2 \ldots b_t}{99\ldots 9}$$

Let $a_1a_2\ldots a_s$ and $b_1b_2\ldots b_t$ equal the integers h and k respectively. Equation (3) becomes

$$\frac{10^s a}{b} = h + \frac{k}{99\ldots9},$$

or

$$\frac{a}{b} = \frac{h}{2^s5^s} + \frac{k}{2^s5^s(99\ldots9)}.$$

Reducing the fractions of their 2 and 5 factors gives,

$$\frac{a}{b} = \frac{h'}{2^u5^v} + \frac{k'}{2^m5^n(99\ldots9)}.$$

Adding the fractions gives

$$\frac{a}{b} = \frac{h'2^{m'}5^{n'} + k'2^{u'}5^{v'}}{2^x5^y(99\ldots9)}, \tag{4}$$

where x is the larger of u and m and y is the larger of v and n. Also, $m' = x-m$, $n' = y-n$, $u' = x-u$, and $v' = x-v$. It should be noted that either $m' = 0$ or $u' = 0$ and either $n' = 0$ or $v' = 0$. Now, in reducing the right-hand side of (4), it is seen that 2 does not divide the numerator, since 2 divides one of the terms but not the other. Similarly, 5 does not divide the numerator. Therefore, in reducing the fraction to a/b, the factors 2^x and 5^y will remain in the denominator. This implies b and 10 have a common factor. This is the contradiction needed.

Since the remainder of the paper will be concerned with pure recurring decimals, a general rational number of the form a/b will be assumed to be in lowest terms and b and 10 have no common factors.

<u>Corollary 2.4.</u> Every pure recurring decimal is equal to the fraction whose numerator is formed by the period and the denominator is composed of as many 9's as there are digits in the period.

Proof: If x is a pure recurring decimal then $s = 0$ in (1). This implies that (2) becomes

$$x = \frac{b_1 b_2 \ldots b_t}{10^t - 1} = \frac{b_1 b_2 \ldots b_t}{99\ldots9} \quad \text{(t digits)}$$

which is precisely what the corollary states.

This theorem can be used to convert any infinite recurring decimal to its fractional equivalent, provided the recurring decimal is first changed to a pure recurring decimal. For example;

$$x = .73\dot{4}2\dot{6}$$

$$10^2 x = 73.\dot{4}2\dot{6} = 73 + \frac{426}{999} = \frac{73 \cdot 999 + 426}{999} = \frac{73 \cdot (1000-1) + 426}{999}$$

$$x = \frac{(73000 - 73) + 426}{99900} = \frac{72927 + 426}{99900} = \frac{73353}{99900}.$$

It is interesting to note that while the method of Theorem 2.2 is neither new nor difficult, it did not appear in elementary textbooks until just recently. A second method found in most college algebra textbooks, which have a section on geometric progressions, is as follows:

$$x = .73\dot{4}2\dot{6} = .73 + .00\dot{4}2\dot{6} = .73 + 426(.00\dot{0}0\dot{1})$$

$$= \frac{73}{100} + \frac{426}{100}(.\dot{0}0\dot{1}) = \frac{73}{100} + \frac{426}{100}\left[\frac{1}{10^3} + \frac{1}{10^6} + \ldots\right]$$

$$= \frac{73}{100} + \frac{426}{100} \cdot \frac{1/1000}{1-1/1000} = \frac{73}{100} + \frac{426}{100} \cdot \frac{1}{999}$$

$$= \frac{73 \cdot 999 + 426}{100 \cdot 999} = \frac{73353}{99900} = \frac{24451}{33300}.$$

While this method gives a good example of an application for infinite geometric progressions, it does not lend itself to use by a student who has not had a course comparable to college algebra.

Terminating Decimals Written as Recurring Decimals

So far, in this chapter it has been established that some rational numbers can be expressed as terminating decimals, whereas other rational numbers become infinite or non-terminating decimals. Curiously enough, every terminating decimal (except zero) can be expressed in a non-terminating form. Of course, this can be done in a very obvious way when .75 is written as .75000···, i.e., with an infinite succession of zeros. But, apart from this obvious process, there is another way that is a little surprising and certainly more interesting.

Consider the following: $1/9 = .111\cdots$, $2/9 = .222\cdots$, $3/9 = 1/3 = .333\cdots$ and so on until $8/9 = .888\cdots$. If the inductive process is carried one more step the strange-looking result is

(1) $$9/9 = 1 = .999\cdots.$$

Now equation (1) can be shown to be true by use of the method found in Theorem 2.2.

$$\begin{aligned} x &= .999\cdots \\ \text{implies } 10x &= 9.\dot{9} \\ x &= .\dot{9} \\ \hline 9x &= 9 \qquad \text{or } x = 1. \end{aligned}$$

Thus equation (1) is true.

This result allows any terminating decimal to be written as an infinite recurring decimal as illustrated by the following example.

$$\begin{aligned} .376 &= .375 + .001(1) \\ &= .375 + .001(.999\cdots) \\ &= .375 + .000999\cdots \\ &= .375999\cdots \end{aligned}$$

Conversely, if an infinite recurring decimal has a period of the single digit 9, then it can be converted to a terminating decimal as illustrated by the following example:

$$.43\dot{9} = .43 + .00\dot{9} = .43 + .01(.\dot{9}) = .43 + .01 = .44$$

The uniqueness of the decimal representation of a given rational number depends on a choice of notation. For, in addition to writing .44 as .43999···, it could also be written as .440, .4400, .4400···. These, however, are such trivial variations of .44 itself, that they would not be considered as different representations. It will be the practice of the writer to use $.43\dot{9}$ as the infinite decimal representation of .44 instead of $.44\dot{0}$ throughout most of the thesis, but will use the alternate notation whenever it will be expedient.

Recurring Decimals in Other Bases

Recall the process for changing 1/2 to its decimal representation, that is, multiply both numerator and denominator by 5 such that

$$1/2 = 1 \cdot 5/2 \cdot 5 = 5/10 = .5.$$

Therefore, .5 can be interpreted as 5 divided by 10, but 10 implies one base in the decimal numeration system. This would mean that .5 implies 5 is divided by one base.

Consider the question: What is the basimal representation of 1/2 in base eight? In order to answer this a person needs to find the number that, when multiplied by 2, will give one base for the product. In this case it would be four since $[2 \cdot 4 = 10]_{eight}$. Therefore, the answer to the question would be found as follows;

$$[1/2 = 1 \cdot 4/2 \cdot 4 = 4/10 = .4]_{eight}$$

Applying the same type of reasoning it is found that

$$.5_{ten} = .4_{eight} = .6_{twelve} = .1_{two}.$$

If a person attempts to change 1/2 in base five to its equivalent basimal representation, he will have difficulty since there does not exist an integer that when multiplied by two will give the base. This would imply that $(1/2)_{five}$ would have an infinite recurring basimal representation in any odd base.

Reflecting on Theorem 2.1, it can be seen that the numbers two and five are factors of the base ten. If the base of the numeration system had been six instead of ten then the factors of "b" would have to be two and three in order for the theorem to hold in base six. Therefore, base twelve would imply factors of 2 and 3, but base eight would imply only the factor 2.

Now consider some of the recurring decimals in base ten and their representation in other bases.

$$\left[\frac{1}{3}\right]_{ten} = \left[\frac{1}{3}\right]_{twelve} = \left[\frac{1}{3} \cdot \frac{4}{4} = \frac{4}{10} = .4\right]_{twelve}$$

Hence, $[.333\cdots]_{ten} = [.4]_{twelve}$.

$$\left[\frac{1}{9}\right]_{ten} = \left[\frac{1}{9}\right]_{twelve} = \left[\frac{1}{3^2} \cdot \frac{4^2}{4^2} = \frac{14}{10^2} = .14\right]_{twelve}$$

Hence, $[.111\cdots]_{ten} = [.14]_{twelve}$

It should be obvious by now, that while some recurring decimals in base ten become terminating basimals in base twelve not all of them will. For example, 1/11 = .090909··· in base ten, but it becomes .111··· in base twelve. Before illustrating this example, the author needs to define what symbols will be used to denote ten and eleven in base twelve. The author's choice is "t" for ten and "e" for eleven.

$$\left[\begin{array}{r} .1 \\ e\overline{)1.0} \\ e \\ \hline 1 \end{array}\right]_{twelve} \qquad \left[\frac{1}{e} = .111\cdots\right]_{twelve}$$

Also the terminating decimal $[.2]_{ten} = [.\dot{2}49\dot{7}]_{twelve}$ as illustrated below;

$$\left[\begin{array}{r} .2497 \\ 5\overline{)1.0000} \\ \underline{t} \\ 20 \\ \underline{18} \\ 40 \\ \underline{39} \\ 30 \\ \underline{2e} \\ 1 \end{array}\right]_{twelve} \qquad \left[\frac{1}{5} = .\dot{2}49\dot{7}\right]_{twelve}$$

A statement is frequently made that a base of twelve, i.e., the duodecimal system, would have made a better numeration system than base ten. What would be the reasoning behind such a statement? One of the strongest arguments is based on the British system of measure. The parts of a foot could be denoted quite simply as

$$[1 \text{ inch} = .1 \text{ foot}]_{twelve}$$

$$[2 \text{ inches} = .2 \text{ foot}]_{twelve}$$

$$\vdots \qquad \vdots$$

$$[9 \text{ inches} = .9 \text{ foot}]_{twelve}$$

$$[t \text{ inches} = .t \text{ foot}]_{twelve}$$

$$[e \text{ inches} = .e \text{ foot}]_{twelve}$$

$$[10 \text{ inches} = 1.0 \text{ foot}]_{twelve}$$

Also, their monetary system of pence and shilling lends itself to the duodecimal system, since twelve pence is a shilling.

From a straight mathematical viewpoint the fact that twelve is an "abundant" number, i.e., it has more divisors than any number less than it gives rise to more fractions less than one, that would have a terminating basimal representation. Other examples of abundant numbers are

24, 36, 60, 120, and 360. The number 144, which would be one base squared, just misses abundancy, being excelled by 120. Compared with 144, the number 100 is relatively poverty-stricken in this respect--which is why the metric system is said to be a poor one by some people [1].

Another interesting example is to compare the conversion of sixty-fourths in decimals to their representation in duodecimals.

Fraction	Decimal	Duodecimal
25/64	.390625	.483
27/64	.421875	.509
29/64	.453125	.553
31/64	.484375	.599
33/64	.515625	.623

In light of Theorem 2.1, the reader would expect the basimal representation to go from six digits to three, since twelve contains two factors of 2.

It is interesting to note that, according to Aitken [1], "... the decimal system might be rated at about 65 or less, if we assign 100 to the duodecimal."

To carry the discussion a step further, why not use 60 for the base? It is an abundant number and also has 5 as a divisor! The obvious reason being the operational tables would be prohibitive in size. Certainly, the more prime factors of the base, the more rational numbers with terminating basimal representation. The "utopia" base would be the integer that is the product of "all" primes. If this were possible then "all" rationals would have a <u>terminating</u> basimal representation!

Consider now the method of Theorem 2.2 in light of some base other than ten. For example, the following process is done in base <u>eight</u>.

$$
\begin{aligned}
x &= .3\dot{6}\dot{2} \\
(10^3)x &= 362.\dot{6}\dot{2} \\
10x &= 3.\dot{6}\dot{2} \\
(1000-10)x &= 357 \\
770x &= 357 \\
x &= 357/770
\end{aligned}
$$

The method of Corollary 2.4 would be as follows:

$$
\begin{aligned}
x &= .3\dot{6}\dot{2} \\
10x &= 3.62 = 3 + 62/77 = (3\cdot 77 + 62)/77 \\
x &= (275 + 62)/770 = 357/770
\end{aligned}
$$

It is the author's hope that the reader can now see how theory developed in base ten can be generalized to any base. For the most part, the remainder of this paper will be written with the understanding that the base is ten. It will be noted if the base is other than ten.

Historical Notes

It is interesting to note that while the facts from the theorems in this chapter are well known, their origin in print has been documented by L. E. Dickson [12]. His three volume work on the history of the theory of numbers is very complete in giving the original source of proofs. Unfortunately, the author was unable to obtain copies of all the original works since they were published during the eighteenth and nineteenth centuries in Europe. However, the proofs have been, for the most part, modernized and published in English during the last fifty years. The author has footnoted those proofs which are not commonly found in books and will give Dickson's reference to the original proof of the properties of recurring decimals discussed in this paper.

The facts of Theorem 2.2 were first noted by John Wallis in 1685. The fact that a pure recurring decimal is equal to the period divided by 99...9 where the period has k digits and there are k digits 9 was noted by John Robertson in 1768. The technique used in the example following Corollary 2.4 was noted by J. H. Lambert ten years before Robertson's proof. A. Filkel in 1785 published a paper concerning recurring decimals in other bases.

CHAPTER III

FUNDAMENTAL PROPERTIES OF RECURRING DECIMALS

Topics from Elementary Number Theory

In this chapter the reader will find it necessary to have some knowledge of congruences and the Euler ϕ-function. The writer has given definitions, theorems, and some examples to enable the reader to understand how congruences and the Euler ϕ-function are used in the remainder of the chapter.

Definition 3.1. If m is positive and m divides (a-b), then a is said to be congruent to b modulo m and is written $a \equiv b \pmod{m}$.

For example, $23 \equiv 7 \pmod{8}$, $9 \equiv -3 \pmod{4}$. If a is divided by m to obtain q and r such that $a = mq + r$ with $0 \leq r < m$, then m divides (a-r) and $a \equiv r \pmod{m}$. Therefore, a number is congruent, modulo m, to its remainder when it is divided by m.

Definition 3.2. If $a = nq + r$ with $0 \leq r < m$, then r is called the least residue of a modulo m.

Definition 3.3. The set of integers 0, 1, 2, ..., m - 1 is called the least residue system modulo m. Any set of m integers, no two of which are congruent modulo m, is called a complete residue system modulo m.

Theorem 3.A. If $a \equiv b \pmod{m}$, then

(i) $a + c \equiv b + c \pmod{m}$, and

(ii) $ac \equiv bc \pmod{m}$ for any integer c.

Theorem 3.B. If $a \equiv b \pmod{m}$, then $a^n \equiv b^n \pmod{m}$ for any positive integer n.

Definition 3.4. If d is the largest common divisor of a and b, it is called the greatest common divisor of a and b and is denoted by (a,b).

Definition 3.5. If (a,b) = 1, then a and b are said to be relatively prime or coprime.

Definition 3.6. The number of positive integers, not exceeding m, which are relatively prime to m is designated by the function $\phi(m)$. $\phi(m)$ is called the Euler ϕ-function after its originator, Leonard Euler (1707-1783).

Definition 3.7. Any set of $\phi(m)$ integers which are relatively prime to m and which are mutually incongruent (no two are congruent), modulo m, is called a reduced residue system modulo m.

According to these definitions $\phi(12) = 4$, and {1, 5, 7, 11}, {-11, 17, -5, 35} are reduced residue systems modulo 12. Also, $\phi(1) = 1$, $\phi(3) = 2$, and $\phi(7) = 6$. Also, $\phi(p) = p - 1$, since all positive integers less than the prime p are relatively prime to p.

Theorem 3.C. (The Euler-Fermat Theorem). If (a, m) = 1, then $a^{\phi(m)} \equiv 1 \pmod{m}$.

Theorem 3.D. If p is prime, p does not divide a, and p - 1 is the smallest positive value of e such that $a^e \equiv 1 \pmod{p}$, then $a, a^2, \ldots, a^{p-1}$ form a reduced residue system modulo p.

Now, consider the example where $a = 10$ and $p = 23$.

$$10 \equiv 10 \pmod{23}$$

$$10^2 \equiv 100 = 4 \cdot 23 + 8 \text{ implies } 10^2 \equiv 8 \pmod{23}$$

$$10^3 \equiv 80 \equiv 11 \pmod{23}$$

$$10^4 \equiv 8^2 \equiv 18 \equiv -5 \pmod{23}$$

$10^5 \equiv 88 \equiv -4 \pmod{23}$	$10^{14} \equiv 12 \equiv -11 \pmod{23}$
$10^6 \equiv -40 \equiv 6 \pmod{23}$	$10^{15} \equiv -18 \equiv 5 \pmod{23}$
$10^7 \equiv -32 \equiv -9 \pmod{23}$	$10^{16} \equiv 2^2 \equiv 4 \pmod{23}$
$10^8 \equiv 48 \equiv 2 \pmod{23}$	$10^{17} \equiv 40 \equiv -6 \pmod{23}$
$10^9 \equiv 20 \equiv -3 \pmod{23}$	$10^{18} \equiv -14 \equiv 9 \pmod{23}$
$10^{10} \equiv -30 \equiv -7 \pmod{23}$	$10^{19} \equiv 44 \equiv -2 \pmod{23}$
$10^{11} \equiv -24 \equiv -1 \pmod{23}$	$10^{20} \equiv -20 \equiv 3 \pmod{23}$
$10^{12} \equiv 36 \equiv -10 \pmod{23}$	$10^{21} \equiv 30 \equiv 7 \pmod{23}$
$10^{13} \equiv 15 \equiv -8 \pmod{23}$	$10^{22} \equiv (-1)^2 \equiv 1 \pmod{23}$

Thus, $10, 10^2, \ldots, 10^{22}$ form a reduced residue system modulo 23, since each of the numbers in the set

$\{10, 8, 11, -5, -4, 6, -9, 2, -3, -7, -1, -10, -8, -11, 5, 4, -6, 9, -2, 3, 7, 1\}$

is mutually incongruent modulo 23 and there are $\phi(23) = 22$ elements in the set.

<u>Definition 3.6.</u> If k is the least positive integer such that $n^k \equiv 1 \pmod{r}$, then it is said that n <u>belongs to the exponent k</u> modulo r.

Note from the definition and the example above, it can be said that 10 belongs to the exponent 22 modulo 23, since $r = 23$, and $k = r - 1 = 22$. But the following example shows that the value of k does not have to be $r - 1$. Consider the powers of 2 modulo 7.

$$2 \equiv 2,\ 2^2 \equiv 4,\ 2^3 \equiv 1,\ 2^4 \equiv 2,\ 2^5 \equiv 4,\ 2^6 \equiv 1,\ 2^7 \equiv 2,\ \cdots.$$

Note, that 2 belongs to 3 modulo 7 and in general, $2^h \equiv 1 \pmod 7$ implies 3 divides h. This example motivates the following theorem.

Theorem 3.E. If n belongs to k modulo r and $n^h \equiv 1 \pmod r$, then k divides h.

The reader will note that when the Euler-Fermat theorem is applied to Theorem 3.E, the following corollary results.

Corollary 3.F. (i) If n belongs to k modulo r, then k divides $\phi(r)$. (ii) If n belongs to k modulo p where p is a prime, then k divides p-1.

Corollary 3.G. If q belongs to $\phi(r)$ mod r, then $q, q^2, q^3, \ldots, q^{\phi(r)}$ form a reduced residue system modulo r.

Theorem 3.H. If $q, q^2, \ldots, q^{p-1}$ forms a reduced residue system modulo p, then $q + q^2 + \ldots + q^{p-1} \equiv 0$ modulo p.

The reader will note in the example after Theorem 3.D, where $q = 10$ and $p = 23$, that $10^j \equiv -10^{j+11} \pmod{23}$ for $j = 1, 2, \ldots, 11$. Therefore,

$$\sum_{j=1}^{22} 10^j \equiv \sum_{j=1}^{11} 10^j + \sum_{j=12}^{22} 10^j \equiv -\sum_{j=12}^{22} 10^j + \sum_{j=12}^{22} 10^j \equiv 0 \pmod{23}.$$

Corollary 3.I. If q belongs to h, modulo p, where p is an odd prime, and $r_0, r_1, \ldots, r_{h-1}$, are the least positive residues of $g^0, g^1, \ldots, g^{h-1}$, then

$$\begin{aligned} r_0 + r_1 + \ldots + r_{h-1} &\equiv g^0 + g^1 + \ldots + g^{h-1} \\ &\equiv (g^h - 1)/(g - 1) \\ &\equiv 0 \pmod p. \end{aligned}$$

Consider the example where q = 2 and p = 7:

$$2^0 \equiv 1,\ 2^1 \equiv 2,\ 2^2 \equiv 4,\ 2^3 \equiv 1 \pmod 7.$$

Therefore,

$$\begin{aligned} 1 + 2 + 4 &\equiv 1 + 2 + 2^2 \\ &\equiv (2^3 - 1)/(2 - 1) \pmod 7 \\ &\equiv 0 \pmod 7 \end{aligned}$$

Theorem 3.J. Integers a and b are relatively prime, i.e., (a, b) = 1 if and only if there exists integers x and y such that 1 = ax + by.

It should be noted that, since the above theorem is an if and only if theorem, 1 = ax + by implies (a, b) = 1 and (x, y) = 1, or (a, y) = 1 and (x, b) = 1.

Theorem 3.K. If m belongs to h modulo r and belongs to k modulo s, and if (r, s) = 1, then m belongs to [h, k][1] modulo rs.

The following properties of the ϕ-function will be needed.

Theorem 3.L. (i) $\phi(p) = p - 1$, and $\phi(p^n) = p^n - p^{n-1} = p^{n-1}(p - 1)$, where p is prime. (ii) $\phi(pq) = (p - 1)(q - 1)$, where p and q are distinct primes. (iii) $\phi(n) = n(1 - 1/p)(1 - 1/q)\ldots(1 - 1/r)$, where $n = p^a q^b \ldots r^c$ and p, q, ..., r are prime and a, b, ..., c are positive integers.

Example: Let $n = 504 = 2^3 \cdot 3^2 \cdot 7$, then

$$\begin{aligned} \phi(504) &= 2^3 \cdot 3^2 \cdot 7(1 - 1/2)(1 - 1/3)(1 - 1/7) \\ &= 2^2 \cdot 3(1)(2)(6) = 144 \end{aligned}$$

---

[1] [h, k] denotes the least common multiple of h and k.

The Period of m/n

The two main objectives of this section will be first, to explain how the number of digits in the period of m/n can be found without actually finding the period and secondly, to show that this number depends only on the denominator and not the numerator. The material introduced in the last section will be used in the explanation.

<u>Theorem 3.1.</u> Two fractions m/n and r/n produce the same mantissa (purely decimal part) if and only if $m \equiv r \pmod{n}$.

Proof: Assume $m \equiv r \pmod{n}$ then n divides $(m - r)$ by definition. Therefore, $(m - r)/n$ is an integer which implies that the mantissae of m/n and r/n are the same. Now assume that m/n and r/n have the same mantissa, i.e.,

$$m/n = h + .\dot{a}_1 a_2 \ldots \dot{a}_s \text{ and } r/n = k + .\dot{a}_1 a_2 \ldots \dot{a}_s .$$

But
$$\frac{m}{n} - \frac{r}{n} = (h + .\dot{a}_1 a_2 \ldots \dot{a}_s) - (k + .\dot{a}_1 a_2 \ldots \dot{a}_s)$$

$$\frac{m-r}{n} = h - k$$

where h - k is an integer. Therefore, n divides $(m - r)$, which implies $m \equiv r \pmod{n}$.

For example, 31/7 and 3/7 differ by an integer, and therefore, $31 \equiv 3 \pmod{7}$. This implies the mantissae of 31/7 and 3/7 are the same, namely, $.\dot{4}2857\dot{1}$.

Consider the question of the number of different mantissae for a given denominator n. Now, if m/n is a fraction where m is greater than n, then by the division algorithm there exists an integer r such that $0 < r < n$ and $m \equiv r \pmod{n}$. Now, consider r/n, where $r < n$. How many fractions can be formed under this condition? The answer is $(n - 1)$.

But how many of them are in lowest terms? Recalling the definition of $\phi(n)$, it is found that $\phi(n)$ of them will be in lowest terms. The original question can now be answered by $\phi(n)$.

Corollary 3.2. There are $\phi(n)$ different mantissae for the same denominator n.

As an example consider $n = 6$. $\phi(6) = 2$, since $\{1, 5\}$ are the only numbers less than 6 and relatively prime to 6.

$$1/6 = .1\dot{6} \qquad 5/6 = .8\dot{3}$$

Therefore, $.1\dot{6}$ and $.8\dot{3}$ are the mantissae associated with a denominator of 6.

As a second example consider $n = 21$, and hence $1/21 = .\dot{0}4761\dot{9}$ is a mantissa. The positive integers less than 21 and relatively prime to 21 are 1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20. Therefore, there are $\phi(21) = 12$ different mantissae for the denominator.

Now, consider the question of the periodicity of non-terminating decimals. The determination of the number of digits in the period by actual division is frequently a long process. Using the theorems on number congruences from the previous section, this question is quickly answered.

It is known that the multiplying of m/n by 10^k will move the decimal point k places to the right. If k is so chosen that $10^k \equiv 1 \pmod{n}$, then the mantissa will not change, i.e., m/n and $10^k \cdot m/n$ will differ by an integer.

$$\frac{10^k m}{n} - \frac{m}{n} = \frac{m(10^k - 1)}{n}$$

$$= \frac{m(nq)}{n}\text{ ; since } 10^k \equiv 1 \pmod{n},$$

$$= mq\text{, which is an integer.}$$

From the example found immediately after Theorem 3.D., it was found that $10^{22} \equiv 1 \pmod{23}$ and 22 was the smallest exponent such that 10 to that power was congruent to 1. Therefore, 10 belongs to 22 modulo 23.

Now, the decimal expansion of $m/23$, $m < 23$, will be a pure recurring decimal, and its mantissa will be the same as the mantissa of $10^{22} \cdot m/23$. Therefore, the number of digits in the period will have to be 22.

The above can be generalized to the following two theorems:

<u>Theorem 3.3.</u> 10 belongs to k modulo the prime p ($p \neq 2$ or 5), if and only if the period of the decimal expansion of $1/p$ has k digits.

Proof: Assume 10 belongs to k modulo p, then $10^k \equiv 1 \pmod{p}$. Therefore, $10^k/p$ and $1/p$ have the same mantissa by Theorem 3.1. This implies the period for $1/p$ has k digits. To complete the proof, assume $1/p$ has a period of k digits. That is,

$$1/p = .\dot{d}_1 d_2 \ldots \dot{d}_k.$$

Therefore, $$10^k/p = [d_1 d_2 \ldots d_k] + .d_1 d_2 \ldots d_k,$$

where $[d_1 d_2 \ldots d_k]$ denotes a k-digit integer.

Now, $$10^k/p - 1/p = [d_1 d_2 \ldots d_k] = q \text{ (integer)},$$

or $$10^k - 1 = pq$$

which implies $10^k \equiv 1 \pmod{p}$. Now, k is the least power of 10 such that this congruency is true, for assume there exists an integer $h < k$ such that $10^h \equiv 1 \pmod{p}$. This implies $10^h/p$ and $1/p$ have the same mantissa by Theorem 3.1. But the mantissa for $1/p$ is $\dot{d}_1 \ldots d_h d_{h+1} \ldots \dot{d}_k$ and the mantissa for $10^h/p$ is $\dot{d}_{h+1} \ldots d_k d_1 \ldots \dot{d}_h$. Therefore,

$$[d_1 \ldots d_h d_{h+1} \ldots d_k] = [d_{h+1} \ldots d_k d_1 \ldots d_h]$$

implies $d_1 = d_{h+1}$, $d_2 = d_{h+2}$, ..., $d_h = d_k$, since each side of the equation is an integer. But this would make the period 1/p have h digits instead of k digits which contradicts the hypothesis. Therefore, 10 belongs to k modulo p.

Theorem 3.4. [15] The number of digits, k, in the period of m/n depends upon n alone, and not upon the value of m.

Proof: Assume 10 belongs to k modulo n, i.e., $10^k \equiv 1 \pmod{n}$. Therefore, $10^k \cdot m/n$ and m/n have the same mantissa. This implies the period has k digits. But k depends only upon n, since $10^{\phi(n)} \equiv 1$ by Euler-Fermat Theorem (Theorem 3.C.), and k divides $\phi(n)$ by Theorem 3.D.

Theorem 3.3. explains why m/7 has the maximum number of digits 6, since 10 belongs to 6 modulo 7.

$$10^1 \equiv 3,\ 10^2 \equiv 2,\ 10^3 \equiv 6,\ 10^4 \equiv 4,\ 10^5 \equiv 5,\ 10^6 \equiv 1 \pmod{7}.$$

Considering m/11, it is found that 10 belongs to 2 modulo 11.

$$10^1 \equiv 10,\ 10^2 \equiv 1 \pmod{11}$$

Note, 2 divides $\phi(11)$, since $\phi(11) = 10$. Therefore, the period of m/11 will have only 2 digits and not the maximum possible of 11.

Periods with Maximum Number of Digits

Consider the question, when will the period of m/n have the maximum number of digits n - 1? Now, $\phi(n) = n - 1$ implies n is prime, since by Theorem 3.L., $\phi(p) = p - 1$, therefore, $10^{p-1} \equiv 1 \pmod{p}$. But does 10 belong to p - 1 modulo p? Not necessarily, as shown above when p = 11.

The problem of finding those values of p such that 10 belongs to p - 1 modulo p is not an easy one and will be deferred until Chapter IV.

Those primes "p" less than 100, whose reciprocals result in periods with p - 1 digits are 7, 17, 19, 23, 29, 47, 59, 61, 97. A complete listing of such primes less than 13,710 can be found in Appendix A.

The Period Length of 1/n

It will be advantageous at this point to define the k-function: $k(n) = k$, where k is the number of digits in the period of the decimal expansion of 1/n.

For example, $k(3) = 1$, $k(7) = 6$, $k(11) = 2$, $k(17) = 16$.

Now, from the proof of Theorem 3.3, it is seen that $k(n)$ divides $\phi(n)$ or $q \cdot k(n) = \phi(n)$. If n is prime, then $\phi(n) = n - 1$, but $k(n)$ may or may not be $n - 1$. This was illustrated for the cases where $n = 7$ and $n = 11$. Assuming the function value, $k(p)$, is known for any prime p, how would this affect $k(n)$, where n is composite? The theorems of this section will answer this question.

Consider the case where $n = a \cdot b$, a and b distinct primes, e.g., $n = 21 = 7 \cdot 3$. From division, it is found that $1/21 = .\dot{0}4761\dot{9}$; therefore, $k(21) = 6$. But $k(7) = 6$ and $k(3) = 1$, so how are these three numbers related? Consider a second example, say, $n = 707$. From division, it is found that $1/707 = .\dot{0}0141442715\dot{7}$; therefore, $k(707) = 12$. But $k(101) = 4$, since $1/101 = .\dot{0}09\dot{9}$, and $k(7) = 6$, so how are these three numbers related? In the first example $k(7) \cdot k(3) = k(21)$, but in the second example $k(7) \cdot k(101) \neq k(707)$, so $k(a) \cdot k(b) \neq k(ab)$ in general. The reader has probably realized that $12 = [6, 4]$ and $6 = [6, 1]$. These examples motivate the following theorem.

Theorem 3.5. [16] If m and n are primes other than 2 or 5, such that

$k(m) = a$ and $k(n) = b$, then $k(mn) = [a,b]$.

Proof: $k(m) = a$ and $k(n) = b$ implies

$$10^a \equiv 1 \pmod{m} \text{ and } 10^b \equiv 1 \pmod{n}$$

by Theorem 3.3. From Theorem 3.K, it follows, that

$$10^{[a,b]} \equiv 1 \pmod{mn},$$

where $[a,b]$ is the least power of 10 that is congruent to 1 modulo mn. That is, 10 belongs to $[a,b]$ modulo mn. Therefore, by Theorem 3.1, $10^{[a,b]}/mn$ and $1/mn$ have the same mantissa. This implies $k(mn) = [a,b]$.

The next case to consider is $n = p^a$, where p is prime. It will be instructive to first look at the special case of $a = 2$.

Lemma 3.6. If $10^t \equiv 1 \pmod{p}$, where $t = uv$, then

(1) $(10^v)^0 + (10^v)^1 + \ldots + (10^v)^{(u-1)} \equiv 0 \pmod{p}$, and

(2) $(10^v)^0 + (10^v)^1 + \ldots + (10^v)^{(p-2)} \equiv 0 \pmod{p}$.

Proof: The first conclusion is a direct application of Corollary 3.I, where $q = 10^v$, $h = u$. Now, $(10^v)^0 \equiv (10^v)^{(p-1)} \equiv 10^t \equiv 1 \pmod{p}$ changes (1) and (2) to

(3) $(10^v) + (10^v)^2 + \ldots + (10^v)^u \equiv 0 \pmod{p}$

(4) $(10^v) + (10^v)^2 + \ldots + (10^v)^{(p-1)} \equiv 0 \pmod{p}$.

Since t divides $(p - 1)$, then $(p - 1) = wt = wuv$. Therefore, the left-hand side of equation (4) can be written as follows:

(5)
$$[(10^v) + (10^v)^2 + \ldots + (10^v)^u] +$$
$$[(10^v)^{(u+1)} + (10^v)^{(u+2)} + \ldots + (10^v)^{2u}] + \ldots +$$
$$[(10^v)^{(wuv-u+1)} + (10^v)^{(wuv-u+2)} + \ldots + (10^v)^{wuv}],$$

where each of the sums inside the brackets will be congruent to zero modulo p, since each sum is just equation (3) multiplied by some power of $(10^v)^u$. Therefore, equation (5) is congruent to zero modulo p.

<u>Theorem 3.7.</u> [5] If p is a prime, not 2 or 5, and $k(p) = t$, then $k(p^2) = t \cdot p^{2-b}$, where $b \leq 2$, and p^b is the highest power of p dividing $10^t - 1$, i.e., $((10^t - 1)/p^b, p) = 1$.

Proof: Assume $k(p^2) = T$, and let $F(t) = 1 + 10^t + \dots + 10^{(p-1)t}$. But $10^t \equiv 1 \pmod p$, therefore,

$$F(t) \equiv 1 + 1 + \dots + 1 = p(1) \equiv 0 \pmod p.$$

Now, F(t) is the sum of an infinite geometric progression, and is equal to $(10^{pt} - 1)/(10^t - 1)$. Therefore

(1) $$10^{pt} - 1 = (10^t - 1)F(t) \equiv 0 \pmod{p^2},$$

since $10^t \equiv 1 \pmod p$ and $F(t) \equiv 0 \pmod p$, implies $10^t - 1 = rp$ and $F(t) = sp$, or $(10^t - 1)F(t) = rsp^2$. Now, (1) implies that T is pt, or a divisor of pt, and since p is prime, T = t or pv, where v is t, or a divisor of t. If $v < t$, then

$$1 + 10^v + 10^{2v} + \dots + 10^{(p-2)v} = (10^{(p-1)v} - 1)/(10^v - 1) \equiv 0 \pmod p.$$

The reasons for the above step are, first, 10^v is not $\equiv 1 \pmod p$, since 10 belongs to t modulo p, and second, the conditions of Lemma 3.6 are satisfied. Therefore, $10^{(p-1)v} \equiv 1 \pmod p$, which gives, $F(v) \equiv 10^{(p-1)v} \equiv 1 \pmod p$. Now, $10^{pv} - 1 = (10^v - 1)F(v)$, and therefore 10^{pv} is not $\equiv 0 \pmod{p^2}$. Consequently, v is not < t and T = t or pt, i.e., $T = t \cdot p^{2-b}$, b = 1 or 2.

Returning to the general case $n = p^a$, the theorem is as follows:

<u>Theorem 3.8.</u> [19] If p is a prime, not 2 or 5, and $k(p) = t$, then $k(p^n) = t \cdot p^{n-b}$, where p^b is the highest power of p dividing $10^t - 1$ and $b \leq n$.

The following examples illustrate the theorem.

(I) $p = 3$ implies $t = 1$ and $b = 2$, since 3^2 divides $10^1 - 1$, but 3^3 does not divide $10^1 - 1$. Therefore, $k(243) = k(3^5) = 1 \cdot 3^{5-2} = 3^3 = 27$.

(II) $p = 7$ implies $t = 6$ and $b = 1$, since 7^1 divides $10^6 - 1$, but 7^2 does not divide $10^6 - 1$, (i.e., $10^6 \equiv 1 \pmod 7$, but $10^6 \not\equiv 1 \pmod{49}$). Therefore, $k(343) = k(7^3) = 6 \cdot 7^{3-1} = 6 \cdot 49 = 294$.

(III) $p = 13$ implies $t = 6$ and $b = 1$, since $10^6 \equiv 1 \pmod{13}$, and $10^6 \not\equiv 1 \pmod{169}$. Therefore, $k(13^2) = 6 \cdot 13^{2-1} = 78$.

The only known cases in which $k(p^2) = t$ are when $p = 3$ or 487.

The groundwork is complete now for handling the most general case where

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}.$$

<u>Theorem 3.9</u>. If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ where $p_1 p_2 \cdots p_r$ are distinct primes, then if $k(p_1^{a_1}) = s_1$, $k(p_2^{a_2}) = s_1$, $k(p_2^{a_2}) = s_2$, ..., $k(p_r^{a_r}) = s_r$, $k(n) = [s_1, s_2, \ldots, s_r]$, i.e., the least common multiple of $s_1, s_2, \ldots, s_r$.

The proof of the theorem would follow from repeated applications of Theorem 3.K.

Cyclic Properties of Recurring Decimals

Returning to the example of $1/7 = .\dot{1}4285\dot{7}$, the remaining sevenths can be found in two different ways. First, since $1/7 = .\dot{1}4285\dot{7}$; then $10/7 = 1.\dot{4}2857\dot{1}$, and subtracting 1 gives $3/7 = .\dot{4}2857\dot{1}$; then $30/7 = 4.\dot{2}8571\dot{4}$ and subtracting 4 gives $2/7 = .\dot{2}8571\dot{4}$; then

$20/7 = 2.\dot{8}5714\dot{2}$ and subtracting 2 gives $6/7 = .\dot{8}5714\dot{2}$; then
$60/7 = 8.\dot{5}7142\dot{8}$ and subtracting 8 gives $4/7 = .\dot{5}7142\dot{8}$; then
$40/7 = 5.\dot{7}142\dot{8}$ and subtracting 5 gives $5/7 = .\dot{7}1428\dot{5}$; then
$50/7 = 7.\dot{1}4285\dot{7}$ and subtracting 7 gives $1/7 = .\dot{1}4285\dot{7}$. Thus, cyclic permutation of the six-figure period of 1/7 gives 1/7, 3/7, 2/7, 6/7, 4/7, 5/7 where each numerator is the residue, modulo 7, of ten times its predecessor, i.e. $3 \equiv 1\cdot 10$, $2 \equiv 3\cdot 10$, $6 \equiv 2\cdot 10$, $4 \equiv 6\cdot 10$, $5 \equiv 4\cdot 10$ $1 \equiv 5\cdot 10 \pmod 7$. If a_r represents the r^{th} numerator, the situation can be generalized by

$$a_{r+1} \equiv 10a_r \pmod n,\ r = 1, 2, \ldots, k,$$

where k belongs to 10 modulo n. The numerators in the example above form what may be referred to as a "cycle" of numbers.

Secondly, reconsider the example from Chapter II where the decimal expansion of 1/7 was found by the division process.

```
  0.142857
7)1.00...
  0
  1 0
    7
    30
    28
     20
     14
      60
      56
       40
       35
        50
        49
         1.
           .
            .
```

The following can be noted; (i) The "initial" remainder 1 recurs later, and then the process will start to repeat, giving a recurring decimal; (ii) 2/7 is found by starting with the remainder 2, and so obtain the same continuous cycle of figures but starting at another point, and similarly for 3/7, 4/7, $\cdots$; (iii) each "starting number", i.e., numerator, is congruent to 10 times the previous one.

An interesting point that can be made at this time is any remainder is determined by its predecessor. Assume the remainder a has two different predecessors a' and a", therefore,

$$10a' \equiv a \pmod{p} \text{ and } 10a'' \equiv a \pmod{p}.$$

This implies $\quad 10a' - a = rp, \text{ and } 10a'' - a = sp.$

Therefore, $10(a' - a'') = (r - s)\,p$, but p does not divide 10; p divides $(a' - a'')$. Now, a' and a" $<$ p implies $a' - a'' = 0$. Thus, any remainder determines its predecessor uniquely, and hence, tracing backward from the two equals from the decimal group, anything like $.58371\dot{3}7\dot{1}$ is impossible since 3 cannot follow both 8 and 1. So a/p is a pure recurring decimal.

Consider the example 1/13:

```
      .076923
  13)1.000
      0
     1 00
      91
       90
       78
       120
       117
         30
         26
          40
          39
           1.
             .
              .
```

It is seen that only six of the possible 12 remainders occur, and the cyclic permutation of the period 076923 give the six fractions 1/13, 10/13, 9/13, 12/13, 3/13, and 4/13. Now, any one of the six missing numerators can be used when dividing by 13 to obtain the other six digit group which forms the periods.

For example, divide 2 by 13:

```
        .15384
    13)2.000...
       1 3
        70
        65
         50
         39
         110
         104
           60
           52
            80
            78
             2.
               .
                .
```

The reader will note that no remainder could occur in both divisions, since their predecessors would be equal also, and so on.

For p = 13, it was found that two six-figure periods are formed and the two sets of numerators corresponding to them form two six-member cycles.

The different cycles of numerators found so far were generated by the division process. This was not necessary, since $a_{r+1} \equiv 10 \cdot a_r$ (mod p). Thus, for p = 41, the cycle containing 1, the unity cycle, will be: 1; 10, since $10 \equiv 10 \cdot 1$ (mod 41); 18, since $18 \equiv 10 \cdot 10$ (mod 41); 16, since $16 \equiv 10 \cdot 18$ (mod 41); 37, since $37 \equiv 10 \cdot 16$ (mod 41). Since $1 \equiv 10 \cdot 37$ (mod 41), 37 is the last numerator in the unity cycle. So 1/41, 10/41, 18/41, 16/41, and 37/41 are given by cyclic permutation of a five-digit period which has not been found.

The above results are not too surprising when considered in light of the material introduced in the earlier sections of this chapter. For it can be shown that 10 belongs to 5 modulo 41. Thus, $a \equiv 10^5 \cdot a$ (mod 41) and the cycle of numerators will repeat after five steps.

Summarizing, it is found that if 10 belongs to k modulo p, then: (i) each of the fractions 1/p, 2/p, 3/p, ..., (p-1)/p is given by a k digit period; (ii) k of these fractions are given by cyclic permutation of one k digit period; and (iii) since no number can occur in more than one cycle, there will be $(p-1)/k = c$ (say) cycles. Thus,

$p = 7$	$10^6 \equiv 1 \pmod{7}$	$d = 6$	$c = 1$
$p = 13$	$10^6 \equiv 1 \pmod{13}$	$d = 6$	$c = 2$
$p = 41$	$10^5 \equiv 1 \pmod{41}$	$d = 5$	$c = 8$
$p = 3$	$10^1 \equiv 1 \pmod{3}$	$d = 1$	$c = 2.$

A complete listing of primes $\leq 13,709$ and their corresponding c-values can be found in Appendix A.

While the discussion to this point has dealt with prime denominators, it should be pointed out that this is not necessary. For consider $p = 77$, then $1/77 = .\dot{0}1298\dot{7}$ and $\phi(77) = 60$. Thus, there is found $\phi(77)/k(77) = 60/6 = 10$ cycles. The unity cycle will be $\{1, 10, 23, 76, 67, 54\}$. A second cycle could be found by finding the residues of $10^r \cdot 2$, $(r = 0, 1, \ldots, 5)$ modulo 77. In the next selection p will be taken to be any composite relatively prime to 10.

Properties of Remainders and Digits

The proofs of the theorems in this section were based on the article by Batty [7].

In the previous section it was noted that $a_{r+1} \equiv 10 \cdot a_r \pmod{p}$ or $10a_r = a_{r+1} + hp$, $r = 1, 2, \ldots, k$. The reader will find that $h = d_r$, i.e., the digit found in the r^{th} place of the period. Thus, the relationship between the remainder and digits can be generalized by

$$(1) \qquad 10a_r = d_r p + a_{r+1}, \; r = 1, 2, \ldots, k$$

where $a_1/p = .\dot{d}_1 d_2 \ldots \dot{d}_k$.

An interesting property is illustrated by considering $1/7 = .\dot{1}4285\dot{7}$.

Its cycle of remainders was shown to be {1, 3, 2, 6, 4, 5}. Now, the sum of the digits is 27 or 3·9 while the sum of the remainders is 21 or 3·7. The reader will note that one sum is a multiple of 7 and the other the same multiple of 9. Is this a property of only primes with maximum period lengths? Checking 1/13, it was found that $1/13 = .\dot{0}7692\dot{3}$ and the cycle of remainders was {1, 10, 9, 12, 3, 4}. Therefore, the sum of the digits is 27 or 3·9 and the sum of the remainders is 39 or 3·13. These examples motivate the following theorem:

Theorem 3.10. If p is not a multiple of 3, the sum of all remainders in a cycle is a multiple of p, and the sum of the corresponding digits is the same multiple of 9. Thus,

$$\sum_{s=1}^{k} a_s = pH, \quad \sum_{s=1}^{k} d_s = pH$$

where H is an integer.

Proof: Summing the equations (1) gives:

$$\sum_{r=1}^{k} 10a_r = \sum_{r=1}^{k} (d_r p + a_{r+1})$$

(2) $$10 \sum_{r=1}^{k} a_r = p \sum_{r=1}^{k} d_r + \sum_{r=1}^{k} a_{r+1},$$

But $a_{k+1} = a_1$, since $k(p) = k$. Therefore,

$$\sum_{r=1}^{k} a_r = \sum_{r=1}^{k} a_{r+1},$$

and substituting this result into (2), it becomes

(3) $$9 \sum_{r=1}^{k} a_r = p \sum_{r=1}^{k} d_r.$$

If p is not a multiple of 3, then p divides $\sum_{r=1}^{k} a_r$ or $\sum_{r=1}^{k} a_r = pH$. Now, substitute this result into (3) giving

$$9pH = p \sum_{r=1}^{k} d_r.$$

Thus,

$$9H = \sum_{r=1}^{k} d_r,$$

and the proof is complete.

Further, if $k(p) = p - 1$ then a_r is an element of $\{1, 2, 3, \ldots, p - 1\}$, and

$$\sum_{r=1}^{k} a_r = 1 + 2 + \ldots + (p - 1) \quad \text{(not necessarily in this order)}$$
$$= \tfrac{1}{2} \cdot p \cdot (p - 1).$$

Therefore, $H = \frac{1}{2} \cdot (p - 1) = \frac{1}{2} k(p)$.

The following result concerning subsets of $\{a_r\}$ and $\{d_r\}$ will be used in Theorem 3.12.

<u>Lemma 3.11.</u> If m, n are any complementary divisors of $k = k(p)$, i.e., $mn = k(p)$, and if $(p, 10^m - 1) = 1$, then for $r = 1, 2, \ldots, m$,

$$(4) \qquad \sum_{s=0}^{n-1} a_{r+sm} = h_r p$$

$$(5) \qquad \sum_{s=0}^{n-1} d_{r+sm} = 10h_r - h_{r+1},$$

where h_r is an integer satisfying $1 \leq h_r \leq n-1$, and $h_{m+1} = h_1$.

Before proving the lemma, it would be instructive to consider a few examples:

(A) If $m = 1$ then $n = k$ and (4) and (5) become

$$\sum_{s=0}^{k-1} a_{r+s} = h_r p \text{ and } \sum_{s=0}^{k-1} d_{r+s} = 10h_r - h_{r+1}.$$

These equations are the same as those of Theorem 3.10, where $H = h_r = h_{r+1}$. The reason being the sums are over the same sets, but the starting points are different.

If k is prime, this is the only case.

(B) Assume $p = 31$, then $k(31) = 15$, since $1/31 = .\dot{0}3225806451612\dot{9}.$

The unity cycle of remainders is

1, 10, 7, 8, 18, 25, 2, 20, 14, 16, 5, 19, 4, 9, 28

Taking $m = 5$, $n = 3$, gives

$$\sum_{s=0}^{2} a_{1+5s} = 1 + 25 + 5 = 31 \qquad (h_1 = 1)$$

$$\sum_{s=0}^{2} a_{2+5s} = 10 + 2 + 19 = 31 \qquad (h_2 = 1)$$

$$\sum_{s=0}^{2} a_{3+5s} = 7 + 20 + 4 = 31 \qquad (h_3 = 1)$$

$$\sum_{s=0}^{2} a_{4+5s} = 8 + 14 + 9 = 31 \qquad (h_4 = 1)$$

$$\sum_{s=0}^{2} a_{5+5s} = 18 + 16 + 28 = 62 \qquad (h_5 = 2)$$

$$\sum_{s=0}^{2} d_{1+5s} = 0 + 8 + 1 = 9 \qquad (h_1 = 1)$$

$$\sum_{s=0}^{2} d_{2+5s} = 3 + 0 + 6 = 9 \qquad (h_2 = 1)$$

$$\sum_{s=0}^{2} d_{3+5s} = 2 + 6 + 1 = 9 \qquad (h_3 = 1)$$

$$\sum_{s=0}^{2} d_{4+5s} = 2 + 4 + 2 = 8 \qquad (h_4 = 1)$$

$$\sum_{s=0}^{2} d_{5+5s} = 5 + 5 + 9 = 19 \qquad (h_5 = 2)$$

Interchanging m and n gives

$$\sum_{s=0}^{2} a_{1+3s} = 1 + 8 + 2 + 16 + 4 = 31 \qquad (h_1 = 1)$$

$$\sum_{s=0}^{4} a_{2+3s} = 10 + 18 + 20 + 5 + 9 = 62 \qquad (h_2 = 2)$$

$$\sum_{s=0}^{4} a_{3+3s} = 7 + 25 + 14 + 19 + 28 = 93 \qquad (h_3 = 3)$$

$$\sum_{s=0}^{4} d_{1+3s} = 0 + 2 + 0 + 5 + 1 = 8 = 10h_1 - h_2$$

$$\sum_{s=0}^{4} d_{2+3s} = 3 + 5 + 6 + 1 + 2 = 17 = 10h_2 - h_3$$

$$\sum_{s=0}^{4} d_{3+3s} = 2 + 8 + 4 + 6 + 9 = 29 = 10h_3 - h_1$$

Proof of Lemma 3.11:

$$10^k - 1 = 10^{mn} - 1 = (10^m - 1)(10^{(n-1)m} + \dots + 1)$$

$$= (10^m - 1)\sum_{s=0}^{n-1} 10^{sm}.$$

Since $k(p) = k$ implies 10 belongs to k modulo p, then p divides $(10^k - 1)$ but not $(10^m - 1)$. Thus, if p and $(10^m - 1)$ are coprime, it follows that p divides $\sum_{s=0}^{n-1} 10^{sm}$ and, therefore, also $a_r \cdot \sum_{s=0}^{n-1} 10^{sm}$. Now, (2) gives $10^m a_r \equiv a_{r+m} \pmod{p}$, since the cyclic property of the remainder allows starting at any remainder. Therefore,

$$a_r \equiv a_r \pmod{p}$$

$$10^m a_r \equiv a_{r+m} \pmod{p}$$

$$10^{2m} a_r \equiv a_{r+2m} \pmod{p}$$

$$\vdots$$

$$10^{(n-1)m} a_r \equiv a_{r+(n-1)m} \pmod{p},$$

and

$$a_r \cdot \sum_{s=0}^{n-1} 10^{sm} \equiv \sum_{s=0}^{n-1} a_{r+sm} \pmod{p},$$

for $1 \leq r \leq m$. Since p divides the left-hand member, it also divides the right-hand member, i.e.

(6) $$\sum_{s=0}^{n-1} a_{r+sm} = h_r p,$$

for some integer h_r. Also, since $0 < a_r < p$,

(7) $$1 \leq h_r \leq n - 1.$$

Now, equation (6) gives

$$\sum_{s=0}^{n-1} 10a_{r+sm} = 10h_r p,$$

and $$\sum_{s=0}^{n-1} a_{r+1+sm} = h_{r+1} p.$$

Therefore, $\sum_{s=0}^{n-1}(10a_{r+sm} - a_{r+1+sm}) = (10h_r - h_{r+1})p.$

Substituting from equation (1) gives

$$\sum_{s=0}^{n-1} d_{r+sm} p = (10h_r - h_{r+1})p$$

or dividing out the p

(8) $$\sum_{s=0}^{n-1} d_{r+sm} = 10h_r - h_{r+1},\ 1 \leq r \leq m,$$

where $h_{m+1} = h_1$.

The relations of Lemma 3.11 will now be used to prove the following:

<u>Theorem 3.12</u>. If M is the greatest (proper) divisor of k, such that $10^M - 1$, p are coprime, then for any cycles of remainders of p, the value of H as defined by Theorem 3.10 satisfies the inequality $M \leq H \leq k - M$.

Proof: If k is composite, any divisor m (satisfying the coprime conditions of Lemma 3.11) may be used to separate the set $\{a_r\}$ into m subsets, each of k/m terms; and for the corresponding set $\{k_r\}$ gives $1 \leq h_r \leq k/m - 1$ (see (7)). Summing over the m sets gives

$$m \leq \sum_{r=1}^{m} h_r \leq k - m.$$

But $\sum_{r=1}^{m} h_r = H$, since summing (6) over r = 1, 2, ..., m gives

$$\sum_{r=1}^{m} \left(\sum_{s=0}^{n-1} a_{r+sm}\right) = \sum_{r=1}^{m} h_r p$$

or

$$\sum_{r=1}^{k} a_r = \sum_{r=1}^{m} h_r p.$$

With Theorem 3.10, this implies $\sum_{r=1}^{m} h_r = H$. Therefore, $m \leq H \leq k - m$.

Since H satisfies this inequality for each m,

(9) $$M \leq H \leq k - M,$$

where M is the greatest (proper) divisor of k.

If H has its least value of M, then in $\{h_r\}_M$ each h_r has its least value, unity; thus, with k = NM, equations (4) and (5) become

(10) $$\sum_{s=0}^{N-1} a_{r+sM} = p$$

(11) $$\sum_{s=0}^{N-1} d_{r+sM} = 9$$

Similarly, if H has its greatest value k - M, each h_r in the set $\{h_r\}_M$ is k/M - 1.

<u>Corollary 3.13</u>. If p and $10^{\frac{1}{2}k} - 1$ are coprime, then

(12) $$H = \tfrac{1}{2}\cdot k$$

(13) $$a_r + a_{r+\frac{1}{2}\cdot k} = p$$

(14) $$d_r + d_{r+\frac{1}{2}\cdot k} = 9.$$

Proof: When k is even, the greatest (proper) divisor of k is $\frac{1}{2}\cdot k$. So $M = \frac{1}{2}k$ forces (9) to become $\frac{1}{2}k \leq H \leq k - \frac{1}{2}k$, which implies $H = \frac{1}{2}k$.

Now, $M = \frac{1}{2}k$ implies $N = 2$. Therefore, (10) and (11) become

$$a_r + a_{r+\frac{1}{2}\cdot k} = p, \text{ and } d_r + d_{r+\frac{1}{2}\cdot k} = 9.$$

Now, two remainders with sum p and two digits with sum 9 may be called <u>complementary</u>; equations (13) and (14) state that corresponding remainders and digits in the two half-periods are complementary. Periods exhibiting these complementary properties will be said to belong to the class "C". A period of class "C" is necessarily of even length; a further condition, which is sufficient, is that p and $10^{\frac{1}{2}\cdot k} - 1$ are coprime. These conditions depend only on p; therefore, if one period of p is of class "C" so are all periods. In such a case it is said that p is of class "C". It follows from the definitions of k, that all primes or powers of prime with even period are of class "C". In particular, all primes "p" such that $k(p) = p - 1$ are of this class and also those prime powers p^a for which $k(p^a) = \phi(p^a)$, since $\phi(p^a) = p^{a-1}(p-1)$, which is even for prime $p > 2$.

The complementary property of the digits in the half-periods gives at once:

<u>Corollary 3.14</u>. The sum of the two half-periods of a decimal in class "C" is 99...9 ($\frac{1}{2}\cdot k$ digits), or $10^{\frac{1}{2}k} - 1$.

The following examples illustrate the above corollaries:

(C) $p = 11$, $k = 2$

$\frac{1}{11} = .\dot{0}\dot{9}$, $\frac{10}{11} = .\dot{9}\dot{0}$; $\frac{2}{11} = .\dot{1}\dot{8}$, $\frac{9}{11} = .\dot{8}\dot{1}$; $\frac{3}{11} = .\dot{2}\dot{7}$, $\frac{8}{11} = .\dot{7}\dot{2}$;

$\frac{4}{11} = .\dot{3}\dot{6}$, $\frac{7}{11} = .\dot{6}\dot{3}$; $\frac{5}{11} = .\dot{4}\dot{5}$, $\frac{6}{11} = \dot{5}\dot{4}$.

This example can be simplified as below where remainders can be thought of as numerators.

Remainders	1 -	10	2 -	9	3 -	8	4 -	7	5 -	6
Periods	0	9	1	8	2	7	3	6	4	5

(D) $p = 13$, $k = 6$

Remainders	1 -	10 -	9 -	12 -	3 -	4	2 -	7 -	5 -	11 -	6 -	8
Periods	0	7	6	9	2	3	1	5	3	8	4	6

(E) $p = 49$, $k = 42$

Remainders	1 -	10 -	2 -	20 -	4 -	40 -	8 -	31 -	16 -	13 -	32 -
Periods	0	2	0	4	0	8	1	6	3	2	6
	26 -	15 -	3 -	30 -	6 -	11 -	12 -	22 -	24 -	44 -	
	5	3	0	6	1	2	2	4	4	8	
	48 -	39 -	47 -	29 -	45 -	9 -	41 -	18 -	33 -	36 -	
	9	7	9	5	9	1	8	3	6	7	
	17 -	23 -	34 -	46 -	19 -	43 -	38 -	37 -	27 -	25 -	5
	3	4	6	9	3	8	7	7	5	5	1

An example to show k even is not sufficient is as follows:

(F) $p = 39$, $k = 6$

Remainders	1 -	10 -	22 -	25 -	16 -	4
Periods	0	2	5	6	4	1
	2 -	20 -	5 -	11 -	32 -	8
	0	5	1	2	8	2
	7 -	31 -	37 -	19 -	34 -	28
	1	7	9	4	8	7
	14 -	23 -	35 -	38 -	29 -	17
	3	5	8	9	7	4

Now, (13) and (14) fail to hold, since $(39, 10^3 - 1) = 3$. This is a contradiction to the sufficient condition of Corollary 3.13.

The characteristic features of a period of class "C" are that k is even and the complementary remainder $p - a_r$ occurs $\frac{1}{2}\cdot k$ stages after a_r in the division process.

Conversely, it is seen that if the remainder at, say, the m^{th} stage is $p - a_1$ then the corresponding digit is $9 - d_1$, and the process continues with each of the first m remainders and each of the first m digits replaced by its complement, until the remainder a_1 recurs after a further m stages. Thus, k = 2m, and the period has two complementary halves.

It is interesting to note that the above statement can be proved [9] without the use of material from this section. Assume

(a) $$\frac{a_1}{p} = .d_1d_2\ldots d_m + \frac{p-a_1}{p}10^{-m}$$

Then, $$\frac{p-a_1}{p} = 1 - \frac{a_1}{p} = 1 - .d_1d_2\ldots d_m - 10^{-m} + \frac{a_1}{p}10^{-m}$$

$$= .(9-d_1)(9-d_2)\ldots(9-d_m) + \frac{a_1}{p}10^{-m},$$

since $1 - 10^{-m} = .99\ldots9$ (m digits). Thus,

$$\frac{p-a_1}{p}10^{-m} = .00\ldots0(9-d_1)(9-d_2)\ldots(9-d_m) + \frac{a_1}{p}10^{-2m},$$

so (a) becomes

$$\frac{a_1}{p} = .d_1d_2\ldots d_m(9-d_1)(9-d_2)\ldots(9-d_m) + \frac{a_1}{p}10^{-2m},$$

where $(9 - d_i)$ is the digit in the $(n + i)^{th}$ decimal place. Hence, the statement above holds.

If the remainder $p - a_1$ does not occur at any stage, then a_1 and $p - a_1$ belong to distinct cycles of remainders, the periods for a_1/p and $(p - a_1)/p$ are distinct, and corresponding digits in each are complementary, the sum being $.\dot{9} = 1$. The various periods separate into complementary pairs; the number of these periods, $\phi(p)/k$, is therefore even.

Summarizing these results:

A necessary condition for p to be of class "C" is that k is even. A sufficient condition is that $10^{\frac{1}{2}\cdot k} - 1$ is an integer prime to p. Another sufficient condition is that $\phi(p)/k$ is odd.

If p is not of class "C", its periods form complementary pairs. If k is even, a necessary condition is that $10^{\frac{1}{2}\cdot k} - 1$ has a common factor with p; another necessary condition in this case is that $\phi(p)$ is a multiple of 4.

Examples are:

(G) $p = 41$, $\phi(41) = 40$, $k(41) = 5$

Remainders	1 - 10 - 18 - 16 - 37	40 - 31 - 23 - 25 - 4
Periods	0 2 4 3 9	9 7 5 6 0
	3 - 30 - 13 - 7 - 29	38 - 11 - 28 - 34 - 12
	0 7 3 1 7	9 2 6 8 2
	6 - 19 - 26 - 14 - 17	35 - 22 - 15 - 27 - 24
	1 4 6 3 4	8 5 3 6 5
	5 - 9 - 8 - 39 - 21	36 - 32 - 33 - 2 - 20
	1 2 1 9 5	8 7 8 0 4

The reader will note: (i) Corresponding entries in the two columns are complementary. (ii) The H-values from Theorem 3.10 are 2 in the first column and 3 in the second column. (iii) The sum of a period with its complementary period is 9k and the sum of their corresponding remainders is pk; thus the two values of H for the pairs are complementary with sum k.

(H) $p = 21$, $\phi(21) = 12$, $k(21) = 6$

Remainders	1 - 10 - 16 - 13 - 4 - 19
Periods	0 4 7 6 1 9
	20 - 11 - 5 - 8 - 17 - 2
	9 5 2 3 8 0

The reader will note that each of the observations above for p = 41 hold for p = 21 except the H - values are both 3. Now, p = 41 failed to be in class "C" since $\phi(41) = 40$ which is a multiple of 4 and also $k(41) = 5$ is not even. p = 21 failed since $(21, 10^3 - 1) = 3$.

To close out this section, a generalization of Corollary 3.14 is given:

<u>Theorem 3.15.</u> If k is composite with a divisor m, then

$$s_m = d_1 d_2 \cdots d_m + d_{m+1} d_{m+2} \cdots d_{2m} + \cdots + d_{k-m+1} \cdots d_k$$

is a multiple of $10^m - 1$.

Proof: $$S_m = 10^{m-1}(d_1 + d_{m+1} + \cdots + d_{k-m+1}) + \cdots + 10^1(d_{m-1} + d_{2m-1} + \cdots + d_{k-1}) + 10^0(d_m + d_{2m} + \cdots + d_k)$$

By using (5) each sum changes to

$$S_m = 10^{m-1}(10h_1 - h_2) + 10^{m-2}(10h_2 - h_3) + \cdots + 10(10h_{m-1} - h_m) + (10h_m - h_1)$$

$$= 10^m h_1 - 10^{m-1}h_2 + 10^{m-1}h_2 - 10^{m-2}h_3 + \cdots + 10^2 h_{m-1} - 10h_m + 10h_m - h_1$$

$$= (10^m - 1)h_1$$

Similarly, if the digits are grouped cyclically starting at d_r, the sum is $(10^m - 1)h_r$.

Examples are:

(I) $p = 31$, $\phi(31) = 30$, $k(31) = 15$. Periods are:

(1) 032258064516129, (H = 6)

$$S_3 = 032 + 258 + 064 + 516 + 129 = 10^3 - 1,$$

$$S_5 = 03225 + 80645 + 16129 = 10^5 - 1.$$

(2) 967741935483870, (H = 9)

$$S_3 = 2(10^3 - 1),$$

$$S_5 = 10^5 - 1.$$

(J) $p = 43$, $\phi(43) = 42$, $k(43) = 21$. Periods are:

(1) 023255813953488372093, (H = 10).

$S_3 = 3(10^3 - 1)$,

$S_7 = 10^7 - 1$.

(2) 976744186046511627906, (H = 11).

$S_3 = 4(10^3 - 1)$,

$S_7 = 2(10^7 - 1)$.

This chapter has by no means exhausted all the properties of recurring decimals, but only those which the writer felt were "basic". The remainder of the chapter will relate the theory developed in the chapter to bases other than ten and the history of the theory will be traced.

Basimals as Applied to This Chapter

The first section of the chapter will hold true regardless of the base system of numeration. For example, consider the definition of congruence; if m divides a - b using base ten numeration, then it will in any base. The example $23 \equiv 7 \pmod 8$ will become $35 \equiv 11 \pmod{12}$ in the base six system of numeration.

The theory developed in the second section was not completely independent of the base system of numeration. An interesting example to consider in another base is Theorem 3.3. This was the theorem that explained why $k(7) = 6$. Now, if the base had been twelve, then the "10" in the theorem would have been the numeral for the number twelve. Therefore, the question becomes, "To what power does 10 (twelve) belong modulo 7?". (The following congruences are in base twelve.)

(1) $10 \equiv 5,\ 10^2 \equiv 4,\ 10^3 \equiv 6,\ 10^4 \equiv 2,\ 10^5 \equiv 3,\ 10^6 \equiv 1 \pmod 7$.

This implies that k(7) = 6 for base twelve as well as base ten. The division process to find the period of 1/7 in base twelve would be as follows:

(2)
```
      .186t35
    7)1.00...
      7
      50
      48
       40
       36
        60
        5t
         20
         19
          30
          2e
           1.
             .
```

The question, "Is k(7) = 6 true for any basimal?", could be asked by the reader at this time. The answer is no, as shown by the following: Six belongs to two modulo seven which in base six would be expressed as

$$10^2 \equiv 1 \pmod{11},$$

thus, $$[k(11) = 2]_{six}.$$

From the division process the period is found to be

(3)
```
        .0̇5̇
    11)1.00...
        0
       1 00
         55
          1.
            .
```

Since $8 \equiv 1 \pmod 7$, then in the base of eight the congruence would be $10 \equiv 1 \pmod 7$ and 10 (eight) belongs to one modulo 7. This implies k(7) = 1. From the division process the period is found to be

(4)
```
      .1̇
    7)1.0
       7
       1..
```

The periods in the two examples lead to an interesting generalization when compared with the analogous situation in base ten. In (3), the period was found for the reciprocal of the number which was one more than the base. The digits of the period were zero and one less than the base. The reader will recall that in base ten $1/11 = .\dot{0}\dot{9}$ and this is identical to the conditions of (3). Can this observation be shown to be true in general? The question is answered by the following theorem.

Theorem 3.16. If b is the base of the system of numeration, then $1/(b+1) = .\dot{0}\dot{(b-1)}$.

Proof: $$\frac{1}{b+1} = \frac{1}{b+1}\cdot\frac{b-1}{b-1} = \frac{b-1}{b^2-1} = (b-1)\cdot\frac{1}{b^2-1}$$

Now, $1/(b^2 - 1)$ can be expressed as an infinite geometric progression as follows:

$$\begin{array}{r|l}
 & 1/b^2 + 1/b^4 + 1/b^6 + \cdots \\
\hline
b^2-1 & 1 \\
 & \underline{1 - 1/b^2} \\
 & \quad 1/b^2 \\
 & \quad \underline{1/b^2 - 1/b^4} \\
 & \qquad 1/b^4 \\
 & \qquad \underline{1/b^4 - 1/b^6} \\
 & \qquad\quad 1/b^6 \ddots
\end{array}$$

Thus, $1/(b+1) = (b-1)\cdot[1/b^2 + 1/b^4 + 1/b^6 + \ldots]$ which implies $1/(b+1) = .\dot{0}\dot{(b-1)}$.

The reader will observe the generalization of (4) leads to the following:

Theorem 3.17. If b is the base of the system of numeration, then $1/(b-1) = .\dot{1}$.

Proof: Now $1/(b-1)$ can be expressed as an infinite geometric progression as follows:

$$
\begin{array}{r l}
 & 1/b + 1/b^2 + 1/b^3 + \ldots \\
b-1 \overline{)\,1} & \\
\underline{1 - 1/b} & \\
1/b & \\
\underline{1/b - 1/b^2} & \\
1/b^2 & \\
\underline{1/b^2 - 1/b^3} & \\
1/b^3 & \\
\vdots &
\end{array}
$$

Thus, $1/(b-1) = 1/b + 1/b^2 + 1/b^3 + \ldots$ which implies $1/(b-1) = .\dot{1}$.

Since only the function values of the k-function are dependent on the base and not the final results, the Theorems 3.5, 3.7, and 3.8 will still hold with the following modifications: (i) p is a prime which does not divide the base. (ii) "10" remains in the theorems, but is interpreted as the numeral for the new base. Theorem 3.9 is true regardless of the base.

The cyclic properties of recurring decimals are independent of the base system. The reader should, by just looking at (2), be able to write the period of any of the sevenths in base twelve. The result concerning the number of cycles is still **true**. Consider in base <u>six</u> the following: $10 \equiv 10 \cdot 1,\ 1 \equiv 10 \cdot 10 \pmod{11}$.
Thus, $d = 2$ and $c = (11-1)/2 = 3$.

The generalization of Theorem 3.10 to any basimal would be to replace the "9" by "b-1" where b is the base, and to replace the "3" by "any divisor of b-1". As an example consider in base twelve the reciprocal of seven (see (2)). The period is $1/7 = .\dot{1}86t3\dot{5}$ and the cycle of remainders is $\{1, 5, 4, 6, 2, 3\}$. Therefore,

$$\sum_{s=1}^{6} d_s = 1 + 8 + 6 + t + 3 + 5 = 29 = e \cdot 3$$

and

$$\sum_{s=1}^{6} a_s = 1 + 5 + 4 + 6 + 2 + 3 = 19 = 7 \cdot 3$$

It is seen that, since $(7, 10^3 - 1) = 1$, the Corollary 3.13 is satisfied:

$$1 + 6 = 5 + 2 = 4 + 3 = 7$$

$$1 + t = 8 + 3 = 6 + 5 = e.$$

Corollary 3.13 is generalized by replacing the 9's by b-1. The discussion on class "C" primes carries through with only the 9's replaced by b-1. Theorem 3.15 is true regardless of the base, since its proof depends only on place value.

The reader should by now see the "obvious" changes necessary in going from one base to another.

Historical Notes

While the results of Theorem 3.3 were known before 1800, the theorem was proved for the first time by C. F. Gauss in 1801. Theorem 3.5 was first noted in 1771 by Jean Bernoulli, but was not proved in the general form until 1875 by T. Muir.

The history of Theorem 3.8 is interesting due to the presence of the case for the prime 487. Bernoulli was the first person to make the false assertion by overlooking 487. Thibault's formulation of the problem in 1843 was correct, but it was proved by E. Prouhet in 1846. The first person to find the case of 487 was E. Deomarest in 1852. Although W. Shanks was familiar with Deomarest's work, he still erroneously stated the theorem in 1874. He corrected his error in 1877. J. W. L. Glaisher gave the full period of $1/487^2$ in 1878.

Bernoulli in his article in 1771 also noted that if the period of 1/p has p-1 digits then the period of q/p will be a cyclic permutation of the digits.

It is interesting to note that Dicksons gives no reference to Theorem 3.10 even in stated form, but W. H. Hudson in 1864 did note that if $k(p) = p-1$ then $\sum_{i=1}^{p-1} d_i = 9(p-1)/2$. Lemma 3.11 in a modified form was illustrated by E. Midy in 1836. The digital properties of the class "C" periods were noted by several different authors starting with H. Goodwyn in 1802. The first reference to the proof of these properties was in 1851 by P. Lafitte. In 1874 P. Mansion gave a "detailed" proof of $d_r + d_{r+\frac{1}{2}k} = 9$.

CHAPTER IV

GENERAL PROPERTIES OF RECURRING DECIMALS

Primes with Maximum Periods for Their Reciprocals

The extensive amount of material available on the subject of recurring decimals is lacking in the discussion of primes with maximum periods for their reciprocals. Hardy and Wright [17] list the first six primes with this property and prefaced their remarks by saying "... very little is known about them".

Two writers who have considered this type of prime are Ayyangar and Kaprekar [4]. The following three theorems are attributed to them.

Theorem 4.1. If $N = p_1^{a_1}p_2^{a_2}\ldots p_r^{a_r}$ where the p_i's are prime integers and the a_i's are positive integers, then the factors of N other than N itself are factors of at least one of the r integers:

$$\frac{N}{p_1}, \frac{N}{p_2}, \ldots, \frac{N}{p_r}.$$

Proof: Consider any factor of N other than itself, it has to be the form

$$p_1^{b_1}p_2^{b_2}\ldots p_r^{b_r}$$

where at least one of the b_i's, say, b_h, is such that $b_h \leq a_h - 1$. Therefore, this number is a factor of N/p_h.

Theorem 4.2. A necessary and sufficient set of conditions that q is a prime with a maximum number of digits in its period is that

$10^{(q-1)/p} \not\equiv 1 \pmod{q}$ for all prime factors p of q-1.

Proof: The conditions are necessary since if $10^{(q-1)/p} \equiv 1 \pmod{q}$ then $k(q) \leq (q-1)/p$ by the definition of the k-function. But this contradicts the hypothesis of $k(q) = q-1$. To prove the sufficiency of the conditions, assume that $10^t \equiv 1 \pmod{q}$ where t is some proper factor of q-1. Now, by Theorem 4.1, it is known that t is a factor of some number of the form $(q-1)/p$. Therefore, $10^{(q-1)/p} \equiv 1$ which implies $k(q) < q-1$. But this contradicts the hypothesis $k(q) = q-1$.

Iyer [18] stated a "theorem" without proof which was similar to the one above, but he considered only the one case where p = 2. Later, Ayyangar [3] pointed out that Iyer was in error but he neglected to give a counterexample. Before looking at a counterexample let us state the "theorem":

> A prime number of the form 1 + 2m, where 2m is any integer other than a power of 10, has a reciprocal with a maximum recurring period provided that $10^m \equiv -1$ (mod (1+2m)).

The "theorem" is necessary but not sufficient since, if 10 belongs to 2m modulo (1+2m), then $10^m \not\equiv 1$, but $10^m \equiv -1$ modulo (1+2m). To show the theorem is not sufficient, let the prime (2m+1) be 73, then m = 36. From Appendix A it is found that the period for 1/73 has 8 digits or 10 belongs to 8 modulo 73. This implies $10^8 \equiv 1$ and $10^4 \equiv -1 \pmod{73}$. Hence, $10^{36} \equiv -1 \pmod{73}$. Now, the prime 73 does not satisfy Theorem 4.2, since $10^{(73-1)/3} \equiv 10^{24} \equiv 1 \pmod{73}$.

The next theorem gives necessary and sufficient conditions for primes of the form $q = 1 + 2^a 3^b$, of which 73 is one.

Theorem 4.3. If q is a prime of the form $1 + 2^a 3^b$ $(a, b > 0)$ and $10^{(q-1)/6} \equiv t \pmod{q}$ where $|t| < q/2$, then a necessary and sufficient set of conditions that $1/q$ has the maximum recurring period of $(q-1)$ digits is

(1) $$|t| \not\equiv 1 \text{ and } t^2 \equiv t - 1 \pmod{q}.$$

Proof: (Note: All congruences in the proof are for modulo q.) The conditions are necessary since, assume $k(q) = q - 1$ then $t^6 \equiv 1$. This implies

$$t^6 - 1 \equiv 0.$$

Hence, $$(t^3 - 1)(t^3 + 1) \equiv 0.$$

Thus, $$t^3 \equiv 1 \text{ or } t^3 \equiv -1,$$

because $(t^3 - 1)(t^3 + 1) = hq$ for some integer h, implies q divides $(t^3 - 1)$ or $(t^3 + 1)$ since q is prime. Now, $t^3 \not\equiv 1$ since this would contradict the hypothesis, therefore $t^3 \equiv -1$ or $t^3 + 1 \equiv 0$. But $t^3 + 1 = (t + 1)(t^2 - t + 1)$ gives $t \equiv -1$ or $t^2 \equiv t - 1$. The first congruence would contradict the hypothesis, therefore, the conditions (1) are necessary.

It remains to show that if (1) holds then $k(q) = q - 1$. By the way t is defined, $t^6 \equiv 10^{q-1} \equiv 1$. If 10 does not belong to $q - 1$ then 10 has to belong to some factor of $q - 1$ by Corollary 3.F. From Theorem 4.1, it can be said that this factor has to be a factor of $(q - 1)/2$ or $(q - 1)/3$. This implies $10^{(q-1)/2} \equiv 1$ or $10^{(q-1)/3} \equiv 1$. But $10^{(q-1)/2} \equiv t^3$ and $10^{(q-1)/3} \equiv t^2$. Hence, $t^2 \equiv 1$ or $t^3 \equiv 1$ which is the same as $|t| \equiv 1$ or $t^3 - 1 \equiv 0$. $|t| \equiv 1$ contradicts (1). Does $t^3 - 1 \equiv 0$ imply $t^2 \not\equiv t - 1$?

$$t^3 - 1 \equiv (t - 1)(t^2 + t + 1) = 0, \text{ implies}$$
$$t \equiv 1 \text{ or } t^2 + t + 1 \equiv 0.$$

Now, $t \equiv 1$ contradicts (1) and $t^2 \equiv -t - 1$ implies $t^2 \not\equiv t-1$ since, if $t^2 \equiv -t - 1 \equiv t-1$, then $t \equiv 0$ which is impossible. So the answer to the question is yes, and the conditions are sufficient.

Ayyangar, in the same article in which he pointed out Iyer's error, also gave a theorem which generalized Theorem 4.3 to cover all primes.

Theorem 4.4. If $q = 1 + 2^{a_0} p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, $P = p_1 p_2 \cdots p_r$, $Q = (p_1 - 1)(p_2 - 1)\cdots(p_r - 1)$ and $10^{(q-1)/2P} \equiv t \pmod q$, where q, p_1, p_2, ..., p_r are odd primes, then a necessary and sufficient condition for q to be a prime with a maximum recurring period for $1/q$ is that t satisfies a cyclotomic congruence equation of degree Q and order $2P$.

He neglected to give a proof of the theorem, but said it followed immediately from his criterion set forth in his previous article (Theorem 4.3). The writer is of the opinion that this may be true if the reader has studied "cyclotomic congruence equations". The writer had not until he encountered this theorem. The writer found the topic in several advanced abstract algebra books. Unfortunately, the explanation about them assumes a knowledge in "Galois" and "splitting field". Consequently, the writer has elected to discuss the theorem intuitively and show how to apply it.

Since t is defined as $10^{(q-1)/2P}$, then it is seen that $t^{2P} = 10^{q-1} \equiv 1 \pmod q$. The question is whether or not $t^h \equiv 1 \pmod q$ where h divides $2P$. This is where the cyclotomic equation comes into use as pointed out in the theorem.

Before considering an example it will be necessary to define the "Möbius Function":

Definition 4.1. The Möbius function is defined by the following equations:

$$\mu(1) = 1,$$

$$\mu(n) = (-1)^r \text{ if } n = p_1 p_2 \cdots p_r,$$

where the p_i's are distince primes,

$$\mu(n) = 0 \text{ if } p^2 \text{ divides } n \text{ for any prime } p.$$

Van der Waerden [35] showed the cyclotomic polynomials of order h are given by

$$F_h(x) = \prod_{d|h} (x^d - 1)^{\mu(h/d)}$$

where $\prod_{d|h}$ means the product over all the divisors of h.

For the first example consider the special case covered by Theorem 4.3. Thus $q = 1 + 2^a 3^b$ implies $P = 3$, $Q = 2$, and

$$F_6(x) = \prod_{d|6} (x^d - 1)^{\mu(6/d)}$$

$$= (x-1)^{\mu(6)}(x^2 - 1)^{\mu(3)}(x^3 - 1)^{\mu(2)}(x^6 - 1)^{\mu(1)}$$

$$= \frac{(x-1)(x^6 - 1)}{(x^2-1)(x^3 - 1)} = \frac{x^3 + 1}{x + 1} = x^2 - x + 1.$$

Hence, $t \equiv 10^{(q-1)/6} \pmod{q}$ has to satisfy the second degree cyclotomic congruence equation

$$x^2 - x + 1 \equiv 0 \pmod{q}.$$

This is the same congruence equation given in (1). It is said that when t satisfies the cyclotomic congruence equation then t is a "2Pth root of unity"[1] modulo q, but is <u>not</u> a "Dth root of unity" modulo q,

---

[1] x is said to be an "nth root of unity" modulo q if and only if $x^n \equiv 1 \pmod{q}$.

where D is any proper divisor of 2P. In the case of $q \equiv 1 + 2^a 3^b$ this means $t^6 \equiv 1 \pmod q$, but $t^2 \not\equiv 1$ and $t^3 \not\equiv 1 \pmod q$.

As the second example, let $q = 1 + 2^a 3^b 5^c$. Thus, $P = 3\cdot 5 = 15$ and $Q = 2\cdot 4 = 8$.

$$F_{30}(x) = \prod_{d|30} (x^d - 1)^{\mu(30/d)}$$

$$= (x-1)^{\mu(30)}(x^2 - 1)^{\mu(15)}(x^3 - 1)^{\mu(10)}(x^5 - 1)^{\mu(6)}$$

$$(x^6 - 1)^{\mu(5)}(x^{10} - 1)^{\mu(3)}(x^{15} - 1)^{\mu(2)}(x^{30} - 1)^{\mu(1)}$$

$$= \frac{(x^2 - 1)(x^3 - 1)(x^5 - 1)(x^{30} - 1)}{(x - 1)(x^6 - 1)(x^{10} - 1)(x^{15} - 1)}$$

$$= \frac{(x + 1)(x^{15} + 1)}{(x^3 + 1)(x^5 + 1)} = \frac{x^{16} + x^{15} + x + 1}{x^8 + x^5 + x^3 + 1}$$

$$= x^8 + x^7 - x^5 - x^4 - x^3 + x + 1.$$

Hence, for $t = 10^{(q-1)/30} \pmod q$, $1/q$ will have a maximum number of digits in its period or not according as t satisfies or does not satisfy the cyclotomic congruence equation

$$x^8 + x^7 - x^5 - x^4 - x^3 + x + 1 \equiv 0 \pmod q.$$

Before discussing the next theorem it is necessary to introduce material from number theory.

Definition 4.2. If q belongs to $\phi(r)$ modulo r, then q is called a primitive root modulo r.

Theorem 4.A. [36] An integer n has primitive roots if and only if it is 2, 4, p^m, or $2p^m$, where p is an odd prime.

Theorem 4.B. [70] If n is a primitive root of the odd prime p, and if $n^{p-1} \not\equiv 1 \pmod{p^2}$, then n is a primitive root of p^m, for any positive exponent m.

Rao [27], after observing Ayyanger's results and using the above properties, proved the following three results:

Theorem 4.5. If n is an odd number with a maximum recurring period $\phi(n)$ for $1/n$, then n must be of the form p^m, where $m \geq 1$ and p is a prime other than 5.

Proof: From Theorem 3.3 it is seen that if 10 belongs to a modulo n, then the recurring period consists of e digits. If $e = \phi(n)$ then the recurring period is the maximum number of digits. It follows that the recurring period of $1/n$ is a maximum when 10 is primitive root of n. But from Theorem 4.A it is known that there exist primitive roots of a number only when it is 2, 4, $2p^m$ or p^m, where p is an odd prime. Since n is odd, it follows that n must be of the form p^m. $p \neq 5$, since $k(5^n) = 1$ and not $\phi(5^n)$.

Corollary 4.6. Every number of the form p^m (p is an odd prime other than 5, and $m \geq 1$) which has 10 for a primitive root has a maximum recurring period.

Theorem 4.7. There are infinitely many odd numbers whose reciprocals have the maximum recurring period.

Proof: It is known that 10 belongs to 6 modulo 7, but $10^6 \not\equiv 1 \pmod{7^2}$. Consequently, by Theorem 4.B, it can be said that 10 is a primitive root of 7^m, for any positive integer m. Hence, the theorem is proved.

The obvious question to ask after Theorem 4.7 is: "Are there infinitely many odd primes p such that $k(p) = p - 1$?" In checking Appendix A, it is found that among the first 150 odd primes there are 53 primes whose reciprocals have the maximum recurring period. The writer could find no record of this question being answered. Rao said that he felt that it was yes, but he was unable to prove it.

The Conditions for $d_r = a_r$

On evaluating the reciprocal of a prime number, one often finds that a digit in the quotient is the same as a corresponding remainder, i.e., $d_r = a_r$. For primes 19 and 29, the above property results 9 times. It is possible to give necessary conditions for this property.

From the Division Algorithm it follows that:

(1) $$\frac{10a_{r-1}}{p} = d_r + \frac{a_r}{p},\ 0 \leq d_r \leq 9;\ 0 < a_r < p.$$

If $d_r = a_r$, then (1) becomes

(2) $$10a_{r-1} = (p+1)a_r.$$

From (2) it is possible to deduce the following theorem:

Theorem 4.8. [31] (i) For primes of the form $p = 10n + s$, where $s = 1, 3, 7$. Then $d_r = a_r$ only if $a_r = 5$ has a solution. (ii) For primes of the form $p = 10n + 9$, $d_r = a_r = b$ is possible for all values of b, $(1 \leq b \leq 9)$, except for those values of b for which $a_{r-1} = b(n+1)$ does not have integral solutions.

Proof: (i) Substituting $p = 10n + s$ into (2) gives

$$10a_{r-1} = (10n + s + 1)a_r, \text{ or}$$

(3) $5a_{r-1} = (5n + d)a_r$, where $2d = s + 1$, $d = 1, 2, 4$.

Now, the left-hand side of (3) is divisible by 5, hence $(5n + d)a_r$ is divisible by 5. Since 5 does not divide $(5n + d)$, then 5 divides a_r or $a_r = 5h$. But $0 < a_r = d_r \leq 9$ implies that $h = 1$ or $a_r = 5$. It is noted that when $k(p) = p-1$ then $d_r = a_r = 5$ only once, since a_r takes on all values between 1 and p-1 one and only one time.

(ii) Substituting $p = 10n + 9$ into (2) gives

$$10a_{r-1} = (10n + 10)a_r, \text{ or}$$

(4) $$a_{r-1} = (n + 1)a_r.$$

Therefore, for all values of a_{r-1} which are divisible by $(n+1)$, equation (4) has integral solutions. Hence, $d_r = a_r = b$, $(1 \leq b \leq 9)$ except for those values of b for which $a_{r-1} = b(n+1)$ does not have integral solutions.

<u>Corollary 4.9</u>. [31] For primes of the form $p = 10n + 9$, $d_r = a_r = b$ for every b such that $1 \leq b \leq 9$, where $k(p) = p-1$.

Proof: If $k(p) = p-1$, then a_r takes on all values between 1 and $(p-1)$ inclusive. Hence, $d_r = a_r = b$ for every b such that $1 \leq b \leq 9$.

Below are given two examples to illustrate the properties discussed above.

(i) For $p = 17$, $k(17) = 16$ and from Theorem 4.8(i) there exists only one pair $d_r = a_r = 5$. This is shown in the table given below.

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
d_r	0	5	8	8	2	3	<u>5</u>	2	9	4	1	1	7	6	4	7
a_r	10	15	14	4	6	9	<u>5</u>	16	7	2	3	13	11	8	12	1

(ii) For $p = 19$, $k(19) = 18$ and from Theorem 4.8(ii) there exist exactly 9 pairs $d_r = a_r = b$ where $1 \leq b \leq 9$.

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
d_r	0	$\underline{5}$	2	$\underline{6}$	$\underline{3}$	1	5	7	8	$\underline{9}$	4	$\underline{7}$	3	6	$\underline{8}$	$\underline{4}$	$\underline{2}$	$\underline{1}$
a_r	10	$\underline{5}$	12	$\underline{6}$	$\underline{3}$	11	15	17	18	$\underline{9}$	14	$\underline{7}$	13	16	$\underline{8}$	$\underline{4}$	$\underline{2}$	$\underline{1}$

Rao [27] made an interesting observation about Daljet Singh's results, namely, for those numbers of the form p^m such that $k(p^m) = \phi(p^m)$, the conclusions of Corollary 4.9 still hold with slight modification. The simplest example to illustrate his observation is for $p^m = 7^2 = 49$.

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14
d_r	0	$\underline{2}$	0	$\underline{4}$	0	$\underline{8}$	1	6	3	2	6	5	$\underline{3}$	0
a_r	10	$\underline{2}$	20	$\underline{4}$	40	$\underline{8}$	31	16	13	32	26	15	$\underline{3}$	30

r	15	16	17	18	19	20	21	22	23	24	25	26	27	28
d_r	$\underline{6}$	1	2	2	4	4	8	9	7	9	5	$\underline{9}$	1	8
a_r	$\underline{6}$	11	12	22	24	44	48	39	47	29	45	$\underline{9}$	41	18

r	29	30	31	32	33	34	35	36	37	38	39	40	41	42
d_r	3	6	7	3	4	6	9	3	8	7	7	5	$\underline{5}$	$\underline{1}$
a_r	33	36	17	23	34	46	19	43	38	37	27	25	$\underline{5}$	$\underline{1}$

In this particular case, there exists exactly 8 pairs $d_r = a_r = b$ where b takes all values from 1 to 9 except the value 7. This should be the case since 7 is not prime to 7^2. For all other values of p and p^m that are of the form 10n + 9 and have 10 for a primitive root, there exists exactly 9 cases in which $d_r = a_r = b$ and $1 \leq b \leq 9$. By Theorem 4.7, it follows that there are infinitely many numbers of the form 10n + 9 having the property stated above, because $7^{4h-2} \equiv 9 \pmod{10}$ for all positive integers h.

The Frequency of Digits

In Chapter III, it was shown that the period length for m/n depended only on n, and if $k(n) = k$, then there exists $(n-1)/k = c$ distinct cycles of numerators or remainders, and with each cycle there was a "distinct" period. By a "distinct" period, it is meant that any cyclic permutation of the digits of a period gives the same "distinct" period. Schiller [30] stated and proved a theorem and corollary about the frequency with which a given digit appears in the collection of distinct periods for a given prime.

Theorem 4.10. Consider the collection of distinct periods of any prime. If t_d is the total number of times the digit d appears in these periods, then $|t_d - t_{d'}| \leq 1$, for $d \neq d'$.

This means simply that, as far as possible, every digit appears just as often as any other digit.

Proof: Consider the set of all m_s such that m_s/p gives the same distinct period for $s = 1, 2, \ldots, k(p)$. Now, each digit in a distinct period will appear once in the first place after the decimal point. This "first" digit, say d, is found by solving the equation $10m_s = dp + r$, for $0 < r < p$. From Chapter III, it is known that $k(p)$ divides $p-1$ and the total number of digits used in all distinct periods is $p-1$. The question now resolves itself into the following: For each d ($0 \leq d \leq 9$), how many m_s's exist over all the distinct periods so that both the equation and the condition on r are satisfied?

If the distinct period associated with a numerator m is not important, then the subscript s will not be used. In order for the condition on r to be satisfied, the following inequalities must hold:

(1) $$dp < 10m < (d + 1)p.$$

Also if m' is the least such m, and $m' + t_d$ is the greatest such m, then

$$dp < 10m' \leq 10(m' + t_d) < (d + 1)p.$$

Hence,

(2) $$dp + 10t_d < (d + 1)p, \text{ and } p > 10t_d$$

On the other hand, since m' is the least m, and $m' + t_d$ is the greatest,

$$10(m' + t_d + 1) > (d + 1)p, \text{ and } 10(m' - 1) < dp.$$

Therefore,

$$10(m' + t_d + 1) > (d + 1)p > 10(m' - 1) + p, \text{ and}$$

(3) $$10(t_d + 2) > p.$$

Finally, (2) and (3) result in

(4) $$p/10 > t_d > p/10 - 2.$$

Now, all the integers between m' and $m' + t_d$ satisfy (1), therefore the total number of m's would be $t_d + 1$. The reader will note that (4) implies t_d can have two possible values. Therefore, let $t_d = h$ or $h + 1$ for the two possible values. Since t_d did not depend on d, then for any two d's, say d and d', $|t_d - t_{d'}| \leq 1$.

Corollary 4.11. If $p = 10n + r$, then $(11 - r)$ digits appear n times, and $(r - 1)$ digits appear $n + 1$ times.

Proof: The set of all digits can be partitioned into two sets, one containing those that appear h times and the other containing those that appear $h + 1$ times. If there are x distinct digits in the first set, then there would be 10 - x distinct digits in the second set. The total number of digits in all the distinct periods is p-1. Thus,

$$p - 1 = xh + (10 - x)(h + 1), \text{ or}$$

$$10n + r - 1 = xh + 10h - xh + 10 - x, \text{ or}$$

$$(5) \qquad 10n + r = 10h + (11 - x).$$

Since $1 \leq r \leq 9$ and $1 \leq x \leq 9$, equation (5) implies $n = h$; $r = 11 - x$ or $x = 11 - r$ and $10 - x = r - 1$. Hence, the conclusion follows.

Schiller defined "deficient" and "excessive" digits by saying that the digits which appear n times are deficient and those which appear n + 1 times are excessive.

If $p = 10n + 1$, then the corollary says that all digits are deficient. For the remaining cases it is found that $r = 3$ implies two excessive digits; $r = 7$ implies four deficient digits; $r = 9$ implies two deficient digits. Can these excessive and deficient digits be identified? By working with (1), the answer is seen to be yes.

Considering the case $r = 3$, the reader will note that for $d = 3$, $10(3n) + 9 < 10m < 10(4n + 1) + 2$ implies $m = 3n + 1, 3n + 2, \ldots, 4n, 4n+1$. Therefore, m can have n + 1 values which implies $d = 3$ is excessive. For $d = 9$, $10(9n + 2) + 7 < 10m < 10(10n + 3)$ implies $m = 9n + 3, 9n + 4, \ldots, 10n + 1, 10n + 2$. Therefore, m can have n values which implies $d = 9$ is deficient. Listed below are all of the inequalities for each of the digits.

$d = 0$, $\quad 0 < 10m < 10n + 3$, implies 0 is deficient;

$d = 1$, $10n + 3 < 10m < 20n + 6$, implies 1 is deficient;

$d = 2$, $20n + 6 < 10m < 30n + 9$, implies 2 is deficient;

$d = 3$, $30n + 9 < 10m < 40n + 12$, implies 3 is excessive;

$d = 4$, $40n + 12 < 10m < 50n + 15$, implies 4 is deficient;

$d = 5,\ 50n + 15 < 10m < 60n + 18,$ implies 5 is deficient;

$d = 6,\ 60n + 18 < 10m < 70n + 21,$ implies 6 is excessive;

$d = 7,\ 70n + 21 < 10m < 80n + 24,$ implies 7 is deficient;

$d = 8,\ 80n + 24 < 10m < 90n + 27,$ implies 8 is deficient;

$d = 9,\ 90n + 27 < 10m < 100n + 30,$ implies 9 is deficient;

In general, it is seen that an excessive digit occurs whenever the constant terms differ in the second digit and the first digit of the larger number is not zero.

The cases of $r = 7$ and 9 follow similarly and for $r = 7$ it is found that 0, 3, 6, and 9 are deficient; for $r = 9$ it is found that 0 and 9 are deficient.

Four examples which illustrate the corollary and the observation about which digits are excessive or deficient are as follows:

(A) For the case $r = 1$ consider $p = 31$, the two distinct periods are

032258064516129; 096774193548387.

The reader will note that each digit appears $n = 3$ times.

(B) For the case $r = 3$ consider $p = 73$, the nine distinct periods are

01369863; 02739726; 04109589; 05479452; 06849315;

08219178; 12328767; 16438356; 24657534.

It is seen that the digits 3 and 6 appear eight times and the other appear 7 times.

(C) For the case $r = 7$ consider $p = 47$, the one distinct period is

0212765957446808510638297872340425531914893617.

The digits 0, 3, 6, 9 appear four times and 1, 2, 4, 5, 7, 8 appear five times.

(D) For the case r = 9 consider p = 29, the one distinct period is

0344827586206896551724137931.

The digits 0, 9 appear twice and the others appear three times.

Other Properties of Recurring Decimals

The theorems of this section are not necessarily interrelated, but just show more properties of recurring decimals.

Theorem 4.12. If $k(n) = 2m$, then

$$\frac{1}{n} = \frac{A + 1}{10^m + 1},$$

where A is the first half of the period.

Proof: By Corollary 2.4,

(1) $$\frac{1}{n} = \frac{[a_1a_2\ldots a_mb_1b_2\ldots b_m]}{10^{2m} - 1}$$

where $a_1a_2\ldots a_m$ and $b_1b_2\ldots b_m$ are the two half periods.

Now, $$[a_1a_2\ldots a_m] + [b_1b_2\ldots b_m] = 10^m - 1,$$

by Corollary 3.14. This implies (1) can be changed as follows:

$$\frac{1}{n} = \frac{10^m[a_1a_2\ldots a_m] + [b_1b_2\ldots b_m]}{(10^m + 1)(10^m - 1)}$$

$$= \frac{10^m[a_1a_2\ldots a_m] + (10^m - 1) - [a_1a_2\ldots a_m]}{(10^m + 1)(10^m - 1)}$$

$$= \frac{(10^m - 1)([a_1a_2\ldots a_m] + 1)}{(10^m + 1)(10^m - 1)}$$

$$= \frac{A+1}{10^m+1}, \quad A = [a_1a_2\ldots a_m],$$ i.e., the first half period.

According to Dickson, A. Ricke proved this theorem in 1887.

Umansky [34] used Theorem 4.12 in proving the following theorem:

<u>Theorem 4.13</u>. If A and B are the first and second halves, respectively, of the period for 1/p, where p is prime and k(p) = 2m, then

$$\frac{B}{A} = p - 1 + \frac{p - 2}{A} .$$

Proof: From Theorem 4.12 it is seen that

$$\frac{1}{p} = \frac{(A + 1)}{10^m + 1}, \text{ or } p = \frac{10^m + 1}{A + 1} .$$

Corollary 3.14 gives $A + B = 10^m - 1$.

Therefore, $$A + B + 2 = 10^m + 1.$$

By substitution, $p = \frac{A + B + 2}{A + 1}$, or $Ap + p = A + B + 2$.

Hence, $$B = A(p - 1) + (p - 2), \text{ or}$$

$$\frac{B}{A} = (p - 1) + \frac{p - 2}{A} .$$

Examples: $1/7 = .\dot{1}4285\dot{7}$ implies $857/142 = 6 + 5/142$. $1/13 = .\dot{0}7692\dot{3}$ implies $923/076 = 12 + 11/076$.

<u>Theorem 4.14</u>. If (n,10) = 1, then there exists a number n' such that every digit in the product nn' is 1.

Proof: Case I; Assume (n, 3) = 1. Now, by Corollary 2.4

$$\frac{1}{n} = \frac{[d_1 d_2 \ldots d_k]}{10^k - 1} , \text{ or}$$

$$10^k - 1 = n[d_1 d_2 \ldots d_k].$$

Since 9 divides $10^k - 1$ and $(n, 3) = 1$, then 9 divides $[d_1 d_2 \ldots d_k]$.
Hence,

$$11\ldots1(\text{k digits}) = n[d_1' d_2' \ldots d_k'] = nn'$$

where

$$[d_1 d_2 \ldots d_k]/9 = [d_1' d_2' \ldots d_k'].$$

Case II; Assume $(n, 3) = 3$. Now,

$$\frac{1}{9n} = \frac{[d_1 d_2 \ldots d_k]}{10^k - 1}, \text{ or}$$

$$10^k - 1 = 9n[d_1 d_2 \ldots d_k].$$

Hence, $$11\ldots1(\text{k digits}) = n[d_1 d_2 \ldots d_k] = nn'.$$

For example, $n = 41$ implies $n' = 271$, since

$$\frac{1}{41} = \frac{2436}{10^6 - 1}, \text{ gives}$$

$$99999 = 41 \cdot 2439, \text{ or}$$

$$11111 = 41 \cdot 271.$$

To illustrate the second case, let $n = 123$, then

$$\frac{1}{9 \cdot 123} = \frac{1}{1107} = .\dot{0}0090334236675\dot{7}, \text{ or}$$

$$10^{15} - 1 = 9 \cdot 123 \cdot 903342366757.$$

Hence, $$11\ldots1(15 \text{ digits}) = 123 \cdot 903342366757.$$

Rollett [29] gave the following list of factors of $11\ldots1$(k digits), $k = 2$ to 21

$$11 = 11$$

$$111 = 3 \cdot 37$$

$$1{,}111 = 11 \cdot 101$$

$$11{,}111 = 41 \cdot 271$$

$$111{,}111 = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$$

$$1{,}111{,}111 = 239 \cdot 4649$$

$$11{,}111{,}111 = 11 \cdot 73 \cdot 101 \cdot 137$$

$$111{,}111{,}111 = 3^2 \cdot 37 \cdot 333667$$

$$1{,}111{,}111{,}111 = 11 \cdot 41 \cdot 271 \cdot 9091$$

$$11{,}111{,}111{,}111 = 21649 \cdot 513239$$

$$111{,}111{,}111{,}111 = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \cdot 101 \cdot 9901$$

$$1{,}111{,}111{,}111{,}111 = 53 \cdot 79 \cdot 265371653$$

$$11{,}111{,}111{,}111{,}111 = 11 \cdot 239 \cdot 4649 \cdot 909091$$

$$111{,}111{,}111{,}111{,}111 = 3 \cdot 31 \cdot 37 \cdot 41 \cdot 271 \cdot 2906161$$

$$1{,}111{,}111{,}111{,}111{,}111 = 11 \cdot 17 \cdot 73 \cdot 101 \cdot 137 \cdot 5882353$$

$$11{,}111{,}111{,}111{,}111{,}111 = 2071723 \cdot 5363222357$$

$$111{,}111{,}111{,}111{,}111{,}111 = 3^2 \cdot 7 \cdot 11 \cdot 12 \cdot 19 \cdot 37 \cdot 52579 \cdot 333667$$

$$1{,}111{,}111{,}111{,}111{,}111{,}111 = \text{a prime}$$

$$11{,}111{,}111{,}111{,}111{,}111{,}111 = 11 \cdot 41 \cdot 101 \cdot 271 \cdot 3541 \cdot 9091 \cdot 27961$$

$$111{,}111{,}111{,}111{,}111{,}111{,}111 = 3 \cdot 37 \cdot 43 \cdot 239 \cdot 1933 \cdot 4649 \cdot 10838689$$

This table can be used to find the k-function values for the prime factors used, and also the periods for the reciprocals of the primes. For example,

(2) $$111{,}111 = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$$

implies $$\frac{1}{7} = \frac{3 \cdot 11 \cdot 13 \cdot 37}{111{,}111} = \frac{9 \cdot 3 \cdot 11 \cdot 13 \cdot 37}{999{,}999}$$

$$\frac{1}{7} = \frac{142{,}857}{999{,}999} = .\dot{1}4285\dot{7}.$$

Also, (2) implies that k(3), k(7), k(11), k(13) and k(37) are less than, or equal to six. Now, since 73 appears for the first time as a factor in 11,111,111, and it is the smallest prime that appears at this time, then 73 is the smallest prime which has period length of 8.

Using the same type of reasoning for each line above, the following table [29] is constructed where p_n is the smallest prime such that $k(p_n) = n$.

n	p_n	n	p_n	n	p_n
1	3	9	333,667	17	2,071,723
2	11	10	9,091	18	19
3	37	11	21,649	19	$(10^{19}-1)/9$
4	101	12	9,901	20	3,541
5	41	13	53	21	43
6	7	14	909,091	22	23
7	239	15	31	23	?
8	73	16	5,882,353		

CHAPTER V

TECHNIQUES FOR FINDING THE PERIOD

Multiplying to Find the Period

The problem of finding the period of certain primes is quite minor if k(p) is small. The procedure to use in this case is the standard division process. The reader is aware that the k-function values follow no set pattern. For example, consider the twin primes 269 and 271. From the appendix it is found that k(269) = 268, but k(271) = 5.

Now, the first method discussed uses the standard division process. If the process is continued until a "small" remainder r_1 is found, then $1/n = q_1 + r_1/n$. Multiply each side of this equation by r_1, obtaining $r_1/n = r_1q_1 + r_1^2/n$. If $r_1^2 < n$, then let $q_2 = r_1q_1$ and $r_2 = r_1^2$. If $r_1^2 > n$, then let $r_1^2/n = t + r_2/n$ and $q_2 = r_1q_1 + t$. Either case results in a new equation $r_1/n = q_2 + r_2/n$. This process of multiplying by r_1 can be repeated as many times as necessary. The period of $1/n$ is the digits of q_1, q_2, q_3, ... taken in succession until the digits start to repeat.

As an illustration of the method consider 1/29. Long division gives the first of these equations and the others are obtained by successive multiplication of each side by 8.

$$\frac{1}{29} = .03448\tfrac{8}{29}$$

$$\frac{8}{28} = .27586\tfrac{6}{29}$$

$$\frac{8^2}{29} \text{ or } 2\tfrac{6}{29} = 2.20689\tfrac{19}{29}$$

$$\frac{8^3}{29} \text{ or } 17\tfrac{19}{29} = 17.65517\tfrac{7}{29}$$

$$\frac{8^4}{29} \text{ or } 141\tfrac{7}{29} = 141.24137\tfrac{27}{29}$$

$$\frac{8^5}{29} \text{ or } 1129\tfrac{27}{29} = 1129.93103\tfrac{13}{29}$$

Therefore, the period for 1/29 is

$$.\dot{0}34482758620689655172413793\dot{1}.$$

In actual practice the left side of each equation would be entirely omitted, as would also the numbers 2, 17, 141, and 1129 that precede the decimal point. In this abridged form the calculation for 1/49 would be

$$.0204081632653\tfrac{3}{49}$$

$$.0612244897959\tfrac{9}{49}$$

$$.1836734693877\tfrac{27}{49}$$

$$.5510204081632\tfrac{32}{49}$$

Hence, the period for 1/49 is

$$.\dot{0}2040816326530612244897959183673469387755\dot{1},$$

since the digits started to repeat after the "551" in the fourth line. It is interesting to note that Nygaard [25] applied this technique to find the period for 1/487. His q_1 turns out to have 45 digits and r_1 is 5.

This method was also discussed by Glaisher [13], in 1879. He points out an interesting observation of what happens when the remainder is 2 or 5. For example;

$$\frac{1}{61} = .0163934426229\underline{5}\frac{5}{61} \qquad (1)$$

and the remainder of the period "...can be obtained by halving the figures from the commencement."

$$\frac{1}{61} = .0163934426229\underline{5} \qquad (2)$$

$$^{1}0819672131147 \qquad (3)$$

$$^{1}\underline{5}409836065573 \qquad (4)$$

$$^{1}7\underline{7}04918032786 \qquad (5)$$

$$8\underline{85}2459016393 \qquad (6)$$

Hence,

$1/61 = .\dot{0}16393442622950819672131147540983606557377049180327868852459\dot{9}$.

(Note: The superscript 1's are the remainders from previous lines.) Glaisher made no attempt to explain why the above example works. The explanation should prove to be instructive for the reader. From (1) it is seen that

$$\frac{5}{61} = \frac{1}{2}\cdot\frac{10}{61} = \frac{1}{2}(0.16393442622950\frac{50}{61})$$

$$= \underline{0}.0819672131147\underline{5}\frac{25}{61} \qquad (7)$$

$$\frac{25}{61} = \frac{1}{2}\cdot\frac{50}{61} = \frac{1}{2}(0.81967213114750\frac{250}{61})$$

$$= \underline{0}.4098360655737\underline{5}\frac{125}{61} \qquad (8)$$

$$\frac{125}{61} = \frac{1}{2}\cdot\frac{250}{61} = \frac{1}{2}\left(4.09836065573750\frac{1250}{61}\right)$$

$$= \underline{2}.049180327868\underline{75}\frac{625}{61} \qquad (9)$$

$$\frac{625}{61} = \frac{1}{2}\cdot\frac{6250}{61} = \frac{1}{2}\left(20.49180327868750\frac{6250}{61}\right)$$

$$= \underline{10}.2459016393\underline{4375}\frac{3125}{61} \qquad (10)$$

The reader will note that the "integral" part of (7) is added to the last digit of (1), and the integral part of (8) is added to the last digit of (7), and the pattern continues through (10). This allows the process to continue from part to part as shown in lines (2) through (6). (Note: It is the digits which are underlined that are affected by the "carry over" from $r_1 r_i$ being greater than n.)

An interesting and simpler example is 1/19. Since

$$\frac{1}{19} = .05\frac{5}{19},$$

the period for 1/19 can be found as follows:

$$1/19 = .\dot{0}5^{1}263^{1}1^{1}5^{1}7^{1}89^{1}47^{1}3^{1}6842\dot{1}.$$

The superscript 1's are the remainders from the division process in the previous digit.

When a remainder of 50 is found as with 1/199, the process becomes, multiply by 100 and divide by 2. This results in taking two digits at a time as shown below:

$$1/199 = .005025^{1}1256281407^{1}03^{1}51^{1}75^{1}87^{1}93^{1}969849\cdots.$$

The work Nygaard went to in finding the period for 487 could certainly have been simplified by the use of Glaisher's method, since it is much easier and faster to divide by two than multiplying by five.

Finding the Period from "Right-to-Left"

In this section it is necessary to recall the relation between a prime "p" and the d_i's and a_i's found in the division process used to determine the period of p, i.e.,

(1) $$10a_r - d_r p = a_{r+1}.$$

The first case to consider is when $p = 10n - 1$. Assume $k(p) = k$, therefore, $a_{k+1} = 1$ and

$$10a_k - d_k(10n-1) = 1, \text{ or}$$

(2) $$10(a_k - nd_k) + d_k = 1$$

Hence,

(3) $$d_k \equiv 1 \pmod{10}.$$

Since $0 \leq d_k \leq 9$, then $d_k = 1$. Let $t_1 = a_k - nd_k$, or

(4) $$a_k = nd_k + t_1.$$

By the division algorithm, the next equation would be

$$10a_{k-1} - d_{k-1}(10n - 1) = a_k$$

$$10(a_{k-1} - nd_{k-1}) + d_{k-1} = a_k.$$

Hence,

(5) $$d_{k-1} \equiv a_k, \pmod{10} \text{ and}$$

(6) $$a_{k-1} = nd_{k-1} + t_2.$$

Consider the general equation

$$10a_{k-j} - d_{k-j}(10n - 1) = a_{k-j+1}, \text{ or}$$

$$10(a_{k-j} - nd_{k-j}) + d_{k-j} = a_{k-j+1}, \; j = 1, 2, \ldots, k - 1.$$

Hence,

(7) $$d_{k-j} \equiv a_{k-j+1} \pmod{10}, \text{ and}$$

(8) $$a_{k-j} = nd_{k-j} + t_{j+1}.$$

The reader will note that once d_k and t_1 are determined from (2) and (3), then the process can be started. It is continued by equations (4), (5), (6), (7), and (8). If k is not known, the process can be continued until the digits start to repeat. Since $k \leq p - 1$, the process never needs to be more than $p - 1$ steps long.

As an example to illustrate this process consider $p = 29 = 3 \cdot 10 - 1$. Hence, $n = 3$ and $d_k = 1$ implies

$$d_{k-1} \equiv 3 \cdot 1 \pmod{10}, \text{ or}$$

$$d_{k-1} = 3 \text{ and } t_1 = 0.$$

$$d_{k-2} \equiv 3 \cdot 3 + 0 \pmod{10}, \text{ implies}$$

$$d_{k-2} = 9 \text{ and } t_2 = 0.$$

Therefore, $$d_{k-3} \equiv 3 \cdot 9 + 0 \pmod{10}, \text{ implies}$$

$$d_{k-3} = 7 \text{ and } t_3 = 2.$$

Therefore, $$d_{k-4} \equiv 3 \cdot 7 + 2 \pmod{10}, \text{ implies}$$

$$d_{k-4} = 3 \text{ and } t_4 = 2, \text{ and so on.}$$

The process can be shortened further to where the process would be as follows:

$$1/29 = .\dot{0}^1 3^1 4^2 4 8^2 2^1 7^2 5^1 8 6 2^2 0^2 6^2 8^1 9^1 6^1 5 5^2 1 7^1 2 4^1 1^2 3^2 7 9 3 \dot{1}$$

The superscripts are the t_j-values which are not zero and are added to the product $a_k d_{k-j}$ to give $d_{k-(j+1)}$.

This process works equally well for composites as well as primes. The reader will note the fact that "p" is a prime was not used. Considering the process for 39, $n = 4$, $d_k = 1$ and

$$1/39 = .\dot{0}^{2}2^{2}5^{1}64\dot{1}.$$

The case of p = 10n - 1 was pointed out by Chartres [10] and two months later Toy [32] discussed the other cases of p = 10n + 7, p = 10n + 3, and p = 10n + 1. For the case p = 10n - 1, it was seen that $d_k = 1$ and n was the multiplier used to generate the digits. Toy, without any explanation, gave the d_k's and multipliers for the other three cases. The remainder of this section will be devoted to explaining Toy's observation.

Let p (not necessarily prime) be 10n + 7. Hence (1) becomes $10a_k - pd_k = 1$, or $10a_k - d_k(10n + 7) = 1$.

(12) $$10(a_k - nd_k) = 7d_k + 1$$

This implies $$7d_k + 1 \equiv 0 \pmod{10}.$$

Since d_k is an integer and $0 \leq d_k \leq 9$, it can be concluded that $d_k = 7$. Substituting this result back into (12) gives

$$10(a_k - 7n) = 49 + 1, \text{ or}$$

$$a_k = 7n + 5.$$

The explanation from this point on follows the process for p = 10n - 1, and $a_k = 7n + 5$ will be the multiplier for this case. As an example, consider $1/17 = .\dot{0}58823529411764\dot{7}$. Hence, n = 1 and $a_k = 12$ and the remaining digits are found from the following congruences modulo 10:

$d_{k-1} \equiv a_k \cdot d_k \equiv 12\cdot 7$, implies $d_{k-1} = 4$ and $t_1 = 8$.

$d_{k-2} \equiv a_k d_{k-1} + t_1 \equiv 12\cdot 4 + 8$ implies $d_{k-2} = 6$ and $t_2 = 5$.

$d_{k-3} \equiv a_k d_{k-2} + t_2 \equiv 12\cdot 6 + 5$ implies $d_{k-3} = 7$ and $t_3 = 7$.

For p = 10n + 3, equation (1) becomes,

$$10a_k - d_k(10n + 3) = 1, \text{ or}$$

$$(13) \qquad 10(a_k - nd_k) = 3d_k + 1.$$

This implies $3d_k + 1 \equiv 0 \pmod{10}$, and $d_k = 3$. Substituting this result into (13) gives,

$$10(a_k - 3n) = 10, \text{ or}$$

$$a_k = 3n + 1.$$

Hence, for the case $p = 10n + 3$ the last digit is 3 and the multiplier is $3n + 1$. For example, $p = 13$ gives $n = 1$ and $a_k = 4$ and the following congruences modulo 10:

$d_{k-1} \equiv 4 \cdot 3$, implies $d_{k-1} = 2$ and $t_1 = 1$.

$d_{k-2} \equiv 4 \cdot 2 + 1$, implies $d_{k-2} = 9$ and $t_2 = 0$.

$d_{k-3} \equiv 4 \cdot 9 + 0$, implies $d_{k-3} = 6$ and $t_3 = 3$.

$d_{k-4} \equiv 4 \cdot 6 + 3$, implies $d_{k-4} = 7$ and $t_4 = 2$.

$d_{k-5} \equiv 4 \cdot 7 + 2$, implies $d_{k-5} = 0$ and $t_5 = 3$.

$d_{k-6} \equiv 4 \cdot 0 + 3$, implies $d_{k-6} = 3$ and $t_6 = 0$.

$d_{k-7} \equiv 4 \cdot 3 + 0$, implies the process can stop, since $d_{k-7} = d_{k-1}$ and steps would start to repeat. Hence,

$$1/13 = .\dot{0}7692\dot{3}.$$

For the last case $p = 10n + 1$, equation (1) becomes,

$$10a_k - d_k(10n + 1) = 1$$

$$(14) \qquad 10(a_k - nd_k) = d_k + 1.$$

This implies, $d_k + 1 \equiv 0 \pmod{0}$, and $d_k = 9$. Substituting this result into (14) gives

$$10(a_k - 9n) = 10, \text{ or}$$

$$a_k = 9n + 1.$$

Hence, for this case the multiplier is $9n + 1$. The following is the process for $p = 21$:

$$d_{k-1} \equiv 19 \cdot 9 \text{ implies } d_{k-1} = 1 \text{ and } t_1 = 17$$

$$d_{k-2} \equiv 19 \cdot 1 + 17 \text{ implies } d_{k-2} = 6 \text{ and } t_2 = 3$$

$$d_{k-3} \equiv 19 \cdot 6 + 3 \text{ implies } d_{k-3} = 7 \text{ and } t_3 = 11$$

$$d_{k-4} \equiv 19 \cdot 7 + 11 \text{ implies } d_{k-4} = 4 \text{ and } t_4 = 14$$

$$d_{k-5} \equiv 19 \cdot 4 + 14 \text{ implies } d_{k-5} = 0 \text{ and } t_5 = 9$$

$$d_{k-6} \equiv 19 \cdot 0 + 9 \text{ implies } d_{k-6} = 9 \text{ and } t_6 = 0$$

$$d_{k-7} \equiv 19 \cdot 9 + 0 \text{ implies } d_{k-7} = d_{k-1} \text{ and the process stops.}$$

Therefore, $1/21 = .\dot{0}4761\dot{9}$.

Two observations can now be made. First, if a person is concerned with only pure recurring decimals, then the four cases discussed are the only cases possible. Second, the process can be used for numerators different than 1 by using the same multiplier. For example, 2/13 implies

$$10a_k - d_k(10n + 3) = 2, \text{ or}$$

$$10(a_k - nd_k) = 3d_k + 2.$$

Hence, $$3d_k + 2 \equiv 0 \pmod{10}, \text{ and } d_k = 6.$$

The multiplier for 13 was 4 and

$$2/13 = .\dot{1}^1 5^2 3^1 8^2 4\dot{6}.$$

In conclusion, it can be said that this "right-to-left" method can be used to convert any rational number of the form a/b where $(b,10) = 1$. Secondly, the process is good in any base, since it was based on modulo "10", where "10" is one base.

"Bose" Numbers

R. C. Das [11] in an article in the American Mathematical Monthly gave an explanation of a method for finding the period of the fraction 1/m. He credits N. C. Bose Majumdar for the method and names one of the numbers in the method after Bose. The reader will note that Bose's method is precisely the process described in the previous section, but Das' explanation is different from the author's.

The first term defined is the "End Number". It is the smallest integer e such that when multiplied by m gives a number ending in 9. The "Bose Number b" is defined by the equation

(1) $$e_1 m = 10b - 1.$$

Bose's method consists of writing the End Number e_1, multiplying it by the Bose Number b, and placing the last digit of the product before e_1, calling it e_2 and carrying over the remaining digits, multiplying e_2 by b and adding the number carried over and writing the last digit of this sum e_3, and so on, until the digits recur.

The reader will note that (1) of this section, comes directly from (1) of the previous section, where $e_1 = d_k$ and $b = a_k$. From the illustration for p = 21, it was found that $a_k = 19 = b$ and $d_k = 9 = e_1$ and the period for 1/21 is found by Bose's method as follows:

> Write down the End Number 9 first, multiply this 9 by the Bose Number 19 and obtain 171. Place 1 before 9 and carry 17; multiply this 1 by the Bose Number 19 and add 17, obtaining 36. Place 6 before 1 and carry 3; multiply this by 19 and adding 3 gives 117. Place 7 before 6 and carry 11 and so on. Continue the process until the numbers recur.

The reader will note that the amount carried each time is precisely the same as the "t" used in the last section.

Das, in his explanation, does not go "backward" in the division process as the writer has done, but considers the entire period at one time.

Suppose $k(m) = k$ and

$$\frac{1}{m} = \frac{[e_k e_{k-1} \cdots e_2 e_1]}{10^k - 1} .$$

Let $r = [e_k e_{k-1} \cdots e_2 e_1]$, then

(2) $$1/m = r/(10^k - 1), \text{ and}$$

(3) $$r = e_1 + 10e_2 + 10^2 e_3 + \ldots + 10^{k-1} e_k.$$

Then Bose's method indicates that the successive digits of r (counting from the right toward the left) may be obtained as follows: the first digit is e_1; the first two digits of $e_1 + 10be_1$ give the first two digits of r; the first three digits $e_1 + 10be_1 + 10^2be_2$ give the first three digits of r; and in general the first i digits of $e_1 + 10be_1 + 10^2be_2 + \ldots + 10^{i-1}be_{i-1}$ are the first i digits of r, i = 2, 3, ..., k.

For example, m = 21, 21·9 = 189 = 10·19-1; $e_1 = 9$, b = 19. Then

10·19·9 is

$$\begin{array}{r} 1710 \\ 9 \\ \hline 1719 \end{array}$$ showing e_2 is 1.

10^2·19·1 is

$$\begin{array}{r} 1900 \\ 1710 \\ 9 \\ \hline 3619 \end{array}$$ showing e_3 is 6.

10^3·19·6 is

$$\begin{array}{r} 114000 \\ 1900 \\ 1710 \\ 9 \\ \hline 117619 \end{array}$$ showing e_4 is 7.

This can be condensed into the form

$$\begin{array}{r}
9\\
1710 = 10^{1}\cdot 19\cdot 9\\
1900 = 10^{2}\cdot 19\cdot 1\\
114000 = 10^{3}\cdot 19\cdot 6\\
1330000 = 10^{4}\cdot 19\cdot 7\\
\underline{7600000} = 10^{5}\cdot 19\cdot 4\\
047619 = r, \text{ so } 1/21 \text{ is } .\dot{0}4761\dot{9},
\end{array}$$

since $k(21) = 6$. The process can be continued until the digits start to repeat if $k(m)$ is not known.

To prove the soundness of the procedure, it must be shown that $e_1 + 10be_1 + 10^2be_2 + \ldots + 10^{i-1}be_i \equiv r \pmod{10^i}$, $i = 2, 3, \ldots, k$. Eliminating m from (1) and (2), gives

$$(10b - 1)r = (10^k - 1)\, e_1, \text{ or}$$

$$10br - r = 10^k e_1 - e_1.$$

Then, multiplying (3) by 10b and substituting it for 10br gives

$$10be_1 + 10^2be_2 + \ldots + 10^kbe_k - r = 10^ke_1 - e_1.$$

Thus,
$$e_1 + 10be_1 + 10^2be_2 + \ldots + 10^{i-1}be_{i-1} - r$$
$$= 10^ke_1 - 10^ibe_i - 10^{i+1}be_{i+1} - \ldots - 10^kbe_k,$$

or $e_1 + 10be_1 + 10^2be_2 + \ldots + 10^{i-1}be_{i-1} \equiv r \pmod{10^i}$, $i = 2, 3, \ldots, k$.

Das made the same observation as the writer when he pointed out that the process can still be used when the numerator has a value other than one. The same Bose number b can be used, but the value for $e_1(d_k)$ has to be changed. The reader can see that if

$$\frac{1}{m} = \frac{e_k \ldots e_1}{10^k - 1},$$

then $$\frac{n}{m} = \frac{[e'_k \cdots e'_1]}{10^k - 1}, \text{ and}$$

$ne_1 \equiv e'_1 \pmod{10}$. Therefore, the "End number" for n/m is congruent, modulo 10, to n times the "End number" for 1/m.

B. Misra [13] showed how the "Bose and End numbers", developed by Das, could be used in a different way to find the period for the fraction 1/m. His method consists of dividing the end number e_1 by b, giving rise to a quotient q_1 and a remainder r_1. Then q_1 is placed after r_1, and the number $10r_1 + q_1$ so formed is divided by b, giving rise to a quotient q_2 and a remainder r_2. The process is repeated. As an example to illustrate the method, consider m = 123. Since $3 \cdot 123 = (37 \cdot 10) - 1$, then b = 37 and $e_1 = 3$. Dividing 3 by 37 and continuing the process as indicated above, gives

```
37)3            (0  0  8  1  3
   0             ↑  ↑  ↑  ↑  ↑
r1→30  ←q1       q1 q2 q3 q4 q5
    0
r2→300  ←q2
   296
r3→ 48  ←q3
    37
r4 → 111 ← q4
     111
r5 →  03 ← q5   (The process repeats after this step.)
```

Hence, $1/123 = .\dot{0}081\dot{3}$.

Misra justified his method of using the Bose and end numbers to find the period for 1/m with the following theorem:

<u>Theorem 5.1.</u> (i) Continuation after k steps of the above process will cause the quotients $q_1, q_2, \ldots, q_k$ $(0 \le q_i \le 9)$ to repeat in the same order; and (ii) $[q_1 q_2 \cdots q_k]$ is the period of the recurring decimal for 1/m, i.e., $[q_1 q_2 \cdots q_k] = [d_1 d_2 \cdots d_k]$.

Proof: The process as it is described gives rise to the following equations

$$
(4)\qquad
\begin{aligned}
e_1 &= bq_1 + r_1 \\
10r_1 + q_1 &= bq_2 + r_2 \\
&\vdots \\
10r_{k-1} + q_{k-1} &= bq_k + r_k.
\end{aligned}
$$

Multiplying the k equations (4) by 10^k, 10^{k-1}, ... , 10 respectively and adding gives

$$10^k e_1 + 10^{k-1}q_1 + \dots + 10q_{k-1} + 10^k r_1 + 10^{k-1}r_2 + \dots + 10^2 r_{k-1}$$

$$= 10^k bq_1 + 10^{k-1}bq_2 + \dots + 10bq_k + 10^k r_1 + 10^{k-1}r_2 + \dots + 10^2 r_{k-1} + 10r_k.$$

Now, this equation can be simplified by removing like terms from both sides and substituting

$$[q_1 q_2 \dots q_k] = 10^{k-1}q_1 + 10^{k-2}q_2 + \dots + q_k.$$

Thus, $\qquad 10^k e_1 + [q_1 q_2 \dots q_k] - q_k = 10b[q_1 q_2 \dots q_k] + 10r_k$, or

$$10^k e_1 = (10b - 1)[q_1 q_2 \dots q_k] + (10r_k + q_k).$$

Subtract e_1 from both sides and using (1) gives

$$(5)\qquad (10^k - 1)\, e_1 - e_1 m[q_1 q_2 \dots q_k] = (10r_k + q_k) - e.$$

The equation (5) shows e_1 divides $(10r_k + q_k)$. Let $se_1 = 10r_k + q_k$. Now, dividing the equation (5) by $e_1 m$ and using Corollary 2.4 gives

$$(6)\qquad [d_1 d_2 \dots d_k] - [q_1 q_2 \dots q_k] = (s-1)/m.$$

Equation (6) implies that m divides s-1 and, <u>if it can be shown</u> that $s \leq m$, then s = 1 and, consequently,

$$[q_1 q_2 \dots q_k] = [d_1 d_2 \dots d_k].$$

The latter relation would establish the second part of the theorem and $s = 1$ implies that $r_k = 0$ and $q_k = e_1$. As a consequence of these values for r_k and q_k the process will repeat after the k^{th} step. So, if it can be established that $s \leq m$, then the theorem will be proved.

To show that $s \leq m$, it will be necessary to develop some intermediate inequalities. First, by the way e_1 was defined, it is seen that

(a) $$1 \leq e_1 \leq 9.$$

Thus, $$1 < m \leq e_1 m \leq 9m < 10m - 1.$$

Since $e_1 m = 10b - 1$, then $10b - 1 < 10m - 1$ or $b < m$. This implies

(b) $$1 \leq b \leq m - 1,$$

since b and m are integers. From the division algorithm, it is known that

(c) $$0 \leq r_i \leq b - 1 \text{ (all i)}.$$

Since $e_1 = bq_1 + r_1$, it can be concluded that $q_1 \leq e_1 \leq 9$. Now, if $q_i \leq 9$, then multiplying (c) by 10 and adding q_i to both sides gives,

(d) $$q_i \leq 10r_i + q_i \leq 10b - 1 + (q_i - 9) \leq 10b - 1.$$

From (4) it is seen that $10r_i + q_i = bq_{i+1} + r_{i+1}$, therefore,

$$bq_{i+1} + r_{i+1} \leq 10b - 1.$$

This result implies, $bq_{i+1} \leq 10b - 1$, since $r_{i+1} \geq 0$. Now, the only way that this result can hold is for $q_{i+1} \leq 9$. This completes the steps necessary to say $q_i \leq 9$ for all i by induction, i.e., $q_i \leq 9$ then $q_{i+1} \leq 9$. Consider the case $i = k$, then (d) becomes

(e) $$10r_k + q_k \leq 10b - 1.$$

Since equation (5) showed that e_1 divides $10r_k + q_k$, then s was defined to be that number such that $se_1 = 10r_k + q_k$.

Thus, (e) becomes $se_1 \leq a_1m$ or $s \leq m$.

Misra completes his method by showing the argument for the case when the numerator is some value other than one. The proof is similar to the case just completed and the result is analogous to Das' method.

Finding the Period as Part of Another Number

D. R. Kaprekar [20] gave a method for finding the period of a recurring decimal by looking at the last "few" digits of a certain type of product. The reader will notice that the method uses some of the same principles as "Bose numbers". Also, this method gives another explanation of the "right-to-left" method discussed earlier in this chapter.

It will be necessary to prove a theorem that is used in the method.

Theorem 5.2. Let N be a number having a zero as its last digit. Then in the product

$$Z = (N - 1)(1 + N + N^2 + \dots + N^r),$$

the last (r+1) digits will be nines.

Proof: Since $(1 + N + N^2 + \dots + N^r)$ is a geometric progression, its sum is $(N^{r+1} - 1)/(N-1)$. Therefore,

$$Z = (N - 1)(1 + N + N^2 + \dots + N^r) \text{ becomes,}$$

$$Z = (N - 1)\cdot\frac{(N^{r+1} - 1)}{(N - 1)} = N^{r+1} - 1.$$

But $N = 10\cdot b$ implies Z has 9's for its last (r+1) digits, since $N^{r+1} - 1 = 10^{r+1}b^{r+1} - 1$.

As an illustration let N = 140, then

$$139(1 + 140 + 140^2 + 140^3 + 140^4) = 50722499999$$

which has 9's for the last five digits.

As an example to motivate the general method consider the fraction 1/7. It is first necessary to find the least number e_1 such that $7e_1 + 1$ is divisible by 10. (It is the same as the "End number" discribed earlier.) It is seen that $e_1 = 7$ and $N = 50$. By Theorem 5.2, it is known that the last (r+1) digits in

$$50^{r+1} - 1 = 49(1 + 50 + 50^2 + \dots + 50^r)$$

must be nines. By Corollary 2.4, it is seen that $10^6 - 1$ is divisible by 7. Therefore, the recurring portion in the fraction 1/7 will be the last 6 digits in

$$\frac{50^{r+1} - 1}{7} = 7(1 + 50 + 50^2 + \dots + 50^r)$$

for $r \geq 6$. This conclusion is verified by the following remarks. If the left-hand side of the equation is written as

$$\frac{50^{r+1} - 1}{7} = [d_1 d_2 d_3 \dots d_h],$$

then

$$50^{r+1} - 1 = 7[d_1 d_2 d_3 \dots d_h].$$

Now, the left-hand side has nine's in at least the first (r+1) digits which implies that

$$7 \cdot [d_{h-5} d_{h-4} d_{h-3} d_{h-2} d_{h-1} d_h] = 999999, \text{ or}$$

$$\frac{1}{7} = \frac{[d_{h-5} d_{h-4} d_{h-3} d_{h-2} d_{h-1} d_h]}{999999} .$$

Therefore, $[d_{h-5} d_{h-4} d_{h-3} d_{h-2} d_{h-1} d_h] = 142857$. Taking $r = 6$ and working out the multiplication $7(1 + 50 + 50^2 + \dots + 50^6)$ gives

$$\begin{array}{r} 7 \\ 350 \\ 17500 \\ 875000 \\ 43750000 \\ 2187500000 \\ \underline{109275000000} \\ 101507142857 \end{array}$$

Hence, the recurring portion of 1/7 is 142857. If the value of r has not been determined and the process is continued indifinitely, then the period 142857 is repeated again and again.

For the general case, let p be any odd prime other than 5 and e_1 be the least integer such that $N = e_1p + 1$ is divisible by 10. Then the number

$$N^{r+1} - 1 = (N - 1)(1 + N + N^2 + \ldots + N^r)$$
$$= e_1p(1 + N + N^2 + \ldots + N^r)$$

is divisible by p and the last r+1 digits are nines due to Theorem 5.2. Hence, if $r \geq p - 1$, the last p - 1 digits in $e_1(1 + N + N^2 + \ldots + N^r)$ are such that multiplication by p yields all nine's for the p-1 digits. This implies that the p-1 digits will either be the actual period of 1/p or will contain the period a whole number of times.

The value of e_1 will change as the unit's digit of the prime changes. The number e_1 will be 9, 3, 7, 1 according as the prime ends in 1, 3, 7, or 9. For the prime 19, e_1 is 1 and N is 20. Hence, the last 18 digits in

$$(1) \qquad 1\cdot(1 + 20 + 20^2 + 20^3 + \cdots)$$

will be the period for 1/19. It is noted that the sum in (1) can be found by starting with the first digit "1" and multiplying it by 2 and this product gives the second digit. Then multiply the second digit by

2 and this product gives the third digit. If the process is continued, it has the same effect as finding the period by the "right-to-left" method described earlier.

The last three sections are interrelated but each of the writers has taken a different approach to the problem of finding the period.

Finding the Period of $(1/m)^2$ from $1/m$

D. R. Kaprekar [19] gave a technique for finding the period of $(1/m)^2$ from $1/m$. Assume $k(m) = k$ and $R = [d_1 d_2 \ldots d_k]$, then

$$\frac{1}{m} = \frac{R}{10^k - 1} .$$

Hence,

$$\frac{1}{m^2} = \frac{R^2}{(10^k - 1)^2} = \frac{R^2}{(10^k - 1)} \cdot \frac{1}{(10^k - 1)} .$$

Now, expressing $\frac{R^2}{10^k - 1}$ as $P + \frac{Q}{10^k - 1}$ gives

$$\frac{1}{m^2} = \frac{P}{10^k - 1} + \frac{Q}{(10^k - 1)^2}$$

$$= \frac{P \cdot 10^{-k}}{1 - 10^{-k}} + \frac{Q \cdot 10^{-2k}}{(1 - 10^{-k})^2}$$

$$= P \cdot 10^{-k}(1 - 10^{-k})^{-1} + Q \cdot 10^{-2k}(1 - 10^{-k})^{-2}$$

$$= P \cdot 10^{-k}(1 + 10^{-k} + 10^{-2k} + \ldots) + Q \cdot 10^{-2k}(1 + 2 \cdot 10^{-k} + 3 \cdot 10^{-2k} + \ldots)$$

$$= P \cdot 10^{-k} + (P + Q)10^{-2k} + (P + 2Q)10^{-3k} + \ldots .$$

In practice, Theorem 3.8 indicates that the process continues until k or kp of the digits are determined. For example, let $m = 11$, then $k(11) = 2$ and b of Theorem 3.8 is 1. Thus, $k(11^2) = k(121) = 2 \cdot 11^{2-1} = 22$. Now, $m = 11$ gives $R = 9$, $P = 0$, and $Q = 81$.

$$1/11^2 = 81 \cdot 10^{-4} + 162 \cdot 10^{-6} + 324 \cdot 10^{-8} + \cdots$$

$$\begin{array}{rl}
= & .0081 \\
 & .000162 \\
 & .00000243 \\
 & .0000000324 \\
 & .000000000405 \\
 & .00000000000486 \\
 & .0000000000000567 \\
 & .000000000000000648 \\
 & .00000000000000000729 \\
 & .0000000000000000000810 \\
 & .000000000000000000000891 \\
 & .00000000000000000000000972 \\
 & .0000000000000000000000001053 \\
 & \cdot \\
 & \cdot \\
 & \cdot \\
\hline
= & .\dot{0}08264462809917355371\dot{9}0082\cdots
\end{array}$$

A second interesting example is 1/81 or $1/9^2$. Using the technique as described above, R = 1, P = 0, and Q = 1. Therefore,

$$1/9^2 = 1/(10 - 1)^2 = 10^{-2}/(1 - 10^{-1})^{-2}$$

$$= 10^{-2}(1 + 2 \cdot 10^{-1} + 3 \cdot 10^{-2} + \cdots)$$

$$\begin{array}{rl}
= & .01 \\
 & .002 \\
 & .0003 \\
 & .00004 \\
 & .000005 \\
 & .0000006 \\
 & .00000007 \\
 & .000000008 \\
 & .0000000009 \\
 & .00000000010 \\
 & \cdot \\
 & \cdot \\
 & \cdot \\
\hline
 & .\dot{0}1234567\dot{9}01\cdots
\end{array}$$

Thus, $1/81 = .\dot{0}1234567\dot{9}$.

CHAPTER VI

RELATED TOPICS

Recurring Decimals and Group Theory

In this sector, the reader will find it helpful to have some knowledge of group theory. The intent of the section is to show how the set of possible remainders (numerators), discussed in Chapter III, relates to certain properties of abelian groups. The writer has given those definitions and theorems that will be illustrated by the set mentioned above.

Definition 6.1. A group is a set $G = \{a, b, c, \cdots\}$ for which a binary operation $*$ is defined. This operation is subject to the following laws:

1. Closure. If a and b are in G, then $a * b$ is in G.
2. Associativity. If a, b, and c are in G, then $(a * b) * c = a * (b * c)$.
3. Identity. There exists a unique element e in G (called the identity element) such that for all a in G, $a * e = e * a = a$.
4. Inverse. For every a in G there exists a unique element a^{-1} in G called the inverse of a, such that $a * a^{-1} = a^{-1} * a = e$.

Definition 6.2. A group is said to be an abelian group if it satisfies the following law:

5. Commutativity. If a and b are in G, then $a * b = b * a$.

The following table illustrates that multiplication, modulo 13, over the set $\{1, 2, \ldots, 12\}$ forms an abelian group.

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	1	3	5	7	9	11
3	3	6	9	12	2	5	8	11	1	4	7	10
4	4	8	12	3	7	11	2	6	10	1	5	9
5	5	10	2	7	12	4	9	1	6	11	3	8
6	6	12	5	11	4	10	3	9	2	8	1	7
7	7	1	8	2	9	3	10	4	11	5	12	6
8	8	3	11	6	1	9	4	12	7	2	10	5
9	9	5	1	10	6	2	11	7	3	12	8	4
10	10	7	4	1	11	8	5	2	12	9	6	3
11	11	9	7	5	3	1	12	10	8	6	4	2
12	12	11	10	9	8	7	6	5	4	3	2	1

Definition 6.3. A collection of elements H in G is said to form a subgroup of G if H forms a group relative to the binary operation defined in G.

Consider the set, $H = \{10, 10^2, 10^3, 10^4, 10^5, 10^6\}$. Using the above table, it is found that the residues modulo 13 of the respective elements are $H = \{10, 9, 12, 3, 4, 1\}$. The reader will note that H is also an abelian subgroup of G. Due to the fact that the elements of H are congruent modulo 13 to 10^n, for $n = 1, 2, \ldots, 6$, respectively, the group H is said to be a cyclic subgroup with the "generator" 10. It is also noted that 4 will generate H, but it is the generator 10 that relates the group to the topic of recurring decimals. Recalling from

Chapter III the example where the period of 1/13 was found by the division process, it is seen that the set H is the set of remainders (numerators) found in the "unity-cycle".

Definition 6.4. If G is a group, H a subgroup of G, and a any element in G, then the set of elements ha, h arbitrary in H, is called the right coset generated by a and H. It is denoted by Ha. Similarly aH is called the left coset. (Since all groups discussed are abelian, the right coset Ha will equal the left coset aH. Hence, the writer will use the term "coset" and denote it by Ha.)

As an example to illustrate this definition consider the groups G and H discussed above. H, itself, is a coset, since if a = 1, then h·1 is in H. Actually, if a is in H then ha is H and Ha = H. To find a second coset, let a be any element in G which is not in H. For example, a = 2 since 2 is the smallest such element. Now, H·2 = {2, 6, 8, 5, 7, 11}, since $2 \equiv 1 \cdot 2$, $6 \equiv 3 \cdot 2$, $8 \equiv 4 \cdot 2$, $5 \equiv 9 \cdot 2$, $7 \equiv 10 \cdot 2$, $11 \equiv 12 \cdot 2$, modulo 13. Therefore, the cosets H and H·2 partition G into two disjoint subsets each with 6 elements. In general, if H is a subgroup of G, then the cosets of H partition G into disjoint subsets, each with the same order (number of elements) as H.

It was noted that 10 generated the subgroup H, since 10^j is in H for j = 1, 2, ..., 6. Therefore, the order of 10 is 6, i.e., 10 belongs to 6. But k(13) = 6, since $10^6 \equiv 1 \pmod{13}$. Is this just a coincidence that the subgroup generated by 10 has order k(13)? The answer is "no", since by their respective definitions they will be the same number.

Returning to the cosets of G generated by H, it is seen that in addition to H being the unity-cycle of remainders, H·2 is the other cycle of remainders associated with p = 13. As a second example the reader can note the cycles of remainders for p = 11, found after Corollary 3.14. In this example, H = {1, 10}, since $10^2 \equiv 1 \pmod{13}$. Consequently, there exists five cosets {1, 10}, {2, 9}, {3, 8}, {4, 7}, and {5, 6}. This discussion and examples of cosets suggest the following theorem.

Theorem 6.A. [22] If G is a finite group of order n and H is a subgroup of order r, then r divides n.

The two examples of groups given so far had the operation of multiplication modulo a prime, and the order of the group was one less than the prime. If the modulus is not prime, then the group will not contain all elements less than it. For example, consider the group G formed by multiplication modulo 39. If 3 and 13 are in G, then $3 \cdot 13 \equiv 0$, and 0 must be in G. But, 0 is not in G, hence 3 and 13 are not in G. Actually, it is found that G consists of only those numbers less than 39 and relatively prime to 39. Therefore, the order of G would be $\phi(39) = 24$. The next question is, "What cyclic subgroup H is generated by 10?" H is found to be {1, 10, 22, 25, 16, 4} and its order is 6. This is another illustration for Theorem 6.A.

The cosets of G generated by H are H, H·2, H·7, and H·14. Comparing these sets with the cycle of remainders for 39 (Example F, page 49), it is seen that they are the same.

The three examples can be generalized by a corollary to Theorem 6.A:

Corollary 6.B. [22] If G is the multiplicative group modulo m, whose order will be $\phi(m)$, with H the subgroup generated by 10, whose order will be k(m); then k(m) divides $\phi(m)$.

The conclusion of the corollary was also reached in the proof of Theorem 3.3.

Since k(m) divides $\phi(m)$, there exists a number, say c(m), such that $\phi(m) = k(m)\cdot c(m)$. In terms of the groups G and H, it is said that c(m) is the index of H in G, i.e., c(m) is the number of cosets generated by H. In terms of recurring decimals, c(m) is the number of distinct cycles of remainders (numerators), each of which has its own "distinct" period associated with it.

For the reader who is familiar with permutation groups, it is noted that the cyclic permutation which moves the digits one place to the right in the period of k digits forms a cyclic subgroup of the group of permutation on k objects. These two groups are not abelian.

Diagonalisation Method and Fibonacci Numbers

The term "diagonalisation" is a word given to an operation on a sequence of numbers. D. R. Kaprekar [19] was the first person to use the term. He used the operation in his development of the concept of a demlo number. A demlo number is a positive integer whose digits have the property that they can be partitioned into three parts such that the sum of the first and third (last) parts is a number whose digits are all the same, and the second (middle) part has this same digit as its only digit. For example, 43329 is a demlo number since 4 + 29 = 33 and the second part is made up of only the digit "3". A second example would be 499995, since 4 + 5 = 9 and the "middle" of the number is made up of all 9's.

The concept of a demlo number has been discussed extensively by Kaprekar and other Indian mathematicians since the 1930's. The writer is of the opinion that they are a study in themselves and their main contribution to recurring decimals is in the use of the method of diagonalising a sequence. It should be noted that the term "diagonalisation" was originally called "demlofication" by Kaprekar[4], but he later changed it to the more descriptive term.

Definition 6.5. (i) Right diagonalisation of the sequence $\{a_1, a_2, a_3, \cdots\}$ is defined by the equation

$$\vec{D}_k = a_1 + 10^{-k} \cdot a_2 + 10^{-2k} \cdot a_3 + \cdots$$

(ii) Left diagonalisation of the sequence $\{a_1, a_2, a_3, \cdots\}$ is defined by the equation

$$\overleftarrow{D}_k = a_1 + 10^{k} \cdot a_2 + 10^{2k} \cdot a_3 + \cdots$$

The sequence $\{a_1, a_2, a_3, \cdots\}$ can be an A. P. (arithmetic progression), G. P. (geometric progression), or any sequence formed according to some fixed rule. As an example, consider the A. P. $\{15, 18, 21, 24\}$:

$$\overleftarrow{D}_1 = 15 + 180 + 2100 + 24000 = 26,295$$

$$\vec{D}_1 = 15 + 1.8 + .21 + .024 = 17.034$$

Since the interest is in the digits of the numbers and not the location of the decimal point, the two sums can be found as follows:

Left Diagonalisation	Right Diagonalisation

<pre>   15
  18
 21
24
----- 	
26295</pre>
	<pre>15
	 18
	  21
	   24
	----- 	
	17034</pre>

The sum of the two diagonalisations is 43329, which is a demlo number. Thus, this is a way of obtaining demlo numbers.

Kaprekar [19] pointed out several different ways in which the period of 1/7 can be found using diagonalisation. For example, $\vec{D}_2$ of {14, 28, 56, ...} gives

```
14
  28
    56
     112
       224
         448
           896
--------------
142857142857...
```

The explanation is as follows:

$$\frac{1}{7} = \frac{14}{100-2} = \frac{14}{100}(1 - .02)^{-1}$$

$$= \frac{14}{100}(1 + .02 + .0004 + \cdots)$$

$$= \frac{1}{100}(14 + .28 + .0056 + \cdots)$$

$$= \vec{D}_2(14,\ 28,\ 56,\ \cdots).$$

Similarly, $\frac{1}{7} = \vec{D}(1,\ 3,\ 9,\ \cdots).$

Now, consider $\overleftarrow{D}_1$ of $(7,\ 35,\ 175,\ \cdots)$:

```
        7
       35
     175
    875
  4375
21875
  .
  .
  .
----------
...142857
```

In this case, the common ratio for the G. P. is 50 and, thus, the infinite sum of G. P. will not converge. The explanation, therefore, must be different from the two previous examples. (Note: S_n will be the sum of the first n terms of a G. P.)

$$50S_n = 7\cdot 50^n + 7\cdot 50^{n-1} + \cdots + 7\cdot 50^2 + 7\cdot 50$$

$$S_n = 7\cdot 50^{n-1} + \cdots + 7\cdot 50^2 + 7\cdot 50 + 7$$

$$49S_n = 7\cdot 50^n - 7$$

$$S_n = \frac{50^n - 1}{7}$$

The reader will recall that this is the same example given after Theorem 5.2 and the reasoning is the same as before.

The recurring periods for fractions like 1/19, 1/31, 1/39, 1/891, etc. can also be derived by diagonalisation of certain geometric progressions. The diagonalisation of certain arithmetic progressions will result in such fractions as $1/9^2$ and $1/11^2$. The reader will see that (1) in the section "Finding the Period of $(1/m)^2$ from 1/m" is $\vec{D}_k$ of $\{P, P+Q, P+2Q, \cdots\}$ where k is the number od digits in Q. For example, from this same section it is seen that

$$1/81 = \vec{D}_1 \text{ of } \{1, 2, 3, \cdots\}, \text{ and}$$

$$1/121 = \vec{D}_2 \text{ of } \{81, 162, 243, \cdots\}.$$

Kaprekar [19] gave several examples where the sequence was other than an A. P. or G. P. His first example was $\vec{D}_1$ of $\{1, 3, 6, 10, 15, 21, 28, \cdots\}$. This sequence, which is denoted as "the triangular" number and whose rule of formulation is $n(n+1)/2$, gives rise to the period of $1/9^3$ or 1/729. Since k(729) = 81, the diagonalisation would have a minimum of 82 steps. He also observed that for n a positive integer

$$\frac{1}{9^4} = \frac{1}{6561} = \vec{D}_1 \text{ of } \left\{\frac{n(n+1)(n+2)}{6}\right\};$$

$$\frac{1}{9^5} = \frac{1}{59049} = \vec{D}_1 \text{ of } \left\{\frac{n(n+1)(n+2)(n+3)}{24}\right\} .$$

Also,

$$\frac{37}{243} = \vec{D}_1 \text{ of } \{1, 4, 10, 19, \cdots\}, \text{ or}$$

$$= \vec{D}_1 \text{ of } \{a_n = a_{n-1} + 3(n-1)\}, \text{ where } a_1 = 1.$$

Kaprekar [4] went to great lengths to show that

$$\frac{1}{109} = \overleftarrow{D}_1 \text{ of } \{1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots\}$$

The writer was able to shorten the explanation by using the reasoning which follows from Theorem 5.2. This particular sequence of numbers is known as the Fibanocci numbers, and they have held the fascination of mathematicians down through the years. Their rule for formulation is $a_n = a_{n-1} + a_{n-2}$, where $a_1 = a_2 = 1$.

The explanation is as follows:

$$S_n = a_1 + 10a_2 + 10^2a_3 + \dots + 10^{n-1}a_n$$

$$10S_n = \quad 10a_1 + 10^2a_2 + \dots + 10^{n-1}a_{n-1} + 10^na_n$$

$$9S_n = -a_1 + 10(a_1-a_2) + 10^2(a_2-a_3) + \dots + 10^{n-1}(a_{n-1}-a_n) + 10^na_n$$

$$= 10^na_n - a_1 + 10(a_1-a_2) - 10^2[a_1 + 10a_2 + \dots + 10^{n-3}a_{n-2}]$$

$$= 10^na_n - 1 + 10(1-1) - 10^2[S_n - 10^{n-2}a_{n-1} - 10^{n-1}a_n].$$

Therefore,

$$109S_n = \left[10^n a_n + 10^n a_{n-1} + 10^{n+1} a_n\right] - 1$$

$$= \left[10^n a_{n+1} + 10^{n+1} a_n\right] - 1$$

$$= 10^n\left[a_{n+1} + 10a_n\right] -1, \text{ or}$$

$$S_n = \frac{10^n[a_{n+1} + 10a_n] - 1}{109}$$

Where the last n digits of $10^n[a_{n+1} + 10a_n] - 1$ will be 9's. Therefore, if $n \geq 108$, then the first 108 digits of S_n will be the period for 1/109, since k(109) = 108.

Kaprekar [4] proposed the question, "What would happen if the Lucas numbers, i.e., {1, 3, 4, 7, 11, 18, 29, 47, ...} were diagonalised?" The reader will note that Lucas numbers follow the same rule as Fibonacci numbers, but have 1 and 3 for the first two terms. Rao [27] answered the question as follows:

> It may be noted that in the recurring period of 1/109, the digits from the 101st onward are the numbers of Lucas. In fact, the series of Lucas is also of the Fibonacci type and all types of Fibonacci series can be found in the recurring period of 1/109, if it is written, twice. Thus we do not get anything new by applying the process to Lucas numbers. It is highly remarkable that by applying this process to all types of Fibonacci series we are led to the recurring period of 109 and it is to be noted that 109 is the only prime having this property.

The Cantor Ternary Set

The interest in this section is focused on the set I and special types of subsets of I, where I is the set of all real numbers between 0 and 1, inclusive. If x is an element of I, then

$$x = .d_1 d_2 d_3 \cdots, \text{ or}$$

$$x = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots.$$

Now, the representation of x is dependent on the base of the numeration system as discussed in Chapter II.

The subset of I, known as the Cantor Ternary set, is an interesting application of the use of different bases for the numeration system. The Cantor Ternary set is discussed in most graduate level analysis and topology textbooks, due to its properties. Some of its properties might seem strange to the neophyte mathematician. For example, it is nowhere dense, but it has the "same" number of elements as I.

Definition 6.6. The Cantor Ternary set, C, is the set of all x in I such that when x is represented in its ternary form, i.e., base three numeration system, $d_h \neq 1$ for any positive integer h.

Before discussing the properties of C, it should be instructive to state an alternate definition for C and show that they are equivalent.

Definition 6.7. Let C_1 be the subset of I consisting of all points of I that do not lie in the open interval (1/3, 2/3). Thus, C_1 is obtained by deleting the open middle third of the interval I. Define C_2 to be the subset of C_1 obtained by deleting the open middle third of each of the two intervals that form C_1. Continuing in this manner, define C_n for each positive integer. The Cantor Ternary set C is then defined by the following:

$$C = \bigcap_{n=1}^{\infty} C_n$$

Theorem 6.1. Definition 6.7 if and only if Definition 6.6.

Proof: Assume Definition 6.7 and show Definition 6.6 follows. Now, the rational number $1/3_{ten} = .0\dot{2}_{three}$, since

$$\left[\frac{1}{3} = \frac{1}{3} + \frac{0}{3^3} + \cdots\right]_{ten}$$

$$\left[\frac{1}{10} = \frac{1}{10} + \frac{0}{100} + \frac{0}{1000} + \cdots\right]_{three}$$

$$[1/10 = .100\cdots]_{three}$$

$$[.1 = .0\dot{2}]_{three}$$

Also, $[2/3]_{ten} = [.2]_{three}$. Assume x_1 is an element of C_1' (complement of C_1). Thus, $[.02 < x_1 < .2]_{three}$. This implies the first digit of x_1 in its ternary expansion is 1.

From the definition of C_2, it is seen that

$$C_2' = (1/9, 2/9) \cup (7/9, 8/9).$$

But

$$[1/9]_{ten} = [.01 = .00\dot{2}]_{three},$$

$$[2/9]_{ten} = [.02]_{three},$$

$$[7/9]_{ten} = [.21 = .20\dot{2}]_{three},$$

$$[8/9]_{ten} = [.22]_{three}$$

implies that if x_2 is element of C_2', then the second digit of the ternary expansion of x_2 must be a 1. Similarly, if x_3 is an element of C_3', then the third digit in the ternary expansion is a 1. In general, if x_n is an element of C_n', then the nth digit in the expansion of x_n will be a 1.

From DeMorgan's Law it is known that

$$C' = \left[\bigcap_{n=1}^{\infty} C_n\right]' = \bigcup_{n=1}^{\infty} C'_n.$$

Hence, from the discussion above it is seen that any element of I which contains the digit 1 in its ternary expansion must belong to C'. Thus, if x belongs to the C of Definition 6.7, then x belongs to the C of Definition 6.6.

The reader will note that the reasoning going one way in the proof will also be valid in the other direction as well. Hence, the two definitions are equivalent.

Theorem 6.2. The sum of the lengths of the intervals that form the complement of the Cantor set is one.

Proof: The length of C'_1 is $2/3 - 1/3 = 1/3$. The length of C'_2 is $(8/9 - 7/9) + (2/9 - 1/9) = 1/9 + 1/9 = 2/9$. The length of C'_3 is $(26/27 - 25/27) + (20/27 - 19/27) + (8/27 - 7/27) + (2/27 - 1/27) = 1/27 + 1/27 + 1/27 + 1/27 = 4/27$. This pattern continues and the sum becomes $1/3 + 2/9 + 4/27 + \cdots$, or

$$\frac{1/3}{1 - 2/3} = 1,$$

since it is the infinite sum of a G. P. with a ratio of 2/3.

In terms of measure theory, Theorem 6.2 leads to the interesting result that the measure of the Cantor set is zero. The Cantor set differs from most sets which have a measure of zero since it is not countable, let alone finite. In fact, it has the same cardinality as I.

Theorem 6.3. The cardinality of C is the same as I.

Proof: First, it should be noted if y is an element of I, then y when represented in binary form will have only 0's and 1's used for its digits. Define the function f whose domain is C and whose range is I, where the points of I are represented in binary form. The rule of correspondence is

$$f(x) = \frac{1}{2} \cdot x.$$

Now, x is always divisible by 2, since the digits used in the ternary expansion of x are only 0's and 2's. It is seen that for every element in C there exists a corresponding element in I. Also, since f is monotone, f is one-to-one. Therefore, C and I have the same cardinality.

Corollary 6.3. The Cantor set is uncountable.

Proof: Since there exists a one-to-one correspondence between C and I (See proof of Theorem 6.2), and I is uncountable, then C must be uncountable.

The Cantor set has several properties, with some of which the reader may be familiar. A few of these properties are discussed in terms of their definitions or characterizations.

(I) C is closed, i.e., it contains all its accumulation points [14].

(II) C is compact, i.e., it is closed and bounded [2].

(III) C is nowhere dense, i.e., it is closed and does not contain any interval [14].

(IV) C is perfect, i.e., it is closed and dense-in-itself [14].

(V) The characteristic function of C is Riemann-integrable on [0, 1]. In fact, the integral is equal to zero [2].

CHAPTER VII

SUMMARY AND EDUCATIONAL IMPLICATIONS

Summary

In this paper, material concerning recurring decimals is discussed. This presentation makes the research concerning this topic more readable and more readily available to the student of elementary number theory. It also provides examples of how some of the basic theorems of number theory can be used to prove theorems about recurring decimals.

In Chapter I the statement of the problem, procedure, scope of the paper, and expected outcome are given. Chapter II includes an elementary introduction to the subject along with how the base of the numeration system affects recurring decimals. In Chapter III most of the properties used from number theory are listed and discussed. The theorems in this chapter prove most of the properties of recurring decimals that the writer feels are fundamental to the subject. The basic result is Theorem 3.3, which states that the period length of $1/n$ is k if and only if 10 belongs to k modulo n. Chapter IV provides other properties of recurring decimals. In general, the properties discussed are not basic to the subject. The most important theorem in the chapter is Theorem 4.4, which gives the necessary and sufficient conditions for the reciprocal of a prime to have the maximum number of digits in its period. Chapter V is devoted to different techniques of finding the period of a recurring decimal. Some of the techniques are similar, but the explanation of the justification differs from writer to writer. In Chapter VI, three topics are discussed in terms of how they relate to the subject of recurring decimals. For the neophyte abstract algebra student, the discussion relating cosets to recurring decimals should be of interest.

Educational Implications

Many of the concepts of mathematics, and number theory in particular, can be understood by the laymen and also by elementary and secondary school students. In the interest of mathematics, it is important that some of the more basic concepts be presented to these groups in a systematic manner. A paper such as this, in addition to consolidating some of the literature, presents the necessary background needed for an understanding of the subject, and should bring to more students a better knowledge of recurring decimals.

As a result of reading this thesis, the student should gain an awareness of some of the elementary concepts of number theory and the current and past research that has been done in the area concerning recurring decimals. It is also of significance that the reader who is a potential teacher of mathematics may find material to motivate his class, and perhaps enlarge on some of the concepts presented.

Undoubtedly, the most immediate result of this paper lies in the knowledge and experience gained by the writer in its preparation.

A SELECTED BIBLIOGRAPHY

1. Aitken, A. C., The Case Against Decimalisation, London: Oliver and Boyd, 1962.

2. Apostol, Tom M., Mathematical Analysis, Reading, Mass.: Addison - Wesley Pub. Co., Inc., 1957.

3. Azyangar, A. A. Krishnaswami, "On Primes with Maximum Recurring Periods for Their Reciprocals," Current Science, Vol. 9, (1948), pp. 261-262.

4. Azyanger, A. A. Krishnaswami and Kaprekar, D. R., "Demlofication of Fibonacci Numbers," Journal of the University of Bombay, Vol. 14, (1945), pp. 6-10.

5. Barnard, S. and Child, J. M., Higher Algebra, London. Macmillian and Co., Limited, 1936.

6. Barnett, I. A., "The Theory of Numbers as a Required Course in the College Curriculum for Majors," The American Mathematical Monthly, Vol. 73 (1966), pp. 1002-1004.

7. Batty, J. S., "Some Properties of Pure Recurring Decimals," The Mathematical Gazette, Vol. 38, (1954), pp. 90-95.

8. Bennett, E. R., "Periodic Decimal Fractions," The American Mathematical Monthly, Vol. 14, (1909), pp. 79-81.

9. Berry, E. M., "Problems and Solutions," The American Mathematical Monthly, Vol. 30, (1923), p. 82.

10. Chartres, R., "Notes", Nature, Vol. 18, (1878), p. 291.

11. Das, R. C., "On Bose Numbers," The American Mathematical Monthly, Vol. 56, (1949), pp. 87-89.

12. Dickson, Leonard E., History of the Theory of Numbers, Washington: Carnegie Institute, Vol. I, 1919.

13. Glaisher, J. W. L., "Circulating Decimals," Nature, Vol. 19, (1879), pp. 208-209.

14. Goffman, Casper, Real Functions, New York: Rinehart and Co., Inc., 1953.

15. Green, R. E., "Primes and Recurring Decimals," _The Mathematical Gazette_, Vol. 46, (1962), pp. 25-33.

16. Grümbaum, Hugo, "Digits of Periodic Decimals," _Scripta Mathematica_, Vol. 16, (1950), p. 185.

17. Hardy, G. H. and Wright, E. M., _An Introduction to the Theory of Numbers_, 4th Edition, 1960.

18. Iyer, R. Venkalachalam, "Prime Numbers Whose Reciprocals have the Maximum Recurring Period," _Journal of the University of Bombay_, Vol. 15, (1947), pp. 9-10.

19. Kaprekar, D. R., _Cycles of Recurring Decimals_, Vol. I, II, Shivaji Nagar, Poona, India: Aryabhushan Press, 1950.

20. _____, "Some Properties of Demlo Numbers and Their Application to Recurring Decimals," _The Mathematics Student_, Vol. 7, (1939), pp. 47-51.

21. Long, Calvin T., _Elementary Introduction to Number Theory_, Boston: D. C. Heath and Co., 1965.

22. Miller, Kenneth S., _Elements of Modern Abstract Algebra_, New York: Harper and Row, 1958.

23. Misra, B., "On Bose Numbers," _The American Mathematical Monthly_, Vol. 60, (1953), pp. 319-321.

24. Niven, Ivan, _Numbers: Rational and Irrational_, New York: The L. W. Singer Co., 1961.

25. Nygaard, P. H., "Repeating Decimals," _The Mathematics Teacher_, Vol. 31, (1938), pp. 316-321.

26. Rademacher, Hans and Toeplitz, Otto, _The Enjoyment of Mathematics_, Princeton, N. J.: Princeton University Press, 1957.

27. Rao, K. Subba, "A Note on the Recurring Period of the Reciprocal of an Odd Number," _The American Mathematical Monthly_, Vol. 62, (1955), pp. 484-487.

28. Reichmann, W. J., _The Fascination of Numbers_, London: Methuen, 1957.

29. Rollett, A. P., "The Factors of 11...11, etc.", _The Mathematical Gazette_, Vol. 21, (1937), pp. 410-411.

30. Schiller, James K., "A Theorem in the Decimal Representation of Rationals," _The American Mathematical Monthly_, Vol. 66, (1959), pp. 707-708.

31. Singh, Daljit, "Concerning the Reciprocal of a Prime," The American Mathematical Monthly, Vol. 61, (1954), pp. 32-34.

32. Toy, Edmund P., "Circulating Decimals," Nature, Vol. 18, (1878), p. 541.

33. Tripp, M. O., "Periodic Decimal Fractions," School Science and Mathematics, Vol. 19, (1919), pp. 110-113.

34. Umansky, Harlan L., "A Note on Periodic Decimals," Scripta Mathematica, Vol. 22, (1956), p. 85.

35. Van der Waerden, B. L., Modern Algebra, New York: Frederick Ungar Publishing Co., Vol. I, English Edition, 1949.

36. Wright, Harry N., First Course in Theory of Numbers, New York: J. Wiley and Sons, Inc., 1939.

APPENDIX A

Primes and Cycle Table

The following table is reprinted from *Cycles of Recurring Decimals* [19]. In the table P is the value of the prime and C is the number of cycles for that prime when it is converted into a recurring decimal. Thus,

$$k(P) = \frac{P - 1}{C} .$$

For example when P = 13, it is found in the table that C = 2 and k(13) = (13-1)/2 = 6. If P = 73 then C = 9 and k(73) = (73-1)/9 = 8.

Primes and Cycle Table

P	C	P	C	P	C	P	C	P	C	P	C
3	2	223	1	487	1	787	2	1093	4	1439	2
7	1	227	2	491	1	797	2	097	1	447	1
11	5	229	1	499	1	809	4	103	1	451	5
13	2	233	1	503	1	811	1	109	1	453	2
17	1	239	34	509	1	821	1	117	2	459	9
19	1	241	8	521	10	823	1	123	2	471	2
23	1	251	5	523	2	827	2	129	2	481	2
29	1	257	1	541		829	3	151	2	483	6
31	2	263	1	547		839	2	153	1	487	1
37	12	269	1	557	2	853	4	163	2	489	6
41	8	271	54	563	2	857	1	171	1	493	4
43	2	277	4	569	2	859	33	181	1	499	7
47	1	281	10	571	1	863	1	187	2	511	2
53	4	283	2	577	1	877	2	193	1	523	2
59	1	293	2	587	2	881	2	201	6	531	1
61	1	307	2	593	1	883	2	213	6	543	1
67	2	311	2	599	2	887	1	217	1	549	1
71	2	313	1	601	2	907	6	223	1	553	1
73	9	317	4	607	3	911	2	229	1	559	2
79	6	331	3	613	12	919	2	231	30	567	1
83	2	337	1	617	7	929	2	237	6	571	1
89	2	347	2	619	1	937	1	249	6	579	1
97	1	349	3	631	2	941	1	259	1	583	1
101	25	353	11	641	20	947	2	277	2	597	12
103	3	359	2	643	6	953	1	279	2	601	8
107	2	367	1	647	1	967	3	283	2	607	1
109	1	373	2	653	2	971	1	289	14	609	8
113	1	379	1	669	1	977	1	291	1	613	4
127	3	383	1	661	3	983	1	297	1	619	1
131	1	389	1	673	3	991	2	301	1	621	1
137	17	397	4	677	2	997	6	303	1	627	6
139	3	401	2	683	2	1009	4	307	2	637	4
149	1	409	2	691	3	013	4	319	2	657	3
151	2	419	1	701	1	019	1	321	24	663	1
157	2	421	3	709	1	021	1	327	1	667	2
163	2	431	2	719	2	031	10	361	2	669	3
167	1	433	1	727	1	033	1	367	1	693	4
173	4	439	2	733	12	039	2	873	2	697	1
179	1	443	2	739	3	049	2	381	1	699	3
181	1	449	14	743	1	051	1	399	2	709	1
191	2	457	3	751	6	061	5	409	44	721	4
193	1	461	1	757	28	063	1	423	9	723	6
197	2	463	3	761	2	069	1	427	2	733	2
199	2	467	2	769	4	087	1	429	1	741	1
211	7	479	2	773	4	1091	1	1433	1	1747	6

P	C	P	C	P	C	P	C	P	C	P	C
1753	3	2111	2	2447	1	2801	2	3203	2	3557	14
759	2	113	1	459	1	803	2	209	2	559	2
777	1	129	4	467	18	819	1	217	3	571	1
783	1	131	3	473	1	833	1	221	1	581	1
787	2	137	1	477	4	837	4	229	3	583	3
789	1	141	1	503	9	843	2	251	1	593	1
801	2	143	1	521	4	851	1	253	6	607	1
811	1	153	1	531	55	857	7	257	1	613	6
823	1	161	72	539	1	861	1	259	1	617	1
831	6	179	1	543	1	879	2	271	2	623	1
847	1	203	2	549	1	887	1	299	1	631	2
861	1	207	1	551	6	897	1	301	1	637	4
867	2	213	4	557	4	903	1	307	2	643	2
871	2	221	1	579	1	909	1	313	1	659	1
873	1	237	2	591	10	917	2	319	6	671	10
877	2	239	2	593	1	927	1	323	2	673	1
879	6	243	2	609	2	939	1	329	4	677	2
889	16	251	1	617	1	953	3	331	1	691	3
901	5	267	2	621	1	957	2	343	1	697	3
907	2	269	1	633	1	963	2	347	2	701	1
913	1	273	1	647	3	969	8	359	2	709	1
931	5	281	10	657	1	971	1	361	2	719	2
933	92	287	3	659	3	999	2	371	1	727	1
949	1	293	2	663	1	3001	2	373	4	733	4
951	10	297	1	671	2	011	1	389	1	739	3
973	2	309	1	677	12	019	1	391	2	761	2
979	1	311	10	683	6	023	1	407	1	767	1
987	6	333	4	687	1	037	12	413	2	769	2
993	3	339	1	689	64	041	8	433	1	779	1
997	2	341	1	693	2	049	6	449	8	793	3
999	2	347	2	699	1	061	15	457	9	797	4
2003	2	351	2	707	2	067	2	461	1	803	2
011	3	357	2	711	2	079	2	463	1	821	1
017	1	371	1	713	1	083	2	467	2	823	3
027	2	377	9	719	2	089	2	469	1	833	1
029	1	381	5	729	4	109	21	491	5	847	1
039	2	383	1	731	1	119	2	499	11	851	5
053	6	389	1	741	1	121	20	511	2	853	4
063	1	393	13	749	3	137	1	517	4	863	1
069	1	399	2	753	1	163	2	527	1	877	4
081	2	411	1	767	1	167	1	529	2	881	2
083	2	417	1	777	1	169	44	533	2	889	2
087	7	423	1	789	1	181	5	539	1	907	2
089	2	437	2	791	90	187	18	541	177	911	2
2099	1	2441	8	2797	4	3191	110	3547	2	3917	2

P	C	P	C	P	C	P	C	P	C	P	C
3919	6	4283	2	4679	2	5077	2	5477	4	5849	4
923	2	289	2	691	1	081	4	479	2	851	3
929	8	297	3	703	1	087	1	483	2	857	1
931	3	327	1	721	2	099	1	501	1	861	1
943	1	337	1	723	2	101	3	503	1	867	2
947	2	339	1	729	4	107	2	507	2	869	1
967	1	349	1	733	4	113	3	519	2	879	2
989	1	357	18	751	2	119	6	521	16	881	2
4001	8	363	2	759	2	147	2	527	1	897	1
003	46	373	4	783	1	153	1	531	1	903	1
007	1	391	2	787	2	167	1	557	6	923	2
013	118	397	14	789	21	171	47	563	2	927	1
019	1	409	8	793	1	179	1	569	4	939	1
021	15	421	1	799	2	189	1	573	2	953	3
027	2	423	1	801	6	197	12	581	1	981	1
049	2	441	2	813	6	209	14	591	2	987	2
051	1	447	1	817	1	227	2	623	1	6007	7
057	1	451	1	831	6	231	2	639	2	011	1
073	1	457	1	861	5	233	1	641	12	029	1
079	2	463	1	871	2	237	68	647	3	037	2
091	1	481	2	877	4	261	5	651	1	043	2
093	186	483	18	889	2	273	1	653	2	047	1
099	1	493	4	903	3	279	2	657	1	053	2
111	2	507	6	909	3	281	2	659	1	067	2
127	1	513	3	919	2	297	1	669	1	073	1
129	2	517	2	931	1	303	1	683	2	079	6
133	4	519	6	933	2	309	1	689	18	089	8
139	1	523	2	937	1	323	2	693	4	091	3
153	1	547	2	943	1	333	4	701	1	101	5
157	2	549	3	951	2	347	2	711	10	113	1
159	6	561	2	957	12	351	2	717	4	121	2
177	1	567	1	967	1	381	1	737	1	131	1
201	56	583	1	969	6	387	2	741	1	133	4
211	1	591	2	973	22	393	1	743	1	143	1
217	1	597	2	987	2	399	2	749	1	151	6
219	1	603	2	993	3	407	3	779	1	163	78
229	1	621	5	999	14	413	2	783	1	173	2
231	2	637	76	5003	2	417	1	791	6	197	2
241	4	639	2	009	8	419	1	801	4	199	2
243	2	643	2	011	3	431	2	807	1	203	14
253	4	649	664	021	1	437	4	813	2	211	1
259	1	651	1	023	3	441	2	821	1	217	1
261	1	657	3	039	2	443	6	827	2	221	1
271	2	663	21	051	101	449	2	839	2	229	3
4273	3	4673	1	059	1	5471	10	5843	2	6247	1

P	C	P	C	P	C	P	C	P	C	P	C
6257	1	6659	1	7019	1	7487	1	7867	2	8273	1
263	1	661	1	027	6	489	4	873	4	287	1
269	1	673	1	039	18	499	1	877	2	291	1
271	6	679	2	043	14	507	2	879	2	293	4
277	4	689	4	057	1	517	2	883	2	297	1
287	1	691	1	069	1	523	2	901	1	311	2
299	67	701	1	079	2	529	4	907	2	317	18
301	1	703	1	103	1	537	3	919	2	329	8
311	2	709	1	109	1	541	1	927	1	353	1
317	2	719	2	121	2	547	2	933	2	363	2
323	2	733	2	127	7	549	3	937	1	369	2
329	2	737	1	129	12	559	2	949	1	377	1
337	1	761	4	151	26	561	4	951	2	387	14
343	1	763	42	159	2	573	12	963	2	389	1
353	1	779	1	177	1	577	1	993	3	419	3
359	2	781	5	187	2	583	1	8009	4	423	1
361	4	791	10	193	1	589	7	011	3	429	1
367	1	793	1	207	1	591	2	017	1	431	2
373	6	803	2	211	7	603	6	039	2	443	2
379	3	823	1	213	4	607	1	053	2	447	1
389	1	827	2	219	1	621	15	059	1	461	3
397	82	829	1	229	1	639	2	069	1	467	2
421	3	833	1	237	18	643	2	081	4	501	1
427	6	841	8	243	2	649	4	087	1	513	1
449	4	857	1	247	1	669	27	089	6	521	12
451	3	863	1	253	98	673	1	093	2	527	3
469	7	869	1	283	2	681	4	101	5	537	1
473	1	871	2	297	3	687	1	111	10	539	3
481	24	883	2	307	2	691	1	117	4	543	1
491	5	899	1	309	1	699	1	123	2	563	2
521	8	907	6	321	2	703	1	147	2	573	2
529	6	911	2	331	5	717	4	161	8	581	3
547	6	917	2	333	12	723	6	167	3	597	4
551	2	947	2	349	1	727	1	171	1	599	6
553	1	949	1	351	6	741	9	179	1	609	8
563	2	959	2	369	4	753	1	191	6	623	1
569	4	961	2	393	1	757	4	209	2	627	2
571	1	967	1	411	1	759	2	219	1	629	3
577	3	971	1	417	3	789	3	221	3	641	2
581	5	977	1	433	1	793	1	231	2	647	1
599	2	983	1	451	1	817	1	233	1	663	1
607	3	991	2	457	1	823	1	237	2	669	1
619	1	997	4	459	1	829	1	243	2	677	12
637	14	7001	4	477	2	841	140	263	1	681	10
6653	2	7013	2	7481	10	7853	2	8269	1	8689	4

P	C	P	C	P	C	P	C	P	C	P	C
8689	4	9067	2	9463	3	9859	3	10289	2	10723	2
693	2	091	909	467	2	871	2	301	1	729	18
699	1	103	1	473	1	883	2	303	3	733	4
707	2	109	1	479	2	887	1	313	1	739	1
713	1	127	3	491	1	901	825	321	4	753	21
719	2	133	6	497	1	907	2	331	4	771	5
731	1	137	1	511	6	923	2	333	2	781	1
737	3	151	6	521	16	929	8	337	1	789	1
741	1	157	2	533	4	931	1	343	1	799	2
747	2	161	40	539	1	941	5	357	2	831	2
753	1	173	2	547	2	949	1	369	4	837	172
761	10	181	3	551	10	967	1	391	2	847	1
779	399	187	2	587	2	973	18	399	6	853	2
783	1	199	2	601	2	10007	1	427	2	859	1
803	6	203	2	613	36	009	2	429	11	861	1
807	1	209	4	619	8	037	26	433	1	867	6
819	1	221	1	623	1	039	2	453	2	883	2
821	1	227	2	629	1	061	1	457	1	889	4
831	2	239	2	631	2	067	2	459	1	891	9
837	2	241	2	643	2	069	1	463	1	903	1
839	2	257	1	649	16	079	2	477	6	909	9
849	16	277	2	661	7	091	1	487	1	937	1
861	1	281	10	677	4	093	4	499	1	939	1
863	1	283	6	679	6	099	3	501	3	949	1
867	2	293	4	689	28	103	1	513	1	957	4
887	1	311	2	697	1	111	2	529	2	973	4
893	4	319	2	719	2	133	4	531	1	979	1
923	6	323	2	721	2	139	1	559	2	987	2
929	62	337	3	733	4	141	1	567	1	993	1
933	4	341	1	739	1	151	2	589	1	11003	2
941	3	343	1	743	1	159	2	597	2	027	2
951	2	349	3	749	1	163	2	601	10	047	1
963	2	371	1	767	1	169	2	607	1	057	1
969	2	377	1	769	2	177	1	613	14	059	1
971	1	391	2	781	1	181	1	627	2	069	1
999	2	397	116	787	2	193	7	631	2	071	18
9001	8	403	6	791	2	211	1	639	2	083	6
007	3	413	2	803	2	223	1	651	1	087	23
011	1	419	17	811	1	243	18	657	1	093	4
013	4	421	1	817	1	247	1	663	1	113	3
029	1	431	2	829	1	253	4	667	2	117	4
041	8	433	9	833	1	259	1	687	1	119	2
043	2	437	2	839	2	267	2	691	1	131	1
049	2	439	6	851	1	271	130	709	1	149	1
9059	1	9461	1	9857	1	10273	1	10711	18	11159	2

P	C	P	C	P	C	P	C	P	C	P	C
11161	36	11617	1	12041	2	12457	1	12853	28	13267	18
171	1	621	1	043	6	473	1	889	4	291	1
173	2	633	1	049	2	479	2	893	4	297	1
177	1	657	1	071	34	487	1	899	1	309	3
197	4	677	2	073	1	491	1	901	2	313	1
213	4	681	2	097	3	497	1	911	2	327	3
239	2	689	24	101	1	503	1	917	2	331	1
243	2	699	1	107	2	511	6	919	6	337	1
251	5	701	1	109	3	517	84	923	2	339	1
257	1	717	4	113	1	527	1	941	1	367	1
261	5	719	2	119	2	539	1	953	1	381	1
273	1	731	1	143	1	541	3	959	2	397	2
279	2	743	1	149	1	547	2	967	3	399	14
287	1	777	1	157	6	553	1	973	6	411	1
299	1	779	3	161	2	569	2	979	1	417	3
311	30	783	1	163	2	577	1	983	1	421	1
317	12	789	1	197	4	583	1	13001	8	441	2
321	10	801	4	203	2	589	1	003	2	451	1
329	6	807	1	211	3	601	2	007	1	457	1
351	2	813	2	227	2	611	1	009	6	463	1
353	1	821	1	239	2	613	2	033	3	469	1
369	14	827	2	241	2	619	3	037	2	477	2
383	1	831	70	251	1	637	4	043	2	487	1
393	1	833	1	253	4	641	4	049	4	499	1
399	2	839	2	263	1	647	1	063	1	513	3
411	5	863	1	269	1	653	2	093	6	523	2
423	1	867	2	277	4	659	1	099	1	537	1
437	4	887	1	281	2	671	70	103	1	553	7
443	6	897	1	289	32	689	16	109	1	567	3
447	1	903	1	301	5	697	1	121	2	577	1
467	2	909	1	323	2	703	3	127	1	591	10
471	2	923	2	329	4	713	1	147	14	597	4
483	2	927	1	343	3	721	6	151	10	613	2
489	4	933	2	347	2	739	3	159	6	619	11
491	15	939	1	373	2	743	1	163	2	627	2
497	1	941	1	377	1	757	6	171	3	633	3
503	1	953	1	379	1	763	18	177	1	649	16
519	2	959	2	391	2	781	1	183	3	669	1
527	3	969	34	401	10	791	2	187	2	679	2
549	1	971	1	409	2	799	6	217	1	681	4
551	6	981	1	413	2	809	2	219	3	687	3
579	1	987	2	421	1	821	1	229	1	691	1
587	6	12007	3	433	3	823	1	241	8	693	42
593	1	011	1	437	2	829	3	249	46	697	1
11597	2	12037	3	12451	1	12841	2	13259	7	13709	1

VITA

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